Triangulation (computing depth)

Given the intrinsic parameters, the projections of scene point on the two image sensors are:

$$\begin{vmatrix} u_l \\ v_l \\ 1 \end{vmatrix} = \begin{vmatrix} f_x^{(l)} & 0 & o_x^{(l)} & 0 \\ 0 & f_y^{(l)} & o_y^{(l)} & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix} \begin{vmatrix} x_l \\ y_l \\ z_l \\ 1 \end{vmatrix} \qquad \begin{vmatrix} u_r \\ v_r \\ 1 \end{vmatrix} = \begin{vmatrix} f_x^{(r)} & 0 & o_x^{(r)} & 0 \\ 0 & f_y^{(r)} & o_y^{(r)} & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix} \begin{vmatrix} x_r \\ y_r \\ z_r \\ 1 \end{vmatrix}$$

left camera imaging equation — right camera imaging equation

We also know the relative position and orientation between the two cameras:

$$\begin{vmatrix} x_l \\ y_l \\ z_l \\ 1 \end{vmatrix} = \begin{vmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} x_r \\ y_r \\ z_r \\ 1 \end{vmatrix}$$

$$\begin{vmatrix} u_l \\ v_l \\ 1 \end{vmatrix} = \begin{vmatrix} f_x^{(l)} & 0 & o_x^{(l)} & 0 \\ 0 & f_y^{(l)} & o_y^{(l)} & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix} \begin{vmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} x_r \\ y_r \\ z_r \\ 1 \end{vmatrix}$$

$\tilde{u}_l = \qquad P_l \qquad \tilde{x}_r$

Substitute in the left imaging equation

$$\begin{vmatrix} u_r \\ v_r \\ 1 \end{vmatrix} = \begin{vmatrix} f_x^{(r)} & 0 & o_x^{(r)} & 0 \\ 0 & f_y^{(r)} & o_y^{(r)} & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix} \begin{vmatrix} x_r \\ y_r \\ z_r \\ 1 \end{vmatrix}$$

$\tilde{u}_r = \qquad M_{int_r} \qquad \tilde{x}_r$

Keep the right imaging equation as it is

$$\begin{vmatrix} u_l \\ v_l \\ 1 \end{vmatrix} = \begin{vmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{vmatrix} \begin{vmatrix} x_r \\ y_r \\ z_r \\ 1 \end{vmatrix}$$

$\tilde{u}_l \qquad P_l \qquad \tilde{x}_r$

Known — Unknown

$$\begin{vmatrix} u_r \\ v_r \\ 1 \end{vmatrix} = \begin{vmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \end{vmatrix} \begin{vmatrix} x_r \\ y_r \\ z_r \\ 1 \end{vmatrix}$$

$\tilde{u}_r \qquad M_r \qquad \tilde{x}_r$

Known — Unknown

Rearranging the terms :

$$\begin{bmatrix} u_r u_{31} - u_{11} & u_r u_{32} - u_{12} & u_r u_{33} - u_{13} \\ v_r u_{31} - u_{21} & v_r u_{32} - u_{22} & v_r u_{33} - u_{23} \\ u_l p_{31} - p_{11} & u_l p_{32} - p_{12} & u_l p_{33} - p_{13} \\ u_r p_{31} - p_{21} & v_l p_{32} - p_{22} & v_l p_{33} - p_{23} \end{bmatrix} \begin{bmatrix} x_r \\ y_r \\ z_r \end{bmatrix} = \begin{bmatrix} u_{14} - u_{34} \\ u_{24} - u_{34} \\ p_{14} - p_{34} \\ p_{24} - p_{34} \end{bmatrix}$$

$A_{4\times3}$ (Known) $\quad x_r$ (Unknown) $\quad b_{4\times1}$ (Known)

Find least squares solution using pseudo-inverse :

$$Ax_r = b$$

$$A^T A x_r = A^T b$$

$$x_r = (A^T A)^{-1} A^T b$$

we repeat this for every pair of corresponding points in the left and right image. This gives us the complete 3-dimensional depth map of the scene.