# Shortest path of bounded curvature in the plane

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# Plus courts chemins de courbure bornée dans le plan

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## Abstract

Given two oriented points in the plane, we determine and compute the shortest paths of bounded curvature joining them. This problem has been solved recently, by Dubins in the no-cusp case, and by Reeds and Shepp otherwise. We propose a new solution based on the minimum principle of Pontryagin. Our approach simplifies the proofs and makes clear the global or local nature of the results.

## Résumé

Etant donnés deux points orientés dans le plan, on caractérise les plus courts chemins de courbure bornée les joignant. Le problème a été traité récemment par Dubins dans le cas sans rebroussement et par Reeds et Shepp dans le cas général. Nous proposons une nouvelle solution basée sur le principe du minimum de Pontryagin. Notre approche simplifie les démonstrations et précise la nature locale ou globale des résultats.

# 1 Introduction

The question considered here is the following : given two oriented points $(M_i, \theta_i)$ and $(M_f, \theta_f)$ in the plane, determine and compute the shortest piecewise regular paths joining them, along which the curvature is everywhere bounded by a given $\frac{1}{R} > 0$. Minimizing the length is meaningful both in the class of paths which are $C^1$ and piecewise $C^2$, and in the slightly larger class of paths admitting a finite number of cusps.

This question appears in many applications : for instance Markov[3] studied the no–cusp case for joining pieces of railways. A 3–dimensional version applies to planning plumbary networks, or the version with cusps to any car driver.

Even without obstacles, characterizing the shortest paths is not simple. This was only done recently, by Dubins[2] in the no–cusp case, and by Reeds and Shepp[5] otherwise.

Our way of solving the question here is entirely different from theirs. It is both much simpler and better adapted to further generalization to the case of obstacles limiting moves. The essential tool we use is the powerful result of optimal control theory known as the "minimum principle of Pontryagin". We recall in Section 2 its basic version which we will use, and refer to classical books in control theory, like [1], [4], and [7], for its quite delicate proof.

In Section 3 we apply the principle to our case, and deduce some general crucial lemmas. Section 4 is devoted to the no–cusp case, and Section 5 to the more difficult case with cusps.

Our results are essentially the same as those of [2] and [5]. The interest of the present work lies in the method of proof, both simplified by the use of a single idea, and as local as the statements will allow. Indeed, we make a clear distinction between local and global proofs, and we insist on local proofs in view of further work dealing with obstacles.

Related results to be reported in a forthcoming publication have been obtained independently by Sussmann and Tang [6]. The results in [2] and [5] are also deduced from the minimum principle and new lights on the piecewise regularity of optimal controls.

# 2 The minimum principle : a basic version

Given are :

- two integers $n$ and $r$, two points $(x^i)$ and $(x^f)$ in $\mathbb{R}^n$, and a compact subset $U$ of $\mathbb{R}^n$,
- a $C^0$ function $f(x,u) : \mathbb{R}^n \times U \to \mathbb{R}^n$,
- a $C^0$ function $f_0(x,u) : \mathbb{R}^n \times U \to \mathbb{R}^n$.

A "control" is a piecewise continuous (but not necessarily continuous) function $u(t) : [0,T] \to U$, for some $T > 0$. $\mathcal{U}$ denotes the set of controls. We want to find a $u \in \mathcal{U}$ which minimizes the integral

$$J(u) = \int_0^T f_0(x(t), u(t))dt\,.$$

Here $x(t) = (x_1(t) \cdots, x_n(t)) : [0,T] \to \mathbb{R}^n$ is a solution of the differential system with boundary conditions

$$\begin{cases} \dfrac{dx_j}{dt} = f_j(x, u(t)) \ \ (j = 1, \ldots, n) \\ \\ x(0) = x^i;\ x(T) = x^f \end{cases} \tag{1}$$

$J(u)$ is called the "cost" of the "path" $x(t)$, solution of (1), given the "control" $u(t)$. The solution $x(t)$ of the system with initial conditions $x(0) = x^i$ is well determined for a given $u$. This is the case even in a much more general setting (see e.g. [7, Theorem II.4.11]). We denote by $\mathcal{U}_i^f$ the subset of "admissible" controls $u \in \mathcal{U}$ such that the associated paths $x$ satisfy also the final conditions $x(T) = x^f$. The optimal control problem is to find controls $u^* \in \mathcal{U}_i^f$ satisfying

$$J(u^*) = \min_{u \in \mathcal{U}_i^f} J(u)$$

Such an $u^*$ is called "optimal", as well as the associated path $x$. Note that $T$ is here arbitrary. But in the particular case $f_0 \equiv 1$, $J(u) = T$ and we want to minimize the "time" to go from $x^i$ to $x^f$.

We give a more geometric interpretation to the approach by adding an extra variable $x_0(t)$, solution of $\frac{dx_0}{dt} = f_0(x, u(t)), x_0(0) = 0$. We are therefore looking for a $u \in \mathcal{U}$ such that the solution of

$$\begin{cases} \dfrac{dx_j}{dt} = f_j(x, u(t)) \ (j = 0, \ldots, n) \\ \\ x_j(0) = x^i \end{cases} \qquad (j = 0, \ldots, n) \tag{2}$$

satisfies the conditions $x_j(T) = x_j^f$ $(j = 1,\dots,n)$ and $J(u) = x_0(T)$ is minimal (for some $T > 0$). A basic idea in mechanics is then to introduce "dual" variables $\psi = (\psi_0,\psi_1,\dots,\psi_n) : [0,T] \longrightarrow I\!R^{n+1}$ which are continuous, piecewise $C^1$, and solution of the "adjoint" system

$$\left\{\begin{array}{l} \dfrac{d\psi_j}{dt} = -\displaystyle\sum_{i=0}^{n} \frac{\partial f_j}{\partial x_i}(x,u(t))\psi_j \ \ (j = 0,\dots,n) \\ \\ \psi(0) = \psi^i \, . \end{array}\right. \tag{3}$$

For a given $u \in \mathcal{U}_i^f$, $x$ the associated solution of (2), and an arbitrary initial condition $\psi^i$, (3) has a unique solution $\psi$.

