Final Rivew - Part I

Basic Graph Theory

Be sure to memorize the terminologies well...

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VE203 - Discrete Mathmatics

Terminology

Graph Definition

A **graph** $G := (V, E)$ consists of set of **vertices** $V(G)$ and **edges** $E(G)$, together with a relation indicating each edge **incident** with one or two vertices. Two vertices are called **adjacent** if they are connected by at least one edge.

Simple Graph

An edge with just one end is called a **loop**. Two distinct edges with the same ends are **parallel**. A graph without loops or parallel edges is called **simple**.

Terminology

There are two important relations and for graphs. Hmm, perhaps give some examples?

Isomorphism

An **isomorphism** between simple graph G and H is a bijection $f\colon V(G) \to V(H)$ such that $uv \in E(G) \Leftrightarrow f(u)\,f(v) \in E(H)$. In other words, f **preserves the structure** of G and H. If such f exists, we say G is isomorphic to H, or $G \cong H$.

Isomorphism is an **equivalence** relation between graphs.

Subgrpah

If $V(H) \subset V(G)$ and $E(H) \subset E(G)$ and G and H shares the same incidence relation, we say H is a **subgraph** of G, or $H \subset G$.

This forms a **partial order** between graphs.

Terminology

Complement

The **complement** $\overline{G}$ of a simple graph G is the simple graph with vertex set $V(G)$ defined by $uv \in E\left(\overline{G}\right)$ iff $uv \notin E(G)$. Note that given graph $G = (V, E)$, we have $\overline{G} = \left(V, \binom{V}{2} - E\right)$.

A graph G is said to be **self-complementary** if $G \cong \overline{G}$.

Null Graph

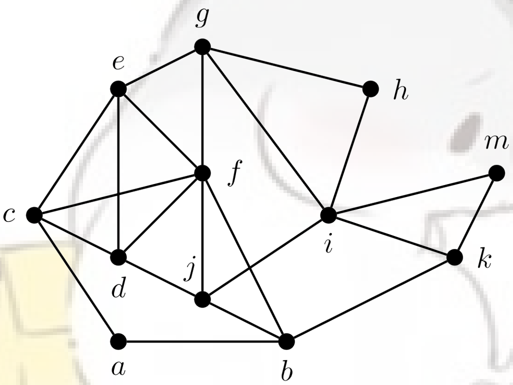
The **null graph** is the graph whose vertex set and edge set are empty, namely $G = (\varnothing, \varnothing)$.

Clique

A **clique** in a graph is a set of pairwise adjacent vertices.

Exercise

1. Given a graph G as follows:



See the questions in last semester's exam!

Standard Graph

Complete Graph

A **complete graph** with n vertices $K_n := (V, E)$ satisfies $E = \binom{V}{2}$. If a subgraph of a graph is complete, its vertices are called a clique in this graph.

Path

A **path** with n vertices is $P_n := \left(\{v_i\}_{i=1}^{n}, \{v_i v_{i+1}\}_{i=1}^{n-1}\right)$, where $i \neq j \Rightarrow v_i \neq v_j$.

Cycle

A **cycle** with n vertices is $C_n := \left(\{v_i\}_{i=1}^{n}, \{v_i v_{i+1}\}_{i=1}^{n-1} \cup \{v_n v_1\}\right)$, where $i \neq j \Rightarrow v_i \neq v_j$.

Are C_1 and C_2 still simple graphs?

The Handshaking Theorem

Undirected graph:

$$2|E| = \sum_{v \in V} \deg(v)$$

Directed graph:

$$|E| = \sum_{v \in V} \deg^{+}(v) = \sum_{v \in V} \deg^{-}(v)$$

Remark:

- A vertex is said to be isolated if it has degree zero.
- A vertex is said to be pendant if it has degree one.
- $\deg^{+}(v)$: in-degree of a vertex v
- $\deg^{-}(v)$: out-degree of a vertex v

Exercise

A little bit tricky exercise! But possible to appear in the exam!

2. Which of the following statements about graphs are correct?

(A) C_5 is self-complementary.

(B) P_4 is self-complementary

(C) $K_{2,2}$ is induced in C_4.

(D) C_1 is induced in K_5.

Answer: A B C

Walks and Connectivity

Definition

A walk W in G is a sequence of vertices $\{v_i\}_{i=0}^{n}$ and edges $\{e_i\}_{i=1}^{n}$ so that e_i is incident with v_{i-1} and v_i.

- W is called **closed** if $v_n = v_0$
- The **length** of W is its number of edges n
- G is connected if $\forall u, v \in V(G)$, there is a walk from u to v
- A walk is generally **not** a graph, why?

Components

Definition

A component of a graph G is a **maximal connected subgraph** in G. In other words, it is not contained in any other connected subgraphs.

The number of components of G is denoted as $\mathsf{comp}(G)$.

Theorem

Every vertex is in a **unique** component.

Note

If a graph G isn't connected, it may be useful to consider its components.

Cuts

Definition(substraction)

Given $G=(V,E)$, $S\subset E$, $X\subset V$, then $G-S:=(V,E\setminus S)$ and $G-X:=(V\setminus X,\{e\in E: e \text{ not incident with } x\in X\})$.

Definition

- $e\in E$ is a **cut-edge** or **bridge** if no cycle contains e
- $v\in V$ is a **cut-vertex** if $\operatorname{comp}(G-v)>\operatorname{comp}(G)$

What happens when we delete an edge or vertex?

