Markov-Dubins Problem: an Optimal Control Approach

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1 Introduction

A *Markov-Dubins path* is the shortest C^1 and piecewise-C^2 curve joining two points in the two dimensional euclidean plane with prescribed initial and terminal tangents to the path and bounded curvature.
The problem of finding such a curve was originally posed by Markov in 1889, and then solved in 1957 by Lester Dubins [1]; since it was apparently impossible to give an explicit formula for the shortest path, Dubins gives a sufficient set of paths, i.e., a set which always contains what he called an R-geodesic, or optimal path. In 1990 Reeds and Sheep extended Dubins' result for a car that goes both forward and backward [5], allowing the presence of cusps in the path.
These problems were initially approached by means of geometrical arguments, and only in subsequent years they have been revisited with optimal control techniques. The main tool for the optimal control approach is the *Pontryagin's maximum principle*, formulated in 1956 by the Russian mathematician Lev Pontryagin and his students; it has been applied on Dubins curves several times over the past 50 years, for different applications across many engineering fields.
Nowadays Dubins curve are commonly used in mobile robotics, aerospace navigation systems, railway design, and more in general wherever path planning is needed.

The aim of this work is to discusses and revisits some well known results of the Dubins' theory. The original Dubins' formulation is based on a long chain of lemmas and theorems. Many of them are not needed in this work and therefore are not treated in what follows, while few key concepts of particular interest are summarized in Section 2.
Section 3 addresses the Markov-Dubins problem from an optimal control point of view; cost function and constraints are derived by means of simple considerations based on the dynamics of a wheeled robot, and then the problem is solved by using the Pontryagin Maximum principle. The last part of this section is devoted to the characterization of both singular and nonsingular solutions.
Section 4 concludes the report with a series of practical examples solved in Matlab environment.

2 Dubins Formulation

Dubins' solution shows that the sequence of concatenated arcs can be only of type CSC, CCC, or a subset of these (we denote with C a circular arc and with S a straight line segment). In order to get to such important result it is necessary to passes through a series of key considerations.

Consider a particle pursue a continuously differentiable path from an initial point $u \in \mathbb{R}^2$ to a final point $v \in \mathbb{R}^2$, with given initial and final velocity vectors $\dot{u}$ and $\dot{v}$. Suppose that the particle maintains the same speed unitary throughout the path. Direction instead can varies, but its rate of change is bounded by the minimum allowed turning radius R. With these premises in mind it is now possible to summarize some important Dubins' notions.

R-geodesic. It is possible to define the *average curvature* of a path as

$$\hat{r} = \frac{\left\| \dot{Z}(s_1) - \dot{Z}(s_2) \right\|}{|s_1 - s_2|} \tag{1}$$

where Z is a curve in $\mathbb{R}^2$ (even if the same concept holds in $\mathbb{R}^n$) and s_1, s_2 belonging to the interval of definition of Z.

Given fixed vectors $u,\ \dot{u},\ v,\ \dot{v}\ \in \mathbb{R}^2$ and a fixed positive R, is called R-geodesic the path of minimal length having average curvature everywhere less than or equal to R^{-1}.

Existence of R-geodesic. Let $u,\ \dot{u},\ v,\ \dot{v}$ be vectors of $\mathbb{R}^2$ with $\|\dot{u}\| = \|\dot{v}\| = 1$, and let R be a positive real number. Let $C = C_2(u,\ \dot{u},\ v,\ \dot{v},\ R)$ be the collection of the all curves X defined on $[0, L]$, where $L = L(X)$ varies with X, such that $X(s)$ belongs to the 2-dimensional Euclidean space for $0 \leq s \leq L$; let the average curvature of X be everywhere less than or equal to R^{-1}, and

$$X(0) = u, \ \ \dot{X}(0) = \dot{u}$$
$$X(L) = v, \ \ \dot{X}(L) = \dot{v}$$

Then for any $u,\ \dot{u},\ v,\ \dot{v}, R,$ there exists an X in $C = C_2(u,\ \dot{u},\ v,\ \dot{v},\ R)$ of minimal length.

