# Post-Quantum Cryptography QIC 891

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# Isogenies

Let $E_1, E_2$ be elliptic curves over a finite field $\mathbb{F}_q$.

### Isogeny

An rational map

$$\phi : E_1 \to E_2$$

that preserves the group structure.

- a nonzero isogeny is surjective
- an isogeny is uniquely determined by its kernel

$$0 \to H \to E \xrightarrow{\phi} E' \to 0$$

# Endomorphisms

- An *endomorphism* is an isogeny $\phi : E \to E$.

**Example**

- Multiplication by an integer

$$\begin{array}{rccc} [m]: & E & \longrightarrow & E \\ & P & \longmapsto & mP \end{array}$$

- Frobenius

$$\begin{array}{rccc} \pi: & E & \longrightarrow & E \\ & (x, y) & \longmapsto & (x^q, y^q) \end{array}$$

# The endomorphism ring

- The endomorphisms form a ring denoted $\mathrm{End}_k(E)$.
- We always have $\mathbb{Z} \subseteq \mathrm{End}_k(E)$.

## Theorem

$\mathbb{Q} \otimes \mathrm{End}_{\bar{k}}(E)$ *is isomorphic to one of the following*

ordinary case: $\mathbb{Q}$ *(only possible if* $\mathrm{char}\, k = 0$*)*,

ordinary case (complex multiplication): *an imaginary quadratic field*,

supersingular case: *a quaternion algebra (only possible if* $\mathrm{char}\, k \neq 0$*)*.

## Corollary

$\mathrm{End}(E)$ *is isomorphic to an order* $\mathcal{O} \subset \mathbb{Q} \otimes \mathrm{End}(E)$.

# Computing isogenies

An isogeny $\phi$ is represented as a rational function

$$\frac{N(x)}{D(x)} = \frac{x^n + \cdots + n_1 x + n_0}{x^{n-1} + \cdots + d_1 x + d_0} \in k(x), \qquad \text{with } n = \deg \phi,$$

and $D(x)$ vanishes on $\ker \phi$.

**The explicit isogeny problem**

Input: A *description* of the isogeny (e.g, its kernel).

Output: The curve $E/H$ and the rational fraction $N/D$.

**The explicit isogeny evaluation problem**

Input: A *description* of the isogeny $\phi$, a point $P \in E(k)$.

Output: The curve $E/H$ and $\phi(P)$.

# Computing isogenies (cryptanalysis)

**The implicit isogeny problem**

Input: Two isogenous curves $E_1, E_2$.

Output: An isogeny $E_1 \to E_2$.

**The implicit isogeny evaluation problem**

Input: Two isogenous curves $E_1, E_2$, and a point $P \in E_1(k)$,

Output: The image $\phi(P) \in E_2(k)$ under the isogeny.

# Isogeny graphs

Consider the graph $G_\ell(k) = (V, D)$ where

- $V$: the set of elliptic curves over a given field $k$
- $D$: the set of $\ell$-isogenies between elements of $V$

We want to study the graph of elliptic curves with isogenies *up to isomorphism.* We say two isogenies $\phi, \phi'$ are isomorphic if:

$$\begin{tikzcd} E \arrow[r, "\phi"] \arrow[dr, "\phi'"'] & E' \arrow[d, leftrightarrow, "\wr"] \\ & E' \end{tikzcd}$$

# Isogenies up to endomorphism

In some cases we want to identify edges between the same vertices. We say two isogenies $\phi, \phi'$ are in the same class if there exist endomorphisms $a$ and $b$ of $E$ and $E'$ such that:

$$\begin{tikzcd} E \arrow[r, bend left, "\phi'"] \arrow[r, bend right, "\phi"'] & E' \end{tikzcd} \qquad \begin{tikzcd} E \arrow[r, "\phi"] \arrow[d, "a"'] & E' \arrow[d, "b"] \\ E \arrow[r, "\phi'"'] & E' \end{tikzcd}$$

## Facts

- This is an equivalence relation.
- Two isogenies are in the same class if and only if they have the *same domain and codomain*.

# Dual isogenies

**Theorem:** for any isogeny $\phi : E \to E'$ there exists $\hat{\phi}$

$$\begin{tikzcd} E \arrow[r, "\phi"] \arrow[d, "{[m]}"'] & E' \arrow[dl, "\hat{\phi}"] \\ E & \end{tikzcd}$$

- $\hat{\phi}$ is called the dual isogeny, $\deg \phi = \deg \hat{\phi} = m$.
- $\hat{\hat{\phi}} = \phi$.

## Obvious corollaries:

- $\phi(E[m]) = \ker \hat{\phi}$.
- Graphs of isogenies are undirected.

# Structure of the graph[1]

### Theorem (Serre-Tate)

*Two curves are isogenous over a finite field $k$ if and only if they have the same number of points on $k$.*

The graph of isogenies of prime degree $\ell \neq p$

*Ordinary case*

- Nodes can have degree $0, 1, 2$ or $\ell + 1$.
- Connected components form so called *volcanoes*.