- If e is a cut-edge, $\operatorname{comp}(G-e)=\operatorname{comp}(G)+1$
- If e is not, $\operatorname{comp}(G-e)=\operatorname{comp}(G)$
- Further, $\operatorname{comp}(G-v)\leq\operatorname{comp}(G)+\deg(v)-1$

Induced subgraph

Please remember to delete both the vertexes and the edges!

Exercise

3. A graph G is called ***k*-regular** if all vertices of G have the same degree k.

(i) Show that a k-regular bipartite graph has no cut-edge for $k \geq 2$.

(ii) Show that a k-regular bipartite graph has a perfect matching for $k \geq 1$.

(Take from Homework 6)

Bipartation

Bipartation

A **bipartation** of G is a partition (A, B) of $V(G)$ so that every edge is **incident** with one vertex in A and one in B. G is called **bipartite** if it admits a bipartation.

A **complete bipartite graph** or **biclique** $K_{m,n}$, is a simple bipartite graph with every edges in A and that in B are adjacent to each other, where $|A| = m, |B| = n$.

Exercise

For which values of n are the following graphs bipartite?

i) K_n ii) C_n iii) W_n iv) Q_n

Bipartation

Theorem

For every graph G, the following are equivalent:

- G is bipartite
- G has no cyle of odd length
- G has no closed walk of odd length
- G has no induced cycle of odd length.

Proof.

(iiii)⇒(ii). We show the contrapositive, i.e., $\neg$(ii) $\Rightarrow \neg$ (iiii). Suppose G has a cycle of odd length, choose a shortest cycle $C \subset G$. Note that C is induced, otherwise $\exists e \in E(G) \setminus E(C)$, with ends x, y. But now either C_1 or C_2 is an odd cycle of shorter length, contradiction.

Matching

Definition

A **matching** in a graph $G = (V, E)$ is a subset of edges M such that M does not contain a loop and no two edges in M are incident with a common vertex.

- A matching is **maximal** if it is not contained in another matching.
- A matching is **maximum** if its size is the largest among all matching.
- A matching M is **perfect** if $\forall v \in G$, $\deg_M(v) \geq 1$.

Group Transversals

This is what xrz wrote...

Definition

For subgroups $H, K \leq G$, the coset intersection graph $\Gamma^G_{H,K}$ contains vertices of all left cosets of H in G and right cosets of K in G. aH and Kb are adjacent iff $aH \cap Kb \neq \varnothing$.

Theorem

A coset intersection graph is always a disjoint union of complete bipartite graphs.

Neighbors and Covers

Neighbors

For $X \subset V(G)$, its **neighbors** $N(X)$ is

$$N(X) := \{v \in V(G) \setminus X \mid v \text{ is adjacent to a vertex in } X\}$$

Furthermore, we denote $N(x) := N(\{x\})$.

Cover

The **edges** $S \subset E(G)$ **covers** $X \subset V(G)$ if every $x \in X$ is incident to some $e \in S$.

The **vertices** $X \subset V(G)$ **covers** $S \subset E(G)$ if every $e \in S$ is incident to some $v \in X$.

Hall's Theorem

Hall's Theorem

Let G be a finite bipartite graph with bipartation (A, B). There exists a matching covering A iff there does not exist $X \subset A$ with $|N(X)| < |X|$.

Interesting Example

Given a sequence of (not necessarily distinct) sets $S_1, S_2, \ldots, S_m$, there exists a sequence of distinct elements $x_1, x_2, \ldots, x_m$ such that $x_i \in S_i$ for all $i = 1, 2, \ldots, m$ if and only if **Hall's condition** holds. State **Hall's condition** in this context.

For every $k = 1, 2, \ldots, m$, the union of any k sets has at least k elements, that is

$$|\bigcup_{i \in I} S_i| \geq |I| \text{ for all } I \subset \{1, \ldots, m\}$$

Exercise

4. Let G be a bipartite graph with bipartation (A, B), and G has no isolated vertices. If the minimum degree of vertices in A is no less than the maximum degree of vertices in B, show that there exists a matching covering A.

Kőnig-Egerváry Theorem

Vertex Cover

A **vertex cover** of a graph G is a set $X \subset V(G)$ if every $e \in E(G)$ is incident with a vertex in X. The vertices in X **cover** $E(G)$.

Kőnig-Egerváry Theorem

Given a finite bipartite graph G, $\alpha'(G) = \beta(G)$, where $\alpha'(G)$ the size of the largest matching and $\beta(G)$ is the size of the smallest vertex cover.

Fulkerson Theorem

Kőnig-Egerváry theorem implies Dilworth theorem (and vice versa).

Farewell

- Take a look at last semester's final paper, but it is less helpful than mid.
- Take quiz!!! If you want that 9 points!!!
- Join piazza!!! If you want that 1 point!!!
- Check homework! No solution this time because we are lazy.
- Be confident! The exam will be easy! But probably a little bit difficult than mid...

The *Discrete Mathematics* is a course about **how to compute**. If you find any thing in this course **abstract**, find a **concrete** example to make an analogy. And, last but not the least, there will be a day for you to translate all things we have covered in this course into **codes**.

Thank you for your support for the whole semester! I really learn a lot within these 10 weeks of retaking this course. Hope to see you offline in the summer semester!

Good Luck For Your Exam!

Reference

- Content from Runze Cai's Slides.
- Content from Ve203-2021-fall Final RC by Xue Runze.
- Exericses from Ve203-2021-fall Final Exam.
- Exercises from Ve203-2021-summer Final Exam.
- Exercises from Ve203-2022-spring Homework 6.
- Exercises from Ve203-2020-fall Assignment 9.
- Cute paintings of Hamham from Wang Ruizhe.