If the "Hamiltonian" is defined by $H(\psi,x,u) : I\!R^{2n+2+r} \longrightarrow I\!R$

$$H(\psi,x,u) =< \psi, f >= \sum_{j=0}^{n} \psi_j f_j(x,u)\,, \tag{4}$$

then, equations (2) and (3) can be rewritten as a Hamilton–Jacobi system with parameter $u$ :

$$\left\{\begin{array}{ll} \dfrac{dx_j}{dt} = \dfrac{\partial H}{\partial \psi_j} \quad , \ x_j(0) = x_j^i & \\ & j = 0,\dots,n \\ \dfrac{d\psi_j}{dt} = -\dfrac{\partial H}{\partial x_j} \quad , \ \psi_j(0) = \psi_j^i\,. & \end{array}\right. \tag{5}$$

Finally, we define

$$M(\psi,x) = \min_{u\in U} H(\psi,x,u)$$

where $\psi, x, u$ are considered as independent variables. The fundamental result of Pontryagin[4] is then :

**Theorem 1 (Minimum principle)** : *If $u^*$ is an optimal admissible control, there exists a non-zero adjoint vector $\psi$, and $T > 0$, such that, $(x(t),\psi(t))$ being the solution of (5) for $u = u^*$, one has :*

1. *$\forall t \in [0,T]$ $H(\psi(t),x(t),u^*(t)) \equiv M(\psi(t),x(t))$*
2. *Furthermore, $\forall t \in [0,T]$, $M(\psi(t),x(t)) \equiv 0$ and $\psi_0(t) \equiv \psi_0(0) \geq 0$.*

This changes the question of minimizing the functional $J(u)$ over $\mathcal{U}$ into a minimum problem for the "scalar" function $H$ on $U$. In some sense, conditions 1 and 2 hereabove insure that $(x(t), \psi(t))$ is "stationary" among the solutions of (5), and the principle asserts that the optimal paths are to be found only among these. The minimum principle of course only gives necessary conditions, and does not even assert that an optimal control exists. Its existence has to be proved independently, and it is usually done in the much larger class of controls which are only assumed to be integrable (see e.g. [7, Theorem V.6.1]). The minimum principle applies just as well in this larger class (see [4, Chapter II]), and the following computations (Section 3) will then prove that the optimal controls in fact belong to our smaller class $\mathcal{U}$. This is why we allow ourselves to restrict our discussion to the class $\mathcal{U}$, as well as for the sake of clarity, and for emphasis on applications.

A new proof of the piecewise regularity of optimal controls can be found in [6], in a general setting which includes our case, based on the properties of subanalytic sets.

Most theorems of existence assume that the range of controls is convex (see e.g. [7, Theorem V.6.1]). This is the technical reason why we consider convex ranges of control in Section 3.

# 3 Application to shortest paths of bounded curvature

Let us recall that we want to minimize the length of continuous and piecewise $C^2$ paths $(x(t), y(t))$ in the plane $\mathbb{R}^2$ joining given initial and final points with given orientations $(x^i, y^i, \theta^i)$ and $(x^f, y^f, \theta^f)$. By assumptions on the control, such paths are formed of a sequence of $C^2$ paths, glued together at isolated commutation points $t_k \in ]0, T[$, where either the curve is $C^1$ (called "inflexion" points) or the orientation is reversed (called "cusps").

Thus the following quantities are well defined.

- The polar angle $\alpha(t)$ of the tangent to the path, considered as a globally continuous $\mathbb{R}/2\pi$–valued function by assuming its continuity at the cusps. Its intuitive meaning is that the tangent to the path is directed as the lights of a car that would follow the path, changing from front to rear gear or conversely at a cusp.

- The arc length along the path, which we denote by $t$ : intuitively, the path is run at constant speed one.

- The curvature $u(t) = \frac{d\alpha}{dt}$, which is defined everywhere except at the commutation points. When $u(t) \neq 0$, its sign depends on whether the point $(x, y)$ runs in clockwise sense ($u < 0$) or in counter–clockwise sense ($u > 0$) as $t$ increases.

The differential system (2) can be written as (from now on we write $\dot{z}$ for $\frac{dz}{dt}$) :

$$\left\{\begin{array}{lll} \dot{x} = \varepsilon \cos\alpha & x(0) = x^i & x(T) = x^f \\ \dot{y} = \varepsilon \sin\alpha & y(0) = y^i & y(T) = y^f \\ \dot{\alpha} = u & \alpha(0) = \alpha^i & \alpha(T) = \alpha^f \\ \dot{x_0} = 1 & x_0(0) = 0 & \end{array}\right. \tag{6}$$

with control functions $(\varepsilon, u) \in U \subset \mathbb{R}^2$, where $U = \{-1, +1\} \times [-\frac{1}{R}, +\frac{1}{R}]$. This means that we control the instantaneous curvature $u$, allowing also changes between front and rear gears (the sign of $\varepsilon$). The cost we want to minimize is equal to the length of the trajectory and defined by :

$$J(u) = \int_0^T dt = T \tag{7}$$

For the technical reasons already mentioned at the end of section 2, we should assume here $U$ convex, more precisely : $U = [-1, +1] \times [-\frac{1}{R}, +\frac{1}{R}]$, that is to say $-1 \leq \varepsilon \leq 1$. We modify (6) accordingly and consider now the system :

$$\left\{\begin{array}{lll} \dot{x} = \varepsilon \cos\alpha & x(0) = x^i & x(T) = x^f \\ \dot{y} = \varepsilon \sin\alpha & y(0) = y^i & y(T) = y^f \\ \dot{\alpha} = |\varepsilon|\, u & \alpha(0) = \alpha^i & \alpha(T) = \alpha^f \\ \dot{x_0} = 1 & x_0(0) = 0\,. & \end{array}\right. \tag{8}$$

