LEMPEL-ZIV COMPLEXITY REFERENCE

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Abstract

The aim of this note is to provide some reference facts for LZW—mostly from Thomas and Cover [1]—adapted to the needs of the Luminous project. LZW is an algorithm to compute a Kolmogorov Complexity estimate derived from a limited programming language that only allows copy and insertion in strings (not Turing complete set).

Despite its delightful simplicity, it is rather powerful and fast. We then focus on definitions of LZW derived complexity metrics consistent with the notion of descriptive length, and discuss different normalizations, which result in a set of metrics we call ρ_0 , ρ_1 and ρ_2 , in addition to the Description Length l_{LZW} and the Entropy Rate.

1. LZW COMPRESSION: THE MAIN CONCEPT

The main idea in LZW is to look for repeating patterns in the data, and instead of rewriting repeating sequences, refer to the last one seen [3]. As Kaspar clearly states, LZW is the Kolmogorov Complexity computed with a limited set of programs that only allow copy and insertion in strings [2, 5].

"We do not profess to offer a new absolute measure for complexity which, as mentioned already, we believe to be nonexistent. Rather, we propose to evaluate the complexity of a finite sequence from the point of view of a simple self- delimiting learning machine which, as it scans a given n- digit sequence $S = s_1 \cdot s_1 \cdot ...s_n$, from left to right, adds a new word to its memory every time it discovers a substring of consecutive digits not previously encountered. The size of the vocabulary, and the rate at which new words are encountered along S, serve as the basic ingredients in the proposed evaluation of the complexity of S."

We consider a string of characters in and alphabet with A symbols (typically binary) of length n. From wikipedia: A high level view of the encoding algorithm is shown here:

- 2. Find the longest string W in the dictionary that matches the current input.
- 3. Emit the dictionary index for W to output and remove W from the input.
- 4. Add W followed by the next symbol in the input to the dictionary.
- 5. Go to Step 2.

After applying LZW, we will end up with a set of words (or phrases, as they are sometimes called) c(n) that go into a dictionary. The length of the compressed string will be $l_{LZW} \leq n$ (the analog of Kolmogorov or algorithmic complexity).

The description length of the sequence encoded by LZW would have length less or equal to the number of phrases times the number of bits needed to identify a seen phrase plus the bits to specify a new symbol (to form a new phrase), hence¹

(1)
$$l_{LZW} \le c(n) \log_2 [c(n) + \log_2 A] \approx c(n) \log_2 [c(n)]$$

2. The process of digitization

When we digitize (e.g., binarize) a signal prior LZW, we are creating a new string from the data, and we make an explicit choice on what aspects of the data we wish to compress. In this process we destroy information—we are going to do lossy compression. Thus, the choice of digitization results in us having access to a subset of the features of the original string.

¹Actually, we can do a bit better than this. In practice, not all dictionary entries are used. We can use the max dictionary key ID and state that "n bits are needed to describe any key entry, and there are m of them (and here they are)", leading to $l_{LZW} \leq \log_2(\log_2 \max(output)) + \operatorname{length}(output) * \log_2[\max(output)]$, since we need $\log_2(\log_2 \max(output))$ bits to describe n. This is how it is implemented in the appended code.

A reasonable strategy is to preserve as much information as possible in the resulting transformed string. In this sense, using methods that maximize the entropy of the resulting series are recommended, such as using the median for thresholding (this is guaranteed to result in $H_0 = 1$)².

On the other hand, other methods that destroy more information may tap and highlight other, also relevant features of the data. At this stage, then, how to binarize or preprocess (e.g., filter) the original string is an empirical question. The same applies to the choice of compression method, of course, as LZW is just one framework for compression.

3. LZW and entropy rate for stochastic processes

The main fact from Thomas and Cover [1] refers to stochastic random processes $\{X_i\}$. A key concept is the **entropy rate** of the stochastic process, given by

(2)
$$\mathcal{H}(X) = \lim_{n \to \infty} \frac{1}{n} H(X_1, ..., X_n),$$

when this limit exists, with H denoting the usual multivariate entropy of X, $H(X) = -E_X[\log(P(X))]$. It is an important theorem that for stationary processes,

(3)
$$\mathcal{H}(X) = \lim_{n \to \infty} H(X_n | X_{n-1}, X_{n-2}, ..., X_1).$$

Let also

$$H_0(p) = -p \log p - (1-p) \log(1-p)$$

denote the **univariate entropy**, with p the probability of a Bernoulli (binary) process (Markov chain³ of order zero).

We note that entropy rate of a stochastic processes is non-increasing as a function of order, that is, $0 \le \mathcal{H} \le ... \le H_q \le ... \le H_0 \le 1$.

