<u>One Pair</u>: 2 of one rank, 3 cards of 3 other <u>different</u> ranks

- Pick rank for the Pair: 13 ways
- Pick 3 other ranks: $\binom{12}{3}$ ways
- Fill the pair: $\binom{4}{2} = 6$ ways
- Choose the 3 singletons $= 4 \times 4 \times 4 = 64$ ways

Total: $13 \times \binom{12}{3} \times 6 \times 4^3 = 1,098,240$

TOTAL No of possible hands $\binom{52}{5} = 2,598,960$

the number of these NOT included in any of the above is $1,302,540$

Random Variables and Prob. Distribution

We are now in a position to answer questions like "If an urn contains w White balls and b Black balls, and n balls are drawn out ~~(of course $n \leq b+w$)~~ (according to some sampling scheme) in such a way that all possible sequences of balls are equally likely, then what is

P (exactly x of the balls are white)

(for various values of x) ?

Whatever the sampling scheme (with replacement, without replacement, Polya scheme), our approach will be to use the multiplication principle as follows.

(*)

A) first count the total no. of possible sequences $\Rightarrow$ DENOMINATOR

B) To compute the NUMERATOR, we consider the construction of the desired type of sequence as follows:

1) Count the no. ways of obtaining a sequence of x white balls

2) .. .. $(n-x)$ black balls

3) Count the no. ways of ORDERING the x whites and the $n-x$ blacks.

The answers to B1) and B2) (and A) above depend on the sampling scheme used, however the answer to B3) (for all 3 schemes) is always $\binom{n}{x} = \frac{n!}{x!(n-x)!}$ (see ~~last~~ lecture 4)

Sampling with replacement

A) The total # of possible sequences can be obtained by noting that there are

1 i) $w+b$ choices for the 1st ball
2 ii) $w+b$ choices for the 2ND ..
⋮
n) $w+b$ .. .. .. n-th ball

So ~~..~~ multiplying these, there are $(w+b)^n$ possible sequences
(this is the denominator.)

B) 1) Regardless of where in the whole sequence the x whites, there are

1) w choices for 1st ball
2) w .. .. .. 2ND ball
⋮
x) w choices for x-th ball

so w^x ways to obtain a sequence of x whites

2) Similarly, there are 1) b ~~ways~~ choices for 1st Black ball
2) b .. .. .. 2ND .. ..
⋮
$n-x$) b .. .. .. $(n-x)$-th Black ball

So ~~is~~ b^{n-x} ways to obtain a seq. of $(n-x)$ Black balls.

NOTE: these numbers apply even in the "edge" cases of $x=0$ and $x=n$

Thus according to the prescription (*) above,

$$P(\overset{\text{exactly}}{x} \text{ whites}) = \frac{\binom{n}{x} w^x b^{n-x}}{(w+b)^n}$$

$$= \binom{n}{x}\left(\frac{w}{w+b}\right)^x \left(\frac{b}{w+b}\right)^{n-x}$$

$$= \binom{n}{x} p^x (1-p)^{n-x}$$

where $p = \frac{w}{w+b}$ is the PROPORTION of whites in the urn.

This is true for ANY $x = 0, 1, \ldots, n$ and of course is the Binomial Distribution. (or $B(n,p)$)

Sampling WITHOUT Replacement

A) To draw out n balls without replacement there are

1) $w+b$ choices for the first ball
2) $w+b-1$ " " " 2^{ND} ball
⋮
n) $w+b-n+1$ " " " n^{th} ball

So ~~there are~~ the total no. possible sequences is

$$(w+b)(w+b-1)\cdots(w+b-n+1) = \begin{cases} (w+b)! & \text{if } n = b+w \\ \frac{(w+b)!}{(w+b-n)!} & \text{if } n < b+w \end{cases}$$

If we are happy to interpret $0! = 1$, the latter formula applies to both cases. [this is the denominator]

B) 1) ~~There are~~ In constructing a sequence of x whites when sampling without replacement, there are

1) w choices for the first white, then
2) $w-1$ " " " 2ND Wh, then

⋮

x) $w-x+1$ " " " x-th " .

Giving $w(w-1)\ldots(w-x+1) = \dfrac{w!}{(w-x)!}$ (for any $x \le w$)

total possible sequence of x whites

2) Similarly, there are

1) b choices for 1ST Black, then
2) $b-1$ " " " 2ND " "

⋮

$n-x$) $b-n+x+1$ " " $(n-x)$-th Bl.

giving $b(b-1)\ldots(b-n+x+1) = \dfrac{b!}{(b-n+x)!}$, for any x s.t. $n-x \le b$

So, applying the prescription (*), we see that

$$P(\text{exactly } x \text{ Whites}) = \frac{\dfrac{n!}{x!\,(n-x)!}\,\dfrac{w!}{(w-x)!}\,\dfrac{b!}{(b-n+x)!}}{\dfrac{(w+b)!}{(w+b-n)!}}$$

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$$= \frac{\dfrac{w!}{x!\,(w-x)!}\;\dfrac{b!}{(n-x)!\,(b-n+x)!}}{\dfrac{(w+b)!}{n!\,(w+b-n)!}} = \frac{\binom{w}{x}\binom{b}{n-x}}{\binom{w+b}{n}}$$

This is true for any integer x such that

- $0 \leq x \leq n$
- $x \leq w$
- $n-x \leq b \iff x \geq n-b$

This is of course the HYPERGEOMETRIC distribution