In this case, $t$ is the time, and no longer the arc length $s$, and we have : $ds = |\varepsilon(t)|\, dt$. The expression (7) of the cost remains unchanged, although it corresponds now to a minimum time problem. Calling $(p, q, \beta, e)$ a set of dual variables to $(x, y, \alpha, x_0 = t)$, the minimum principle applies here. The Hamiltonian is

$$H = e + \varepsilon\, p \cos\alpha + \varepsilon\, q \sin\alpha + |\varepsilon|\, \beta\, u\,, \tag{9}$$

and the adjoint system

$$\left\{\begin{array}{l} \dot{p} = 0 \\ \dot{q} = 0 \\ \dot{\beta} = \varepsilon p \sin\alpha - \varepsilon q \cos\alpha \\ \dot{e} = 0 \end{array}\right. \tag{10}$$

with arbitrary (but not all zero) initial values. Thus $p, q, e$ are constant on $[0, T]$. Putting $p = \lambda \cos\phi$, $q = \lambda \sin\phi$, with $\lambda = \sqrt{p^2 + q^2} \geq 0$, determines an angle $\phi$ modulo $2\pi$, such that $\tan\phi = \frac{q}{p}$. We rewrite (9) and (10) as :

$$H = e + \varepsilon\lambda\cos(\alpha - \phi) + |\varepsilon|\beta u \tag{11}$$

$$\dot{\beta} = \varepsilon\lambda\sin(\alpha - \phi) \tag{12}$$

As $H$ is a piecewise affine function in $\varepsilon$, it cannot reach its minimum w.r.t. $\varepsilon$ elsewhere than at $\varepsilon = 0, \pm 1$. But $\varepsilon \equiv 0$ on some interval is obviously irrelevant since it corresponds to zero velocity which is trivially not an optimal control for a minimum time problem. Thus, condition 1 of the minimum principle asserts that optimal controls can only be obtained for $\varepsilon = \pm 1$, so that we set $|\varepsilon| = 1$ in the following of the discussion. Hence, the arc length becomes equal to $t$, and minimum time solutions provide minimum length paths. Moreover, $H$ rewrites

$$H = e + \varepsilon\lambda\cos(\alpha - \phi) + \beta u \tag{13}$$

and systems (6) and (8) are identical. Condition 1 also states that for an optimal control $u$, along any $C^2$ piece of the optimal path, we have

$$\lambda\varepsilon\cos(\alpha - \phi) \leq 0 \text{ and } \beta u \leq 0\,. \tag{14}$$

Furthermore, either one of the following two cases holds :

- $\frac{\partial H}{\partial u} = \beta \equiv 0$, thus $\dot{\beta} \equiv 0$, and $\alpha \equiv \phi$ or $\alpha \equiv \phi + \pi$, and the path is a line segment with direction $\phi$,
- or $\frac{\partial H}{\partial u} \not\equiv 0$ and thus, by condition 1, $u = \pm\frac{1}{R}$, and the path is an arc of circle of radius $R$.

The preceding discussion leads immediately to :

**Proposition 2** *Any optimal path is the concatenation of arcs of circles of radius $R$, and line segments all parallel to some fixed direction $\phi$.*

Also condition 2 of the minimum principle ($H \equiv 0$ and $e \geq 0$) implies :

**Lemma 3** *$\beta = 0$ at the inflexion points and on the line segments.*

*Proof*: That $\beta = 0$ on the line segments was already mentioned. It implies $\beta = 0$ at the inflexions between a line segment and an arc of circle, since $\beta$ is continuous. But at an inflexion between two arcs, $u$ changes sign, and so should $\beta$ in order that $H \equiv 0$ ; thus the same continuity argument applies. □

**Lemma 4** *If $\lambda = 0$, the optimal path is either a line segment, or a sequence of arcs of radius $R$ joined by cusps.*

*Proof*: (13) and condition 2 imply $\beta\, u \equiv -e$. If $e = 0$, $\beta$ cannot vanish since $\psi = (p, q, \beta, e)$ would then vanish, which is forbidden by the principle. Thus $u \equiv 0$, and the whole path is one line segment. If $e > 0$, then neither $\beta$ nor $u$ can vanish, and the path contains neither line segments nor inflexions, by Lemma 3. □

In the following, we assume $\lambda \neq 0$.

**Lemma 5** $e > 0$

*Proof*: Assume that $e = 0$. Then, by (13), $\beta\, u + \varepsilon\, \lambda\, \cos(\alpha - \phi) \equiv 0$. But the two parts of the sum are of the same sign by (14) so that they have to be zero. Since $|\varepsilon| = 1$ and $\lambda \neq 0$, this implies (i) : $\cos(\alpha - \phi) = 0$ and (ii) : $\beta\, u = 0$. By (i), $\alpha = \phi \pm \frac{\pi}{2}$ is constant, and this path is a line segment along which, necessarily, $\beta = u = 0$ and $\dot{\beta} = 0$. This implies $\alpha = \phi$ or $\alpha = \phi + \pi$, and leads to a contradiction. Hence $e \neq 0$ and the result follows from the condition 2 ($e \geq 0$) of Theorem 1. □

**Lemma 6** *$\beta - p\,y + q\,x$ is constant along any optimal path. Consequently, all the points of an optimal path where $\beta$ takes the same value are on the same straight line of direction $\phi$.*

*Proof*: According to (6) and (10) $\dot{\beta} - p\,\dot{y} + q\,\dot{x} \equiv 0$ , and thus $\beta - p\,y + q\,x \equiv c$ for some $c \in \mathbb{R}$, on the whole optimal path. □

Lemma 6 has the two following consequences :

**Lemma 7** *On any optimal path, the line segments and the inflexion points are all on a straight line $D_0$ with direction $\phi$ (of equation $p\,y - q\,x + c = 0$).*