The fundamental relation is that description length is closely related to entropy rate,

(4)
$$l_{LZW} = c(n) \log_2 \left[c(n) + \log_2 A \right] \approx c(n) \log_2 \left[c(n) \right] \longrightarrow n\mathcal{H}$$

Another important result in what follows is that with probability 1 (Thomas and Cover Theorem 13.5.3)

$$\lim_{n\to\infty}\sup l_{LZW}\leq n\mathcal{H}$$

²Can we generalize this idea? Can we, e.g., binarize the data so that it has maximal H_0 and H_1 ?

³We denote a **Markov chain of order** m to be one where the future state depends on the past m states (time-translation invariantly), $P(X_n|X_{n-1},...,X_1) = P(X_n|X_{n-1},...,X_{n-m})$ for n > m.

which can rewrite as

$$\lim_{n \to \infty} \sup c(n) \log_2 c(n) \le n\mathcal{H} \le nH_0$$

and use to rewrite (in the limit above)

(5)
$$c(n) \le \frac{n\mathcal{H}}{\log_2 c(n)} \le \frac{n\mathcal{H}}{\log_2 \frac{n\mathcal{H}}{\log_2 c(n)}} \sim \frac{n\mathcal{H}}{\log_2 n} \le \frac{nH_0}{\log_2 n}$$

which we use below for normalization purposes.

4. Metrics

Two metrics are used in the field, one is c(n) and the other l_{LZW} . Of the two the latter is more closely related to Kolmogorov complexity or description length. Both contain similar information (in fact one is a monotonic function of the other).

5. Fundamental Normalization of LZW

The purest way to normalize this metric is to normalize by the original string length n

$$\rho_0 = l_{LZW}/n = \frac{c(n)\log_2[c(n) + A]}{n} \to \mathcal{H}$$

with units of bits per character. This is the LZW compression ratio.

6. Other normalizations or measures

A typical normalization adopted by the literature is to "divide by entropy". By this we mean $\rho_1 = l_{LZW}/H_0$. In the literature this is usually defined through c(n),

$$ho_1 = rac{c(n)}{rac{nH_0}{\log_2 n}} \sim rac{\mathcal{H}}{H_0} \sim rac{l_{LZW}}{nH_0}
ightarrow rac{\mathcal{H}}{H_0}$$

(with units of bits per character). Essentially the same can be computed from the randomly reshuffled data series, which with high probability forces $l_{LZW} \sim nH_0$ by destroying 2nd order interactions. Hence,

$$\rho_1 \approx \frac{l_{LZW}}{H_0} \approx \frac{l_{LZW}}{l_{LZW}^{shuf}}$$

This ratio tells us how much information density is hidden in 2nd and higher order entropy rate as compared to first order one.

We can think of this a being the comparison of "first order apparent complexity" (entropy) and an estimate of the entropy rate (which provides and upper bound to algorithmic complexity). This is an important comparison, as the proposal in [4] is that conscious level may be associated to systems that exhibit high apparent entropy with low algorithmic complexity.

Alternatively, we could define

$$\rho_2 = H_0 - \rho_0 > 0$$

which can be interpreted as the extra apparent extra entropy (bits/char) incurred by using first order methods instead estimating the true entropy rate.

At any rate, from a machine learning point of view it is probably best to compute ρ_0 , H_0 , ..., H_q as separate measures.

Also, it is known that LZW or entropy estimates are sensitive to string length. When comparing metrics across datasets make sure you keep string length constant and as long as possible.

7. LZW AND KOLMOGOROV COMPLEXITY

As mentioned above, LZW description length is only an estimate of algorithmic complexity. We can easily provide examples of sequences that have low algorithmic complexity, but which are rather hard to compress using LZW. For example, consider the firs n digits of π . Or consider "the digits of the smallest prime with 10^9 digits". The algorithmic complexity of these numbers is very low, but LZW won't compress them much if at all. Does this mean that LZW is useless? No, but we should keep in mind that it provides only an upper bound on algorithmic complexity. In order to get better bounds, we may consider a family of compressors and use the lowest complexity that any of them can find as a better upper bound.

References

- [1] Thomas M. Cover and Joy A. Thomas. *Elements of information theory*. John Wiley & sons, 2 edition, 2006.
- [2] F. Kaspar and H. G. Schuster. Easily calculable measure for the complexity of spatiotemporal patterns. Phys. Rev. A, 36(2):842–848, 1987.
- [3] A Lempel and J Ziv. On the complexity of finite sequences. IEEE Transactions on Information Theory, IT-22(1):75-81, January 1976.
- [4] G. Ruffini. An algorithmic information theory of consciousness. Submitted to the Neuroscience of Consciousness. August 2016.
- [5] Giulio Ruffini, David Iba nez, Marta Castellano, Stephen Dunne, and Aureli Soria-Frisch. Eeg-driven rnn classification for prognosis of neurodegeneration in at-risk patients. *ICANN 2016*, 2016.

