4.1.

$$\begin{aligned} x[n] &= x_c(nT) \\ &= \sin\left(2\pi(100)n\frac{1}{400}\right) \\ &= \sin\left(\frac{\pi}{2}n\right) \end{aligned}$$

4.2. The discrete-time sequence

$$x[n] = \cos(\frac{\pi n}{4})$$

results by sampling the continuous-time signal

$$x_c(t) = \cos(\Omega_o t).$$

Since $\omega = \Omega T$ and $T = 1/1000$ seconds, the signal frequency could be:

$$\Omega_o = \frac{\pi}{4} \cdot 1000 = 250\pi$$

or possibly:

$$\Omega_o = (2\pi + \frac{\pi}{4}) \cdot 1000 = 2250\pi.$$

4.3. (a) Since $x[n] = x_c(nT)$,

$$\begin{aligned} \frac{\pi n}{3} &= 4000\pi n T \\ T &= \frac{1}{12000} \end{aligned}$$

(b) No. For example, since

$$\cos(\frac{\pi}{3}n) = \cos(\frac{7\pi}{3}n),$$

T can be 7/12000.

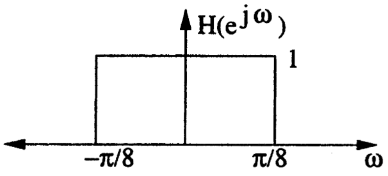
4.4. (a) Letting $T = 1/100$ gives

$$\begin{aligned} x[n] &= x_c(nT) \\ &= \sin\left(20\pi n \frac{1}{100}\right) + \cos\left(40\pi n \frac{1}{100}\right) \\ &= \sin\left(\frac{\pi n}{5}\right) + \cos\left(\frac{2\pi n}{5}\right) \end{aligned}$$

(b) No, another choice is $T = 11/100$:

$$\begin{aligned} x[n] &= x_c(nT) \\ &= \sin\left(20\pi n \frac{11}{100}\right) + \cos\left(40\pi n \frac{11}{100}\right) \\ &= \sin\left(\frac{11\pi n}{5}\right) + \cos\left(\frac{22\pi n}{5}\right) \\ &= \sin\left(\frac{\pi n}{5}\right) + \cos\left(\frac{2\pi n}{5}\right) \end{aligned}$$

4.5. A plot of $H(e^{j\omega})$ appears below.



(a)

$$x_c(t) = 0, \quad ,|\Omega| \geq 2\pi \cdot 5000$$

The Nyquist rate is 2 times the highest frequency. $\Rightarrow T = \frac{1}{10{,}000}$ sec. This avoids all aliasing in the C/D converter.

(b)

$$\begin{aligned}
\frac{1}{T} &= 10kHz \\
\omega &= T\Omega \\
\frac{\pi}{8} &= \frac{1}{10,000}\Omega_c \\
\Omega_c &= 2\pi \cdot 625\text{rad/sec} \\
f_c &= 625Hz
\end{aligned}$$

(c)

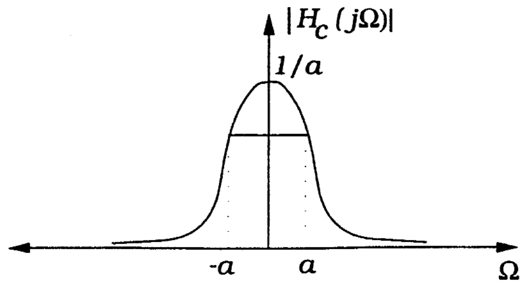
$$\begin{aligned}
\frac{1}{T} &= 20kHz \\
\omega &= T\Omega \\
\frac{\pi}{8} &= \frac{1}{20,000}\Omega_c \\
\Omega_c &= 2\pi \cdot 1250\text{rad/sec} \\
f_c &= 1250Hz
\end{aligned}$$

4.6. (a) The Fourier transform of the filter impulse response

$$\begin{aligned} H_c(j\Omega) &= \int_{-\infty}^{\infty} h_c(t)e^{-j\Omega t}\,dt \\ &= \int_{0}^{\infty} a^{-at}e^{-j\Omega t}\,dt \\ &= \frac{1}{a+j\Omega} \end{aligned}$$

So, we take the magnitude

$$|H_c(j\Omega)| = \left(\frac{1}{a^2+\Omega^2}\right)^{\frac{1}{2}}$$



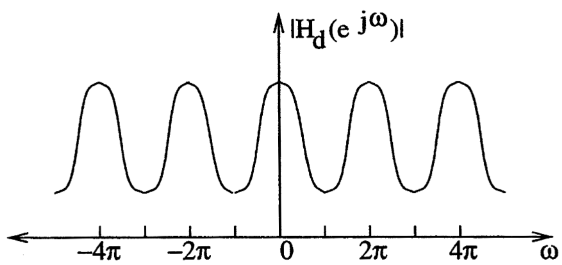
(b) Sampling the filter impulse response in (a), the discrete-time filter is described by

$$\begin{aligned} h_d[n] &= Te^{-anT}u[n] \\ H_d(e^{j\omega}) &= \sum_{n=0}^{\infty} Te^{-anT}e^{-j\omega n} \\ &= \frac{T}{1-e^{-aT}e^{-j\omega}} \end{aligned}$$

Taking the magnitude of this response

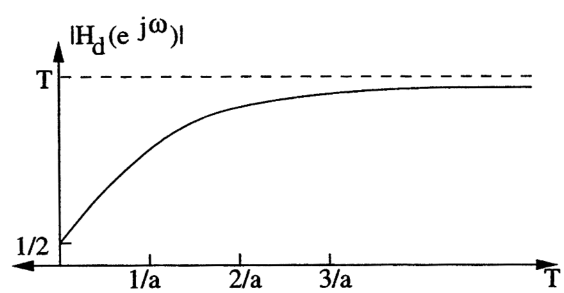
$$|H_d(e^{j\omega})| = \frac{T}{(1-2e^{-aT}\cos(\omega)+e^{-2aT})^{\frac{1}{2}}}.$$

Note that the frequency response of the discrete-time filter is periodic, with period 2π.



(c) The minimum occurs at $\omega = \pi$. The corresponding value of the frequency response magnitude is

$$\begin{aligned} |H_d(e^{j\pi})| &= \frac{T}{(1+2e^{-aT}+e^{-2aT})^{\frac{1}{2}}} \\ &= \frac{T}{1+e^{-aT}}. \end{aligned}$$



4.7. The continuous-time signal contains an attenuated replica of the original signal with a delay of τ_d.

$$x_c(t) = s_c(t) + \alpha s_c(t - \tau_d)$$

(a) Taking the Fourier transform of the analog signal:

$$X_c(j\Omega) = S_c(j\Omega) \cdot (1 + \alpha e^{-j\tau_d \Omega})$$

Note that $X_c(j\Omega)$ is zero for $|\Omega| > \pi/T$. Sampling the continuous-time signal yields the discrete-time sequence, $x[n]$. The Fourier transform of the sequence is

$$\begin{aligned} X(e^{j\omega}) &= \frac{1}{T} \sum_{r=-\infty}^{\infty} S_c(\frac{j\omega}{T} + j\frac{2\pi r}{T}) \\ &+ \frac{\alpha}{T} \sum_{r=-\infty}^{\infty} S_c(\frac{j\omega}{T} + j\frac{2\pi r}{T}) e^{-j\tau_d(\frac{\omega}{T} + \frac{2\pi r}{T})} \end{aligned}$$

(b) The desired response:

$$H(j\Omega) = \begin{cases} 1 + \alpha e^{-j\tau_d \Omega}, & \text{for } |\Omega| \le \frac{\pi}{T} \\ 0, & \text{otherwise} \end{cases}$$

Using $\omega = \Omega T$, we obtain a discrete-time system which simulates the above response:

$$H(e^{j\omega}) = 1 + \alpha e^{-j\frac{\tau_d \omega}{T}}$$

(c) We need to take the inverse Fourier transform of the discrete-time impulse response of part (b).

$$\begin{aligned} h[n] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} \, d\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (1 + \alpha e^{-j\frac{\tau_d \omega}{T}}) e^{j\omega n} \, d\omega \end{aligned}$$

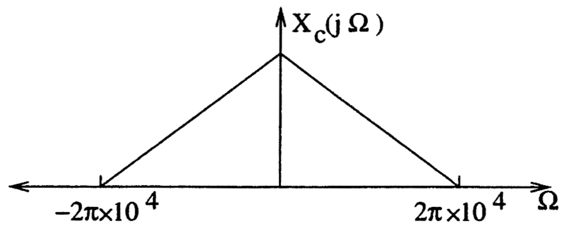
(i) Consider the case when $\tau_d = T$:

$$\begin{aligned} h[n] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (e^{j\omega n} + \alpha e^{j\omega(n-1)}) \, d\omega \\ &= \frac{\sin(\pi n)}{\pi n} + \frac{\alpha \sin[\pi(n-1)]}{\pi(n-1)} \\ &= \delta[n] + \alpha \delta[n-1] \end{aligned}$$

(ii) For $\tau_d = T/2$:

$$\begin{aligned} h[n] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (e^{j\omega n} + \alpha e^{j\omega(n-\frac{1}{2})}) \, d\omega \\ &= \frac{\sin(\pi n)}{\pi n} + \frac{\alpha \sin[\pi(n-\frac{1}{2})]}{\pi(n-\frac{1}{2})} \\ &= \delta[n] + \frac{\alpha \sin[\pi(n-\frac{1}{2})]}{\pi(n-\frac{1}{2})} \end{aligned}$$

4.8. A plot of $X_c(j\Omega)$ appears below.



(a) For $x_c(t)$ to be recoverable from $x[n]$, the transform of the discrete signal must have no aliasing. When sampling, the radian frequency is related to the analog frequency by

$$\omega = \Omega T.$$

No aliasing will occur if the sampling interval satisfies the Nyquist Criterion. Thus, for the band-limited signal, $x_c(t)$, we should select T as:

$$T \leq \frac{1}{2 \times 10^4}.$$

(b) Assuming that the system is linear and time-invariant, the convolution sum describes the input-output relationship.

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

We are given

$$\begin{aligned} y[n] &= T \sum_{k=-\infty}^{n} x[k] \\ &= T \sum_{k=-\infty}^{\infty} x[k]u[n-k] \end{aligned}$$

Hence, we may infer that the impulse response of the system

$$h[n] = T \cdot u[n].$$

(c) We use the expression for $y[n]$ as given and examine the limit

$$\begin{aligned} \lim_{n\to\infty} y[n] &= \lim_{n\to\infty} T \cdot \sum_{k=-\infty}^{n} x[k] \\ &= T \cdot \sum_{k=-\infty}^{\infty} x[k] \end{aligned}$$

Recall the analysis equation for the Fourier transform:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

Hence,

$$\lim_{n\to\infty} y[n] = T \cdot X(e^{j\omega})|_{\omega=0}$$

(d) We use the result from part (c). Noting that

$$X(e^{j\omega}) = \frac{1}{T}\sum_{r=-\infty}^{\infty} X_c(\frac{j\omega}{T} + \frac{j2\pi r}{T}).$$

Thus, we have

$$T \cdot X(e^{j\omega})|_{\omega=0} = \sum_{r=-\infty}^{\infty} X_c(\frac{j2\pi r}{T})$$

From the given information, we seek a value of T such that:

$$\begin{aligned}\sum_{r=-\infty}^{\infty} X_c(\frac{j2\pi r}{T}) &= \int_{-\infty}^{\infty} x_c(t)\,dt \\ &= X_c(j\Omega)|_{\Omega=0}\end{aligned}$$

For the final equality to be true, there must be no contribution from the terms for which $r \neq 0$. That is, we require no aliasing at $\Omega = 0$. Since we are only interested in preserving the spectral component at $\Omega = 0$, we may sample at a rate which is lower than the Nyquist rate. The maximum value of T to satisfy these conditions is

$$T \leq \frac{1}{1 \times 10^4}.$$

4.9. (a) Since $X(e^{j\omega}) = X(e^{j(\omega-\pi)})$, $X(e^{j\omega})$ is periodic with period π.

(b) Using the inverse DTFT,

$$\begin{aligned} x[n] &= \frac{1}{2\pi}\int_{\langle 2\pi\rangle} X(e^{j\omega})e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\int_{\langle 2\pi\rangle} X(e^{j(\omega-\pi)})e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\int_{\langle 2\pi\rangle} X(e^{j\omega})e^{j(\omega+\pi)n}d\omega \\ &= \frac{1}{2\pi}e^{j\pi n}\int_{\langle 2\pi\rangle} X(e^{j\omega})e^{j\omega n}d\omega \\ &= (-1)^n x[n]. \end{aligned}$$

All odd samples of $x[n] = 0$, because $x[n] = -x[n]$. Hence $x[3] = 0$.

(c) Yes, $y[n]$ contains all even samples of $x[n]$, and all odd samples of $x[n]$ are 0.

$$x[n] = \begin{cases} y[n/2], & n \text{ even} \\ 0, & \text{otherwise} \end{cases}$$

4.10. Use $x[n] = x_c(nT)$, and simplify:

(a) $x[n] = \cos(2\pi n/3)$.

(b) $x[n] = \sin(4\pi n/3) = -\sin(2\pi n/3)$

(c) $x[n] = \frac{\sin(2\pi n/5)}{\pi n/5000}$

4.11. (a) Pick T such that

$$x[n] = x_c(nT) = \sin(10\pi nT) = \sin(\pi n/4) \qquad \Longrightarrow T = 1/40$$

There are other choices. For example, by realizing that $\sin(\pi n/4) = \sin(9\pi n/4)$, we find $T = 9/40$.

(b) Choose $T = 1/20$ to make $x[n] = x_c(nT)$. This is unique.

4.12. (a) Notice first that $H(e^{j\omega}) = 10j\omega, -\pi \leq \omega < \pi$.

(i) After sampling,

$$\begin{aligned} x[n] &= \cos(\frac{3\pi}{5}n), \\ y[n] &= |H(e^{j\frac{3\pi}{5}})|\cos(\frac{3\pi}{5}n + \angle H(e^{j\frac{3\pi}{5}})) \\ &= 6\pi\cos(\frac{3\pi}{5}n + \frac{\pi}{2}) \\ &= -6\pi\sin(\frac{3\pi}{5}n) \\ y_c(t) &= -6\pi\sin(6\pi t). \end{aligned}$$

(ii) After sampling, $x[n] = \cos(\frac{7\pi}{5}n) = \cos(\frac{3\pi}{5}n)$, so again, $y_c(t) = -6\pi\sin(6\pi t)$.

(b) $y_c(t)$ is what you would expect from a differentiator in the first case but not in the second case. This is because aliasing has occurred in the second case.

4.13. (a)

$$\begin{aligned} x_c(t) &= \sin(\frac{\pi}{20}t) \\ y_c(t) &= \sin(\frac{\pi}{20}(t-5)) \\ &= \sin(\frac{\pi}{20}t - \frac{\pi}{4}) \\ y[n] &= \sin(\frac{\pi n}{2} - \frac{\pi}{4}) \end{aligned}$$

(b) We get the same result as before:

$$\begin{aligned} x_c(t) &= \sin(\frac{\pi}{10}t) \\ y_c(t) &= \sin(\frac{\pi}{10}(t-2.5)) \\ &= \sin(\frac{\pi}{10}t - \frac{\pi}{4}) \\ y[n] &= \sin(\frac{\pi n}{2} - \frac{\pi}{4}) \end{aligned}$$

(c) The sampling period T is not limited by the continuous time system $h_c(t)$.

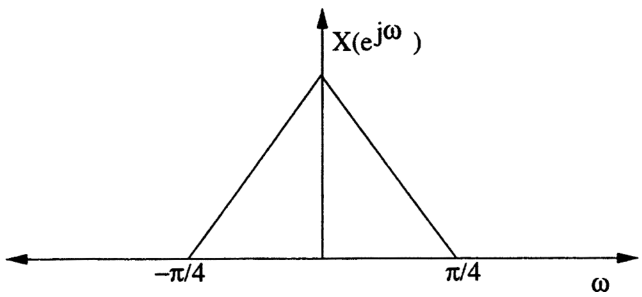
4.14. There is no loss of information if $X(e^{j\omega/2})$ and $X(e^{j(\omega/2-\pi)})$ do not overlap. This is true for (b), (d), (e).

4.15. The output $x_r[n] = x[n]$ if no aliasing occurs as result of downsampling. That is, $X(e^{j\omega}) = 0$ for $\pi/3 \leq |\omega| \leq \pi$.

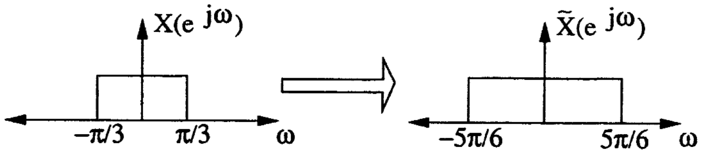
(a) $x[n] = \cos(\pi n/4)$. $X(e^{j\omega})$ has impulses at $\omega = \pm\pi/4$, so there is no aliasing. $x_r[n] = x[n]$.

(b) $x[n] = \cos(\pi n/2)$. $X(e^{j\omega})$ has impulses at $\omega = \pm\pi/2$, so there is aliasing. $x_r[n] \neq x[n]$.

(c) A sketch of $X(e^{j\omega})$ is shown below. Clearly there will be no aliasing and $x_r[n] = x[n]$.



4.16. (a) In the frequency domain, we have



$$\frac{M}{L} = \frac{5\pi/6}{\pi/3} = \frac{5}{2}$$

This is unique.

(b) One choice is

$$\frac{M}{L} = \frac{\pi/2}{3\pi/4} = \frac{2}{3}$$

However, this is not unique. We can also write $\tilde{x}_d[n] = \cos(\frac{5\pi}{2}n)$, so another choice is

$$\frac{M}{L} = \frac{5\pi/2}{3\pi/4} = \frac{10}{3}$$

4.17. (a) In the frequency domain,

$$X(e^{j\omega}) = \begin{cases} 1, & |\omega| < 2\pi/3 \\ 0, & 2\pi/3 < |\omega| < \pi \end{cases}$$

After the sampling rate change,

$$\tilde{X}_d(e^{j\omega}) = \begin{cases} 4/3, & |\omega| < \pi/2 \\ 0, & \pi/2 < |\omega| < \pi \end{cases},$$

which leads to

$$x[n] = \frac{4}{3}\frac{\sin(\pi n/2)}{\pi n}$$

(b) Upsampling by 3 and low-pass filtering $x[n] = \sin(3\pi n/4)$ results in $\sin(\pi n/4)$. Downsampling by 5 gives us $\tilde{x}_d[n] = \sin(5\pi n/4) = -\sin(3\pi n/4)$.

4.18. For the condition to be satisfied, we have to ensure that $\omega_0/L \leq \min(\pi/L, \pi/M)$, so that the lowpass filtering does not cut out part of the spectrum.

(a) $\omega_0/2 \leq \pi/3 \Longrightarrow \omega_{0,max} = 2\pi/3$.

(b) $\omega_0/3 \leq \pi/5 \Longrightarrow \omega_{0,max} = 3\pi/5$.

(c) Since $L > M$, there is no chance of aliasing. Hence $\omega_{0,max} = \pi$.

4.19. The nyquist sampling property must be satisfied: $T \leq \pi/\Omega_0$.

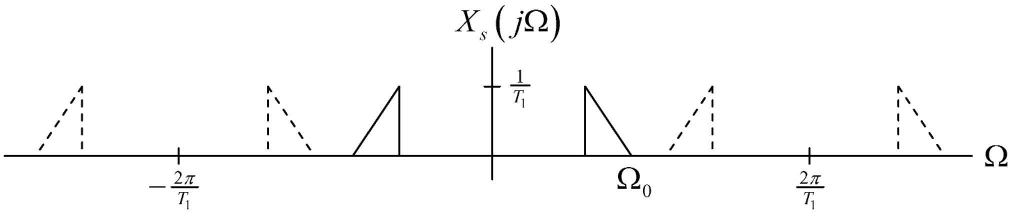
4.20. (a) The Nyquist sampling property must be satisfied: $T \leq \pi/\Omega_0 \Longrightarrow F_s \geq 2000$.

(b) We'd have to sample so that $X(e^{j\omega})$ lies between $|\omega| < \pi/2$. So $F_s \geq 4000$.

4.21. A. The impulse-train signal $x_s(t)$ has spectrum $X_s(j\Omega)$ given by

$$X_s(j\Omega)=\frac{1}{T}\sum_{k=-\infty}^{\infty}X\left[j\left(\Omega-k\frac{2\pi}{T_1}\right)\right].$$

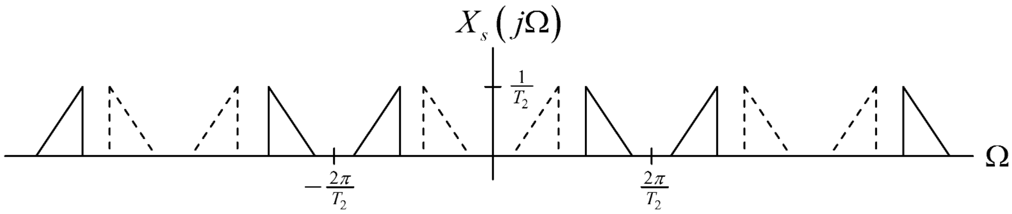
An example is shown below.



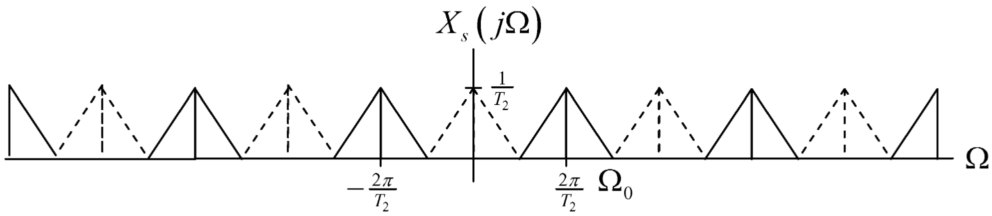
We will have $x_r(t)=x_c(t)$ provided $T_1\le\frac{\pi}{\Omega_0}$.

B. We will have $x_o(t)=x_c(t)$ under any of the following circumstances:

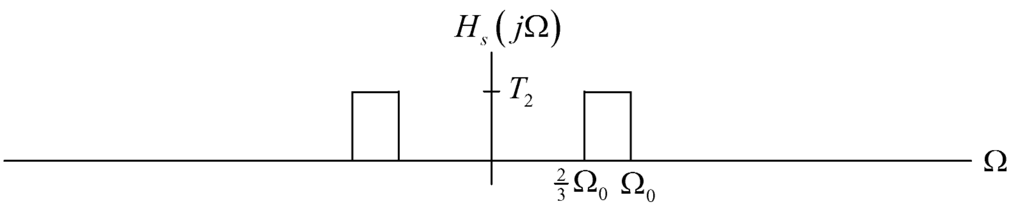
1. As illustrated above, $T_2\le\frac{\pi}{\Omega_0}$.
2. As illustrated below, $\frac{1.5\pi}{\Omega_0}\le T_2\le\frac{2\pi}{\Omega_0}$.



3. As illustrated below, $T_2=\frac{3\pi}{\Omega_0}$.



The frequency response of the filter that is needed to recover $x_c(t)$ is shown below.



4.22. (a) Given

$$x[n] = \cos(\omega_0 n), \quad \omega_0 = \Omega_0 T < \pi,$$

we have from Table 2.3,

$$X(e^{j\omega}) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0), \quad |\omega| < \pi.$$

(b) Eq. (4.46) gives $H(e^{j\omega}) = \dfrac{j\omega}{T}, \quad |\omega| < \pi$. Then

$$\begin{aligned} Y(e^{j\omega}) &= H(e^{j\omega})X(e^{j\omega}) \\ &= \frac{j\omega}{T}\left[\pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)\right] \\ &= \frac{j\omega_0}{T}\pi\delta(\omega - \omega_0) - \frac{j\omega_0}{T}\pi\delta(\omega + \omega_0), \quad |\omega| < \pi. \end{aligned}$$

(c) From Eq. (4.32),

$$\begin{aligned} Y_r(j\Omega) = H_r(j\Omega)Y(e^{j\Omega T}) &= \begin{cases} TY(e^{j\Omega T}), & |\Omega| < \dfrac{\pi}{T} \\ 0, & \text{otherwise} \end{cases} \\ &= j\omega_0\pi\delta(\Omega T - \omega_0) - j\omega_0\pi\delta(\Omega T + \omega_0) \\ &= j\omega_0\pi\delta\left[(\Omega - \omega_0/T)T\right] - j\omega_0\pi\delta\left[(\Omega + \omega_0/T)T\right]. \end{aligned}$$

(d) The inverse Fourier transform of $\delta(\Omega T)$ is the constant $1/(2\pi T)$. We then have

$$\begin{aligned} y_r(t) &= \frac{j\omega_0}{2T}e^{j\omega_0 t/T} - \frac{j\omega_0}{2T}e^{-j\omega_0 t/T} \\ &= -\Omega_0\left(\frac{e^{j\omega_0 t/T} - e^{-j\omega_0 t/T}}{j2}\right) \\ &= -\Omega_0 \sin(\Omega_0 t), \end{aligned}$$

as was to have been shown.