*Proof*: Apply Lemma 6 with $\beta = 0$ □

**Lemma 8** *All the cusps of any optimal path are on two straight lines $D_\pm$ parallel and equidistant to $D_0$, of equation $p\,y - q\,x + c = \mp e\, R$. Furthermore, (a) there are no cusps between a line segment and an arc of circle, (b) the "positive" cusps (that is where $u > 0$ both before and after) are on $D_+$, and the others on $D_-$, (c) the half-tangent at a cusp is perpendicular to $D_\pm$.*

*Proof*: By condition 2 of the principle, $0 \equiv e + \varepsilon\,\lambda\,\cos(\alpha - \phi) + \beta\,u$. Near a cusp, $u$ remains constant and equal to $\pm\frac{1}{R}$, while $\varepsilon$ changes from $+1$ or $-1$ or the converse. Hence, by (14), $\cos(\alpha - \phi) = 0$; thus $\alpha = \phi \pm \frac{\pi}{2}$ and $\beta = \pm e\,R$. The first assertion follows from Lemma 6, then (a) and (c) from Lemma 7, and (b) from (14). □

Moreover, we have :

**Lemma 9** *Any $C^2$ arc of circle of an optimal path, between two points where $\beta = 0$, has length $> \pi R$.*

*Proof*: Assume the length of such an arc, between $t = t_1$ and $t = t_2$, to be $\leq \pi R$. As $\beta(t_1) = \beta(t_2) = 0$, we have by (13) and since $H \equiv 0$,

$$\varepsilon\,\cos(\alpha(t_1) - \phi) = \varepsilon\,\cos(\alpha(t_2) - \phi) = -\frac{e}{\lambda} \tag{15}$$

$\beta$ is $C^1$ on $[t_1, t_2]$ and it keeps a constant sign on an arc, by (14). So, $\beta$ reaches an extremum at some point $t_3 \in ]t_1, t_2[$ where $\dot{\beta}(t_3) = \varepsilon\,\lambda\,\sin(\alpha(t_3) - \phi) = 0$ by (12). Now, $\varepsilon\,\cos(\alpha(t_3) - \phi) = -1$, by (15) and since the sign of $\cos(\alpha(t_3) - \phi)$ cannot change on $[t_1, t_2]$, according to our assumption on the lentgh. But then

$$0 \equiv H = e - \lambda + \beta(t_3)\,u(t_3) \leq e - \lambda\,,$$

by (14), and thus $\frac{e}{\lambda} \geq 1$, which compared with (15) implies $e = \lambda$. Since $|u^*(t_3)| = \frac{1}{R} > 0$, $\beta(t_3) = 0$. Hence $\beta \equiv 0$ on $[t_1, t_2]$ ; this means by (12) that this piece of path would be a line segment, not an arc of circle. □

We remark now that we may simplify our equations by using obvious symetries. The rectangular coordinate system in $\mathbb{R}^2$ is arbitrary. So, we can assume $\phi = 0$ by a rotation of axes, $\lambda = 1$ or 0 since the change from $\frac{\psi}{\lambda}$ to $\psi$ leaves the minimum principle invariant when $\lambda > 0$, and the constant $c$ in Lemmas 6, 8 to be zero by a translation of the origin in $\mathbb{R}^2$. Under these assumptions we have $p = 1$, $q = 0$. The lines $D_0$, $D_\pm$ have now the equations $y = 0$ and $y = \mp e\,R$, and $\beta \equiv y$ is the signed distance of a point on the path to $D_0$. Furthermore, we have

$$H = e + \varepsilon\,\cos\alpha + u\,\beta \equiv 0\,. \tag{16}$$

Consider now an arc of circle with a cusp at one of its endpoints. If this endpoint is on $D_-$, $u = -\frac{1}{R}$ and, since at a cusp $\alpha = \pm\frac{\pi}{2}$, by (16), $0 \leq \beta =$

$e\,R + \varepsilon\,R\cos\alpha \leq e\,R$ since $\varepsilon\cos\alpha \leq 0$ by (14). The same argument shows that on an arc with a cusp on $D_+$, one has $0 \geq \beta \geq -e\,R$. Summarizing this discussion, we get :

**Proposition 10** *For any optimal path with $\lambda \neq 0$, there exists a rectangular coordinate system in $\mathbb{R}^2$ such that (see figure 1)*

- *all the line segments and inflexion points on the path lie on the first axis $D_0 : \; y = 0$,*
- *all the cusps where $u > 0$ lie on the line $D_+ : \; y = -e\,R$,*
- *all the cusps where $u < 0$ lie on the line $D_- : \; y = +e\,R$,*
- *an arc of circle with a cusp as one of its endpoints on $D_+$ ($D_-$) lies between $D_+$ and $D_0$( between $D_0$ and $D_-$),*
- *(16) is satisfied at any point on the path.*

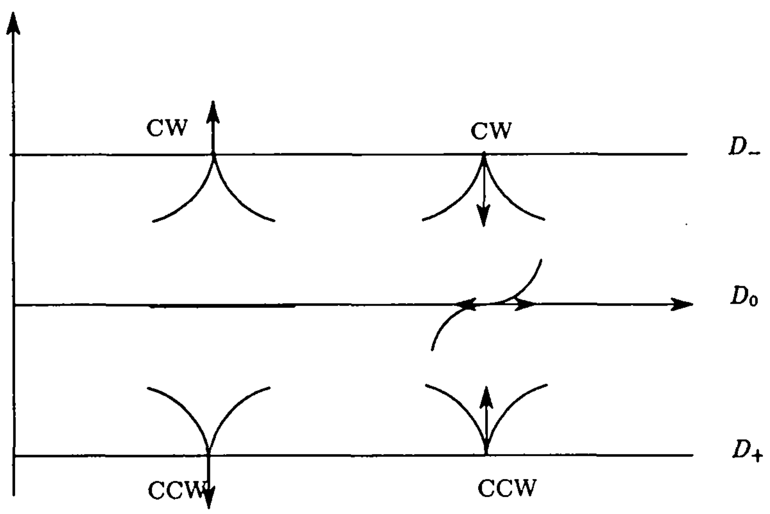


Figure 1: : $D_0$, $D_+$, $D_-$

**Remark 1** (a) Outside the three lines $D_0$, $D_+$, $D_-$ there can be no commutation and no line segment. A piece of an optimal path not intersecting these lines can only consist of one arc of circle, and is contained in one of the strips $[D_+, D_0]$ and $[D_0, D_-]$ as soon as it is not an initial or a final arc. (b) By (16) and since $u \neq 0$ outside $D_0$, the geometric angle of the tangent line to an optimal path with $D_0$ at a point $(x, y)$ is almost determined by the "distance" $y$ of the point to $D_0$. Indeed $\alpha$ satisfies

