On counting firing signals

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1 Firing signals

We first consider firing signals of size $n \in \mathbb{Z}_{\geq 0}$ with refractory period $T_r \in \mathbb{Z}_{\geq 0}$. Those signals can be represented by binary sequences of length n such that there are at least T_r zeros between two consecutive ones. Let $\tilde{\mathcal{Y}}_{T_r}^n$ represent the set of such sequences. We formally define this set as

$$\tilde{\mathcal{Y}}_{T_r}^n = \left\{ (y_1, \ldots, y_n) \in \{0,1\}^n : \sum_{m=0}^{\min\{T_r, n-m\}} y_{k+m} \in \{0,1\}, k \in \{1, \ldots, n\} \right\}. \tag{1}$$

Obviously, if $T_r = 0$, $\tilde{\mathcal{Y}}_{T_r}^n$ simplifies to

$$\tilde{\mathcal{Y}}_{T_r}^n = \{0,1\}^n, \tag{2}$$

and this trivial case is irrelevant for our purpose. That is why, from now on, we will assume $T_r > 0$.

Theorem 1.1. *Let $\tilde{\ell}_{T_r}^n$ denotes the cardinality of $\tilde{\mathcal{Y}}_{T_r}^n$, with $n \in \mathbb{Z}_{\geq 0}$ and $T_r \in \mathbb{Z}_{>0}$. We have:*

$$\tilde{\ell}_{T_r}^n = \begin{cases} 0 & \text{if } n = 0 \\ 2 & \text{if } n = 1 \\ \tilde{\ell}_{T_r}^{n-1} + 1 & \text{if } 1 < n \leq T_r + 1 \\ \tilde{\ell}_{T_r}^{n-1} + \tilde{\ell}_{T_r}^{n-T_r-1} & \text{otherwise.} \end{cases}. \tag{3}$$

Proof. If $n = 0$, it is obvious that $\tilde{\ell}_{T_r}^0 = 0$. If $n = 1$, both 1 and 0 are sequences in $\tilde{\mathcal{Y}}_{T_r}^1$ and thus $\tilde{\ell}_{T_r}^1 = 2$ If $2 \leq n \leq T_r + 1$, there are $\tilde{\ell}_{T_r}^{n-1}$ sequences in $\tilde{\mathcal{Y}}_{T_r}^n$ with the form

$$\underbrace{y_1 \cdots y_{n-1}}_{\in \tilde{\mathcal{Y}}_{T_r}^{n-1}} 0,$$

and 1 sequence with the form

$$\overbrace{0 \cdots 0}^{\leq T_r} 1.$$

If $n \geq T_r + 2$, there are $\ell_{T_r}^{n-1}$ sequences in $\tilde{\mathcal{Y}}_{T_r}^n$ with the form

$$\underbrace{y_1 \cdots y_{n-1}}_{\in \tilde{\mathcal{Y}}_{T_r}^{n-1}} 0,$$

and $\tilde{\ell}_{T_r}^{n-T_r-1}$ sequences with the form

$$\underbrace{y_1 \cdots y_{n-T_r-1}}_{\in \tilde{\mathcal{Y}}_{T_r}^{n-T_r-1}} \overbrace{0 \cdots 0}^{T_r} 1.$$

$\square$

Theorem 1.2. *Let $\tilde{\ell}^n_{T_r}$ denotes the cardinality of $\tilde{\mathcal{Y}}^n_{T_r}$, with $n \in \mathbb{Z}_{\geq 0}$ and $T_r \in \mathbb{Z}_{>0}$ and let $\varphi_{T_r+1,i} \in \mathbb{C}$ with $i \in \{1, \ldots, T_r+1\}$ be as defined in Definition A.1. Then*

$$\tilde{\ell}^n_{T_r} = \begin{cases} \sum_{i=1}^{T_r+1} \frac{\varphi^{2T_r+n+1}_{T_r+1,i}}{\varphi^{T_r+1}_{T_r+1,i}+T_r} & \textit{if } n > 0 \\ 0 & \textit{otherwise} \end{cases}, \tag{4}$$

with $\mathrm{ROC}(\tilde{\ell}^n_{T_r}) = \{z \in \mathbb{C} : |z| > \max_{i \in \{1,\ldots,T_r+1\}} |\varphi_{T_r+1,i}|\}$.

Proof. Inverting the Z-Transform of Theorem B.1, we get

$$\tilde{\ell}^n_{T_r} = \begin{cases} \sum_{i=1}^{T_r+1} \frac{\varphi^{2T_r+n+1}_{T_r+1,i}}{\varphi^{T_r+1}_{T_r+1,i}+T_r} & \text{if } n > 0 \\ -1 + \sum_{i=1}^{T_r+1} \frac{\varphi^{2T_r+n+1}_{T_r+1,i}}{\varphi^{T_r+1}_{T_r+1,i}+T_r} & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}, \tag{5}$$

with $\mathrm{ROC}(\tilde{\ell}^n_{T_r}) = \{z \in \mathbb{C} : |z| > \max_{i \in \{1,\ldots,T_r+1\}} |\varphi_{T_r+1,i}|\}$.

But because of Theorem 1.1, for $n = 0$, we have:

$$\tilde{\ell}^n_{T_r} = -1 + \sum_{i=1}^{T_r+1} \frac{\varphi^{2T_r+n+1}_{T_r+1,i}}{\varphi^{T_r+1}_{T_r+1,i}+T_r} = 0, \tag{6}$$

and substituting $p = T_r + 1$, we prove Corollary A.7 and complete the current proof. □

	T_r									
n	1	2	3	4	5	6	7	8	9	10
1	2	2	2	2	2	2	2	2	2	2
2	3	3	3	3	3	3	3	3	3	3
3	5	4	4	4	4	4	4	4	4	4
4	8	6	5	5	5	5	5	5	5	5
5	13	9	7	6	6	6	6	6	6	6
6	21	13	10	8	7	7	7	7	7	7
7	34	19	14	11	9	8	8	8	8	8
8	55	28	19	15	12	10	9	9	9	9
9	89	41	26	20	16	13	11	10	10	10
10	144	60	36	26	21	17	14	12	11	11
11	233	88	50	34	27	22	18	15	13	12
12	377	129	69	45	34	28	23	19	16	14
13	610	189	95	60	43	35	29	24	20	17
14	987	277	131	80	55	43	36	30	25	21
15	1597	406	181	106	71	53	44	37	31	26
16	2584	595	250	140	92	66	53	45	38	32
17	4181	872	345	185	119	83	64	54	46	39
18	6765	1278	476	245	153	105	78	64	55	47
19	10946	1873	657	325	196	133	96	76	65	56
20	17711	2745	907	431	251	168	119	91	76	66

Table 1: $\tilde{\ell}^n_{T_r}$ for $0 < n \leq 20$ and $0 < T_r \leq 10$.

Given a sample of $\tilde{\mathcal{Y}}^n_{T_r}$, it could also be interesting to know the probability that it contains exactly n_0 zeros and n_1 ones. Let $\tilde{\mathcal{Y}}^{n_0,n_1}_{T_r}$ represent the set of sequences in $\tilde{\mathcal{Y}}^n_{T_r}$ with n_0 zeros and n_1 ones, $n_0 + n_1 = n$, $n_0, n_1 \geq 0$. It can be formally expressed as

$$\tilde{\mathcal{Y}}^{n_0,n_1}_{T_r} = \left\{ (y_1, \ldots, y_n) \in \tilde{\mathcal{Y}}^n_{T_r} : \sum_{i=1}^{n} y_i = n_1, n = n_0 + n_1 \right\}. \tag{7}$$

Theorem 1.3. *Let $\tilde{\ell}_{T_r}^{n_0,n_1}$ denotes the cardinality of $\tilde{\mathcal{Y}}_{T_r}^{n_0,n_1}$ with $n_0, n_1 \in \mathbb{Z}_{\geq 0}$ and $T_r \in \mathbb{Z}_{>0}$. We have:*

$$\tilde{\ell}_{T_r}^{n_0,n_1} = \begin{cases} \binom{n-T_rn_1+T_r}{n_1} & \text{if } n_0 \geq T_r(n_1-1) \\ 0 & \text{otherwise} \end{cases}, \tag{8}$$

with $n = n_0 + n_1$.