4.23. Appears in: Fall04 PS1, Fall02 PS1.

Problem

Note that in OSB and 6.341 Ω denotes continuous-time frequency and ω denotes discrete-time frequency.

Figure 1 shows a continuous-time filter that is implemented using an LTI discrete-time filter with frequency response $H(e^{j\omega})$.

(a) If the CTFT of $x_c(t)$, namely $X_c(j\Omega)$, is as shown in Figure 2 and $\omega_c = \frac{\pi}{5}$, sketch and label $X(e^{j\omega})$, $Y(e^{j\omega})$ and $Y_c(j\Omega)$ for each of the following cases:

- (i) $1/T_1 = 1/T_2 = 2 \times 10^4$
- (ii) $1/T_1 = 4 \times 10^4, \;\; 1/T_2 = 10^4$
- (iii) $1/T_1 = 10^4, \;\; 1/T_2 = 3 \times 10^4$

(b) For $1/T_1 = 1/T_2 = 6 \times 10^3$, and for input signals $x_c(t)$ whose spectra are bandlimited to $|\Omega| < 2\pi \times 5 \times 10^3$ (but otherwise unconstrained), what is the maximum choice of the cutoff frequency ω_c of the filter $H(e^{j\omega})$ for which the overall system is LTI? For this maximum choice of ω_c, specify $H_c(j\Omega)$.

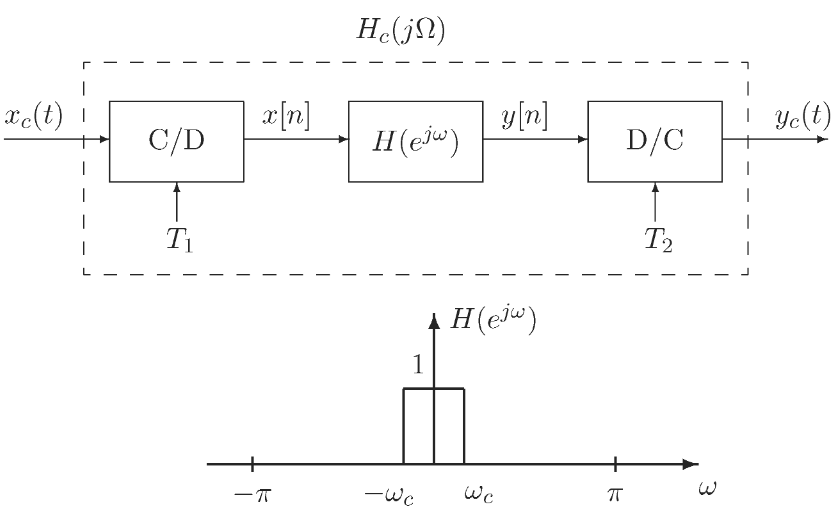


Figure 1: Problem .

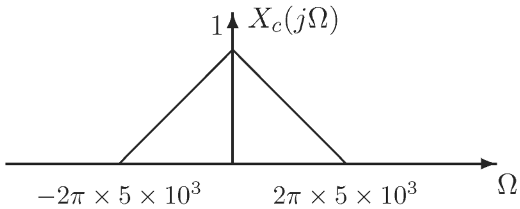
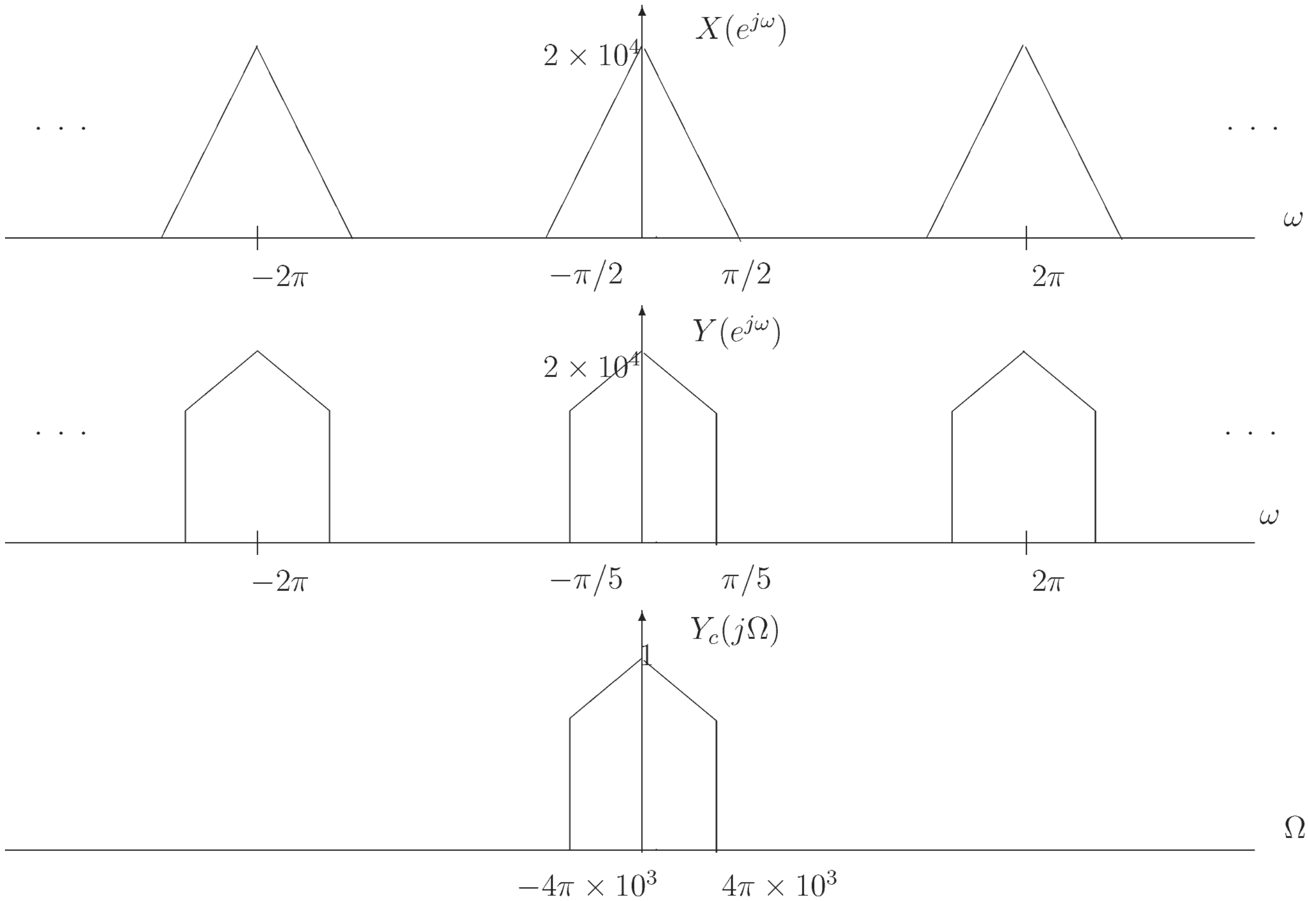


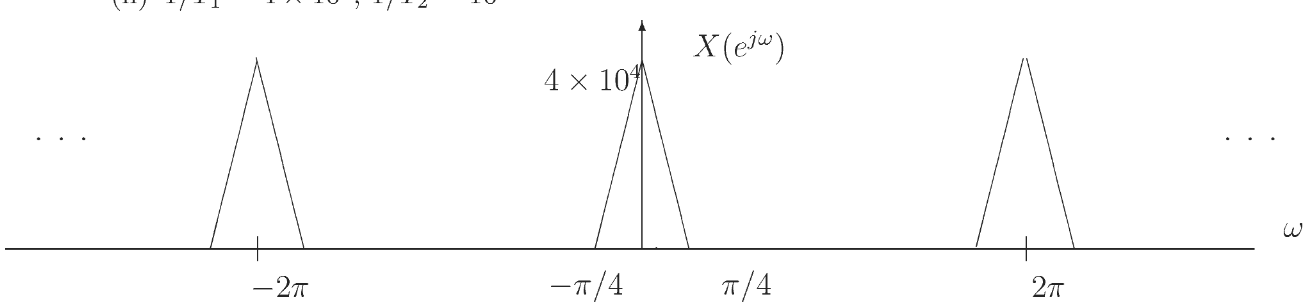
Figure 2: Problem , part (a).

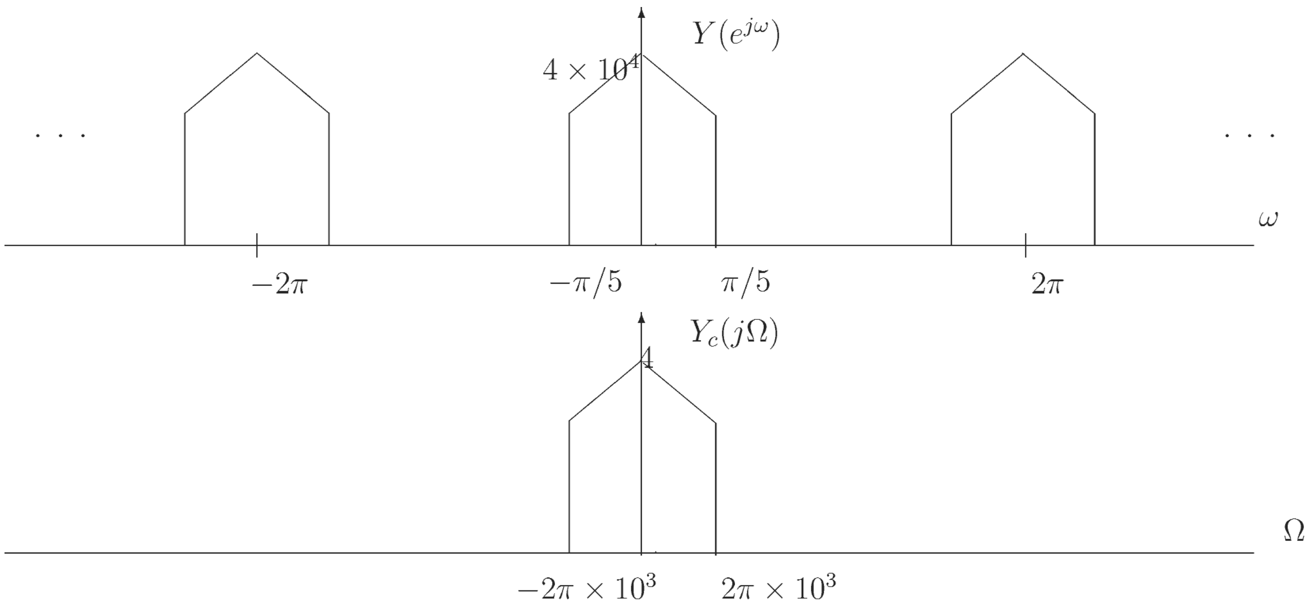
Solution from Fall04 PS1

(a) (i) $1/T_1 = 1/T_2 = 2 \times 10^4$

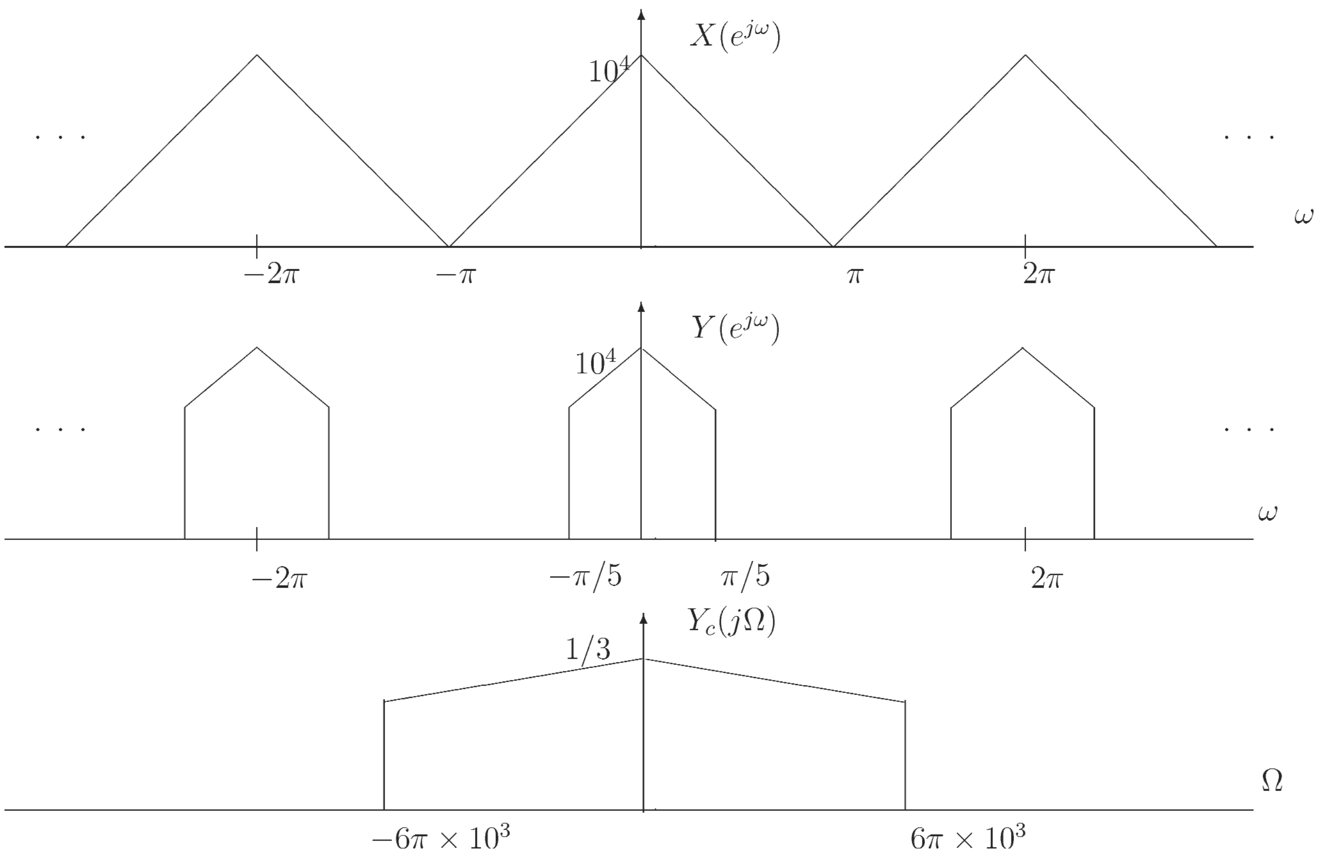


(ii) $1/T_1 = 4 \times 10^4$, $1/T_2 = 10^4$



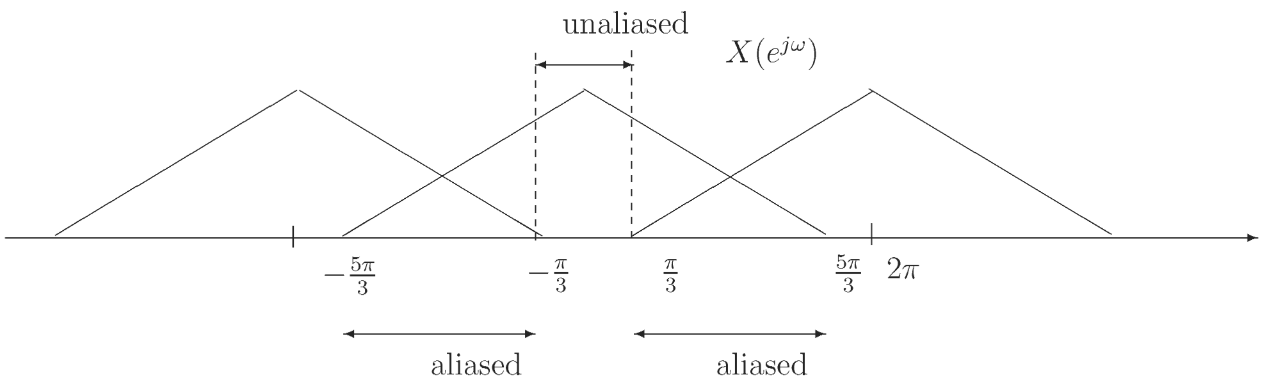


(iii) $1/T_1 = 10^4$, $1/T_2 = 3 \times 10^4$



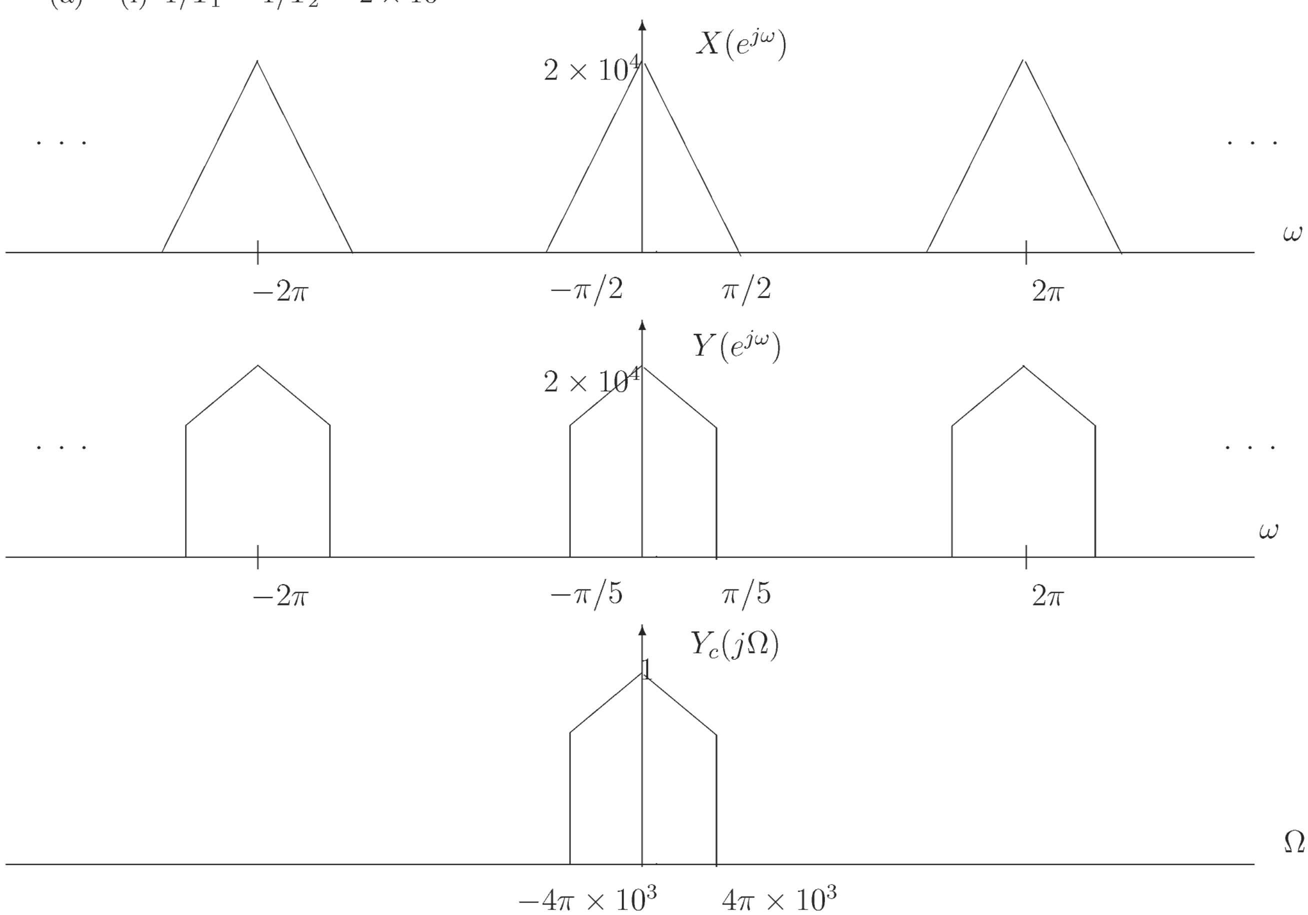
(b) From the figure below, it can be seen that the only portion of the spectrum which remains unaffected by the aliasing is $|\omega| < \pi/3$. So if we choose $\omega_c < \pi/3$, the overall system is LTI with a frequency response of

$$H_c(j\Omega) = \begin{cases} 1 & \text{for } |\Omega| < \omega_c \times 6 \times 10^3 \\ 0 & \text{otherwise.} \end{cases}$$

