Review III(Slides 177 - 225)

Poset & Cardinality

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VE203 - Discrete Mathmatics

Definition

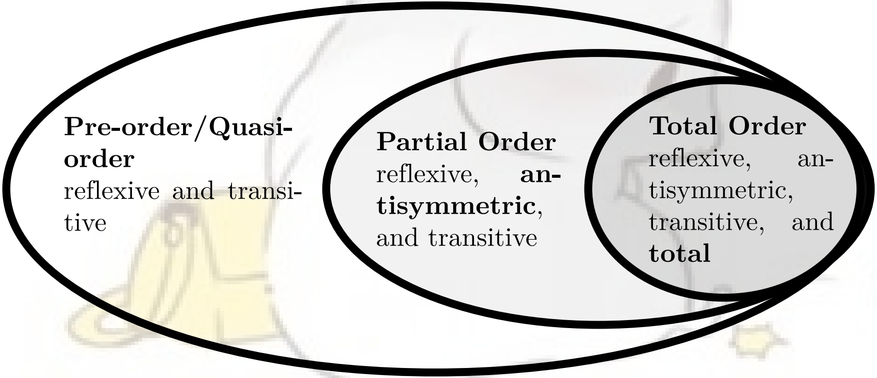
A (binary) relation R on A, *i.e.*, $R \subset A \times A$, is

- **reflexive** if $aRa \Rightarrow \top$.
- **symmetric** if $aRb \Leftrightarrow bRa$.
- **transitive** if $aRb \wedge bRc \Rightarrow aRc$.
- **anti-symmetric** if $aRb \wedge bRa \Rightarrow a = b$.
- asymmetric if $aRb \wedge bRa \Rightarrow \perp$.
- total if $aRb \vee bRa \Rightarrow \top$.

(Non-strict) Partial order: reflexive, antisymmetric, and transitive.
Equivalence relation: reflexive, symmetric, and transitive.
Total order: Partial order + total.

Partial Order

The term partial order typically refers to a **non-strict** partial order relation.



Concept Checking List

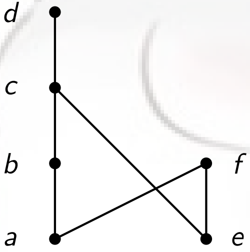
Be familiar with the following:

- covers/adjacent
- minimal/minimum element
- maximal/maximum element
- (in)comparability graph
- chain/antichain
- lattice: join and meet

Let's see one example!

Example

Hasse/Order Diagram: edges are the cover pairs (x, y) with x covered by y.



Question

Fill in CCP03: ex2-ex6. Do be careful!

Exercise

1. In the poset $(\mathbb{Z}^+, |)$ (where $\mathbb{Z}^+$ is the set of all positive integers and $|$ is the divides relation), are the integers 3 and 9 comparable? Are 7 and 10 comparable?

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Solution:

3 and 9 are comparable since $3 \mid 9$, *i.e.*, 3 divides 9. But 7 and 10 are not comparable since $7 \nmid 10$ and $10 \nmid 7$.

Exercise

2. A relation R is defined on ordered pairs of integers as follows: $(x, y)R(u, v)$ if $x < u$ and $y > v$. Then R is:

- A. Neither a partial order nor an equivalence relation
- B. A partial order but not a total order
- C. A total order
- D. An equivalence relation

Answer: A

Exercise

3. Given a set $S = \{a, b, c, d\}$. Consider the following 4 partitions $\pi_1, \pi_2, \pi_3, \pi_4$ on S:

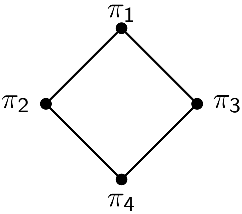
$$\pi_1 = \{\overline{abcd}\}, \pi_2 = \{\overline{ab}, \overline{cd}\}, \pi_3 = \{\overline{abc}, \bar{d}\}, \pi_4 = \{\bar{a}, \bar{b}, \bar{c}, \bar{d}\}.$$

Let p be a **strict** partial order on the set of partitions $S' = \{\pi_1, \pi_2, \pi_3, \pi_4\}$ defined as follows:

$$\pi_i p \pi_j \text{ if and only if } \pi_i \text{ refines } \pi_j$$

Find the poset diagram for (S', p).

Solution



A partition is said to refine another partition if it splits the sets in the second partition to a larger number of sets.

Therefore, the partial order contains the following ordered pairs:

$$\{(\pi_4, \pi_1), (\pi_4, \pi_2), (\pi_4, \pi_3), (\pi_3, \pi_1), (\pi_2, \pi_1)\}$$

Definition

For any set A, we will define a set card A such that

- For any sets A and B, $\operatorname{card} A = \operatorname{card} B \Leftrightarrow A \approx B$.
- For a finite set A, card A is the natural number for which $A \approx n$.

Cantor-Schröder-Bernstein Theorem:

$$(A \preceq B) \wedge (B \preceq A) \Rightarrow A \approx B.$$

A injection $f\colon A \to B$ and another injection $g\colon B \to A \Rightarrow A \approx B$.

