

# Homework 9

April 3, 2017

## Problem 1.

a) Let  $\mathbf{A} \in \mathbb{C}^{n \times n}$  be diagonalizable with eigenvalues  $0 \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ . Express in the terms of the spectral decomposition of  $\mathbf{A}$  the set of non zero vectors  $\mathbf{x}$ , and  $\mathbf{y} \in \mathbb{C}^{n \times 1}$  for which

$$\lambda_1 \leq \frac{\mathbf{x}^* \cdot \mathbf{A} \cdot \mathbf{y}}{\mathbf{x}^* \cdot \mathbf{y}} \leq \lambda_n$$

Give a detail proof of this fact and describe the vectors  $\mathbf{x}$ , and  $\mathbf{y}$  which achieve the minimum and the maximum.

b) Let  $\mathbf{A} \in \mathbb{C}^{n \times n}$  such that  $\mathbf{A}^* = \mathbf{A}$  with eigenvalues  $0 \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ . Express in the terms of the spectral decomposition of  $\mathbf{A}$  the set of non zero vectors  $\mathbf{x} \in \mathbb{C}^{n \times 1}$  for which

$$\lambda_1 \leq \frac{\mathbf{x}^* \cdot \mathbf{A} \cdot \mathbf{x}}{\mathbf{x}^* \cdot \mathbf{x}} \leq \lambda_n.$$

Give a detail proof of this fact and describe the vectors  $\mathbf{x}$  which achieve the minimum and the maximum.

## Problem 2.

Let  $\mathbf{A} \in \mathbb{C}^{n \times n}$  be a Hermitian matrix with strictly positive e-values, find the set of all vectors for which  $\mathbf{x} \in \mathbb{C}^{n \times 1}$

$$0 = \det(\mathbf{A} + \mathbf{x} \cdot \mathbf{x}^*).$$

Give a detail proof of your claim.

## Problem 3.

a) Compute the distance from

$$\mathbf{v} = \begin{pmatrix} i \\ 2+i \\ 3 \\ 4i \end{pmatrix}$$

to

$$W = \text{Span} \left\{ \mathbf{u}_1 = \begin{pmatrix} 1 \\ i \\ 1-i \\ 1 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} 1 \\ 1+i \\ i \\ 1 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} 0 \\ i \\ 0 \\ 1 \end{pmatrix} \right\}.$$

#### Problem 4.

Find the maximum and minimum value attained by the function  $f(x_0, x_1, x_2, x_3)$  for  $x_0, x_1, x_2, x_3 \in \mathbb{R}$  where

$$f(x_0, x_1, x_2, x_3) = \frac{6x_0^2 - 4x_0x_1 + 6x_1^2 - 8x_0x_2 + 6x_2^2 - 8x_1x_3 - 4x_2x_3 + 6x_3^2}{x_0^2 + x_1^2 + x_2^2 + x_3^2}$$

when the variables  $x_0, x_1, x_2, x_3$  are not all simultaneously zero.

#### Problem 5.

Express the spectral decomposition of the Hermitian matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1+i & 0 & i \\ 1-i & 4 & -i & 0 \\ 0 & i & 2 & 2+i \\ -i & 0 & 2-i & 7 \end{pmatrix}$$

in terms of a unitary matrix.