An Analytic and Numerical Study on the Goodwin and Goodwin-Keen Economics Models

Romi Lifshitz [1] Arthur Méndez-Rosales [2] Sara Saad [3]
Grant Forsythe [4] Gheeda Mourtada [4] Jacob Keffer [5]

[1]Department of Arts and Science, McMaster University

[2]Department of Engineering Physics, McMaster University

[3]Department of Electrical and Computer Engineering, McMaster University

[4]Department of Mathematics and Statistics, McMaster University

[5]Department of Chemistry and Chemical Biology, McMaster University

Overview

Introduction

- Are economies primarily stable?
- An introduction to internal factors of the Goodwin and Goodwin-Keen models.
- How can these models be used?

Purpose of Study

Studying the long-term equilibrium impact that endogenous economic variables have on the simplified macro-economy, modeled by the Goodwin model as well as the extensions made by Keen.

Goodwin Model

The Goodwin model (Goodwin, 1982) describes the evolution of the employment rate λ and the wage share ω as

$$\begin{aligned} \dot{\lambda} &= \lambda \cdot \left(\frac{1-\omega}{\nu} - \alpha - \beta - \delta \right), \\ \dot{\omega} &= \omega \cdot (\Phi(\lambda) - \alpha). \end{aligned} \tag{1}$$

The Phillips curve connects the employment rate to the wage share. It is defined as

$$\Phi(\lambda) = \frac{\Phi_1}{(1-\lambda)^2} - \Phi_0. \tag{2}$$

Jacobian of the Goodwin Model

The Jacobian matrix for the Goodwin system of equations is defined by

$$J = \begin{bmatrix} \frac{\partial}{\partial \lambda}\dot{\lambda} & \frac{\partial}{\partial \omega}\dot{\lambda} \\ \frac{\partial}{\partial \lambda}\dot{\omega} & \frac{\partial}{\partial \omega}\dot{\omega} \end{bmatrix}. \tag{3}$$

Model Parameters

Table 1: Model Parameters (Grasseli, 2012).

Parameter	**Value**
α	0.025
β	0.02
δ	0.01
Φ_0	$\frac{0.04}{1-0.04^2}$
Φ_1	$\frac{0.04^3}{1-0.04^2}$
κ_0	-0.0065
κ_1	e^{-5}
κ_2	20
r	0.03
ν	3

Methods Overview

1. Generate symbolic model using SymPy Python Solver.
2. Determine equilibrium points using symbolic equations.
3. Calculate Jacobian and evaluate stability at equilibrium points.
4. Validate equilibrium points by graphical and numerical means.

Goodwin-Keen Model

The Goodwin-Keen model (Keen, 1995) describes the impact of three parameters on a simplified macro-economy as

$$\begin{aligned}
\dot{\lambda} &= \lambda \cdot \left(\frac{\kappa(\pi)}{\nu} - \alpha - \beta - \delta \right), \\
\dot{\omega} &= \omega \cdot (\Phi(\lambda) - \alpha), \\
\dot{d} &= d \cdot \left(r - \frac{\kappa(\pi)}{\nu} + \delta \right) + \kappa(\pi) - (1 - \omega), \\
\pi &= 1 - \omega - rd,
\end{aligned} \tag{4}$$

where λ is the employment rate, ω is wage share, and d is private debt. $\kappa(\pi)$ now represents the non-linear rate of new investment, and π represents the net profits share from capital investments.

Jacobian of the Goodwin-Keen Model

The Jacobian matrix for the system is now defined by

$$J = \begin{bmatrix} \frac{\partial}{\partial \lambda}\dot{\lambda} & \frac{\partial}{\partial \omega}\dot{\lambda} & \frac{\partial}{\partial d}\dot{\lambda} \\ \frac{\partial}{\partial \lambda}\dot{\omega} & \frac{\partial}{\partial \omega}\dot{\omega} & \frac{\partial}{\partial d}\dot{\omega} \\ \frac{\partial}{\partial \lambda}\dot{d} & \frac{\partial}{\partial \omega}\dot{d} & \frac{\partial}{\partial d}\dot{d} \end{bmatrix}. \tag{5}$$

Goodwin-Keen Model Sensitivity Analysis

We ran 1000 simulations, across the following domains.

