# On counting firing signals

Hugo Aguettaz

February 9, 2022

## 1 Firing signals

We first consider firing signals of size $n \in \mathbb{Z}_{\geq 0}$ with refractory period $T_r \in \mathbb{Z}_{\geq 0}$. Those signals can be represented by binary sequences of length $n$ such that there are at least $T_r$ zeros between two consecutive ones. Let $\tilde{\mathcal{Y}}^n_{T_r}$ represent the set of such sequences. We formally define this set as

$$\tilde{\mathcal{Y}}^n_{T_r} = \left\{ (y_1, \ldots, y_n) \in \{0,1\}^n : \sum_{m=0}^{\min\{T_r, n-m\}} y_{k+m} \in \{0,1\}, k \in \{1, \ldots, n\} \right\}. \tag{1}$$

Obviously, if $T_r = 0$, $\tilde{\mathcal{Y}}^n_{T_r}$ simplifies to

$$\tilde{\mathcal{Y}}^n_{T_r} = \{0,1\}^n, \tag{2}$$

and this trivial case is irrelevant for our purpose. That is why, from now on, we will assume $T_r > 0$.

**Theorem 1.1.** *Let $\tilde{\ell}^n_{T_r}$ denotes the cardinality of $\tilde{\mathcal{Y}}^n_{T_r}$, with $n \in \mathbb{Z}_{\geq 0}$ and $T_r \in \mathbb{Z}_{>0}$. We have:*

$$\tilde{\ell}^n_{T_r} = \begin{cases} 0 & \text{if } n = 0 \\ 2 & \text{if } n = 1 \\ \tilde{\ell}^{n-1}_{T_r} + 1 & \text{if } 1 < n \leq T_r + 1 \\ \tilde{\ell}^{n-1}_{T_r} + \tilde{\ell}^{n-T_r-1}_{T_r} & \text{otherwise.} \end{cases} \tag{3}$$

*Proof.* If $n = 0$, it is obvious that $\tilde{\ell}^0_{T_r} = 0$. If $n = 1$, both 1 and 0 are sequences in $\tilde{\mathcal{Y}}^1_{T_r}$ and thus $\tilde{\ell}^1_{T_r} = 2$ If $2 \leq n \leq T_r + 1$, there are $\tilde{\ell}^{n-1}_{T_r}$ sequences in $\tilde{\mathcal{Y}}^n_{T_r}$ with the form

$$\underbrace{y_1 \cdots y_{n-1}}_{\in \tilde{\mathcal{Y}}^{n-1}_{T_r}} 0,$$

and 1 sequence with the form

$$\overbrace{0 \cdots 0}^{\leq T_r} 1.$$

If $n \geq T_r + 2$, there are $\ell^{n-1}_{T_r}$ sequences in $\tilde{\mathcal{Y}}^n_{T_r}$ with the form

$$\underbrace{y_1 \cdots y_{n-1}}_{\in \tilde{\mathcal{Y}}^{n-1}_{T_r}} 0,$$

and $\tilde{\ell}^{n-T_r-1}_{T_r}$ sequences with the form

$$\underbrace{y_1 \cdots y_{n-T_r-1}}_{\in \tilde{\mathcal{Y}}^{n-T_r-1}_{T_r}} \overbrace{0 \cdots 0}^{T_r} 1.$$

$\square$

**Theorem 1.2.** *Let $\tilde{\ell}^n_{T_r}$ denotes the cardinality of $\tilde{\mathcal{Y}}^n_{T_r}$, with $n \in \mathbb{Z}_{\geq 0}$ and $T_r \in \mathbb{Z}_{>0}$ and let $\varphi_{T_r+1,i} \in \mathbb{C}$ with $i \in \{1, \ldots, T_r + 1\}$ be as defined in Definition A.1. Then*

$$
\tilde{\ell}^n_{T_r} = \begin{cases} \sum_{i=1}^{T_r+1} \frac{\varphi_{T_r+1,i}^{2T_r+n+1}}{\varphi_{T_r+1,i}^{T_r+1} + T_r} & \text{if } n > 0 \\ 0 & \text{otherwise} \end{cases}, \tag{4}
$$

*with $\mathrm{ROC}(\tilde{\ell}^n_{T_r}) = \{z \in \mathbb{C} : |z| > \max_{i \in \{1,\ldots,T_r+1\}} |\varphi_{T_r+1,i}|\}$.*

*Proof.* Inverting the Z-Transform of Theorem B.1, we get

$$
\tilde{\ell}^n_{T_r} = \begin{cases} \sum_{i=1}^{T_r+1} \frac{\varphi_{T_r+1,i}^{2T_r+n+1}}{\varphi_{T_r+1,i}^{T_r+1} + T_r} & \text{if } n > 0 \\ -1 + \sum_{i=1}^{T_r+1} \frac{\varphi_{T_r+1,i}^{2T_r+n+1}}{\varphi_{T_r+1,i}^{T_r+1} + T_r} & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}, \tag{5}
$$

with $\mathrm{ROC}(\tilde{\ell}^n_{T_r}) = \{z \in \mathbb{C} : |z| > \max_{i \in \{1,\ldots,T_r+1\}} |\varphi_{T_r+1,i}|\}$.

But because of Theorem 1.1, for $n = 0$, we have:

$$
\tilde{\ell}^n_{T_r} = -1 + \sum_{i=1}^{T_r+1} \frac{\varphi_{T_r+1,i}^{2T_r+n+1}}{\varphi_{T_r+1,i}^{T_r+1} + T_r} = 0, \tag{6}
$$

and substituting $p = T_r + 1$, we prove Corollary A.7 and complete the current proof. $\square$

| | | | | | $T_r$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 1 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 2 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| 3 | 5 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |
| 4 | 8 | 6 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| 5 | 13 | 9 | 7 | 6 | 6 | 6 | 6 | 6 | 6 | 6 |
| 6 | 21 | 13 | 10 | 8 | 7 | 7 | 7 | 7 | 7 | 7 |
| 7 | 34 | 19 | 14 | 11 | 9 | 8 | 8 | 8 | 8 | 8 |
| 8 | 55 | 28 | 19 | 15 | 12 | 10 | 9 | 9 | 9 | 9 |
| 9 | 89 | 41 | 26 | 20 | 16 | 13 | 11 | 10 | 10 | 10 |
| 10 | 144 | 60 | 36 | 26 | 21 | 17 | 14 | 12 | 11 | 11 |
| 11 | 233 | 88 | 50 | 34 | 27 | 22 | 18 | 15 | 13 | 12 |
| 12 | 377 | 129 | 69 | 45 | 34 | 28 | 23 | 19 | 16 | 14 |
| 13 | 610 | 189 | 95 | 60 | 43 | 35 | 29 | 24 | 20 | 17 |
| 14 | 987 | 277 | 131 | 80 | 55 | 43 | 36 | 30 | 25 | 21 |
| 15 | 1597 | 406 | 181 | 106 | 71 | 53 | 44 | 37 | 31 | 26 |
| 16 | 2584 | 595 | 250 | 140 | 92 | 66 | 53 | 45 | 38 | 32 |
| 17 | 4181 | 872 | 345 | 185 | 119 | 83 | 64 | 54 | 46 | 39 |
| 18 | 6765 | 1278 | 476 | 245 | 153 | 105 | 78 | 64 | 55 | 47 |
| 19 | 10946 | 1873 | 657 | 325 | 196 | 133 | 96 | 76 | 65 | 56 |
| 20 | 17711 | 2745 | 907 | 431 | 251 | 168 | 119 | 91 | 76 | 66 |

Table 1: $\tilde{\ell}^n_{T_r}$ for $0 < n \leq 20$ and $0 < T_r \leq 10$.

Given a sample of $\tilde{\mathcal{Y}}^n_{T_r}$, it could also be interesting to know the probability that it contains exactly $n_0$ zeros and $n_1$ ones. Let $\tilde{\mathcal{Y}}^{n_0,n_1}_{T_r}$ represent the set of sequences in $\tilde{\mathcal{Y}}^n_{T_r}$ with $n_0$ zeros and $n_1$ ones, $n_0 + n_1 = n$, $n_0, n_1 \geq 0$. It can be formally expressed as

$$
\tilde{\mathcal{Y}}^{n_0,n_1}_{T_r} = \left\{ (y_1, \ldots, y_n) \in \tilde{\mathcal{Y}}^n_{T_r} : \sum_{i=1}^{n} y_i = n_1, n = n_0 + n_1 \right\}. \tag{7}
$$

**Theorem 1.3.** *Let $\tilde{\ell}_{T_r}^{n_0,n_1}$ denotes the cardinality of $\tilde{\mathcal{Y}}_{T_r}^{n_0,n_1}$ with $n_0, n_1 \in \mathbb{Z}_{\geq 0}$ and $T_r \in \mathbb{Z}_{>0}$. We have:*

$$\tilde{\ell}_{T_r}^{n_0,n_1} = \begin{cases} \binom{n-T_r n_1 + T_r}{n_1} & \text{if } n_0 \geq T_r(n_1 - 1) \\ 0 & \text{otherwise} \end{cases}, \tag{8}$$

