Bayesian Linear Regression

Hypo. Model: $\underset{n\times 1}{y} = \underset{n\times k}{X}\underset{k\times 1}{\beta} + \underset{n\times 1}{\varepsilon}, \quad \varepsilon \sim N(0, \sigma^2 I_n)$

$\hookrightarrow y \sim N(X\beta, \sigma^2 I_n) \Rightarrow$ normal likelihood

(informative) Conjugate Priors:
$$\begin{cases} \underset{k\times 1}{\beta} \sim N(\beta_0, \underset{k\times k}{\Sigma_0}) \\ \underset{1\times 1}{\sigma^2} \sim \text{invGamma}(\alpha_1, \alpha_2) \end{cases}$$

Bayes Updating: $\underbrace{P((\beta,\sigma^2)\mid y, X)}_{\text{Posterior}} \propto \underbrace{P(y\mid(\beta,\sigma^2),X)}_{\text{Likelihood}}\ \underbrace{P(\beta)\,P(\sigma^2)}_{\text{Priors}}$

Conditional Posteriors:

$\hookrightarrow \begin{cases} \text{Data}: (X, y) \\ \text{Priors}: (\beta, \sigma^2) \end{cases}$

$$\begin{cases} \beta_{post} \mid (X, y, \sigma^2) \sim N(\tilde{\mu}, \sigma^2\tilde{\Sigma}) \\ \quad \hookrightarrow \begin{cases} \tilde{\mu} = (X'X + \Sigma^{-1})^{-1}(X'y + \Sigma^{-1}\beta) \\ \tilde{\Sigma} = (X'X + \Sigma^{-1})^{-1} \end{cases} \\ \sigma^2_{post} \mid (X, y, \beta) \sim \text{Gamma}(\tilde{\alpha}_1, \tilde{\alpha}_2) \\ \quad \hookrightarrow \begin{cases} \tilde{\alpha}_1 = \alpha_1 + \frac{n}{2} \\ \tilde{\alpha}_2 = \alpha_2 + \frac{(y - X\beta)'(I + X\Sigma_0 X')^{-1}(y - X\beta)}{2} \end{cases} \end{cases}$$

True Model: $y = X^{*}\alpha + u$

$\hookrightarrow$ Used to generate realized outcome

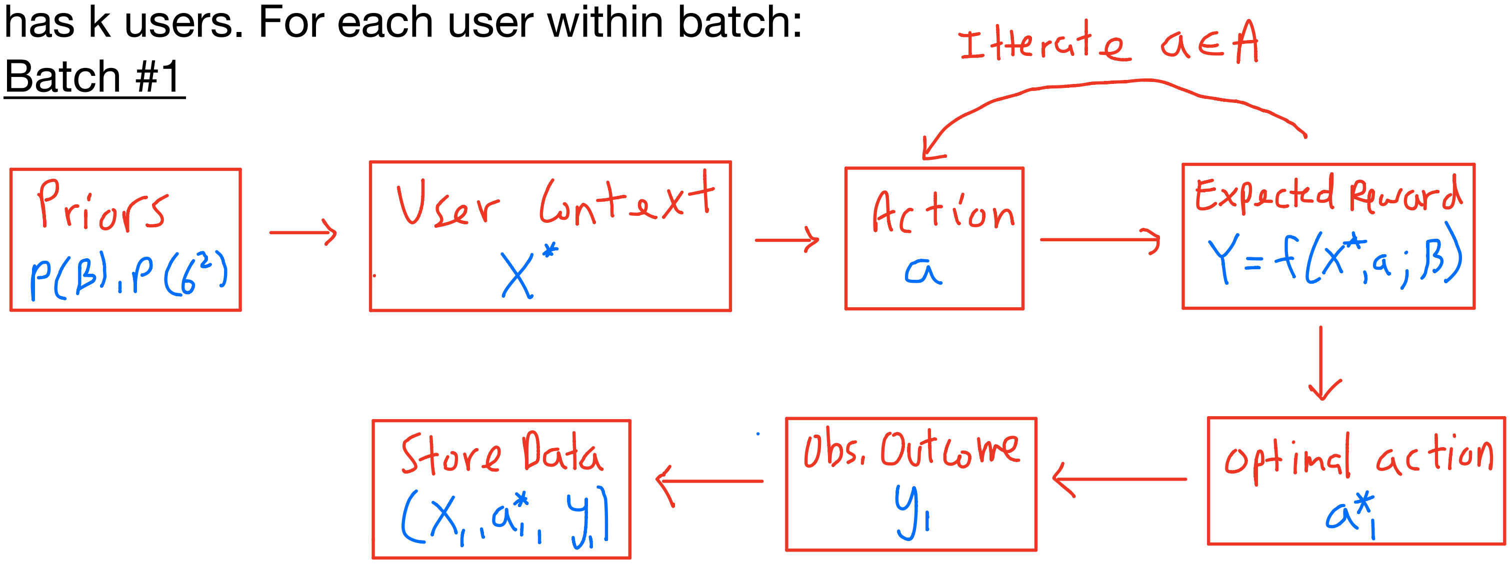
Note: Python can be used to draw from multivariate normal and gamma distribution using standard libraries:

