Project Report

# Global Approximation of Functions on Manifolds via Partitions of Unity

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# 1. Introduction

In this work, we empirically test a hypothesis that locally defined (fitted) functions on the charts of a 2 - dimensional manifold can be extended to be globally defined over the whole surface of the manifold. We consider linear/ polynomial functions defined locally and compute the global approximation using Partitions of Unity Method (PUM). We observe that using our approach, one can closely approximate it within an upper bound of $< 1$ on the average loss. We make use of compactly supported bump functions as local approximants in PUM.

# 2. Background

## 2.1 Manifold:

A manifold is a topological space that is locally Euclidean. To be precise, each point of a n-dimensional manifold has a neighbourhood that is homeomorphic (has similar properties) to a Euclidean space of dimension n. Properties that are defined locally on a manifold may not be defined globally. For example, the whole manifold may not be homeomorphic or may not have topological properties similar to a Euclidean space. Along those lines, the goal of this project is to define local functions over the whole surface of manifold.

## 2.2 Charts of a manifold:

A manifold can also be described as a collection of overlapping charts. We define a chart (or coordinate chart) to be a pair of an open subset of Euclidean space $U$ and a continuous invertible map $\phi$ (called homeomorphism) between the manifold and $U$. The dimension of this Euclidean space is the dimension of the manifold. The homeomorphism between the two topological spaces preserves all its properties such as connectedness and compactness. The ordered pair of the chart is represented by $(U, \phi)$.

## 2.3 Atlas:

An atlas of a topological space $M$ is a collection of $\{(U_\alpha, \phi_\alpha) | \alpha \in A\}$ indexed by a set $A$ of charts on $M$ such that $\bigcup_{\alpha \in A} U_\alpha = M$ i.e. union of the charts covers the whole space.

# 3. Partition of Unity

A partition of unity is a useful tool that helps us work in local coordinates. This can be a tricky matter when we're doing things all over our manifold, since its almost never the case that the entire manifold fits into a single coordinate patch. A (smooth) partition of unity is a way of breaking the function with the constant value 1 up into a bunch of (smooth) pieces that are easier to work with.

More precisely, for any covering of a topological space $X$ (here it is a manifold) by open subsets $\{U_m\}$ (here they are the charts of the manifold), there exists a sequence of smooth nonnegative functions $\{\psi_n\}$, on $X$, called a partition of unity, subordinate to the open cover $\{U_m\}$ such that for every point, x $\in X$ :

- the sum of all function values at x is 1, i.e, $\sum_n \psi_n(x) = 1$. Each $\psi_n = (\varphi_n / \sum_j \varphi_j)$ where we have taken $\varphi_n$ to be a bump function. By this normalization, the summation $\sum_n \psi_n(x)$ comes out to be 1. As we'll see later, we need to introduce 3 bump functions corresponding to 3 charts. Hence n = 3.
- $\forall n,\ 0 \leq \varphi_n(x) \leq 1$

- $\forall n$, supp $\varphi_n \subseteq U_n$, where 'supp' is the support of the function (subset of domain which is not mapped to 0) and $U_n$ is the chart in which $\varphi_n$ is defined and assigned to.

## 4. Bump Functions

A bump function is a function $f\colon R^n \to R$ defined on some euclidean space $R^n$ and has the following properties : smooth (of class $C^\infty$ i.e. it has derivatives of all orders or also called infinitely differentiable), non-negative and compact (closed and bounded) support. The bump function is required to have compact support within the open subset (chart) it is defined in.

One of the bump functions we use:

$$\varphi_1(x) = \begin{cases} e^{(\frac{1}{x^2-1})} & for\ x \in (-1,1), \\ 0 & otherwise \end{cases}$$

