Expected Values of functions of (possibly several) RVs

Suppose we have a classical prob scenario; a sample space S has M "equally likely" outcomes and suppose $X(\cdot)$ is a RV defined on S. Let $g(\cdot)$ be any function. Then $Y = g(X)$ (i.e. $Y(s_i) = g(X(s_i))$) is just another RV defined with expected value

$$E(Y) = E[g(X)] = \frac{1}{M}\sum_{i=1}^{M} g(X(s_i))$$

Note that if X only takes k distinct values $x_1, \dots, x_k$ then this can be expressed as

$$\frac{1}{M}\left\{ m_1 g(x_1) + m_2 g(x_2) + \dots + m_k g(x_k) \right\}$$

(where, as in Lec. 9, for $j = 1, 2, \dots, k$, m_j is the no. of s_i's such that $X(s_i) = x_j$)

$$= \left\{ \frac{m_1}{M} g(x_1) + \frac{m_2}{M} g(x_2) + \dots + \frac{m_k}{M} g(x_k) \right\}$$

$$= g(x_1)P(X = x_1) + g(x_2)P(X = x_2) + \dots + g(x_k)P(X = x_k)$$

$$= \sum_{j=1}^{k} g(x_j)P(X = x_j)$$

Important Examples

1) **Moments**: the r-th moment of a RV X is simply $E(X^r)$ (so the ordinary expected value is the 1st moment)

Example: Let X be the no. showing on a roll of a "fair" 6-sided die. Then $P(X=1)=P(X=2)=\cdots=P(X=6)=\frac{1}{6}$

and $E(X^2) = 1^2.P(X=1) + 2^2P(X=2)+\cdots+6^2P(X=6)$

$$= \frac{1}{6}\{1+4+9+16+25+36\} = \frac{91}{6}.$$

2) **Factorial Moments**: The r-th factorial moment of a RV X is $E[\underbrace{X(X-1)\cdots(X-r+1)}_{r \text{ factors}}]$ $\quad {}_XP_r = \frac{X!}{(X-r)!}$

Again, the case $r=1$ gives the ordinary expected value.

Examples: $X \sim B(n,p)$ [Assume $n \geq 2$) otherwise $X(X-1)$ is identically 0.]

$$E[X(X-1)] = \sum_{x=0}^{n} x(x-1)P(X=x) = [0(-1).P(X=0) + 1.0.P(X=1) + 2.1.P(X=2)+\cdots+n(n-1)P(X=n)]$$

$$= \sum_{x=2}^{n} \cancel{x(x-1)} \frac{n(n-1)(n-2)\cdots(n-x+1)}{\cancel{x(x-1)}(x-2)\cdots 1} \cdot p^x(1-p)^{n-x}$$

$$= n(n-1)p^2 \sum_{x=2}^{n} \binom{n-2}{x-2} p^{x-2} (1-p)^{[n-2]-[x-2]}$$

Write $y = x-2$

$$= n(n-1)p^2 \underbrace{\sum_{y=0}^{n-2} \binom{n-2}{y} p^y (1-p)^{(n-2)-y}}_{\text{sum of all } B(n-2,p) \text{ probs.}}$$

$$= n(n-1)p^2$$

<u>NOTE:</u> By definition

$$E[X(X-1)] = E[X^2 - X] = \frac{1}{M} \sum_{i=1}^{M} [X(s_i)^2 - X(s_i)]$$

$$= \frac{1}{M} \sum_{i=1}^{M} [X(s_i)^2] - \frac{1}{M} \sum_{i=1}^{M} X(s_i)$$

$$= E(X^2) - E(X)$$

So once we know 2 of the quantities

$$E(X), \quad E(X^2) \text{ and } E[X(X-1)]$$

we can work out the <u>other one</u>.

So, for $X \sim B(n,p)$, $E(X^2) = E[X(X-1)] + E(X)$

$$= n(n-1)p^2 + np.$$

<u>The Variance</u>: If (in a classical prob scenario) a RV X has $E(X) = \frac{1}{M}\sum_{i=1}^{M} X(s_i) = \mu$. Then its <u>Variance</u> is its expected squared distance from μ:

$$\mathrm{Var}(X) = \frac{1}{M}\sum_{i=1}^{M}\left\{\left[X(s_i)-\mu\right]^2\right\} \quad \left[= E\left[(X-\mu)^2\right]\right]$$

$$= \frac{1}{M}\sum_{i=1}^{M}\left\{X(s_i)^2 - 2\mu X(s_i) + \mu^2\right\}$$

$$= \frac{1}{M}\Big\{ X(s_1)^2 - 2\mu X(s_1) + \mu^2$$
$$+ X(s_2)^2 - 2\mu X(s_2) + \mu^2$$
$$+ \cdots\cdots$$
$$+ X(s_M)^2 - 2\mu X(s_M) + \mu^2 \Big\}$$

$$= \frac{1}{M}\left\{\sum_{i=1}^{M}\left[X(s_i)\right]^2 - 2\mu\sum_{i=1}^{M}X(s_i) + M\mu^2\right\}$$

$$= E(X^2) - 2\mu E(X) + \mu^2$$

$$= E(X^2) - \left[E(X)\right]^2$$

This is the so-called "computing formula for the Variance".

<u>Example</u>: For X = no. showing on fair 6-sided die, $E(X) = 3.5$ [Exercise]

$$\mathrm{Var}(X) = E(X^2) - \left[E(X)\right]^2 = \frac{91}{6} - \left[\frac{7}{2}\right]^2 = \frac{91}{6} - \frac{49}{4} = \frac{182-147}{12} = \frac{35}{12}.$$

Variance of a Binomial

We have shown already that if $X \sim B(n,p)$,

$$E(X) = np \quad \text{and}$$

$$E(X^2) = n(n-1)p^2 + np$$

SO,

$$\begin{aligned} Var(X) &= E(X^2) - [E(X)]^2 \\ &= n(n-1)p^2 + np - n^2p^2 \\ &= \cancel{n^2p^2} - np^2 + np \cancel{- n^2p^2} \\ &= np(1-p). \end{aligned}$$

Variance of ~~Hypergeomtor~~ Hypergeometric

The strategy is

1) find $E(X)$
2) find $E[X(X-1)]$
3) HENCE find $Var(X) = E[X(X-1)] + E(X) - [E(X)]^2$