Research Meeting

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PhD course plan

12 ECTS are required :

- Deep Learning in Artificial and Biological Neuronal Networks (4 ECTS) : done!
- Algebra and Error Correcting Codes (6 ECTS) : Spring 2022
- Learning to Teach (2 ECTS) : Fall 2022

Erratum
Dale's Principle

Encyclopaedia Britannica

"[Sir Henry Dale] stated that a neurotransmitter releases at one axon terminal of a neuron can be presumed to be released at other axon terminals of the same neuron. Dale's principle refers only to he presynaptic neuron, as the responses of different postsynaptic receptors to a single neurotransmitter can vary in the same or different neurons."

Principles of Neural Science, Sixth Edition by Kandel et al.

"There are two major classes of transmitters receptors. Ionotropic receptors are ligand-gated ion channels. Binding of transmitter to an extracellular binding site triggers a conformational change that opens the channel pore, generating an ionic current that excites (depolarizes) or inhibits (hyperpolarizes) the postsynaptic cells, depending on the receptor."

Conjecture

A RSNN with $N > 0$ neurons and a causal impulse response h with (finite) duration $T_h > 0$, is able to memorize any sequence in $\mathcal{M}^N_{L,Tr,Th}$.

Definition (Firing sequence - single channel)

A (single channel) firing sequence with length $L > 0$ and refractory period $T_r \geq 0$ is a binary sequence with at least T_r zeros between two consecutifs ones, including between the last one and the first one of the sequence.

We denote by $\mathcal{F}_{L,T_r}$ the set of all (single-channel) firing sequences with length L and refractory period T_r. We can also define the set of all (single-channel) firing sequences with length L and refractory period T_r, where we relax the constraint on the minimum number of zeros between the last one and the first one of the sequence. We denote this set by $\tilde{\mathcal{F}}_{L,T_r}$.

Definition (Generalized golden ratio)

Let $P(x) := x^p - x^{p-1} - 1$ with $p > 0$. The roots of $P(x)$ are denoted by $\varphi_{p,i}$ for $i = 1, \ldots, p$. Moreover, we call generalized golden ratio and denote it by $\hat{\varphi}_p$ the root with the greatest modulus.

p	$\hat{\varphi}_p$	p	$\hat{\varphi}_p$	p	$\hat{\varphi}_p$	p	$\hat{\varphi}_p$
1	2	6	1.285199	11	1.184276	16	1.140034
2	1.618034	7	1.255423	12	1.172951	17	1.133902
3	1.465571	8	1.232055	13	1.163120	18	1.128356
4	1.380278	9	1.213150	14	1.154494	19	1.123311
5	1.324718	10	1.197491	15	1.146854	20	1.118699

One can show that for any $p > 0$, $\hat{\varphi}_p$ is real and in the range $(1, 2]$.

Theorem (Cardinality)

The cardinality of $\mathcal{F}_{L,T_r}$ is given by:

$$|\mathcal{F}_{L,T_r}| = \sum_{i=1}^{T_r+1} \varphi_{T_r+1,i}^{L},$$

where $\varphi_{T_r+1,i}$ for $i = 1, \ldots, T_r+1$ are the roots of $P(x) := x^{T_r+1} - x^{T_r} - 1$.

It follows that

$$\lim_{L\to\infty} |\mathcal{F}_{L,T_r}| = \hat{\varphi}_{T_r+1}^{L},$$

where $\hat{\varphi}_{T_r+1}$ is the generalized golden ratio of order $T_r + 1$ as defined before. Moreover, for $L \gg T_r$, $\hat{\varphi}_{T_r+1}^{L}$ is actually an excellent approximate for $|\mathcal{F}_{L,T_r}|$.

Firing sequences
Single channel

Proof (1/4).

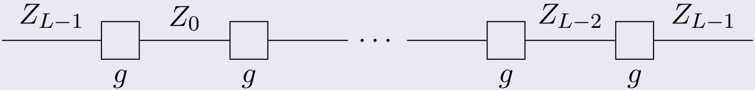
For $k = 0, \dots, L-1$, we define

$$Z_k := (X_{k-T_r}, \dots, X_k) \in \tilde{\mathcal{F}}_{T_r+1, T_r}$$

and

$$g_{k,k+1}(z_k, z_{k+1}) := \mathbb{1}_{\{x_{k-T_r}, \dots, x_{k+1} \in \tilde{\mathcal{F}}_{T_r+2, T_r}\}} = g(z_k, z_{k+1}),$$

where all indices are taken modulo L, and we draw the following factor graph.



Proof (2/4).

