Permutations / Combinations

Permutations (ordered selections or "sequences") and combinations (unordered selections or "subsets") are fundamental concepts in any introductory probability/statistics course. We can derive the necessary results using the "multiplication principle" applied to certain Urn Models.

Permutations

Suppose an urn contains N (>1) balls, numbered 1 to N; and that n (for some $1 \le n \le N$) balls are to be drawn out in sequence. How many outcomes are in the sample space?

This is simply addressed using the mult'n principle. The process is an n-step procedure:

1) N choices for 1^{ST} ball
2) Whatever the first ball drawn, $(N-1)$ choices remain for 2^{ND} ball.

⋮

n) Whatever the first $(n-1)$ choices, $(N-n+1)$ choices remain for n^{th} ball

(note: sums of $(1,N), (2,N-1), \ldots, (n, N-n+1)$ are each $N+1$).

So, according to the mult. principle, the total no. possible sequences is

$$\underbrace{N(N-1)\cdots(N-n+1)}_{n \text{ factors}} = \begin{cases} N! & \text{if } n=N \text{ - (all balls being drawn)} \\ \dfrac{N!}{(N-n)!} & \text{if } n<N \end{cases}$$

If we are happy to DEFINE $0! = 1$ then the 2^{ND} formula makes sense in both cases:

$$N_{P_n} = \frac{N!}{(N-n)!} = \text{no. permutations of size/length } n \text{ from } \{1,2,\dots,N\}$$

($N \geq 1$, ~~$n \geq 0$~~ $0 \leq n \leq N$ integers)

Combinations

Suppose an urn contains w white balls and b black balls (suppose w, b are integers ≥ 1). - each uniquely numbered. Suppose ALL are to be drawn out in order and placed in sequence, in such a way that all possible sequences are equally likely. What is the probability that all white balls are drawn out BEFORE any of the black balls?

Another way to think of this is to associate each possible sequence of colours with the corresponding "word" e.g. "all Wh's first" = "WW…WBB…B"
So what is $P(WW\cdots WBB\cdots B)$?

<u>Firstly</u>: How many elements/outcomes in the sample space? Suppose whites are numbered $1, 2, \ldots, w$ and the blacks are numbered $w+1, w+2, \ldots, w+b$. According to the previous section, the total no. outcomes is simply <u>$(w+b)!$</u> So how many of these have all W's before all B's?

To construct such an outcome, we have

1) w choices for first pos'n, then
2) $w-1$ " " second " "
⋮
w) 1 choice for w-th pos'n

THEN

$w+1$) b choices for $(w+1)$ pos'n, then
$w+2$) ~~[illegible]~~ $b-1$ " " $(w+2)$-th " " " "
⋮
$w+b$) 1 choice " " $(w+b)$-th pos'n.

So, the total no. outcomes is $w!\,b!$

thus $P(WW\cdots WBB\cdots B) = \frac{w!\,b!}{(w+b)!}$

How about $P(BB\cdots BWW\cdots W)$?

1) b choices 1^{st} pos'n
2) $b-1$ .. .. 2^{ND} ..
⋮
b) 1 choices b-th pos'n

THEN

$b+1$) w choices for $(b+1)$th pos'n
$b+2$) $w-1$ .. .. .. $(b+2)$th pos'n
⋮
$b+w$) 1 choice .. .. $(b+w)$th pos'n,

so again there are $b!\,w!$ outcomes ~~with the word~~ $BB\cdots BWW\cdots W$. and so $\boxed{P(BB\cdots BWW\cdots W) = \frac{b!\,w!}{(b+w)!}}$

ZEN MOMENT #1: A ~~little~~ little reflection shows that ANY prespecified "word" containing b B's and w W's will have the same probability.

ZEN MOMENT # 2: Sum of probs for all possible words adds to 1. They ~~are all~~ each ~~eq~~ have prob $\frac{b!\,w!}{(b+w)!}$, so there must be $\frac{(b+w)!}{b!\,w!}$ of them!

We can associate each such word with a <u>subset</u> of size w from $\{1, 2, \ldots, b+w\}$ (i.e. subset values correspond to positions of W's in the word).
(or equivalently, a subset of size b)

Making the correspondence

$$N \longleftrightarrow b+w$$

$$n \longleftrightarrow w$$

we have that for integers $N > 1$, $0 < n < N$,

$${}^{N}C_{n} = \binom{N}{n} = \frac{N!}{n!\,(N-n)!} = \binom{N}{N-n}$$

> # combinations / subsets of size n from $\{1, 2, \ldots, N\}$

Again, allowing $0! = 1$ this formula also applies for $N = 1$, $n = 0$ and $n = N$

i.e. $\binom{N}{0} = \binom{N}{N} = 1$