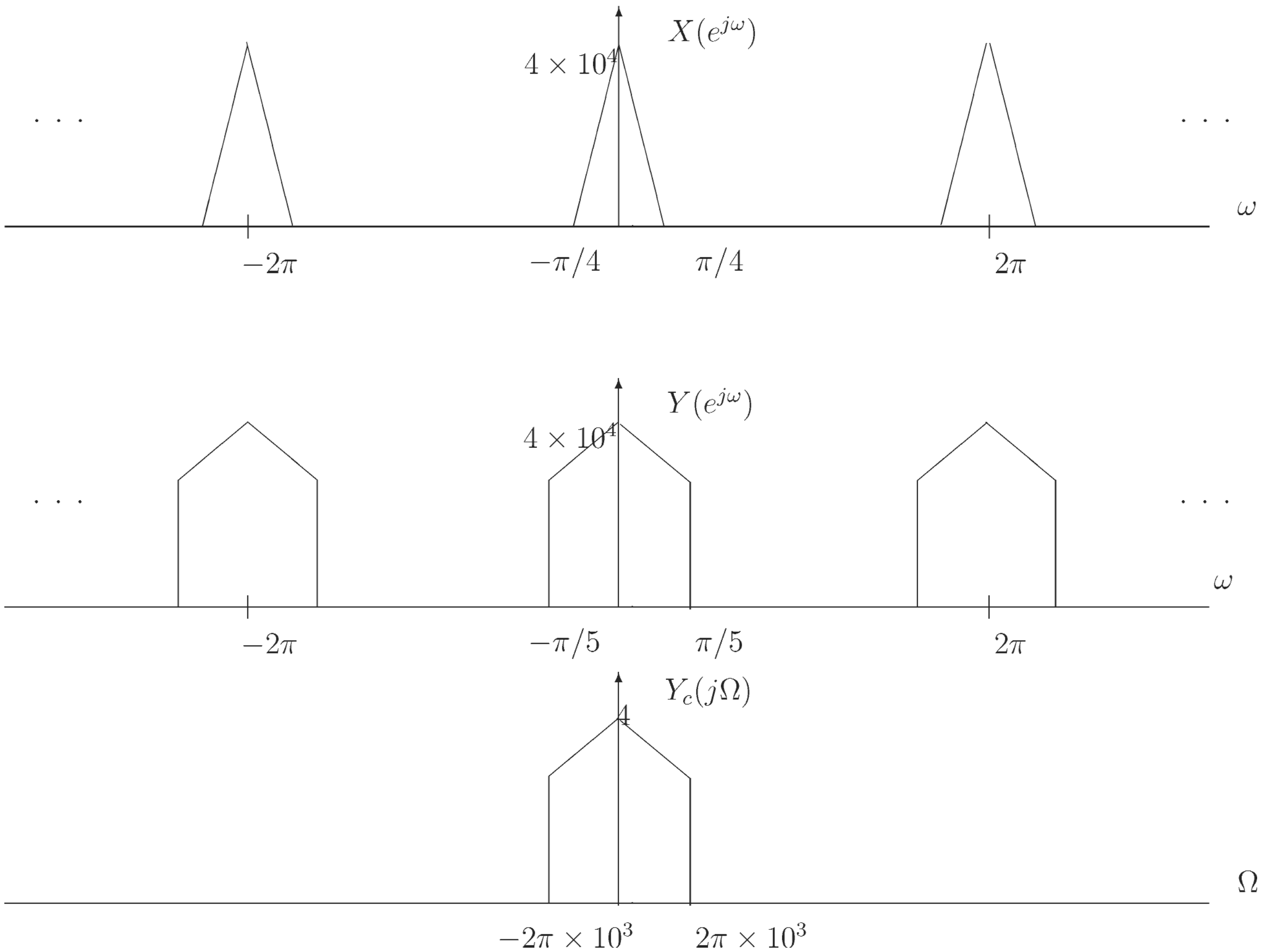


Solution from Fall02 PS1

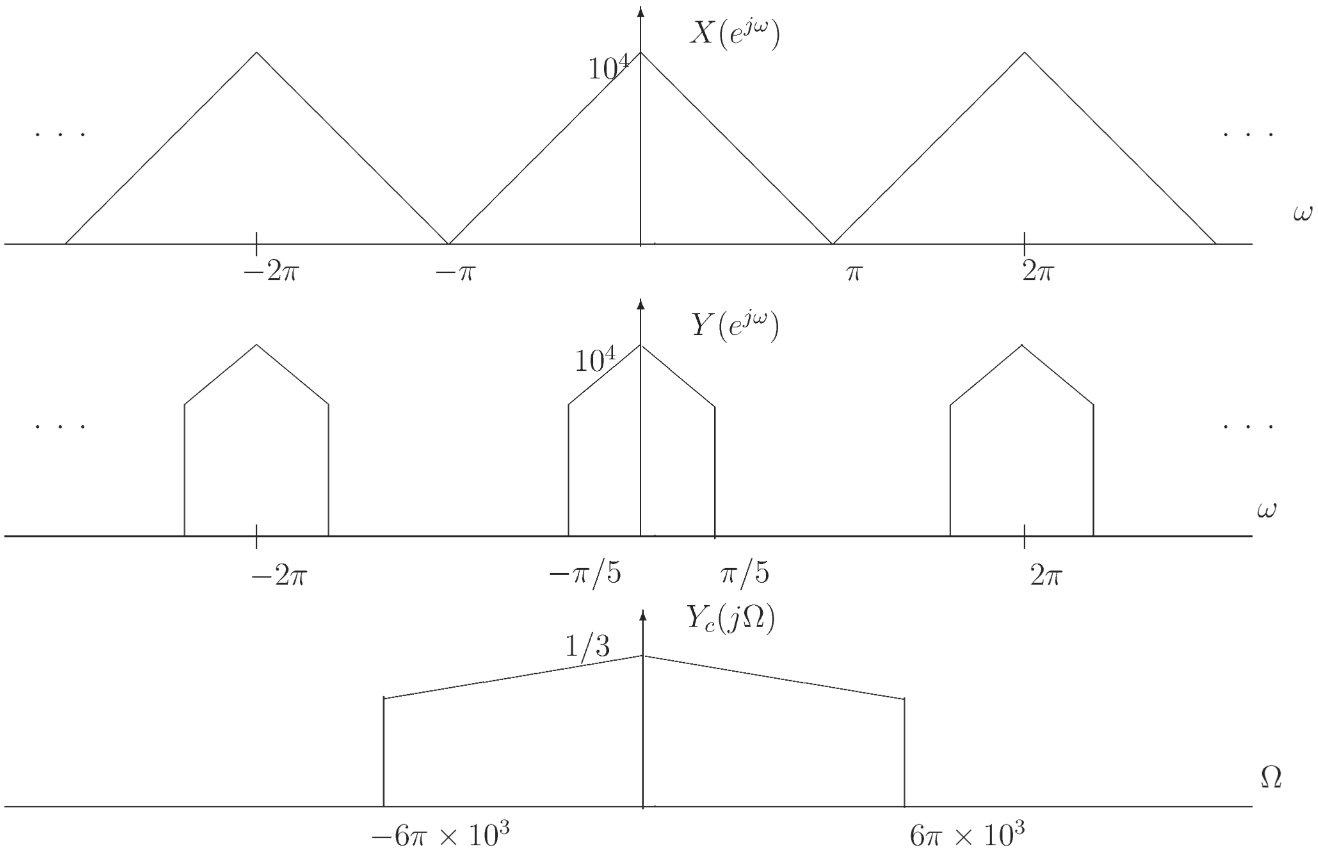
(a) (i) $1/T_1 = 1/T_2 = 2 \times 10^4$



(ii) $1/T_1 = 4 \times 10^4$, $1/T_2 = 10^4$

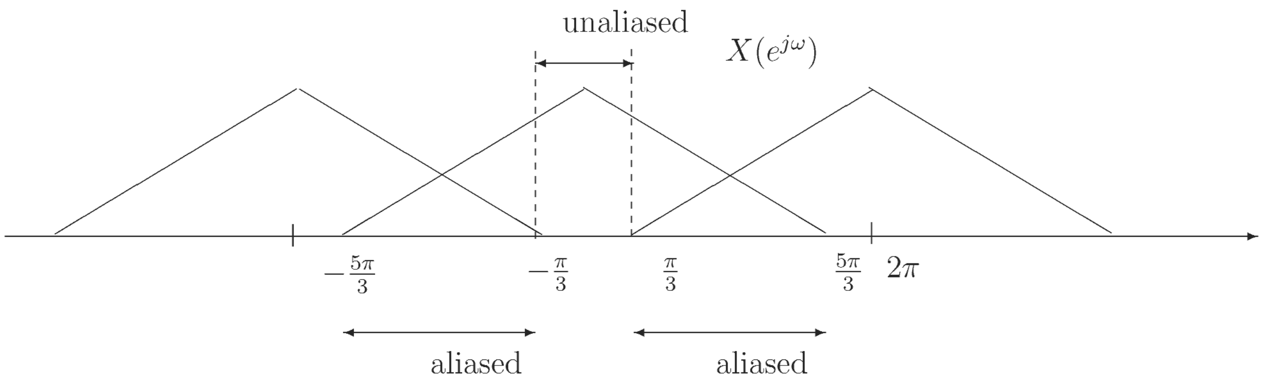


(iii) $1/T_1 = 10^4$, $1/T_2 = 3 \times 10^4$



(b) From the figure below, it can be seen that the only portion of the spectrum which remains unaffected by the aliasing is $|\omega| < \pi/3$. So if we choose $\omega_c < \pi/3$, the overall system is LTI with a frequency response of

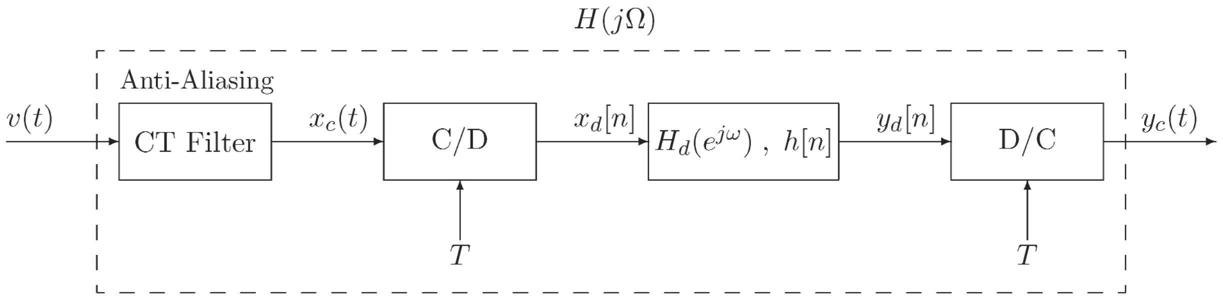
$$H_c(j\Omega) = \begin{cases} 1 & \text{for } |\Omega| < \omega_c \times 6 \times 10^3 \\ 0 & \text{otherwise.} \end{cases}$$



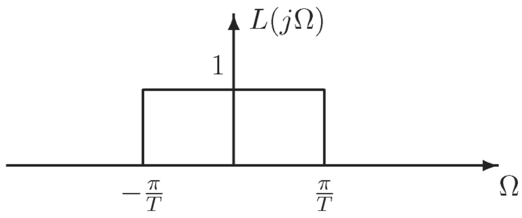
4.24. Problem 13 from Fall 2005 Background exam

Problem

Consider the following system:



The anti-aliasing filter is a continuous-time filter with the frequency response $L(j\Omega)$ shown below:



The frequency response of the LTI discrete-time system between the converters is given by:

$$H_d(e^{j\omega}) = e^{-j\frac{\omega}{3}}\ , \quad |\omega| < \pi$$

(a) What is the effective continuous-time frequency response of the overall system, $H(j\Omega)$?

(b) Choose the most accurate statement:

A. $y_c(t) = \frac{d}{dt}x_c(3t)$
B. $y_c(t) = x_c(t - \frac{T}{3})$
C. $y_c(t) = \frac{d}{dt}x_c(t - 3T)$
D. $y_c(t) = x_c(t - \frac{1}{3})$

(c) Express $y_d[n]$ in terms of $y_c(t)$.

(d) Determine the impulse response $h[n]$ of the discrete-time LTI system.

Solution from Fall05 background exam

(a) $H(j\Omega) = e^{-j\Omega\frac{T}{3}}$ for $|\Omega| < \frac{\pi}{T}$, 0 otherwise.

(b) (Circle one) A $\boxed{\text{B}}$ C D E

(c) $y_d[n] = y_c(nT)$

(d) $h[n] = \frac{\sin(\pi(n-\frac{1}{3}))}{\pi(n-\frac{1}{3})}$

4.25. We are given $w(t) = x_1(t)x_2(t)$. From the properties of the continuous-time Fourier transform we have $W(j\Omega) = \frac{1}{2\pi}X_1(j\Omega) * X_2(j\Omega)$. This implies that $w(t)$ is bandlimited to $\Omega_1 + \Omega_2$; that is,

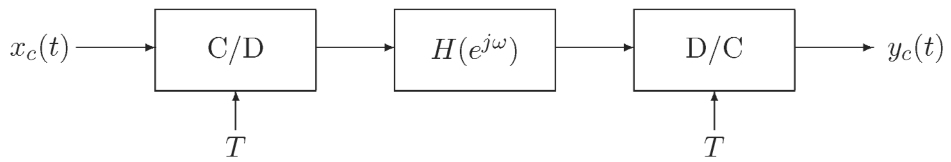
$$W(j\Omega) = 0, \quad |\Omega_1 + \Omega_2| \geq 0.$$

Now $w(t)$ will be recoverable from $w_p(t)$ through the use of an ideal lowpass filter provided $\dfrac{1}{T} > 2(\Omega_1 + \Omega_2)$, or $T < \dfrac{1}{2(\Omega_1 + \Omega_2)}$.

4.26. Problem 10 from Fall 2002 Background exam

Problem

We are trying to design a discrete time system to filter continuous time music signals, sampled at 16kHz, as shown below:



$H(e^{j\omega})$ is an ideal lowpass filter with a cutoff of $\frac{\pi}{2}$rads/sample.
What should the input $x_c(t)$ be bandlimited to so that the overall system is LTI?

Solution from Fall02 background exam

For the overall system to be LTI, $x_c(t)$ should be bandlimited to: 12kHz (aliased components beyond 8kHz are rejected by the low-pass filter).

4.27. A. The input signal is sampled at a rate high enough to avoid aliasing. Then

$$X_d\left(e^{j\omega}\right) = \frac{1}{T} X_c\left(j\frac{\omega}{T}\right), \quad |\omega| < \pi .$$

Now

$$\begin{aligned} Y_d\left(e^{j\omega}\right) &= H_d\left(e^{j\omega}\right) X_d\left(e^{j\omega}\right) \\ &= \frac{1}{T} H_d\left(e^{j\omega}\right) X_c\left(j\frac{\omega}{T}\right), \quad |\omega| < \pi \end{aligned}$$

The D/C converter includes an ideal lowpass filter of bandwidth $\frac{\pi}{T}$ and gain T. Therefore

$$\begin{aligned} Y_c(j\Omega) &= \begin{cases} TY_d\left(e^{j\Omega T}\right), & |\Omega| < \dfrac{\pi}{T} \\ 0, & \text{otherwise} \end{cases} \\ &= H_d\left(e^{j\Omega T}\right) X_c(j\Omega). \end{aligned}$$

The continuous-time frequency response of the end-to-end system is given by

$$\begin{aligned} H_c(j\Omega) &= \begin{cases} \dfrac{Y_c(j\Omega)}{X_c(j\Omega)}, & |\Omega| < \dfrac{\pi}{T} \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} H_d\left(e^{j\Omega T}\right), & |\Omega| < \dfrac{\pi}{T} \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} \dfrac{e^{j\Omega T/2} - e^{-j\Omega T/2}}{T}, & |\Omega| < \dfrac{\pi}{T} \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} \dfrac{j2}{T}\sin(\Omega T/2), & |\Omega| < \dfrac{\pi}{T} \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

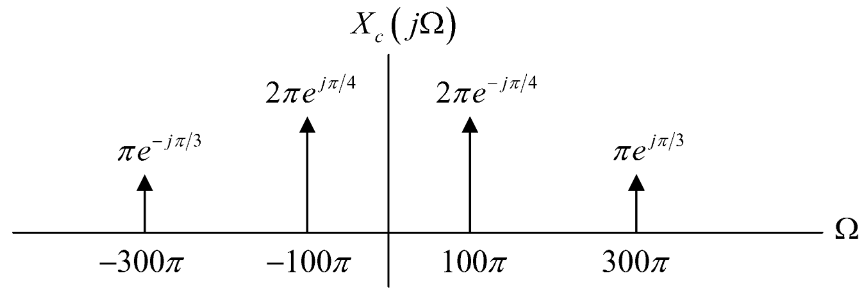
B. We are given $x_c(t) = \dfrac{\sin(\Omega_M t)}{\Omega_M t}$, with $\Omega_M = \dfrac{\pi}{T}$. Then

$$\begin{aligned} x_d[n] &= \frac{\sin(\Omega_M nT)}{\Omega_M nT} \\ &= \frac{\sin(\pi n)}{\pi n} \\ &= 0. \end{aligned}$$

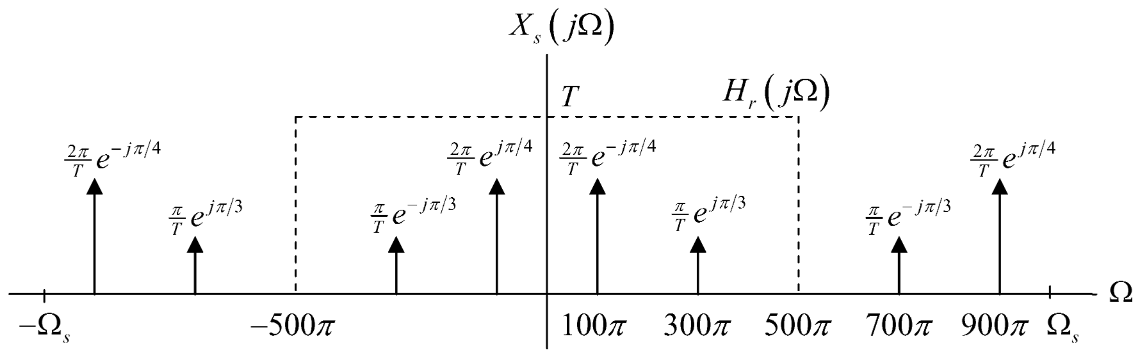
But then $y_d[n] = 0$ and $y_c(t) = 0$ as well.

4.28. A.

$$X_c(j\Omega) = 2\pi e^{-j\pi/4}\delta(\Omega - 100\pi) + 2\pi e^{j\pi/4}\delta(\Omega + 100\pi)$$
$$+ \pi e^{j\pi/3}\delta(\Omega - 300\pi) + \pi e^{-j\pi/3}\delta(\Omega + 300\pi).$$



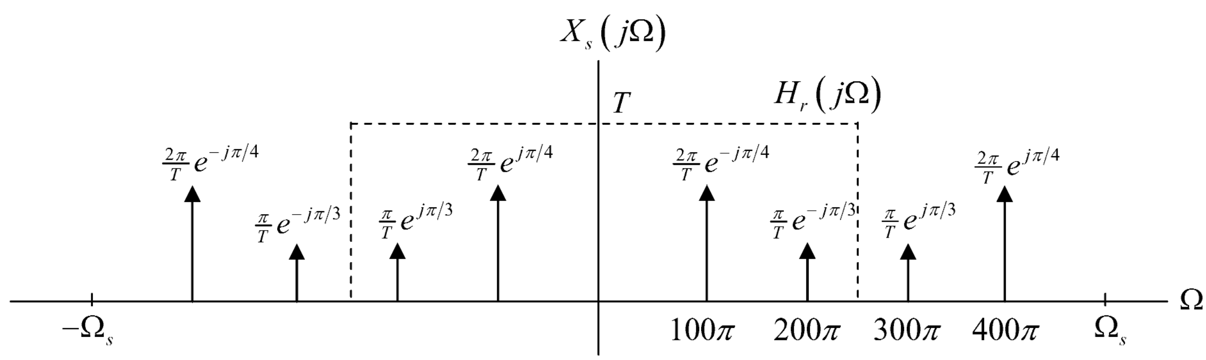
B. If $f_s = 1/T = 500$ samples/s then $\Omega_s = 2\pi/T = 1000\pi$ rad/s.



There is no aliasing, so $x_r(t) = x_c(t)$; that is,

$$x_r(t) = 2\cos(100\pi t - \pi/4) + \cos(300\pi t + \pi/3).$$

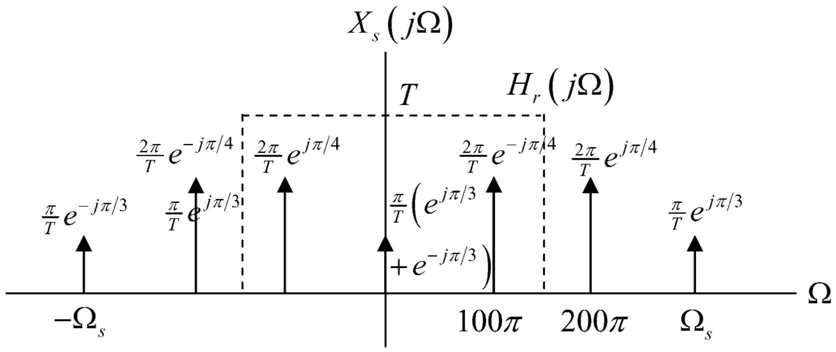
C. If $f_s = 1/T = 250$ samples/s then $\Omega_s = 2\pi/T = 500\pi$ rad/s.



Now there is aliasing and

$$x_r(t) = 2\cos(100\pi t - \pi/4) + \cos(200\pi t - \pi/3).$$

D. We want to sample the component at 300π rad/s exactly once per cycle, so that all the samples have the same value. At $\Omega_s = 300\pi$ rad/s we have

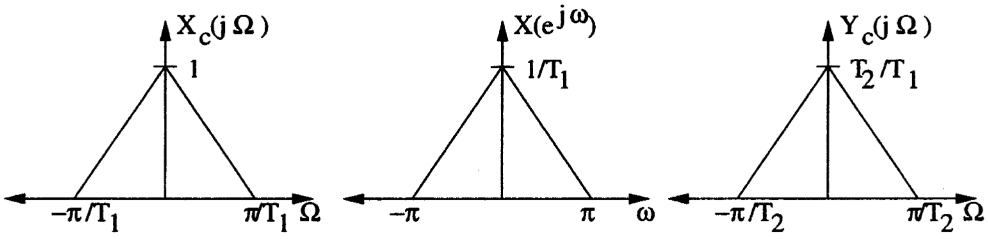


Now

$$
\begin{aligned}
x_r(t) &= \cos(\pi/3) + 2\cos(100\pi - \pi/4) \\
&= 1/2 + 2\cos(100\pi - \pi/4).
\end{aligned}
$$

We have $A = 1/2$.

4.29. In the frequency domain, we have



$$x_c(t) = 0, \qquad |\Omega| \geq \frac{\pi}{T_1}$$

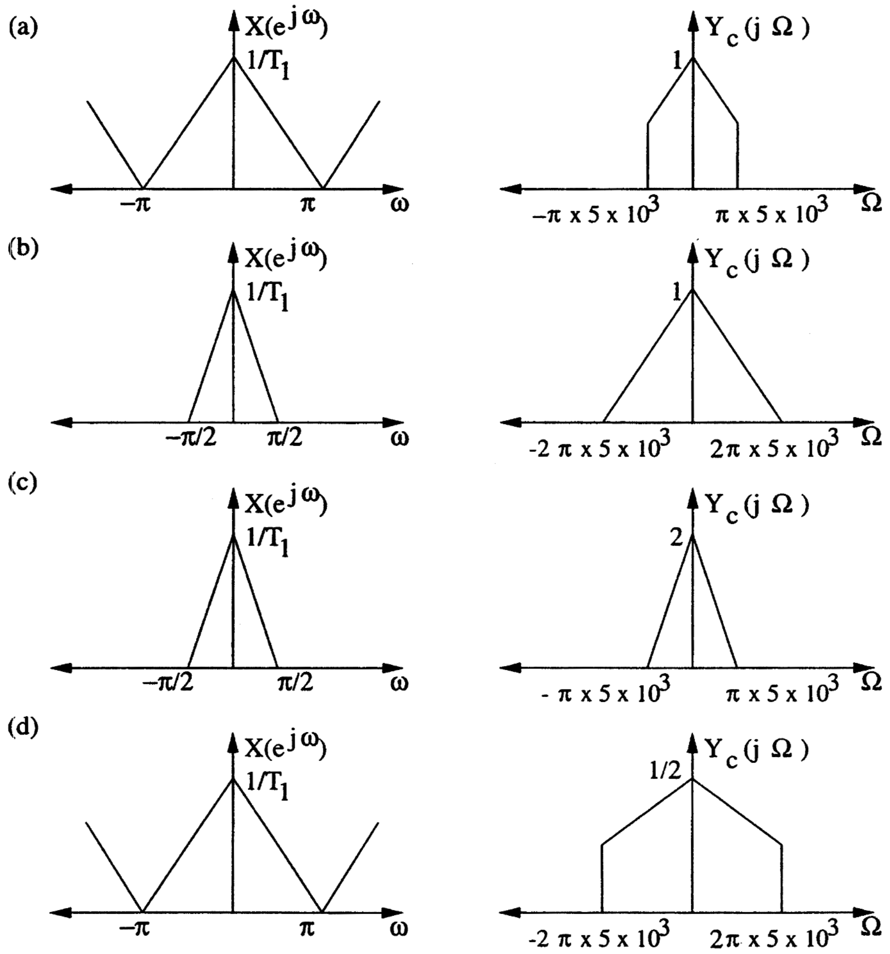
Therefore, since we are sampling this $x_c(t)$ at the Nyquist frequency $x[n]$ will be full band and unaliased.

$$x[n] = x_c(nT_1)$$

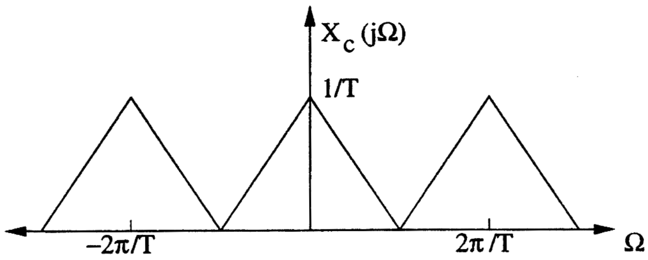
$y_c(t)$ is a band-limited interpolation of $x[n]$ at a different period. Since no aliasing occurs at $x[n]$, the spectrum of $y_c(t)$ will be a frequency axis scaling of the spectrum of $x_c(t)$ for $T_1 > T_2$ or $T_1 < T_2$. As we show in the figure,

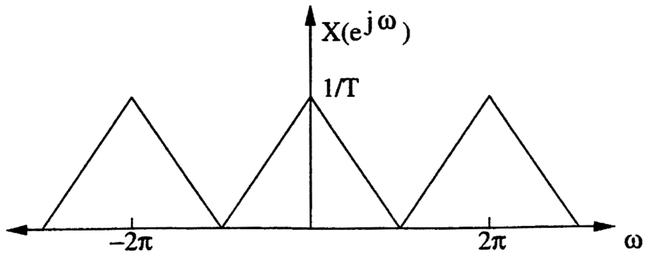
$$y_c(t) = \frac{T_2}{T_1} x_c\left(\frac{T_2}{T_1} t\right)$$

4.30. The Fourier transform of $y_c(t)$ is sketched below for each case.



4.31. (a) $x_s(t) = x_c(t)s(t) \Rightarrow X_s(j\Omega) * s(j\Omega)$





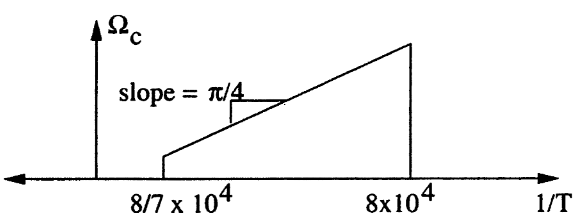
(b) Since $H_d(e^{j\omega})$ is an ideal lowpass filter with $\omega_c = \frac{\pi}{4}$, we don't care about any signal aliasing that occurs in the region $\frac{\pi}{4} \le \omega \le \pi$. We require:

$$\begin{aligned} \frac{2\pi}{T} - 2\pi \cdot 10000 &\ge \frac{\pi}{4T} \\ \frac{1}{T} &\ge \frac{8}{7} \cdot 10000 \\ T &\le \frac{7}{8} \times 10^{-4}\text{sec} \end{aligned}$$

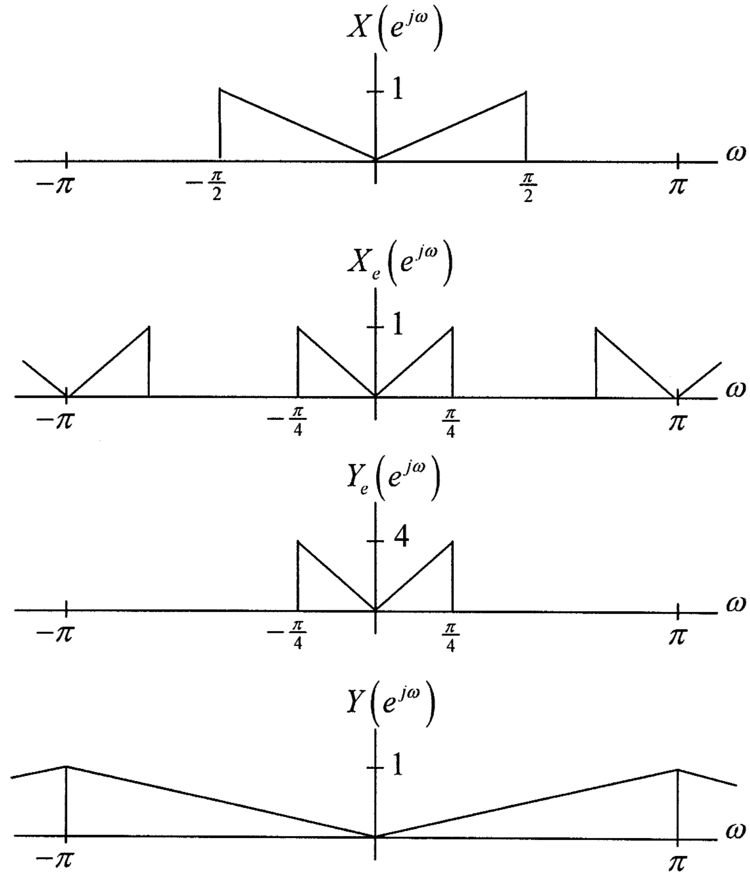
Also, once all of the signal lies in the range $|\omega| \le \frac{\pi}{4}$, the filter will be ineffective, i.e., $\frac{\pi}{4} \le T(2\pi \times 10^4)$. So, $T \ge 12.5\mu$sec.