Uniqueness of R-geodesic. Let $u,\ \dot{u},\ v,\ \dot{v}$ be vectors of $\mathbb{R}^2$ with $\|\dot{u}\| = \|\dot{v}\| = 1$, and let R be a positive real number. Suppose Y differentiable curve of type CSC defined for $0 \leq s \leq d$, and suppose

$$Y(0) = u, \ \ \dot{Y}(0) = \dot{u}$$
$$Y(d) = v, \ \ \dot{Y}(d) = \dot{v}$$

Then Y is the unique R-geodesic in $C = C_2(u,\ \dot{u},\ v,\ \dot{v},\ R)$.

Structure of an R-geodesic. Every planar R-geodesic is necessarily a continuously differentiable curve which is either:

- an arc of a circle of radius R, followed by a line segment, followed by an arc of a circle of radius R.
- a sequence of three arcs of circles of radius R.
- a subpath of the previous ones.

3 Optimal Control Approach

Model derivation. Dubins path is commonly used in robotics and control theory as a way to plan paths for wheeled robots. To motivate the importance of the Markov-Dubins problem we are going to consider the practical example of a wheeled unicycle robot (*Figure 1*). In this case the state of the system can be represented as an element of the special Euclidean group SE(2), while the control inputs are *curvature input* $\dot{\theta}(t)$ which controls the rate of change of the heading angle and a *velocity input* $v(t)$ which controls the rate of change of the robot position in the direction of the heading angle.
The unicycle kinamatic equations are the following:

$$\begin{aligned} \dot{x}(t) &= v(t)\cos\theta(t) \\ \dot{y}(t) &= v(t)\sin\theta(t) \\ \dot{\theta}(t) &= u(t) \end{aligned} \tag{2}$$

where $x(t), y(t) \in \mathbb{R}^2$ is the position of the unicycle, and $\theta(t) \in \mathbb{R}$.

Suppose that the robot, starting from an initial state (or pose) $p(t_0) = (x(t_0),\, y(t_0),\, \theta(t_0))^T$, needs to move up to a final state $p(t_f) = (x(t_f),\, y(t_f),\, \theta(t_f))^T$ in minimum time and without exceeding the bounds on the control inputs $v(t) \leq v_{max}$ and $\dot{\theta}(t) \leq \dot{\theta}_{max}$, $\forall t \in [t_0, t_f]$.

In addressing this problem it is natural to fix $v(t) = v_{max}, \forall t \in [t_0, t_f]$. Being that said, it's now clear that the problem of minimizing the path length can be re-written as a *minimum-time control problem*, where the only control input is $u(t)$.

$$\min\ t_f = \int_0^{t_f} L(\cdot)\,dt = \int_0^{t_f} (1)\,dt \tag{3}$$

s.t.

$$\begin{aligned} \dot{x}(t) = &\ \cos\theta(t),\ x(0) = x_0,\ x(t_f) = x_f \\ \dot{y}(t) = &\ \sin\theta(t),\ y(0) = y_0,\ y(t_f) = y_f \\ \dot{\theta}(t) = &\ u(t), \qquad \theta(0) = \theta_0,\ \theta(t_f) = \theta_f \\ |u(t)| \leq &\ a\,,\ \text{for a.e. } t \in [0,t_f] \end{aligned} \tag{4}$$