*with $n = n_0 + n_1$.*

*Proof.* First of all, notice that if $n_0 < T_r(n_1 - 1)$, it is not possible that there are at least $T_r$ zeros between two consecutives ones. Hence $\tilde{\ell}_{T_r}^{n_0,n_1} = 0$. Now assume $n_0 \geq T_r(n_1 - 1)$ and consider the following *bijective mapping*:

$$\underbrace{\overbrace{0\cdots0}^{k_0 \geq 0}1\overbrace{0\cdots0}^{k_1 \geq T_r}1\cdots\ \overbrace{0\cdots0}^{k_{n_1-1} \geq T_r}1\overbrace{0\cdots0}^{k_{n_1} \geq 0}}_{\in \tilde{\mathcal{Y}}_{T_r}^n} \mapsto \overbrace{0\cdots0}^{k_0 \geq 0}1\overbrace{0\cdots0}^{k_1' \geq 0}1\cdots\ \overbrace{0\cdots0}^{k_{n_1-1}' \geq 0}1\overbrace{0\cdots0}^{k_{n_1} \geq 0} \tag{9}$$

with $k_0 + k_1 + \cdots + k_{n_1} = n_0$ and $k_i' = k_i - T_r$ for $1 \leq i < n_1$. In other words, this mapping maps any binary sequence in $\tilde{\mathcal{Y}}_{T_r}^{n_0,n_1}$ to another binary sequence of lenght $n - T_r(n_1 - 1)$ with $n_0 - T_r(n_1 - 1)$ zeros and $n_1$ ones. Now, it suffices to notice that there exists

$$\binom{n - T_r n_1 + T_r}{n_1} \tag{10}$$

such sequences to complete the proof. $\square$

| | | | | | $n_1$ | | | | | | | |
| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | $\ell_{T_r}^{\tilde{n}}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | - | - | - | - | - | - | - | - | - | 2 |
| 2 | 1 | 2 | 0 | - | - | - | - | - | - | - | - | 3 |
| 3 | 1 | 3 | 1 | 0 | - | - | - | - | - | - | - | 5 |
| 4 | 1 | 4 | 3 | 0 | 0 | - | - | - | - | - | - | 8 |
| 5 | 1 | 5 | 6 | 1 | 0 | 0 | - | - | - | - | - | 13 |
| 6 | 1 | 6 | 10 | 4 | 0 | 0 | 0 | - | - | - | - | 21 |
| 7 | 1 | 7 | 15 | 10 | 1 | 0 | 0 | 0 | - | - | - | 34 |
| 8 | 1 | 8 | 21 | 20 | 5 | 0 | 0 | 0 | 0 | - | - | 55 |
| 9 | 1 | 9 | 28 | 35 | 15 | 1 | 0 | 0 | 0 | 0 | - | 89 |
| 10 | 1 | 10 | 36 | 56 | 35 | 6 | 0 | 0 | 0 | 0 | 0 | 144 |

Table 2: $\tilde{\ell}_{T_r}^{n_0,n_1}$ for $0 < n \leq 10$, $0 \leq n_1 \leq n$ and $T_r = 1$.

| | | | | | $n_1$ | | | | | | | |
| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | $\ell_{T_r}^{\tilde{n}}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | - | - | - | - | - | - | - | - | - | 2 |
| 2 | 1 | 2 | 0 | - | - | - | - | - | - | - | - | 3 |
| 3 | 1 | 3 | 0 | 0 | - | - | - | - | - | - | - | 4 |
| 4 | 1 | 4 | 1 | 0 | 0 | - | - | - | - | - | - | 6 |
| 5 | 1 | 5 | 3 | 0 | 0 | 0 | - | - | - | - | - | 9 |
| 6 | 1 | 6 | 6 | 0 | 0 | 0 | 0 | - | - | - | - | 13 |
| 7 | 1 | 7 | 10 | 1 | 0 | 0 | 0 | 0 | - | - | - | 19 |
| 8 | 1 | 8 | 15 | 4 | 0 | 0 | 0 | 0 | 0 | - | - | 28 |
| 9 | 1 | 9 | 21 | 10 | 0 | 0 | 0 | 0 | 0 | 0 | - | 41 |
| 10 | 1 | 10 | 28 | 20 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 60 |

Table 3: $\tilde{\ell}_{T_r}^{n_0,n_1}$ for $0 < n \leq 10$, $0 \leq n_1 \leq n$ and $T_r = 2$.

| | | | | | $n_1$ | | | | | | | |
| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | $\ell_{T_r}^{\tilde{n}}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | - | - | - | - | - | - | - | - | - | 2 |
| 2 | 1 | 2 | 0 | - | - | - | - | - | - | - | - | 3 |
| 3 | 1 | 3 | 0 | 0 | - | - | - | - | - | - | - | 4 |
| 4 | 1 | 4 | 0 | 0 | 0 | - | - | - | - | - | - | 5 |
| 5 | 1 | 5 | 1 | 0 | 0 | 0 | - | - | - | - | - | 7 |
| 6 | 1 | 6 | 3 | 0 | 0 | 0 | 0 | - | - | - | - | 10 |
| 7 | 1 | 7 | 6 | 0 | 0 | 0 | 0 | 0 | - | - | - | 14 |
| 8 | 1 | 8 | 10 | 0 | 0 | 0 | 0 | 0 | 0 | - | - | 19 |
| 9 | 1 | 9 | 15 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | - | 26 |
| 10 | 1 | 10 | 21 | 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 36 |

Table 4: $\tilde{\ell}_{T_r}^{n_0,n_1}$ for $0 < n \leq 10$, $0 \leq n_1 \leq n$ and $T_r = 3$.

## 2 Firing signals with cycles

We now consider firing signals of size $n \in \mathbb{Z}_{\geq 0}$ with refractory period $T_r \in \mathbb{Z}_{\geq 0}$ and cycled constrained. Those signals can be represented by binary sequences of length $n$ such that there are at least $T_r$ zeros between two consecutive ones even between the last one and the first one of the sequence. Let $\mathcal{Y}_{T_r}^n$ represent the set of such sequences. We formally define this set as

$$\mathcal{Y}_{T_r}^n = \left\{ (y_1, \ldots, y_n) \in \{0,1\}^n : \sum_{m=0}^{T_r} y_{(k+m \bmod n)+1} \in \{0,1\}, k \in \{1, \ldots, n\} \right\}. \tag{11}$$

Obviously, if $T_r = 0$, $\mathcal{Y}_{T_r}^n$ simplifies to

$$\mathcal{Y}_{T_r}^n = \{0,1\}^n, \tag{12}$$

and this trivial case is irrelevant for our purpose. That is why, from now on, we further assume $T_r > 0$.

**Theorem 2.1.** *Let $\ell_{T_r}^n$ denotes the cardinality of $\mathcal{Y}_{T_r}^n$, with $n \in \mathbb{Z}_{\geq 0}$ and $T_r \in \mathbb{Z}_{>0}$. We have:*

$$\ell_{T_r}^n = \begin{cases} 0 & \text{if } n = 0 \\ \ell_{T_r}^{n-1} + 1 & \text{if } n = 1 \\ \ell_{T_r}^{n-1} & \text{if } 2 \leq n \leq T_r \\ \ell_{T_r}^{n-1} + 1 + T_r & \text{if } n = T_r + 1 \\ \ell_{T_r}^{n-1} + 1 & \text{if } T_r + 2 \leq n \leq 2T_r + 1 \\ \ell_{T_r}^{n-1} + \tilde{\ell}_{T_r}^{n-2T_r-1} + T_r & \text{if } 2T_r + 2 \leq n \leq 3T_r + 2 \\ \ell_{T_r}^{n-1} + \tilde{\ell}_{T_r}^{n-2T_r-1} + T_r \tilde{\ell}_{T_r}^{n-3T_r-2} & \text{otherwise} \end{cases} \tag{13}$$

*Proof.* If $n = 0$, it is obvious that $\ell_{T_r}^n = 0$. If $n = 1$, the only periodic-valid sequence is the one containing one zero and zero one and $\ell_{T_r}^n = \ell_{T_r}^{n-1} + 1 = 0 + 1$. If $2 \leq n \leq T_r$, there are $\ell_{T_r}^{n-1}$ sequences in $\mathcal{Y}_{T_r}^n$ with the form

$$\underbrace{y_1 \cdots y_{n-1}}_{\in \mathcal{Y}_{T_r}^{n-1}} 0.$$

If $n = T_r + 1$, there are $\ell_{T_r}^{n-1}$ sequences in $\mathcal{Y}_{T_r}^n$ with the form

$$\underbrace{y_1 \cdots y_{n-1}}_{\in \mathcal{Y}_{T_r}^{n-1}} 0,$$

$T_r$ sequences in $\mathcal{Y}_{T_r}^n$ with the form

$$\underbrace{\overbrace{0 \cdots 0}^{k_1 \geq 0} 1 \overbrace{0 \cdots 0}^{k_2 \geq 0}}_{\notin \mathcal{Y}_{T_r}^{n-1}} 0,$$

with $k_1 + k_2 = k = T_r - 1$, and 1 sequence with the form

$$\overbrace{0 \cdots 0}^{T_r} 1.$$

If $T_r + 2 \leq n \leq 2T_r + 1$, there are $\ell_{T_r}^{n-1}$ sequences in $\mathcal{Y}_{T_r}^n$ with the form

$$\underbrace{y_1 \cdots y_{n-1}}_{\in \mathcal{Y}_{T_r}^{n-1}} 0,$$

and 1 sequence with the form

$$\overbrace{0 \cdots 0}^{n-1 > T_r} 1.$$

5

If $2T_r + 2 \leq n \leq 3T_r + 2$, there are $\ell_{T_r}^{n-1}$ sequences in $\mathcal{Y}_{T_r}^n$ with the form

$$\underbrace{y_1 \cdots y_{n-1}}_{\in \mathcal{Y}_{T_r}^{n-1}} 0,$$

$T_r$ sequences with the form

$$\underbrace{0 \cdots 0\, 1}^{k_1 \geq 0}\ \underbrace{0 \cdots 0}^{n - T_r - 2 \geq T_r}\ \underbrace{1\, 0 \cdots 0}^{k_2 \geq 0} 0,$$
$$\not\in \mathcal{Y}_{T_r}^{n-1}$$

with $k_1 + k_2 = k = T_r - 1$ and $\tilde{\ell}_{T_r}^{n-2T_r-1}$ sequences with the form

$$\underbrace{0 \cdots 0}^{T_r} \underbrace{y_{T_r+1} \cdots y_{n-T_r-1}}_{\in \tilde{\mathcal{Y}}_{T_r}^{n-2T_r-1}} \underbrace{0 \cdots 0}^{T_r} 1.$$