Proof. First of all, notice that if $n_0 < T_r(n_1 - 1)$, it is not possible that there are at least T_r zeros between two consecutives ones. Hence $\tilde{\ell}_{T_r}^{n_0,n_1} = 0$. Now assume $n_0 \geq T_r(n_1 - 1)$ and consider the following *bijective mapping*:

$$\underbrace{\overbrace{0\cdots0}^{k_0\geq 0}1\overbrace{0\cdots0}^{k_1\geq T_r}1\cdots\overbrace{0\cdots0}^{k_{n_1-1}\geq T_r}1\overbrace{0\cdots0}^{k_{n_1}\geq 0}}_{\in\tilde{\mathcal{Y}}_{T_r}^{n}} \mapsto \overbrace{0\cdots0}^{k_0\geq 0}1\overbrace{0\cdots0}^{k'_1\geq 0}1\cdots\overbrace{0\cdots0}^{k'_{n_1-1}\geq 0}1\overbrace{0\cdots0}^{k_{n_1}\geq 0} \tag{9}$$

with $k_0 + k_1 + \cdots + k_{n_1} = n_0$ and $k'_i = k_i - T_r$ for $1 \leq i < n_1$. In other words, this mapping maps any binary sequence in $\tilde{\mathcal{Y}}_{T_r}^{n_0,n_1}$ to another binary sequence of lenght $n - T_r(n_1 - 1)$ with $n_0 - T_r(n_1 - 1)$ zeros and n_1 ones. Now, it suffices to notice that there exists

$$\binom{n - T_rn_1 + T_r}{n_1} \tag{10}$$

such sequences to complete the proof. $\square$

						n_1						
n	0	1	2	3	4	5	6	7	8	9	10	$\ell_{T_r}^{\tilde{n}}$
1	1	1	-	-	-	-	-	-	-	-	-	2
2	1	2	0	-	-	-	-	-	-	-	-	3
3	1	3	1	0	-	-	-	-	-	-	-	5
4	1	4	3	0	0	-	-	-	-	-	-	8
5	1	5	6	1	0	0	-	-	-	-	-	13
6	1	6	10	4	0	0	0	-	-	-	-	21
7	1	7	15	10	1	0	0	0	-	-	-	34
8	1	8	21	20	5	0	0	0	0	-	-	55
9	1	9	28	35	15	1	0	0	0	0	-	89
10	1	10	36	56	35	6	0	0	0	0	0	144

Table 2: $\tilde{\ell}_{T_r}^{n_0,n_1}$ for $0 < n \leq 10$, $0 \leq n_1 \leq n$ and $T_r = 1$.

						n_1						
n	0	1	2	3	4	5	6	7	8	9	10	$\ell_{T_r}^{\tilde{n}}$
1	1	1	-	-	-	-	-	-	-	-	-	2
2	1	2	0	-	-	-	-	-	-	-	-	3
3	1	3	0	0	-	-	-	-	-	-	-	4
4	1	4	1	0	0	-	-	-	-	-	-	6
5	1	5	3	0	0	0	-	-	-	-	-	9
6	1	6	6	0	0	0	0	-	-	-	-	13
7	1	7	10	1	0	0	0	0	-	-	-	19
8	1	8	15	4	0	0	0	0	0	-	-	28
9	1	9	21	10	0	0	0	0	0	0	-	41
10	1	10	28	20	1	0	0	0	0	0	0	60

Table 3: $\tilde{\ell}_{T_r}^{n_0,n_1}$ for $0 < n \leq 10$, $0 \leq n_1 \leq n$ and $T_r = 2$.

						n_1						
n	0	1	2	3	4	5	6	7	8	9	10	$\ell_{T_r}^{\tilde{n}}$
1	1	1	-	-	-	-	-	-	-	-	-	2
2	1	2	0	-	-	-	-	-	-	-	-	3
3	1	3	0	0	-	-	-	-	-	-	-	4
4	1	4	0	0	0	-	-	-	-	-	-	5
5	1	5	1	0	0	0	-	-	-	-	-	7
6	1	6	3	0	0	0	0	-	-	-	-	10
7	1	7	6	0	0	0	0	0	-	-	-	14
8	1	8	10	0	0	0	0	0	0	-	-	19
9	1	9	15	1	0	0	0	0	0	0	-	26
10	1	10	21	4	0	0	0	0	0	0	0	36

Table 4: $\tilde{\ell}_{T_r}^{n_0,n_1}$ for $0 < n \leq 10$, $0 \leq n_1 \leq n$ and $T_r = 3$.

2 Firing signals with cycles

We now consider firing signals of size $n \in \mathbb{Z}_{\geq 0}$ with refractory period $T_r \in \mathbb{Z}_{\geq 0}$ and cycled constrained. Those signals can be represented by binary sequences of length n such that there are at least T_r zeros between two consecutive ones even between the last one and the first one of the sequence. Let $\mathcal{Y}_{T_r}^n$ represent the set of such sequences. We formally define this set as

$$\mathcal{Y}_{T_r}^n = \left\{ (y_1, \ldots, y_n) \in \{0,1\}^n : \sum_{m=0}^{T_r} y_{(k+m \bmod n)+1} \in \{0,1\}, k \in \{1, \ldots, n\} \right\}. \tag{11}$$

Obviously, if $T_r = 0$, $\mathcal{Y}_{T_r}^n$ simplifies to

$$\mathcal{Y}_{T_r}^n = \{0,1\}^n, \tag{12}$$

and this trivial case is irrelevant for our purpose. That is why, from now on, we further assume $T_r > 0$.

Theorem 2.1. *Let $\ell_{T_r}^n$ denotes the cardinality of $\mathcal{Y}_{T_r}^n$, with $n \in \mathbb{Z}_{\geq 0}$ and $T_r \in \mathbb{Z}_{>0}$. We have:*

$$\ell_{T_r}^n = \begin{cases} 0 & \text{if } n = 0 \\ \ell_{T_r}^{n-1} + 1 & \text{if } n = 1 \\ \ell_{T_r}^{n-1} & \text{if } 2 \leq n \leq T_r \\ \ell_{T_r}^{n-1} + 1 + T_r & \text{if } n = T_r + 1 \\ \ell_{T_r}^{n-1} + 1 & \text{if } T_r + 2 \leq n \leq 2T_r + 1 \\ \ell_{T_r}^{n-1} + \tilde{\ell}_{T_r}^{n-2T_r-1} + T_r & \text{if } 2T_r + 2 \leq n \leq 3T_r + 2 \\ \ell_{T_r}^{n-1} + \tilde{\ell}_{T_r}^{n-2T_r-1} + T_r \tilde{\ell}_{T_r}^{n-3T_r-2} & \text{otherwise} \end{cases}. \tag{13}$$

Proof. If $n = 0$, it is obvious that $\ell_{T_r}^n = 0$. If $n = 1$, the only periodic-valid sequence is the one containing one zero and zero one and $\ell_{T_r}^n = \ell_{T_r}^{n-1} + 1 = 0 + 1$. If $2 \leq n \leq T_r$, there are $\ell_{T_r}^{n-1}$ sequences in $\mathcal{Y}_{T_r}^n$ with the form

$$\underbrace{y_1 \cdots y_{n-1}}_{\in \mathcal{Y}_{T_r}^{n-1}} 0.$$

If $n = T_r + 1$, there are $\ell_{T_r}^{n-1}$ sequences in $\mathcal{Y}_{T_r}^n$ with the form

$$\underbrace{y_1 \cdots y_{n-1}}_{\in \mathcal{Y}_{T_r}^{n-1}} 0,$$

T_r sequences in $\mathcal{Y}_{T_r}^n$ with the form

$$\underbrace{\overbrace{0 \cdots 0}^{k_1 \geq 0} 1 \overbrace{0 \cdots 0}^{k_2 \geq 0}}_{\notin \mathcal{Y}_{T_r}^{n-1}} 0,$$

with $k_1 + k_2 = k = T_r - 1$, and 1 sequence with the form

$$\overbrace{0 \cdots 0}^{T_r} 1.$$

If $T_r + 2 \leq n \leq 2T_r + 1$, there are $\ell_{T_r}^{n-1}$ sequences in $\mathcal{Y}_{T_r}^n$ with the form

$$\underbrace{y_1 \cdots y_{n-1}}_{\in \mathcal{Y}_{T_r}^{n-1}} 0,$$

and 1 sequence with the form

$$\overbrace{0 \cdots 0}^{n-1 > T_r} 1.$$

If $2T_r+2 \le n \le 3T_r+2$, there are $\ell_{T_r}^{n-1}$ sequences in $\mathcal{Y}_{T_r}^n$ with the form

$$\underbrace{y_1 \cdots y_{n-1}}_{\in \mathcal{Y}_{T_r}^{n-1}} 0,$$

T_r sequences with the form

$$\underbrace{\overbrace{0\cdots0}^{k_1\ge0}1\;\overbrace{0\cdots0}^{n-T_r-2\ge T_r}\;1\overbrace{0\cdots0}^{k_2\ge0}}_{\notin \mathcal{Y}_{T_r}^{n-1}}0,$$

with $k_1+k_2=k=T_r-1$ and $\tilde{\ell}_{T_r}^{n-2T_r-1}$ sequences with the form

$$\overbrace{0\cdots0}^{T_r}\underbrace{y_{T_r+1}\cdots y_{n-T_r-1}}_{\in \tilde{\mathcal{Y}}_{T_r}^{n-2T_r-1}}\overbrace{0\cdots0}^{T_r}1.$$