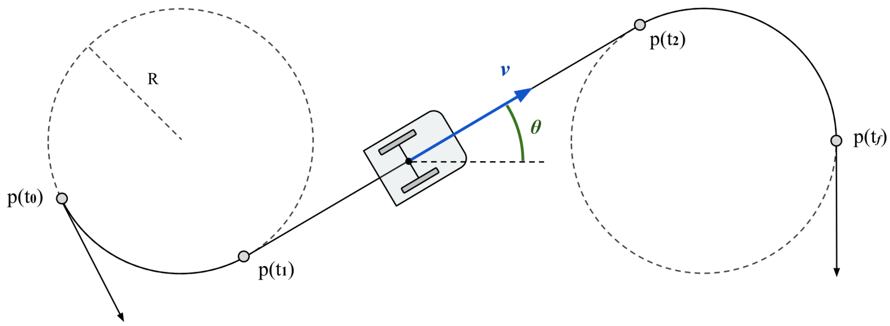


Figure 1: Unicycle robot moving along a Dubins path

Pontryagin Maximum Principle. The goal of this paragraph is to find an optimal control function $u^*(t)$ and, with it, an optimal trajectory of the state varaible $\psi^*(t)$; by *Pontryagin Maximum principle* these are the arguments that maximize the so called *Hamiltonian function*.
Let us denote the state of the system as the vector field $\psi = (x(t), y(t), \theta(t))^T$, and the so called *costate vector* as $\lambda = (\lambda_1(t), \lambda_2(t), \lambda_3(t))^T$. The Hamiltonin associated to (3)-(4) is expressed as follows:

$$\begin{aligned} H(t, \psi, u, \lambda_0, \lambda) &= \lambda_0 \, L(t, \psi, u) + \lambda^T \, f(t, \psi, u) = \\ &= \lambda_0 + \lambda_1 \, \cos\theta + \lambda_2 \, \sin\theta + \lambda_3 \, u \end{aligned} \tag{5}$$

ψ and λ need to satisfy the *Hamiltonian system*:

$$\dot{\psi}^* = \frac{\partial H}{\partial \lambda}(\psi^*, u^*, \lambda^*, \lambda_0^*) \tag{6a}$$

$$\dot{\lambda}^* = -\frac{\partial H}{\partial \psi}(\psi^*, u^*, \lambda^*, \lambda_0^*) \tag{6b}$$

More precisely, (6a) is satisfied by construction; developing (6b)

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial x} = 0 \tag{7a}$$

$$\dot{\lambda}_2 = -\frac{\partial H}{\partial y} = 0 \tag{7b}$$

$$\dot{\lambda}_3 = -\frac{\partial H}{\partial \theta} = \lambda_1 \sin\theta - \lambda_2 \cos\theta \tag{7c}$$

it is easy to verify that λ_1 and λ_2 necessarily assume constant values for all $t \in [0, t_f]$. Finally, the Pontryagin maximum principle is expressed by the following necessary conditions for global optimality:

$$u(t) \in \arg\min_{\|v\| \leq a} H(\psi(t), \, \lambda(t), \lambda_0, \, v) \tag{8a}$$

$$H(\psi(t)^*, \, \lambda(t)^*, \lambda_0^*, \, v^*) = 0 \tag{8b}$$

taking into account the fact that $\lambda_0, \lambda_1, \lambda_2$ are constants, one can simply rewrite

$$\boxed{u(t) \in \arg\min_{\|v\| \leq a} \lambda_3(t)\, v} \tag{9}$$

leading to the the optimal control

$$u(t) = \begin{cases} a, & \text{if } \lambda_3(t) < 0 \\ -a, & \text{if } \lambda_3(t) > 0 \\ \text{undetermined}, & \text{if } \lambda_3(t) = 0 \end{cases} \tag{10}$$

where $\lambda_3(t)$ is called *switching function* since it determines the value of the optimal control $u(t)$.

Some remarks need to be done:

- if $\lambda_3(t) \neq 0$ for a.e. $t \in [0, t_f]$ the control $u(t)$ is said to be *nonsingular control*, and more in particular it is *bang-bang* type. This holds true even if $\lambda_3(t) = 0$ for isolated values of t. The fact that the control is either a or $-a$ coincides to the fact that every arc of circle in an optimal path can only have minimum turning radius R.
- if $\lambda_3(t) = 0$ for a.e. $t \in [\zeta_1, \zeta_2] \subset [0, t_f]$, then $u(t)$ is said *singular control*; in this case the Hamiltonian function (5) results to be independent from the control for non isolated instants of time, and Pontryagin minimum principle doesn't helps in finding an optimal solution.