If $n \geq 3T_r + 3$, the are $\ell_{T_r}^{n-1}$ sequences in $\mathcal{Y}_{T_r}^n$ with the form

$$\underbrace{y_1 \cdots y_{n-1}}_{\in \mathcal{Y}_{T_r}^{n-1}} 0, \tag{14}$$

$T_r \tilde{\ell}_{T_r}^{n-3T_r-2}$ sequences with the form

$$\underbrace{0 \cdots 0\, 1}^{k_1 \geq 0} \underbrace{0 \cdots 0}^{T_r} \underbrace{\underbrace{y_{k_1+T_r+T_r} \cdots y_{n-k_2-T_r-1}}_{\in \tilde{\mathcal{Y}}_{T_r}^{n-3T_r-2}} \underbrace{0 \cdots 0\, 1}^{T_r} \underbrace{0 \cdots 0}^{k_2 \geq 0} 0,}_{\not\in \mathcal{Y}_{T_r}^{n-1}} \tag{15}$$

with $k_1 + k_2 = k = T_r - 1$, and $\tilde{\ell}_{T_r}^{n-2T_r-1}$ sequences with the form

$$\underbrace{0 \cdots 0}^{T_r} \underbrace{y_{T_r+1} \cdots y_{n-T_r-1}}_{\in \tilde{\mathcal{Y}}_{T_r}^{n-2T_r-1}} \underbrace{0 \cdots 0}^{T_r} 1. \tag{16}$$

$\square$

**Theorem 2.2.** *Let $\ell_{T_r}^n$ be as introduced before, with $n \in \mathbb{Z}_{\geq 0}$ and $T_r \in \mathbb{Z}_{>0}$. Then:*

$$\ell_{T_r}^n = \begin{cases} 0 & \text{if } n \leq 0, \\ 1 & \text{if } 1 \leq n \leq T_r, \\ n+1 & \text{if } T_r + 1 \leq n \leq 2T_r, \\ \sum_{i=1}^{T_r+1} \frac{\varphi_{T_r+1,i}^{T_r+1+n}}{\varphi_{T_r+1,i}^{T_r+1} + T_r} - 2T_r^2 + nT_r & \text{if } 2T_r + 1 \leq n \leq 3T_r + 1, \\ \sum_{i=1}^{T_r+1} \varphi_{T_r+1,i}^n & \text{if } n > 3T_r + 1. \end{cases} \tag{17}$$

*with* $\mathrm{ROC}(\ell_{T_r}^n) = \{z \in \mathbb{C} : |z| > \max_{i \in \{1,\dots,T_r+1\}} |\varphi_{T_r+1,i}|\}$.

*Proof.* Inverting the Z-Transform of Theorem B.2, we get

$$\ell_{T_r}^n = \begin{cases} 0 & \text{if } n \leq 0, \\ 1 & \text{if } 1 \leq n \leq T_r, \\ n+1 & \text{if } T_r + 1 \leq n \leq 2T_r - 1, \\ T_r + \sum_{i=1}^{T_r+1} \frac{\varphi_{T_r+1,i}^{3T_r+1}}{\varphi_{T_r+1,i}^{T_r+1} + T_r} & \text{if } n = 2T_r, \\ T_r n - 2T_r^2 + \sum_{i=1}^{T_r+1} \frac{\varphi_{T_r+1,i}^{T_r+1+n}}{\varphi_{T_r+1,i}^{T_r+1} + T_r} & \text{if } 2T_r + 1 \leq n \leq 3T_r, \\ \sum_{i=1}^{T_r+1} \frac{\varphi_{T_r+1,i}^{4T_r+2}}{\varphi_{T_r+1,i}^{T_r+1} + T_r} + T_r \sum_{i=1}^{T_r+1} \frac{\varphi_{T_r+1,i}^{3T_r+1}}{\varphi_{T_r+1,i}^{T_r+1} + T_r} & \text{if } n = 3T_r + 1, \\ \sum_{i=1}^{T_r+1} \varphi_{T_r+1,i}^n & \text{if } n > 3T_r + 1. \end{cases} , \tag{18}$$

with $\text{ROC}(\tilde{\ell}_{T_r}^n) = \{z \in \mathbb{C} : |z| > \max_{i \in \{1,\ldots,T_r+1\}} |\varphi_{T_r+1,i}|\}$.

Let $n = 2T_r$. There is 1 sequence in $\mathcal{Y}_{T_r}^{2T_r}$ with the form

$$\overbrace{0 \cdots 0}$$

and $2T_r$ sequences with the form

$$\overbrace{0 \cdots 0}^{k_1 \geq 0} 1 \overbrace{0 \cdots 0}^{k_2 \geq 0},$$

with $k_1 + k_2 = 2T_r - 1 \geq T_r$, $k_1, k_2 \geq 0$. Thus, we necessarily have

$$\sum_{i=1}^{T_r+1} \frac{\varphi_{T_r+1,i}^{3T_r+1}}{\varphi_{T_r+1,i}^{T_r+1} + T_r} = T_r + 1, \tag{19}$$

and substituting $p = T_r + 1$, we prove Corollary A.8.

Now, we can use this result in the case $n = 3T_r + 1$ to get

$$\sum_{i=1}^{T_r+1} \frac{\varphi_{T_r+1,i}^{4T_r+2}}{\varphi_{T_r+1,i}^{T_r+1} + T_r} + T_r \sum_{i=1}^{T_r+1} \frac{\varphi_{T_r+1,i}^{3T_r+1}}{\varphi_{T_r+1,i}^{T_r+1} + T_r} = \sum_{i=1}^{T_r+1} \frac{\varphi_{T_r+1,i}^{T_r+1+3T_r+1}}{\varphi_{T_r+1,i}^{T_r+1} + T_r} + T_r(3T_r + 1) - 2T_r^2, \tag{20}$$

and merge it with $2T_r + 1 \leq n \leq 3T_r$. $\qquad\square$

|  |  |  |  |  | $T_r$ |  |  |  |  |  |
|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 3 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 3 | 4 | 4 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 4 | 7 | 5 | 5 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 5 | 11 | 6 | 6 | 6 | 1 | 1 | 1 | 1 | 1 | 1 |
| 6 | 18 | 10 | 7 | 7 | 7 | 1 | 1 | 1 | 1 | 1 |
| 7 | 29 | 15 | 8 | 8 | 8 | 8 | 1 | 1 | 1 | 1 |
| 8 | 47 | 21 | 13 | 9 | 9 | 9 | 9 | 1 | 1 | 1 |
| 9 | 76 | 31 | 19 | 10 | 10 | 10 | 10 | 10 | 1 | 1 |
| 10 | 123 | 46 | 26 | 16 | 11 | 11 | 11 | 11 | 11 | 1 |
| 11 | 199 | 67 | 34 | 23 | 12 | 12 | 12 | 12 | 12 | 12 |
| 12 | 322 | 98 | 47 | 31 | 19 | 13 | 13 | 13 | 13 | 13 |
| 13 | 521 | 144 | 66 | 40 | 27 | 14 | 14 | 14 | 14 | 14 |
| 14 | 843 | 211 | 92 | 50 | 36 | 22 | 15 | 15 | 15 | 15 |
| 15 | 1364 | 309 | 126 | 66 | 46 | 31 | 16 | 16 | 16 | 16 |
| 16 | 2207 | 453 | 173 | 89 | 57 | 41 | 25 | 17 | 17 | 17 |
| 17 | 3571 | 664 | 239 | 120 | 69 | 52 | 35 | 18 | 18 | 18 |
| 18 | 5778 | 973 | 331 | 160 | 88 | 64 | 46 | 28 | 19 | 19 |
| 19 | 9349 | 1426 | 457 | 210 | 115 | 77 | 58 | 39 | 20 | 20 |
| 20 | 15127 | 2090 | 630 | 276 | 151 | 91 | 71 | 51 | 31 | 21 |

Table 5: $\ell_{T_r}^n$ with $1 \leq n \leq 20$ and $0 < T_r \leq 10$.

**Theorem 2.3.** *Let $\ell_{T_r}^n$ be as introduced before, with $n \in \mathbb{Z}_{\geq 0}$ and fixed $T_r \in \mathbb{Z}_{>0}$. Then:*

$$\lim_{n \to \infty} \ell_{T_r}^n = \hat{\varphi}_{T_r+1}^n \tag{21}$$

*with $\hat{\varphi}_{T_r+1}$ as defined in Theorem A.10.*

7

*Proof.* The proof follows directly from Theorem 2.2 and Theorem A.10. □

It is interesting to notice that as $n$ goes large, the logarithmic of the number of valid sequences increases linearly in the length of the sequence as $\log \ell^n_{T_r} \simeq n \log \hat\varphi_{T_r+1}$. The rate of increase corresponds to the number of bits per seconde required to specify the particular signal used.

