Post-quantum cryptography QIC 891

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CryptoWorks21

Agenda

- Multivariate quadratic Public Key Cryptosystems (MPKC)
  - Mathematical Problems: the $\mathcal{MQ}$ and $\mathcal{IP}$ problems
  - Building crypto from the $\mathcal{IP}$ problem
  - The UOV and Rainbow signature schemes

Multivariate quadratic public key cryptosystems (MPKC)

Why quadratic?

Given a system of cubic (or higher degree) equations

$$x_1\mathbf{x_2}\mathbf{x_3} + 2x_1x_3 - 3\mathbf{x_2^2}\mathbf{x_3} - 7 = 0$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$-x_1x_2 + x_1x_3 + 6\mathbf{x_3^3} - 7 = 0$$

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- Solve the quadratic system for the new variables and backtrack.

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Notation

Let $\mathbb{F}_q$ denote a finite field of q elements where q is a prime power. A generic quadratic map $p : (\mathbb{F}_q)^n \to \mathbb{F}_q$ can be represented by a quadratic polynomial

$$p(x_1, \cdots, x_n) = \sum_{1 \le i \le j \le n} \alpha_{ij} x_i x_j + \sum_{1 \le i \le n} \beta_i x_i + \gamma$$

where $\alpha_{ij}, \beta_i, \gamma \in \mathbb{F}_q$.

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- A useful (implementation-friendly) matrix representation is generally used:

$$[x_1, \cdots, x_n] \begin{bmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ 0 & \alpha_{22} & \dots & \alpha_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \alpha_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \sum_{1 \le i \le j \le n} \alpha_{ij} x_i x_j$$

Multivariate quadratic Public Key Cryptosystems (MPKC)

$\mathcal{MQ}$ problem

- Given a system of m **random** quadratic equations in n variables over a finite field $\mathbb{F}_q$ (q is a prime power) of any characteristic:

$$\begin{cases} p_1(x_1, \cdots, x_n) = y_1 \\ p_2(x_1, \cdots, x_n) = y_2 \\ \cdots \cdots \cdots \cdots = \cdots \\ p_m(x_1, \cdots, x_n) = y_m \end{cases}$$

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  finding a solution $\mathbf{x} \in (\mathbb{F}_q)^n$ is **NP-hard** (a reduction from the 3-SAT to the $\mathcal{MQ}$ problem exists [Patarin and Goubin 1997]).

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- For the **average case**, it is believed to be hard due to experiments although no reduction exist.

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(Generic) Encryption

- Public key:

$$P = \{p_1(x_1, \cdots, x_n), \cdots, p_m(x_1, \cdots, x_n)\}$$

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- To decrypt, a trapdoor is needed to solve the system for $\mathbf{x}$:

$$x_1, \cdots, x_n = P^{-1}(c_1, \cdots, c_m)$$

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- To sign a document $D \in \{0,1\}^*$, compute the hash $h_1, \ldots, h_m := H(D) \in (\mathbb{F}_q)^m$ solve for $\mathbf{x}$ to get a signature σ

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- To verify σ, recompute $\{h_1, \cdots, h_m\} = H(D)$ and check

$$p_i(\sigma) \stackrel{?}{=} h_i, \;\; 1 \le i \le m$$

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General **Trapdoor** construction

- Let $\mathcal{S} : \mathbb{F}_q^n \to \mathbb{F}_q^n$ and $\mathcal{T} : \mathbb{F}_q^m \to \mathbb{F}_q^m$ be two invertible linear maps

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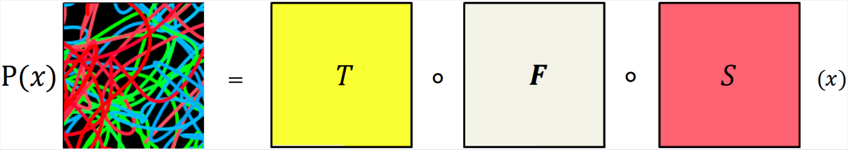

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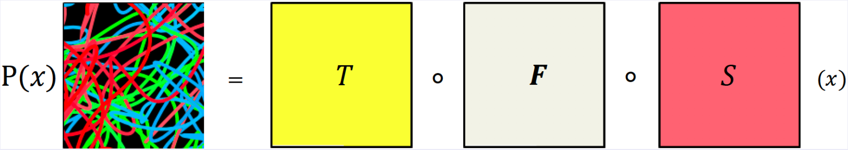