If $n \ge 3T_r+3$, the are $\ell_{T_r}^{n-1}$ sequences in $\mathcal{Y}_{T_r}^n$ with the form

$$\underbrace{y_1 \cdots y_{n-1}}_{\in \mathcal{Y}_{T_r}^{n-1}} 0, \tag{14}$$

$T_r\tilde{\ell}_{T_r}^{n-3T_r-2}$ sequences with the form

$$\underbrace{\overbrace{0\cdots0}^{k_1\ge0}1\overbrace{0\cdots0}^{T_r}\underbrace{y_{k_1+T_r+T_r}\cdots y_{n-k_2-T_r-1}}_{\in \tilde{\mathcal{Y}}_{T_r}^{n-3T_r-2}}\overbrace{0\cdots0}^{T_r}1\overbrace{0\cdots0}^{k_2\ge0}0,}_{\notin \mathcal{Y}_{T_r}^{n-1}} \tag{15}$$

with $k_1+k_2=k=T_r-1$, and $\tilde{\ell}_{T_r}^{n-2T_r-1}$ sequences with the form

$$\overbrace{0\cdots0}^{T_r}\underbrace{y_{T_r+1}\cdots y_{n-T_r-1}}_{\in \tilde{\mathcal{Y}}_{T_r}^{n-2T_r-1}}\overbrace{0\cdots0}^{T_r}1. \tag{16}$$

□

Theorem 2.2. *Let $\ell_{T_r}^n$ be as introduced before, with $n \in \mathbb{Z}_{\ge 0}$ and $T_r \in \mathbb{Z}_{>0}$. Then:*

$$\ell_{T_r}^n = \begin{cases} 0 & \text{if } n \le 0, \\ 1 & \text{if } 1 \le n \le T_r, \\ n+1 & \text{if } T_r+1 \le n \le 2T_r, \\ \sum_{i=1}^{T_r+1} \frac{\varphi_{T_r+1,i}^{T_r+1+n}}{\varphi_{T_r+1,i}^{T_r+1}+T_r} - 2T_r^2 + nT_r & \text{if } 2T_r+1 \le n \le 3T_r+1, \\ \sum_{i=1}^{T_r+1} \varphi_{T_r+1,i}^n & \text{if } n > 3T_r+1. \end{cases} \tag{17}$$

with $\mathrm{ROC}(\ell_{T_r}^n) = \{z \in \mathbb{C} : |z| > \max_{i\in\{1,\dots,T_r+1\}} |\varphi_{T_r+1,i}|\}$.

Proof. Inverting the Z-Transform of Theorem B.2, we get

$$\ell_{T_r}^n = \begin{cases} 0 & \text{if } n \le 0, \\ 1 & \text{if } 1 \le n \le T_r, \\ n+1 & \text{if } T_r+1 \le n \le 2T_r-1, \\ T_r + \sum_{i=1}^{T_r+1} \frac{\varphi_{T_r+1,i}^{3T_r+1}}{\varphi_{T_r+1,i}^{T_r+1}+T_r} & \text{if } n = 2T_r, \\ T_r n - 2T_r^2 + \sum_{i=1}^{T_r+1} \frac{\varphi_{T_r+1,i}^{T_r+1+n}}{\varphi_{T_r+1,i}^{T_r+1}+T_r} & \text{if } 2T_r+1 \le n \le 3T_r, \\ \sum_{i=1}^{T_r+1} \frac{\varphi_{T_r+1,i}^{4T_r+2}}{\varphi_{T_r+1,i}^{T_r+1}+T_r} + T_r \sum_{i=1}^{T_r+1} \frac{\varphi_{T_r+1,i}^{3T_r+1}}{\varphi_{T_r+1,i}^{T_r+1}+T_r} & \text{if } n = 3T_r+1, \\ \sum_{i=1}^{T_r+1} \varphi_{T_r+1,i}^n & \text{if } n > 3T_r+1. \end{cases}, \tag{18}$$

with $\mathrm{ROC}(\tilde{\ell}^n_{T_r}) = \{z \in \mathbb{C} : |z| > \max_{i\in\{1,\dots,T_r+1\}} |\varphi_{T_r+1,i}|\}$.

Let $n = 2T_r$. There is 1 sequence in $\mathcal{Y}^{2T_r}_{T_r}$ with the form

$$\overbrace{0\cdots 0}$$

and $2T_r$ sequences with the form

$$\overbrace{0\cdots 0}^{k_1\geq 0}1\overbrace{0\cdots 0}^{k_2\geq 0},$$

with $k_1 + k_2 = 2T_r - 1 \geq T_r$, $k_1, k_2 \geq 0$. Thus, we necessarily have

$$\sum_{i=1}^{T_r+1} \frac{\varphi^{3T_r+1}_{T_r+1,i}}{\varphi^{T_r+1}_{T_r+1,i} + T_r} = T_r + 1, \tag{19}$$

and substituting $p = T_r + 1$, we prove Corollary A.8.

Now, we can use this result in the case $n = 3T_r + 1$ to get

$$\sum_{i=1}^{T_r+1} \frac{\varphi^{4T_r+2}_{T_r+1,i}}{\varphi^{T_r+1}_{T_r+1,i} + T_r} + T_r \sum_{i=1}^{T_r+1} \frac{\varphi^{3T_r+1}_{T_r+1,i}}{\varphi^{T_r+1}_{T_r+1,i} + T_r} = \sum_{i=1}^{T_r+1} \frac{\varphi^{T_r+1+3T_r+1}_{T_r+1,i}}{\varphi^{T_r+1}_{T_r+1,i} + T_r} + T_r(3T_r+1) - 2T_r^2, \tag{20}$$

and merge it with $2T_r + 1 \leq n \leq 3T_r$. □

	T_r									
n	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2	3	1	1	1	1	1	1	1	1	1
3	4	4	1	1	1	1	1	1	1	1
4	7	5	5	1	1	1	1	1	1	1
5	11	6	6	6	1	1	1	1	1	1
6	18	10	7	7	7	1	1	1	1	1
7	29	15	8	8	8	8	1	1	1	1
8	47	21	13	9	9	9	9	1	1	1
9	76	31	19	10	10	10	10	10	1	1
10	123	46	26	16	11	11	11	11	11	1
11	199	67	34	23	12	12	12	12	12	12
12	322	98	47	31	19	13	13	13	13	13
13	521	144	66	40	27	14	14	14	14	14
14	843	211	92	50	36	22	15	15	15	15
15	1364	309	126	66	46	31	16	16	16	16
16	2207	453	173	89	57	41	25	17	17	17
17	3571	664	239	120	69	52	35	18	18	18
18	5778	973	331	160	88	64	46	28	19	19
19	9349	1426	457	210	115	77	58	39	20	20
20	15127	2090	630	276	151	91	71	51	31	21

Table 5: $\ell^n_{T_r}$ with $1 \leq n \leq 20$ and $0 < T_r \leq 10$.

Theorem 2.3. *Let $\ell^n_{T_r}$ be as introduced before, with $n \in \mathbb{Z}_{\geq 0}$ and fixed $T_r \in \mathbb{Z}_{>0}$. Then:*

$$\lim_{n\to\infty} \ell^n_{T_r} = \hat{\varphi}^n_{T_r+1} \tag{21}$$

with $\hat{\varphi}_{T_r+1}$ as defined in Theorem A.10.

Proof. The proof follows directly from Theorem 2.2 and Theorem A.10. $\square$

It is interesting to notice that as n goes large, the logarithmic of the number of valid sequences increases linearly in the length of the sequence as $\log \ell^n_{T_r} \simeq n \log \hat{\varphi}_{T_r+1}$. The rate of increase corresponds to the number of bits per seconde required to specify the particular signal used.