**Remark.** *As $T_r$ becomes large (i.e. $T_r \gg n$), we have:*

$$\ell^n_{T_r} = 1, \tag{22}$$

*as the only valid sequence is the one without any spike.*

**Remark.** *For fixed $T_r > 0$ and for any $n \geq 0$, we have:*

$$\ell^{n+1}_{T_r} > \ell^n_{T_r}. \tag{23}$$

**Remark.** *For fixed $n \geq 0$ and for any $T_r > 0$, we have:*

$$\ell^n_{T_r+1} < \ell^n_{T_r}. \tag{24}$$

**Remark** (Capacity)**.** *The capacity $C$ of a channel transmitting signals in $\mathcal{Y}^n_{T_r}$ is:*

$$C \triangleq \lim_{n\to\infty} \frac{\log \ell^n_{T_r+1}}{n} = \log \hat\varphi_{T_r+1}. \tag{25}$$

Given a sample of $\mathcal{Y}^n_{T_r}$, it could also be interesting to know the probability that it contains exactly $n_0$ zeros and $n_1$ ones. Let $\mathcal{Y}^{n_0,n_1}_{T_r}$ represent the set of sequences in $\mathcal{Y}^n_{T_r}$ with $n_0$ zeros and $n_1$ ones, $n_0 + n_1 = n$, $n_0, n_1 \geq 0$. It can be formally expressed as

$$\mathcal{Y}^{n_0,n_1}_{T_r} = \left\{ (y_1, \ldots, y_n) \in \mathcal{Y}^n_{T_r} : \sum_{i=1}^{n} y_i = n_1 \right\}, \tag{26}$$

with $n = n_0 + n_1 > 0$.

**Theorem 2.4.** *Let $\ell^{n_0,n_1}_{T_r}$ denotes the cardinality of $\mathcal{Y}^{n_0,n_1}_{T_r}$ with $n_0, n_1 \in \mathbb{Z}_{\geq 0}$ and $T_r \in \mathbb{Z}_{>0}$. We have:*

$$\ell^{n_0,n_1}_{T_r} = \begin{cases} 0 & \text{if } n_0 < T_r n_1 \\ 1 & \text{if } n_1 = 0 \\ n & \text{if } n_1 = 1 \\ \binom{n-T_r n_1 + T_r}{n_1} - \sum_{k=2}^{T_r+1} \binom{n-T_r(n_1-1)-k}{n_1-2}(k-1) & \text{otherwise} \end{cases}, \tag{27}$$

*Proof.* First of all, it is obvious that if $n_0 < T_r n_1$ we have $\ell^{n_0,n_1}_{T_r} = 0$. Let's now consider the non trivial cases with $n_0, n_1$ such that $n_0 \geq T_r n_1$ If $n_1 = 0$, there is 1 sequence in $\mathcal{Y}^{n_0,n_1}_{T_r}$ with the form

$$\overbrace{0 \cdots 0}^{n}$$

If $n_1 = 1$, there are $n$ sequence in $\mathcal{Y}^{n_0,n_1}_{T_r}$ with the form

$$\overbrace{0 \cdots 0}^{k_1} 1 \overbrace{0 \cdots 0}^{k_2},$$

with $k_1 + k_2 = n - 1 = n_0 \geq T_r$.

If $n_1 \geq 2$, we have to consider sequences that are in $\tilde{\mathcal{Y}}^{n_0,n_1}_{T_r} \setminus \mathcal{Y}^{n_0,n_1}_{T_r}$ and then compute $\ell^{n_0,n_1}_{T_r}$ using $\left| \mathcal{Y}^{n_0,n_1}_{T_r} \right| = \left| \tilde{\mathcal{Y}}^{n_0,n_1}_{T_r} \right| - \left| \tilde{\mathcal{Y}}^{n_0,n_1}_{T_r} \setminus \mathcal{Y}^{n_0,n_1}_{T_r} \right|$. If $n_1 = 2$, there are $\sum_{k=0}^{T_r-1} k+1$ sequences in $\tilde{\mathcal{Y}}^{n_0,n_1}_{T_r} \setminus \mathcal{Y}^{n_0,n_1}_{T_r}$ with the form

$$\overbrace{0 \cdots 0}^{k_1} 1 \overbrace{0 \cdots 0}^{\geq T_r} 1 \overbrace{0 \cdots 0}^{k_2},$$

8

with $k_1, k_2 \geq 0$ and $0 \leq k_1 + k_2 \leq T_r - 1$.

If $n_1 > 2$, any sequence in $\tilde{\mathcal{Y}}_{T_r}^{n_0,n_1} \setminus \mathcal{Y}_{T_r}^{n_0,n_1}$ has the form

$$\overbrace{0\cdots0}^{k_1}1\overbrace{0\cdots0}^{T_r}\underbrace{y_{k_1+T_r+1}\cdots y_{n-k_2-T_r}}_{\in \tilde{\mathcal{Y}}_{T_r}^{n_0-2T_r-k+2,n_1-2}}\overbrace{0\cdots0}^{T_r}1\overbrace{0\cdots0}^{k_2}, \qquad (28)$$

with $k_1, k_2 \geq 0$ and $0 \leq k_1 + k_2 \leq T_r - 1$. Hence, there are

$$\sum_{k=0}^{T_r-1} \tilde{\ell}_{T_r}^{n_0-2T_r-k+2,n_1-2}(k+1) = \sum_{k=2}^{T_r+1} \binom{n - T_r(n_1-1) - k}{n_1 - 2}(k+1). \qquad (29)$$

such sequences.

Now it suffices to notice that $\tilde{\ell}_{T_r}^{n_0-2T_r-k+2,0} = 1$ to merge case $n_1 = 2$ with case $n_1 > 2$ and complete the proof. $\qquad \square$

| | | | | | | $n_1$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | $\ell^n_{T_r}$ |
| 1 | 1 | 0 | - | - | - | - | - | - | - | - | - | 1 |
| 2 | 1 | 2 | 0 | - | - | - | - | - | - | - | - | 3 |
| 3 | 1 | 3 | 0 | 0 | - | - | - | - | - | - | - | 4 |
| 4 | 1 | 4 | 2 | 0 | 0 | - | - | - | - | - | - | 7 |
| 5 | 1 | 5 | 5 | 0 | 0 | 0 | - | - | - | - | - | 11 |
| 6 | 1 | 6 | 9 | 2 | 0 | 0 | 0 | - | - | - | - | 18 |
| 7 | 1 | 7 | 14 | 7 | 0 | 0 | 0 | 0 | - | - | - | 29 |
| 8 | 1 | 8 | 20 | 16 | 2 | 0 | 0 | 0 | 0 | - | - | 47 |
| 9 | 1 | 9 | 27 | 30 | 9 | 0 | 0 | 0 | 0 | 0 | - | 76 |
| 10 | 1 | 10 | 35 | 50 | 25 | 2 | 0 | 0 | 0 | 0 | 0 | 123 |

Table 6: $\ell^{n_0,n_1}_{T_r}$ for $0 < n \leq 10$, $0 \leq n_1 \leq n$ and $T_r = 1$.

| | | | | | | $n_1$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | $\ell^n_{T_r}$ |
| 1 | 1 | 0 | - | - | - | - | - | - | - | - | - | 1 |
| 2 | 1 | 0 | 0 | - | - | - | - | - | - | - | - | 1 |
| 3 | 1 | 3 | 0 | 0 | - | - | - | - | - | - | - | 4 |
| 4 | 1 | 4 | 0 | 0 | 0 | - | - | - | - | - | - | 5 |
| 5 | 1 | 5 | 0 | 0 | 0 | 0 | - | - | - | - | - | 6 |
| 6 | 1 | 6 | 3 | 0 | 0 | 0 | 0 | - | - | - | - | 10 |
| 7 | 1 | 7 | 7 | 0 | 0 | 0 | 0 | 0 | - | - | - | 15 |
| 8 | 1 | 8 | 12 | 0 | 0 | 0 | 0 | 0 | 0 | - | - | 21 |
| 9 | 1 | 9 | 18 | 3 | 0 | 0 | 0 | 0 | 0 | 0 | - | 31 |
| 10 | 1 | 10 | 25 | 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 46 |

Table 7: $\ell^{n_0,n_1}_{T_r}$ for $0 < n \leq 10$, $0 \leq n_1 \leq n$ and $T_r = 2$.

| | | | | | | $n_1$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | $\ell^n_{T_r}$ |
| 1 | 1 | 0 | - | - | - | - | - | - | - | - | - | 1 |
| 2 | 1 | 0 | 0 | - | - | - | - | - | - | - | - | 1 |
| 3 | 1 | 0 | 0 | 0 | - | - | - | - | - | - | - | 1 |
| 4 | 1 | 4 | 0 | 0 | 0 | - | - | - | - | - | - | 5 |
| 5 | 1 | 5 | 0 | 0 | 0 | 0 | - | - | - | - | - | 6 |
| 6 | 1 | 6 | 0 | 0 | 0 | 0 | 0 | - | - | - | - | 7 |
| 7 | 1 | 7 | 0 | 0 | 0 | 0 | 0 | 0 | - | - | - | 8 |
| 8 | 1 | 8 | 4 | 0 | 0 | 0 | 0 | 0 | 0 | - | - | 13 |
| 9 | 1 | 9 | 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | - | 19 |
| 10 | 1 | 10 | 15 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 26 |

Table 8: $\ell^{n_0,n_1}_{T_r}$ for $0 < n \leq 10$, $0 \leq n_1 \leq n$ and $T_r = 3$.

