# On sampling firing signals

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In this paper, we present a backward filtering forward sampling algorithm to uniformly sample sequences on the set of firing sequences of length $n \in \mathbb{Z}_{\geq 0}$ and with a refratory period $T_r \in \mathbb{Z}_{\geq 0}$.

We recall the mathematical definition of this set :

$$\mathcal{Y}_{T_r}^n = \left\{ (y_1, \ldots, y_n) \in \{0,1\}^n : \sum_{m=0}^{T_r} y_{(k+m \bmod n)+1} \in \{0,1\}, k \in \{1, \ldots, n\} \right\}. \tag{1}$$

We would like to create samples $(x_1, \ldots, x_n)$ such that:

$$p(x_1, \ldots, x_n) = \begin{cases} \left(\ell_{T_r}^n\right)^{-1} & \text{if } (x_1, \ldots, x_n) \in \mathcal{Y}_{T_r}^n \\ 0 & \text{otherwise} \end{cases}. \tag{2}$$

First of all, notice that if $n \leq 2T_r$, the only valid sequence is the all-zeros one and if $n = 2T_r + 1$, a valid sequence contains at most one spike. In these two cases, we can easily sample a valid sequence uniformly. From now on, we consider the non-trivial cases, and we thus assume $n > 2T_r + 1$.

Let $Z_k = (X_{k-T_r+1}, \ldots, X_k)$ represents random sequences of length $T_r$ with $k \in \{1, \ldots, n\}$ and where $X_i, i \in \{k - T_r + 1, \ldots, k\}$ takes value in $\{0,1\}$. Obviously, this sequence is a Markov chain. Using the constraint functions

$$g_{k-1,k}(z_{k-1}, z_k) = \begin{cases} 1 & \text{if } (x_{k-T_r}, \ldots, x_k) \in \mathcal{Y}_{T_r}^{T_r+1} \\ 0 & \text{otherwise} \end{cases}, \tag{3}$$

we represent the construction of a valid sequence in $\mathcal{Y}_{T_r}^n$ as a factor graph with loop (Figure 1). The

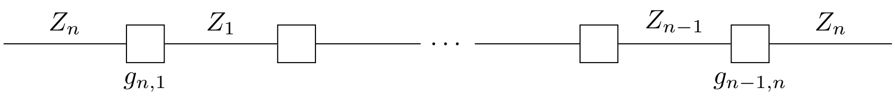


Figure 1: Factor graph

sampling algorithm consists in iteratively constructing a sequence from this factor graph, in three steps:

1. Loop removing (c.f. Section 1)
2. Backward filtering (c.f. Section 2)
3. Forward sampling (c.f. Section 3)

## 1 Loop-free factor graph

Fixing $Z_n$, we can transform the factor graph in Figure 1 into the loop-free factor graph in Figure 2.

To uniformly sample $z_n = (x_{n-T_r+1}, \ldots, x_n)$ it suffices to notice that it contains at most one spike. If $z_n$ is all-zeros, then we have

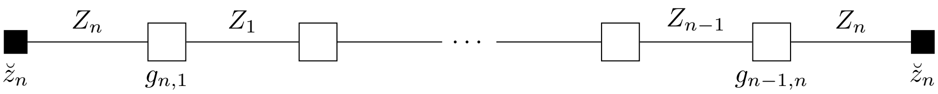


Figure 2: Factor graph with fixed $Z_n = \breve{z}_n$

$$\underbrace{x_1 \cdots x_{n-T_r}}_{\in \tilde{\mathfrak{Y}}_{T_r}^{n-T_r}} \overbrace{0 \cdots 0}^{T_r},$$

and there are $\tilde{\ell}_{T_r}^{n-T_r}$ such sequences.

If $z_n$ contains exactly one spike, then we have

$$\overbrace{0 \cdots 0}^{T_r-k_2} \underbrace{x_{T_r-k_2+1} \cdots x_{n-2T_r+k_1-1}}_{\in \tilde{\mathfrak{Y}}_{T_r}^{n-2T_r-1}} \overbrace{0 \cdots 0}^{T_r-k_1} \overbrace{0 \cdots 0}^{k_1} 1 \overbrace{0 \cdots 0}^{k_2},$$

with $k_1 + k_2 = T_r - 1, k_1, k_2 \geq 0$, and there are $\tilde{\ell}_{T_r}^{n-2T_r-1}$ such sequences.

