Post-quantum cryptography II

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CryptoWorks21

Agenda

- Multivariate quadratic Public Key Cryptosystems (MPKC)

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Why quadratic?

Given a system of cubic (or higher degree) equations

$$x_1\mathbf{x_2}\mathbf{x_3} + 2x_1x_3 - 3\mathbf{x_2^2}\mathbf{x_3} - 7 = 0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$-x_1x_2 + x_1x_3 + 6\mathbf{x_3^3} - 7 = 0$$

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- One can always do **degree reduction** or **linearization** by introducing new variables $\color{red}{x_4} = x_2x_3$ and $\color{blue}{x_5} = x_3^2$

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- Solve the quadratic system for the new variables and backtrack.

Multivariate quadratic public key cryptosystems (MPKC)

Notation

Let $\mathbb{F}_q$ denote a finite field of q elements where q is a prime power.
A generic quadratic map $p : (\mathbb{F}_q)^n \to \mathbb{F}_q$ can be represented by a quadratic polynomial

$$p(x_1, \cdots, x_n) = \sum_{1 \le i \le j \le n} \alpha_{ij} x_i x_j + \sum_{1 \le i \le n} \beta_i x_i + \gamma$$

where $\alpha_{ij}, \beta_i, \gamma \in \mathbb{F}_q$.

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- A useful matrix representation can also be used:

$$[x_1, \cdots, x_n] \begin{bmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ 0 & \alpha_{22} & \dots & \alpha_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \alpha_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \sum_{1 \le i \le j \le n} \alpha_{ij} x_i x_j$$

Multivariate quadratic public key cryptosystems (MPKC)

$\mathcal{MQ}$ problem: **NP-hard** [Patarin and Goubin 1997]

- Given a system of m random quadratic equations in n variables over a finite field $\mathbb{F}_q$ of any characteristic

$$\begin{cases} p_1(x_1, \cdots, x_n) = y_1 \\ p_2(x_1, \cdots, x_n) = y_2 \\ \cdots\cdots\cdots\cdots = \cdots \\ p_m(x_1, \cdots, x_n) = y_m \end{cases}$$

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  finding any solution $\mathbf{x} \in (\mathbb{F}_q)^n$ is NP-hard (red. from 3-SAT).
- It is believed to be hard **on average** from experiments, although no reduction exist.

Multivariate quadratic public key cryptosystems (MPKC)

Public key sizes

- Usually, public keys P consist of m quadratic polynomials of shape:

$$p_i(x) = [x_1, \cdots, x_n] \begin{bmatrix} \alpha_{11}^{(i)} & \alpha_{12}^{(i)} & \dots & \alpha_{1n}^{(i)} \\ 0 & \alpha_{22}^{(i)} & \dots & \alpha_{2n}^{(i)} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \alpha_{nn}^{(i)} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

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  - Stick to $d = 2$.

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(Generic) Encryption

- Public key:

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- To decrypt, a trapdoor is needed to solve the system for **x**:

$$x_1, \cdots, x_n = P^{-1}(c_1, \cdots, c_m)$$

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- To sign a document $D \in \{0,1\}^*$, compute the hash $\{h_1, \cdots, h_m\} = H(D)$ and

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- To verify σ, recompute $\{h_1, \cdots, h_m\} = H(D)$ and check

$$p_i(\sigma) \stackrel{?}{=} h_i, \quad 1 \leq i \leq m$$

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MPKC **Trapdoor**

- Let $\mathcal{S} : \mathbb{F}_q^n \to \mathbb{F}_q^n$ and $\mathcal{T} : \mathbb{F}_q^m \to \mathbb{F}_q^m$ be two invertible linear maps

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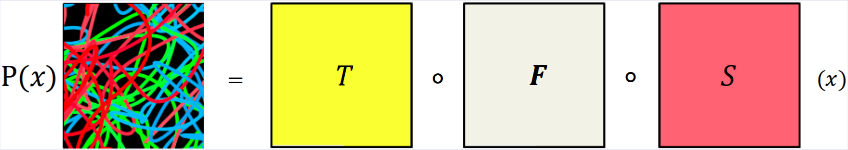

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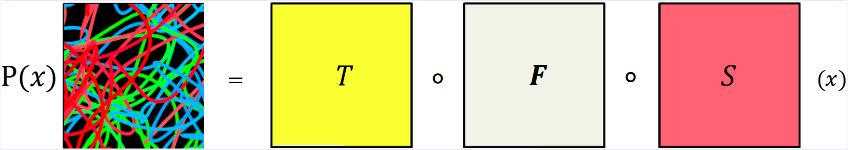