Singular curves. It is natural to ask which kind of path one should expect when a singular control is applied; the answer to this question is not straightforward, indeed it requires a further investigation.
Let us denote the constant costate variables as $\lambda_1(t) = \bar{\lambda}_1$, $\lambda_2(t) = \bar{\lambda}_2$, $\forall t \in [0, t_f]$, and introduce the variables

$$\rho := \sqrt{\bar{\lambda}_1^2 + \bar{\lambda}_2^2}\,, \qquad \tan\phi := \frac{\bar{\lambda}_2}{\bar{\lambda}_1} \tag{11}$$

with these the last equation of the Hamiltonian system (7c) can be rewritten as

$$\dot{\lambda}_3 = \rho\, \sin(\theta(t) - \phi) \tag{12}$$

and the Hamiltonian (5) become

$$H(t, \theta, u, \lambda_0, \lambda) = \lambda_0 \,+\, \rho\, \cos(\theta(t) - \phi) \,+\, \lambda_3(t)\, u(t) \tag{13}$$

Finally, the previous one and (8b) bring to

$$\lambda_0 \,+\, \rho\, \cos(\theta(t) - \phi) \,+\, \lambda_3(t)\, u(t) = 0 \tag{14}$$

Now, suppose that the optimal control $u(t)$ is singular for a.e. $t \in [\zeta_1, \zeta_2] \subset [0, t_f]$, i.e., $\lambda_3(t) = 0$ for a.e. $t \in [\zeta_1, \zeta_2]$. In view of (12) follows that $\cos(\theta(t) - \phi) = \pm 1$.
Substituting $\cos(\theta(t) - \phi) = +1$ in (14) one obtains $\rho = -\lambda_0 \leq 0$. But $\lambda_0 < 0$ violates the Pontryagin principle. It also should be noticed that $\rho\,(\bar{\lambda}_1, \bar{\lambda}_2)$ by construction is a positive semidefinite function, assuming null value only for $\bar{\lambda}_1 = \bar{\lambda}_2 = 0$.

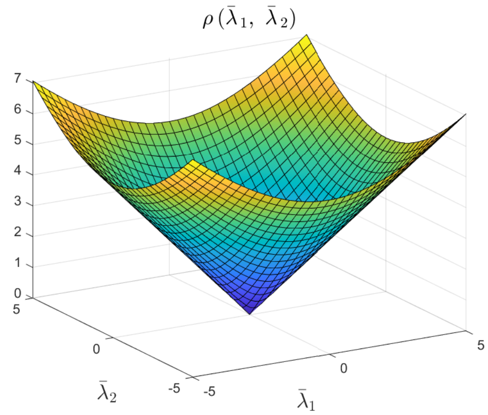


Figure 2: $\rho\,(\bar{\lambda}_1,\ \bar{\lambda}_2)$ function

$\lambda_0 = 0$ is not allowed too, because otherwise there will be a null costate vector. It follows that $\cos(\theta(t) - \phi) = -1$ is the unique possibility, leading to $\rho = \lambda_0 > 0$.

More compactly:
$u(t)$ singular $\Rightarrow\ \lambda_3(t) = 0\ \Rightarrow\ sin(\theta(t) - \phi) = 0\ \Rightarrow\ cos(\theta(t) - \phi) = -1\ \Rightarrow\ \rho > 0$