As a consequence, we can simply sample $z_n$ according to

$$p(z_n) = p(x_{n-T_r+1}, \ldots, x_n) = \begin{cases} \frac{\tilde{\ell}_{T_r}^{n-T_r}}{\tilde{\ell}_{T_r}^{n-T_r} + \tilde{\ell}_{T_r}^{n-2T_r-1}} & \text{if } \sum_{k=n-T_r+1}^{n} x_k = 0 \\ \frac{\tilde{\ell}_{T_r}^{n-2T_r-1}}{\tilde{\ell}_{T_r}^{n-T_r} + \tilde{\ell}_{T_r}^{n-2T_r-1}} & \text{if } \sum_{k=n-T_r+1}^{n} x_k = 1 \\ 0 & \text{otherwise} \end{cases}. \tag{4}$$

## 2 Backward filtering

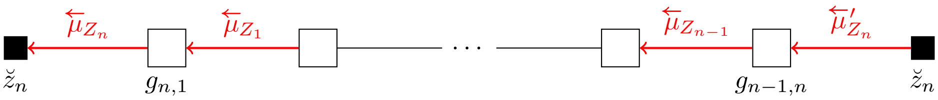


Figure 3: Backward filtering

Once, $z_n$ is fixed, we can compute the backward messages $\overleftarrow{\mu}_{Z_k}$ by sum-product message passing as illustrated in Figure 3. Starting with the message

$$\overleftarrow{\mu}'_{Z_n}(z_n) = \begin{cases} 1 & \text{if } z_n = \breve{z}_n \\ 0 & \text{otherwise} \end{cases}, \tag{5}$$

we can recursively compute all backward messages, from right to left using:

$$\overleftarrow{\mu}_{Z_{k-1}}(z_{k-1}) = \sum_{z_k} g_{k-1,k}(z_{k-1}, z_k) \overleftarrow{\mu}_{Z_k}(z_k), \quad k \in \{1, \ldots, n\}, \tag{6}$$

with $z_0 = z_n$.

It can also be expressed in matrix form as

$$\overleftarrow{\boldsymbol{\mu}}_{\boldsymbol{Z_{k-1}}} = \overleftarrow{\boldsymbol{\mu}}_{\boldsymbol{Z_k}} \boldsymbol{A} \tag{7}$$

with

$$\left\{ \overleftarrow{\boldsymbol{\mu}}_{\boldsymbol{Z_k}} \right\}_{i_{z_k}} = \overleftarrow{\mu}_{Z_k}(z_k) \tag{8}$$

for $i_{z_k} \in \{1, \ldots, T_r + 1\}$ and

$$z_k = \mathbf{0} \mapsto i_{z_k} = 1 \tag{9}$$

$$z_k = \boldsymbol{e_i} \mapsto i_{z_k} = i + 1 \tag{10}$$

and with

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \in \{0, 1\}^{T_r+1 \times T_r+1}. \tag{11}$$

# 3 Forward sampling

Using backward messages, we can sample by forward sampling as shown in Figure 4. We sample $z_k$ for $k \in \{1, \ldots, n-1\}$ according to:

$$\begin{aligned} p(z_k|z_{k-1}) &= \frac{p(z_k, z_{k-1})}{p(z_{k-1})} && (12) \\ &= \frac{g_{k-1,k}(z_{k-1}, z_k)\overrightarrow{\mu}_{Z_{k-1}}(z_{k-1})\overleftarrow{\mu}_{Z_k}(z_k)}{\overrightarrow{\mu}_{Z_{k-1}}(z_{k-1})\overleftarrow{\mu}_{Z_{k-1}}(z_{k-1})} && (13) \\ &= \frac{g_{k-1,k}(z_{k-1}, z_k)\overleftarrow{\mu}_{Z_k}(z_k)}{\overleftarrow{\mu}_{Z_{k-1}}(z_{k-1})}, && (14) \end{aligned}$$

with $z_0 = z_n$.

Again, this can be expressed in the matrix form

$$\boldsymbol{p}_{\boldsymbol{Z_k}|\boldsymbol{Z_{k-1}}} = \left\{\overleftarrow{\boldsymbol{\mu}}_{\boldsymbol{Z_{k-1}}}\right\}^{-1}_{i_{z_{k-1}}} \overleftarrow{\boldsymbol{\mu}}_{\boldsymbol{Z_k}} \boldsymbol{A}_{:,i_{\breve{z}_{k-1}}}. \tag{15}$$

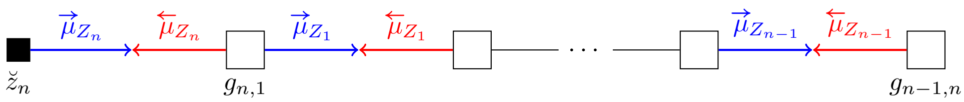


Figure 4: Forward sampling