Remark. *As T_r becomes large (i.e. $T_r \gg n$), we have:*

$$\ell^n_{T_r} = 1, \tag{22}$$

as the only valid sequence is the one without any spike.

Remark. *For fixed $T_r > 0$ and for any $n \geq 0$, we have:*

$$\ell^{n+1}_{T_r} > \ell^n_{T_r}. \tag{23}$$

Remark. *For fixed $n \geq 0$ and for any $T_r > 0$, we have:*

$$\ell^n_{T_r+1} < \ell^n_{T_r}. \tag{24}$$

Remark (Capacity). *The capacity C of a channel transmitting signals in $\mathcal{Y}^n_{T_r}$ is:*

$$C \triangleq \lim_{n\to\infty} \frac{\log \ell^n_{T_r+1}}{n} = \log \hat{\varphi}_{T_r+1}. \tag{25}$$

Given a sample of $\mathcal{Y}^n_{T_r}$, it could also be interesting to know the probability that it contains exactly n_0 zeros and n_1 ones. Let $\mathcal{Y}^{n_0,n_1}_{T_r}$ represent the set of sequences in $\mathcal{Y}^n_{T_r}$ with n_0 zeros and n_1 ones, $n_0 + n_1 = n$, $n_0, n_1 \geq 0$. It can be formally expressed as

$$\mathcal{Y}^{n_0,n_1}_{T_r} = \left\{ (y_1, \ldots, y_n) \in \mathcal{Y}^n_{T_r} : \sum_{i=1}^{n} y_i = n_1 \right\}, \tag{26}$$

with $n = n_0 + n_1 > 0$.

Theorem 2.4. *Let $\ell^{n_0,n_1}_{T_r}$ denotes the cardinality of $\mathcal{Y}^{n_0,n_1}_{T_r}$ with $n_0, n_1 \in \mathbb{Z}_{\geq 0}$ and $T_r \in \mathbb{Z}_{>0}$. We have:*

$$\ell^{n_0,n_1}_{T_r} = \begin{cases} 0 & \text{if } n_0 < T_r n_1 \\ 1 & \text{if } n_1 = 0 \\ n & \text{if } n_1 = 1 \\ \binom{n-T_r n_1 + T_r}{n_1} - \sum_{k=2}^{T_r+1} \binom{n-T_r(n_1-1)-k}{n_1-2}(k-1) & \text{otherwise} \end{cases}, \tag{27}$$

Proof. First of all, it is obvious that if $n_0 < T_r n_1$ we have $\ell^{n_0,n_1}_{T_r} = 0$. Let's now consider the non trivial cases with n_0, n_1 such that $n_0 \geq T_r n_1$ If $n_1 = 0$, there is 1 sequence in $\mathcal{Y}^{n_0,n_1}_{T_r}$ with the form

$$\overbrace{0 \cdots 0}^{n}$$

If $n_1 = 1$, there are n sequence in $\mathcal{Y}^{n_0,n_1}_{T_r}$ with the form

$$\overbrace{0 \cdots 0}^{k_1} 1 \overbrace{0 \cdots 0}^{k_2},$$

with $k_1 + k_2 = n - 1 = n_0 \geq T_r$.

If $n_1 \geq 2$, we have to consider sequences that are in $\tilde{\mathcal{Y}}^{n_0,n_1}_{T_r} \setminus \mathcal{Y}^{n_0,n_1}_{T_r}$ and then compute $\ell^{n_0,n_1}_{T_r}$ using $\left|\mathcal{Y}^{n_0,n_1}_{T_r}\right| = \left|\tilde{\mathcal{Y}}^{n_0,n_1}_{T_r}\right| - \left|\tilde{\mathcal{Y}}^{n_0,n_1}_{T_r} \setminus \mathcal{Y}^{n_0,n_1}_{T_r}\right|$. If $n_1 = 2$, there are $\sum_{k=0}^{T_r-1} k+1$ sequences in $\tilde{\mathcal{Y}}^{n_0,n_1}_{T_r} \setminus \mathcal{Y}^{n_0,n_1}_{T_r}$ with the form

$$\overbrace{0 \cdots 0}^{k_1} 1 \overbrace{0 \cdots 0}^{\geq T_r} 1 \overbrace{0 \cdots 0}^{k_2},$$

with $k_1, k_2 \geq 0$ and $0 \leq k_1 + k_2 \leq T_r - 1$.

If $n_1 > 2$, any sequence in $\tilde{\mathcal{Y}}_{T_r}^{n_0,n_1} \setminus \mathcal{Y}_{T_r}^{n_0,n_1}$ has the form

$$\overbrace{0\cdots 0}^{k_1} 1 \overbrace{0\cdots 0}^{T_r} \underbrace{y_{k_1+T_r+1}\cdots y_{n-k_2-T_r}}_{\in\tilde{\mathcal{Y}}_{T_r}^{n_0-2T_r-k+2,n_1-2}} \overbrace{0\cdots 0}^{T_r} 1 \overbrace{0\cdots 0}^{k_2}, \tag{28}$$

with $k_1, k_2 \geq 0$ and $0 \leq k_1 + k_2 \leq T_r - 1$. Hence, there are

$$\sum_{k=0}^{T_r-1} \tilde{\ell}_{T_r}^{n_0-2T_r-k+2,n_1-2}(k+1) = \sum_{k=2}^{T_r+1} \binom{n - T_r(n_1-1) - k}{n_1 - 2}(k+1). \tag{29}$$

such sequences.

Now it suffices to notice that $\tilde{\ell}_{T_r}^{n_0-2T_r-k+2,0} = 1$ to merge case $n_1 = 2$ with case $n_1 > 2$ and complete the proof. $\square$

	n_1											
n	0	1	2	3	4	5	6	7	8	9	10	$\ell^n_{T_r}$
1	1	0	-	-	-	-	-	-	-	-	-	1
2	1	2	0	-	-	-	-	-	-	-	-	3
3	1	3	0	0	-	-	-	-	-	-	-	4
4	1	4	2	0	0	-	-	-	-	-	-	7
5	1	5	5	0	0	0	-	-	-	-	-	11
6	1	6	9	2	0	0	0	-	-	-	-	18
7	1	7	14	7	0	0	0	0	-	-	-	29
8	1	8	20	16	2	0	0	0	0	-	-	47
9	1	9	27	30	9	0	0	0	0	0	-	76
10	1	10	35	50	25	2	0	0	0	0	0	123

Table 6: $\ell^{n_0,n_1}_{T_r}$ for $0 < n \leq 10$, $0 \leq n_1 \leq n$ and $T_r = 1$.

	n_1											
n	0	1	2	3	4	5	6	7	8	9	10	$\ell^n_{T_r}$
1	1	0	-	-	-	-	-	-	-	-	-	1
2	1	0	0	-	-	-	-	-	-	-	-	1
3	1	3	0	0	-	-	-	-	-	-	-	4
4	1	4	0	0	0	-	-	-	-	-	-	5
5	1	5	0	0	0	0	-	-	-	-	-	6
6	1	6	3	0	0	0	0	-	-	-	-	10
7	1	7	7	0	0	0	0	0	-	-	-	15
8	1	8	12	0	0	0	0	0	0	-	-	21
9	1	9	18	3	0	0	0	0	0	0	-	31
10	1	10	25	10	0	0	0	0	0	0	0	46

Table 7: $\ell^{n_0,n_1}_{T_r}$ for $0 < n \leq 10$, $0 \leq n_1 \leq n$ and $T_r = 2$.

	n_1											
n	0	1	2	3	4	5	6	7	8	9	10	$\ell^n_{T_r}$
1	1	0	-	-	-	-	-	-	-	-	-	1
2	1	0	0	-	-	-	-	-	-	-	-	1
3	1	0	0	0	-	-	-	-	-	-	-	1
4	1	4	0	0	0	-	-	-	-	-	-	5
5	1	5	0	0	0	0	-	-	-	-	-	6
6	1	6	0	0	0	0	0	-	-	-	-	7
7	1	7	0	0	0	0	0	0	-	-	-	8
8	1	8	4	0	0	0	0	0	0	-	-	13
9	1	9	9	0	0	0	0	0	0	0	-	19
10	1	10	15	0	0	0	0	0	0	0	0	26

Table 8: $\ell^{n_0,n_1}_{T_r}$ for $0 < n \leq 10$, $0 \leq n_1 \leq n$ and $T_r = 3$.