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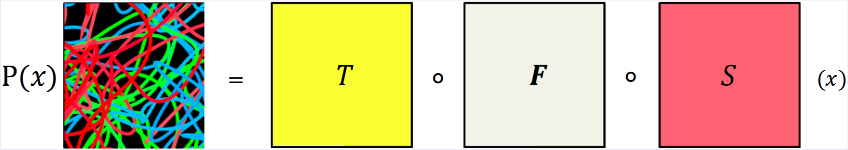

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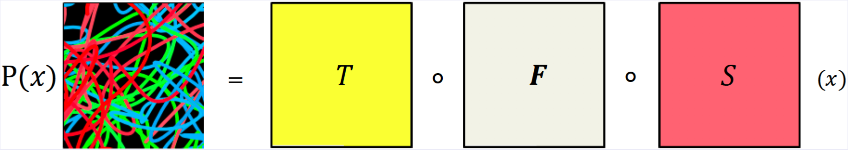

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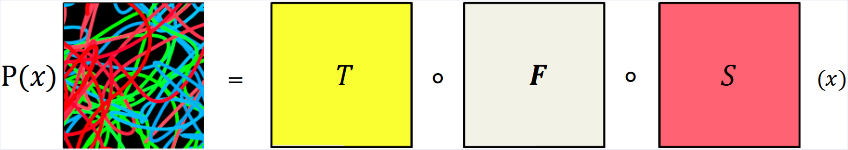

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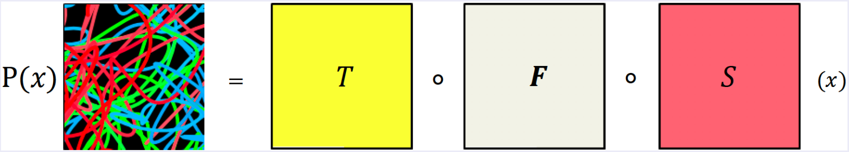

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- Inverting $\mathcal{P}$ is related to the Isomorphism of Polynomials problem.

Multivariate quadratic Public Key Cryptosystems (MPKC)

Isomorphism of Polynomials Problem ($\mathcal{IP}$-**problem**)

Two systems of equations/polynomials $\mathcal{U}, \mathcal{V} : \mathbb{F}_q^n \to \mathbb{F}_q^m$ are called **isomorphic** (up to linear transforms) iif $\exists$ linear maps $\mathcal{L}_1, \mathcal{L}_2$ s.t.

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Extended Isomorphism of Polynomials (**EIP-problem**)

Given a public key $\mathcal{P} = \mathcal{T} \circ \mathcal{F} \circ \mathcal{S}$, find a map $\overline{\mathcal{F}}$ isomorphic to $\mathcal{P}$, i.e.,

$$\mathcal{P} = \overline{\mathcal{T}} \circ \overline{\mathcal{F}} \circ \overline{\mathcal{S}}$$

for some invertible $\overline{\mathcal{T}}$ and $\overline{\mathcal{S}}$ s.t. $\overline{\mathcal{F}}$ inherits the trapdoor structure of $\mathcal{F}$

Multivariate quadratic Public Key Cryptosystems (MPKC)

Public key sizes

- Usually, public keys P consist of m quadratic polynomials of shape:

$$p_i(x) = [x_1, \cdots, x_n] \begin{bmatrix} \alpha_{11}^{(i)} & \alpha_{12}^{(i)} & \dots & \alpha_{1n}^{(i)} \\ 0 & \alpha_{22}^{(i)} & \dots & \alpha_{2n}^{(i)} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \alpha_{nn}^{(i)} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

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  - Stick to $d = 2$.

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  for the signature $x \in \mathbb{F}_q^n$.
  - Best time complexity is exponential when $m \approx n$
  - Gröbner bases complexity $f(q, m, n)$:
  $$f(q, m, n) = O\left(m \cdot \binom{n + d_{reg} - 1}{d_{reg}}^{\omega}\right)$$
  where d_{reg} is degree of regularity of the system and $2 < \omega \leq 3$.

(cont. $\cdots$)

- **Minrank** attack: find a low rank quadratic map.

MinRank

Given a set of m matrices M_i, find a nontrivial solution $a_1, \cdots a_m$ s.t.

$$\sum_{i=1}^{m} a_i M_i$$

is of minimum rank.

Finding a low rank matrix implies that we have less independent equations $\Rightarrow$ more variables per equation can make the system easier to solve.