$$\varepsilon \cos\alpha = \pm\frac{y}{R} - e \leq 0 \tag{17}$$

which gives only two possible values of $\alpha$ for any piece of path with no commutation (that is for given $\varepsilon$ and $u$).
(c) As soon as an optimal path contains a line segment, the tangent at the inflexion points is $D_0$ itself. Indeed, at a point on a line segment, $y = 0$, $\alpha = 0 \bmod \pi$ by Proposition 2, thus by (16) $e = \pm\varepsilon = 1$, and (17) then gives $\cos\alpha = \pm 1$ at any point on $D_0$.

Proposition 10 and Remark 1 invite us to classify the different possible kinds of optimal paths according to whether $e > 1$, $e = 1$, or $0 < e < 1$. We will denote each concatenation of segments and arcs by a word like for instance "$C|CSC|C$", each $C$ meaning an arc of circle of radius $R$, $S$ a line segment of direction $\phi = 0$, and $|$ a cusp ; $C_v$ will mean an arc of circle of length $R\,v$.

# 4 The $C^1$ (no cusp) case

The preceding study applies to the characterization of $C^1$ shortest paths between given initial and final positions and orientations in the free plane. It is enough to set $\varepsilon$ to be 1 and allow no cusps in all the statements above (but here $\cos\alpha$ can take arbitrary sign as we do not minimize anymore with respect to $\varepsilon$).

As soon as an optimal path contains a line segment, it has to be of $CSC$ type (or the degenerate forms $CS, SC, S$). Indeed, since $S$ is on $D_0$, any other event would mean another commutation on $D_0$, thus a full circle, which is obviously not optimal.

That an optimal path without segment is necessarily of the $CCC$ kind (or the degenerate forms $CC$ or $C$) is a consequence of the remark that a portion of an optimal path is itself an optimal path, and from the following lemma :

**Lemma 11** *No path of type $CCCC$ is optimal.*

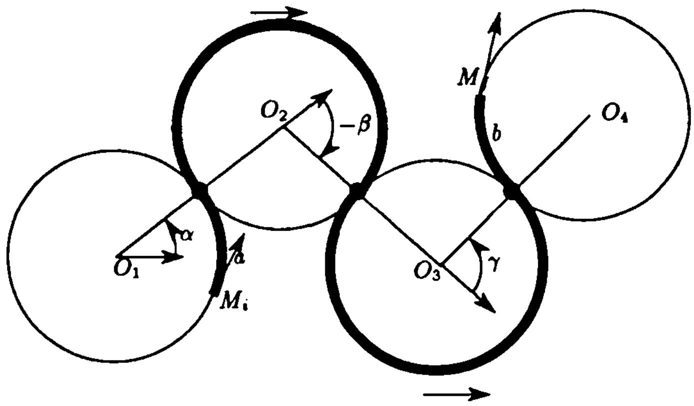


Figure 2: : The case $CCCC$

We give a direct local proof of this lemma, by showing that there is always a strictly shorter path arbitrarily close to the initial one, having the same endpoints, and of the same type $CCCC$.

*Proof*: The notations being clear on Figure 2, let us write $\vec{v}_\theta$ for the unitary vector of polar angle $\theta$, and assume $R = 1$ for simplicity.
If such a path of the $CCCC$ kind were optimal for some values of $a, b$ and for $\alpha = \alpha_0$, $\beta = \beta_0$, $\gamma = \gamma_0$, Lemmas 7 and 9 would imply $\beta_0 = \gamma_0$ and that the intermediary arcs are each of length $> \pi$. Thus

$$\left\{ \begin{array}{l} O_4 = O_1 + 2\vec{v}_{\alpha_0} + 2\vec{v}_{\alpha_0-\beta_0} + 2\vec{v}_{\alpha_0-\beta_0+\gamma_0} \\ L_0 = a + (\pi + \beta_0) + (\pi + \gamma_0) + b \end{array} \right. \tag{18}$$

where $L_0$ is the length of the path. We can deform this path into a similar one with the same endpoints but slightly different angles $\alpha, \beta, \gamma$, and we get in the same way (noting $L$ its length)

$$\left\{ \begin{array}{l} O_4 = O_1 + 2\vec{v}_\alpha + 2\vec{v}_{\alpha-\beta} + 2\vec{v}_{\alpha-\beta+\gamma} \\ L = a + (\alpha - \alpha_0) + (\pi + \beta_0) + (\pi + \gamma_0) + b \\ +(\alpha - \beta + \gamma - \alpha_0 + \beta_0 - \gamma_0) \end{array} \right. \tag{19}$$

so that

$$L - L_0 = 2\,(\alpha - \alpha_0) + 2\,(\gamma - \gamma_0)\,, \text{ thus } dL = 2\,(d\alpha + d\gamma) \tag{20}$$

$$\begin{cases} \cos\alpha + \cos(\alpha-\beta) + \cos(\alpha-\beta+\gamma) \\ = \cos\alpha_0 + \cos(\alpha_0-\beta_0) + \cos(\alpha_0-\beta_0+\gamma_0) \\ \sin\alpha + \sin(\alpha-\beta) + \sin(\alpha-\beta+\gamma) \\ = \sin\alpha_0 + \sin(\alpha_0-\beta_0) + \sin(\alpha_0-\beta_0+\gamma_0) \end{cases} \tag{21}$$