Table 2: Sensitivity Simulation Domains.

Initial Condition	**Lower Bound**	**Upper Bound**	**Step Size**
λ_0	0.6	1.0	$0.04\overline{4}$
ω_0	0.5	1.0	$0.05\overline{5}$
d_0	0	10	$1.1\overline{1}$

Goodwin Model Symbolic Solution

The Jacobian of the Goodwin model is given by

$$\mathsf{J} = \begin{bmatrix} \frac{1-\omega}{\nu} - \alpha - \beta - \delta & -\frac{\lambda}{\nu} \\ \frac{2\Phi_1 \omega}{(1-\lambda)^3} & \frac{\Phi_1}{(1-\lambda)^2} - \Phi_0 - \alpha \end{bmatrix}. \tag{6}$$

Its long-term equilibria (disregarding the trivial solution) are given by

$$(\lambda^*_{\pm}, \omega^*) = \left(1 \pm \sqrt{\frac{\Phi_1}{\alpha + \Phi_0}}, \quad 1 - \nu \cdot (\alpha + \beta + \delta)\right). \tag{7}$$

Goodwin Model Symbolic Solution

With the economically realistic equilibrium point

$$(\lambda^*, \omega^*) = \left(1 - \sqrt{\frac{\Phi_1}{\alpha + \Phi_0}},\quad 1 - \nu \cdot (\alpha + \beta + \delta)\right), \tag{8}$$

the system's Jacobian matrix yields the following eigenvalues

$$\pm\sqrt{2} \cdot \sqrt{\xi_1 - \xi_2}, \tag{9}$$

where

$$\begin{aligned} \xi_1 &= \frac{\Phi_0 + \alpha}{\nu} + \frac{\sqrt{\Phi_1 \cdot (\Phi_0 + \alpha)}(\Phi_0 + \alpha)(\alpha + \beta + \delta)}{\Phi_1}, \\ \xi_2 &= \frac{\sqrt{\Phi_1 \cdot (\Phi_0 + \alpha)}(\Phi_0 + \alpha)}{\Phi_1 \nu} + (\Phi_0 + \alpha)(\alpha + \beta + \delta). \end{aligned} \tag{10}$$

Goodwin Model Symbolic Solution

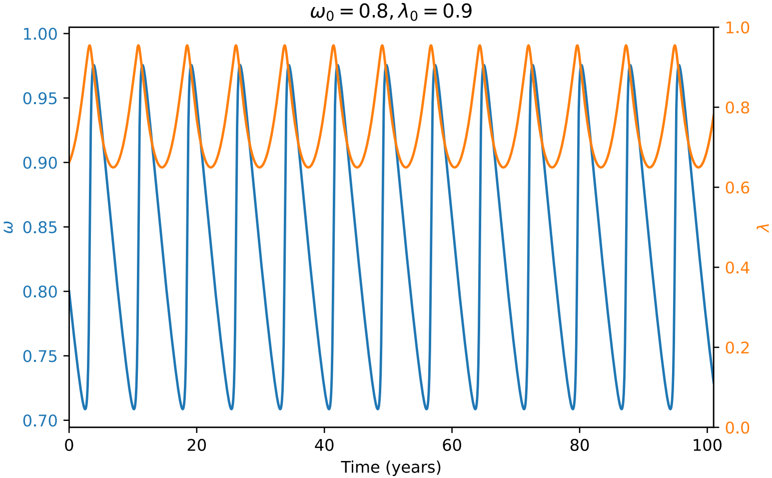
We can define two potential cases for these eigenvalues

$$\pm\sqrt{2}\cdot\sqrt{\xi_1-\xi_2},$$

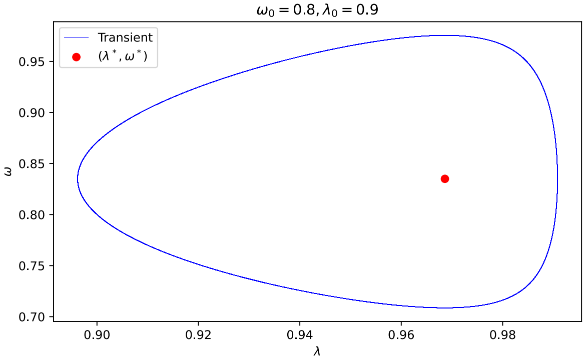
$$\xi_1 = \frac{\Phi_0+\alpha}{\nu} + \frac{\sqrt{\Phi_1\cdot(\Phi_0+\alpha)}(\Phi_0+\alpha)(\alpha+\beta+\delta)}{\Phi_1},$$
$$\xi_2 = \frac{\sqrt{\Phi_1\cdot(\Phi_0+\alpha)}(\Phi_0+\alpha)}{\Phi_1\nu} + (\Phi_0+\alpha)(\alpha+\beta+\delta).$$