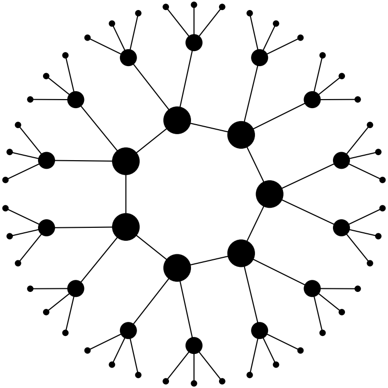
*Supersingular case*

- The graph is $(\ell + 1)$-regular.
- There is a unique connected component made of all supersingular curves with the same number of points.

[1] Kohel 1996; Fouquet and Morain 2002.

# Isogeny graphs (ordinary curves)

*Example:* Finite field, graph of 3-isogenies.



# Isogeny graphs (ordinary curves)

Let $\mathrm{End}(E) = \mathcal{O} \subset \mathbb{Q}(\sqrt{d})$ be the endomorphism ring of $E$. Define

- $\mathcal{I}(\mathcal{O})$, the group of *invertible fractional ideals*,
- $\mathcal{P}(\mathcal{O})$, the group of *principal ideals*,

**Definition (The class group)**

The class group of $\mathcal{O}$ is

$$\mathrm{Cl}(\mathcal{O}) = \mathcal{I}(\mathcal{O})/\mathcal{P}(O).$$

- It is a finite abelian group.
- It arises as the Galois group of an abelian extension of $\mathbb{Q}(\sqrt{d})$.

# Isogeny (classes) = ideal (classes)

## Definition

Let

- $\mathfrak{a}$ be a fractional ideal of $\mathcal{O}$;
- $E[\mathfrak{a}]$ be the the subgroup of $E(\bar{k})$ annihilated by $\mathfrak{a}$;
- $\phi : E \to E/E[\mathfrak{a}]$.

Then $\deg \phi = \mathcal{N}(\mathfrak{a})$. We denote by $*$ the action on the set of elliptic curves.

$$\mathfrak{a} * j(E) = j(E/E[\mathfrak{a}]).$$

## Theorem

*The action* $*$ *factors through* $\mathrm{Cl}(\mathcal{O})$*. It is faithful and transitive.*

# DH-like key exchange based on (semi)-group actions

Let $G$ be an abelian group acting (faithfully and transitively) on a set $X$.

$$\begin{tikzcd}
 & x_0 \arrow[dl, "g"'] \arrow[dr, "h"] & \\
g \cdot x_0 \arrow[dr, "h"'] & & h \cdot x_0 \arrow[dl, "g"] \\
 & gh \cdot x_0 = hg \cdot x_0 &
\end{tikzcd}$$

# DH using class groups[2]

*Public data:*

- $E/\mathbb{F}_p$ ordinary elliptic curve with complex multiplication field $\mathbb{K}$,
- primes $\ell_1$,$\ell_2$ not dividing $\mathrm{Disc}(E)$ and s.t. $\left(\frac{D_{\mathbb{K}}}{\ell_i}\right) = 1$.
- A *direction* on the isogeny graph (a Frobenius eigenvalue).

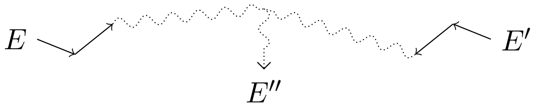
*Secret data:* Random walks $\mathfrak{a}, \mathfrak{b}$ in the $\ell_i$-isogeny graphs.

$$\begin{tikzcd}
 & E \arrow[dl, "\ell_1^{a_1}\ell_2^{a_2} = \mathcal{N}(\mathfrak{a})"'] \arrow[dr, "\mathcal{N}(\mathfrak{b}) = \ell_1^{b_1}\ell_2^{b_2}"] & \\
\mathfrak{a} * E \arrow[dr] & & \mathfrak{b} * E \arrow[dl] \\
 & \mathfrak{a}\mathfrak{b} * E = \mathfrak{b}\mathfrak{a} * E &
\end{tikzcd}$$

[2] Rostovtsev and Stolbunov 2006.

# Isogeny walks and cryptanalysis

Classic meet in the middle attack[3]




## Fourth root attacks

- Start two random walks from the two curves and wait for a collision.
- Over $\mathbb{F}_q$, the average size of an isogeny class is $h_\Delta \sim \sqrt{q}$.
- A collision is expected after $O(\sqrt{h_\Delta}) = O(q^{\frac{1}{4}})$ steps.

[3] Galbraith 1999; Galbraith, Hess, and Smart 2002; Charles, Lauter, and Goren 2009; Bisson and Sutherland 2011.

# Hidden Subgroup Problem (quantum)

Let $G$ be a group, $X$ a set and $f : G \to X$. We say that $f$ *hides* a subgroup $H \subset G$ if

$$f(g_1) = f(g_2) \Leftrightarrow g_1 H = g_2 H.$$

**Definition (Hidden Subgroup Problem (HSP))**

Input: $G, X$ as above, an oracle computing $f$.

Output: generators of $H$.