# A   Generalized golden number

**Definition A.1** (Generalized Golden Number of order $p$). *Let $\varphi_{p,i} \in \mathbb{C}$ be the $p$ complex roots of the polynomials $P(x) = x^p - x^{p-1} - 1$ with $p \in \mathbb{Z}_{\geq 2}$ and $i \in \{1, \ldots, p\}$. The largest real root of $P(x)$ is called the* Generalized Golden Number of order $p$.

**Remark.** *Moreover, from the complex conjugate root Theorem, since $P$ is a polynomial in one variable with real coefficients, the roots are real or come in complex conjugate pairs.*

**Remark.** *From Definition A.1, it follows that for any $p_1, p_2 > 2$, the following holds:*

$$\varphi_{p_1,i}^{p_1} - \varphi_{p_1,i}^{p_1-1} = \varphi_{p_1,i}^{p_1-1}(\varphi_{p_1,i} - 1) = \varphi_{p_2,i}^{p_2-1}(\varphi_{p_2,i} - 1) = \varphi_{p_2,i}^{p_2} - \varphi_{p_2,i}^{p_2-1}, \tag{30}$$

**Theorem A.1.** *Any root of $P(x)$ has multiplicity one.*

*Proof.* Let $\varphi_{p,i}$ be a root of $P(x)$ and assume it has multiplicity $l > 1$. Then, $\varphi_{p,i}$ should also be a root of $P'(x)$, that is:

$$P'(\varphi_{p,i}) = p\varphi_{p,i}^{p-1} - (p-1)\varphi_{p,i}^{p-2} = 0. \tag{31}$$

Using the fact that $\varphi_{p,i} \neq 0$, for any $p > 1$ and $i \in \{1, \ldots, p\}$, we get:

$$\varphi_{p,i} = \frac{p-1}{p}, \tag{32}$$

It yields:

$$P(\varphi_{p,i}) = \left(\frac{p-1}{p}\right)^p - \left(\frac{p-1}{p}\right)^{p-1} - 1 < 0, \tag{33}$$

which is in contradiction with $P(\varphi_{p,i}) = 0$ and the proof is complete. $\qquad\square$

**Theorem A.2.** *Let $\varphi_{p,i} \in \mathbb{C}$ with $p \in \mathbb{Z}_{\geq 2}$ and $i \in \{1, \ldots, p\}$ be as in Definition A.1. Then, the following hold:*

$$\sum_{i=1}^{p} \varphi_{p,i} = 1, \tag{34}$$

$$\prod_{i=1}^{p} \varphi_{p,i} = (-1)^{p-1}, \tag{35}$$

$$\sum_{1 \leq i_1 < i_2 < \cdots < i_{p-m} \leq p} \prod_{j=1}^{p-m} \varphi_{p,i_j} = 0, \tag{36}$$

*with $0 < m < p - 1$.*

*Proof.* Using Vieta's formulas with $P(x) = x^n - x^{n-1} - 1$, we directly get the result. $\qquad\square$

**Corollary A.3.** *Let $\varphi_{p,i} \in \mathbb{C}$ with $p \in \mathbb{Z}_{\geq 2}$ and $i \in \{1, \ldots, p\}$ be as in Definition A.1 and $0 < m < p$. Then, the following hold:*

$$(-1)^{p-m}\varphi_{p,i}^m \sum_{\substack{1 \leq i_1 < i_2 < \cdots < i_{p-m} \leq p \\ i_1, i_2, \ldots, i_{p-1-m} \neq i}} \prod_{j=1}^{p-m} \varphi_{p,i_j} = 1 \tag{37}$$

*Proof.* We prove it by induction. First of all, for $m = p - 1$, we have:

$$\varphi_{p,i}^{p-1} \sum_{\substack{j=1 \\ j \neq i}}^{p} \varphi_{p,i_j} = \varphi_{p,i}^{p-1}(1 - \varphi_{p,i}) \tag{38}$$

$$= \varphi_{p,i}^{p-1} - \varphi_{p,i}^{p} \tag{39}$$

$$= -1 \tag{40}$$

$$\tag{41}$$

Assume, the property is true for $m + 1 \le p - 1$ and prove it is also true for $m > 0$. We have:

$$\varphi_{p,i}^m \sum_{\substack{1 \le i_1 < i_2 < \cdots < i_{p-m} \le p \\ i_1, i_2, \ldots, i_{p-1-m} \ne i}} \prod_{j=1}^{p-m} \varphi_{p,i_j} = -\varphi_{p,i}^m \sum_{\substack{1 \le i_1 < i_2 < \cdots < i_{p-m} \le p \\ i \in \{i_1, i_2, \ldots, i_{p-m}\}}} \prod_{j=1}^{p-m} \varphi_{p,i_j} \tag{42}$$

$$= -\varphi_{p,i}^{m+1} \sum_{\substack{1 \le i_1 < i_2 < \cdots < i_{p-m-1} \le p \\ i_1, i_2, \ldots, i_{p-m-1} \ne i}} \prod_{j=1}^{p-m-1} \varphi_{p,i_j} \tag{43}$$

$$= -(-1)^{p-m-1} \tag{44}$$

$$= (-1)^{p-m}, \tag{45}$$

and the proof is complete. $\qquad\square$

**Corollary A.4.** *Let $\varphi_{p,i} \in \mathbb{C}$ with $p \in \mathbb{Z}_{\ge 2}$ and $i \in \{1, \ldots, p\}$ be as in Definition A.1. Then, the following hold:*

$$\sum_{i=1}^p \varphi_{p,i}^{p-1} = 1 \tag{46}$$

*Proof.* If $p = 2$, we have $\sum_{i=1}^p \varphi_{p,i}^{p-1} = \sum_{i=1}^p \varphi_{p,i} = 1$. Now, assuming $p > 2$, we have:

$$\left( \sum_{i=1}^p \varphi_{p,i} \right)^{p-1} = (p-2)! \sum_{i=1}^p \varphi_{p,i} \sum_{\substack{1 \le i_1 < \cdots < i_{p-2} \le p \\ i_1, \ldots, i_{p-2} \ne i}} \prod_{j=1}^{p-2} \varphi_{p,i_j} \tag{47}$$

$$+ \sum_{m=2}^{p-2} \sum_{i=1}^p \frac{(p-1)!}{m!} \varphi_{p,i}^m \sum_{\substack{1 \le i_1 < \cdots < i_{p-1-m} \le p \\ i_1, \ldots, i_{p-1-m} \ne i}} \prod_{j=1}^{p-1-m} \varphi_{p,i_j}$$

$$+ \sum_{i=1}^p \varphi_{p,i}^{p-1}$$

$$= (p-2)! \sum_{i=1}^p \sum_{\substack{1 \le i_1 < \cdots < i_{p-1} \le p \\ i \in \{i_1, \ldots, i_{p-1}\}}} \prod_{j=1}^{p-1} \varphi_{p,i_j} \tag{48}$$

$$+ \sum_{m=2}^{p-2} \frac{(p-1)!}{m!} \sum_{i=1}^p \varphi_{p,i}^{m-1} \sum_{\substack{1 \le i_1 < \cdots < i_{p-m} \le p \\ i \in \{i_1, \ldots, i_{p-m}\}}} \prod_{j=1}^{p-m} \varphi_{p,i_j}$$

$$+ \sum_{i=1}^p \varphi_{p,i}^{p-1}$$

$$= (p-2)! \sum_{1 \le i_1 < \cdots < i_{p-1}} \prod_{j=1}^{p-1} \varphi_{p,i_j} \tag{49}$$

$$+ \sum_{m=2}^{p-2} \frac{(p-1)!}{m!} \varphi_{p,i}^{m-1} \sum_{1 \le i_1 < \cdots < i_{p-m} \le p} \prod_{j=1}^{p-m} \varphi_{p,i_j}$$

$$+ \sum_{i=1}^p \varphi_{p,i}^{p-1}$$

$$= \sum_{i=1}^p \varphi_{p,i}^{p-1}, \tag{50}$$

It suffices now to notice that $\sum_{i=1}^p \varphi_{p,i}^{p-1} = \left( \sum_{i=1}^p \varphi_{p,i} \right)^{p-1}$ to complete the proof. $\qquad\square$

**Corollary A.5.** *Let $\varphi_{p,i} \in \mathbb{C}$ with $p \in \mathbb{Z}_{\geq 2}$ and $i \in \{1, \ldots, p\}$ be as in Definition A.1. Then, the following hold:*

$$\varphi_{p,i} \prod_{\substack{j=1 \\ j \neq i}}^{p} (\varphi_{p,i} - \varphi_{p,j}) = \varphi_{p,i}^p + p - 1 = p + \varphi_{p,i}^{p-1}. \tag{51}$$