A Generalized golden number

Definition A.1 (Generalized Golden Number of order p). *Let $\varphi_{p,i} \in \mathbb{C}$ be the p complex roots of the polynomials $P(x) = x^p - x^{p-1} - 1$ with $p \in \mathbb{Z}_{\geq 2}$ and $i \in \{1, \dots, p\}$. The largest real root of $P(x)$ is called the* Generalized Golden Number of order p.

Remark. *Moreover, from the complex conjugate root Theorem, since P is a polynomial in one variable with real coefficients, the roots are real or come in complex conjugate pairs.*

Remark. *From Definition A.1, it follows that for any $p_1, p_2 > 2$, the following holds:*

$$\varphi_{p_1,i}^{p_1} - \varphi_{p_1,i}^{p_1-1} = \varphi_{p_1,i}^{p_1-1}(\varphi_{p_1,i} - 1) = \varphi_{p_2,i}^{p_2-1}(\varphi_{p_2,i} - 1) = \varphi_{p_2,i}^{p_2} - \varphi_{p_2,i}^{p_2-1}, \tag{30}$$

Theorem A.1. *Any root of $P(x)$ has multiplicity one.*

Proof. Let $\varphi_{p,i}$ be a root of $P(x)$ and assume it has multiplicity $l > 1$. Then, $\varphi_{p,i}$ should also be a root of $P'(x)$, that is:

$$P'(\varphi_{p,i}) = p\varphi_{p,i}^{p-1} - (p-1)\varphi_{p,i}^{p-2} = 0. \tag{31}$$

Using the fact that $\varphi_{p,i} \neq 0$, for any $p > 1$ and $i \in \{1, \dots, p\}$, we get:

$$\varphi_{p,i} = \frac{p-1}{p}, \tag{32}$$

It yields:

$$P(\varphi_{p,i}) = \left(\frac{p-1}{p}\right)^p - \left(\frac{p-1}{p}\right)^{p-1} - 1 < 0, \tag{33}$$

which is in contradiction with $P(\varphi_{p,i}) = 0$ and the proof is complete. $\square$

Theorem A.2. *Let $\varphi_{p,i} \in \mathbb{C}$ with $p \in \mathbb{Z}_{\geq 2}$ and $i \in \{1, \dots, p\}$ be as in Definition A.1. Then, the following hold:*

$$\sum_{i=1}^{p} \varphi_{p,i} = 1, \tag{34}$$

$$\prod_{i=1}^{p} \varphi_{p,i} = (-1)^{p-1}, \tag{35}$$

$$\sum_{1 \leq i_1 < i_2 < \dots < i_{p-m} \leq p} \prod_{j=1}^{p-m} \varphi_{p,i_j} = 0, \tag{36}$$

with $0 < m < p - 1$.

Proof. Using Vieta's formulas with $P(x) = x^n - x^{n-1} - 1$, we directly get the result. $\square$

Corollary A.3. *Let $\varphi_{p,i} \in \mathbb{C}$ with $p \in \mathbb{Z}_{\geq 2}$ and $i \in \{1, \dots, p\}$ be as in Definition A.1 and $0 < m < p$. Then, the following hold:*

$$(-1)^{p-m}\varphi_{p,i}^m \sum_{\substack{1 \leq i_1 < i_2 < \dots < i_{p-m} \leq p \\ i_1, i_2, \dots, i_{p-1-m} \neq i}} \prod_{j=1}^{p-m} \varphi_{p,i_j} = 1 \tag{37}$$

Proof. We prove it by induction. First of all, for $m = p - 1$, we have:

$$\begin{aligned} \varphi_{p,i}^{p-1} \sum_{\substack{j=1 \\ j \neq i}}^{p} \varphi_{p,i_j} &= \varphi_{p,i}^{p-1}(1 - \varphi_{p,i}) && (38) \\ &= \varphi_{p,i}^{p-1} - \varphi_{p,i}^{p} && (39) \\ &= -1 && (40) \\ & && (41) \end{aligned}$$

Assume, the property is true for $m+1 \leq p-1$ and prove it is also true for $m > 0$. We have:

$$\begin{aligned}
\varphi_{p,i}^{m} \sum_{\substack{1 \leq i_1 < i_2 < \cdots < i_{p-m} \leq p \\ i_1, i_2, \ldots, i_{p-1-m} \neq i}} \prod_{j=1}^{p-m} \varphi_{p,i_j} &= -\varphi_{p,i}^{m} \sum_{\substack{1 \leq i_1 < i_2 < \cdots < i_{p-m} \leq p \\ i \in \{i_1, i_2, \ldots, i_{p-m}\}}} \prod_{j=1}^{p-m} \varphi_{p,i_j} && (42) \\
&= -\varphi_{p,i}^{m+1} \sum_{\substack{1 \leq i_1 < i_2 < \cdots < i_{p-m-1} \leq p \\ i_1, i_2, \ldots, i_{p-m-1} \neq i}} \prod_{j=1}^{p-m-1} \varphi_{p,i_j} && (43) \\
&= -(-1)^{p-m-1} && (44) \\
&= (-1)^{p-m}, && (45)
\end{aligned}$$

and the proof is complete. $\square$

Corollary A.4. *Let $\varphi_{p,i} \in \mathbb{C}$ with $p \in \mathbb{Z}_{\geq 2}$ and $i \in \{1, \ldots, p\}$ be as in Definition A.1. Then, the following hold:*

$$\sum_{i=1}^{p} \varphi_{p,i}^{p-1} = 1 \tag{46}$$

Proof. If $p = 2$, we have $\sum_{i=1}^{p} \varphi_{p,i}^{p-1} = \sum_{i=1}^{p} \varphi_{p,i} = 1$. Now, assuming $p > 2$, we have:

$$\begin{aligned}
\left(\sum_{i=1}^{p} \varphi_{p,i}\right)^{p-1} &= (p-2)! \sum_{i=1}^{p} \varphi_{p,i} \sum_{\substack{1 \leq i_1 < \cdots < i_{p-2} \leq p \\ i_1, \ldots, i_{p-2} \neq i}} \prod_{j=1}^{p-2} \varphi_{p,i_j} && (47) \\
&+ \sum_{m=2}^{p-2} \sum_{i=1}^{p} \frac{(p-1)!}{m!} \varphi_{p,i}^{m} \sum_{\substack{1 \leq i_1 < \cdots < i_{p-1-m} \leq p \\ i_1, \ldots, i_{p-1-m} \neq i}} \prod_{j=1}^{p-1-m} \varphi_{p,i_j} \\
&+ \sum_{i=1}^{p} \varphi_{p,i}^{p-1} \\
&= (p-2)! \sum_{i=1}^{p} \sum_{\substack{1 \leq i_1 < \cdots < i_{p-1} \leq p \\ i \in \{i_1, \ldots, i_{p-1}\}}} \prod_{j=1}^{p-1} \varphi_{p,i_j} && (48) \\
&+ \sum_{m=2}^{p-2} \frac{(p-1)!}{m!} \sum_{i=1}^{p} \varphi_{p,i}^{m-1} \sum_{\substack{1 \leq i_1 < \cdots < i_{p-m} \leq p \\ i \in \{i_1, \ldots, i_{p-m}\}}} \prod_{j=1}^{p-m} \varphi_{p,i_j} \\
&+ \sum_{i=1}^{p} \varphi_{p,i}^{p-1} \\
&= (p-2)! \sum_{1 \leq i_1 < \cdots < i_{p-1}} \prod_{j=1}^{p-1} \varphi_{p,i_j} && (49) \\
&+ \sum_{m=2}^{p-2} \frac{(p-1)!}{m!} \varphi_{p,i}^{m-1} \sum_{1 \leq i_1 < \cdots < i_{p-m} \leq p} \prod_{j=1}^{p-m} \varphi_{p,i_j} \\
&+ \sum_{i=1}^{p} \varphi_{p,i}^{p-1} \\
&= \sum_{i=1}^{p} \varphi_{p,i}^{p-1}, && (50)
\end{aligned}$$

It suffices now to notice that $\sum_{i=1}^{p} \varphi_{p,i}^{p-1} = \left(\sum_{i=1}^{p} \varphi_{p,i}\right)^{p-1}$ to complete the proof. $\square$

Corollary A.5. *Let $\varphi_{p,i} \in \mathbb{C}$ with $p \in \mathbb{Z}_{\geq 2}$ and $i \in \{1, \dots, p\}$ be as in Definition A.1. Then, the following hold:*

$$\varphi_{p,i} \prod_{\substack{j=1 \\ j \neq i}}^{p} (\varphi_{p,i} - \varphi_{p,j}) = \varphi_{p,i}^{p} + p - 1 = p + \varphi_{p,i}^{p-1}. \tag{51}$$