(c)

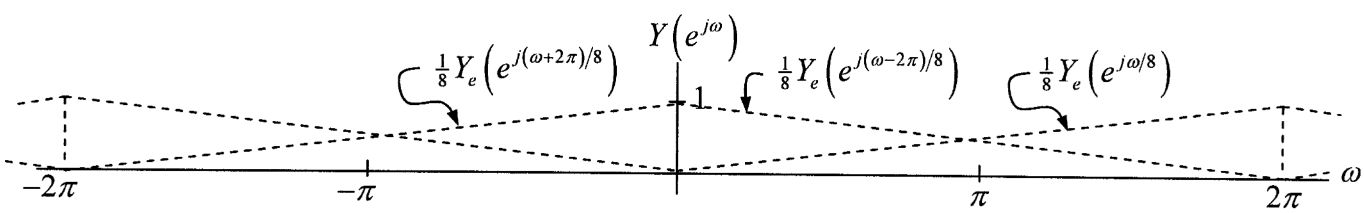
$$\Omega = \frac{\omega}{T} \Rightarrow \Omega_c = \frac{\pi}{4T}$$



4.32. A. With $L = 2$ and $M = 4$,



B. With $L = 2$ and $M = 8$, $X_e(e^{j\omega})$ and $Y_e(e^{j\omega})$ remain as in part A, except that $Y_e(e^{j\omega})$ now has a peak value of 8. After expanding we have



We see that $Y(e^{j\omega}) = 1$ for all ω. Inverse transforming gives $y[n] = \delta[n]$ in this case.

4.33.

$$x[n] \to \boxed{\uparrow 5} \to \boxed{\downarrow 3} \to \boxed{\downarrow 5} \to \boxed{\uparrow 3} \to y[n]$$

Since 3 and 5 are relatively prime, the order of the two operations in the center can be interchanged. This gives

$$x[n] \to \boxed{\uparrow 5} \to \boxed{\downarrow 5} \to \boxed{\downarrow 3} \to \boxed{\uparrow 3} \to y[n]$$

Expanding by 5 and immediately compressing by 5 produces no net effect. We have

$$x[n] \to \boxed{\downarrow 3} \xrightarrow{x_1[n]} \boxed{\uparrow 3} \to y[n]$$

Compressing by 3 produces

$$x_1[n] = x[3n].$$

Expanding by 3 now gives

$$y[n] = \begin{cases} x_1[n/3], & n = 3k, \quad k \text{ any integer} \\ 0, & \text{otherwise.} \end{cases}$$

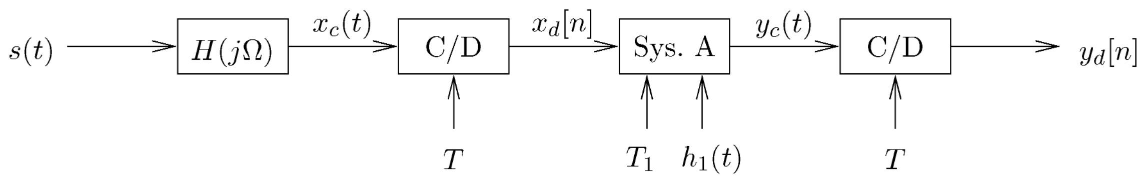
That is,

$$y[n] = \begin{cases} x[n], & n = 3k, \quad k \text{ any integer} \\ 0, & \text{otherwise.} \end{cases}$$

4.34. Problem 2 from Spring 2005 Midterm
Appears in: Fall05 PS2.

Problem

In the system shown below, the individual blocks are defined as indicated.



$$H(j\Omega): \quad H(j\Omega) = \begin{cases} 1, & |\Omega| < \pi \cdot 10^{-3} \text{ rad/sec} \\ 0, & |\Omega| > \pi \cdot 10^{-3} \text{ rad/sec} \end{cases}$$

$$\text{First C/D:} \quad x_d[n] = x_c(nT)$$

$$\text{System A:} \quad y_c(t) = \sum_{k=-\infty}^{\infty} x_d[k] h_1(t - kT_1)$$

$$\text{Second C/D:} \quad y_d[n] = y_c(nT)$$

(a) Specify a choice for T, T_1 and $h_1(t)$ so that $y_c(t)$ and $x_c(t)$ are guaranteed to be equal for any choice of $s(t)$.

(b) State whether your choice in (a) is unique or whether there are other choices for T, T_1 and $h_1(t)$ that will guarantee that $y_c(t)$ and $x_c(t)$ are equal. As usual, clearly show your reasoning.

(c) For this part, we are interested in what is often referred to as *consistent resampling*. Specifically, the system A constructs a continuous-time signal $y_c(t)$ from $x_d[n]$ the sequence of samples of $x_c(t)$ and is then resampled to obtain $y_d[n]$. The resampling is referred to as consistent if $y_d[n] = x_d[n]$. Determine the most general conditions you can on T, T_1 and $h_1(t)$ so that $y_d[n] = x_d[n]$.

Solution from Fall05 PS2

(a) Let's rewrite System A as the cascade of two systems: an S/I (sample to impulse) converter followed by a CT filter. The S/I converter turns the DT signal $x_d[n]$ into a CT impulse train. If we allow the output of the S/I converter to be $x_s(t)$, then we have

$$x_s(t) = \sum_{k=-\infty}^{\infty} x_d[k]\delta(t - kT_1)$$

Then, the output $y_c(t)$ of the CT filter is the convolution of $x_s(t)$ and $h_1(t)$, or

$$y_c(t) = x_s(t) * h_1(t).$$

We see that by combining the above two equations, we get the equation that describes the behavior of System A.

$x_c(t)$ is bandlimited to $\Omega_c = \pi \cdot 10^{-3}$ rad/sec. Thus, we know that we can guarantee the equality of $x_c(t)$ and $y_c(t)$ when T is sufficiently small (*i.e.* no aliasing from the first C/D converter) and System A is an ideal D/C converter with the same sampling period.

We have no aliasing when $\Omega_c T < \pi$, or when $T < 1000$. System A is an ideal D/C converter when $h_1(t)$ is an appropriate sinc function.

Thus, the following conditions work:

$$\begin{aligned} T &= 500. \\ T_1 &= 500. \\ h_1(t) &= \frac{\sin(\pi t/T)}{\pi t/T} \end{aligned}$$

(b) As we saw in the solution to part (a), our choices are not unique. We can choose any T such that $T < 1000$. However, we see that we need $T_1 = T$. We also have a choice regarding $h_1(t)$. Since $X_d(e^{j\omega})$, or the Fourier transform of $x_d[n]$, is zero for

$$\frac{T\pi}{1000} < |\omega| < \pi,$$

$X_s(j\Omega)$, or the Fourier transform of $x_s(t)$, is zero for

$$\frac{\pi T}{1000 T_1} = \frac{\pi}{1000} < |\Omega| < \frac{\pi}{T_1}.$$

Thus, $H_1(j\Omega)$, or the Fourier transform of $h_1(t)$, can be anything in that frequency range (and, by extension, any "copy" of this section of the frequency spectrum). If it is a constant of T for $|\Omega| < \frac{\pi}{1000}$ and zero for $|\Omega| > \frac{\pi}{T}$, then we have $y_c(t) = x_c(t)$.

(c) Since we are interested only in the operations between $x_d[n]$ and $y_d[n]$, we need not worry about aliasing from the first C/D converter in the whole system destroying our hopes for achieving consistent resampling. Thus, there are no absolute restrictions on T and T_1 like we had in parts (a) and (b); they may, however, be related to each other.

In other words, what is going on between $x_d[n]$ and $y_d[n]$? System A is taking each sample of $x_d[n]$ and replacing it with $h_1(t)$ delayed by nT_1 and scaled by $x_d[n]$ at that point. Then, the C/D converter resamples the result.

Let's consider what happens with $x_d[n] = \delta[n - n_0]$ for an integer n_0. Then, $y_c(t) = h_1(t - n_0T_1)$. The sampled version of $y_c(t)$ is

$$\begin{aligned} y_d[n] &= y_c(nT) \\ &= h_1(nT - n_0T_1). \end{aligned}$$

A condition for consistent resampling is thus

$$h_1(nT - n_0T_1) = \delta[n - n_0].$$

Because of the linearity of the mapping from $x_d[n]$ to $y_d[n]$, this is actually the only condition that must be checked. To simplify the condition further, we have

$$\text{evaluating at } n = n_0\text{:} \quad 1 = h_1(n(T - T_1))$$

$$\text{evaluating at } n \neq n_0\text{:} \quad 0 = h_1(nT - n_0T_1))$$

The case of practical significance is to have $T = T_1$, in which case we find that $h_1(t)$ should satisfying an *interpolating condition*: $h_1(0) = 1$ and $h_1(t) = 0$ for all multiples of T. (It doesn't matter what $h_1(t)$ is for other values of t.)

Solution from Spring05 Midterm

(7%) (a) Specify a choice for T, T_1 and $h_1(t)$ so that $y_c(t)$ and $x_c(t)$ are guaranteed to be equal for any choice of $s(t)$.

Let's rewrite System A as the cascade of two systems: an S/I (sample to impulse) converter followed by a CT filter. The S/I converter turns the DT signal $x_d[n]$ into a CT impulse train. If we allow the output of the S/I converter to be $x_s(t)$, then we have

$$x_s(t) = \sum_{k=-\infty}^{\infty} x_d[k]\delta(t - kT_1)$$

Then, the output $y_c(t)$ of the CT filter is the convolution of $x_s(t)$ and $h_1(t)$, or

$$y_c(t) = x_s(t) * h_1(t).$$

We see that by combining the above two equations, we get the equation that describes the behavior of System A.

$x_c(t)$ is bandlimited to $\Omega_c = \pi \cdot 10^{-3}$ rad/sec. Thus, we know that we can guarantee the equality of $x_c(t)$ and $y_c(t)$ when T is sufficiently small (*i.e.* no aliasing from the first C/D converter) and System A is an ideal D/C converter with the same sampling period.

We have no aliasing when $\Omega_c T < \pi$, or when $T < 1000$. System A is an ideal D/C converter when $h_1(t)$ is an appropriate sinc function.

Thus, the following conditions work:

$$\begin{aligned} T &= 500. \\ T_1 &= 500. \\ h_1(t) &= \frac{\sin(\pi t/T)}{\pi t/T} \end{aligned}$$

Technically speaking, we can allow $T = 1000$. The impulse response $h(t)$ of the LPF is a sinc. If we calculate its Fourier transform $H(j\Omega)$ exactly at the cutoff frequency, we will get the average of the function from both sides, or 0.5. Likewise, $h_1(t)$ will have the same "halving" effect on the other side. So, even though there is aliasing at a single frequency (the cutoff frequency), System A will cut that in half so that we get the exactly the same values at the cutoff frequency.

(7%) (b) State whether your choice in (a) is unique or whether there are other choices for T, T_1 and $h_1(t)$ that will guarantee that $y_c(t)$ and $x_c(t)$ are equal. As usual, clearly show your reasoning.

As we saw in the solution to part (a), our choices are not unique. We can choose any T such that $T < 1000$. However, we see that we need $T_1 = T$. We also have a choice regarding $h_1(t)$. Since $X_d(e^{j\omega})$, or the Fourier transform of $x_d[n]$, is zero for

$$\frac{T\pi}{1000} < |\omega| < \pi,$$

$X_s(j\Omega)$, or the Fourier transform of $x_s(t)$, is zero for

$$\frac{\pi T}{1000 T_1} = \frac{\pi}{1000} < |\Omega| < \frac{\pi}{T_1}.$$

Thus, $H_1(j\Omega)$, or the Fourier transform of $h_1(t)$, can be anything in that frequency range (and, by extension, any "copy" of this section of the frequency spectrum). If it is a constant of T for $|\Omega| < \frac{\pi}{1000}$ and zero for $|\Omega| > \frac{\pi}{T}$, then we have $y_c(t) = x_c(t)$.

(8%) (c) For this part, we are interested in what is often referred to as *consistent resampling.* Specifically, the system A constructs a continuous-time signal $y_c(t)$ from $x_d[n]$ the sequence of samples of $x_c(t)$ and is then resampled to obtain $y_d[n]$. The resampling is referred to as consistent if $y_d[n] = x_d[n]$. Determine the most general conditions you can on T, T_1 and $h_1(t)$ so that $y_d[n] = x_d[n]$.

Since we are interested only in the operations between $x_d[n]$ and $y_d[n]$, we need not worry about aliasing from the first C/D converter in the whole system destroying our hopes for achieving consistent resampling. Thus, there are no absolute restrictions on T and T_1 like we had in parts (a) and (b); they may, however, be related to each other.

In words, what's going on between $x_d[n]$ and $y_d[n]$? System A is taking each sample of $x_d[n]$ and replacing it with $h_1(t)$ delayed by nT_1 and scaled by $x_d[n]$ at that point. Then, the C/D converter resamples the result.

Let's consider what happens with $x_d[n] = \delta[n - n_0]$ for an integer n_0. Then, $y_c(t) = h_1(t - n_0 T_1)$. The sampled version of $y_c(t)$ is

$$\begin{aligned} y_d[n] &= y_c(nT) \\ &= h_1(nT - n_0 T_1). \end{aligned}$$

A condition for consistent resampling is thus

$$h_1(nT - n_0 T_1) = \delta[n - n_0].$$

Because of the linearity of the mapping from $x_d[n]$ to $y_d[n]$, this is actually the only condition that must be checked. To simplify the condition further, we have

$$\text{evaluating at } n = n_0: \quad 1 = h_1(n(T - T_1))$$

$$\text{evaluating at } n \neq n_0: \quad 0 = h_1(nT - n_0 T_1))$$

The case of practical significance is to have $T = T_1$, in which case we find that $h_1(t)$ should satisfying an *interpolating condition*: $h_1(0) = 1$ and $h_1(t) = 0$ for all multiples of T. (It doesn't matter what $h_1(t)$ is for other values of t.)

A thorough answer also considers what happens with $T \neq T_1$. It turns out that if T/T_1 is rational, the equations above are inconsistent. If T/T_1 is irrational, then we find that a necessary and sufficient condition on $h_1(t)$ is

$$h_1(t) = \begin{cases} 1, & \text{if } t \text{ is an integer multiple of } T - T_1; \\ 0, & \text{if } t \text{ is expressible as } nT - n_0T_1 \text{ for integers } n \text{ and } n_0; \\ \text{anything}, & \text{otherwise.} \end{cases}$$

4.35. Problem 2 in Fall2005 Midterm exam.

Problem

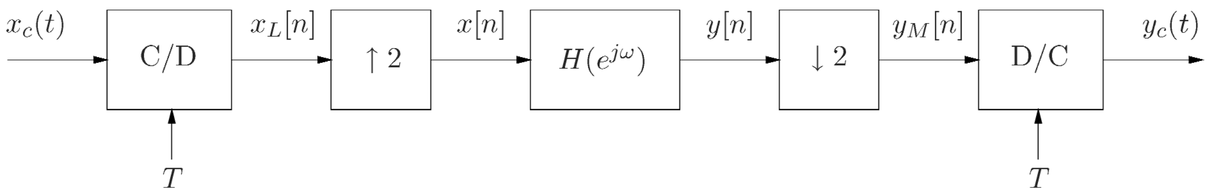


Figure 1:

For parts (a) and (b) only, $X_c(j\Omega) = 0$ for $|\Omega| > 2\pi \times 10^3$ and $H(e^{j\omega})$ is as shown in Figure **??** (and of course periodically repeats).

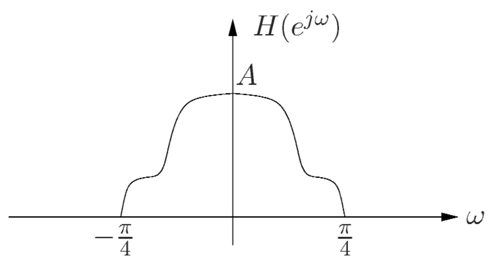


Figure 2:

(a) Determine the most general condition on T, if any, so that the overall continuous-time system from $x_c(t)$ to $y_c(t)$ is LTI.

(b) Sketch and clearly label the overall equivalent continuous-time frequency response $H_{\text{eff}}(j\Omega)$ that results when the condition determined in (a) holds.

(c) **For this part only** assume that $X_c(j\Omega)$ in Figure **??** is bandlimited to avoid aliasing, i.e. $X_c(j\Omega) = 0$ for $|\Omega| \geq \frac{\pi}{T}$. For a general sampling period T, we would like to choose the system $H(e^{j\omega})$ in Figure **??** so that the overall continuous-time system from $x_c(t)$ to $y_c(t)$ is LTI for any input $x_c(t)$ bandlimited as above.

Determine the most general conditions on $H(e^{j\omega})$, if any, so that the overall CT system is LTI. Assuming that these conditions hold, also specify in terms of $H(e^{j\omega})$ the overall equivalent continuous-time frequency response $H_{\text{eff}}(j\Omega)$.

Solution from Fall05 midterm

We can simplify Figure **??** by replacing the expander, $H(e^{j\omega})$ and compressor with a single DT LTI system. The impulse response of the equivalent DT LTI system is $h[2n]$, where $h[n]$ is the inverse DTFT of $H(e^{j\omega})$. We saw this result in Problem Set 6 where we derived it by polyphase decomposition and subsequent simplification. The frequency response of the equivalent DT LTI system is $\frac{1}{2}H(e^{j\frac{\omega}{2}}) + \frac{1}{2}H(e^{j(\frac{\omega}{2}-\pi)})$.

Alternatively, we can reach the same conclusion by relating $Y_M(e^{j\omega})$ to $X_L(e^{j\omega})$:

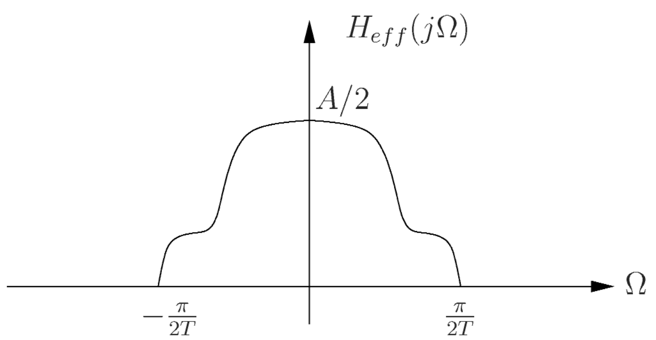
$$Y_M(e^{j\omega}) = \frac{1}{2}\left[H(e^{j\frac{\omega}{2}})X_L(e^{j2\frac{\omega}{2}}) + H(e^{j(\frac{\omega}{2}-\pi)})X_L(e^{j2(\frac{\omega}{2}-\pi)})\right]$$

Both $X_L(\cdot)$ terms reduce to $X_L(e^{j\omega})$, the latter because of 2π periodicity, leaving behind the same equivalent frequency response.

The overall CT system will be LTI if there is no aliasing at the C/D converter, or if the DT filter rejects any frequency content contaminated by aliasing. Since the equivalent DT frequency response is 0 for $\frac{\pi}{2} < |\omega| \le \pi$, the copy of $X_c(j\Omega)$ centred at $\omega = 0$ can extend to $\omega = 2\pi - \frac{\pi}{2} = \frac{3\pi}{2}$ without passing any aliased components. Applying the $\omega = \Omega T$ mapping:

$$\omega_{max} = \Omega_{max}T = (2\pi \times 10^3)T < \frac{3\pi}{2}$$

and so we require that $T < \frac{3}{4} \times 10^{-3}$.



The simplification in part (a) yields an equivalent DT LTI system between the C/D and D/C converters. Thus the overall system will be LTI if $x_c(t)$ is appropriately bandlimited to avoid aliasing, as assumed in this part. There are no conditions on $H(e^{j\omega})$.

4.36. Problem 3 in Spring2004 Midterm exam.

Problem

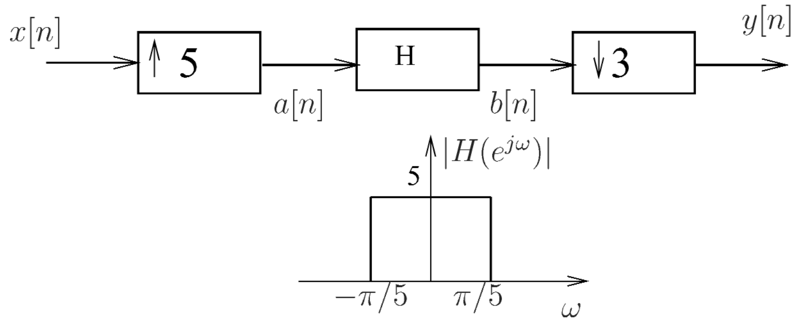
We have a discrete-time signal, $x[n]$, arriving from a source at a rate of $\frac{1}{T_1}$ samples per second. We want to digitally resample it to create a signal $y[n]$ that has $\frac{1}{T_2}$ samples per second, where $T_2 = \frac{3}{5}T_1$.

(a) Draw a block diagram of a discrete-time system to perform the resampling. Specify the input/output relationship for all the boxes in the Fourier domain.

(b) For an input signal $x[n] = \delta[n] = \begin{cases} 1, & n = 0 \\ 0, & otherwise \end{cases},$ determine y[n].

Solution from Spring04 midterm

Part(a):



The input-output relationships for the boxes in the above figure are as follows:

$$A(e^{j\omega}) = X(e^{j5\omega})$$

$$B(e^{j\omega}) = \begin{cases} 5A(e^{j\omega}), & |w| < \pi/5 \\ 0, & \text{otherwise} \end{cases}$$

$$Y(e^{j\omega}) = \frac{1}{3}\sum_{k=0}^{2} B(e^{\frac{j\omega - 2\pi k}{3}})$$

Part(b):

$$A(e^{j\omega}) = 1$$

$$B(e^{j\omega}) = \begin{cases} 5, & |w| < \pi/5 \\ 0, & \text{otherwise} \end{cases}$$

$$Y(e^{j\omega}) = \begin{cases} 5/3, & |w| < 3\pi/5 \\ 0, & \text{otherwise} \end{cases}$$

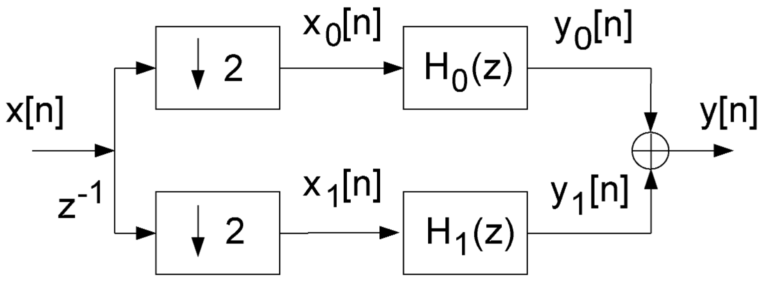
Taking the inverse Fourier Transform:

$$y[n] = \frac{5}{3}\frac{\sin(3\pi/5n)}{\pi n} \tag{1}$$

4.37. Appears in: Fall05 PS6, Fall04 PS5, Fall02 PS5.

Problem

Consider the decimation filter structure shown below:

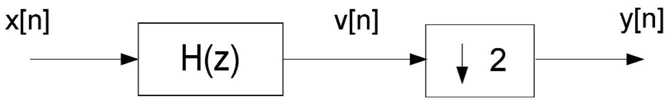


where $y_0[n]$ and $y_1[n]$ are generated according to the following forward recursions:

$$\begin{aligned} y_0[n] &= \frac{1}{4}y_0[n-1] - \frac{1}{3}x_0[n] + \frac{1}{8}x_0[n-1] \\ y_1[n] &= \frac{1}{4}y_1[n-1] + \frac{1}{12}x_1[n] \end{aligned}$$

(a) How many multiplies per output sample does the implementation of the filter structure require? Consider a divide to be equivalent to a multiply.

The decimation filter can also be implemented as shown below,



where $v[n] = av[n-1] + bx[n] + cx[n-1]$.

(b) Determine a, b, and c.

(c) How many multiplies per output sample does this second implementation require?

Solution from Fall05 PS6

(a) There is one output sample generated for every pair of input samples. Even input samples require 3 multiplies and odd input samples require 2 multiplies. Thus each pair requires 5 multiplies.