For any $k = 1, \dots, L$, Z_k only takes values in:

$$\tilde{\mathcal{F}}_{T_r+1,T_r} = \{ \overbrace{\,0\,}^{\text{1st el.}}, \overbrace{\boldsymbol{e_1}}^{\text{2nd el.}}, \overbrace{\boldsymbol{e_2}}^{\text{3rd el.}}, \dots, \overbrace{\boldsymbol{e_{T_r}}}^{T_r+1\text{-th el.}}, \overbrace{\boldsymbol{e_{T_r+1}}}^{T_r+2\text{-th el.}} \},$$

with

$$\mathbf{0} := [\overbrace{0, \cdots, 0}^{T_r+1}]^\intercal \text{ and } \boldsymbol{e_i} := [\overbrace{0, \cdots, 0}^{i-1}, 1, \overbrace{0, \cdots, 0}^{T_r+1-i}]^\intercal.$$

We express the messages in vector notation using the same ordering, i.e., 1st element corresponds to $\mathbf{0}$, 2nd element to $\boldsymbol{e_1}$, 3rd element to $\boldsymbol{e_2}$, and so on.

Proof (3/4).

The sum-product message passing can be expressed in matrix form:

$$\overleftarrow{\boldsymbol{\mu}}_{\boldsymbol{Z_k}} = \overleftarrow{\boldsymbol{\mu}}_{\boldsymbol{Z_{k+1}}} \boldsymbol{A} \text{ with } \boldsymbol{A} = \begin{bmatrix} 1 & & \\ 0 & & \\ \vdots & & \boldsymbol{I_{T_r+1}} \\ \vdots & & \\ 0 & & \\ 1 & 1 & 0 \cdots\cdots 0 \end{bmatrix}.$$

One can notice that the partition sum of the graph is then given by:

$$|\mathcal{F}_{L,T_r}| = \operatorname{tr}\left(\boldsymbol{A}^L\right).$$

Proof (4/4).

By inspection on $\boldsymbol{A}$, one can determine its characteristic polynomial:

$$P_A(x) = x(x^{T_r+1} - x^{T_r} - 1) = x \prod_{i=1}^{T_r+1} (x - \varphi_{T_r+1,i})$$

whose roots are the eigenvalues of $\boldsymbol{A}$. The result is now obvious as

$$\operatorname{tr}\left(\boldsymbol{A}^L\right) = \sum_{i=1}^{T_r+1} \varphi_{T_r+1,i}^L.$$

$\square$

Definition (Firing sequence - multiple channels)

An N-channels firing sequence with length $L > 0$ and refractory period $T_r \geq 0$ is a N-dimensional binary sequence such that each channel is a single-channel firing sequence with length L and refractory period T_r.

We denote by $\mathcal{F}^N_{L,T_r}$ the set of all N-dimensional firing sequence with length L and refractory period T_r.

Theorem (Cardinality)

The cardinality of $\mathcal{F}^N_{L,T_r}$ is given by

$$\left|\mathcal{F}^N_{L,T_r}\right| = \left|\mathcal{F}_{L,T_r}\right|^N .$$

Definition (Predictable firing sequence)

Let $L > 0$ and $T_w > T_r \geq 0$. An N-channels firing sequence $\boldsymbol{f^N} = (\boldsymbol{f_0^N}, \boldsymbol{f_1^N}, \ldots, \boldsymbol{f_{L-1}^N}) \in \mathcal{F}_{L,T_r}^N$ is said T_w predictable, if $\boldsymbol{f_k^N}$ is fully determined by $(\boldsymbol{f_{k-T_w}^N}, \ldots, \boldsymbol{f_{k-1}^N})$, for any $0 \leq k < L$.

We denote by $\mathcal{P}_{L,T_r,T_w}^N$ the set of all N-dimensional T_W-predictable firing sequences.

Example

With $L = 12$, $T_r = 2$, $N = 2$, and $T_w = 4$, we consider the following firing sequences :

$$\boldsymbol{f_1} = \begin{bmatrix} 0 & 1 & 0 & 0 & \color{purple}{0} & 1 & 0 & 0 & \color{purple}{1} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \color{purple}{0} & 0 & 1 & 0 & \color{purple}{0} & 0 & 1 & 0 \end{bmatrix},$$

$$\boldsymbol{f_2} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

and

$$\boldsymbol{f_3} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

$\boldsymbol{f_1}$ is not T_w-predictable but $\boldsymbol{f_2}$ and $\boldsymbol{f_3}$ are.

Finite State Machine

It is obvious that any predictable sequence can be stored by an FSM. Basically, starting from any T_w-window of such a sequence, on can perfectly reconstruct it by following the transition of the FSM.