**Theorem**

*If $G$ is abelian, then*

- $HSP \in poly_{BQP}(\log |G|)$,
- *using $poly(\log |G|)$ queries to the oracle.*

# Hidden Subgroup Problem (quantum)

## Known reductions

- Discrete Log on $G$ of size $p \to$ HSP on $(\mathbb{Z}/p\mathbb{Z})^2$,
- hence DH, ECDH, etc. are broken by quantum computers.
- Semigroup-DH on $G \to$ HSP on the dihedral group $G \ltimes \mathbb{Z}/2\mathbb{Z}$.

## Quantum algorithms for dihedral HSP

Kuperberg[a]: $2^{O(\sqrt{\log|G|})}$ quantum time, space and query complexity.

Regev[b]: $L_{|G|}(\frac{1}{2}, \sqrt{2})$ quantum time and query complexity,
$\mathrm{poly}(\log(|G|)$ quantum space.

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[a] Kuperberg 2005.
[b] Regev 2004.

# R&S key exchange

Key generation: compose small degree isogenies
polynomial in the lenght of the random walk.

Attack: find an isogeny between two curves
polynomial in the degree, exponential in the length.

Quantum[4]: HSP + isogeny evaluation
subexponential in the length of the walk.

[4] Childs, Jao, and Soukharev 2010.

# Supersingular curves

$\mathbb{Q} \otimes \mathrm{End}(E)$ is a quaternion algebra (non-commutative)

## Facts

- Every supersingular curve is defined over $\mathbb{F}_{p^2}$.
- $E(\mathbb{F}_{p^2}) \simeq (\mathbb{Z}/(p+1)\mathbb{Z})^2$ (up to twist, and overly simplifying!).
- There are $g(X_0(p)) + 1 \sim \frac{p+1}{12}$ supersingular curves up to isomorphism.
- There is a unique isogeny class of supersingular curves over $\bar{\mathbb{F}}_p$ (there are two over any finite field).
- The graph of $\ell$-isogenies is $\ell + 1$-regular.

# R&S key exchange with supersingular curves

- there is no action of a commutative class group.
- left ideals of $\mathrm{End}(E)$ still act on the isogeny graph:

$$\begin{tikzcd} E \arrow[r, "\mathfrak{a}"] \arrow[d, "\mathfrak{b}"'] & E' \arrow[d, "\mathfrak{b}_{\mathfrak{a}}"] \\ E'' \arrow[r, "\mathfrak{a}_{\mathfrak{b}}"'] & E''' \end{tikzcd}$$

- The action factors through the *right-isomorphism* equivalence of ideals.

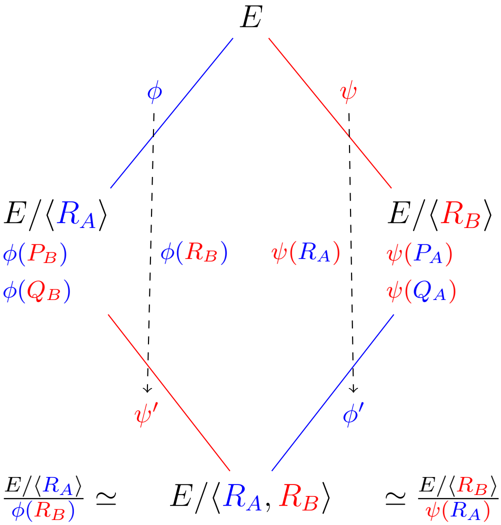
# The SIDH System[5]

*Public data:*

- Prime $p$ such that $p+1 = \ell_A^a \ell_B^b$;
- Supersingular curve $E \simeq (\mathbb{Z}/(p+1)\mathbb{Z})^2$;
- $E[\ell_A^a] = \langle P_A, Q_A \rangle$;
- $E[\ell_B^b] = \langle P_B, Q_B \rangle$.

*Secret data:*

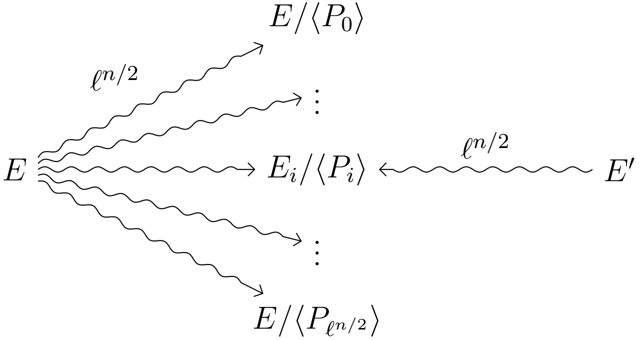
- $R_A = m_A P_A + n_A Q_A$,
- $R_B = m_B P_B + n_B Q_B$,




[5] Jao and De Feo 2011.

# Generic attacks

*Problem:* Given $E, E'$, isogenous of degree $\ell^n$, find $\phi : E \to E'$.




- With high probability $\phi$ is the unique collision (or *claw*).
- A *quantum claw finding*[6] algorithm solves the problem in $O(\ell^{n/3})$.

[6] Tani 2008.

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