(b) Applying the compressor identity to the previous structure results in:

$$H(z) = H_0(z^2) + z^{-1}H_1(z^2).$$

From the difference equations in the previous part we have:

$$H_0(z) = \frac{-\frac{1}{3} + \frac{1}{8}z^{-1}}{1 - \frac{1}{4}z^{-1}},$$

and

$$H_1(z) = \frac{\frac{1}{12}}{1 - \frac{1}{4}z^{-1}}.$$

Thus,

$$H(z) = \frac{-\frac{1}{3} + \frac{1}{8}z^{-2} + \frac{1}{12}z^{-1}}{1 - \frac{1}{4}z^{-2}} = \frac{-\frac{1}{3}(1 - \frac{3}{4}z^{-1})(1 + \frac{1}{2}z^{-1})}{(1 - \frac{1}{2}z^{-1})(1 + \frac{1}{2}z^{-1})} = \frac{-\frac{1}{3} + \frac{1}{4}z^{-1}}{1 - \frac{1}{2}z^{-1}}.$$

Therefore, $a = 1/2, b = -1/3$ and $c = 1/4$.

(c) In this implementation 3 multiplies are required for every input sample. For every output sample we need to calculate 2 values of $v[n]$. Altogether we need 6 multiplies per output sample.

Solution from Fall04 PS5

(a) There is one output sample generated for every pair of input samples. Even input samples require 3 multiplies and odd input samples require 2 multiplies. Thus each pair requires 5 multiplies.

(b) Applying the compressor identity to the previous structure results in:

$$H(z) = H_0(z^2) + z^{-1}H_1(z^2).$$

From the difference equations in the previous part we have:

$$H_0(z) = \frac{-\frac{1}{3} + \frac{1}{8}z^{-1}}{1 - \frac{1}{4}z^{-1}},$$

and

$$H_1(z) = \frac{\frac{1}{12}}{1 - \frac{1}{4}z^{-1}}.$$

Thus,

$$H(z) = \frac{-\frac{1}{3} + \frac{1}{8}z^{-2} + \frac{1}{12}z^{-1}}{1 - \frac{1}{4}z^{-2}} = \frac{-\frac{1}{3}(1 - \frac{3}{4}z^{-1})(1 + \frac{1}{2}z^{-1})}{(1 - \frac{1}{2}z^{-1})(1 + \frac{1}{2}z^{-1})} = \frac{-\frac{1}{3} + \frac{1}{4}z^{-1}}{1 - \frac{1}{2}z^{-1}}.$$

Therefore, $a = 1/2, b = -1/3$ and $c = 1/4$.

(c) In this implementation 3 multiplies are required for every input sample. For every output sample we need to calculate 2 values of $v[n]$. Altogether we need 6 multiplies per output sample.

Solution from Fall02 PS5

1. There is one output sample generated for every pair of input samples. Even input samples require 3 multiplies and odd input samples require 2 multiplies. Thus each pair requires 5 multiplies.

2. Applying the compressor identity to the previous structure results in:

$$H(z) = H_0(z^2) + z^{-1}H_1(z^2).$$

From the difference equations in the previous part we have:

$$H_0(z) = \frac{-\frac{1}{3} + \frac{1}{8}z^{-1}}{1 - \frac{1}{4}z^{-1}},$$

and

$$H_1(z) = \frac{\frac{1}{12}}{1 - \frac{1}{4}z^{-1}}.$$

Thus,

$$H(z) = \frac{-\frac{1}{3} + \frac{1}{8}z^{-2} + \frac{1}{12}z^{-1}}{1 - \frac{1}{4}z^{-2}} = \frac{-\frac{1}{3}(1 - \frac{3}{4}z^{-1})(1 + \frac{1}{2}z^{-1})}{(1 - \frac{1}{2}z^{-1})(1 + \frac{1}{2}z^{-1})} = \frac{-\frac{1}{3} + \frac{1}{4}z^{-1}}{1 - \frac{1}{2}z^{-1}}.$$

Therefore, $a = 1/2, b = -1/3$ and $c = 1/4$.

3. In this implementation 3 multiplies are required for every input sample. For every output sample we need to calculate 2 values of $v[n]$. Altogether we need 6 multiplies per output sample.

4.38. Appears in: Spring05 PS3.

Problem

Consider the two systems of Figure **??**.

(a) For $M = 2$, $L = 3$, and any arbitrary $x[n]$, will $y_A[n] = y_B[n]$? If your answer is yes, justify your answer. If your answer is no, clearly explain or give a counterexample.

(b) (Optional) How must M and L be related to guarantee $y_A[n] = y_B[n]$ for arbitrary $x[n]$?

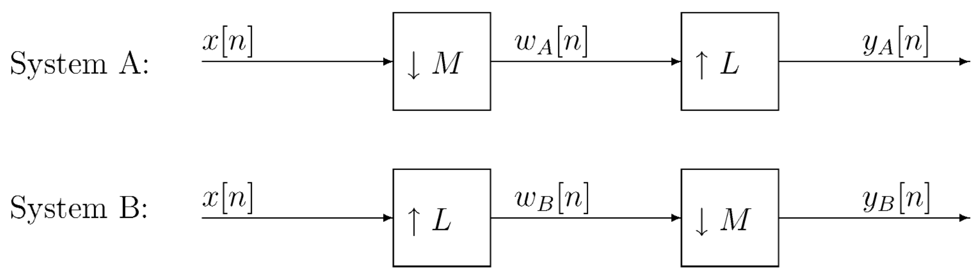


Figure 1: Systems compared in Problem 3.8.

Solution from Spring05 PS3

(a) The following equations describe the stages of System A:

$$w_A[n] = x[2n]$$

$$y_A[n] = \begin{cases} w_A[\frac{n}{3}] & \text{if } \frac{n}{3} \text{ is an integer,} \\ 0 & \text{otherwise.} \end{cases}$$

The following equations describe the stages of System B:

$$w_B[n] = \begin{cases} x[\frac{n}{3}] & \text{if } \frac{n}{3} \text{ is an integer,} \\ 0 & \text{otherwise.} \end{cases}$$

$$y_B[n] = w_B[2n]$$

Therefore,

$$y_A[n] = \begin{cases} x[\frac{2n}{3}] & \text{if } \frac{n}{3} \text{ is an integer,} \\ 0 & \text{otherwise.} \end{cases}$$

and

$$y_B[n] = \begin{cases} x[\frac{2n}{3}] & \text{if } \frac{2n}{3} \text{ is an integer,} \\ 0 & \text{otherwise.} \end{cases}$$

Because for all integer values of n for which $\frac{n}{3}$ is an integer, $\frac{2n}{3}$ is also an integer and vice-versa, the systems are equivalent.

(b) More generally, the systems can be described by the following equations:

$$y_A[n] = \begin{cases} x[\frac{Mn}{L}] & \text{if } \frac{n}{L} \text{ is an integer,} \\ 0 & \text{otherwise.} \end{cases}$$

$$y_B[n] = \begin{cases} x[\frac{Mn}{L}] & \text{if } \frac{Mn}{L} \text{ is an integer,} \\ 0 & \text{otherwise.} \end{cases}$$

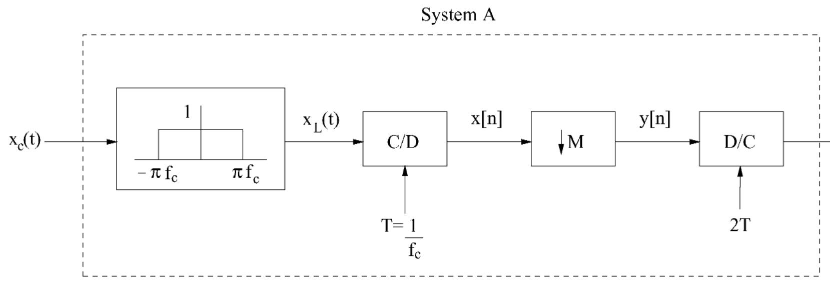
Therefore the two systems are equivalent if for all integer values of n where $\frac{Mn}{L}$ is an integer, $\frac{n}{L}$ is also an integer, and if for all integer values of n where $\frac{n}{L}$ is an integer, $\frac{Mn}{L}$ is also an integer. Since we are guaranteed that for each n which gives integer values of $\frac{n}{L}$, $\frac{Mn}{L}$ must also be an integer (since we're only considering integer M and L), we need only to show that every time $\frac{Mn}{L}$ is an integer, $\frac{n}{L}$ is an integer in order to have an equivalence between the two systems.

For arbitrary integer n, $\frac{Mn}{L}$ is an integer if and only if Mn is an integral multiple of L. This only occurs whenever Mn contains all of L's prime factors. Likewise, $\frac{n}{L}$ is an integer if and only if n contains all of L's prime factors. It is therefore true that in order for the systems to be equivalent, Mn containing all of L's prime factors must imply that n contains all of L's prime factors. This is guaranteed to be true if M and L share no prime factors in common besides 1. (This condition will ensure that any prime factors which Mn has in common with L, besides 1, must have come exclusively from n.) Therefore, the two systems are equivalent if the greatest common factor of M and L is 1 (M and L are co-prime).

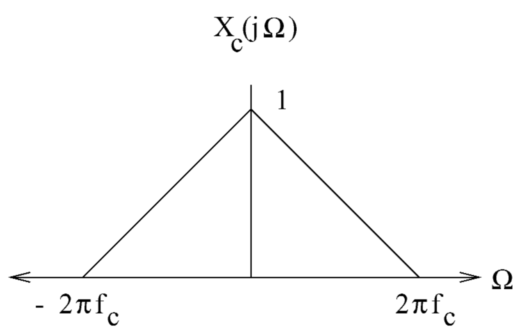
4.39. Problem 5 in Fall 2002 Midterm exam.
Appears in: Fall03 PS3, Spring03 PS3.

Problem

In system A a continuous-time signal $x_c(t)$ is processed as indicated.



a) If M=2 and $x_c(t)$ has the Fourier transform shown in next figure, determine $y[n]$. Clearly show your work on this part.

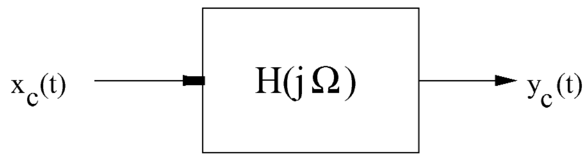


We would now like to modify system A by appropriately mounting additional processing modules in the cascade chain of system A (i.e. blocks can be added at any point in the cascade chain – at the beginning, at the end, or even in between existing blocks). All of the current blocks in system A must be kept. We would like the modified system to be an ideal LTI lowpass filter as indicated in the following figure.

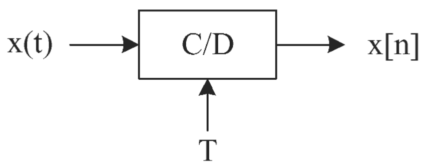
We have available an unlimited number of the six modules specified in the table given in the next page. The per unit cost for each module is indicated and we would like the final cost to be as low as possible. **Note that the D/C converter is running at a rate of "2T".**

b) Design the lowest cost modified system if M=2 in System A. Specify the parameters for all the modules used.

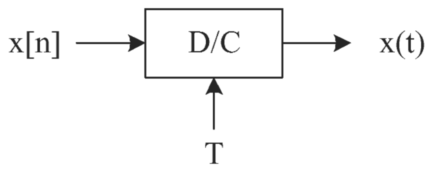
c) Design the lowest cost modified system if M=4 in System A. Specify the parameters for all the modules used.



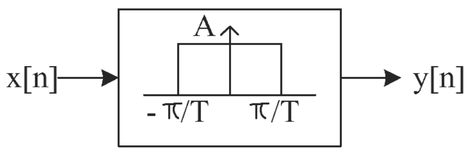
$$H(j\Omega) = \begin{cases} 1 & |\Omega| < \frac{2\pi f_c}{5} \\ 0 & o.w. \end{cases}$$



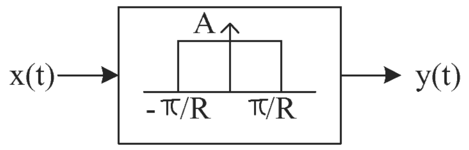
Continuous to Discrete Time Converter
Parameters: T
Cost : 10



Discrete to Continuous Time Converter
Parameters: T
Cost : 10



Discrete Time Low Pass Filter
Parameters: A, T
Cost : 10



Continuous Time Low Pass Filter
Parameters: A, R
Cost : 20

Expander
Parameters: L
Cost : 5

Compressor
Parameters: M
Cost : 5

Solution from Fall03 PS3

(a) Due to aliasing, the spectrum has a constant value at all frequencies. In the time domain, $y[n] = \frac{3}{4T}\delta[n]$.

(b) Between the C/D block and the downsample-by-2 block, insert a discrete time low pass filter; $A = 1$, $T = 5/2$, total cost $= 10$. Note that the downsample-by-2 block causes spectral spreading, but that is acceptable since the D/C block has a lower sampling frequency, and therefore, the CT output, $y_c(t)$, will have the same spectral spread as the low-passed input, $x_L(t)$.

(c) In this case the downsample-by-4 block causes so much spectral spreading that we need to upsample for spectral compression beforehand (to correct the sampling frequency). Between the C/D block and the downsample-by-4 block, insert an upsample-by-2 block ($L = 2$). The upsampler is immediately followed by a discrete time low pass filter; set $A = 2$ to compensate for the additional spectral amplitude scaling in the downsampler, compared to part (b), and set $T = 5$. The total cost is 15.

Solution from Spring03 PS3

(a) Due to aliasing, the spectrum has a constant value at all frequencies. In the time domain, $y[n] = \frac{3}{4T}\delta[n]$.

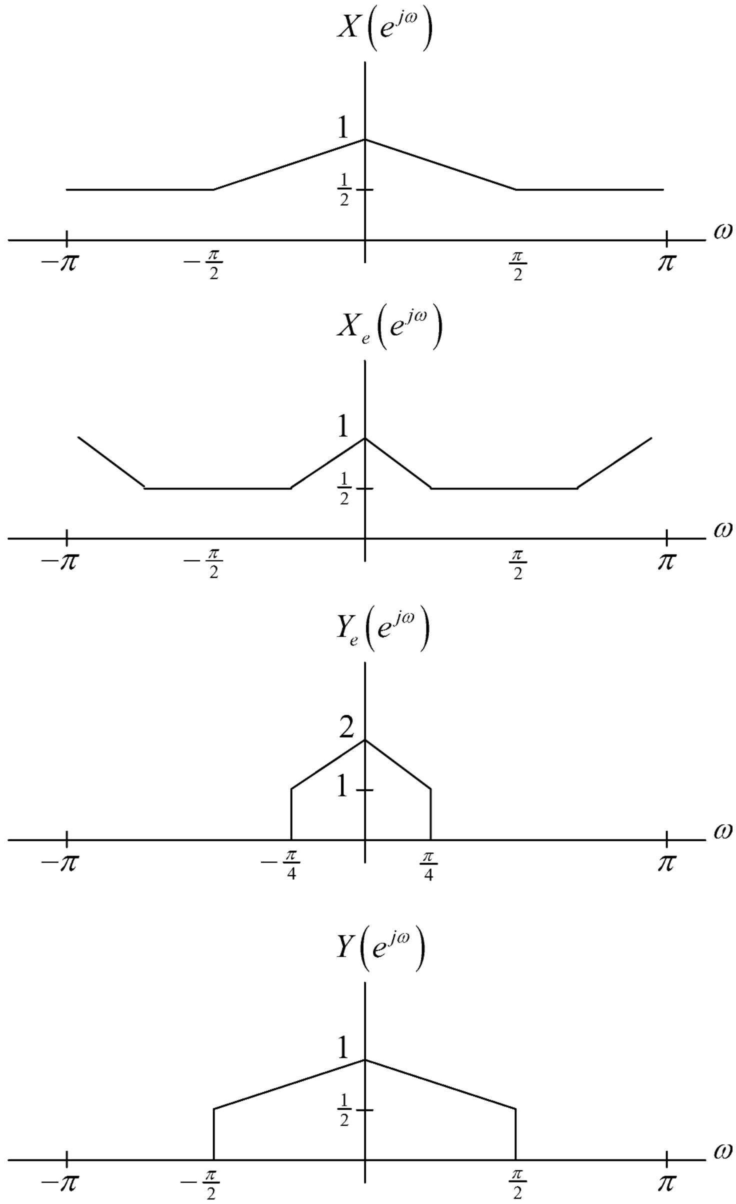
(b) Between the C/D block and the downsample-by-2 block, insert a discrete time low pass filter; $A = 1$, $T = 5/2$, total cost $= 10$. Note that the downsample-by-2 block causes spectral spreading, but that is acceptable since the D/C block has a lower sampling frequency, and therefore, the CT output, $y_c(t)$, will have the same spectral spread as the low-passed input, $x_L(t)$.

(c) In this case the downsample-by-4 block causes so much spectral spreading that we need to upsample for spectral compression beforehand (to correct the sampling frequency). Between the C/D block and the downsample-by-4 block, insert an upsample-by-2 block ($L = 2$). The upsampler is immediately followed by a discrete time low pass filter; set $A = 2$ to compensate for the additional spectral amplitude scaling in the downsampler, compared to part (b), and set $T = 5$. The total cost is 15.

Solution from Fall02 Midterm

N/A

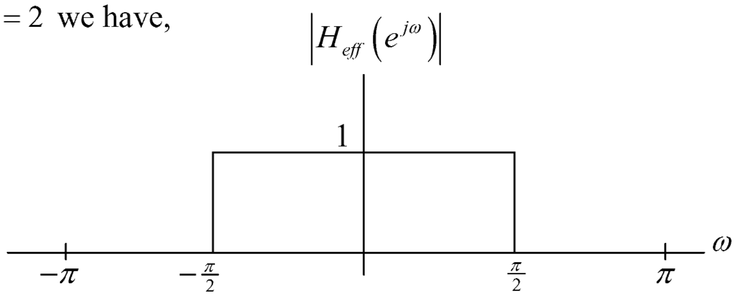
4.40. A.



B.

$$
\begin{aligned}
\varepsilon &= \sum_{n=-\infty}^{\infty} \left| x[n] - y[n] \right|^2 \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| X\left(e^{j\omega}\right) - Y\left(e^{j\omega}\right) \right|^2 d\omega \\
&= \frac{1}{2\pi} 2 \int_{\pi/2}^{\pi} (1/2)^2 \, d\omega \\
&= 1/8.
\end{aligned}
$$

C. For $M = 2$ we have,



D. When $M = 6$ we have $X_e\left(e^{j\omega}\right) = X\left(e^{j6\omega}\right)$. Then

$$\begin{aligned} Y_e\left(e^{j\omega}\right) &= H\left(e^{j\omega}\right)X_e\left(e^{j\omega}\right) \\ &= H\left(e^{j\omega}\right)X\left(e^{j6\omega}\right) \\ &= \begin{cases} 6X\left(e^{j6\omega}\right), & |\omega| \le \dfrac{\pi}{4} \\ 0, & \dfrac{\pi}{4} < |\omega| \le \pi. \end{cases} \end{aligned}$$

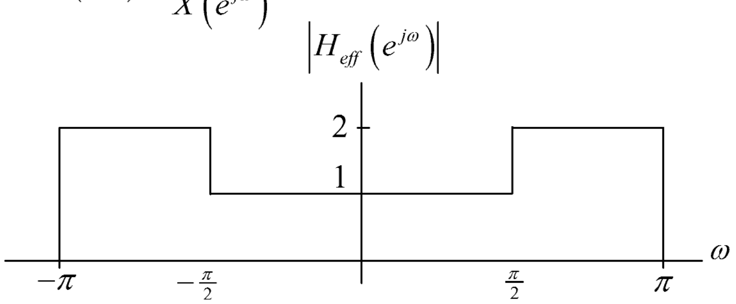
Now

$$Y\left(e^{j\omega}\right) = \begin{cases} X\left(e^{j6(\omega+2\pi)/6}\right) + X\left(e^{j6\omega/6}\right), & -\pi \le \omega \le -\dfrac{\pi}{2} \\ X\left(e^{j6\omega/6}\right), & -\dfrac{\pi}{2} < \omega < \dfrac{\pi}{2} \\ X\left(e^{j6\omega/6}\right) + X\left(e^{j6(\omega-2\pi)/6}\right), & \dfrac{\pi}{2} \le \omega \le \pi. \end{cases}$$

But $X\left(e^{j(\omega\pm 2\pi)}\right) = X\left(e^{j\omega}\right)$. Thus

$$Y\left(e^{j\omega}\right) = \begin{cases} 2X\left(e^{j\omega}\right), & -\pi \le \omega \le -\dfrac{\pi}{2} \\ X\left(e^{j\omega}\right), & -\dfrac{\pi}{2} < \omega < \dfrac{\pi}{2} \\ 2X\left(e^{j\omega}\right), & \dfrac{\pi}{2} \le \omega \le \pi. \end{cases}$$

Finally, $H_{eff}\left(e^{j\omega}\right) = \dfrac{Y\left(e^{j\omega}\right)}{X\left(e^{j\omega}\right)}$. The magnitude is plotted below.



4.41. A. We are given

$$h[n] = \begin{cases} \left(\frac{1}{2}\right)^n, & 0 \le n \le 11 \\ 0, & \text{otherwise.} \end{cases}$$

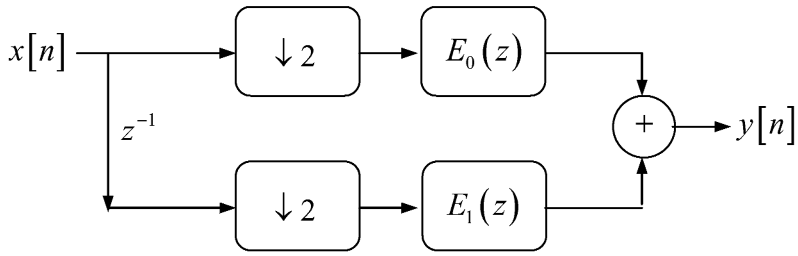
Then

$$E_0(z) = 1 + \frac{1}{2^2}z^{-1} + \frac{1}{2^4}z^{-2} + \frac{1}{2^6}z^{-3} + \frac{1}{2^8}z^{-4} + \frac{1}{2^{10}}z^{-5},$$

and

$$E_1(z) = \frac{1}{2} + \frac{1}{2^3}z^{-1} + \frac{1}{2^5}z^{-2} + \frac{1}{2^7}z^{-3} + \frac{1}{2^9}z^{-4} + \frac{1}{2^{11}}z^{-5}.$$

A compressor structure with two polyphase components is shown below.



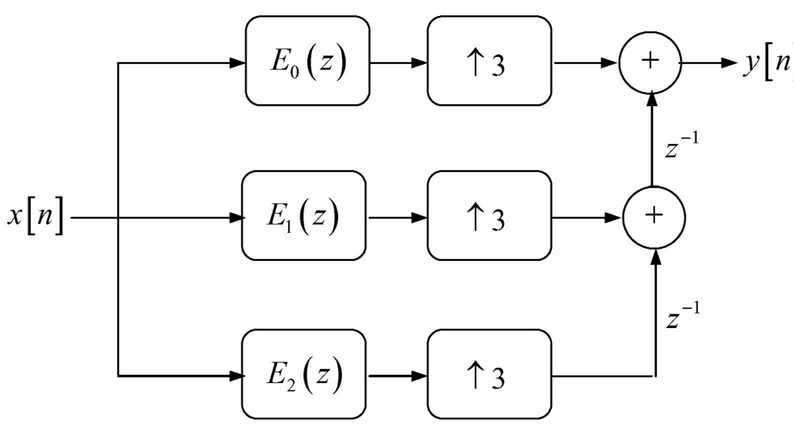
B. Using the same $h[n]$,

$$E_0(z) = 1 + \frac{1}{2^3}z^{-1} + \frac{1}{2^6}z^{-2} + \frac{1}{2^9}z^{-3}$$

$$E_1(z) = \frac{1}{2} + \frac{1}{2^4}z^{-1} + \frac{1}{2^7}z^{-2} + \frac{1}{2^{10}}z^{-3}$$

$$E_2(z) = \frac{1}{2^2} + \frac{1}{2^5}z^{-1} + \frac{1}{2^8}z^{-2} + \frac{1}{2^{11}}z^{-3}.$$

An expander structure with three polyphase components is shown below.