*Proof.* We can directly develop the product to get the desire result:

$$\varphi_{p,i} \prod_{\substack{j=1 \\ j \neq i}}^{p} (\varphi_{p,i} - \varphi_{p,j}) = \varphi_{p,i}^p + \sum_{m=1}^{p-1} (-1)^{p-m} \varphi_{p,i}^m \sum_{\substack{1 \leq i_1 < \cdots < i_{p-m} \leq p \\ i_1, \ldots, i_{p-m} \neq i}} \prod_{j=1}^{p-m} \varphi_{p,i_j} \tag{52}$$

$$= \varphi_{p,i}^p + \sum_{m=1}^{p-1} 1 \tag{53}$$

$$= \varphi_{p,i}^p + p - 1 \tag{54}$$

$$= \varphi_{p,i}^{p-1} + p. \tag{55}$$

$\square$

**Corollary A.6.** *Let $\varphi_{p,i} \in \mathbb{C}$ with $p \in \mathbb{Z}_{\geq 2}$ and $i \in \{1, \ldots, p\}$ be as in Definition A.1. Then, the following hold:*

$$\prod_{i=1}^{p} (1 - \varphi_{p,i}) = -1. \tag{56}$$

*Proof.* We can directly develop the product to get the desire result:

$$\prod_{i=1}^{p} (1 - \varphi_{p,i}) = \sum_{m=0}^{p} (-1)^{p-m} \sum_{1 \leq i_1 < \cdots < i_{p-m} \leq p} \prod_{j=1}^{p-m} \varphi_{p,i_j} \tag{57}$$

$$= (-1)^p \prod_{i}^{p} \varphi_{p,i} \tag{58}$$

$$+ \sum_{m=1}^{p-2} (-1)^m \sum_{1 \leq i_1 < \cdots < i_{p-m} \leq p} \prod_{j=1}^{p-m} \varphi_{p,i_j}$$

$$- \sum_{i}^{p} \varphi_{p,i}$$

$$+ 1$$

$$= (-1)^p \prod_{i}^{p} \varphi_{p,i} \tag{59}$$

$$= -1. \tag{60}$$

$\square$

**Corollary A.7.** *Let $\varphi_{p,i} \in \mathbb{C}$ with $p \in \mathbb{Z}_{\geq 2}$ and $i \in \{1, \ldots, p\}$ be as in Definition A.1. Then, the following holds:*

$$\sum_{i=1}^{p} \frac{\varphi_{p,i}^{2p-1}}{\varphi_{p,i}^p + p - 1} = 1. \tag{61}$$

*Proof.* Proved in Theorem 1.2.

$\square$

**Corollary A.8.** *Let $\varphi_{p,i} \in \mathbb{C}$ with $p \in \mathbb{Z}_{\geq 2}$ and $i \in \{1, \ldots, p\}$ be as in Definition A.1. Then, the following holds:*

$$\sum_{i=1}^{p} \frac{\varphi_{p,i}^{3p-2}}{\varphi_{p,i}^p + p - 1} = p. \tag{62}$$

*Proof.* Proved in Theorem 2.2. □

Of particular interest in our case is the distribution of $\varphi_{p,i}$ for $i \in \{1, \ldots, p\}$ in the complex plan.

**Theorem A.9.** *Let $\varphi_{p,1}$, $\varphi_{p,2}$ be as in Definition A.1 with $p \in \mathbb{Z}_{\geq 2}$ and such that $\varphi_{p,1} = re^{i\theta_1}$ and $\varphi_{p,2} = re^{i\theta_2}$ with $r \in \mathbb{R}_{>0}$ and $\theta_1 \neq \theta_2$. Then we necessarily have $\theta_1 = -\theta_2$.*

*Proof.* From Definition A.1, we have

$$r^p e^{i\theta_1 p} - r^{p-1} e^{i\theta_1 (p-1)} = r^p e^{i\theta_2 p} - r^{p-1} e^{i\theta_2 (p-1)} \tag{63}$$

Now using the fact that $\theta_1 \neq \theta_2$, we have

$$r = \frac{e^{i\theta_1 (p-1)} - e^{i\theta_2 (p-1)}}{e^{i\theta_1 p} - e^{i\theta_2 p}} = \frac{e^{i\frac{\theta_1+\theta_2}{2}(p-1)} \left( e^{i\frac{\theta_1-\theta_2}{2}(p-1)} - e^{-i\frac{\theta_1-\theta_2}{2}(p-1)} \right)}{e^{i\frac{\theta_1+\theta_2}{2}p} \left( e^{i\frac{\theta_1-\theta_2}{2}p} - e^{-i\frac{\theta_1-\theta_2}{2}p} \right)} \tag{64}$$

After some calculations, we can verify that $\arg(r) = 0$ (because $r \in \mathbb{R}_{>0}$) implies that $\theta_1 = -\theta_2$. □

**Remark.** *A direct consequence of this theorem is that there cannot exist more than two roots with the same modulus. If we are interested in the set of roots with the largest modulus, then either this set contains only one real number, either it contains a pair of complex conjugate numbers.*

**Theorem A.10** (Existence and uniqueness of the Golden Number of order $p$)**.** *Let $\varphi_{p,i} \in \mathbb{C}$ with $p \in \mathbb{Z}_{\geq 2}$ and $i \in \{1, \ldots, p\}$ be as in Definition A.1, and denote by $\hat{\varphi}_p$ the root with the largest modulus. Then, the following hold:*

1. *$\hat{\varphi}_p$ exists;*

2. *$\hat{\varphi}_p$ is unique;*

3. *$\hat{\varphi}_p$ is a stricly positive real number.*

*Proof.* The fact that $\hat{\varphi}_p$ exists is obvious.

From Theorem A.9, it follows that either $\hat{\varphi}_p$ is real, either it defines a pair of complex conjugate numbers. From Theorem 2.2, as $n$ goes large and with $p = T_r + 1$, either $\ell_{p-1}^n \simeq (\hat{\varphi}_p)^n + (\overline{\hat{\varphi}_p})^n$ if $\hat{\varphi}_p$ defines a pair of complex conjugate numbers either $\ell_{p-1}^n \simeq (\hat{\varphi}_p)^n$ if $\hat{\varphi}_p$ is real.

But since $\ell_{p-1}^n > 0$ for any $n > 0$, we also should have $(\hat{\varphi}_p)^n + (\overline{\hat{\varphi}_p})^n = 2\hat{r}^n \cos n\hat{\theta} > 0$ which is not true for an arbitrary $n$ as $\hat{\theta} \mod \pi \neq 0$. In the same manner, we cannot have $\hat{\varphi}_p \in \mathbb{R}_{<0}$ as $(\hat{\varphi}_p)^n = (-1)^n |\hat{\varphi}_p|^n$ in this case which can be negative. Hence, necessarily, $\hat{\varphi}_p \in \mathbb{R}_{>0}$ and we have proven that $\hat{\varphi}_p$ is unique and a strictly positive real number. □

**Remark.** *It follows from Theorem A.10 that for any $p > 1$, $|\hat{\varphi}_p| = \hat{\varphi}_p$*

**Theorem A.11.** *Let $\hat{\varphi}_p \in \mathbb{R}_{>0}$ with $p \in \mathbb{Z}_{\geq 2}$ be as in Theorem A.10. Then, the following holds:*

$$\hat{\varphi}_p > 1. \tag{65}$$

*Proof.* Using Theorem A.2, we have:

$$\hat{\varphi}_p^p = |\hat{\varphi}_p|^p = \left( \max_{i \in \{1,\ldots,p\}} |\varphi_{p,i}| \right)^p \geq \prod_{i=1}^p |\varphi_{p,i}| = \left| \prod_{i=1}^p \varphi_{p,i} \right| = 1, \tag{66}$$

and thus:

$$\hat{\varphi}_p \geq 1^{1/p} = 1, \tag{67}$$

Noticing that for any $p > 1$, $\hat{\varphi}_p \neq 1$ as $1^p - 1^{p-1} - 1 = -1 \neq 0$, we complete the proof. □

