Homework 5

February 5, 2017

Problem 1.

Let $\mathbf{A}$ denote an $(n-1) \times n$ matrix with complex number entries for which the RREF($\mathbf{A}$) has $n-1$ pivots show that we can prepend $\mathbf{A}$ with an additional row so as to ensure that

$$\det(\mathbf{A}) \in \mathbb{R} \quad \text{and} \quad \det(\mathbf{A}) > 0.$$

Describe how to obtain that vector using the Laplace expansion of the determinant.

Problem 2.

Let $\mathbf{A} \in \mathbb{C}^{n \times n}$ such that $\det(\mathbf{A}) \neq 0$. Consider the system $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$. We showed in classe that row linear combinations are performed on the augmented matrix to put the coefficient matrix in REF (thus obtaining the $\mathrm{REF}_{\star}(\mathbf{A})$). Assume that we further perform the following row operations to the augmented matrix associated with the $\mathrm{REF}_{\star}(\mathbf{A})$

$$\text{for all } 1 \leq i < n, \quad \left(\mathrm{REF}_{\star}(\mathbf{A})[i,i]\right)^{-1} \times \mathrm{Row}_i \to \mathrm{Row}_i$$

and

$$\left(\prod_{1 \leq i < n} \mathrm{REF}_{\star}(\mathbf{A})[i,i]\right) \times \mathrm{Row}_n \to \mathrm{Row}_n$$

Describe the non zero entries of the last row of the of the resulting augmented matrix as determinants of matrices that you should explicitly describe. Explain in detail your derivation.

Problem 3.

Let $\mathbf{A}$ and $\mathbf{B}$ denote a $m \times n$ and $p \times n$ matrices where $n > m > p > 2$. Describe how to find a basis for the subspaces :
a)

$$\text{NullSpace}(\mathbf{A}) \cup \text{NullSpace}(\mathbf{B}).$$

b)

$$\text{NullSpace}(\mathbf{A}) \cap \text{NullSpace}(\mathbf{B}).$$

Problem 4.

a) Find the rank of an $n \times n$ matrix $\mathbf{A}$ whose entries are given by

$$\mathbf{A}[i,j] = 1 + n \cdot (i-1) + (j-1),$$

for all $1 \leq i, j \leq n$, where $n > 2$.
b) What can you conclude about the dimension of the Nullspace of $\mathbf{A}$ for all $n > 1$.
c) For what value of n is the matrix $\mathbf{A}$ invertible ?

Problem 5.

Let $\mathbf{A}$ denote the $n \times n$ matrix whose entries are given by

$$a_{uv} = \exp\left\{i \cdot \frac{2\pi}{n} \cdot (u-1) \cdot (v-1)\right\}$$

for all $1 \leq u, v \leq n$, where $n \geq 2$. Using the standard inner-product discussed in class for elements of the vector space $\mathbb{C}^{n \times 1}$, Find the angle between any two columns of $\mathbf{A}$.