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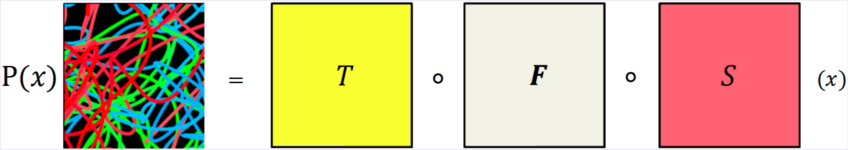

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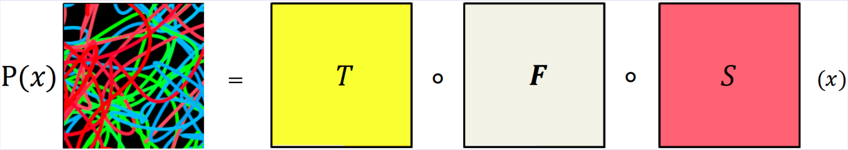

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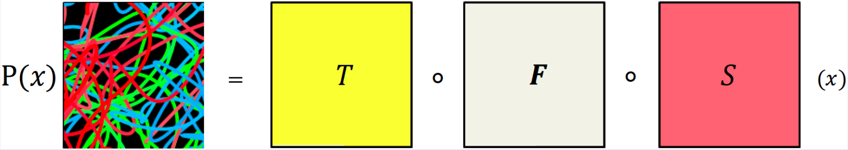

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- Related to the Isomorphism of Polynomials problem.

Multivariate quadratic public key cryptosystems (MPKC)

Isomorphism of Polynomials Problem (**IP-problem**)

Two systems of equations/polynomials $\mathcal{U}, \mathcal{V} : \mathbb{F}_q^n \to \mathbb{F}_q^m$ are called **isomorphic** (up to linear transforms) iif $\exists$ linear maps $\mathcal{L}_1, \mathcal{L}_2$ s.t.

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IP is in **NP** $\cap$ **co-NP** !

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Extended Isomorphism of Polynomials (**EIP-problem**)

Given a public key $\mathcal{P} = \mathcal{T} \circ \mathcal{F} \circ \mathcal{S}$, find a map $\overline{\mathcal{F}}$ isomorphic to $\mathcal{P}$, i.e.,

$$\mathcal{P} = \overline{\mathcal{T}} \circ \overline{\mathcal{F}} \circ \overline{\mathcal{S}}$$

for some invertible $\overline{\mathcal{T}}$ and $\overline{\mathcal{S}}$, and $\overline{\mathcal{F}}$ inherits the trapdoor structure of $\mathcal{F}$

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- Attacks are exponential when $m \approx n$
- Grobner bases complexity $f(q, m, n)$:

$$f(q, m, n) = O\left(m \cdot \binom{n + d_{reg} - 1}{d_{reg}}^{\omega}\right)$$

where d_{reg} is degree of regularity of the system and $2 < \omega \leq 3$.

(cont. Important attacks $\cdots$)

- **Minrank** attacks. Find a low rank quadratic map.

MinRank

Given a set of n matrices M_i, find a nontrivial solution $a_1, \cdots a_n$ s.t.

$$\sum_{i=1}^{n} a_i M_i$$

is of minimum rank.

Finding a low rank matrix implies that we have less independent equations $\Rightarrow$ more variables per equation can make the system easier to solve.

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Encryption: Requires $\mathbf{m} \geq \mathbf{n}$

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- If $m \geq n$ (not undetermined) then we ensure that $\mathcal{F}$ is more or less injective and decryption is not mapped to many different plaintexts.

Encryption: The Matsumoto-Imai'88 **trapdoor**

- Define the central map to be

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- Notice that $X^{q^\theta+1}$ is a quadratic transformation since q^θ is linear (q^θ-Frobenius).
- The quadratic transformation takes place in the big "hidden field" $\mathbb{F}_{q^n}$ instead of the vector space over the smaller field, i.e., $\mathbb{F}_q^n$

(cont. $\cdots$)

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- Therefore the quadratic map is invertible. The KeyGen, encryption and decryption for MI are done as generically explained before.

(cont. $\cdots$)

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  - Later, Kipnis and Shamir showed that A cannot be too small otherwise minrank+linearization attacks apply.
  - But if A is increased decryption becomes too slow.
  - 2003, Faugere and Joux improved the attacks using **F4** and made the system impractical.

