Review II(Slides 103 - 159)

Relations

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VE203 - Discrete Mathmatics

Relation

A subset $R \subset A \times B$ is called a (binary) relation from A to B. If $A = B$, we say that R is a relation on A.

Check the following:

- $\text{domain}(R) = \{x \mid \exists y(xRy)\}$
- $\text{range}(R) = \{y \mid \exists x(xRy)\}$
- $R = \emptyset$: the empty relation
- $A = B$: identity relation
- The relation $A \times B$ itself?

Functions

A function is a relation F such that

$$\forall x \in \text{dom } F(\exists! y(xFy))$$

. Check the following:

- For a function F and a point $x \in \text{dom}(F)$, the unique y such that xFy is called the value of F at x and is denoted $F(y)$.
- Given function $F : A \to B$, then $\forall x, y \in A(x = y \Rightarrow F(x) = F(y))$.
- Partial Function/ Total Tunction.

Operations on Functions

For arbitrary sets A, relations F, and functions G,

- Inverse: $F^T = F^{-1} = \{(y, x) \mid xFy\}$.
- Composition: $F \circ G = \{(x, z) \mid \exists y \in A(xFy \wedge yGz)\}$.
- Restriction: $F \mid A = \{(x, y) \in F \mid x \in A\}$.
- Image: $F(A) = \mathrm{ran}(F \mid A) = \{y \mid (\exists x \in A)xFy\}$.

If F is a function, then $F(A) = \{F(x) \mid x \in A\}$.

Exercise

1. Let $A : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}.$$

Let

$$U = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

Question

- What's the restriction of A to U?
- What's the image of the restriction?
- What's the inverse of the restriction?

(Take from vv286 lecture slides)

-jectivity

Given a function $F : A \to B$, with $\text{dom } F = A$ and $\text{ran}(F) \subset B$, then

- F is injective or one-to-one if $\forall x, y \in A(F(x) = F(y) \Rightarrow x = y)$;
- F is surjective or onto if $\text{ran}(F) = B$;
- F is bijective if it is both injective and surjective.

Given a function $F : A \to B$, $A \neq \varnothing$, then

- There exists a function $G : B \to A$ (a "left inverse") such that $G \circ F = id_A \Leftrightarrow F$ is one-to-one;
- There exists a function $G : B \to A$ (a "right inverse") such that $F \circ G = id_B \Leftrightarrow F$ is onto.

Let $f : A \to B, g : B \to C$,

- If $g \circ f$ is injective, then f is injective.
- If $g \circ f$ is surjective, then g is surjective

Exercise

2. Recall that $\mathbb{Z}$ denotes the set of integers, $\mathbb{Z}^+$ the set of positive integers, and $\mathbb{Q}$ the set of rational numbers. Define a function:

$$f : \mathbb{Z} \times \mathbb{Z}^+ \to \mathbb{Q}, \qquad f(p, q) = \frac{p}{q}.$$

1. Is f an injection? Why?
2. Is f an surjection? Why?
3. Is f an bijection? Why?

Definition

A (binary) relation R on A, *i.e.*, $R \subset A \times A$, is

- **reflexive** if $aRa \Rightarrow \top$.
- **symmetric** if $aRb \Leftrightarrow bRa$.
- **transitive** if $aRb \wedge bRc \Rightarrow aRc$.
- **anti-symmetric** if $aRb \wedge bRa \Rightarrow a = b$.
- asymmetric if $aRb \wedge bRa \Rightarrow \perp$.
- total if $aRb \vee bRa \Rightarrow \top$.

(Non-strict) Partial order: reflexive, antisymmetric, and transitive.
Equivalence relation: reflexive, symmetric, and transitive.
Total order: Partial order + total.

Exercise

3. Recall that $\mathbb{R}$ denotes the set of real numbers, while $\mathbb{Z}$ denotes the set of integers. Define a relation $\sim$ on $\mathbb{R}$ by

$$x \sim y \Leftrightarrow x - y \in \mathbb{Z}$$

for any $x, y \in \mathbb{R}$. Prove that $\sim$ is an equivalence relation.

Equivalence Class

Given an equivalence relation R on A,

- Equivalence class containing x

$$[x]_R = \{t \in A \mid xRt\}.$$

- This is also a partition for A.
- For $x, y \in A$,

$$[x]_R = [y]_R \Leftrightarrow xRy.$$

- Quotient set is given by

$$A/R = \{[x]R \mid x \in A\}.$$

Equinumerosity

Definition:

A set A is equinumerous to a set B (written $A \approx B$) if there is a bijection from A to B.