Now, since $\lambda_3(t) = 0$ for a.e $t \in [\zeta_1, \zeta_2]$, it is also true that $\dot{\lambda}_3(t) = 0$ for a.e $t \in [\zeta_1, \zeta_2]$. Being that said, it is evident from (12) that

$$\rho\ \sin(\theta(t) - \phi) = 0 \tag{15}$$

and keeping in mind that $\rho > 0$, we conclude that $\theta(t) = \phi$ for a.e. $t \in [\zeta_1, \zeta_2]$, and then $\theta(t)$ constant, i.e., $u(t) = 0$.
This proves the fact that a singular control corresponds to straight line segment. The associated optimal control is called *bang-singular* control:

$$\boxed{u(t) = -a\ \mathrm{sgn}\,(\lambda_3(t)),\ \text{a.e. } \forall t \in [0, t_f]} \tag{16}$$

Nonsingular curves. At this point it is still unknown how the system behave in case of *nonsingular control*. Different cases arise in the analysis of the *nonsingular curves*.

(i) Suppose $\rho = 0$. Then from (14) and (16) follows

$$\lambda_0 = -\lambda_3(t)\, u(t) = a\, \lambda_3(t)\, \mathrm{sgn}(\lambda_3(t)) = a\, |\lambda_3(t)| \geq 0 \tag{17}$$

But $\lambda_0 = 0$ implies that $\lambda_3(t)\, u(t) = 0$, which cannot be true because both $\lambda_3(t)$ and $u(t)$ are different from zero (the costate vector cannot be null and $u(t) \neq 0$ by nonsingularity hypothesis). Then the only possibility is $\lambda_0 > 0$, i.e., only *normal* optimal solutions exist when $\rho = 0$. $\lambda_3(t)$ is constrained to be

$$\lambda_3(t) = \pm \frac{\lambda_0}{a} \tag{18}$$

Being that said, clearly the control $u(t)$ takes values $\pm a$ $\forall t \in [0, t_f]$.

(ii) Suppose $\rho > 0$ and $\lambda_0 > 0$ (i.e. *normal case*), with $\rho \neq \lambda_0$. It was shown previously that if $0 < \rho \neq \lambda_0 > 0$, then the control is *bang-bang* type.

(iii) Suppose $\rho > 0$ and $\lambda_0 = 0$ (i.e. *abnormal case*), with $\rho \neq \lambda_0$. Introduce the differential equation

$$\dot{\lambda}_3^2(t) + (a\, |\lambda_3(t)| - \lambda_0)^2 = \rho^2 \tag{19}$$

which is solved by the costate variable λ_3. This equation, under the previous hypothesis, reduces to

$$\dot{\lambda}_3^2(t) + a^2\, \lambda_3^2(t) = \rho^2 \tag{20}$$

But if $\lambda_3(t) = 0$, then (20) implies that $\dot{\lambda}_3(t) \neq 0$, and therefore optimal control cannot be of singular type.
This case is the only case of abnormal optimal solutions; in this context is worth to remark two facts related to this kind of solutions:

- An abnormal optimal path is either of type CC or C.
- Its length is at most $2\pi/a$ if CC, or π/a if C.
- An abnormal solution is also a normal solution. The converse does not holds.

Cases (i), (ii) and (iii) show that the optimal control associated to nonsingular curves can only be of bang-bang type. If this is the case, the corresponding optimal path does not contains straight line segments, but it results to be the concatenation of arcs of minimum turning radius.

4 Simulations

Before diving into the examples it is necessary to spend some words to explain how the switching function is computed in practice. Suppose first the optimal control problem to be normal ($\lambda_0 > 0$), and without loss of generality fix $\lambda_0 = 1$. The cases of singular and nonsingular curves need to be treated separately. Consider first the singular case. Allowed paths of this kind are only of type

$$\{CSC,\, CS,\, SC, S\}$$

This fact was fully proven in both [1] (with geometrical arguments) and [2] (with optimal control arguments). As shown in section 3, along the subarc S $\lambda_3(t) = 0$ and $\rho = \lambda_0 = 1$. Equation (14) reduces to $\cos(\theta(t) - \phi)$=-1, with $\theta(t) = \bar{\theta}$ constant in time, and clearly, $\phi = \bar{\theta} \pm \pi$. Being that said, the switching function $\lambda_3(t)$ can be uniquely computed as follows:

$$\lambda_3(t) = \begin{cases} -\dfrac{[\cos(\theta(t) - \bar{\theta} + \pi) + 1]}{u(t)} & \text{if C arc} \\ 0\,, & \text{if S arc} \end{cases} \tag{21}$$