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  - $\mathbf{x} = \mathcal{T}^{-1}(\mathbf{c}) \in F_q^m$
  - $\mathbf{w} = \mathcal{F}^{-1}(\mathbf{x})$
  - $\mathbf{z} = \mathcal{S}^{-1}(\mathbf{w})$

Multivariate quadratic Public Key Cryptosystems (MPKC)

Encryption: Requires $\mathbf{m} \geq \mathbf{n}$

- To encrypt a document $\mathbf{d} \in (\mathbb{F}_q)^n$, evaluate

$$\mathbf{c} = \mathcal{P}(\mathbf{d}) \in \mathbb{F}_q^m$$

- To decrypt, compute
  - $\mathbf{x} = \mathcal{T}^{-1}(\mathbf{c}) \in F_q^m$
  - $\mathbf{w} = \mathcal{F}^{-1}(\mathbf{x})$
  - $\mathbf{z} = \mathcal{S}^{-1}(\mathbf{w})$
- If $m \geq n$ (not undetermined) then we ensure that $\mathcal{F}$ is more or less injective and decryption is not mapped to many different plaintexts.

Encryption: The Matsumoto and Imai 1988 **trapdoor**

- Central map is defined as

$$\mathcal{F} = \phi \circ \overline{\mathcal{F}} \circ \phi^{-1}$$

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with $a_i \in \mathbb{F}_q$ and $\overline{\mathcal{F}} : \mathbb{F}_{q^n} \to \mathbb{F}_{q^n}$ is defined as

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- Notice that $X^{q^\theta+1}$ is a quadratic transformation since q^θ is linear (q^θ-Frobenius).
- The quadratic transformation takes place in the big "hidden field" $\mathbb{F}_{q^n}$ instead of the vector space over the smaller field $(\mathbb{F}_q)^n$.

(cont. $\cdots$)

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- Notice that for the quadratic map

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to be invertible, ones has to take $q^\theta + 1$-th roots and thus $q^\theta + 1 \in \mathbb{F}_{q^n}$ must invertible

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- Thus, the quadratic map can be made invertible. The keygen, encryption and decryption for MI can be done as explained before.

(cont. $\cdots$)

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  - 2000, Patarin suggests **Hidden Field Equations (HFE)** encryption with a slightly modified trapdoor with $\overline{\mathcal{F}} : \mathbb{F}_{q^n} \to \mathbb{F}_{q^n}$ redefined to:

$$X \mapsto \sum_{0 \le i,j \le d} \mathcal{A}_{ij} X^{q^i + q^j} + \sum_k \mathcal{B}_k X^{q^k} + \gamma$$

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    - Later, Kipnis and Shamir 1999 showed that degree d cannot be too small otherwise minrank + linearization attacks apply.
    - But if d is increased decryption becomes too slow.
    - Finally, Faugere and Joux 2003 improved the attacks using **F4** algorithm which made the system impractical.

Multivariate quadratic public key cryptosystems (MPKC)

Signature: The UOV **trapdoor**, Kipnis, Patarin, and Goubin 1999

- Goal: $\mathcal{F}^{-1}(h) = (x_1, \cdots, x_n)$

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- Let $o, v \in \mathbb{N}$, define $n = o + v$ and $m = o$
- Write the quadratic polynomials as follows:

$$p(x_1, \cdots, x_n) = \underbrace{\sum_{i=1}^{v}\sum_{j=i}^{v} \alpha_{ij} x_i x_j}_{v \times v \text{ terms}} + \underbrace{\sum_{i=1}^{v}\sum_{j=v+1}^{n} \beta_{ij} x_i x_j}_{v \times o \text{ terms}} + \sum_{i=1}^{n} \gamma_i x_i + \delta$$

where $(x_1, \cdots, x_v)$ are the *vinegar* variables and are $(x_{v+1}, \cdots, x_n)$ are the **oil** variables.

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- Notice *oil* variables are **not mixed** with themselves. // Easier to see using matrix notation, or even easier mixing the ingredients!!

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How to invert the UOV trapdoor

- To invert guess at random the vinegars $(x_1, \cdots, x_v) \in_R (\mathbb{F}_q)^v$

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$$p_k(x_1, \cdots, x_n) = \underbrace{\sum_{i=1}^{v} \sum_{j=i}^{v} \alpha_{ij}^{(k)} x_i x_j}_{v \times v \text{ terms}} + \underbrace{\sum_{i=1}^{v} \sum_{j=v+1}^{n} \beta_{ij}^{(k)} x_i x_j}_{v \times o \text{ terms}} + \sum_{i=1}^{n} \gamma_i x_i^{(k)}$$

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- If the system has no solution, try another guess for $(x_1, \cdots, x_v)$