Proof. We can directly develop the product to get the desire result:

$$\begin{aligned}
\varphi_{p,i} \prod_{\substack{j=1 \\ j \neq i}}^{p} (\varphi_{p,i} - \varphi_{p,j}) &= \varphi_{p,i}^{p} + \sum_{m=1}^{p-1} (-1)^{p-m} \varphi_{p,i}^{m} \sum_{\substack{1 \leq i_1 < \cdots < i_{p-m} \leq p \\ i_1, \dots, i_{p-m} \neq i}} \prod_{j=1}^{p-m} \varphi_{p,i_j} && (52) \\
&= \varphi_{p,i}^{p} + \sum_{m=1}^{p-1} 1 && (53) \\
&= \varphi_{p,i}^{p} + p - 1 && (54) \\
&= \varphi_{p,i}^{p-1} + p. && (55)
\end{aligned}$$

□

Corollary A.6. *Let $\varphi_{p,i} \in \mathbb{C}$ with $p \in \mathbb{Z}_{\geq 2}$ and $i \in \{1, \dots, p\}$ be as in Definition A.1. Then, the following hold:*

$$\prod_{i=1}^{p} (1 - \varphi_{p,i}) = -1. \tag{56}$$

Proof. We can directly develop the product to get the desire result:

$$\begin{aligned}
\prod_{i=1}^{p} (1 - \varphi_{p,i}) &= \sum_{m=0}^{p} (-1)^{p-m} \sum_{1 \leq i_1 < \cdots < i_{p-m} \leq p} \prod_{j=1}^{p-m} \varphi_{p,i_j} && (57) \\
&= (-1)^{p} \prod_{i}^{p} \varphi_{p,i} && (58) \\
&\quad + \sum_{m=1}^{p-2} (-1)^{m} \sum_{1 \leq i_1 < \cdots < i_{p-m} \leq p} \prod_{j=1}^{p-m} \varphi_{p,i_j} \\
&\quad - \sum_{i}^{p} \varphi_{p,i} \\
&\quad + 1 \\
&= (-1)^{p} \prod_{i}^{p} \varphi_{p,i} && (59) \\
&= -1. && (60)
\end{aligned}$$

□

Corollary A.7. *Let $\varphi_{p,i} \in \mathbb{C}$ with $p \in \mathbb{Z}_{\geq 2}$ and $i \in \{1, \dots, p\}$ be as in Definition A.1. Then, the following holds:*

$$\sum_{i=1}^{p} \frac{\varphi_{p,i}^{2p-1}}{\varphi_{p,i}^{p} + p - 1} = 1. \tag{61}$$

Proof. Proved in Theorem 1.2. □

Corollary A.8. *Let $\varphi_{p,i} \in \mathbb{C}$ with $p \in \mathbb{Z}_{\geq 2}$ and $i \in \{1, \dots, p\}$ be as in Definition A.1. Then, the following holds:*

$$\sum_{i=1}^{p} \frac{\varphi_{p,i}^{3p-2}}{\varphi_{p,i}^{p} + p - 1} = p. \tag{62}$$

Proof. Proved in Theorem 2.2. $\square$

Of particular interest in our case is the distribution of $\varphi_{p,i}$ for $i \in \{1, \ldots, p\}$ in the complex plan.

Theorem A.9. *Let $\varphi_{p,1}$, $\varphi_{p,2}$ be as in Definition A.1 with $p \in \mathbb{Z}_{\geq 2}$ and such that $\varphi_{p,1} = re^{i\theta_1}$ and $\varphi_{p,2} = re^{i\theta_2}$ with $r \in \mathbb{R}_{>0}$ and $\theta_1 \neq \theta_2$. Then we necessarily have $\theta_1 = -\theta_2$.*

Proof. From Definition A.1, we have

$$r^p e^{i\theta_1 p} - r^{p-1} e^{i\theta_1(p-1)} = r^p e^{i\theta_2 p} - r^{p-1} e^{i\theta_2(p-1)} \tag{63}$$

Now using the fact that $\theta_1 \neq \theta_2$, we have

$$r = \frac{e^{i\theta_1(p-1)} - e^{i\theta_2(p-1)}}{e^{i\theta_1 p} - e^{i\theta_2 p}} = \frac{e^{i\frac{\theta_1+\theta_2}{2}(p-1)}\left(e^{i\frac{\theta_1-\theta_2}{2}(p-1)} - e^{-i\frac{\theta_1-\theta_2}{2}(p-1)}\right)}{e^{i\frac{\theta_1+\theta_2}{2}p}\left(e^{i\frac{\theta_1-\theta_2}{2}p} - e^{-i\frac{\theta_1-\theta_2}{2}p}\right)} \tag{64}$$

After some calculations, we can verify that $\arg(r) = 0$ (because $r \in \mathbb{R}_{>0}$) implies that $\theta_1 = -\theta_2$. $\square$

Remark. *A direct consequence of this theorem is that there cannot exist more than two roots with the same modulus. If we are interested in the set of roots with the largest modulus, then either this set contains only one real number, either it contains a pair of complex conjugate numbers.*

Theorem A.10 (Existence and uniqueness of the Golden Number of order p)**.** *Let $\varphi_{p,i} \in \mathbb{C}$ with $p \in \mathbb{Z}_{\geq 2}$ and $i \in \{1, \ldots, p\}$ be as in Definition A.1, and denote by $\hat{\varphi}_p$ the root with the largest modulus. Then, the following hold:*

1. *$\hat{\varphi}_p$ exists;*
2. *$\hat{\varphi}_p$ is unique;*
3. *$\hat{\varphi}_p$ is a stricly positive real number.*

Proof. The fact that $\hat{\varphi}_p$ exists is obvious.

From Theorem A.9, it follows that either $\hat{\varphi}_p$ is real, either it defines a pair of complex conjugate numbers. From Theorem 2.2, as n goes large and with $p = T_r + 1$, either $\ell^n_{p-1} \simeq (\hat{\varphi}_p)^n + \left(\overline{\hat{\varphi}_p}\right)^n$ if $\hat{\varphi}_p$ defines a pair of complex conjugate numbers either $\ell^n_{p-1} \simeq (\hat{\varphi}_p)^n$ if $\hat{\varphi}_p$ is real.

But since $\ell^n_{p-1} > 0$ for any $n > 0$, we also should have $(\hat{\varphi}_p)^n + \left(\overline{\hat{\varphi}_p}\right)^n = 2\hat{r}^n \cos n\hat{\theta} > 0$ which is not true for an arbitrary n as $\hat{\theta} \bmod \pi \neq 0$. In the same manner, we cannot have $\hat{\varphi}_p \in \mathbb{R}_{<0}$ as $(\hat{\varphi}_p)^n = (-1)^n |\hat{\varphi}_p|^n$ in this case which can be negative. Hence, necessarily, $\hat{\varphi}_p \in \mathbb{R}_{>0}$ and we have proven that $\hat{\varphi}_p$ is unique and a strictly positive real number. $\square$

Remark. *It follows from Theorem A.10 that for any $p > 1$, $|\hat{\varphi}_p| = \hat{\varphi}_p$*