The Jacobian of (21) with respect to $\beta$ and $\gamma$ is equal to $-\sin\gamma$ thus not zero near $(\alpha_0, \beta_0, \gamma_0)$ since $0 < \beta_0 = \gamma_0 < \pi$. So system (21) defines implicitly $\beta$ and $\gamma$ as functions of $\alpha$ in a neighborhood of $(\alpha_0, \beta_0, \gamma_0)$, which proves that such a deformation is possible. Differentiating twice (21) with respect to $\alpha$, we get

$$\begin{cases} \dfrac{d\beta}{d\alpha} = \dfrac{\sin(\beta-\gamma) - \sin\gamma}{-\sin\gamma}, \; \dfrac{d\gamma}{d\alpha} = \dfrac{\sin(\beta-\gamma) + \sin\beta}{-\sin\gamma} \\ \dfrac{d^2\gamma}{d\alpha^2} = \dfrac{-1}{\sin\gamma}[\cos\beta + \cos(\beta-\gamma) + (1+\cos\gamma)(1-\dfrac{d\beta}{d\alpha})^2 \\ +(1+\cos\gamma)(1-\dfrac{d\beta}{d\alpha}+\dfrac{d\gamma}{d\alpha})^2] \end{cases}$$

In particular, $\frac{d\gamma}{d\alpha}(\alpha_0) = -1$, and thus $\frac{dL}{d\alpha}(\alpha_0) = 0$, by (20) as expected. But further

$$\frac{d^2L}{d\alpha^2}(\alpha_0) = 2\frac{d^2\gamma}{d\alpha^2}(\alpha_0) = -4\frac{1+\cos\beta_0}{\sin\beta_0} = -4\cot\frac{\beta_0}{2} < 0\,,$$

and this proves $L(\alpha) < L_0$ in a neighborhood of $\alpha_0$. □

Thus we get the result of [2] :

**Theorem 12 (Dubins)** *Any $C^1$ and piecewise $C^2$ shortest path of bounded curvature in the free plane between given endpoints and orientations is either of type $CSC$, or of type $CC_vC$ with $v > \pi$, or a degenerate form of these.*

**Remark 2** Dubins[2] (and Reeds and Shepp[5] in case of cusps) seem to get a more general result, as they minimize the length in a larger class of paths. But our conclusion amounts to the same, as we already explained at the end of Section 2.

## 5 The general case (allowing cusps)

We go back to the general statements and results of Section 3. The case of an optimal path for which $\lambda = 0$ has already been settled by lemma 4.

Notice that in this case, any path of type $S$ is obviously optimal, as well as any path of type $C_{v_1}|C_{v_2}|\ldots|C_{v_p}|\ldots$ when $v_1+\ldots+v_p+\ldots \leq \pi$, since all these paths have the same length $R(v_1+\ldots+v_p+\ldots) = R|\alpha^f-\alpha^i|$ ; but the number of cusps is unbounded and may even be infinite, since we can always replace a sequence $C_{v_1}|C_{v_2}|C_{v_3}$ by $C_{v_1'}|C_v|C_{v_3'}|C_w|C_{v_2'}$ with $v_1+v_2+v_3 = v_1'+v+v_3'+w+v_2'$. It can be shown that the converse is also true. Thus there are in this case infinitely many shortest paths, in striking contrast to section 4. Notice however that there always exists one with at most two cusps, as proved by [5].

In the following we focus on the case of an optimal path for which $\lambda \neq 0$, where the whole of Section 3 applies, and in particular Proposition 10 (with $\lambda = 1$, $\phi = c = 0$). We assume that there is at least one cusp on the path (the no–cusp case being already settled) and we lead the discussion according to the values of $e > 0$.

### 5.1 $e > 1$

No arc of circle of radius $R$ perpendicular to $D_\pm$ can reach $D_0$. Thus, there is no inflexion, and the path is of a type $C|\,(C_\pi|)^k\,C$ for some $k \geq 0$.

### 5.2 $0 < e < 1$

Any circle of radius $R$ centered on $D_\pm$ intersects $D_0$ at angles satisfying $\cos\alpha = -\varepsilon e \neq 0$. Thus the possible values for $\alpha$ are $\pm v, \pm v+\pi$ for some $0 < v < \frac{\pi}{2}$.

But then part (c) of Remark 1 asserts that the path contains no line segment and an arc of circle starting at a cusp $A$, on $D_+$ ($D_-$), can only

1. either be final : $|C$,
2. or end at an inflexion point $I_1$ : $|C_vC\ldots$ or $I_2$ : $|C_{\pi-v}C\ldots$,
3. or end at a cusp : $|C_\pi|$.

An inflexion at $I_2$ (the inflexion point farthest from $A$) is impossible, since the arc $I_1I_2$ would be an arc between two points where $\beta = y = 0$ and shorter than $\pi R$, contrary to Lemma 9. The third case is also impossible by the same argument since the arc intersects $D_0$ in two points. Thus we are left with only $|C$ or $|C_vC$, the last arc being again either final or ending at a new cusp on $D_-$ ($D_+$), and thus also of length $vR$.

We conclude that any portion of an optimal path starting at a cusp is of one of the three following types $|C$, $|C_vC$ or $|C_vC_v|\dots$ with $0 < v < \frac{\pi}{2}$, and the same $v$ along the whole path.

Hence the possible types of optimal paths in the case $0 < e < 1$ are

$$C|C,\ C|C_vC,\ CC_v|C,\ CC_v|C_vC,\ C|C_vC_v|C$$

with $0 < v < \frac{\pi}{2}$ (compare to the list in [5]), and also others of type

$$\dots CC_v|C_vC_v|C\dots\ ,\ \text{or}\ \dots C|C_vC_v|C_vC\dots$$

The following lemma shows that the two last cases cannot be part of an optimal path by computing a local deformation of the paths, similarly to the proof of Lemma 11.