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- This system $o \times o$ has a solution with very high probability

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  - The case where $o \approx v$ (balanced oil and vinegar) was broken by Kipnis and Shamir 1998.
  - For $v > o$ the complexity of the attack becomes $O(q^{v-o} \cdot o^4)$

UOV parameter sizes [from A. Petzoldt, 2017]

security level (bit)	scheme	public key size (kB)	private key size (kB)	hash size (bit)	signature (bit)
80	UOV(GF(16),40,80)	144.2	135.2	160	480
	UOV(GF(256),27,54)	89.8	86.2	216	648
100	UOV(GF(16),50,100)	280.2	260.1	200	600
	UOV(GF(256), 34,68)	177.8	168.3	272	816
128	UOV(GF(16),64,128)	585.1	538.1	256	768
	UOV(GF(256),45,90)	409.4	381.8	360	1,080
192	UOV(GF(16),96,192)	1,964.3	1,786.7	384	1,152
	UOV(GF(256),69,138)	1,464.6	1,344.0	552	1,656
256	UOV(GF(16),128,256)	4,644.1	4,200.3	512	1,536
	UOV(GF(256),93,186)	3,572.9	3,252.2	744	2,232

Summary of UOV

- UOV was proposed in 1999 and has not suffured major attacks.
- Faster than ECDSA to sign. $2 - 4\times$ faster to sign, $10 - 20\times$ faster for verifying.
- Signature sizes are less than $1KiB$
- Public keys are large: tens or hundreds KiB
  - Potential topic for research!

Multivariate quadratic Public Key Cryptosystems (MPKC)

Rainbow signature

- Proposed by Ding and Schmidt 2005.
- It is a generalization of UOV.
- Idea: split private quadratic maps into layers.
  - Solve more but smaller systems of equations.
  - *Vinegars* for the next layer will be the the *vinegars* + *oils* from the previous one.

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- Let $V_i := \{1, \cdots, v_i\}$ and $O_i := \{v_i + 1, \cdots, v_{i+1}\}$, be sets and $o_i := v_{i+1} - v_i$ the numbers of oils.

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- Central map $\mathcal{F}$ consists of $m = n - v_1$ polynomials $f^{v_1+1}, \cdots, f^{(n)}$

$$f^{(k)} = \sum_{i,j \in V_\ell} \alpha_{ij}^{(k)} x_i x_j + \sum_{i \in V_\ell, j \in O_\ell} + \beta_{ij}^{(k)} x_i x_j + \sum_{i \in V_\ell \cup O_\ell} \gamma_i^{(k)} x_i$$

where ℓ is the only integer s.t. $k \in O_\ell$.

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- Choose invertible linear maps $\mathcal{T} : \mathbb{F}_q^m \to \mathbb{F}_q^m$ and $\mathcal{S} : \mathbb{F}_q^n \to \mathbb{F}_q^n$

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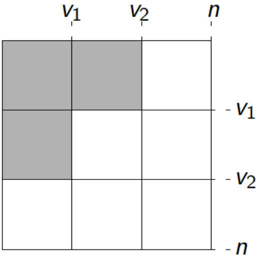
- Choose invertible linear maps $\mathcal{T} : \mathbb{F}_q^m \to \mathbb{F}_q^m$ and $\mathcal{S} : \mathbb{F}_q^n \to \mathbb{F}_q^n$
- Public key is $\mathcal{P} = \mathcal{T} \circ \mathcal{F} \circ \mathcal{S}$.

Rainbow central map with 2 layers

$F^{(k)} =$



$v_1 + 1 \leq k \leq v_2$

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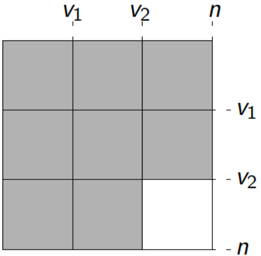

$v_2 + 1 \leq k \leq n$

- $x_1, \cdots, x_{v_1}, x_{v_1+1}, \cdots, x_{v_1+o_1}$ will be the vinegars $x_1, \cdots, x_{v_2}$ for the second layer.

