Practice Final, MA 351 Section: Spring 2017

Name:

PUID:

Exam

No calculators may be used on this exam.

Problem Number	Possible Points	Points Earned
I	50pts	
II	20pts	
III	30pts	
Total Points	100pts	

I (50pts)

Answer the following questions by clearly circling your answers. No partial credit is awarded for the questions in Part I

(1) Find the eigenvalues and eigenvectors of

$$\mathbf{A} = \begin{pmatrix} 1 & -4 \\ 1 & 1 \end{pmatrix}$$

(a) $\lambda_1, \lambda_2 \in \{-1, 3\}$ and $\mathbf{v}_1,\ \mathbf{v}_2 \in \left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$.

(b) $\lambda_1, \lambda_2 \in \{-1, 3\}$ and $\mathbf{v}_1,\ \mathbf{v}_2 \in \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right\}$.

(c) $\lambda_1, \lambda_2 \in \{1 - 2i,\ 1 + 2i\}$ and $\mathbf{v}_1,\ \mathbf{v}_2 \in \left\{ \begin{pmatrix} 2i \\ 1 \end{pmatrix}, \begin{pmatrix} -2i \\ 1 \end{pmatrix} \right\}$.

(d) $\lambda_1, \lambda_2 \in \{1 - 2i,\ 1 + 2i\}$ and $\mathbf{v}_1,\ \mathbf{v}_2 \in \left\{ \begin{pmatrix} i \\ 2 \end{pmatrix}, \begin{pmatrix} -i \\ 2 \end{pmatrix} \right\}$.

(e) $\lambda_1, \lambda_2 \in \{1 - 4i,\ 1 + 4i\}$ and $\mathbf{v}_1,\ \mathbf{v}_2 \in \left\{ \begin{pmatrix} i \\ 1 \end{pmatrix}, \begin{pmatrix} -i \\ 1 \end{pmatrix} \right\}$.

(2) Let W denote the vector space spanned by

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$$

and let $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$. Find the distance from $\mathbf{v}$ to W.

(a) 0
(b) 1
(c) 2
(d) $\sqrt{2}$
(e) 4

(3) Let W be the vector space spanned by the vectors:

$\mathbf{u}_1 = \begin{pmatrix} 1 & 0 & 1 & 0 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} -1 & 1 & 1 & 1 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 1 & 1 & 1 & 0 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 0 & 2 & 2 & 3 \end{pmatrix}$.

Apply the Gram-Schmidt process to the vectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, $\mathbf{u}_4$ (in the order specified) to derive an orthonormal basis $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ of W. What is $\mathbf{v}_3$.

(a) $\mathbf{v}_3 = \begin{pmatrix} 1/2 & 1/2 & -1/2 & 1/2 \end{pmatrix}$
(b) $\mathbf{v}_3 = \begin{pmatrix} 1/\sqrt{12} & 3/\sqrt{12} & -1/\sqrt{12} & -1/\sqrt{12} \end{pmatrix}$
(c) $\mathbf{v}_3 = \begin{pmatrix} 1/4 & 3/4 & -1/4 & 1/4 \end{pmatrix}$
(d) $\mathbf{v}_3 = \begin{pmatrix} (3-2\sqrt{2})/4 & 1/4 & (1-\sqrt{2})/4 & 1/4 \end{pmatrix}$
(e) $\mathbf{v}_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}$

(4) Let $\mathbf{A}$ be an $n \times n$ matrix of rank $(n-1)$ and

$$\mathbf{A} = \sum_{1 \leq k \leq n-1} \mathbf{u}_k \cdot \mathbf{v}_k^\top$$

where $\{\mathbf{u}_k\}_{1 \leq k \leq n} \subset \mathbb{C}^{n \times 1}$ and $\{\mathbf{v}_k\}_{1 \leq k \leq n} \subset \mathbb{C}^{n \times 1}$.
Then which of the following statements is necessarily true ? (circle all that apply)

(a) $\{\mathbf{u}_k\}_{1 \leq k \leq n-1}$ is a basis for the vector space $\mathbb{C}^{n \times 1}$.
(b) The vectors $\{\mathbf{u}_k\}_{1 \leq k \leq n-1}$ can be linearly dependent as long as $\{\mathbf{v}_k\}_{1 \leq k \leq n-1}$ are linearly independent or vice versa.
(c) The vectors $\{\mathbf{u}_k\}_{1 \leq k \leq n-1}$ must be linearly independent and the same holds for the vectors $\{\mathbf{v}_k\}_{1 \leq k \leq n-1}$.
(d) The union $\{\mathbf{u}_k\}_{1 \leq k \leq n} \cup \{\mathbf{v}_k\}_{1 \leq k \leq n}$ can include the zero vector.
(e) None of the above.

(5) Let $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ be an orthogonal basis of $\mathbb{C}^{4\times 1}$, using the standard inner product. Let

$$W = \text{Span}\left\{\mathbf{v}_1, \mathbf{v}_2\right\},$$

and let $\mathbf{u}$ be a vector in $W^{\perp}$. Then which of the following is NOT necessarily true ? (circle all that apply)

(a) $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 + \mathbf{u}, \mathbf{v}_4 + \mathbf{u}$ is a basis for $\mathbb{C}^{4\times 1}$.
(b) $\mathbf{v}_1 + \mathbf{u}, \mathbf{v}_2 + \mathbf{u}, \mathbf{v}_3, \mathbf{v}_4$ is a basis for $\mathbb{C}^{4\times 1}$.
(c) $(\mathbf{v}_1 + \mathbf{v}_3)$ is orthogonal to $(\mathbf{v}_2 + \mathbf{v}_4)$.
(d) $(\mathbf{v}_1 + \mathbf{v}_2)$ is orthogonal to $(\mathbf{v}_3 + \mathbf{v}_4 + \mathbf{u})$
(e) None of the above.

II (20pts)

Let

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

a) Find if possible the spectral and singular value decomposition of $\mathbf{A}$. Show all your work in detail and clarity.

b) Use the Sylvesterian elimination to determine the existence or non-existence of a 2×2 matrix $\mathbf{Q}$ subject to .

$$\mathbf{Q} = \mathbf{Q}^\top, \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \mathbf{Q} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } \operatorname{Tr}(\mathbf{Q}) = \det \mathbf{Q}$$

Show all your work in detail and clarity

Answer:

III (30 pts)

a) Let $\mathbf{A} \in \mathbb{C}^{n \times n}$ be Hermitian with eigenvalues $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ Give detail proof of the following statements

$$\text{For all } \mathbf{x} \in \mathbb{C}^{n \times 1}, \quad \lambda_1 \leq \frac{\mathbf{x}^* \cdot \mathbf{A} \cdot \mathbf{x}}{\mathbf{x}^* \cdot \mathbf{x}} \leq \lambda_n$$

b) For which assignment of $\mathbf{x}$ does $\frac{\mathbf{x}^* \cdot \mathbf{A} \cdot \mathbf{x}}{\mathbf{x}^* \cdot \mathbf{x}}$ attains it maximum and minimum respectively ?