Multivariate quadratic public key cryptosystems (MPKC)

Signature: The UOV **trapdoor**, Kipnis, Patarin, and Goubin 1999

- Goal: $\mathcal{F}^{-1}(h) = (x_1, \cdots, x_n)$

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- Goal: $\mathcal{F}^{-1}(h) = (x_1, \cdots, x_n)$
- Let $o, v \in \mathbb{N}$, define $n = o + v$ and $m = o$
- Write the quadratic polynomials as the following:

$$p(x_1, \cdots, x_n) = \underbrace{\sum_{i=1}^{v} \sum_{j=i}^{v} \alpha_{ij} x_i x_j}_{v \times v \textit{ terms}} + \underbrace{\sum_{i=1}^{v} \sum_{j=v+1}^{n} \beta_{ij} x_i x_j}_{v \times o \textit{ terms}} + \sum_{i=1}^{n} \gamma_i x_i + \delta$$

where $(x_1, \cdots, x_v)$ are the *vinegar* variables and are $(x_{v+1}, \cdots, x_n)$ are the **oil** variables.

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- Notice *oil* variables are **not mixed** with themselves. // Easier to see using matrix notation.

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How to invert the UOV trapdoor

- To invert guess at random the vinegars $(x_1, \cdots, x_v) \in_R \mathbb{F}_q^v$

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- If the system has no solution, try another guess for $(x_1, \cdots, x_v)$

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- Security note: in practice pick $\mathbf{v} \approx \mathbf{2o}$.
  - The case where $o \leq v$ (balanced oil and vinegar) was broken by Kipnis and Shamir in 1998.
  - Complexity of the attack for $v > o$ is $O(q^{v-o} \cdot o^4)$

UOV parameter sizes [from A. Petzoldt, 2017]

security level (bit)	scheme	public key size (kB)	private key size (kB)	hash size (bit)	signature (bit)
80	UOV(GF(16),40,80)	144.2	135.2	160	480
	UOV(GF(256),27,54)	89.8	86.2	216	648
100	UOV(GF(16),50,100)	280.2	260.1	200	600
	UOV(GF(256), 34,68)	177.8	168.3	272	816
128	UOV(GF(16),64,128)	585.1	538.1	256	768
	UOV(GF(256),45,90)	409.4	381.8	360	1,080
192	UOV(GF(16),96,192)	1,964.3	1,786.7	384	1,152
	UOV(GF(256),69,138)	1,464.6	1,344.0	552	1,656
256	UOV(GF(16),128,256)	4,644.1	4,200.3	512	1,536
	UOV(GF(256),93,186)	3,572.9	3,252.2	744	2,232

How to invert the UOV trapdoor

- UOV was proposed in 1999 and has not suffured major attacks.
- Faster than ECDSA to sign. $2 - 4\times$ faster to sign, $10 - 20\times$ faster for verifying.
- Signature sizes are less than $1 KiB$
- Public keys are large: tens or hundreds KiB

Multivariate quadratic public key cryptosystems (MPKC)

Rainbow signature

- Proposed by Ding and Schmidt 2005.
- It is a generalization of UOV.
- Idea: split private quadratic maps into layers.
  - Solve more but smaller systems of equation.
  - *Vinegars* for the next layer will be the the *vinegars* + *oils* from the previous one.

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- The central map $\mathcal{F}$ consists of $m = n - v_1$ polynomials $f^{v_1+1}, \cdots, f^{(n)}$

$$f^{(k)} = \sum_{i,j \in V_\ell} \alpha_{ij}^{(k)} x_i x_j + \sum_{i \in V_\ell, j \in O_\ell} + \beta_{ij}^{(k)} x_i x_j + \sum_{i \in V_\ell \cup O_\ell} \gamma_i^{(k)} x_i + \delta^{(k)}$$

where ℓ is the only integer s.t. $k \in O_\ell$.

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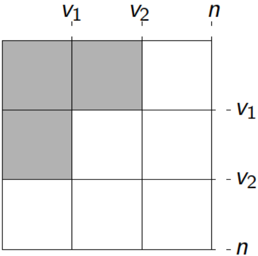
- Choose invertible linear maps $\mathcal{T} : \mathbb{F}_q^m \to \mathbb{F}_q^m$ and $\mathcal{S} : \mathbb{F}_q^n \to \mathbb{F}_q^n$
- Public key is $\mathcal{P} = \mathcal{T} \circ \mathcal{F} \circ \mathcal{S}$.

Rainbow central map with 2 layers

$F^{(k)} =$



$v_1 + 1 \leq k \leq v_2$

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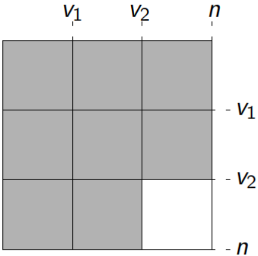

$v_2 + 1 \leq k \leq n$

- $x_1, \cdots, x_{v_1}, x_{v_1+1}, \cdots, x_{v_1+o_1}$ will be the vinegars $x_1, \cdots, x_{v_2}$ for the second layer.