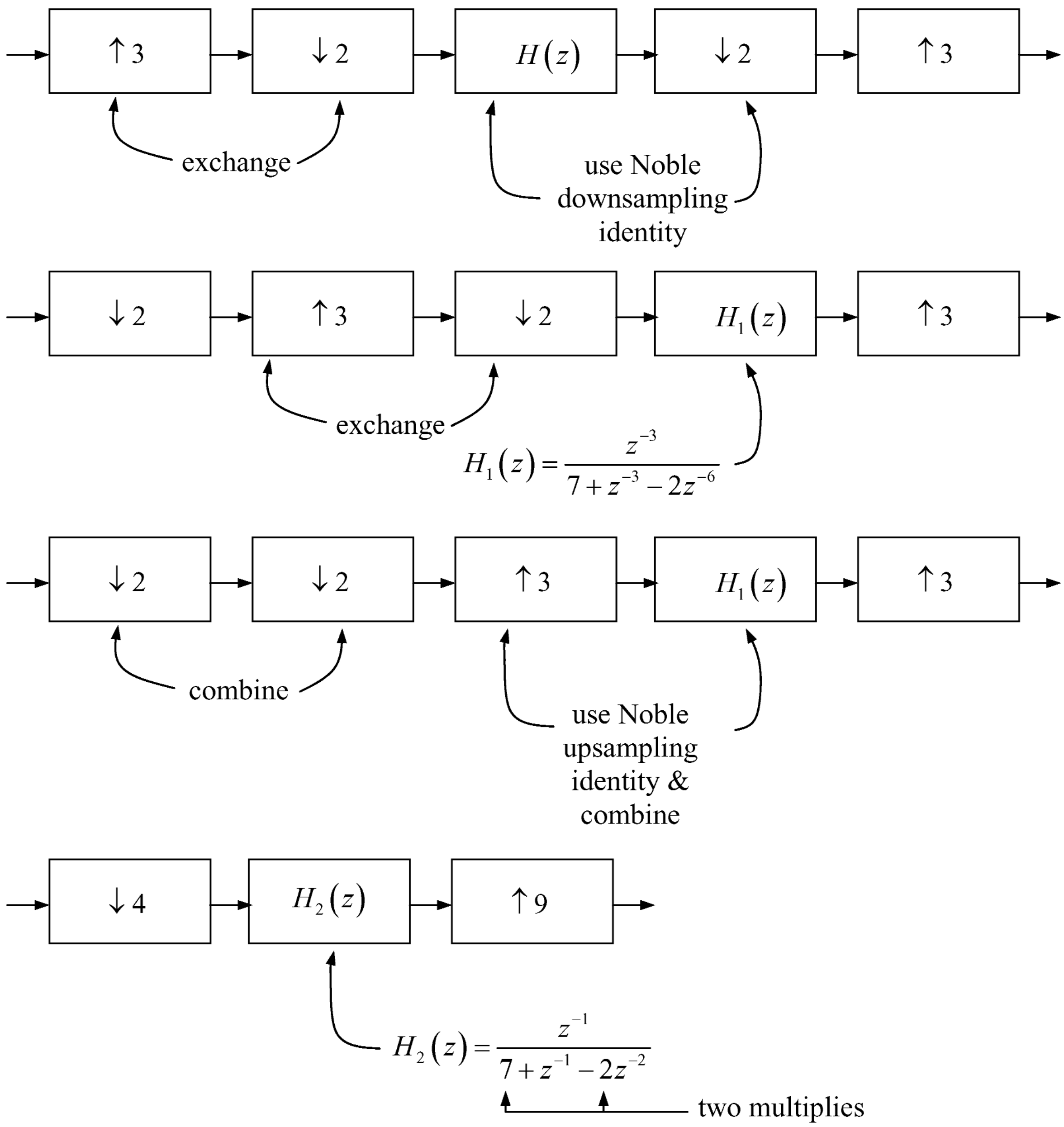
4.42. It is not possible to find a choice for $H_2(z)$ that will guarantee that $y_2[n] = y_1[n]$ whenever $x_2[n] = x_1[n]$. One way to show this is to observe that $H_1(z) = 1 + z^{-3}$, so that $H_1(e^{j\omega}) = 1 + e^{-j3\omega}$. This implies that $H_1(e^{j0}) = 2$, while $H_1(e^{j\pi}) = 0$. This in turn means that $Y_1(e^{j0})$ may not equal 0, but $Y_1(e^{j\pi}) = 0$.

Now consider System 2. For this system

$$\begin{aligned} Y_2(e^{j\omega}) &= W_2(e^{j2\omega}) \\ &= H_2(e^{j2\omega}) X_2(e^{j2\omega}). \end{aligned}$$

To guarantee $Y_2(e^{j\omega}) = Y_1(e^{j\omega})$ for all ω we must allow $Y_2(e^{j0})$ to possibly have a nonzero value. This implies that $H_2(e^{j0}) \neq 0$. We must also insure that $Y_2(e^{j\pi}) = 0$, and this implies that $H_2(e^{j2\pi}) = 0$. The frequency resoponse $H_2(e^{j\omega})$ must be periodic in ω, however, with period 2π. Consequently the required conditions on $H_2(e^{j\omega})$ cannot be satisfied.

4.43. We need to compress as much as possible, filter, and then expand. To start, the expand by 3 and compress by 2 blocks at the input can be exchanged.



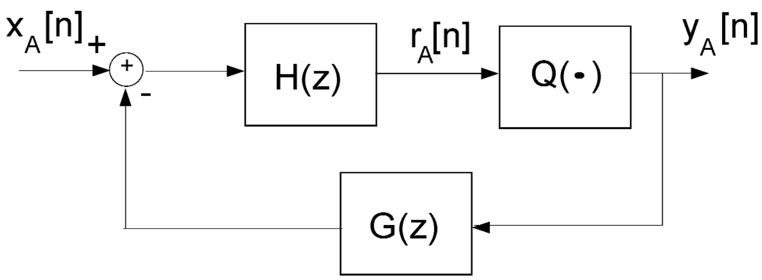
Total: 2/9 multiplications per output sample.

4.44. Appears in: Fall02 PS3, Spring00 PS4.

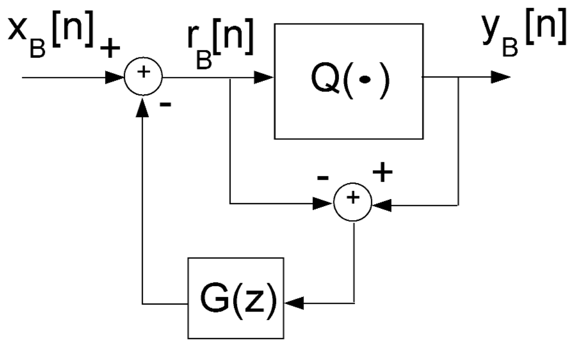
Problem

Consider the following two systems:

System A:



System B:



where $Q(\cdot)$ represents a quantizer which is the same in both systems. For any given $G(z)$, can $H(z)$ always be specified so that the two systems are equivalent (i.e. $y_A[n] = y_B[n]$ when $x_A[n] = x_B[n]$) for any arbitrary quantizer $Q(\cdot)$? If so, specify $H(z)$. If not, clearly explain your reasoning.

Solution from Fall02 PS3

Since $Q(\cdot)$ can do anything, we need to let the transfer function from $x[n]$ to $r[n]$ and from $y[n]$ to $r[n]$ be the same in both systems. In other words $H_{x_A r_A}(z) = H_{x_B r_B}(z)$ and $H_{y_A r_A}(z) = H_{y_B r_B}(z)$, where

$$\begin{aligned}
H_{x_A r_A}(z) &= H(z) \\
H_{x_B r_B}(z) &= \frac{1}{1 - G(z)} \\
H_{y_A r_A}(z) &= -G(z)H(z) \\
H_{y_B r_B}(z) &= -\frac{G(z)}{1 - G(z)}.
\end{aligned}$$

Both conditions are satisfied if we let $H(z) = \frac{1}{1-G(z)}$. Thus the two systems are equivalent if we choose $H(z)$ appropriately.

Solution from Spring00 PS4

Since $Q(\cdot)$ can do anything, we need to let the transfer function from $x[n]$ to $r[n]$ and from $y[n]$ to $r[n]$ be the same in both systems. In other words $H_{x_A r_A}(z) = H_{x_B r_B}(z)$ and $H_{y_A r_A}(z) = H_{y_B r_B}(z)$, where

$$\begin{aligned}
H_{x_A r_A}(z) &= H(z) \\
H_{x_B r_B}(z) &= \frac{1}{1 - G(z)} \\
H_{y_A r_A}(z) &= -G(z)H(z) \\
H_{y_B r_B}(z) &= -\frac{G(z)}{1 - G(z)}.
\end{aligned}$$

Both conditions are satisfied if we let $H(z) = \frac{1}{1-G(z)}$. Thus the two systems are equivalent if we choose $H(z)$ appropriately.

4.45. Problem 3 in Spring2003 midterm exam.

Problem

The quantizer $Q(\cdot)$ in the system S_1 (Figure **??**) can be modeled with an additive noise. Figure **??** shows system S_2 which is a model for system S_1

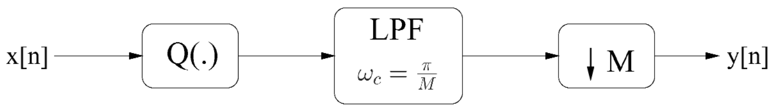


Figure 1: System S_1

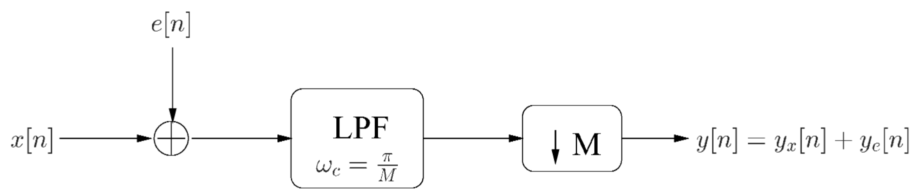


Figure 2: System S_2

The input $x[n]$ is a zero-mean, wide-sense stationary (WSS) random process with power spectral density $\Phi_{xx}(e^{j\omega})$ which is band-limited to π/M and we have $E\left[x^2[n]\right] = 1$. The additive noise $e[n]$ is WSS white noise with zero mean and variance σ_e^2. Input and additive noise are uncorrelated. The frequency response of the low-pass filter in all the diagrams has a unit gain.

(a) For system S_2 find the signal to noise ratio: $SNR = 10 \log \frac{E[y_x^2[n]]}{E[y_e^2[n]]}$. Note that $y_x[n]$ is the output due to $x[n]$ alone and $y_e[n]$ is the output due to $e[n]$ alone.

(b) To improve the SNR due to quantization, the following system is proposed:

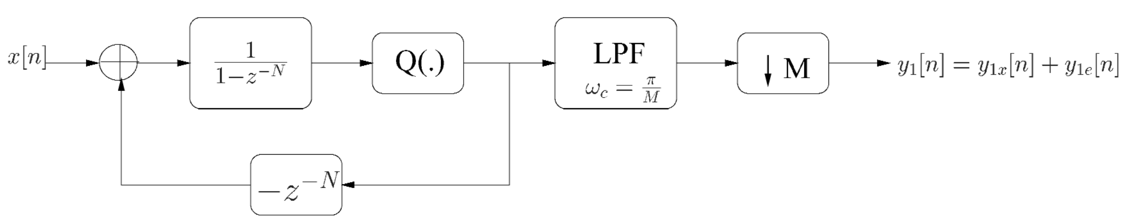


Figure 3:

where $N > 0$ is an integer such that $\pi N << M$. Replace the quantizer with the additive model as in Figure **??**. Express $y_{1x}[n]$ in terms of $x[n]$ and $y_{1e}[n]$ in terms of $e[n]$.

Figure 4:

(c) Assume that $e[n]$ is a zero mean WSS white noise which is uncorrelated with input $x[n]$. Is $y_{1e}[n]$ a WSS signal? How about $y_1[n]$? Explain.

(d) Is the proposed method in part (b) improving the SNR? For which value of N is the SNR of the system in part (b) maximized?

Useful tools:

$\sin\alpha \approx \alpha$, for $\alpha << 1$

$|1 - e^{-j\omega}|^2 = \left(2\sin\frac{\omega}{2}\right)^2$

Solution from Spring03 midterm

(a) Much of the white noise is blocked by the low pass filter; SNR $= 10\log_{10}(M/\sigma_e^2)$

(b) Define $q[n]$ as the output from the quantizer, and write $q[n] = q_x[n] + q_e[n]$, where the first term is due to $x[n]$ alone and the second term is due to $e[n]$ alone. Then $q_x[n] = x[n]$ and $q_e[n] = e[n] - e[n-N]$.

Then define $f[n]$ as the output of the low pass filter, and write $f[n] = f_x[n] + f_e[n]$, with terms due to $x[n]$ and $e[n]$ as above. Then:

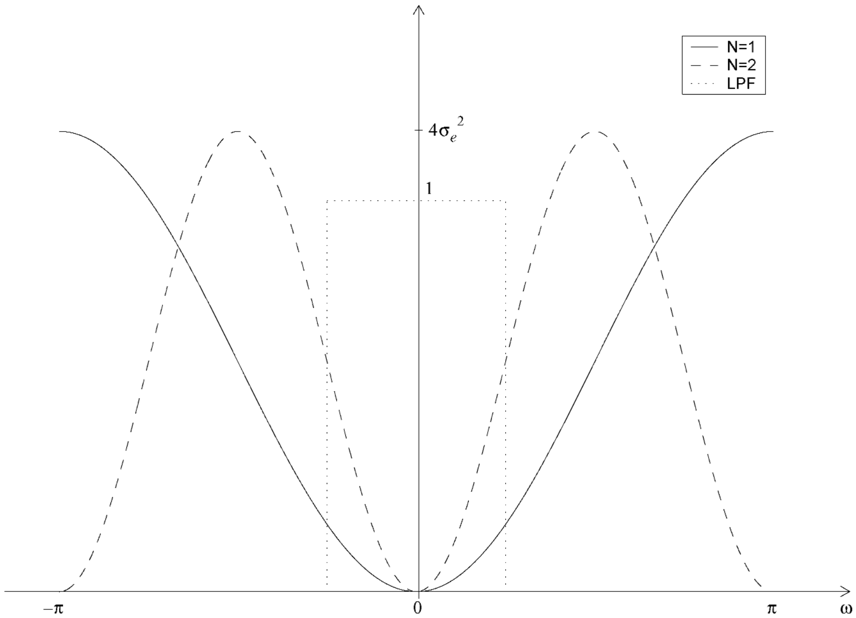
$$
\begin{aligned}
f_x[n] &= q_x[n] * \left(\frac{\sin\left(\frac{\pi}{M}n\right)}{\pi n}\right) \\
f_e[n] &= q_e[n] * \left(\frac{\sin\left(\frac{\pi}{M}n\right)}{\pi n}\right)
\end{aligned}
$$

Finally, $y_{1x}[n] = f_x[Mn]$ and $y_{1e}[n] = f_e[Mn]$.

(c) If $x[n]$ and $e[n]$ are uncorrelated and each is WSS, then any scaled and time-shifted version of $x[n]$ is WSS and uncorrelated with any scaled and time-shifted version of $e[n]$, which is also WSS. Extending that idea, applying any LTI system to $x[n]$ produces a signal that is WSS; applying any LTI system to $e[n]$ produces a signal that is WSS, and these two signals are uncorrelated with each other. Therefore $q_e[n]$ and $q_x[n]$ are uncorrelated and each is WSS. The same is true for $f_x[n]$ and $f_e[n]$. Finally, the outputs of the downsampler, $y_{1x}[n]$ and $y_{1e}[n]$, are also WSS and uncorrelated with each other.

Since $y_1[n]$ is the sum of two uncorrelated WSS signals, $y_1[n]$ is also WSS.

(d) The power spectral density $\Phi_{Q_eQ_e}\left(e^{j\omega}\right) = \sigma_e^2\left(1 - e^{-jN\omega}\right)\left(1 - e^{jN\omega}\right) = 2\sigma_e^2\left(1 - \cos(N\omega)\right)$. It is plotted below, for two different values of N.



Values of N larger than 1 just pull more of the noise into the filter's pass band. For best noise shaping, choose $N = 1$. This can also be seen by calculating the exact signal to noise ratio after the low pass filter:

$$\begin{aligned} \phi_{f_e f_e}[0] &\approx \frac{1}{2\pi} \cdot 2 \int_0^{\frac{\pi}{M}} 4\sigma_e^2 \frac{N^2\omega^2}{4} d\omega \\ &= \sigma_e^2 N^2 \frac{1}{3} \frac{\pi^2}{M^3} \end{aligned}$$

The SNR is $10 \log_{10} \frac{3M^3}{\pi^2 \sigma_e^2 N^2}$.

4.46. A. This proposed identity is not valid.
Consider as an input $\delta[n]$.
If $\delta[n]$ is compressed by a factor of two, the result is $\delta[n]$. If $\delta[n]$ is applied to a half-sample delay, the result is nonzero for some values of n.

On the other hand, if $\delta[n]$ is delayed by a single sample, the result is $\delta[n-1]$. Then compressing by a factor of two yield zero for all values of n.

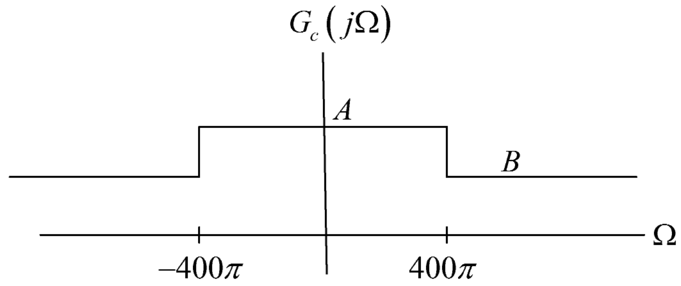
B. This proposed identity is not valid.
Consider as input $\delta[n-1]$ and suppose $h[n]=\delta[n-1]$.
If $\delta[n-1]$ is delayed one sample, the result is $\delta[n-2]$. Compressing by a factor of two yields $\delta[n-1]$. The response of the filter is $\delta[n-2]$. Expanding by a factor of two gives $\delta[n-4]$. Finally, advancing one sample produces $\delta[n-3]$.

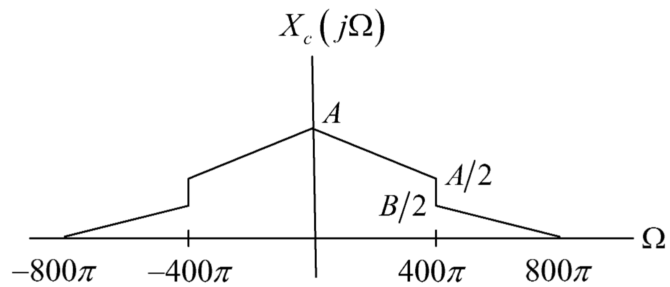
On the other hand, if $\delta[n-1]$ is advanced one sample, the result is $\delta[n]$. Compressing by a factor of two yields $\delta[n]$ again. Since $h[n+1]=\delta[(n+1)-1]=\delta[n]$, the filter response is $\delta[n]$. Expanding by a factor of two gives $\delta[n]$, and delaying by two samples produces $\delta[n-2]$.

C. This proposed identity is valid. The validity is demonstrated by looking in the frequency domain.
Let the input to the first system be $v[n]$ with DTFT $V(e^{j\omega})$. Expanding by a factor of L produces $V(e^{j\omega L})$. The response of system A is then $\left(V(e^{j\omega L})\right)^L$.

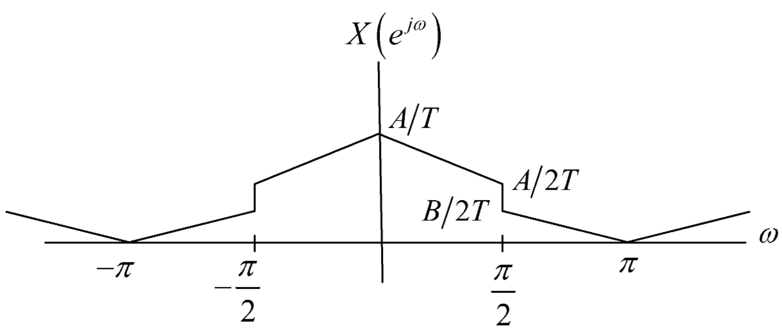
Now consider the second system with the same input $v[n]$. The response of block A is $\left(V(e^{j\omega})\right)^L$. Expanding by a factor of L produces $\left(V(e^{j\omega L})\right)^L$.

4.47.

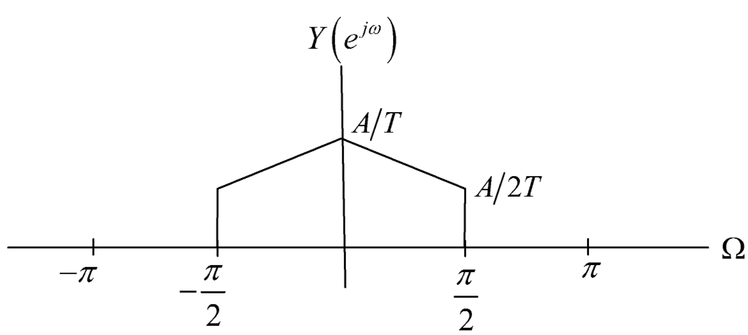


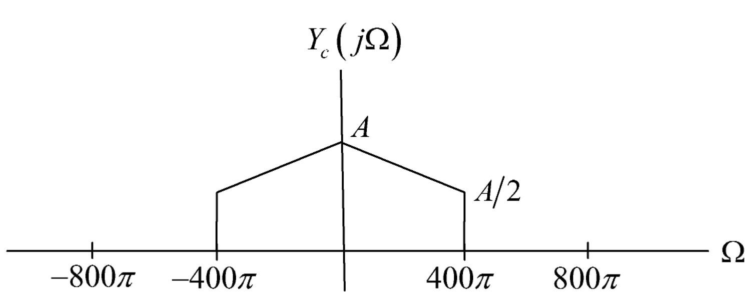


A. Since the sampling rate is $2\pi/T = 1600\pi$ there is no aliasing.



After applying $H_1(e^{j\omega})$, we get $Y(e^{j\omega})$.





B. We want $y_c(t) = f_c(t-0.1)$. As in part A, $2\pi/T = 1600\pi$, or $1/T = 800$ samples/s. In the frequency domain, $Y_c(j\Omega) = e^{-j0.1\Omega}F_c(j\Omega)$. We therefore require

$$H_1(j\Omega) = \begin{pmatrix}\text{Cancel the effect}\\ \text{of anti-alias}\end{pmatrix}\begin{pmatrix}\text{Give a}\\ \text{delay}\end{pmatrix}\begin{pmatrix}\text{Lowpass filter}\\ \text{to pick } F_c(j\Omega) \text{ and}\\ \text{remove } E_c(j\Omega)\end{pmatrix}$$

Let

$$H_a(e^{j\omega}) = \frac{1}{H_{aa}(j\omega/T)} = \frac{1}{1-\dfrac{|\omega 800|}{800\pi}} = \frac{1}{1-\dfrac{|\omega|}{\pi}}.$$

Let

$$H_b(e^{j\omega}) = e^{-j0.1\omega/T} = e^{-j80\omega}.$$

Let

$$H_c(e^{j\omega}) = \begin{cases} 1, & |\omega| \le \pi/2 \\ 0, & |\omega| > \pi/2. \end{cases}$$

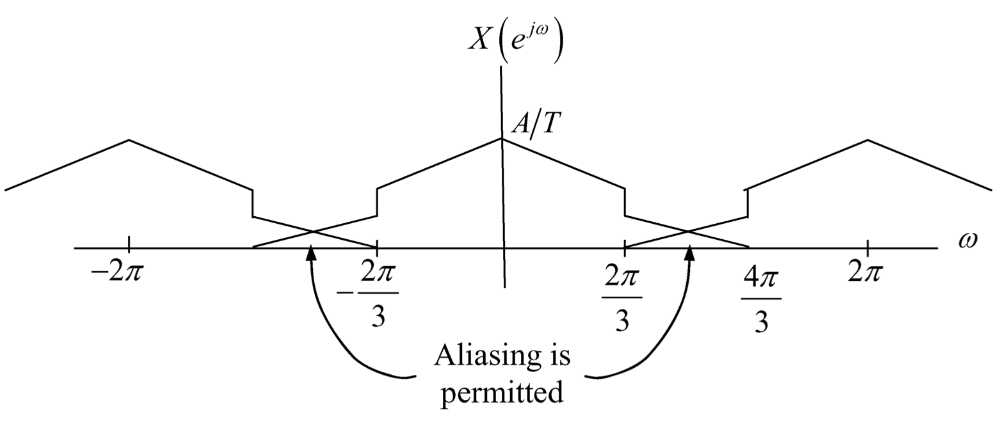
Then

$$H_1(e^{j\omega}) = H_a(e^{j\omega})H_b(e^{j\omega})H_c(e^{j\omega}) = \frac{e^{-j80\omega}}{1-\dfrac{|\omega|}{\pi}}, \quad |\omega| \le \frac{\pi}{2}.$$

C. Since we want $y_c(t) = f_c(t)$ we require

$$H_1(j\Omega) = \begin{pmatrix}\text{Cancel the effect}\\ \text{of anti-alias}\end{pmatrix}\begin{pmatrix}\text{Lowpass filter}\\ \text{to pick } F_c(j\Omega) \text{ and}\\ \text{remove } E_c(j\Omega)\end{pmatrix}.$$

Also, since we are going to lowpass filter in the discrete domain, we can afford to alias a little, provided that the aliased part of the spectrum is filtered out.



At the minimum allowable sampling rate, $400\pi = \dfrac{2\pi}{3T}$. That is, $1/T = 600$ samples/s.