Theorem A.11. *Let $\hat{\varphi}_p \in \mathbb{R}_{>0}$ with $p \in \mathbb{Z}_{\geq 2}$ be as in Theorem A.10. Then, the following holds:*

$$\hat{\varphi}_p > 1. \tag{65}$$

Proof. Using Theorem A.2, we have:

$$\hat{\varphi}_p^p = |\hat{\varphi}_p|^p = \left(\max_{i\in\{1,\ldots,p\}} |\varphi_{p,i}|\right)^p \geq \prod_{i=1}^{p} |\varphi_{p,i}| = \left|\prod_{i=1}^{p} \varphi_{p,i}\right| = 1, \tag{66}$$

and thus:

$$\hat{\varphi}_p \geq 1^{1/p} = 1, \tag{67}$$

Noticing that for any $p > 1$, $\hat{\varphi}_p \neq 1$ as $1^p - 1^{p-1} - 1 = -1 \neq 0$, we complete the proof. $\square$

p	$\hat{\varphi}_p$	p	$\hat{\varphi}_p$	p	$\hat{\varphi}_p$	p	$\hat{\varphi}_p$
1	2	51	1.058439	101	1.034301	151	1.024919
2	1.618034	52	1.057571	102	1.034035	152	1.024788
3	1.465571	53	1.056732	103	1.033774	153	1.024659
4	1.380278	54	1.05592	104	1.033517	154	1.02453
5	1.324718	55	1.055133	105	1.033265	155	1.024404
6	1.285199	56	1.054371	106	1.033017	156	1.024278
7	1.255423	57	1.053632	107	1.032772	157	1.024154
8	1.232055	58	1.052915	108	1.032532	158	1.024032
9	1.21315	59	1.052219	109	1.032296	159	1.023911
10	1.197491	60	1.051544	110	1.032063	160	1.023791
11	1.184276	61	1.050888	111	1.031834	161	1.023672
12	1.172951	62	1.05025	112	1.031609	162	1.023555
13	1.16312	63	1.04963	113	1.031387	163	1.023439
14	1.154494	64	1.049027	114	1.031169	164	1.023325
15	1.146854	65	1.04844	115	1.030954	165	1.023211
16	1.140034	66	1.047869	116	1.030742	166	1.023099
17	1.133902	67	1.047312	117	1.030534	167	1.022988
18	1.128356	68	1.04677	118	1.030328	168	1.022878
19	1.123311	69	1.046241	119	1.030126	169	1.022769
20	1.118699	70	1.045726	120	1.029927	170	1.022662
21	1.114465	71	1.045223	121	1.02973	171	1.022555
22	1.110562	72	1.044733	122	1.029536	172	1.02245
23	1.10695	73	1.044254	123	1.029345	173	1.022346
24	1.103598	74	1.043787	124	1.029157	174	1.022242
25	1.100476	75	1.04333	125	1.028972	175	1.02214
26	1.097561	76	1.042884	126	1.028789	176	1.022039
27	1.094833	77	1.042448	127	1.028608	177	1.021939
28	1.092272	78	1.042022	128	1.02843	178	1.02184
29	1.089863	79	1.041605	129	1.028255	179	1.021742
30	1.087593	80	1.041198	130	1.028082	180	1.021644
31	1.08545	81	1.040799	131	1.027911	181	1.021548
32	1.083422	82	1.040409	132	1.027743	182	1.021453
33	1.0815	83	1.040027	133	1.027576	183	1.021358
34	1.079675	84	1.039653	134	1.027412	184	1.021265
35	1.077941	85	1.039286	135	1.02725	185	1.021172
36	1.07629	86	1.038927	136	1.027091	186	1.021081
37	1.074717	87	1.038576	137	1.026933	187	1.02099
38	1.073215	88	1.038231	138	1.026777	188	1.0209
39	1.071779	89	1.037893	139	1.026623	189	1.020811
40	1.070406	90	1.037562	140	1.026472	190	1.020723
41	1.069091	91	1.037237	141	1.026322	191	1.020635
42	1.06783	92	1.036918	142	1.026174	192	1.020548
43	1.066619	93	1.036606	143	1.026027	193	1.020463
44	1.065457	94	1.036299	144	1.025883	194	1.020378
45	1.064339	95	1.035998	145	1.02574	195	1.020293
46	1.063264	96	1.035702	146	1.025599	196	1.02021
47	1.062228	97	1.035412	147	1.02546	197	1.020127
48	1.06123	98	1.035127	148	1.025323	198	1.020045
49	1.060267	99	1.034846	149	1.025187	199	1.019964
50	1.059337	100	1.034571	150	1.025052	200	1.019883

Table 9: Golden number $\hat{\varphi}_p$ for various p.

p	Duration
10	$41.1\,\mu\text{s} \pm 724\,\text{ns}$
100	$1.02\,\text{ms} \pm 10.9\,\mu\text{s}$
1000	$1.5\,\text{s} \pm 46.3\,\text{ms}$

Table 10: Computation time to find the p roots of $P(x) = x^p - x^{p-1} - 1$.

Theorem A.12. *Let $\hat{\varphi}_p \in \mathbb{R}_{>0}$ with $p \in \mathbb{Z}_{\geq 2}$ be as in Theorem A.10. Then, the following holds:*

$$\hat{\varphi}_p > \hat{\varphi}_{p+1}. \tag{68}$$

Proof. As previously mentionned, we have:

$$\hat{\varphi}_{p+1}^p(\hat{\varphi}_{p+1} - 1) = \hat{\varphi}_p^{p-1}(\hat{\varphi}_p - 1) \tag{69}$$

Now assume $\hat{\varphi}_{p+1} = \hat{\varphi}_p + \varepsilon$, with $\varepsilon \geq 0$ and let's show it yields to a contradiction. We have:

$$\begin{aligned}
\hat{\varphi}_{p+1}^p(\hat{\varphi}_{p+1} - 1) &= (\hat{\varphi}_p + \varepsilon)^p(\hat{\varphi}_p + \varepsilon - 1) && (70)\\
&\geq \hat{\varphi}_p^p(\hat{\varphi}_p + \varepsilon - 1) && (71)\\
&> \hat{\varphi}_p^{p-1}(\hat{\varphi}_p + \varepsilon - 1) && (72)\\
&\geq \hat{\varphi}_p^{p-1}(\hat{\varphi}_p - 1) && (73)
\end{aligned}$$

where we used Theorem A.11 in the third step. This is a contradiction with $\hat{\varphi}_{p+1}^p(\hat{\varphi}_{p+1} - 1) = \hat{\varphi}_p^{p-1}(\hat{\varphi}_p - 1)$. Hence we necessarily have $\hat{\varphi}_{p+1} = \hat{\varphi}_p + \varepsilon$, with $\varepsilon < 0$, or in other terms, $\hat{\varphi}_{p+1} < \hat{\varphi}_p$. □

Remark. *It follows that for any $p > 2$, $\hat{\varphi}_p < \hat{\varphi}_2 = \frac{1+\sqrt{5}}{2}$.*

B Z-Transforms

Theorem B.1. *Let $\tilde{\ell}_{T_r}^n$ be as introduced in Theorem 1.1, with $T_r > 0$ and $n \geq 0$. Its Z-Transform satisfies:*

$$\frac{\tilde{\mathcal{L}}_{T_r}(z)}{z} = \frac{\tilde{A}}{z} + \sum_{i=1}^{T_r+1} \frac{\tilde{B}_i}{z - \varphi_{T_r+1,i}}. \tag{74}$$

with

$$\tilde{A} = -1, \tag{75}$$

$$\tilde{B}_i = \frac{\varphi_{T_r+1,i}^{2T_r+1}}{\varphi_{T_r+1,i}^{T_r+1} + T_r}, \tag{76}$$

and $\varphi_{T_r+1,i}$ with $i \in \{1, \ldots, T_r + 1\}$ as defined above.

Proof. By directly computing the Z-Transform of Theorem 1.1, we get

$$\begin{aligned}
\tilde{\mathcal{L}}_{T_r}(z) &\triangleq \sum_{n\geq 0} \tilde{\ell}_{T_r}^n z^{-n} && (77)\\
&= 2z^{-1} + \sum_{n=2}^{T_r+1} z^{-n} + \sum_{n\geq 2} \tilde{\ell}_{T_r}^{n-1} z^{-n} + \sum_{n\geq T_r+2} \tilde{\ell}_{T_r}^{n-1-T_r} z^{-n} && (78)\\
&= z^{-1} + \sum_{n=1}^{T_r+1} z^{-n} + z^{-1}\sum_{n\geq 1} \tilde{\ell}_{T_r}^n z^{-n} + z^{-1-T_r}\sum_{n\geq 1} \tilde{\ell}_{T_r}^n z^{-n} && (79)\\
&= z^{-1} + z^{-T_r-1}\frac{1 - z^{T_r+1}}{1 - z} + z^{-1}\tilde{\mathcal{L}}_{T_r}(z) + z^{-1-T_r}\tilde{\mathcal{L}}_{T_r}(z), && (80)
\end{aligned}$$

and then

$$\frac{\tilde{\mathcal{L}}_{T_r}(z)}{z} = \frac{z^{T_r} + \frac{1-z^{T_r+1}}{1-z}}{z\prod_{i=1}^{T_r+1}(z-\varphi_{T_r+1,i})}, \tag{81}$$

where $\varphi_{T_r+1,i}$ for $i \in \{1,\dots,T_r+1\}$ as in Definition A.1. Since every poles have multiplicity one and the degree of the denominator is larger than the degree of the numerator, we can make partial fraction decomposition to get:

$$\frac{\tilde{\mathcal{L}}_{T_r}(z)}{z} = \frac{\tilde{A}}{z} + \sum_{i=1}^{T_r+1}\frac{\tilde{B}_i}{z-\varphi_{T_r+1,i}}, \tag{82}$$

