Tutorial Week 4

1. If two events A and B are independent, then show that the pairs (A, B^c), (A^c, B) and (A^c, B^c) are also pairs of independent events.

2. When we derived $E(X)$ and $E[X(X-1)]$ in the hypergeometric case, we made the simplifying assumption that both $w \geq n$ and $b \geq n$, so that all values $x = 0, 1, \ldots, n$ were possible. We now verify that the same results hold in the general case where we only assume that the sample size $n \leq w + b$ (the "population size").

   Suppose X denotes the number of white balls drawn, without replacement, in a sample of size n from an urn that initially contains w white balls and b black balls. Then we immediately have that

$$0 \leq X \leq n\,.$$

   However, note also that X can be no bigger than w. Similarly, the number of black balls in the sample $n - X$ can be no bigger than b. That is,

$$n - X \leq b \quad \Rightarrow \quad X \geq n - b\,.$$

   The two sets of inequalities

$$\begin{aligned} 0 \leq X &\leq n \\ n - b \leq X &\leq w \end{aligned}$$

   can be expressed as

$$\max(0, n-b) \leq X \leq \min(n, w)\,.$$

   Thus if each possible sequence of balls is equally likely we may write

$$E(X) \quad = \quad \sum_{x=\max(0,n-b)}^{\min(n,w)} x \, \frac{\binom{w}{x}\binom{b}{n-x}}{\binom{w+b}{n}}\,.$$

   Show by

   (a) (possibly) removing the $x = 0$ term;
   (b) rewriting $\binom{w}{x}$ and $\binom{w+b}{n}$;
   (c) taking appropriate constants outside the summation;
   (d) re-expressing the summation in terms of a new dummy variable $y = x - 1$;

   that the resulting summation adds to 1 and hence that $E(X) = nw/(w + b)$ even in this more general case.

3. Repeat the previous question but for $E[X(X - 1)]$.

4. Show that if (for real $-1 \leq \Delta \leq 1$ and positive integers $b \geq 1$, $w \geq 1$ and $n \leq w + b$) a random variable X has $E(X) = \dfrac{nw}{w+b}$ and

$$E[X(X-1)] = \frac{n(n-1)w(w+\Delta)}{(w+b)(w+b+\Delta)}\,,$$

   then

$$\mathrm{Var}(X) = n\left(\frac{w}{w+b}\right)\left(\frac{b}{w+b}\right)\frac{(w+b+\Delta n)}{(w+b+\Delta)}$$

   Note: the cases $\Delta = 0, -1$ and $+1$ correspond to binomial, hypergeometric and beta-binomial random variables, respectively. **Hint:** recall the computing formula for the variance.