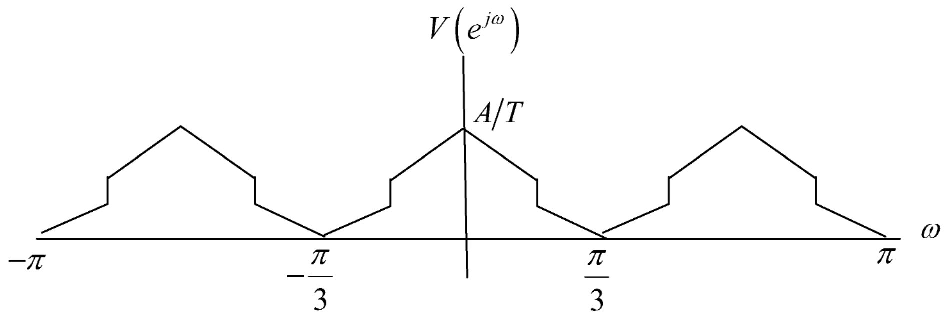
If we sample at this sampling rate, $F(e^{j\omega})$ lies in the band $|\omega| \le \dfrac{2\pi}{3}$. We want

$$H_1(e^{j\omega}) = \frac{1}{1-\dfrac{|\omega 600|}{800\pi}} = \frac{1}{1-\dfrac{3}{4\pi}|\omega|}, \quad |\omega| \le \frac{2\pi}{3}.$$

Then,

$$H_1\left(e^{j\omega}\right)=\begin{cases}\dfrac{1}{1-\dfrac{3}{4\pi}|\omega|}, & |\omega|\le\dfrac{2\pi}{3}\\[2ex] 0, & |\omega|>\dfrac{2\pi}{3}.\end{cases}$$

D. We have $X\left(e^{j\omega}\right)$ from part A. After expanding, the ω axis is compressed.



Since we want $y[n]=f_c(nT/3)$, we need only pick the $F_c(j\Omega)$ from the center "house-like" part of the spectrum. Also, $H_2\left(e^{j\omega}\right)$ should cancel out the effects of the anti-alias filter and multiply $V\left(e^{j\omega}\right)$ by the upsampling factor of 3. We require

$$H_2\left(e^{j\omega}\right)=\frac{3}{1-\dfrac{|3\omega/T|}{800\pi}}=\frac{3}{1-\dfrac{3}{\pi}|\omega|},\quad |\omega|\le\frac{\pi}{6}.$$

Then,

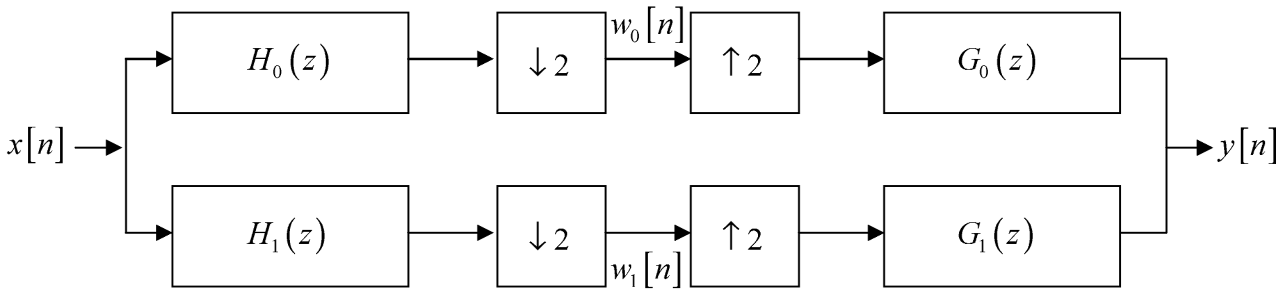
$$H_2\left(e^{j\omega}\right)=\begin{cases}\dfrac{3}{1-\dfrac{3}{\pi}|\omega|}, & |\omega|\le\dfrac{\pi}{6}\\[2ex] 0, & |\omega|>\dfrac{\pi}{6}.\end{cases}$$

4.48. (a) $B(z)+B(-z)=2c$ means that $b[n]+(-1)^n b[n]=2c\,\delta[n]$, for all n. For $n=0$, $b[0]+b[0]=2c$. For $n\neq 0$, $b[n]+(-1)^n b[n]=0$. For n even, $(-1)^n=1$, so $b[n]+(-1)^n b[n]=b[n]+b[n]=2b[n]$. For n odd, $b[n]+(-1)^n b[n]=b[n]-b[n]=0$.

So, we have the following constraints: $b[n]=0$ for even $n\neq 0$, $b[0]=c$, and $b[n]$ can be anything for odd n.

(b) It is possible. One example is $h[n]=\delta[n]+\delta[n-1]$. Thus $b[n]=h[n]*h[-n]=\delta[n+1]+2\delta[n]+\delta[n-1]$.

(c) Let $H_0(z)=H(z)$, and $H_1(z)=z^{N-1}H(-z^{-1})$.



Then

$$W_0(e^{j\omega})=\frac{1}{2}\left[X(e^{j\omega/2})H_0(e^{j\omega/2})+X(e^{j(\omega-2\pi)/2})H_0(e^{j(\omega-2\pi)/2})\right]$$

$$W_1(e^{j\omega})=\frac{1}{2}\left[X(e^{j\omega/2})H_1(e^{j\omega/2})+X(e^{j(\omega-2\pi)/2})H_1(e^{j(\omega-2\pi)/2})\right]$$

Now, $Y(e^{j\omega})=W_0(e^{j2\omega})G_0(e^{j\omega})+W_1(e^{j2\omega})G_1(e^{j\omega})$:

$$\begin{aligned}
Y(e^{j\omega})&=\frac{1}{2}\left[X(e^{j\omega})H_0(e^{j\omega})+X(e^{j(\omega-\pi)})H_0(e^{j(\omega-\pi)})\right]G_0(e^{j\omega})\\
&\quad+\frac{1}{2}\left[X(e^{j\omega})H_1(e^{j\omega})+X(e^{j(\omega-\pi)})H_1(e^{j(\omega-\pi)})\right]G_1(e^{j\omega})\\
&=\frac{1}{2}\left[H_0(e^{j\omega})G_0(e^{j\omega})+H_1(e^{j\omega})G_1(e^{j\omega})\right]X(e^{j\omega})\\
&\quad+\frac{1}{2}\left[H_0(e^{j(\omega-\pi)})G_0(e^{j\omega})+H_1(e^{j(\omega-\pi)})G_1(e^{j\omega})\right]X(e^{j(\omega-\pi)}).
\end{aligned}$$

We want to get rid of the term in the output which is multiplied by $X(e^{j(\omega-\pi)})$, which is the result of aliasing. With this term present the system will not be LTI. Thus we must have

$$H_0(e^{j(\omega-\pi)})G_0(e^{j\omega})+H_1(e^{j(\omega-\pi)})G_1(e^{j\omega})=0.$$

This condition can be satisfied with the following choice for $G_0(z)$ and $G_1(z)$: $G_0(e^{j\omega}) = 2H_1(e^{j(\omega-\pi)})$ and $G_1(e^{j\omega}) = -2H_0(e^{j(\omega-\pi)})$. Looking up our definition of $H_1(z)$, we have

$$G_0(z) = 2H_1(-z) = 2(-z)^{N-1} H(z^{-1})$$
$$G_1(z) = -2H_0(z) = -2H(-z).$$

With this choice of $G_0(z)$ and $G_1(z)$, the aliasing resulting from decimation in the analysis section of the QMF filter bank is perfectly cancelled by the synthesis part. The factors of 2 are optional – they compensate for $\frac{1}{2}$ introduced by the downsampler.

(d)

$$\begin{aligned}
Y(z) &= \frac{1}{2}\left[H_0(z)G_0(z) + H_1(z)G_1(z)\right]X(z) \\
&= \frac{1}{2}\left[2H(z)(-z)^{N-1} H(z^{-1}) + 2z^{N-1}H(-z^{-1})(-H(-z))\right]X(z) \\
&= \left[(-1)^{N-1} z^{N-1}H(z)H(z^{-1}) - z^{N-1}H(-z)H(-z^{-1})\right]X(z) \\
&= z^{N-1}\left[(-1)^{N-1} H(z)H(z^{-1}) - H(-z)H(-z^{-1})\right]X(z).
\end{aligned}$$

Recall that we are given that $H(z)H(z^{-1}) + H(-z)H(-z^{-1}) = c$. For N even $(-1)^{N-1} = -1$, and we have

$$Y(z) = z^{N-1}\left[-H(z)H(z^{-1}) - H(-z)H(-z^{-1})\right]X(z) = -cz^{N-1}X(z).$$

Thus for even N, the output of the QMF is simply a scaled shifted version of the input. Therefore, the overall system does indeed reconstruct the input perfectly, but only for even N.

4.49. Suppose $x[n]$ has spectrum $X(e^{j\omega})$. At the output of the first $H_0(e^{j\omega})$ stage, the spectrum is

$$H_0(e^{j\omega})X(e^{j\omega}) = \begin{cases} X(e^{j\omega}), & |\omega| < \dfrac{\pi}{2} \\ 0, & \dfrac{\pi}{2} < |\omega| < \pi. \end{cases}$$

After downsampling, we have $\frac{1}{2}X(e^{j\omega/2}), \quad |\omega| < \pi$. At the output of the $Q_0(e^{j\omega})$ stage we have $\frac{1}{2}X(e^{j\omega/2})Q_0(e^{j\omega})$. The next step is upsampling. This produces $\frac{1}{2}X(e^{j\omega})Q_0(e^{j2\omega}), \quad |\omega| < \dfrac{\pi}{2}$. The upsampled signal is passed through a second $H_0(e^{j\omega})$ stage. At the output of the upper branch we have the spectrum

$$\begin{cases} \frac{1}{2}X(e^{j\omega})Q_0(e^{j2\omega}), & |\omega| < \dfrac{\pi}{2} \\ 0, & \dfrac{\pi}{2} < |\omega| < \pi. \end{cases}$$

At the output of the first $H_1(e^{j\omega})$ stage in the lower branch, the spectrum is

$$H_1(e^{j\omega})X(e^{j\omega}) = \begin{cases} 0, & |\omega| < \dfrac{\pi}{2} \\ X(e^{j\omega}), & \dfrac{\pi}{2} < |\omega| < \pi. \end{cases}$$

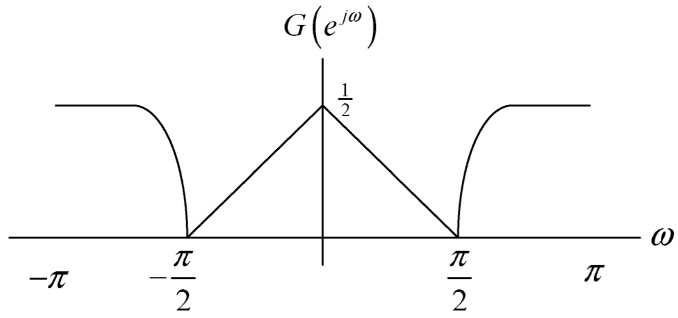
After downsampling, we have $\frac{1}{2}X\left(e^{j\omega/2}\right), \quad \pi<|\omega|<2\pi$. At the output of the $Q_1\left(e^{j\omega}\right)$ stage we have $\frac{1}{2}Q_1\left(e^{j\omega}\right)X\left(e^{j\omega/2}\right), \quad \pi<|\omega|<2\pi$. The next stage is upsampling; upsampling produces $\frac{1}{2}Q_1\left(e^{j2\omega}\right)X\left(e^{j\omega}\right), \quad \frac{\pi}{2}<|\omega|<\pi$. The upsampled signal is passed through a second $H_1\left(e^{j\omega}\right)$ stage. At the output of the lower branch we have the spectrum

$$\begin{cases} 0, & 0<|\omega|<\frac{\pi}{2} \\ \frac{1}{2}Q_1\left(e^{j2\omega}\right)X\left(e^{j\omega}\right), & \frac{\pi}{2}<|\omega|<\pi. \end{cases}$$

Finally, we combine the two branches and divide by $X\left(e^{j\omega}\right)$ to obtain the frequency response

$$G\left(e^{j\omega}\right)=\begin{cases} \frac{1}{2}Q_0\left(e^{j2\omega}\right), & 0<|\omega|<-\frac{\pi}{2} \\ \frac{1}{2}Q_1\left(e^{j2\omega}\right), & \frac{\pi}{2}<|\omega|<\pi. \end{cases}$$

The result is sketched below.



4.50. (a)

(a1) If $H_0(z)$ is linear phase, then $H_0(e^{j\omega}) = A(e^{j\omega})e^{-j\alpha\omega + j\beta}$, where $A(e^{j\omega})$ is real-valued and α and β are constants. In this case we have $H_0^2(e^{j\omega}) = A^2(e^{j\omega})e^{-j2\alpha\omega + j2\beta}$ and $H_0(e^{j(\omega-\pi)}) = A(e^{j(\omega-\pi)})e^{-j\alpha(\omega-\pi)+j\beta}$, so that $H_0^2(e^{j(\omega-\pi)}) = A^2(e^{j(\omega-\pi)})e^{-j2\alpha(\omega-\pi)+j2\beta} = A^2(e^{j(\omega-\pi)})e^{-j2\alpha\omega+j2(\beta-\alpha\pi)}$. Now $T(e^{j\omega})$ is given by

$$\begin{aligned}T(e^{j\omega}) &= \tfrac{1}{2}\left(H_0^2(e^{j\omega}) - H_0^2(e^{j(\omega-\pi)})\right)\\ &= \tfrac{1}{2}A^2(e^{j\omega})e^{-j2\alpha\omega+j2\beta} - \tfrac{1}{2}A^2(e^{j(\omega-\pi)})e^{-j2\alpha\omega+j2(\beta-\alpha\pi)}\\ &= \left[\tfrac{1}{2}A^2(e^{j\omega}) - \tfrac{1}{2}A^2(e^{j(\omega-\pi)})e^{-j2\pi\alpha}\right]e^{-j2\alpha\omega+j2\beta}.\end{aligned}$$

We see that $T(z)$ will be linear phase if $\left[\tfrac{1}{2}A^2(e^{j\omega}) - \tfrac{1}{2}A^2(e^{j(\omega-\pi)})e^{-j2\pi\alpha}\right]$ is real-valued. A sufficient condition is that α is an integer multiple of $\tfrac{1}{2}$.

(a2) If $E_0(z)$ and $E_1(z)$ are linear phase then $E_0(e^{j\omega}) = A_0(e^{j\omega})e^{-j\alpha_0\omega+j\beta_0}$ and $E_1(e^{j\omega}) = A_1(e^{j\omega})e^{-j\alpha_1\omega+j\beta_1}$. In this case $T(e^{j\omega})$ is given by

$$\begin{aligned}T(e^{j\omega}) &= 2e^{-j\omega}E_0(e^{j2\omega})E_1(e^{j2\omega})\\ &= 2e^{-j\omega}A_0(e^{j2\omega})e^{-j2\alpha_0\omega+j2\beta_0}A_1(e^{j2\omega})e^{-j2\alpha_1\omega+j2\beta_1}\\ &= 2A_0(e^{j2\omega})A_1(e^{j2\omega})e^{-j[2(\alpha_0+\alpha_1)+1]\omega+j2(\beta_0+\beta_1)}.\end{aligned}$$

We see in this case that $T(z)$ is linear phase.

(b) Given $h_0[n] = \delta[n] + \delta[n-1] + \tfrac{1}{4}\delta[n-2]$,

(b1) We have $H_0(z) = 1 + z^{-1} + \tfrac{1}{4}z^{-2}$. Then $H_1(z) = H_0(-z) = 1 - z^{-1} + \tfrac{1}{4}z^{-2}$. Also, $G_0(z) = H_0(z)$ and $G_1(z) = -H_1(z)$. These give $h_1[n] = \delta[n] - \delta[n-1] + \tfrac{1}{4}\delta[n-2]$, $g_0[n] = \delta[n] + \delta[n-1] + \tfrac{1}{4}\delta[n-2]$, and $g_1[n] = -\delta[n] + \delta[n-1] - \tfrac{1}{4}\delta[n-2]$.

(b2) We have $e_0[n] = h_0[2n] = \delta[n] + \tfrac{1}{4}\delta[n-1]$ and $e_1[n] = h_0[2n+1] = \delta[n]$.

(b3) Now $E_0(z) = 1 + \tfrac{1}{4}z^{-1}$ and $E_1(z) = 1$. Therefore

$$\begin{aligned}T(z) &= 2z^{-1}E_0(z^2)E_1(z^2)\\ &= 2z^{-1}\left(1 + \tfrac{1}{4}z^{-2}\right)\cdot 1\\ &= 2z^{-1} + \tfrac{1}{2}z^{-3}.\end{aligned}$$

Also, $t[n] = 2\delta[n-1] + \tfrac{1}{2}\delta[n-3]$.

4.51.

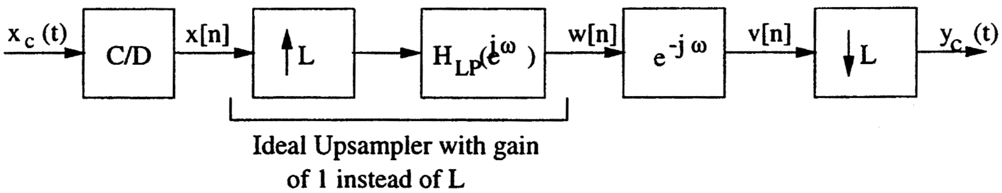
$$\begin{aligned} y[n] &= x^2[n] \\ Y(e^{j\omega}) &= X(e^{j\omega}) * X(e^{j\omega}) \end{aligned}$$

therefore, $Y(e^{j\omega})$ will occupy twice the frequency band that $X(e^{j\omega})$ does if no aliasing occurs.
If $Y(e^{j\omega}) \neq 0, \quad -\pi < \omega < \pi$, then $X(e^{j\omega}) \neq 0, \quad -\frac{\pi}{2} < \omega < \frac{\pi}{2}$ and so $X(j\Omega) = 0, \quad |\Omega| \geq 2\pi(1000)$.
Since $\omega = \Omega T$,

$$\begin{aligned} \frac{\pi}{2} &\geq T \cdot 2\pi(1000) \\ T &\leq \frac{1}{4000} \end{aligned}$$

4.52. Split $H(e^{j\omega})$ into a lowpass and a delay.

$$
\begin{aligned}
H(e^{j\omega}) &= H_{LP}(e^{j\omega})e^{-j\omega} \\
H_{LP}(e^{j\omega}) &= \begin{cases} 1, & |\omega| < \frac{\pi}{L} \\ 0, & \frac{\pi}{L} < |\omega| \le \pi \end{cases}
\end{aligned}
$$



Then we analyze the system as follows:

$$
\begin{aligned}
x[n] &= x_c(nT) \quad \text{no aliasing assumed} \\
w[n] &= \frac{1}{L}x_c(n\frac{T}{L}) \quad \text{rate change} \\
v[n] &= w[n-1] = \frac{1}{L}x_c\left(n\frac{T}{L} - \frac{T}{L}\right), \quad \text{delay at higher rate} \\
y[n] &= v[nL] = \frac{1}{L}x_c\left(nT - \frac{T}{L}\right)
\end{aligned}
$$

4.53. Sampling random processes

$$\phi_{x_cx_c}(\tau) = E(x_c(t)x_c^*(t+\tau)) \Leftrightarrow P_{x_cx_c}(\Omega) = \int_{-\infty}^{\infty} \phi_{x_cx_c}(\tau)e^{-j\Omega\tau}d\tau$$

(a)

$$\begin{aligned} \phi_{xx}[m] &= E(x[n]x^*[n+m]) = E(x_c(nT)x_c^*(nT+mT)) \\ &= \phi_{x_cx_c}(mT), \quad \text{i.e., sampled autocovariance} \end{aligned}$$

(b) Since $\phi_{xx}[m]$ is a sampled $\phi_{x_cx_c}(\tau)$

$$P_{xx}(\omega) = \frac{1}{T}\sum_{K=-\infty}^{\infty} P_{x_cx_c}\left(\frac{\omega}{T} + \frac{2\pi k}{T}\right)$$

(c) If

$$P_{x_cx_c} = 0, \text{ for } |\omega| \geq \pi$$

then

$$P_{xx}(\omega) = \frac{1}{T}P_{x_cx_c}\left(\frac{\omega}{T}\right), \quad |\omega| \leq \pi$$

4.54. (a)

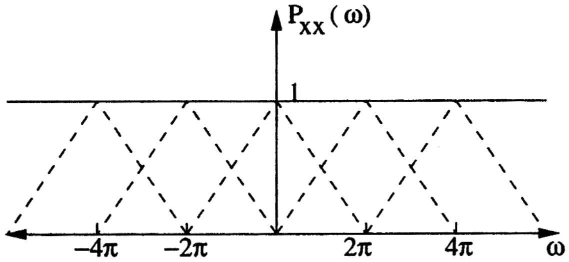
$$\begin{aligned}\phi_{x_cx_c}(\tau) &= E(x_c(t)x_c(t+\tau)\\ \phi_{xx}[m] &= E(x[n]x[n+m]) = E(x_c(nT)x_c(nT+mT))\\ &= \phi_{x_cx_c}(mT)\end{aligned}$$

(b)

$$P_{xx}(\omega) = \frac{1}{T}\sum_{r=-\infty}^{\infty} P_{x_cx_c}\left(\frac{\omega}{T} + \frac{2\pi r}{T}\right)$$

Therefore, we require that $\frac{\pi}{T} \geq \Omega_0$.

(c) For the spectrum of Fig P3.8-2 it is clear that if $T = \frac{2\pi}{\Omega_0}$ then the discrete-time power spectrum will be white, as shown in the figure above.

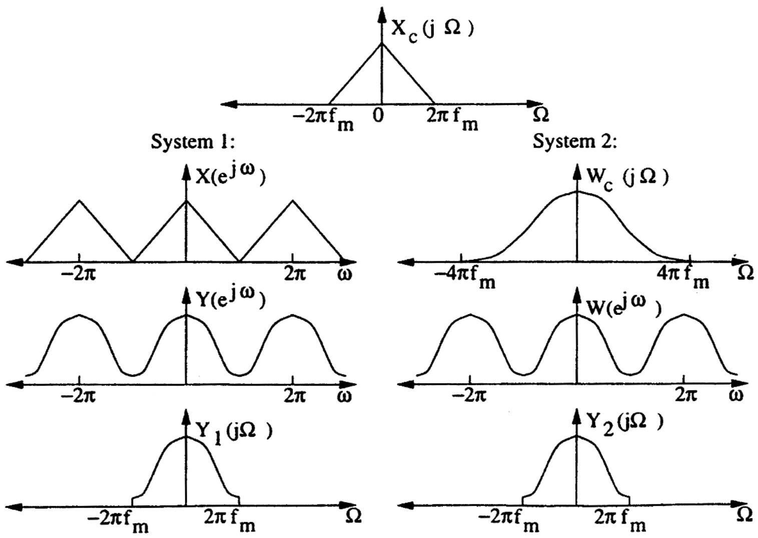


(d) For white discrete-time signal $\Rightarrow \phi_{xx}[m] = 0, \quad m \neq 0$ but $\phi_{xx}[m] = \phi_{x_cx_c}(mT)$. Therefore, any analog signal whose autocorrelation function has zeros equally spaced at intervals of T will yield a white discrete-time sequence is sampled with sampling period T. For example, for Fig P3.8-1:

$$\phi_{x_cx_c}(\tau) = \frac{\sin\Omega_0 T}{T\pi} \Rightarrow \phi_{xx}[m] = \frac{\sin\Omega_0 mT}{\pi m T}$$

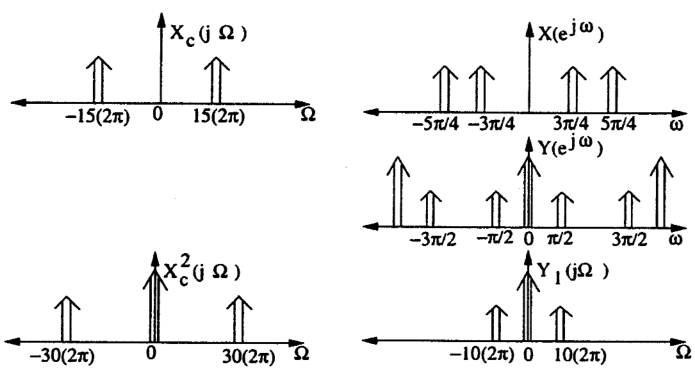
$$\text{if } T = \frac{\pi}{\Omega_0} \qquad \phi_{xx}[m] = \frac{\sin \pi m}{\pi^2 m/\Omega_o} = 0, \qquad m \neq 0$$

4.55. (a) Consider the following plots.



$y_1(t) = y_2(t)$. Convolution is a linear process. Aliasing is a linear process. Periodic convolution is equivalent to convolution followed by aliasing.
$y_1(t) \neq x^2(t)$: System 2 at Step 1 shows $X_c^2(j\Omega)$. This is clearly not $Y_1(j\Omega)$. $Y_1(j\Omega)$ is an aliased version of $X_c(j\Omega)$

(b) Now,



(c)

$$\begin{aligned}
x(t) &= A\cos(30\pi t)\\
x^3(t) &= \frac{3}{4}A\cos(30\pi t) + \frac{1}{4}A\cos(3\cdot 30\pi t),\\
v[n] &= \frac{3}{4}A\cos\left(\frac{3}{4}\pi n\right) + \frac{1}{4}A\cos\left(\frac{1}{4}\pi n\right)\\
v[n] &= x^3[n]\\
y[n] &= x[n]
\end{aligned}$$

We can see here that sometimes aliasing won't be destructive. When aliased sections do not overlap they can be reconstructed.