1) Multivariate normal = np.random.multivariate_normal(mean, cov, size)
2) Inverse Gamma = invgamma.rvs(alpha1, alpha2, size)

Thompson Sampling with Bayesian Linear Regression

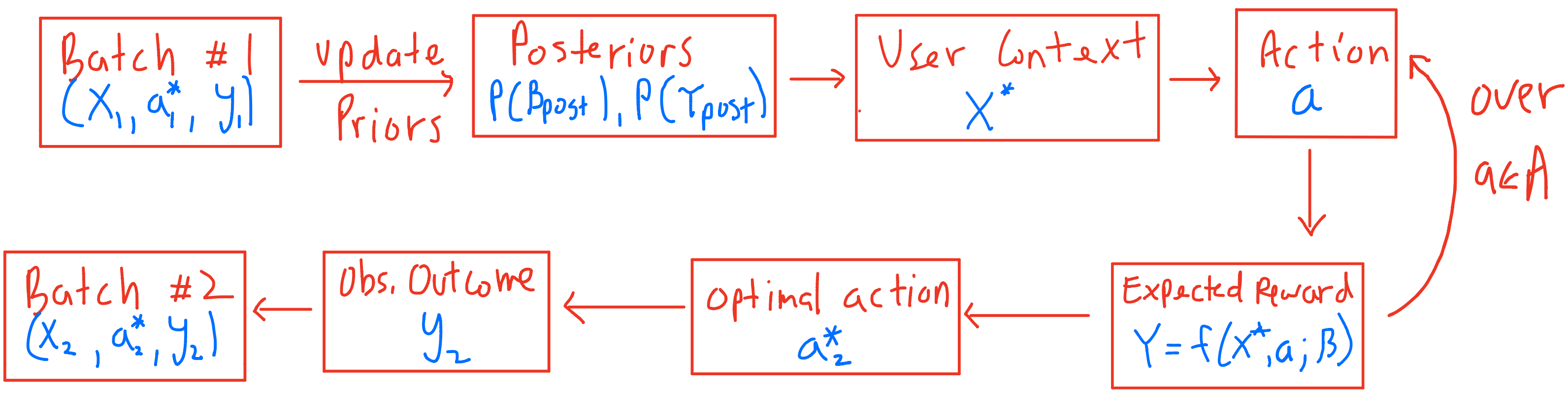
Suppose the data is collected in several batches over time, and each batch has k users. For each user within batch:

Batch #1

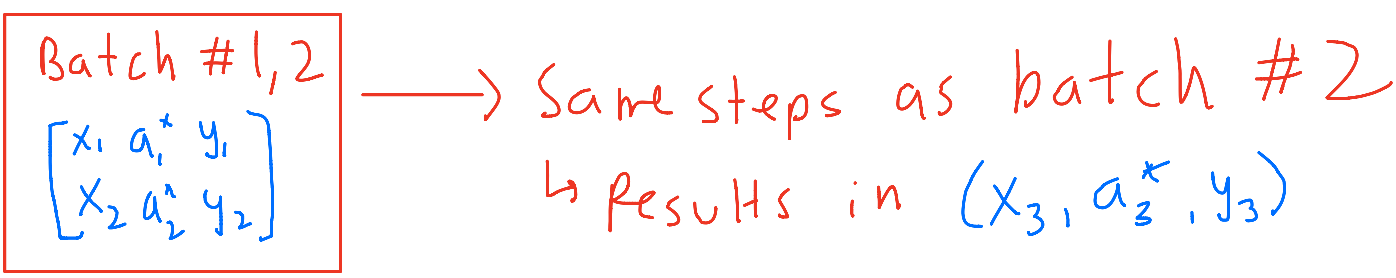


Note observed outcome generated by true linear regression model. Whereas the expected reward given by hypothesized linear regression model. No updating of priors in batch #1.