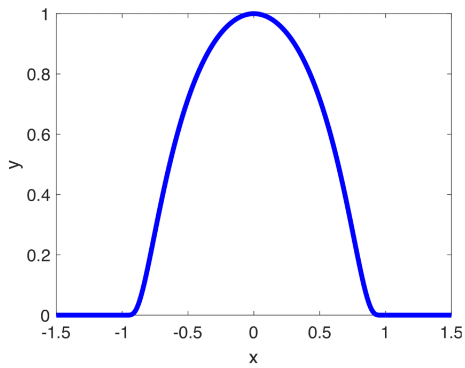


Figure 1: Graph of $\varphi_1$

$\varphi_1$ has a compact support in [-1,1] and is naturally 0 outside it. Also, we have sampled points from a unit sphere centered at 0. This ensures that the condition supp $\varphi_1 \subseteq U_1$ (where $U_1$ is some chart) is satisfied since the points lying in $U_1$ will also have a range between -1 and 1. A bump function for a n-dimensional input has the advantage of being expressible as the product of its values at each dimension:

$$f(x_1, x_2, .., x_n) = f(x_1)f(x_2)...f(x_n)$$

In our case n = 2, since the bump functions are defined on charts consisting of 2 - dimensional points. Further details given in Section 6.

## 5. Linear Regression

Linear regression is a form of regression analysis that takes a linear approach to modelling the relationship between dependent and multiple (or single) independent variables. For a training sample $i$ in the dataset $\{y_i, x_{i1}, ..., x_{ip}\}_{i=1}^{N}$, common form of the linear model :

$$\begin{aligned} y_i &= \beta_0 1 + \beta_1 x_{i1} + .... + \beta_p x_{ip} + \varepsilon_i \\ &= \boldsymbol{x_i^T} \boldsymbol{\beta} + \varepsilon_i \end{aligned}$$

where $y_i$ is the dependent variable, $\boldsymbol{x_i}$ is the p-dimensional input/independent variables vector, $\boldsymbol{\beta}$ is a vector consisting of the parameters/ weights of the linear model to be estimated from the data and $\varepsilon_i$ is some noise. As we can see, $\boldsymbol{\beta}$ is a (p + 1) dimensional vector, extra dimension being $\beta_0$ which is also called the intercept of the curve.

In our case, we perform linear regression for both 3-dimensional as well as 2-dimensional inputs and sample the noise from a Gaussian distribution. We use Ordinary Least Squares (OLS) to estimate the parameter vector $\hat{\beta}$ :

$$\hat{\beta} = \arg\min_{\beta} \ S(\beta)$$

where $S(\beta)$ is the sum of squared error (loss function) to measure the overall fit of the model :

$$S(\beta) = \sum_{i=1}^{N} \left(y_i - x_i^T \beta\right)^2$$

## 6. Our Approach

We consider a spherical manifold embedded in a 3 - dimensional Euclidean space. This unit sphere ($x^2 + y^2 + z^2 = 1$) can be covered by an atlas of six charts, that are basically $R^2$ discs. This is because, for example, the plane $z = 0$ divides the sphere into two half spheres ($z > 0$ and $z < 0$) which can be mapped to the discs defined by $x^2 + y^2 < 1$ by projecting the points lying on it on the $xy$-plane. The map used for this would be : $\phi(x, y, z) = (x, y)$. This furnishes us two charts, and by repeating this procedure for other two coordinate plates $x = 0$ and $y = 0$ gives us in total 6 charts.

The intrinsic dimension of a manifold is always less than the dimension of surrounding space it is embedded in and is equal to the dimension of the euclidean space it's charts lie in. Hence, in our case it is 2. The manifold is denoted by $S^2$. Now, steps involved in computing the approximation :

Step 1: We randomly sample 1000 3 - dimensional points from a unit $R^3$ sphere (which is our manifold $S^2$).

Step 2: We then divide $S^2$ into the six charts by projecting the 3 - dimensional points into their respective 2 - dimensional discs. Each 3-D point, therefore, now has a 2-D representation in 3 different charts (one each in $xy$, $yz$ and $xz$-planes).

Step 3: Corresponding to each of these 3 charts, we assign a smooth bump function whose support lies compactly in that particular chart.

Step 4: We then take the set of points lying in a chart, randomly sample target values for them from a Gaussian distribution ($\mu = 0$, $\sigma = 1$), and fit linear/ polynomial functions. We repeat this for each chart.