$\bar{\theta}$ can be thought as the orientation of the unicycle robot through the straight line segment. With similar reasoning one can easily find the expressions of the switching function for nonsingular curves. Allowed paths of this kind are only of type

$$\{CCC,\, CC,\, C\}$$

All cases are summarized here:

$$\text{CCC:} \qquad \lambda_3(t) = -\frac{\sec((\theta_1 - \theta_2)/2)[\cos(\theta(t) - (\theta_1 + \theta_2)/2) + 1]}{u(t)} \tag{22}$$

$$\text{CC normal:} \qquad \lambda_3(t) = -\rho\, \frac{\cos(\theta(t) - \theta_1 + 2\pi/3) + 1}{u(t)} \tag{23}$$

$$\text{CC abnormal:} \qquad \lambda_3(t) = -\rho\, \frac{\cos(\theta(t) - \theta_1 + \frac{1}{2}\pi)}{u(t)} \tag{24}$$

$$\text{C normal:} \qquad \lambda_3(t) = -\rho\, \frac{\cos(\theta(t)) + 1}{u(t)} \tag{25}$$

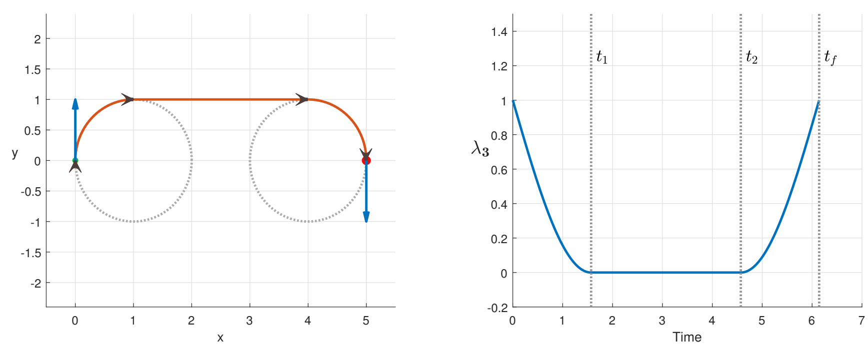
$$\text{C abnormal:} \qquad \lambda_3(t) = -\rho\, \frac{\cos(\theta(t) - \theta_1 + \frac{1}{2}\pi)}{u(t)} \tag{26}$$

Where θ_1 and θ_2 represent the orientation of the robot in t_1 and t_2 (switching instants).

A series of practical examples follow. Simulations are done in Matlab (version R2019b). Note that the dynamics of the unicycle robot is nonlinear, and therefore the optimal control problem (3)-(4) is a nonlinear (nonconvex) optimization problem. This motivates the usage of the Navigation Toolbox (version 1.0) which offers built-in methods for Dubins path computations.
Example 1 is a basic example of singular CSC optimal paths, while 2 and 3 replicate the examples already addressed in [2] (using Matlab instead of AMPL). Example 4 is devoted to the computation a nonsingular CC optimal path.

Additional information on the notation used:
ξ_i is the length of the i-th arc.
t_i is the time instant at which the i-th arc ends. t_f is the total time.
R, L, S refer respectively to a right, left or straight line arc curves.

Example 1. We are looking for the shortest path from the initial state $p_0 = (0,\, 0,\, \frac{1}{2}\pi)^T$ to the final state $p_f = (5,\, 0,\, \frac{3}{2}\pi)^T$; the minimum turning radius is $R = 1$. Speed is assumed to be unitary.



(a) Dubins path (b) Switching function

Figure 3: Dubins path and switching function associated to the *example 1*.