Annex: StarLZW.py

```
# LZW Update May 2017, Feb 12 2017 to match TN00344.
2 # G. Ruffini with Eleni Kroupi / Starlab + Neuroelectrics (2017)
3 # Updated May 2017 with Entropy rate calculators
   # Part of this code from sources as indicated, but modified to
   # be used with binary data.
   # Permission is hereby granted, free of charge, to any person obtaining a copy of
   # this software and associated documentation files (the "Software"), to deal in
   # the Software without restriction, including without limitation the rights to
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   # the Software, and to permit persons to whom the Software is furnished to do so,
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15 # copies or substantial portions of the Software.
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17\, # THE SOFTWARE IS PROVIDED "AS IS", WITHOUT WARRANTY OF ANY KIND, EXPRESS OR
   # IMPLIED, INCLUDING BUT NOT LIMITED TO THE WARRANTIES OF MERCHANTABILITY, FITNESS
18
  # FOR A PARTICULAR PURPOSE AND NONINFRINGEMENT. IN NO EVENT SHALL THE AUTHORS OR
20
  # COPYRIGHT HOLDERS BE LIABLE FOR ANY CLAIM, DAMAGES OR OTHER LIABILITY, WHETHER
21
   # IN AN ACTION OF CONTRACT, TORT OR OTHERWISE, ARISING FROM, OUT OF OR IN
   # # CONNECTION WITH THE SOFTWARE OR THE USE OR OTHER DEALINGS IN THE SOFTWARE.
22
23
24
25
26 import numpy as np
27
   28
29
  def binarizeby(data, method="median"):
        """ Binarize 1D numpy data array using different methods: mean or median"""
30
31
32
       d=data*0
33
       if method == "median":
34
           thr=np.median(data)
           d[np.where(data>= thr)]=1
35
           print " Using median as threshold"
36
37
           print " The median is:", str(thr)
38
39
       else: # use mean
40
           thr=np.mean(data)
41
           d[np.where(data>= thr)]=1
42
           print " Using mean as threshold"
           print " The mean is:", str(thr)
43
44
       print " Binarized data average", str(np.mean(d.astype(np.int8)))
45
       return d.astype(np.int8)
46
47
48
   # adapted from https://rosettacode.org/wiki/LZW_compression#Python
   def compress(theString, mode='binary', verbose=True):
49
50
        """Compress a *string* to a list of output symbols. Starts from two sympols, 0 and 1.
51
          Returns the compressed string and the length of the dictionary
          If you need to, convert first arrays to a string ,
52
53
          e.g., entry="".join([np.str(np.int(x)) for x in theArray]) """
54
       if mode == 'binary':
55
56
           dict_size=2
           dictionary={'0':0, '1':1}
57
58
        elif mode=='ascii':
           # Build the dictionary for generic ascii.
```

```
60
             dict_size = 256
 61
             dictionary = dict((chr(i), i) for i in xrange(dict_size))
 62
            print "unrecognized mode, please use binary or ascii"
 63
 64
 65
         result = []
 66
         for c in theString:
 67
             WC = W + C
 68
 69
             if wc in dictionary:
 70
                 w = wc
 71
             else:
 72
                 result.append(dictionary[w])
 73
                 # Add wc to the dictionary.
 74
                 dictionary[wc] = dict_size
 75
                 dict_size += 1
 76
                 w = c
 77
 78
         # Output the code for w.
 79
         if w:
 80
             result.append(dictionary[w])
 81
         if verbose:
             print "length of input string:", len(theString)
 82
             print "length of dictionary:", len(dictionary)
 84
             print "length of result:", len(result)
 85
         return result,len(dictionary)
 86
 87
 88
     def decompress(compressed, mode='binary'):
 89
 90
         """Decompress a list of output ks to a string."""
         from cStringIO import StringIO
 91
 92
 93
         if mode == 'binary':
 94
             dict_size=2
 95
             dictionary={0:'0', 1:'1'}
 96
             w = str(compressed.pop(0))
 97
 98
         elif mode=='ascii':
99
100
             \mbox{\tt\#} Build the dictionary for generic ascii.
101
             dict_size = 256
102
             dictionary = dict((i, chr(i)) for i in xrange(dict_size))
103
             w = chr(compressed.pop(0))
104
105
             print "unrecognized mode, please use binary or ascii"
106
107
108
         # use StringIO, otherwise this becomes O(N^2)
109
         # due to string concatenation in a loop
110
         result = StringIO()
111
         result.write(w)
112
113
         for k in compressed:
114
             if k in dictionary:
115
                 entry = dictionary[k]