Examples:

- $\mathbb{R} \approx (0,1)$
- $\mathbb{N} \not\approx \mathbb{R}$
- $\mathbb{N} \approx \mathbb{N}^2$
- $\mathbb{N} \approx \mathbb{N}^3$
- $\mathbb{N} \approx \mathbb{N}^{\mathbb{N}}$?

Question

- Why isn't it an equivalence relation?
- How to prove/disprove a equinumerosity?
- How is $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ constructed respectively?

Cantor's Theorem

Cantor's Theorem:

- $\mathbb{R} \not\approx \mathbb{N}$.
- For every set A, $A \not\approx \mathcal{P}(A)$.

Top asked questions

- $\mathcal{P}(\mathbb{N}) = \mathbb{R}$?
- Why is $\mathbb{R}$ not countable?
- How to prove cantor's theorem?

Example

Let $A = \{a, b, c, d, e\}$. The mapping $f\colon A \to \mathcal{P}(A)$ is defined by

$f(a) = \{a, d, e\}$, $f(b) = \{a, c\}$, $f(c) = \{a, b, d, e\}$, $f(d) = \varnothing$, $f(e) = \{b, c, e\}$

Determine the set $B = \{x \in A \mid x \notin f(x)\}$.

Solution

This set is used in the proof of Cantor's theorem. We see that

$$a \in f(a),\ b \notin f(b),\ c \notin f(c),\ d \notin f(d),\ e \in f(e).$$

Hence, $B = \{x \in A \mid x \notin f(x)\} = \{b, c, d\}$.

Exercise

4. The mapping function $f\colon \mathbb{N} \to \mathcal{P}(\mathbb{N})$ is defined by

$$f(n) = \mathbb{N} \setminus \left\{ n^2 - (2m-1)\, n \right\},$$

where $n, m \in \mathbb{N}$. Determine the set $B = \{x \in A \mid x \notin f(x)\}$.

5. Prove that the set of all sets does not exist.

Pigeonhole Principle

Two versions¿:

xxs Let $r, s \in \mathbb{N} \backslash \{0\}$, if a set containing at least $rs + 1$ elements is partitioned into r subsets, then some subsets contains at least $s + 1$ elements.

dxs No set of the form $[n] = \{1, \cdots, n\}$ is equinumerous to a proper subset of itself, where $n \in \mathbb{N}$.

Exercise

6. Consider a sequence $\{\sqrt{3}, 2\sqrt{3}, 3\sqrt{3}, \cdots\}$. Prove that there are infinite number of terms have a mantissa of less than 0.01.

Sorting Algorithms

We have lots lots of sorting algorithms!

- Demostration: `https://www.byteflying.com/archives/6171`
- Classification
- Costs
  - Time complexity (Worst/Averge/Best)
  - Space complexity
  - Stability

Question

How to choose a sorting algorithms that most satisfy your need?

Quick Sort with Second key word

```
void QuickSort(vector<int>& v,int left,int right,vector<int>& o){
    if (left>=right)  return;
    int key = (left+right)/2;// you can also choose right/left
    while (left<right){
        while (left<right&& (v[right] > v[key] || v[right]==v[key] && o[right]>o[key]))  right--;
        while (left<right&& (v[left] <= v[key] || v[left]==v[key] && o[left]<o[key]))   left++;
        if (left < right) swap(v[left], v[right]);
    }
    swap(v[left], v[key]);   //left== right
    int meet= key;      //divide into two parts
    QuickSort(v, left, meet-1);
    QuickSort(v, meet+1, right);
}
```

Modified Bubble Sort

```java
	public static void bubble_sort(int[] intArr) {
		int max = intArr.length - 1;
		int secondCount = max;
		//record where the exchange happen last time
		for (int i = 0; i < max; i++) {
			System.out.println( (i + 1) + "times");
			boolean flag = true;int lastChangeIndex = 0;
			for (int j = 0; j < secondCount; j++) {
				if (intArr[j] > intArr[j + 1]) {
					swap(Arr[j],Arr[j+1]);
					flag = false; lastChangeIndex = j;
				}
				System.out.println("Compare:"+(j+1)+", Result:"+Arrays.toString(intArr));
			}
			if (flag) break;//already well ordering
			secondCount = lastChangeIndex;//update
		}
	}
```

Exercise

7. Try to implement three-way merge sort in C++. If you know Master Theorem, try to calculate the time complexity and compare with the original merge sort.

8. Finish the following exercise regarding sorting algorithms:

- https://vijos.org/p/1398
- https://vijos.org/p/1257

Reference

- Examples from Vv286 Lecture Slides.
- Exercises from 2021-Fall-Ve203 TA Zhao Jiayuan
- Stable Quick Sort, https://blog.csdn.net/liuchenjane/article/details/72902325
- Modified Bubble Sort, https://blog.csdn.net/weixin_43168559/article/details/88873585