STAT2011 – Tutorial 3 – Solutions

1) There are 36 possible outcomes, with corresponding sums shown in table below

Green

Red \ Green	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

sum	2	3	4	5	6	7	8	9	10	11	12
# outcomes	1	2	3	4	5	6	5	4	3	2	1
prob	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$.

$$\text{Expectation} = \frac{1}{36}(2+12) + \frac{2}{36}(3+11) + \frac{3}{36}(4+10) + \frac{4}{36}(5+9) + \frac{5}{36}(6+8) + \frac{6}{36}\times 7$$

$$= \frac{14}{36}\left[1+2+3+4+5+3\right] = \frac{14\times 18}{36} = \underline{\underline{7}}$$

2) The sum defining the expectation can be written as

$$\mu P(Z=\mu) + \cancel{[\mu-1+\mu+1]P} \; (\mu+1)P(Z=\mu+1)+\dots+(\mu+m)P(Z=\mu+m)$$

$$+ (\mu-1)P(Z=\mu-1)+\dots+(\mu-m)P(Z=\mu-m)$$

$$= \mu q_0 + [(\mu+1)+(\mu-1)]q_1 + [(\mu+2)+(\mu-2)]q_2+\dots+[(\mu+m)+(\mu-m)]q_m$$

$$= \mu q_0 + 2\mu q_1 + 2\mu q_2 + \dots + 2\mu q_m$$

$$= \mu[q_0 + 2q_1 + 2q_2 + \dots + 2q_m] \quad \text{———} \quad (*)$$

But the sum of all probs is 1:

$$P(Z=\mu) + P(Z=\mu+1)+\dots+P(Z=\mu+m)$$

$$+ P(Z=\mu-1)+\dots+P(Z=\mu-m) = q_0 + 2q_1+\dots+2q_m$$

$$= 1$$

So $E(Z)=\mu$ (since the sum in $[\cdot]$ at $(*)$ above is 1).

3) There are n^2 possible outcomes:

$$S = \{ (x,y) : x = 1,2,..,n ; \ y = 1,2,..,n \}$$

there is 1 with the sum = 2 : $(1,1)$

there are 2 .. .. 3 : $(1,2),(2,1)$

⋮

there are $n-1$ .. .. n : $(1,n-1),(2,n-2),..,(n-1,1)$

there are n .. .. $n+1$: $(1,n),(2,n-1),...,(n,1)$

there .. $n-1$ .. .. $n+2$ $(2,n),(3,n-1),..,(n,2)$

.. .. 2 .. .. .. $2n-1$: $(n-1,n),(n,n-1)$

there is 1 .. .. .. $2n$: (n,n).

If each outcome is equally likely then

$$P(\text{sum} = 2) = P(\text{sum} = 2n)$$
$$P(\text{sum} = 3) = P(\text{sum} = 2n-1)$$
$$\vdots$$
$$P(\text{sum} = n) = P(\text{sum} = n+2).$$

i.e. the distribution is symmetric about $\underline{n+1}$.

Thus the expected value is $\underline{n+1}$. (by Q2).

4) a) The term in $\sum_{x=0}^{n} x P(X=x)$ corresp. to $x=0$ is 0,

so can be omitted, i.e. $E(X) = \sum_{x=1}^{n} x P(X=x)$.

b) $$\binom{w-1+x}{x} = \overbrace{\frac{w}{x} \underbrace{\frac{w+1}{x-1} \cdots \frac{w-1+x}{1}}_{x-1 \text{ factors}}}^{x \text{ factors}}$$

$$= \frac{w}{x} \binom{w-1+x}{x-1}$$

c) $$\binom{w+b+n-1}{n} = \overbrace{\frac{w+b}{n} \underbrace{\frac{w+b+1}{n-1} \cdots \frac{w+b+n-1}{1}}_{n-1 \text{ factors}}}^{n \text{ factors}}$$

$$= \frac{w+b}{n} \binom{w+b+n-1}{n-1}.$$

d) $$E(X) = \sum_{x=1}^{n} x \cdot \frac{\binom{w-1+x}{x}\binom{b-1+n-x}{n-x}}{\binom{w+b+n-1}{n}}$$

$$= \sum_{x=1}^{n} \frac{\cancel{x} \cdot \frac{w}{\cancel{x}} \cdot \binom{w-1+x}{x-1}\binom{b-1+n-x}{n-x}}{\frac{w+b}{n}\binom{w+b+n-1}{n-1}}$$

$$= \frac{nw}{w+b} \sum_{x=1}^{n} \frac{\binom{w-1+x}{x-1}\binom{b-1+n-x}{n-x}}{\binom{w+b+n-1}{n-1}}$$

e) Writing $x = y+1$, $n = m+1$ inside the sum, it becomes.

$$\sum_{y+1=1}^{m+1} \frac{\binom{(w+1)+y-1}{y}\binom{b-1+(m+1)-(y+1)}{(m+1)-(y+1)}}{\binom{w+b+m+1-1}{(m+1)-1}}$$

$$= \sum_{y=0}^{m} \frac{\binom{(w+1)+y-1}{y}\binom{b+m-y-1}{m-y}}{\binom{(w+1)+b+m-1}{m}}$$

which is the sum of all ~~poss~~ Beta-binomial probs for the distribution of the no. whites when sampling from an urn containing $(w+1)$ whites and b blacks, m times, following the Polya sampling scheme.

Thus the expectation ~~as~~ $E(X) = \frac{nw}{w+b} = np$

where (again) p is the proportion of whites in the urn.