             elif k == dict_size:
116
117
                 entry = w + w[0]
118
                 raise ValueError('Bad compressed k: %s' % k)
119
120
             result.write(entry)
```

```
121
122
             # Add w+entry[0] to the dictionary.
123
             dictionary[dict_size] = w + entry[0]
124
            dict_size += 1
125
126
             w = entry
127
         return result.getvalue()
128
129
130
131
    132
133
    def Compute_rho0(theArray):
         """Computes rho0 metric (bits/Sample) as described in TN000344 Starlab / Luminous """
134
135
136
                   ComputeDescriptionLength(theArray) / len(theArray)*1.0
         return
137
138
    def Compute_rho1(theArray):
139
         """Computes rho1 metric (bits/Sample) as described in TN000344 /Starlab Luminous"""
140
141
         return
                   ComputeDescriptionLength(theArray) / ShannonEntropy(theArray) / len(theArray)
142
143 def Compute_rho2(theArray):
         """Computes rho2 metric (bits/Sample) as described in TN00044 """
144
145
146
                   Compute_rho0(theArray) - ShannonEntropy(theArray)
147
148
149
    def ComputeDescriptionLength(theArray,classic=False):
           "Computes description lenght l_{LZW} as described in TN000344"""
150
151
152
         entry="".join([np.str(np.int(x)) for x in theArray])
153
154
         compressedstring,len_dict=compress(entry) # returns
155
         ndigits=len(np.unique(theArray))
156
         print "distinct digits:", ndigits
157
158
159
160
         # you need n bits log(max(comp))) to describe this sequence. And there are this many: len(a)...
161
         # DL: "you need n bits. There are these (m) of them".
162
163
         \texttt{DL=np.log2(np.log2(max(compressed string)))+ np.log2(max(compressed string))*len(compressed string))}
164
165
         if classic:
166
         \# old way ... more for LZ than LZW:
167
          DL= len_dict*np.log2(len_dict)
168
169
         return DL
170
171
    def ShannonEntropy(labels):
172
         """ Computes entropy of label distribution. numpy array int in """
         n_{\text{labels}} = len(labels)
173
174
         if n_labels <= 1:</pre>
175
            return 0
176
177
         counts = np.bincount(labels)
178
         probs = counts*1.0 / n_labels*1.0
179
         n_classes = np.count_nonzero(probs)
180
181
         if n_classes <= 1:</pre>
```

```
182
            return 0
183
184
        # Compute standard entropy.
185
        ent = 0.
186
        for i in probs:
187
            ent -= i * np.log2(i)
188
189
        return ent
190
191
193 # Entropy rate of English ...
194
195 import re, codecs, random, math, textwrap
   from collections import defaultdict, deque, Counter
196
197
198
    def ComputeER(theArray, markov_order=4):
199
             """ Compute entropy rate of array with some memomory / markov_order"""
200
            # compute entropy rate ...
201
            #np.savetxt("rule.txt", theArray.astype(int),fmt='%i', delimiter="")
202
            #model, stats = markov_model(chars("rule.txt"), markov_order)
203
            datastream=streamArray(theArray.astype(int))
            model, stats = markov_model(datastream, markov_order)
204
205
            del datastream
206
            return entropy_rate(model, stats)
207
208
209 def streamArray(theArray):
210
         """Generator for array elements (convenient)"""
        for element in theArray:
211
212
            yield element
213
214
215
   # The following 3 are adapted from:
    # Clement Pit--Claudel (http://pit-claudel.fr/clement/blog)
216
217
    def markov_model(stream, model_order):
218
         """model describes prob of a digit appearing following n prior ones"""
        """ Stats provide frequencies of groups of n symbols"""
219
220
        model, stats = defaultdict(Counter), Counter()
221
        circular_buffer = deque(maxlen = model_order)
222
223
        for token in stream:
224
            prefix = tuple(circular_buffer)
225
            circular_buffer.append(token)
            if len(prefix) == model_order:
226
                stats[prefix] += 1.0
227
228
                model[prefix][token] += 1.0
229
        return model, stats
230
    def entropy(stats, normalization_factor):
231
232
        return -sum(proba / normalization_factor * math.log(proba
233
                        / normalization_factor, 2) for proba in stats.values())
234
235
    def entropy_rate(model, stats):
236
        return sum(stats[prefix] * entropy(model[prefix], stats[prefix])
237
238
                   for prefix in stats) / sum(stats.values())
```