Step 5: Final approximation for a point 'x' lying on the manifold :
$f_{approx}(x) = \sum_i f_i(x_i) * \psi_i(x_i)$, where : $x_i$ is the 2-dimensional projection of 'x' in the $i^{th}$ chart, $f_i(x_i)$ is the linear/ polynomial fitted function value and $\psi_i(x_i)$ is the normalized bump function value of $x_i$ in the $i^{th}$ chart, respectively. The summation is taken over all the $i$ (here 3) charts a point lies in.

What step 5 intuitively does is - it takes functions that are defined locally, bumps them off by taking its product with a normalized bump function so that they are 0 outside their domain, and then adds them up across the charts to approximate it globally.

## 7. Evaluation and Results

We fit linear/ polynomial curves to the $n$ (= 1000) 3 - dimensional points lying on the spherical manifold as well, which acts as our true function ($f_{true}$). For fitting, we again sample target values from the same Gaussian distribution ($\mu = 0$, $\sigma = 1$) as that for the charts. Our goal is to approximate the local functions (obtained from steps 2, 3 and 4 from previous section) to $f_{true}$, as closely as possible.

We use k-fold cross validation method to evaluate the performance of our approach, where we set k = 10 and divide the train and test set in 9 : 1 ratio. We then accumulate the difference between $f_{true}$ and $f_{approx}$ for test sets across all 10 rounds of cross validation and report its average. Average test loss comes out to be $\approx 0.20$. [Code : https://github.com/greninja/PoU-Manifold].

## 8. Conclusion and future work:

Hence we show that using our approach, one can empirically evaluate and test the hypothesis that local functions defined on charts of manifold can be approximated to the global function value within some predefined bound on loss. Future work could include experimenting with a scalar valued function other than linear or polynomial function. Also, we can look into how this approach can potentially be used for learning the manifold of the dataset.

## 9. Acknowledgement

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## Dataset and charts :

The following python code is used to sample points from a unit $R^3$ sphere:

```
import numpy as np
def sample_spherical(npoints, ndim=3):
    vec = np.random.randn(ndim, npoints)
    vec /= np.linalg.norm(vec, axis=0)
    return vec
xi, yi, zi = sample_spherical(1000)
```

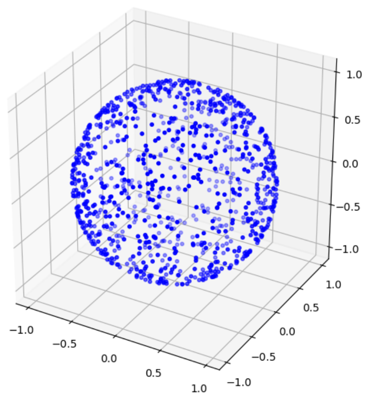


Figure 2: 1000 sample points lying on the spherical manifold of unit radius centered at 0

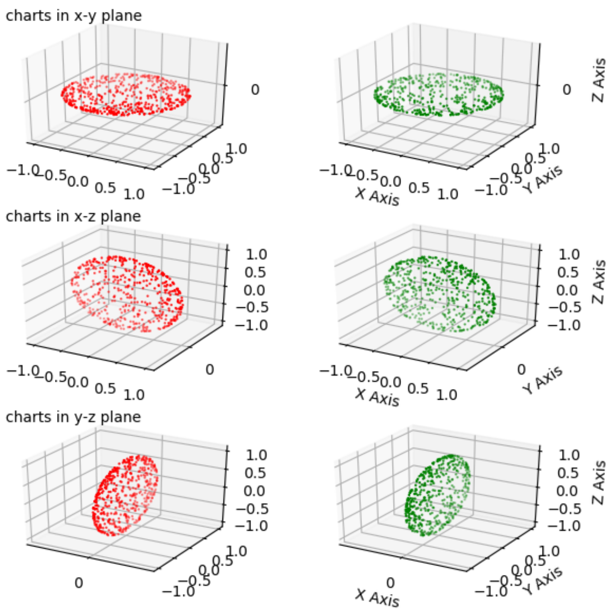


Figure 3: Six charts of the spherical manifold. For each of the 3 planes, the points colored in red and green are projections from 2 possible hemispheres, respectively.