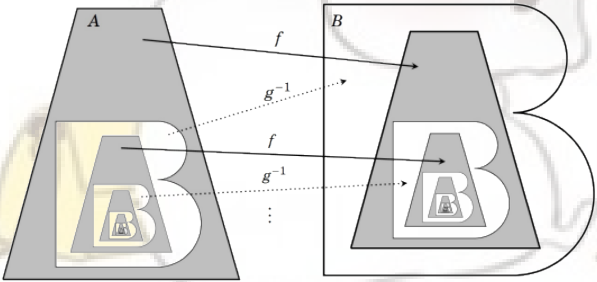
Why?

$\{X \mid \operatorname{card} X = \kappa\}$ is not a set, except for $\kappa = 0$.

Explanation for Slides

Iterate to get a bijection $h : A \to B$:

$$h(x) = \begin{cases} f(x), & x \in \bigcup_{k\in\mathbb{N}} (g \circ f)^k(A - g(B)) \\ g^{-1}(x), & \text{otherwise} \end{cases}$$



Exercise

4. Prove the following equinumerosity:
   - $\mathbb{Z} \approx \mathbb{N}$
   - $\mathbb{N} \times \mathbb{N} \approx \mathbb{N}$
   - $(0, 1) \approx \mathbb{R}$
   - $[0, 1] \approx (0, 1)$
   - $\mathcal{P}(\mathbb{N}) \approx \mathbb{R}$
   - $\mathbb{N}^{\mathbb{N}} \approx \mathbb{R}$

Pay attention to the way that you build the bijection!

Proof: $\mathbb{N}^{\mathbb{N}} \approx \mathbb{R}$

We have $2^{\aleph_0} \le 3^{\aleph_0} \le 4^{\aleph_0} \le \cdots \le \aleph_0^{\aleph_0}$, because of the inclusions $\{0,1\}^{\mathbb{N}} \subset \{0,1,2\}^{\mathbb{N}} \subset \cdots \subset \mathbb{N}^{\mathbb{N}}$. So if we prove that $\aleph_0^{\aleph_0} \le 2^{\aleph_0}$, then we see that all of these cardinalities are in fact equal.

To show this, we need to find some injection $f\colon \mathbb{N}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$. There are many ways to do this; my favorite is as follows. Let $a = (a_n)$ be some sequence of natural numbers. Then we define $f(a)$ to be the sequence consisting of first a_0 ones, followed by a zero, then a_1 ones, followed by a zero, then a_2 ones, followed by a zero, and so on. This gives a sequence of zeroes and ones, and if $b = (b_n)$ is another sequence of natural numbers, then $f(a) = f(b)$ if and only if $a_n = b_n$ for all indices n if and only if $a = b$. So f is indeed injective, and therefore $\aleph_0^{\aleph_0} \le 2^{\aleph_0}$.

So indeed $2^{\aleph_0} = 3^{\aleph_0} = \cdots = \aleph_0^{\aleph_0}$.

Thinking

A strange thought, why the previous one is wrong?

- Consider $\mathbb{N}^2, \mathbb{N}^3, \ldots$ is all countable, so $\mathbb{N}^{\mathbb{N}}$ is also countable.
- Consider $2^{\mathbb{N}}, 3^{\mathbb{N}}, \ldots$ is all equinumerous to $\mathbb{R}$, so $\mathbb{N}^{\mathbb{N}}$is also equinumerous to $\mathbb{R}$.

What does this mean?

$$\operatorname{card} \mathbb{N} = \aleph_0 \qquad \operatorname{card} \mathbb{R} = \aleph_1 \qquad \operatorname{card} \mathbb{R}^{\mathbb{R}} = \aleph_2$$

Finite Set

Important results:

- Let A be any finite set. $f\colon A \to A$, f injective $\Leftrightarrow$ f surjective.
- No finite set is equinumerous to a proper subset of itself.

Try to understand them! This may appear in the exam!

Longest Increasing Subsequence

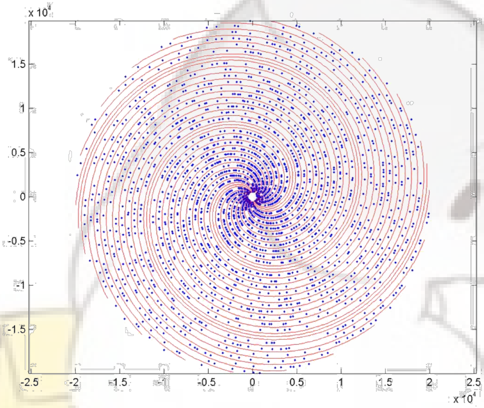
5. Find the longest increasing/decreasing sequence.

$$\text{nums} = [10, 9, 2, 5, 3, 7, 101, 18]$$

Methodology

num.	0	1	2	3	4	5	6
val.	10	9	2	5	7	101	18
len.	1	1	1	2	3	4	4
pre.	-	-	-	2	3	4	4

Prime spiral



Links:

- Zhihu: https://www.zhihu.com/question/24236455
- Bilibili: https://www.bilibili.com/video/BV1tE411h7x4

Reference

- Examples from Dr. Cai Runze's Sildes.
- Exercises/graphics from 2021-Fall-Ve203 TA Zhao Jiayuan
- What-is-the-result-of-a-number-greater-than-2-raised-to-the-power-of-aleph-0
  https://math.stackexchange.com/questions/1646830