Batch #2

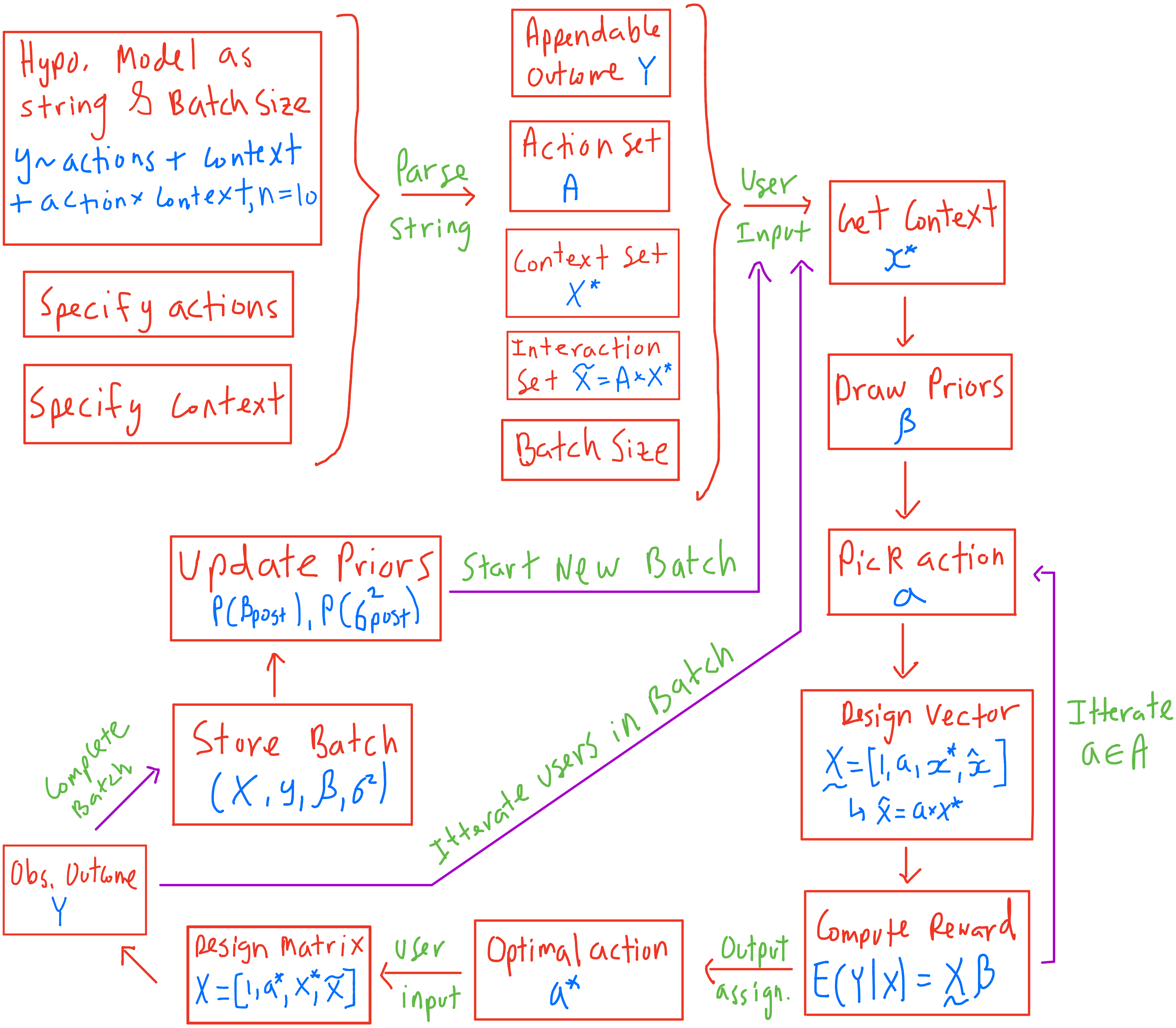


Batch #3



Iterate above procedure for all batches. Priors are updated using the formulas above. The distributions of the coefficients and error variance gets updated at each step (using data from all previous batches). The expected rewards depend on draws from the posterior distributions. <u>For conjugate priors we only need to use data from previous batch</u> (do not need to append data from all previous batches).

Detailed Work Flow Chart



- Action set contains all possible actions. Similarly context set is all possible contexts. <u>Note that the interaction set only contains desired interaction possibilities.</u> For each user, one combination of (action, context, interaction) will be realized, this will go into the regression model.
- Note that batch is defined by the batch size. So if the batch is of size 10, then the updating of priors occurs in multiples of 10.
- Design matrix contains all the data from the batch, each row is data from a user in the batch. Whereas the design vector is a single row.
- For simulation, the observed outcome needs to generated from true model. In a real experiment, this output will be provided by user.

Input and outputs

Inputs
- Hypothesized Model (String)
- Batch Size (numeric)
- Action Variables (String)
- Context Variables (String)
- NIG prior param. $(\underset{R\times 1}{\mu}, \underset{R\times R}{V}, \underset{1\times 1}{a}, \underset{1\times 1}{b})$

↳ R = # of regression paremeters

Outputs
- Optimal actions (contextual policy)
- Observed outcome (Reward)
- Regret

→ Function of user itterations

↳ Can be converted to batch itterations

Concrete Example

$$Y = \beta_0 + \beta_1 \text{charity2} + \beta_2 \text{Charity3} + \beta_3 \text{Matching} + \beta_4 \text{Republican} + \beta_5 \text{Republican} \times \text{Matching} + \varepsilon \Rightarrow R = 6$$

Possible Hypo. Models

$$A = \underbrace{[0,1]}_{C2} \times \underbrace{[0,1]}_{C3} \times \underbrace{[0,1]}_{M} - \underbrace{\{[1,1,0],[1,1,1]\}}_{\text{Cannot be both } C2, C3}$$ (Action Set)

↳ Charity2 + Charity3 < 2

len(A) = 6

$$X^* = \underbrace{[0,1]}_{R} \Rightarrow \text{len}(X^*) = 2, \quad \tilde{X} = \underbrace{[0,1]}_{R} \times \underbrace{[0,1]}_{M} \Rightarrow \text{len}(\tilde{X}) = 4$$

(Context Set) (Interaction set)

↳ $\hat{X} = \{ r \times m \mid (r,m) \in \tilde{X} \}$

User Example

$$\begin{cases} \text{Realized Context: } \underset{1\times 1}{X^*} = 1 \\ \text{Choose action: } \underset{1\times 3}{a} = (0,0,1) \\ \text{Interaction: } \underset{1\times 2}{\tilde{x}} = (1,1) \to \underset{1\times 1}{1} = \hat{x} \end{cases} \Longrightarrow \underbrace{\underset{1\times 6}{\underset{\sim}{X}} = [\overset{\text{Constant term}}{1}, \underbrace{0,0,1,1,1}_{(X^*,\, a,\, \hat{x})}]}_{\text{Design Vector}}$$