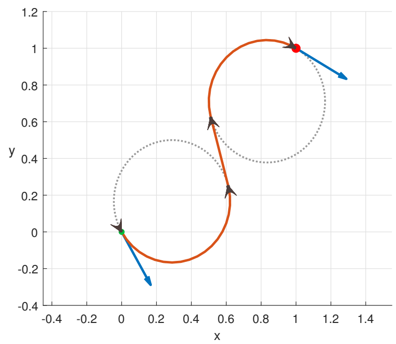
The resulting optimal curve is of type CSC, or more precisely $R_{\xi_1} S_{\xi_2} R_{\xi_3}$.

$$\xi_1 = 1.5708, \quad \xi_2 = 3, \quad \xi_3 = 1.5708$$

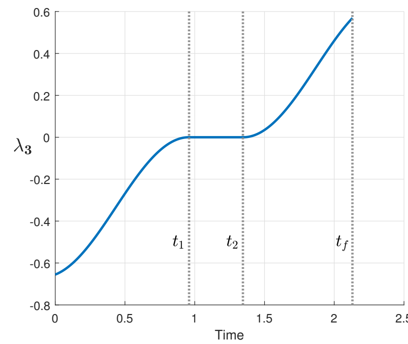
$$t_1 = 1.5708, \quad t_2 = 4.5708, \quad t_f = 6.1416$$

Figure 3a shows the Dubins path, while *Figure 3b* illustrates how the switching function $\lambda_3(t)$ evolves while the robot moves along the path. The presence of a straight line segment tell us that the control is necessary of bang-singular type. As expected from (16), the function assumes zero values throughout the straight arc segment. In both first and third arcs $\lambda_3(t) > 0$ implies $u(t) = -a$, corresponding to right curves.

Example 2. Now the initial state is assumed to be $p_0 = (0,\ 0,\ -\frac{1}{3}\pi)^T$, the final state $p_f = (1,\ 1,\ -\frac{1}{6}\pi)^T$, $R = 1/3$.



(a) Dubins path



(b) Switching function

Figure 4: Dubins path and switching function associated to the *example 2*.

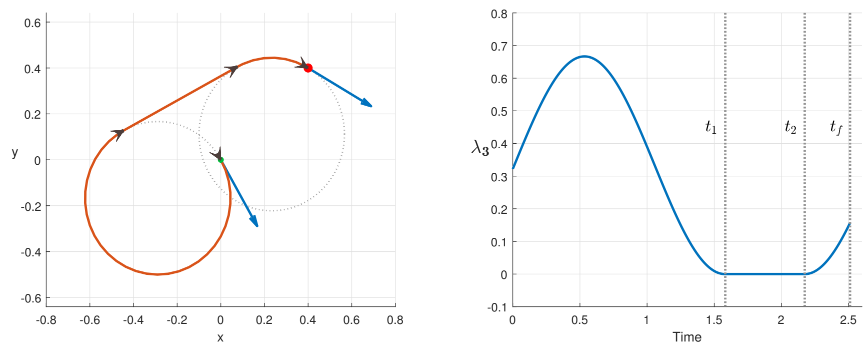
The resulting Dubins path is still of type CSC, this time $L_{\xi_1} S_{\xi_2} R_{\xi_3}$.

$$\xi_1 = 0.9596, \quad \xi_2 = 0.3858, \quad \xi_3 = 0.7851$$

$$t_1 = 0.9596, \quad t_2 = 1.3454, \quad t_f = 2.1305$$

Again, the switching function behave as expected, assuming negative values when the robot turns left, positive values when the robot turns right, and zero corresponding to straight line segments.

Example 3. Initial state is $p_0 = (0,\ 0,\ -\frac{1}{3}\pi)^T$, the final state $p_f = (0.4,\ 0.4,\ -\frac{1}{6}\pi)^T$, $R = 1/3$. This example is similar to the previous one, but the starting and final positions are placed closer. The resulting Dubins path in *Figure 5a* appears to be completely different in both shape and length.