**Lemma 13** *No path of type $CC_v|C_vC_v|C$ (or $C|C_vC_v|C_vC$) with $0 < v < \frac{\pi}{2}$ can be optimal.*

***Proof*** : We consider only the case $CC_v|C_vC_v|C$, the computations for the other one being identical. The notations being clear on Figure 3, let again $\vec{v}_\theta$ denote the unitary vector with polar angle $\theta$ and $L$ the length of the path. Assume $R = 1$ for simplicity.
For any nearby path of the same type and with the same endpoints $CC_\beta|C_\gamma C_\delta|C$, $O_1$ and $O_5$ are unchanged and we can take $\vec{O_1O_5}$ as the $x$-axis, so that

$$\begin{cases} O_5 = O_1 + 2\vec{v}_{\pi-\alpha} + 2\vec{v}_{-\alpha+\beta} + 2\vec{v}_{\pi-\alpha+\beta+\gamma} + 2\vec{v}_{-\alpha+\beta+\gamma-\delta} \\ L = a + (\alpha - v) + \beta + \gamma + \delta + b - (\alpha - \beta - \gamma + \delta) \end{cases}$$

while for the initial path $\alpha = \beta = \gamma = \delta = v$ by Proposition 10. Hence

$$\begin{cases} O_5 = O_1 + 2\vec{v}_{\pi-v} + 2\vec{v}_0 + 2\vec{v}_{\pi+v} + 2\vec{v}_0 \\ L_0 = a + v + v + v + b \ \text{ for some positive } a, b \end{cases}$$

This yields

$$L - L_0 = 2(\beta - v) + 2(\gamma - v),\ \text{thus } dL = 2d\beta + 2d\gamma \tag{22}$$

$$\begin{cases} -\cos\alpha + \cos(\beta-\alpha) - \cos(\beta+\gamma-\alpha) + \cos(\beta+\gamma-\alpha-\delta) \\ = 2 - 2\cos v \\ \sin\alpha + \sin(\beta-\alpha) - \sin(\beta+\gamma-\alpha) + \sin(\beta+\gamma-\alpha-\delta) = 0 \end{cases} \tag{23}$$

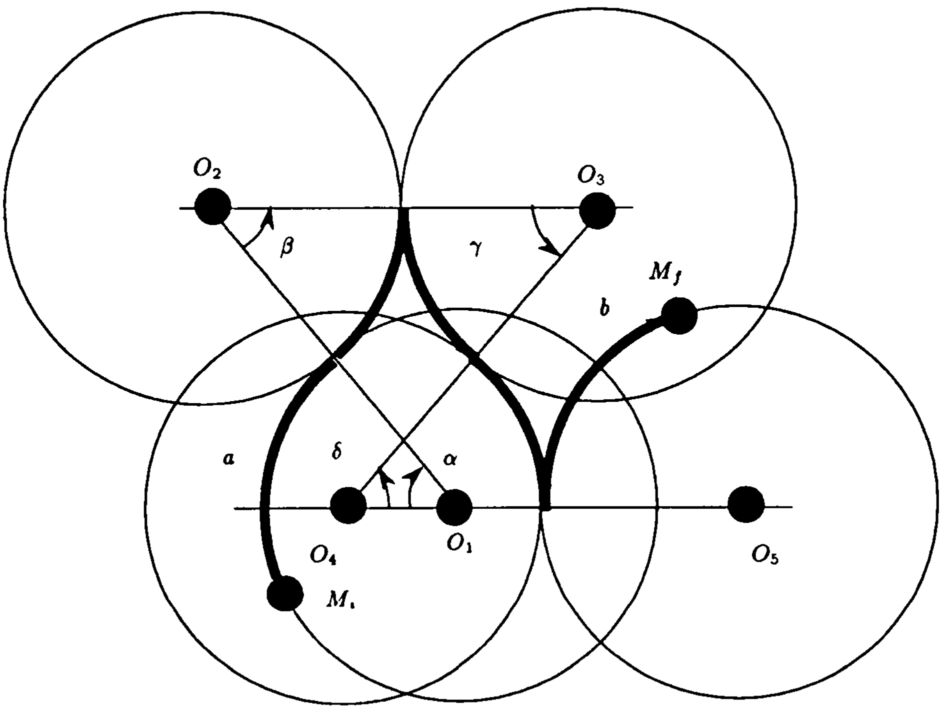


Figure 3: : The $CC_v|C_vC_v|C$ case

Now we look for shorter paths assuming furthermore $\delta = \gamma$. Equations (22) and (23) become

$$L - L_0 = 2(\beta + \gamma - 2v) \tag{24}$$

$$\begin{cases} -\cos\alpha + 2\cos(\beta - \alpha) - \cos(\beta - \alpha + \gamma) = 2 - 2\cos v \\ \sin\alpha + 2\sin(\beta - \alpha) - \sin(\beta - \alpha + \gamma) = 0 \end{cases} \tag{25}$$

The Jacobian of (25) with respect to $\beta$ and $\gamma$ is equal to $-2\sin\gamma \neq 0$ for $\gamma$ sufficiently close to $v$, since $0 < v < \frac{\pi}{2}$. So (25) defines implicitly $\beta$ and $\gamma$ as functions of $\alpha$ in a neighborbood of $(v, v, v)$. This proves that such a deformation is possible, and differentiating (25) w.r.t. $\alpha$ yields

$$\frac{d\beta}{d\alpha} = 1 - \frac{\sin(\beta + \gamma)}{2\sin\gamma} \text{ and } \frac{d\gamma}{d\alpha} = \frac{2\sin\beta - \sin(\beta + \gamma)}{-2\sin\gamma}$$

In particular,

$$\frac{dL}{d\alpha} = 2(\frac{d\beta}{d\alpha} + \frac{d\gamma}{d\alpha}) = 0 \text{ at } \beta = \gamma = v.$$

But further,

$$\frac{d^2L}{d\alpha^2}(0) = -4\frac{\cos v}{\sin^2 v}(1 - \cos v) < 0$$

which proves $L < L_0$ for any small enough $\alpha \neq 0$. □

### 5.3 $e = 1$

Any circle of radius $R$ centered on $D_\pm$ is tangent to $D_0$. Thus any arc of an optimal path starting at a cusp, say $A$ on $D_+$

- either is final : $|C$,
- or ends at a cusp on $D_+$ : $|C_\pi|$,
- or has length $\frac{\pi}{2}R$ and ends at an inflexion point $B$, and is followed by
  - either a segment $BB'$ of $D_0$. If segment $BB'$ is not terminal, then $B'$ is another inflexion point, followed by an arc of circle which, again, is either terminal or of length $\frac{\pi}{2}R$ and ends at a cusp on either $D_+$ or $D_-$.
  - or a final arc of circle,
  - or another $C_{\pi/2}$ ending at a cusp $A'$ on $D_-$.