Using Theorem A.2, we obtain

$$\tilde{A} = z\frac{\tilde{\mathcal{L}}_{T_r}(z)}{z}\bigg|_{z=0} = \frac{(-1)^{T_r+1}}{\prod_{i=1}^{T_r+1}\varphi_{T_r+1,i}} = -1. \tag{83}$$

Using Definition A.1 and Corollary A.5, we get

$$\begin{aligned}
\tilde{B}_i &= (z-\varphi_{T_r+1,i})\frac{\tilde{\mathcal{L}}_{T_r}(z)}{z}\bigg|_{z=\varphi_{T_r+1,i}} && (84)\\
&= \frac{\varphi_{T_r+1,i}^{T_r} + \frac{1-\varphi_{T_r+1,i}^{T_r+1}}{1-\varphi_{T_r+1,i}}}{\varphi_{T_r+1,i}\prod_{\substack{j=1\\ j\neq i}}^{T_r+1}(\varphi_{T_r+1,i}-\varphi_{T_r+1,j})} && (85)\\
&= \frac{\varphi_{T_r+1,i}^{T_r} - \frac{\varphi_{T_r+1,i}^{T_r}}{1-\varphi_{T_r+1,i}}}{\varphi_{T_r+1,i}^{T_r+1}+T_r} && (86)\\
&= \frac{\varphi_{T_r+1,i}^{T_r} - \varphi_{T_r+1,i}^{T_r+1} - \varphi_{T_r+1,i}^{T_r}}{(\varphi_{T_r+1,i}^{T_r+1}+T_r)(1-\varphi_{T_r+1,i})} && (87)\\
&= \frac{\varphi_{T_r+1,i}^{T_r+1}}{(\varphi_{T_r+1,i}^{T_r+1}+T_r)\varphi_{T_r+1,i}^{-T_r}} && (88)\\
&= \frac{\varphi_{T_r+1,i}^{2T_r+1}}{\varphi_{T_r+1,i}^{T_r+1}+T_r}, && (89)
\end{aligned}$$

for $i \in \{1,\dots,T_r+1\}$. □

Theorem B.2. *Let $\ell_{T_r}^n$ be as introduced in Theorem 2.1, with $T_r > 0$ and $n \geq 0$. Its Z-Transform satisfies:*

$$\begin{aligned}
\mathcal{L}_{T_r}(z) = {}& \frac{z}{z-1}(z^{-1}+T_r z^{-T_r-1}) && (90)\\
&+ \frac{z}{(z-1)^2}(z^{-T_r} - z^{-2T_r-1} + T_r z^{-2T_r-2} - T_r z^{-3T_r-2})\\
&+ \frac{\tilde{\mathcal{L}}_{T_r}(z)}{z-1}(z^{-2T_r} + T_r z^{-3T_r-1}).
\end{aligned}$$

Moreover, we have:

$$\bar{\ell}_{T_r}^n \triangleq \mathcal{Z}^{-1}\left\{\frac{\tilde{\mathcal{L}}_{T_r}(z)}{z-1}\right\}[n] = \begin{cases} \sum_{i=1}^{T_r+1}\frac{\varphi_{T_r+1,i}^{3T_r+1+n}}{\varphi_{T_r+1,i}^{T_r+1}+T_r} - (T_r+2) & \text{if } n > 0\\ 0 & \text{otherwise}\end{cases}, \tag{91}$$

with $\mathrm{ROC}(\bar{\ell}_{T_r}^n) = \{z \in \mathbb{C} : |z| > \max_{i\in\{1,\dots,T_r+1\}}|\varphi_{T_r+1,i}|\}$.

Proof. By directly computing the Z-Transform of Theorem 2.1, we get

$$
\begin{aligned}
\mathcal{L}_{T_r}(z) &\triangleq \sum_{n\geq 0} \ell_{T_r}^n z^{-n} && (92)\\
&= \sum_{n\geq 1} \ell_{T_r}^{n-1} z^{-n} + \sum_{n\geq 2T_r+2} \tilde{\ell}_{T_r}^{n-2T_r-1} z^{-n} + T_r \sum_{n\geq 3T_r+3} \tilde{\ell}_{T_r}^{n-3T_r-2} z^{-n} && (93)\\
&+ T_r z^{-T_r-1} + z^{-1} + \sum_{n=T_r+1}^{2T_r+1} z^{-n} + T_r \sum_{n=2T_r+2}^{3T_r+2} z^{-n}\\
&= z^{-1} \sum_{n\geq 1} \ell_{T_r}^n z^{-n} + \left(z^{-2T_r-1} + T_r z^{-3T_r-2}\right) \sum_{n\geq 1} \tilde{\ell}_{T_r}^n z^{-n} && (94)\\
&+ \sum_{n=T_r+1}^{2T_r+1} z^{-n} + T_r z^{-T_r-1} + z^{-1} + T_r \sum_{n=2T_r+2}^{3T_r+2} z^{-n}\\
&= z^{-1}\mathcal{L}_{T_r}(z) + \left(z^{-2T_r-1} + T_r z^{-3T_r-2}\right) \tilde{\mathcal{L}}_{T_r}(z) && (95)\\
&+ T_r z^{-T_r-1} + z^{-1} + \frac{z^{-T_r} - z^{-2T_r-1}}{z-1} + T_r \frac{z^{-2T_r-1} - z^{-3T_r-2}}{z-1}\\
& && (96)
\end{aligned}
$$

and thus:

$$
\begin{aligned}
\mathcal{L}_{T_r}(z) = &\frac{z}{z-1}(z^{-1} + T_r z^{-T_r-1}) && (97)\\
&+ \frac{z}{(z-1)^2}(z^{-T_r} - z^{-2T_r-1} + T_r z^{-2T_r-1} - T_r z^{-3T_r-2})\\
&+ \frac{\tilde{\mathcal{L}}_{T_r}(z)}{z-1}(z^{-2T_r} + T_r z^{-3T_r-1}).
\end{aligned}
$$

Let's now compute the following inverse Z-Transform

$$
\bar{\ell}_{T_r}^n \triangleq \mathcal{Z}^{-1}\left\{\frac{\tilde{\mathcal{L}}_{T_r}(z)}{z-1}\right\}[n]. \tag{98}
$$

Using Theorem B.1, we have:

$$
\frac{\tilde{\mathcal{L}}_{T_r}(z)}{z(z-1)} = \frac{A}{z} + \sum_{i=1}^{T_r+1} \frac{B_i}{z - \varphi_{T_r+1,i}} + \frac{C}{z-1}, \tag{99}
$$

with

$$
A = z\frac{\tilde{\mathcal{L}}_{T_r}(z)}{z(z-1)}\bigg|_{z=0} = -\tilde{A} = 1, \tag{100}
$$

$$
B_i = (z - \varphi_{T_r+1,i})\frac{\tilde{\mathcal{L}}_{T_r}(z)}{z(z-1)}\bigg|_{z=\varphi_{T_r+1,i}} = \frac{\tilde{B}_i}{\varphi_{T_r+1,i} - 1} = \frac{\varphi_{T_r+1,i}^{3T_r+1}}{\varphi_{T_r+1,i}^{T_r+1} + T_r}, \tag{101}
$$

and

$$
C = (z-1)\frac{\tilde{\mathcal{L}}_{T_r}(z)}{z(z-1)}\bigg|_{z=1} = -(T_r+2), \tag{102}
$$

using Corollary A.6.

Using Theorem A.11, we obtain:

$$
\bar{\ell}_{T_r}^n = \begin{cases} \sum_{i=1}^{T_r+1} \frac{\varphi_{T_r+1,i}^{3T_r+1+n}}{\varphi_{T_r+1,i}^{T_r+1} + T_r} - T_r - 2 & \text{if } n > 0 \\ \sum_{i=1}^{T_r+1} \frac{\varphi_{T_r+1,i}^{3T_r+1+n}}{\varphi_{T_r+1,i}^{T_r+1} + T_r} - T_r - 1 & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}, \tag{103}
$$

with $\mathrm{ROC}(\bar{\ell}^n_{T_r}) = \{z \in \mathbb{C} : |z| > \max_{i \in \{1,\ldots,T_r+1\}} |\varphi_{T_r+1,i}|\}$.

$\square$