Homework 10

April 10, 2017

Problem 1.

a) Let $\mathbf{U} \in \mathbb{C}^{n\times n}$ be a unitary matrix show that the family of n^2 matrices $\{\mathbf{Q}_{kl}\}_{1\le k,l\le n}$ constructed from the entries $\mathbf{U}$ as follows

$$\mathbf{Q}_{kl} = \sum_{1\le j\le n} \left(\mathbf{e}_j \cdot \mathbf{e}^{\top}_{\{(j+k) \mod n\}}\right) \cdot \mathbf{U}\left[j,l\right]$$

form an orthogonal basis. Note that the vectors $\{\mathbf{e}_j\}_{1\le j\le n} \subset \mathbb{C}^{n\times 1}$ correspond to column vectors of the identity matrix $\mathbf{I}_n$.

b) Let

$$\mathbf{U} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{pmatrix}$$

where $\omega = \exp\left\{i\,\frac{2\pi}{3}\right\}$. Express the projection of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

onto each of the matrices in the set $\{\mathbf{Q}_{kl}\}_{1\le k,l\le n}$.

Problem 2.

a) Use the Gram-Schmidt process to find an orthonormal basis for

$$\text{Span}\left\{\mathbf{U}_1 = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}, \mathbf{U}_2 = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \mathbf{U}_3 = \begin{pmatrix} 0 & i \\ 0 & 1 \end{pmatrix}, \mathbf{U}_4 = \begin{pmatrix} 1 & 0 \\ i & 0 \end{pmatrix}\right\}.$$

b) Compute the distance between

$$\mathbf{B} = \begin{pmatrix} 1 & 2i \\ 3 & 4i \end{pmatrix}$$

and $\text{Span}\{\mathbf{U}_1, \mathbf{U}_2, \mathbf{U}_4\}$.

Problem 3.

Let $\mathbf{A} \in \mathbb{C}^{n\times n}$ be Hermitian with distinct eigenvalues, prove that eigevectors with distinct eigenvalues must be orthogonal.

Problem 4.

Find a basis for $W^\perp$ for

$$W = \mathrm{Span}\left\{\begin{pmatrix} 1 & i \\ -1 & -i \end{pmatrix}, \begin{pmatrix} 1 & -i \\ -1 & i \end{pmatrix}\right\}$$

Problem 5.

Give a detail proof of the general Cauchy interlacing theorem as discussed in class when the submatrix $\mathbf{A}_{11}$ in the partition

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix} \in \mathbb{C}^{n\times n},$$

has size $m \times m$ where $m < n$.