Homework 9

April 3, 2017

Problem 1.

a) Let $\mathbf{A} \in \mathbb{C}^{n\times n}$ be diagonalizable with eigenvalues $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Express in the terms of the spectral decomposition of $\mathbf{A}$ the set of non zero vectors $\mathbf{x}$, and $\mathbf{y} \in \mathbb{C}^{n\times 1}$ for which

$$\lambda_1 \leq \frac{\mathbf{x}^* \cdot \mathbf{A} \cdot \mathbf{y}}{\mathbf{x}^* \cdot \mathbf{y}} \leq \lambda_n$$

Give a detail proof of this fact and describe the vectors $\mathbf{x}$, and $\mathbf{y}$ which achieve the minimum and the maximum.
b) Let $\mathbf{A} \in \mathbb{C}^{n\times n}$ such that $\mathbf{A}^* = \mathbf{A}$ with eigenvalues $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Express in the terms of the spectral decomposition of $\mathbf{A}$ the set of non zero vectors $\mathbf{x} \in \mathbb{C}^{n\times 1}$ for which

$$\lambda_1 \leq \frac{\mathbf{x}^* \cdot \mathbf{A} \cdot \mathbf{x}}{\mathbf{x}^* \cdot \mathbf{x}} \leq \lambda_n.$$

Give a detail proof of this fact and describe the vectors $\mathbf{x}$ which achieve the minimum and the maximum.

Problem 2.

Let $\mathbf{A} \in \mathbb{C}^{n\times n}$ be a Hermitian matrix with strictly positive e-values, find the set of all vectors for which $\mathbf{x} \in \mathbb{C}^{n\times 1}$

$$0 = \det\left(\mathbf{A} - \mathbf{x} \cdot \mathbf{x}^*\right).$$

Give a detail proof of your claim.

Problem 3.

a) Compute the distance from

$$\mathbf{v} = \begin{pmatrix} i \\ 2+i \\ 3 \\ 4i \end{pmatrix}$$

to

$$W = \mathrm{Span}\left\{ \mathbf{u}_1 = \begin{pmatrix} 1 \\ i \\ 1-i \\ 1 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} 1 \\ 1+i \\ i \\ 1 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} 0 \\ i \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Problem 4.

Find the maximum and minimum value attained by the function $f(x_0, x_1, x_2, x_3)$ for x_0,x_1,x_2,$x_3 \in \mathbb{R}$ where

$$f(x_0, x_1, x_2, x_3) = \frac{6\,x_0^2 - 4\,x_0x_1 + 6\,x_1^2 - 8\,x_0x_2 + 6\,x_2^2 - 8\,x_1x_3 - 4\,x_2x_3 + 6\,x_3^2}{x_0^2 + x_1^2 + x_2^2 + x_3^2}$$

when the variables x_0,x_1,x_2,x_3 are not all simultaneously zero.

Problem 5.

Express the spectral decomposition of the Hermitian matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1+i & 0 & i \\ 1-i & 4 & -i & 0 \\ 0 & i & 2 & 2+i \\ -i & 0 & 2-i & 7 \end{pmatrix}$$

in terms of a unitary matrix.