More Examples

$n=2$, $b = w \geq 2$

<u>Sampling with Replacement</u> the distribution is <u>unchanged</u> from the $b=w=1$ case (only depends on n and $p = \frac{w}{w+b} = \frac{1}{2}$ if $b=w$)

(the distribution is again $B(2,\frac{1}{2})$) – $P(0 \text{ wh}) = \frac{1}{4}$, $P(1 \text{ wh}) = \frac{1}{2}$, $P(2 \text{ wh}) = \frac{1}{4}$.

<u>Sampling WITHOUT Replacement</u>

If $b = w$ then $P(x \text{ Wh}) = \dfrac{\binom{w}{x}\binom{b}{n-x}}{\binom{w+b}{n}}$

and $n=2$

$$= \frac{\binom{w}{x}\binom{w}{2-x}}{\binom{2w}{2}}$$

So $P(0 \text{ wh}) = \dfrac{\binom{w}{0}\binom{w}{2}}{\binom{2w}{2}} = \dfrac{\binom{w}{2}}{\binom{2w}{2}} = \dfrac{\frac{w(w-1)}{2\cdot 1}}{\frac{2w(2w-1)}{2\cdot 1}} = \cancel{\dfrac{w-1}{2w}}$

$$= \frac{w-1}{2(2w-1)} < \frac{w-\frac{1}{2}}{2(2w-1)} = \frac{1}{4}$$

$$P(1 \text{ wh}) = \frac{\binom{w}{1}\binom{w}{1}}{\binom{2w}{2}} = \frac{w^2}{\frac{2w(2w-1)}{2\cdot 1}} = \frac{w}{2w-1} > \frac{w-\frac{1}{2}}{2w-1} = \frac{1}{2}$$

$$P(2 \text{ wh}) = \dots = \frac{w-1}{2(2w-1)}$$

w	$P(0\text{ wh})$	$P(1\text{ wh})$	$P(2\text{ wh})$
2	$1/6$	$2/3$	$1/6$
3	$1/5$	$3/5$	$1/5$
4	$3/14$	$4/7$	$3/14$
5	$2/9$	$5/9$	$2/9$
6	$5/22$	$6/11$	$5/22$
7	$3/13$	$7/13$	$3/13$
8	$7/30$	$8/15$	$7/30$
	$\nearrow \frac{1}{4}$	$\searrow \frac{1}{2}$	$\nearrow \frac{1}{4}$

So as we would expect, the more balls in the urn at the beginning, the more "binomial-like" the no. whites behaves; the effect of not replacing diminishes.

Polya Scheme

$$P(0\ \text{wh}) = \frac{\binom{w-1}{0}\binom{w-1+2-0}{2}}{\binom{w+b-1+2}{2}}$$

$$= \frac{\binom{w+1}{2}}{\binom{2w+1}{2}} = \frac{(w+1)(w)/\cancel{2}}{(2w+1)\,2\cancel{w}/\cancel{2}} = \frac{w+1}{2(2w+1)}$$

$$> \frac{w+\frac{1}{2}}{2(2w+1)} = \frac{1}{4}$$

$$P(1\ \text{wh}) = \cancel{\frac{\binom{w}{1}\binom{w}{1}}{\binom{2w}{2}} = 2w^2}$$

$$= \frac{\binom{w}{1}\binom{w}{1}}{\binom{2w+1}{2}} = \frac{w^2}{\frac{(2w+1)\,2w}{\cancel{2}}} = \frac{w}{2w+1} < \frac{w+\frac{1}{2}}{2w+1} = \frac{1}{2}$$

$$P(2\ \text{wh}) = \ldots = \frac{w+1}{2(2w+1)}$$

$w =$	$P(0\ \text{wh})$	$P(1\ \text{wh})$	$P(2\ \text{wh})$
2	3/10	4/10	3/10
3	2/7	3/7	2/7
4	5/18	4/9	5/18
5	3/11	5/11	3/11
6	7/26 $\downarrow \frac{1}{4}$	6/13 $\nearrow \frac{1}{2}$	7/26 $\downarrow \frac{1}{4}$

So again, the no. whites becomes more "binomial-like" (but from the "opposite direction")

Random Variables and Probability Distributions

In a classical probability scenario, a random variable (RV) can be thought of as a 2-column table ~~(i.e. a fcn~~ assigning a (real) number to each outcome (i.e. a function!)

Outcome s_i	Value $X(s_i)$
s_1	$X(s_1)$
s_2	$X(s_2)$
$\vdots$	$\vdots$
s_M	$X(s_M)$

If the RV takes k distinct values $\{x_1, \ldots, x_k\}$, say ($k \leq M$ necessarily) then it PARTITIONS the sample space S into k mutually exclusive events whose UNION is S itself.

$$\{s_i \mid X(s_i) = x_1\}, \{s_i \mid X(s_i) = x_2\}, \ldots, \{s_i \mid X(s_i) = x_k\}$$

OR $(X = x_1), (X = x_2), \ldots, (X = x_k)$ for short

Furthermore, it induces a <u>probability distribution</u>

Value.	Prob.
x_1	$P(X=x_1)$
x_2	$P(X=x_2)$
$\vdots$	$\vdots$
x_k	$P(X=x_k)$
	Total $=1$

(really $P(\{s_i \mid X(s_i)=x_1\})$)

The binomial, hypergeometric and beta-binomial distn's are all examples of this. The outcomes in each case there are (numerical) sequences of balls drawn out. The RV is the no. of white balls drawn out / in the sequence.