Rainbow (2 layers): Toy example [from A. Petzoldt, 2017]

- $\mathbb{F}_7$, $(v_1, o_1, o_2) = (2, 2, 2)$, $m = n - v_1 = 4$
- Central map $\mathcal{F} = (f^{(3)}, f^{(4)}, f^{(5)}, f^{(6)})$ with

$$
\begin{aligned}
f^{(3)} &= x_1^2 + 3x_1x_2 + 5x_1x_3 + 6x_1x_4 + 2x_2^2 + 6x_2x_3 + 4x_2x_4 + 2x_2 + 6x_3 \\
&\quad + 2x_4 + 5 \\
f^{(4)} &= 2x_1^2 + x_1x_2 + x_1x_3 + 3x_1x_4 + 4x_1 + x_2^2 + x_2x_3 + 4x_2x_4 + 6x_2 + x_4 \\
f^{(5)} &= 2x_1^2 + 3x_1x_2 + 3x_1x_3 + 3x_1x_4 + x_1x_5 + 3x_1x_6 + 6x_1 + 4x_2^2 \\
&\quad + x_2x_3 + 4x_2x_4 + x_2x_5 + 3x_2x_6 + 3x_2 + 3x_3x_4 + x_3x_5 + 2x_3x_6 \\
&\quad + 2x_3 + 3x_4x_5 + x_5 + 6x_6 \\
f^{(6)} &= 2x_1^2 + 5x_1x_2 + x_1x_3 + 5x_1x_4 + 5x_1x_6 + 6x_1 + 5x_2^2 + 3x_2x_3 \\
&\quad + 5x_2x_5 + 4x_2x_6 + x_2 + 3x_3^2 + 5x_3x_4 + 4x_3x_5 + 2x_3x_6 + 4x_3 + x_4^2 \\
&\quad + 6x_4x_5 + 3x_4x_6 + 4x_4 + 4x_5 + x_6 + 2
\end{aligned}
$$

- Goal: Compute the preimage $\mathbf{x} \in \mathbb{F}_7^6$ for $\mathbf{y} = (6, 2, 0, 5)$ under $\mathcal{F}$.

Rainbow: Toy example [from A. Petzoldt, 2017] (cont. $\cdots$)

- Choose random values for the Vinegar variables x_1 and x_2, e.g. $(x_1, x_2) = (0, 1)$ and substitute them into the polynomials $f^{(3)}, \ldots, f^{(6)}$.

$$
\begin{aligned}
\tilde{f}^{(3)} &= 5x_3 + 6x_4 + 2, \tilde{f}^{(4)} = x_3 + 5x_4, \\
\tilde{f}^{(5)} &= 3x_3x_4 + x_3x_5 + 2x_3x_6 + 3x_3 + 3x_4x_5 + 4x_4 + 2x_5 + 2x_6, \\
\tilde{f}^{(6)} &= 3x_3^2 + 5x_3x_4 + 4x_3x_5 + 2x_3x_6 + x_4^2 + 6x_4x_5 + 3x_4x_6 + 4x_4 + 2x_5 + 5x_6 + 1.
\end{aligned}
$$

- Set $\tilde{f}^{(3)} = y_1 = 6$ and $\tilde{f}^{(4)} = y_2 = 2$ and solve for x_3, x_4 $\Rightarrow (x_3, x_4) = (3, 4)$
- Substitute into $\tilde{f}^{(5)}$ and $\tilde{f}^{(6)}$ $\Rightarrow \tilde{\tilde{f}}^{(5)} = 3x_5 + x_6 + 5, \tilde{\tilde{f}}^{(6)} = 3x_5 + 2x_6 + 1$
- Set $\tilde{\tilde{f}}^{(5)} = y_3 = 0$ and $\tilde{\tilde{f}}^{(6)} = y_4 = 5$, solve for x_5 and x_6 $\Rightarrow (x_5, x_6) = (0, 2)$

A pre image of $\mathbf{y} = (6, 2, 0, 5)$ is given by $\mathbf{x} = (0, 1, 3, 4, 0, 2)$.

Rainbow parameter sizes [from A. Petzoldt, 2017]

(cont. $\cdots$)

security level (bit)	parameters $\mathbb{F}, v_1, o_1, o_2$	public key size (kB)	private key size (kB)	hash size (bit)	signature (bit)
80	GF(16),17,20,20	33.4	22.3	160	228
	GF(256),19,12,13	25.3	19.3	200	352
100	GF(16),22,25,25	65.9	43.2	200	288
	GF(256), 27,16,16	57.2	44.3	256	472
128	GF(16),28,32,32	136.6	87.6	256	368
	GF(256),36,21,22	136.0	102.5	344	632
192	GF(16),45,48,48	475.9	301.8	384	564
	GF(256),58,33,34	523.5	385.5	536	1,000
256	GF(16),66,64,64	1,194.4	763.9	512	776
	GF(256),86,45,46	1,415.7	1,046.3	728	1,416

References I

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