We conclude that any portion of an optimal path starting at a cusp is of one of the four following types $|C$, $|C_{\pi/2}SC$, or $|C_{\pi/2}SC_{\pi/2}|\dots$ and its degenerate cases $|C_\pi|\dots$ and $|C_{\pi/2}C_{\pi/2}|\dots$

Hence any optimal path, in the case $e = 1$, is of one of the following types (together with their degenerate forms) :

$$C|C, C|C_{\pi/2}SC, CSC_{\pi/2}|C, C|C_{\pi/2}SC_{\pi/2}|C$$

(compare to the list in [5]), plus others of type

$$\dots CSC_{\pi/2}|C_{\pi/2}SC\dots\ ,$$

for instance

$$\dots C_wSC_{\pi/2}|C_{\pi/2}SC_{\pi/2}|\dots\ ,\ \dots|C_{\pi/2}SC_{\pi/2}|C_{\pi/2}SC_w\dots$$

Summarizing the above discussion, we state :

**Theorem 14** *Any shortest path in the plane, piecewise $C^2$, and either $C^1$ or with cusps at junction points, between two given oriented points, is of one of the types listed below, together with their degenerate forms :*

$CSC$, $CC_vC$ (with $v > \pi$),
$C|C|\dots|C$, $C|C_vC, CC_v|C, CC_v|C_vC$, $C|C_vC_v|C$ (with $0 < v < \frac{\pi}{2}$),
$CSC_{\pi/2}|C_{\pi/2}SC$, $C|(C_{\pi/2}SC_{\pi/2}|)^kC$, (with $k \geq 0$)

It is worth noticing that all the arguments used so far were of a local nature. In the sequel, we will use global arguments and further restrict the number of possible types of shortest paths.

**Lemma 15** *A path of type $CSC_{\pi/2}|C_{\pi/2}SC_{\pi/2}|C$ (or $C|C_{\pi/2}SC_{\pi/2}|C_{\pi/2}SC$) cannot be optimal.*

*Proof :* We consider only the case $CSC_{\pi/2}|C_{\pi/2}SC_{\pi/2}|C$, the computations for the other one being identical. The four possible cases of type $CSC_{\pi/2}|C_{\pi/2}SC_{\pi/2}|C$ (see (1) to (4), figure 4) can be discarded by global arguments. We apply the following length preserving transformation to any of the paths (1) to (4). First, if the two cusps are not already on the same line (case (2) and (4)), reverse a $C_{\pi/2}|C_{\pi/2}$ portion, in order to get the two cusps on the same line, say $D_+$. Now, transform $SC_{\pi/2}|C_{\pi/2}SC_{\pi/2}|$

(1)

(2)

(3)

(4)

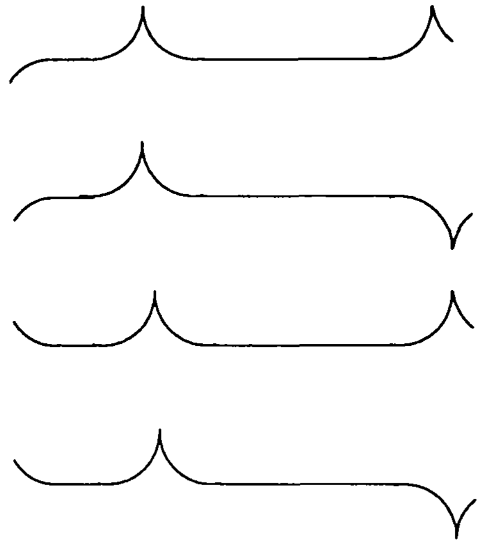

Figure 4: : The $CSC_{\pi/2}|C_{\pi/2}SC_{\pi/2}|C$ case

into $SC_{\pi/2}|C_\pi|$, by sliding a cusp along $D_+$. Finally, reverse $|C_\pi|$ in order to remove the cusp. By this transformation, we obtain a path without cusp, which is of the same length than the initial one, and which contains a $SC_{\pi/2}C_\pi$ section. Then, Lemma 9 applies to prove that such a path is not optimal. □

Finally, observe that a path of type $CC_vC$ with $v > \pi$ is no longer optimal. Indeed, by reversing $C_v$, one obtains a path of type $C|C_w|C$ where $w = 2\pi - v < \pi$, so that it is shorter than the initial path.

We conclude with the following theorem :

**Theorem 16 (Reeds and Shepp)** *Any shortest path in the plane, piecewise $C^2$, and either $C^1$ or with cusps at junction points, joining two given oriented points, is of one of the types listed below, together with their degenerate forms :*

$CSC$,
$C|C|\dots|C$, $C|C_vC, CC_v|C, CC_v|C_vC$, $C|C_vC_v|C$ (with $0 < v < \frac{\pi}{2}$),
$CSC_{\pi/2}|C_{\pi/2}SC$, $C|C_{\pi/2}SC_{\pi/2}|C$

Furthermore, the only two cases where there is an infinity of shortest paths are $C|C|\ldots|C$ and $CSC_{\pi/2}|C_{\pi/2}SC$. But one of them can always be found of the type $C|C|C$, or $C|C_{\pi/2}SC$ (or $CSC_{\pi/2}|C$) respectively. This is Reeds and Shepp's result in [5].

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