Example

With $N = 2$, $L = 4$, and $T_r = 1$, let's consider the sequence

$$\boldsymbol{f} = \left[\begin{array}{cccc} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{array} \right] \in \mathcal{F}_{L,T_r}^{N}$$

For $T_w = 1$, this sequence is obviously not T_w-predictable since the window $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ can be followed either by $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, either by $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. For $T_w = 2$, it is predictable and we can construct the FSM of this sequence.

Finite State Machine

Example

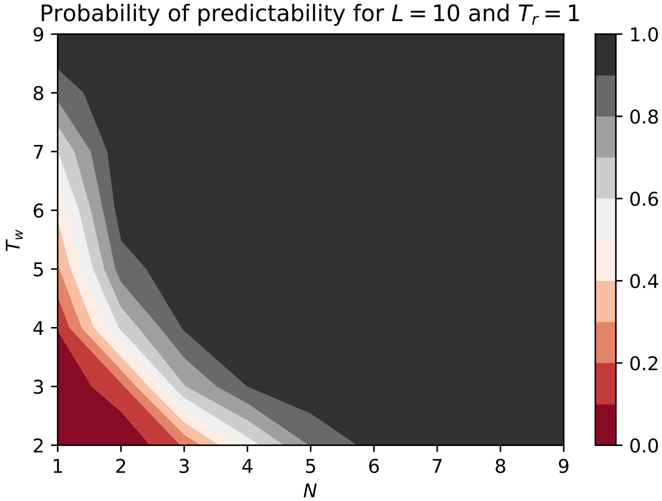
Current state	Next state	Output
$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$
$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Starting from any state, one can perfectly reproduce a periodic extension of the sequence.

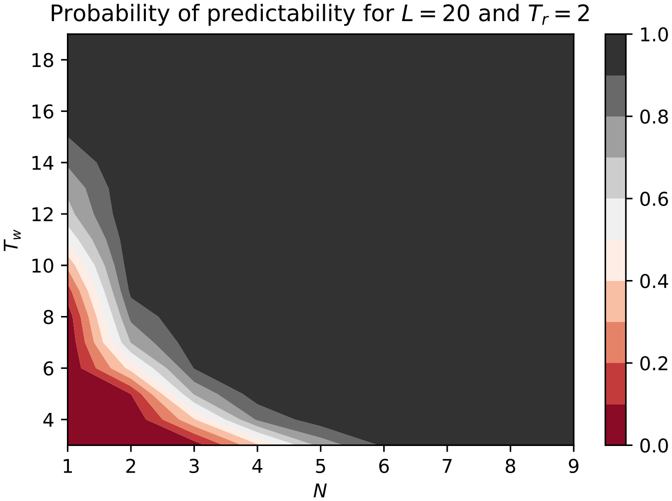
For any $N > 0$, $L > 0$ and $T_r \geq 0$, we would like to determine the lowest $T_w > T_r$ such that, for any arbitrarily small $\varepsilon > 0$, we have

$$p := \mathbb{P}\big(\big\{\boldsymbol{f} \in \mathcal{P}^N_{L,T_r,T_w} | \boldsymbol{f} \in \mathcal{F}^N_{L,T_r}\big\}\big) \geq 1 - \varepsilon$$

Computing this probability is the same as computing the cardinality of $\mathcal{F}^N_{L,T_r} \setminus \mathcal{P}^N_{L,T_r,T_w}$. Finding a closed-form expression is out of the scope of our work. However, one can observe that p increases exponentially with both T_w and N but decreases with L and T_r.



Probability of predictability for $L = 10$ and $T_r = 1$
T_w
N
9
8
7
6
5
4
3
2
1
2
3
4
5
6
7
8
9
1.0
0.8
0.6
0.4
0.2
0.0


Probability of predictability for $L = 20$ and $T_r = 2$
18
16
14
12
10
8
6
4
T_w
1
2
3
4
5
6
7
8
9
N
1.0
0.8
0.6
0.4
0.2
0.0

Let $L = 10$, $T_r = 2$, $N = 2$, and $T_w = T_c = T_d = 5$.

- $\boldsymbol{f_1} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$ is a 2 channels firing sequence, that is 5-redundant and 5-memorizable.
- $\boldsymbol{f_2} = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$ is a 2 channels firing sequence, that is 5-memorizable.
- $\boldsymbol{f_3} = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$ is a 2 channels firing sequence, that is not 5-memorizable.

Energy-efficiency of the human brain

BrainFacts.org

- The brain consumes about 20% of the body's energy for the average adult in a resting state.
- About 25% of the energy is used to maintain the neurons and glial cells and 75% is used for sending and processing electrical signals accros the brain's circuits.
- Different parts of the brain require different quantities of energy.
- The brain requires a relatively steady amount of energy.

Cosmo

The human brain can run on the same amount of power as other mammal brains while performing more complex procedures thanks to a low-density of ion channels.