14

| $p$ | $\hat{\varphi}_p$ | $p$ | $\hat{\varphi}_p$ | $p$ | $\hat{\varphi}_p$ | $p$ | $\hat{\varphi}_p$ | $p$ | $\hat{\varphi}_p$ |
|---|---|---|---|---|---|---|---|---|---|
| 2 | 1.618034 | 52 | 1.057571 | 102 | 1.034035 | 152 | 1.024788 | 202 | 1.024788 |
| 3 | 1.465571 | 53 | 1.056732 | 103 | 1.033774 | 153 | 1.024659 | 203 | 1.024659 |
| 4 | 1.380278 | 54 | 1.055920 | 104 | 1.033517 | 154 | 1.024530 | 204 | 1.024530 |
| 5 | 1.324718 | 55 | 1.055133 | 105 | 1.033265 | 155 | 1.024404 | 205 | 1.024404 |
| 6 | 1.285199 | 56 | 1.054371 | 106 | 1.033017 | 156 | 1.024278 | 206 | 1.024278 |
| 7 | 1.255423 | 57 | 1.053632 | 107 | 1.032772 | 157 | 1.024154 | 207 | 1.024154 |
| 8 | 1.232055 | 58 | 1.052915 | 108 | 1.032532 | 158 | 1.024032 | 208 | 1.024032 |
| 9 | 1.213150 | 59 | 1.052219 | 109 | 1.032296 | 159 | 1.023911 | 209 | 1.023911 |
| 10 | 1.197491 | 60 | 1.051544 | 110 | 1.032063 | 160 | 1.023791 | 210 | 1.023791 |
| 11 | 1.184276 | 61 | 1.050888 | 111 | 1.031834 | 161 | 1.023672 | 211 | 1.023672 |
| 12 | 1.172951 | 62 | 1.050250 | 112 | 1.031609 | 162 | 1.023555 | 212 | 1.023555 |
| 13 | 1.163120 | 63 | 1.049630 | 113 | 1.031387 | 163 | 1.023439 | 213 | 1.023439 |
| 14 | 1.154494 | 64 | 1.049027 | 114 | 1.031169 | 164 | 1.023325 | 214 | 1.023325 |
| 15 | 1.146854 | 65 | 1.048440 | 115 | 1.030954 | 165 | 1.023211 | 215 | 1.023211 |
| 16 | 1.140034 | 66 | 1.047869 | 116 | 1.030742 | 166 | 1.023099 | 216 | 1.023099 |
| 17 | 1.133902 | 67 | 1.047312 | 117 | 1.030534 | 167 | 1.022988 | 217 | 1.022988 |
| 18 | 1.128356 | 68 | 1.046770 | 118 | 1.030328 | 168 | 1.022878 | 218 | 1.022878 |
| 19 | 1.123311 | 69 | 1.046241 | 119 | 1.030126 | 169 | 1.022769 | 219 | 1.022769 |
| 20 | 1.118699 | 70 | 1.045726 | 120 | 1.029927 | 170 | 1.022662 | 220 | 1.022662 |
| 21 | 1.114465 | 71 | 1.045223 | 121 | 1.029730 | 171 | 1.022555 | 221 | 1.022555 |
| 22 | 1.110562 | 72 | 1.044733 | 122 | 1.029536 | 172 | 1.022450 | 222 | 1.022450 |
| 23 | 1.106950 | 73 | 1.044254 | 123 | 1.029345 | 173 | 1.022346 | 223 | 1.022346 |
| 24 | 1.103598 | 74 | 1.043787 | 124 | 1.029157 | 174 | 1.022242 | 224 | 1.022242 |
| 25 | 1.100476 | 75 | 1.043330 | 125 | 1.028972 | 175 | 1.022140 | 225 | 1.022140 |
| 26 | 1.097561 | 76 | 1.042884 | 126 | 1.028789 | 176 | 1.022039 | 226 | 1.022039 |
| 27 | 1.094833 | 77 | 1.042448 | 127 | 1.028608 | 177 | 1.021939 | 227 | 1.021939 |
| 28 | 1.092272 | 78 | 1.042022 | 128 | 1.028430 | 178 | 1.021840 | 228 | 1.021840 |
| 29 | 1.089863 | 79 | 1.041605 | 129 | 1.028255 | 179 | 1.021742 | 229 | 1.021742 |
| 30 | 1.087593 | 80 | 1.041198 | 130 | 1.028082 | 180 | 1.021644 | 230 | 1.021644 |
| 31 | 1.085450 | 81 | 1.040799 | 131 | 1.027911 | 181 | 1.021548 | 231 | 1.021548 |
| 32 | 1.083422 | 82 | 1.040409 | 132 | 1.027743 | 182 | 1.021453 | 232 | 1.021453 |
| 33 | 1.081500 | 83 | 1.040027 | 133 | 1.027576 | 183 | 1.021358 | 233 | 1.021358 |
| 34 | 1.079675 | 84 | 1.039653 | 134 | 1.027412 | 184 | 1.021265 | 234 | 1.021265 |
| 35 | 1.077941 | 85 | 1.039286 | 135 | 1.027250 | 185 | 1.021172 | 235 | 1.021172 |
| 36 | 1.076290 | 86 | 1.038927 | 136 | 1.027091 | 186 | 1.021081 | 236 | 1.021081 |
| 37 | 1.074717 | 87 | 1.038576 | 137 | 1.026933 | 187 | 1.020990 | 237 | 1.020990 |
| 38 | 1.073215 | 88 | 1.038231 | 138 | 1.026777 | 188 | 1.020900 | 238 | 1.020900 |
| 39 | 1.071779 | 89 | 1.037893 | 139 | 1.026623 | 189 | 1.020811 | 239 | 1.020811 |
| 40 | 1.070406 | 90 | 1.037562 | 140 | 1.026472 | 190 | 1.020723 | 240 | 1.020723 |
| 41 | 1.069091 | 91 | 1.037237 | 141 | 1.026322 | 191 | 1.020635 | 241 | 1.020635 |
| 42 | 1.067830 | 92 | 1.036918 | 142 | 1.026174 | 192 | 1.020548 | 242 | 1.020548 |
| 43 | 1.066619 | 93 | 1.036606 | 143 | 1.026027 | 193 | 1.020463 | 243 | 1.020463 |
| 44 | 1.065457 | 94 | 1.036299 | 144 | 1.025883 | 194 | 1.020378 | 244 | 1.020378 |
| 45 | 1.064339 | 95 | 1.035998 | 145 | 1.025740 | 195 | 1.020293 | 245 | 1.020293 |
| 46 | 1.063264 | 96 | 1.035702 | 146 | 1.025599 | 196 | 1.020210 | 246 | 1.020210 |
| 47 | 1.062228 | 97 | 1.035412 | 147 | 1.025460 | 197 | 1.020127 | 247 | 1.020127 |
| 48 | 1.061230 | 98 | 1.035127 | 148 | 1.025323 | 198 | 1.020045 | 248 | 1.020045 |
| 49 | 1.060267 | 99 | 1.034846 | 149 | 1.025187 | 199 | 1.019964 | 249 | 1.019964 |
| 50 | 1.059337 | 100 | 1.034571 | 150 | 1.025052 | 200 | 1.019883 | 250 | 1.019883 |
| 51 | 1.058439 | 101 | 1.034301 | 151 | 1.024919 | 201 | 1.019803 | 251 | 1.019803 |

Table 9: Golden number $\hat{\varphi}_p$ for various $p$.

| $p$ | Duration |
|---|---|
| 10 | $41.1\,\mu\text{s} \pm 724\,\text{ns}$ |
| 100 | $1.02\,\text{ms} \pm 10.9\,\mu\text{s}$ |
| 1000 | $1.5\,\text{s} \pm 46.3\,\text{ms}$ |

Table 10: Computation time to find the $p$ roots of $P(x) = x^p - x^{p-1} - 1$.

**Theorem A.12.** *Let $\hat{\varphi}_p \in \mathbb{R}_{>0}$ with $p \in \mathbb{Z}_{\geq 2}$ be as in Theorem A.10. Then, the following holds:*

$$\hat{\varphi}_p > \hat{\varphi}_{p+1}. \tag{68}$$

*Proof.* As previously mentionned, we have:

$$\hat{\varphi}_{p+1}^p(\hat{\varphi}_{p+1} - 1) = \hat{\varphi}_p^{p-1}(\hat{\varphi}_p - 1) \tag{69}$$

Now assume $\hat{\varphi}_{p+1} = \hat{\varphi}_p + \varepsilon$, with $\varepsilon \geq 0$ and let's show it yields to a contradiction. We have:

$$\hat{\varphi}_{p+1}^p(\hat{\varphi}_{p+1} - 1) = (\hat{\varphi}_p + \varepsilon)^p(\hat{\varphi}_p + \varepsilon - 1) \tag{70}$$
$$\geq \hat{\varphi}_p^p(\hat{\varphi}_p + \varepsilon - 1) \tag{71}$$
$$> \hat{\varphi}_p^{p-1}(\hat{\varphi}_p + \varepsilon - 1) \tag{72}$$
$$\geq \hat{\varphi}_p^{p-1}(\hat{\varphi}_p - 1) \tag{73}$$

where we used Theorem A.11 in the third step. This is a contradiction with $\hat{\varphi}_{p+1}^p(\hat{\varphi}_{p+1} - 1) = \hat{\varphi}_p^{p-1}(\hat{\varphi}_p - 1)$. Hence we necessarily have $\hat{\varphi}_{p+1} = \hat{\varphi}_p + \varepsilon$, with $\varepsilon < 0$, or in other terms, $\hat{\varphi}_{p+1} < \hat{\varphi}_p$. $\square$

**Remark.** *It follows that for any $p > 2$, $\hat{\varphi}_p < \hat{\varphi}_2 = \frac{1+\sqrt{5}}{2}$.*

# B  Z-Transforms

**Theorem B.1.** *Let $\tilde{\ell}_{T_r}^n$ be as introduced in Theorem 1.1, with $T_r > 0$ and $n \geq 0$. Its Z-Transform satisfies:*

$$\frac{\tilde{\mathcal{L}}_{T_r}(z)}{z} = \frac{\tilde{A}}{z} + \sum_{i=1}^{T_r+1} \frac{\tilde{B}_i}{z - \varphi_{T_r+1,i}}. \tag{74}$$

*with*

$$\tilde{A} = -1, \tag{75}$$

$$\tilde{B}_i = \frac{\varphi_{T_r+1,i}^{2T_r+1}}{\varphi_{T_r+1,i}^{T_r+1} + T_r}, \tag{76}$$

*and $\varphi_{T_r+1,i}$ with $i \in \{1, \ldots, T_r + 1\}$ as defined above.*

*Proof.* By directly computing the Z-Transform of Theorem 1.1, we get

$$\tilde{\mathcal{L}}_{T_r}(z) \triangleq \sum_{n \geq 0} \tilde{\ell}_{T_r}^n z^{-n} \tag{77}$$

$$= 2z^{-1} + \sum_{n=2}^{T_r+1} z^{-n} + \sum_{n \geq 2} \tilde{\ell}_{T_r}^{n-1} z^{-n} + \sum_{n \geq T_r+2} \tilde{\ell}_{T_r}^{n-1-T_r} z^{-n} \tag{78}$$

$$= z^{-1} + \sum_{n=1}^{T_r+1} z^{-n} + z^{-1} \sum_{n \geq 1} \tilde{\ell}_{T_r}^n z^{-n} + z^{-1-T_r} \sum_{n \geq 1} \tilde{\ell}_{T_r}^n z^{-n} \tag{79}$$

$$= z^{-1} + z^{-T_r-1} \frac{1 - z^{T_r+1}}{1 - z} + z^{-1} \tilde{\mathcal{L}}_{T_r}(z) + z^{-1-T_r} \tilde{\mathcal{L}}_{T_r}(z), \tag{80}$$