$$\Longrightarrow \text{Thompson Sample (Priors)} \begin{cases} \underset{6\times 1}{\beta} \sim N(\underset{6\times 1}{\mu}, \underset{6\times 6}{V}) \\ \sigma^2 \sim \text{invgamma}(\underset{1\times 1}{\alpha_1}, \underset{1\times 1}{\alpha_2}) \end{cases}$$

$$\Longrightarrow \underbrace{\underset{1\times 1}{E(Y \mid \underset{\sim}{X})}}_{\text{Reward}} = \underset{1\times 6}{\underset{\sim}{X}} \, \underset{6\times 1}{\beta} = \beta_0 + \beta_3 + \beta_4 + \beta_5$$

After iterating through all the users in the batch, we will eventually get a design matrix containing information from that batch. Suppose batch size is 10, and the 10th user is (charity2, no matching, democratic) then the design matrix will be:

$$\text{Design Matrix: } \underset{10\times 6}{X} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{Observed Outcome: } \underset{10\times 1}{y} = \begin{pmatrix} 11 \\ \vdots \\ 4 \end{pmatrix}$$

(Donations in \$)

Update Priors

↳ Uses Batch Data

$$(\underset{10\times 6}{X}, \underset{10\times 1}{y}, \underset{6\times 1}{\beta}, \underset{1\times 1}{\alpha_1}, \underset{1\times 1}{\alpha_2})$$

$$\begin{cases} \underset{6\times 1}{\tilde{\mu}} = (X'X + V^{-1})^{-1}(X'y + V^{-1}\beta) \\ \underset{6\times 6}{\tilde{V}} = (X'X + V^{-1})^{-1} \\ \underset{1\times 1}{\tilde{\alpha}_1} = \alpha_1 + \frac{n}{2} \\ \underset{1\times 1}{\tilde{\alpha}_2} = \alpha_2 + \frac{(y - X\beta)'(I + XVX')^{-1}(y - X\beta)}{2} \end{cases}$$

Handling Missing Data (Missing Indicator Approach)

$$Y_i = \beta_0 + \beta_1 X_i + \beta_2 X_{miss_i} + \varepsilon_i$$

$$\hookrightarrow X_i = \begin{cases} X_i, & i \text{ not missing} \\ 0, & i \text{ missing} \end{cases}, \quad X_{miss_i} = \begin{cases} 1, & X_i \text{ missing} \\ 0, & X_i \text{ not miss.} \end{cases}$$

Uninformative Prior

Jeffrey's Prior: $P(\beta, \sigma^2) = \frac{1}{\sigma^2}$

$\hookrightarrow$ Initialize $\sigma^2 \sim IG(0.1, 0.1) \Rightarrow \beta \sim U\left[-\frac{\sigma^2}{2}, \frac{\sigma^2}{2}\right]$

Posterior: $\beta_{post} | (\sigma^2, X, y) \sim N(\tilde{\mu}, \sigma^2\tilde{\Sigma})$ and $\sigma^2 | (X, y) \sim IG(\tilde{\alpha}_1, \tilde{\alpha}_2)$

$$\begin{cases} \tilde{\mu} = (X'X)^{-1}X'y \\ \tilde{\Sigma} = (X'X)^{-1} \\ \tilde{\alpha}_1 = \frac{n-k}{2}, \ k = \text{\# of regression parameters} \\ \tilde{\alpha}_2 = \frac{1}{2}(y - X\mu)'(y - X\mu) \end{cases}$$

- Batch 1: Coefficients are drawn from a uniform distribution. Hence the reward function won't be meaningful, and the actions will essentially be randomly assigned for users in batch 1.
- Batch 2: The coefficients are drawn from the posterior (updated from data collected in batch 1). The posterior beta distribution is analogous to the OLS sampling distribution.
- Batch 3: Append data from batch1 and batch2 to obtain the posterior distributions using the above formulas. The reward function on average will resemble the OLS regression.