(a) Dubins path (b) Switching function

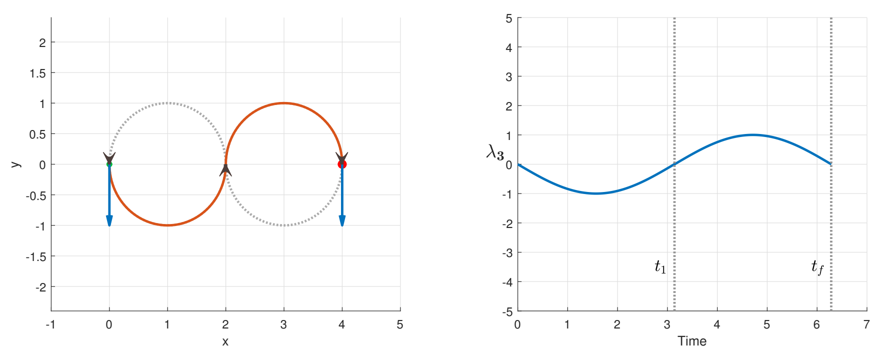
Figure 5: Dubins path and switching function associated to the *example 3*.

The optimal path is of type $R_{\xi_1} S_{\xi_2} R_{\xi_3}$.

$$\xi_1 = 1.5822, \quad \xi_2 = 0.5914, \quad \xi_3 = 0.3376$$

$$t_1 = 1.5822, \quad t_2 = 2.1736, \quad t_f = 2.5113$$

Example 4. Initial and final states are $p_0 = (0,\, 0,\, \frac{3}{2}\pi)^T$, $p_0 = (4,\, 0,\, \frac{3}{2}\pi)^T$; the minimum turning radius is $R = 1$. The resulting optimal path is of type CC, more precisely $L_{\xi_1} L_{\xi_2}$.



(a) Dubins path (b) Switching function

Figure 6: Dubins path and switching function associated to the *example 4*.

$$\xi_1 = 3.1416, \quad \xi_2 = 3.1416$$

$$t_1 = 3.1416, \quad t_f = 6.2832$$

The resulting control is of bang-bang type, and the solution is abnormal. The switching function for an abnormal CC path is computed using equation 24. Notice from *Figure 6b* that $\lambda_3(t)$ assumes null value only in an isolated instant of time, when $\theta(t) = \theta(t_1) = \theta_1 = \frac{1}{2}\pi$.

5 Conclusions

This study has revisited some aspects of the Dubins' theory from an optimal control viewpoint. The optimal control has been found by means of Pontryagin maximum principle, and then the focus has shifted towards the characterization and classification of optimal paths. Finally, simulation outcomes resulted to be coherent with the exposed theoretical concepts.

References

[1] L. E. Dubins, *On Curves of Minimal Length with a Constraint on Average Curvature, and with Prescribed Initial and Terminal Positions and Tangents*, American Journal of Mathematics, Vol. 79, No. 3, 1957.

[2] C. Yalçın Kaya, *Markov-Dubins Path via Optimal Control Theory*, 2017.

[3] U. Halder, U. Kalabic, *Time-Optimal Solution for a Unicycle Path on SE(2) with a Penalty on Curvature*, 2017.

[4] H. H. Johnson, *An Application of the Maximum Principle to the Geometry of Plane Curves*, American Mathematical Society, Vol. 44, No. 2, June 1974.

[5] J. A. Reeds, L. A. Sheep, *Optimal Paths for a Car that goes both Forwards and Backwards*, Pacific Journal of Mathematics, Vol. 142, No. 2, 1990.

[6] D. Liberzon. *Calculus of Variations and Optimal Control Theory*, Princeton university press, 2012.

[7] C. Bruni, G. Di Pillo. *Metodi variazionali per il controllo ottimo*, Aracne Editrice, 2007.