16

and then

$$\frac{\tilde{\mathcal{L}}_{T_r}(z)}{z} = \frac{z^{T_r} + \frac{1-z^{T_r+1}}{1-z}}{z \prod_{i=1}^{T_r+1}(z - \varphi_{T_r+1,i})}, \tag{81}$$

where $\varphi_{T_r+1,i}$ for $i \in \{1, \ldots, T_r + 1\}$ as in Definition A.1. Since every poles have multiplicity one and the degree of the denominator is larger than the degree of the numerator, we can make partial fraction decomposition to get:

$$\frac{\tilde{\mathcal{L}}_{T_r}(z)}{z} = \frac{\tilde{A}}{z} + \sum_{i=1}^{T_r+1} \frac{\tilde{B}_i}{z - \varphi_{T_r+1,i}}, \tag{82}$$

Using Theorem A.2, we obtain

$$\tilde{A} = z\frac{\tilde{\mathcal{L}}_{T_r}(z)}{z}\bigg|_{z=0} = \frac{(-1)^{T_r+1}}{\prod_{i=1}^{T_r+1} \varphi_{T_r+1,i}} = -1. \tag{83}$$

Using Definition A.1 and Corollary A.5, we get

$$\tilde{B}_i = (z - \varphi_{T_r+1,i})\frac{\tilde{\mathcal{L}}_{T_r}(z)}{z}\bigg|_{z=\varphi_{T_r+1,i}} \tag{84}$$

$$= \frac{\varphi_{T_r+1,i}^{T_r} + \frac{1-\varphi_{T_r+1,i}^{T_r+1}}{1-\varphi_{T_r+1,i}}}{\varphi_{T_r+1,i} \prod_{\substack{j=1 \\ j \neq i}}^{T_r+1}(\varphi_{T_r+1,i} - \varphi_{T_r+1,j})} \tag{85}$$

$$= \frac{\varphi_{T_r+1,i}^{T_r} - \frac{\varphi_{T_r+1,i}^{T_r}}{1-\varphi_{T_r+1,i}}}{\varphi_{T_r+1,i}^{T_r+1} + T_r} \tag{86}$$

$$= \frac{\varphi_{T_r+1,i}^{T_r} - \varphi_{T_r+1,i}^{T_r+1} - \varphi_{T_r+1,i}^{T_r}}{(\varphi_{T_r+1,i}^{T_r+1} + T_r)(1 - \varphi_{T_r+1,i})} \tag{87}$$

$$= \frac{\varphi_{T_r+1,i}^{T_r+1}}{(\varphi_{T_r+1,i}^{T_r+1} + T_r)\varphi_{T_r+1,i}^{-T_r}} \tag{88}$$

$$= \frac{\varphi_{T_r+1,i}^{2T_r+1}}{\varphi_{T_r+1,i}^{T_r+1} + T_r}, \tag{89}$$

for $i \in \{1, \ldots, T_r + 1\}$. □

**Theorem B.2.** *Let $\ell_{T_r}^n$ be as introduced in Theorem 2.1, with $T_r > 0$ and $n \geq 0$. Its Z-Transform satisfies:*

$$\mathcal{L}_{T_r}(z) = \frac{z}{z-1}(z^{-1} + T_r z^{-T_r-1}) \tag{90}$$

$$+ \frac{z}{(z-1)^2}(z^{-T_r} - z^{-2T_r-1} + T_r z^{-2T_r-2} - T_r z^{-3T_r-2})$$

$$+ \frac{\tilde{\mathcal{L}}_{T_r}(z)}{z-1}(z^{-2T_r} + T_r z^{-3T_r-1}).$$

*Moreover, we have:*

$$\bar{\ell}_{T_r}^n \triangleq \mathcal{Z}^{-1}\left\{\frac{\tilde{\mathcal{L}}_{T_r}(z)}{z-1}\right\}[n] = \begin{cases} \sum_{i=1}^{T_r+1} \frac{\varphi_{T_r+1,i}^{3T_r+1+n}}{\varphi_{T_r+1,i}^{T_r+1} + T_r} - (T_r + 2) & \text{if } n > 0 \\ 0 & \text{otherwise} \end{cases}, \tag{91}$$

*with* $\text{ROC}(\bar{\ell}_{T_r}^n) = \{z \in \mathbb{C} : |z| > \max_{i \in \{1,\ldots,T_r+1\}} |\varphi_{T_r+1,i}|\}$.

17

*Proof.* By directly computing the Z-Transform of Theorem 2.1, we get

$$\mathcal{L}_{T_r}(z) \triangleq \sum_{n \geq 0} \ell_{T_r}^n z^{-n} \tag{92}$$

$$= \sum_{n \geq 1} \ell_{T_r}^{n-1} z^{-n} + \sum_{n \geq 2T_r+2} \tilde{\ell}_{T_r}^{n-2T_r-1} z^{-n} + T_r \sum_{n \geq 3T_r+3} \tilde{\ell}_{T_r}^{n-3T_r-2} z^{-n} \tag{93}$$

$$+ T_r z^{-T_r-1} + z^{-1} + \sum_{n=T_r+1}^{2T_r+1} z^{-n} + T_r \sum_{n=2T_r+2}^{3T_r+2} z^{-n}$$

$$= z^{-1} \sum_{n \geq 1} \ell_{T_r}^n z^{-n} + \left( z^{-2T_r-1} + T_r z^{-3T_r-2} \right) \sum_{n \geq 1} \tilde{\ell}_{T_r}^n z^{-n} \tag{94}$$

$$+ \sum_{n=T_r+1}^{2T_r+1} z^{-n} + T_r z^{-T_r-1} + z^{-1} + T_r \sum_{n=2T_r+2}^{3T_r+2} z^{-n}$$

$$= z^{-1} \mathcal{L}_{T_r}(z) + \left( z^{-2T_r-1} + T_r z^{-3T_r-2} \right) \tilde{\mathcal{L}}_{T_r}(z) \tag{95}$$

$$+ T_r z^{-T_r-1} + z^{-1} + \frac{z^{-T_r} - z^{-2T_r-1}}{z-1} + T_r \frac{z^{-2T_r-1} - z^{-3T_r-2}}{z-1}$$

$$\tag{96}$$

and thus:

$$\mathcal{L}_{T_r}(z) = \frac{z}{z-1} \left( z^{-1} + T_r z^{-T_r-1} \right) \tag{97}$$

$$+ \frac{z}{(z-1)^2} \left( z^{-T_r} - z^{-2T_r-1} + T_r z^{-2T_r-1} - T_r z^{-3T_r-2} \right)$$

$$+ \frac{\tilde{\mathcal{L}}_{T_r}(z)}{z-1} \left( z^{-2T_r} + T_r z^{-3T_r-1} \right).$$

Let's now compute the following inverse Z-Transform

$$\bar{\ell}_{T_r}^n \triangleq \mathcal{Z}^{-1} \left\{ \frac{\tilde{\mathcal{L}}_{T_r}(z)}{z-1} \right\}[n]. \tag{98}$$

Using Theorem B.1, we have:

$$\frac{\tilde{\mathcal{L}}_{T_r}(z)}{z(z-1)} = \frac{A}{z} + \sum_{i=1}^{T_r+1} \frac{B_i}{z - \varphi_{T_r+1,i}} + \frac{C}{z-1}, \tag{99}$$

with

$$A = z \frac{\tilde{\mathcal{L}}_{T_r}(z)}{z(z-1)} \bigg|_{z=0} = -\tilde{A} = 1, \tag{100}$$

$$B_i = (z - \varphi_{T_r+1,i}) \frac{\tilde{\mathcal{L}}_{T_r}(z)}{z(z-1)} \bigg|_{z=\varphi_{T_r+1,i}} = \frac{\tilde{B}_i}{\varphi_{T_r+1,i} - 1} = \frac{\varphi_{T_r+1,i}^{3T_r+1}}{\varphi_{T_r+1,i}^{T_r+1} + T_r}, \tag{101}$$

and

$$C = (z - 1) \frac{\tilde{\mathcal{L}}_{T_r}(z)}{z(z-1)} \bigg|_{z=1} = -(T_r + 2), \tag{102}$$

using Corollary A.6.

Using Theorem A.11, we obtain:

$$\bar{\ell}_{T_r}^n = \begin{cases} \sum_{i=1}^{T_r+1} \frac{\varphi_{T_r+1,i}^{3T_r+1+n}}{\varphi_{T_r+1,i}^{T_r+1} + T_r} - T_r - 2 & \text{if } n > 0 \\ \sum_{i=1}^{T_r+1} \frac{\varphi_{T_r+1,i}^{3T_r+1+n}}{\varphi_{T_r+1,i}^{T_r+1} + T_r} - T_r - 1 & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}, \tag{103}$$

18

with $\text{ROC}(\bar{\ell}^n_{T_r}) = \{z \in \mathbb{C} : |z| > \max_{i \in \{1,\ldots,T_r+1\}} |\varphi_{T_r+1,i}|\}$.

$\square$