1. $\xi_1 > \xi_2$ Eigenvalues are *real* and the system is unstable.
2. $\xi_1 < \xi_2$ Eigenvalues are purely *imaginary*, *real* parts are 0, and the system converges to a limit cycle.

Behaviour of the Goodwin Model



Goodwin Equilibrium and Cyclical Behaviour



$$(\lambda^*, \omega^*) = (0.968612, 0.835000).$$

Goodwin-Keen Model Symbolic Solution

The Jacobian matrix for the Goodwin-Keen model makes use of both (2) and

$$\kappa = \kappa(\pi) = \kappa_0 + \kappa_1 \mathrm{e}^{\kappa_2 \pi}.$$

Before we present the matrix, we note the use of the notational simplification

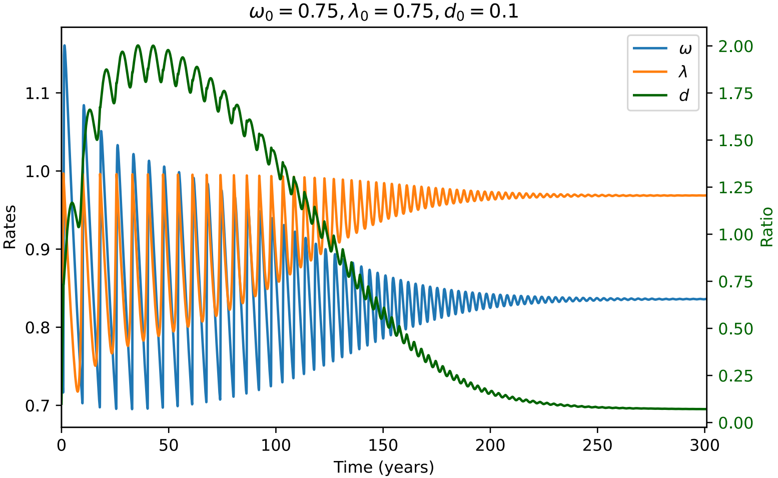
$$\kappa' = -\frac{\partial \kappa}{\partial \omega} = -\frac{1}{r}\frac{\partial \kappa}{\partial d} = \kappa_1 \kappa_2 \mathrm{e}^{\kappa_2 \pi}.$$

Goodwin-Keen Model Symbolic Solution

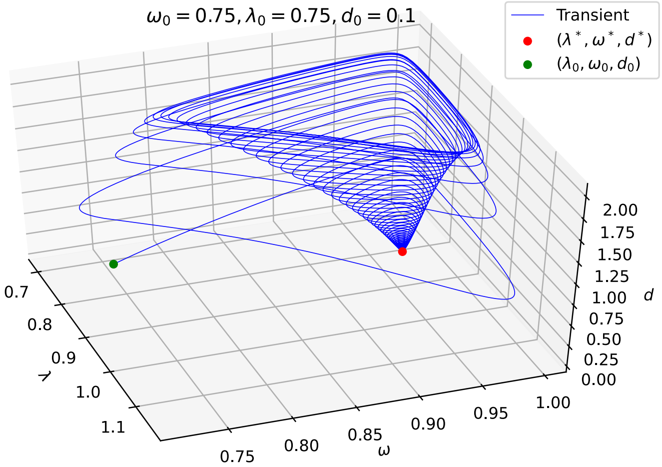
Thus, the Jacobian matrix for the Goodwin-Keen model is given by

$$J = \begin{bmatrix} \frac{\kappa-\nu(\alpha+\beta+\delta)}{\nu} & -\frac{\lambda\kappa'}{\nu} & -\frac{\lambda r\kappa'}{\nu} \\ \frac{2\Phi_1\omega}{(1-\lambda)^3} & \frac{\Phi_1}{(1-\lambda)^2} - \Phi_0 - \alpha & 0 \\ 0 & \frac{(d-\nu)\kappa'+\nu}{\nu} & \frac{r\cdot(d-\nu)\kappa'}{\nu} + \delta + r \end{bmatrix}. \quad (11)$$