Rainbow (2 layers): Toy example [from A. Petzoldt, 2017]

- $\mathbb{F}_7$, $(v_1, o_1, o_2) = (2, 2, 2)$, $m = n - v_1 = 4$
- Central map $\mathcal{F} = (f^{(3)}, f^{(4)}, f^{(5)}, f^{(6)})$ with

$$
\begin{aligned}
f^{(3)} &= x_1^2 + 3x_1x_2 + 5x_1x_3 + 6x_1x_4 + 2x_2^2 + 6x_2x_3 + 4x_2x_4 + 2x_2 + 6x_3 \\
&\quad + 2x_4 + 5 \\
f^{(4)} &= 2x_1^2 + x_1x_2 + x_1x_3 + 3x_1x_4 + 4x_1 + x_2^2 + x_2x_3 + 4x_2x_4 + 6x_2 + x_4 \\
f^{(5)} &= 2x_1^2 + 3x_1x_2 + 3x_1x_3 + 3x_1x_4 + x_1x_5 + 3x_1x_6 + 6x_1 + 4x_2^2 \\
&\quad + x_2x_3 + 4x_2x_4 + x_2x_5 + 3x_2x_6 + 3x_2 + 3x_3x_4 + x_3x_5 + 2x_3x_6 \\
&\quad + 2x_3 + 3x_4x_5 + x_5 + 6x_6 \\
f^{(6)} &= 2x_1^2 + 5x_1x_2 + x_1x_3 + 5x_1x_4 + 5x_1x_6 + 6x_1 + 5x_2^2 + 3x_2x_3 \\
&\quad + 5x_2x_5 + 4x_2x_6 + x_2 + 3x_3^2 + 5x_3x_4 + 4x_3x_5 + 2x_3x_6 + 4x_3 + x_4^2 \\
&\quad + 6x_4x_5 + 3x_4x_6 + 4x_4 + 4x_5 + x_6 + 2
\end{aligned}
$$

- Goal: Compute the preimage $\mathbf{x} \in \mathbb{F}_7^6$ for $\mathbf{y} = (6, 2, 0, 5)$ under $\mathcal{F}$.

Rainbow: Toy example [from A. Petzoldt, 2017] (cont. $\cdots$)

- Choose random values for the Vinegar variables x_1 and x_2, e.g. $(x_1, x_2) = (0, 1)$ and substitute them into the polynomials $f^{(3)}, \ldots, f^{(6)}$.

$$\begin{aligned}
\tilde{f}^{(3)} &= 5x_3 + 6x_4 + 2, \tilde{f}^{(4)} = x_3 + 5x_4, \\
\tilde{f}^{(5)} &= 3x_3x_4 + x_3x_5 + 2x_3x_6 + 3x_3 + 3x_4x_5 + 4x_4 + 2x_5 + 2x_6, \\
\tilde{f}^{(6)} &= 3x_3^2 + 5x_3x_4 + 4x_3x_5 + 2x_3x_6 + x_4^2 + 6x_4x_5 + 3x_4x_6 + 4x_4 + 2x_5 + 5x_6 + 1.
\end{aligned}$$

- Set $\tilde{f}^{(3)} = y_1 = 6$ and $\tilde{f}^{(4)} = y_2 = 2$ and solve for x_3, x_4
  $\Rightarrow (x_3, x_4) = (3, 4)$
- Substitute into $\tilde{f}^{(5)}$ and $\tilde{f}^{(6)}$
  $\Rightarrow \tilde{\tilde{f}}^{(5)} = 3x_5 + x_6 + 5, \tilde{\tilde{f}}^{(6)} = 3x_5 + 2x_6 + 1$
- Set $\tilde{\tilde{f}}^{(5)} = y_3 = 0$ and $\tilde{\tilde{f}}^{(6)} = y_4 = 5$, solve for x_5 and x_6
  $\Rightarrow (x_5, x_6) = (0, 2)$

A pre image of $\mathbf{y} = (6, 2, 0, 5)$ is given by $\mathbf{x} = (0, 1, 3, 4, 0, 2)$.

Rainbow parameter sizes [from A. Petzoldt, 2017]

(cont. $\cdots$)

security level (bit)	parameters $\mathbb{F}, v_1, o_1, o_2$	public key size (kB)	private key size (kB)	hash size (bit)	signature (bit)
80	GF(16),17,20,20	33.4	22.3	160	228
	GF(256),19,12,13	25.3	19.3	200	352
100	GF(16),22,25,25	65.9	43.2	200	288
	GF(256), 27,16,16	57.2	44.3	256	472
128	GF(16),28,32,32	136.6	87.6	256	368
	GF(256),36,21,22	136.0	102.5	344	632
192	GF(16),45,48,48	475.9	301.8	384	564
	GF(256),58,33,34	523.5	385.5	536	1,000
256	GF(16),66,64,64	1,194.4	763.9	512	776
	GF(256),86,45,46	1,415.7	1,046.3	728	1,416

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