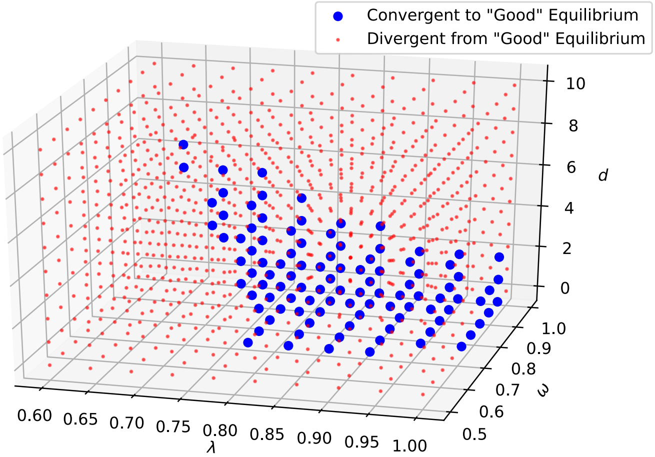
Goodwin-Keen Plot



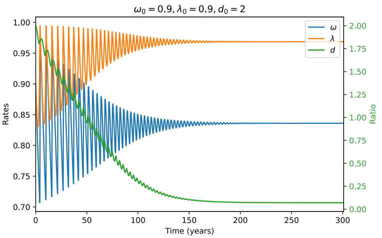
Goodwin-Keen “Good” Equilibrium

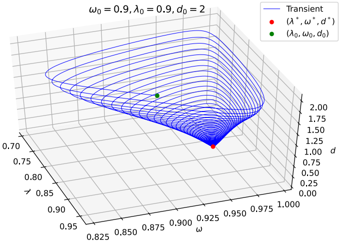


Goodwin-Keen Basin of Attraction



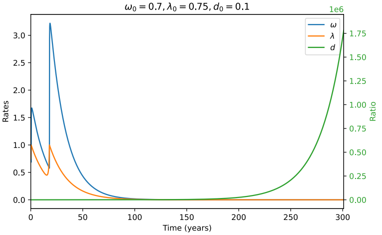
Goodwin-Keen "Good" Equilibrium

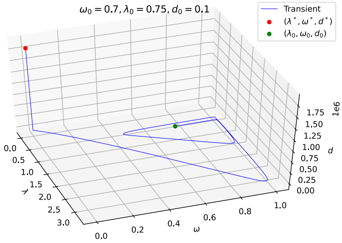




$$(\lambda^*, \omega^*, d^*) = (0.968612, 0.86053, 0.070191).$$

Goodwin-Keen “Bad” Equilibrium





$$(\lambda^{\times}, \omega^{\times}, d^{\times}) = (0, 0, +\infty).$$

Model Comparison

The Goodwin Model

1. Dynamically analogous to the prey-predator model.
2. *One* economically feasible equilibrium points.
3. Has served as a stepping stone for more complete models (Grasseli, 2012; Giraud, et al., 2018).

The Goodwin-Keen Model

1. Capable of modeling a stable and unstable economy.
2. *Two* economically feasible equilibrium point.
3. As an endogenous model, shows that economies can collapse solely due to internal factors (Keen, 1995).

Conclusion

- Qualitative and mathematical presentation of Goodwin and Goodwin-Keen models.
  - Employment and wage share.
  - Employment, wage share, debt ratio, and net profits share.
- Equilibrium found and behaviours plotted.
  - Goodwin is too simplistic, but is foundational.
  - Goodwin-Keen provides more realistic uses.

Conclusion

- Additional Works and Considerations
  - Full Keen model with all factors of *Minsky Instability Hypothesis*
    - *Ponzi-esque* financing and regulatory effects.
  - Enviro-economical model constructed by Giraud, et al. (2018)
    - Connects the rising CO_2 levels to the Goodwin-Keen model, in hopes of an environmentally motivated economic model.
  - Real world impact of economic modelling, and its importance.

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