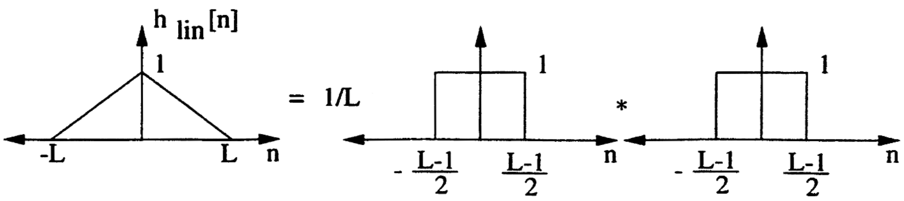
(d) This is the inverse to part (c). Since multiplication in time corresponds to convolution in frequency, a signal $x^2(t)$ has at most two times the bandwidth of $x(t)$. Therefore, $x^{1/2}$ will have at least $\frac{1}{2}$ the bandwidth of $x(t)$. If we run our signal through a box that will raise it to the $1/M$ power, then the sampling rate can be decreased by a factor of M.

4.56. (a)

$$
\begin{aligned}
x_i[n] &= x_u[n] * h_{zoh}[n] \\
h_{zoh}[n] &= \begin{cases} 1, & 0 \le n \le L-1 \\ 0, & \text{else} \end{cases}
\end{aligned}
$$

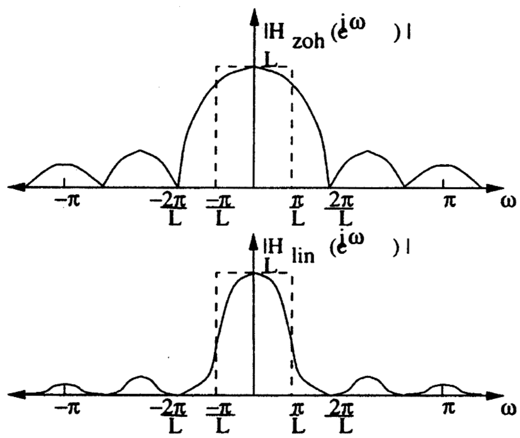
$$H_{zoh}(e^{j\omega}) = \frac{\sin(\omega L/2)}{\sin(\omega/2)} e^{-j(L-1)\omega/2}$$

(b) The impulse response $h_{lin}[n]$ corresponds to the convolution of two rectangular sequences, as shown below.



$$H_{lin}(e^{j\omega}) = \frac{1}{L}\left(\frac{\sin(\omega L/2)}{\sin(\omega/2)}\right)^2$$

(c) The frequency response of zero-order-hold is flatter in the region $[-\pi/L, \pi/L]$, but achieves less out-of-band attenuation.



4.57.

$$\begin{aligned}\phi_{xx}[n] &= x[n] * x[-n] \\ \Phi_{xx}(e^{j\omega}) &= X(e^{j\omega}) * X^*(e^{j\omega})\end{aligned}$$

The bandwidth of $\Phi_{xx}(e^{j\omega})$ is no larger than the bandwidth of $X(e^{j\omega})$. Therefore, the outputs of the systems will be the same if $H_2(e^{j\omega})$ is an ideal lowpass filter with a cutoff of π/L.

4.58. The idea here is to exploit the fact that every other sample supplied to $h[n]$ in Fig 3.27-1 is zero. That is,

$$\begin{aligned}
y_1[n] &= h[n] * w[n] = \sum_{n=-\infty}^{\infty} w[n-k]h[k] \\
&= aw[n] + bw[n-1] + cw[n-2] + dw[n-3] + ew[n-4] \\
&= \begin{cases} ax[n/2] + cx[(n/2)-1] + e[(n/2)-2], & n \text{ even} \\ bx[(n/2)-(1/2)] + dx[(n/2)-(3/2)], & n \text{ odd} \end{cases}
\end{aligned}$$

$$\begin{aligned}
w_1[n] &= \begin{cases} h_1[n/2] * x[n/2], & n \text{ even} \\ 0, & n \text{ odd} \end{cases} \\
&= \begin{cases} h_1[0]x[n/2] + h_1[1]x[(n/2)-1] + h_1[2]x[(n/2)-2], & n \text{ even} \\ 0, & n \text{ odd} \end{cases}
\end{aligned}$$

$$\begin{aligned}
w_2[n] &= \begin{cases} h_2[n/2] * x[n/2], & n \text{ even} \\ 0, & n \text{ odd} \end{cases} \\
&= \begin{cases} h_2[0]x[n/2] + h_2[1]x[(n/2)-1] + h_2[2]x[(n/2)-2], & n \text{ even} \\ 0, & n \text{ odd} \end{cases}
\end{aligned}$$

Comparing $w_1[n], w_2[n]$ with $y_1[n]$ above:

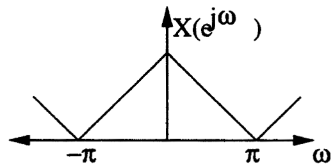
$w[n]$ can give even samples if $h_1[0] = a, h_1[1] = c, h_2[2] = e$. Similarly, $w_2[n]$ can give the odd samples if $h_3[n]$ delays $w_2[n]$ by one sample, i.e., $h_3[0] = 0, h_3[1] = 0, h_3[2] = 0$. Thus

$$w_3[n] = \begin{cases} h_2[0]x[(n-1)/2] + h_2[1]x[(n-1)/2-1] + h_2[2]x[(n-1)/2-2], & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

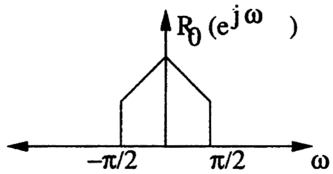
$$h_2[0] = b, \quad h_2[1] = d, \quad h_2[2] = 0$$

4.59. Sketches appear below.

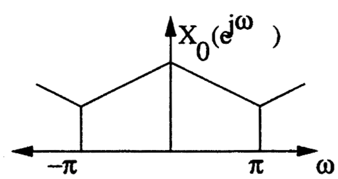
(a) First, $X(e^{j\omega})$ is plotted.



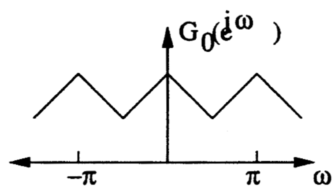
The lowpass filter cuts off at $\frac{\pi}{2}$.



The downsampler expands the frequency axis. Since $R_0(e^{j\omega})$ is bandlimited to $\frac{\pi}{M}$, no aliasing occurs.



The upsampler compresses the frequency axis by a factor of 2.



The lowpass filter cuts off at $\frac{\pi}{2} \Rightarrow Y_0(e^{j\omega}) = R_0(e^{j\omega})$ as sketched above.

(b) $G_0(e^{j\omega}) = \frac{1}{2}\left(X(e^{j\omega})H_0(e^{j\omega}) + X(e^{j(\omega+\pi)})H_0(e^{j(\omega+\pi)})\right)$

(c)

$$\begin{aligned}
Y_0(e^{j\omega}) &= \frac{1}{2}H_0(e^{j\omega})\left(X(e^{j\omega})H_0(e^{j\omega}) + X(e^{j(\omega+\pi)})H_0(e^{j(\omega+\pi)})\right)\\
Y_1(e^{j\omega}) &= \frac{1}{2}H_1(e^{j\omega})\left(X(e^{j\omega})H_1(e^{j\omega}) + X(e^{j(\omega+\pi)})H_1(e^{j(\omega+\pi)})\right)\\
Y(e^{j\omega}) &= Y_0(e^{j\omega}) - Y_1(e^{j\omega})\\
&= \frac{1}{2}X(e^{j\omega})\left[H_0^2(e^{j\omega}) - H_1^2(e^{j\omega})\right]\\
&\quad +\frac{1}{2}X(e^{j(\omega+\pi)})\underbrace{\left[H_0(e^{j\omega})H_0(e^{j(\omega+\pi)}) - H_1(e^{j\omega})H_1(e^{j(\omega+\pi)})\right]}_{=0}
\end{aligned}$$

The aliasing terms always cancel. $Y(e^{j\omega})$ is proportional to $X(e^{j\omega})$ if $[H_0^2(e^{j\omega}) - H_1^2(e^{j\omega})]$ is a constant.

$X(e^{j\omega}) = 0, \pi/3 \leq |\omega| \leq \pi$. $x[n]$ can be thought of as an oversampled signal. The approach is to determine whether n_0 is odd or even, then sample so that n_0 is avoided, upsampled and lowpass filter. This recovers $x[n_0]$.

4.60. (a) In the case where n_0 is not known, we determine whether it is even or odd as follows:

$$\begin{aligned}
\hat{x}[n] &= x[n] - A\delta[n-n_0] \\
\hat{X}(e^{j\omega}) &= X(e^{j\omega}) - Ae^{-j\omega n_0} \\
\hat{X}(e^{j\omega})|_{\omega=\frac{\pi}{2}} &= \sum_n x[n](-j)^n \\
\hat{X}(e^{j(\pi/2)}) &= -A(-j)^{n_0}
\end{aligned}$$

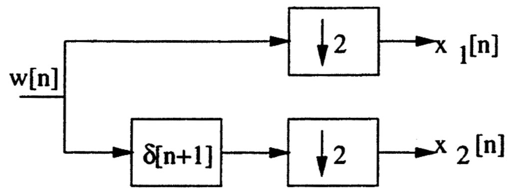
If the result is real, n_0 is even. If the result is imaginary, n_0 is odd.

(b) If n_0 is even, sample $\hat{x}[n]$ so that the even-numbered sequence values are set to zero. If n_0 is odd, sample so the odd-numbered samples are set to zero

(c) Filter the sampled sequence with a lowpass filter with cutoff frequency $\pi/3$, and gain 2. This is an exact procedure if ideal filters are used.

4.61. (a)

$$
\begin{aligned}
w[n] &= \begin{cases} x_1[n/2], & n \text{ even} \\ x_2[(n-1)/2], & n \text{ odd} \end{cases} \\
x_1[n] &= w[2n] \\
x_2[n] &= w[2n+1]
\end{aligned}
$$



The system is linear, time-varying (due to downsampling), non-causal (due to $\delta[n+1]$), and stable.

(b)

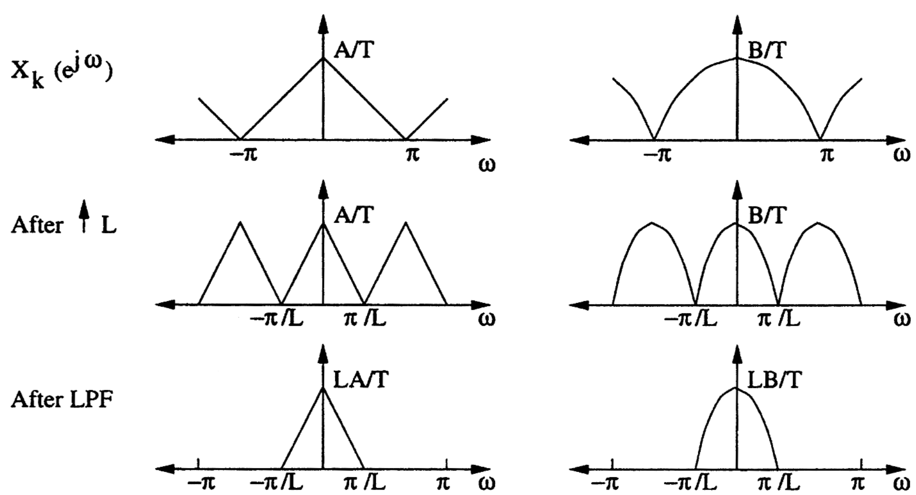
$$T = \frac{\pi}{\Omega_N} = \frac{\pi}{2\pi \times 5000} = 10^{-4}\text{sec}, \qquad \frac{L\omega_1}{T} = 2\pi \times 10^5$$

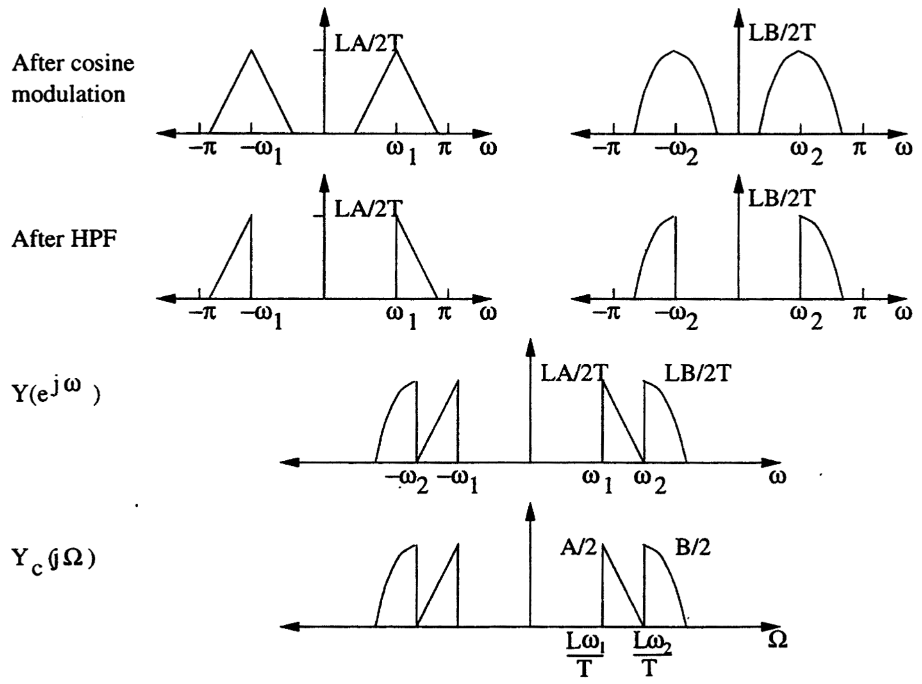
To avoid aliasing in $y_c(t)$:

$$
\begin{aligned}
\frac{L\omega_1}{T} + \frac{2\pi}{T} &\leq \frac{L\pi}{T} \\
\omega_1 &= \frac{20\pi}{L} \\
20\pi + 2\pi &= L\pi
\end{aligned}
$$

$$L = 22, \qquad \omega_1 = 2\pi\left(\frac{10}{22}\right)$$

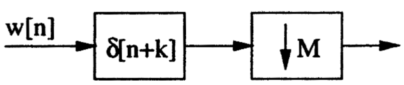
(c) The Fourier transforms are sketched below.



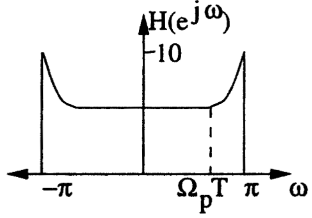


(d) To generalize for M channels, we would use the same modulators, but we would choose a larger value of L to make room for additional spectra above the lower frequency bound. If the lower bound remained $2\pi \cdot 10^5$, L would become $L = 20 + M$ for M channels.

A branch of the TDM demultiplexing system would be:



4.62. Since we want $W(e^{j\omega})$ to equal $X(e^{j\omega})$, then $H(e^{j\omega})$ must compensate for the drop offs in $H_{aa}(j\Omega)$.



4.63. (a)

$$E(e) = \int ep(e)de = \frac{1}{\Delta}\int_{-\Delta/2}^{\Delta/2} ede = \frac{e^2}{2\Delta}\Big|_{-\Delta/2}^{\Delta/2} = 0$$

$$\sigma_e^2 = E(e^2 - 0) = \frac{1}{\Delta}\int_{-\Delta/2}^{\Delta/2} e^2 de = \frac{e^3}{3\Delta}\Big|_{-\Delta/2}^{\Delta/2} = \frac{\Delta^2}{12}$$

$$r[m,n] = E(e[m]e[n]) = \begin{cases} E(e[m])E(e[n]), & m \neq n \\ E(e^2[n]), & m = n \end{cases}$$

$$r[n,m] = r[n-m] = \frac{\Delta^2}{12}\delta[n-m]$$

(b)

$$\text{SNR} = \frac{\sigma_x^2}{\sigma_e^2} = \frac{12\sigma_x^2}{\Delta^2}$$

(c) Let $e_y[n]$ be the output noise.

$$\begin{aligned}
e_y[n] &= \sum_k h[k]e[n-k] \\
E(e_y^2[n]) &= E\left(\sum_k h[k]e[n-k]\sum_l h[l]e[n-l]\right) = \sum_k\sum_l h[k]h[l]\underbrace{E(e[n-k]e[n-l])}_{\sigma_e^2[k-l]} \\
\sigma_{e_y}^2 &= \sigma_e^2\sum_k h^2[k] \\
&= \sigma_e^2\sum_{k=0}^{\infty}\frac{1}{4}\left(a^k + (-a)^k\right)^2 = \frac{\sigma_e^2}{4}\sum_{k=0}^{\infty}\left(a^{2k} + 2a^k(-a)^k + (-a)^{2k}\right) \\
&= \frac{\sigma_e^2}{2}\left(\sum_{k=0}^{\infty}a^{2k} + \sum_{k=0}^{\infty}(-a^2)^k\right) = \frac{\sigma_e^2}{2}\left(\frac{1}{1-a^2} + \frac{1}{1+a^2}\right) \\
&= \sigma_e^2\left(\frac{1}{1-a^4}\right) = \frac{\Delta^2}{12(1-a^4)}
\end{aligned}$$

The variance of $x[n]$ is weighted similarly so the SNR does not change. $\text{SNR}_{\text{out}} = 12\frac{\sigma_x^2}{\Delta^2}$.

(d) $f[n] = x[n]e[n]$

$$E(f[n]) = E(x[n]e[n]) = E(x[n])E(e[n]) = 0$$

$$\sigma_f^2 = E(f^2[n]) = E(x^2[n]e^2[n]) = E(x^2[n])E(e^2[n]) = \sigma_x^2\sigma_e^2$$

$$r_f[n,m] = E(x[n]x[m]e[n]e[m]) = \underbrace{E(x[n]x[m])}_{\sigma_x^2\delta[n-m]}\cdot\underbrace{E(e[n]e[m])}_{\sigma_e^2\delta[n-m]}$$

(e)

$$\text{SNR} = \frac{\sigma_x^2}{\sigma_f^2} = \frac{\sigma_x^2}{\sigma_x^2\sigma_e^2} = \frac{1}{\sigma_e^2} = \frac{12}{\Delta^2}$$

(f) Using the results of part (c).

$$\sigma_{e_y}^2 = \sigma_f^2\left(\frac{1}{1-a^4}\right) = \frac{\sigma_x^2\sigma_e^2}{1-a^4}$$

Again, the variance of $x[n]$ is weighted by the same factor, so the SNR does not change.

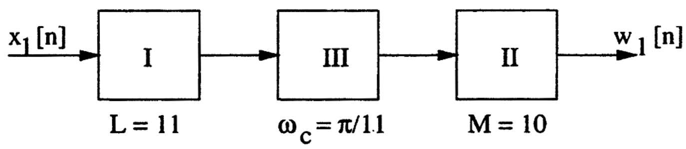
$$\text{SNR}_{\text{out}} = \frac{12}{\Delta^2}.$$

4.64. First, notice that since $y_c(t) = x_1(t)x_2(t)$, $Y_c(j\Omega) = \frac{1}{2\pi}(X_1(j\Omega) * X_2(j\Omega))$, and so $Y_c(j\Omega) = 0$ for $|\Omega| \geq 11\pi/2 \times 10^4$. Hence the Nyquist rate $T = 1/55000$s.

Choose System A and B such that $w_1[n] = ax_1(nT)$ and $w_2[n] = bx_2(nT)$.

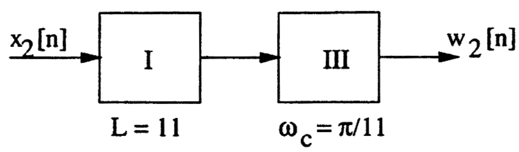
For System A, we need to resample such that

$$\frac{M}{L} = \frac{T}{T_1} = \frac{2 \times 10^{-5}}{1/55000} = \frac{10}{11}$$



For System B, we need to resample such that

$$\frac{M}{L} = \frac{T}{T_1} = \frac{2 \times 10^{-4}}{1/55000} = \frac{1}{11}$$



System C is simply the identity system.

4.65. The speech is first sampled at 44.1 kHz, and we wish to resample it so that the sampling rate is at 8 kHz. There are no aliasing effects anywhere in the system. Hence

$$\frac{L}{M} = \frac{44.1}{8} = \frac{441}{80}$$

We simply make $L = 441$, $M = 80$, and $\omega_c = \pi/441$.

4.66. Ω_p, and Ω_s has to be chosen such that

(a) The region $|\Omega| \leq \Omega_p$ maps to $|\omega| \leq \pi/4$:

$$\Omega_p T = \frac{\pi}{4} \Longrightarrow \Omega_p = 44\pi$$

(b) No aliasing occurs in the region $|\Omega| \leq \Omega_p$ during sampling:

$$\frac{2\pi}{T} - \Omega_s = \Omega_p \Longrightarrow \Omega_s = 2\pi(4 \cdot 44) - 44\pi = 308\pi$$

4.67. (a)

$$\begin{aligned}
V(z) &= H_1(z)(X(z) - Y(z)) \\
U(z) &= H_2(z)(V(z) - Y(z)) \\
Y(z) &= U(z) + E(z) \\
&= \frac{H_1(z)H_2(z)}{1 + H_2(z)(1 + H_1(z))}X(z) + \frac{1}{1 + H_2(z)(1 + H_1(z))}E(z)
\end{aligned}$$

Substituting $H_1(z) = 1/(1 - z^{-1})$ and $H_2(z) = z^{-1}/(1 - z^{-1})$, we find

$$\begin{aligned}
H_{xy}(z) &= z^{-1} \\
H_{ey}(z) &= (1 - z^{-1})^2
\end{aligned}$$

Hence the difference equation is $y[n] = x[n - 1] + f[n]$, where

$$f[n] = e[n] - 2e[n - 1] + e[n - 2].$$

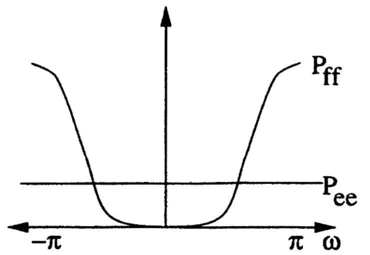
(b)

$$\begin{aligned}
P_{ff}(e^{j\omega}) &= \sigma_e^2 |H_{ey}(e^{j\omega})|^2 \\
&= \sigma_e^2 |(1 - e^{-j\omega})^2|^2 \\
&= \sigma_e^2 (1 - e^{-j\omega})^2 (1 - e^{j\omega})^2 \\
&= \sigma_e^2 (2 - 2\cos(\omega))^2 \\
&= \sigma_e^2 (4\sin^2(\omega/2))^2 \\
&= 16\sigma_e^2 \sin^4(\omega/2)
\end{aligned}$$

The total noise power σ_f^2 is the autocorrelation of $f[n]$ evaluated at 0:

$$\begin{aligned}
\sigma_f^2 &= E[(e[n] - 2e[n - 1] + e[n - 2])^2] \\
&= E[e^2[n]] + E[-2e^2[n - 1]] + E[e[n - 2]^2] \\
&= 6\sigma_e^2,
\end{aligned}$$

where we have used linearity of expectations, and the fact that since $e[n]$ is white, $E[e[n]e[n-k]] = 0$ for $k \neq 0$.



(c) Since $X(e^{j\omega})$ is bandlimited, $x[n] * h_3[n] = x[n]$. Hence,

$$w[n] = y[n] * h_3[n] = (x[n-1] + f[n]) * h_3[n] = x[n-1] + g[n],$$

where $g[n]$ is the quantization noise in the region $|\omega| < \pi/M$.

(d) For a small angle x, $\sin x \approx x$. Therefore,

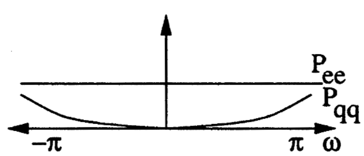
$$\begin{aligned}
\sigma_g^2 &= \frac{1}{2\pi}\int_{\pi/M}^{\pi/M} \sigma_e^2 (2\sin\omega/2)^4 d\omega \\
&\approx \frac{1}{2\pi}\int_{\pi/M}^{\pi/M} \sigma_e^2 (2\omega/2)^4 d\omega \\
&= \frac{\sigma_e^2}{2\pi}\frac{\omega^5}{5}\Big|_{-\pi/M}^{\pi/M} \\
&= \frac{\sigma_e^2 \pi^4}{5M^5}
\end{aligned}$$

(e) $X_c(j\Omega)$ must be sufficiently bandlimited that $X(e^{j\omega}) = X_c(j\Omega T)$ is zero for $|\omega| > \pi/M$. Hence $X_c(j\Omega) = 0$ for $|\Omega| > \pi/MT$.

Assuming that is satisfied, $v_x[n] = x[Mn-1] = x_c(MTn - T)$.

Downsampling does not change the variance of the noise, and hence $\sigma_q^2 = \sigma_g^2$.

$$\begin{aligned}
P_{qq}(e^{j\omega}) &= P_{gg}(e^{j\omega/M}) \\
&= 16\sigma_e^2 \sin^4(\omega/2M)
\end{aligned}$$



4.68. (a) (i) The transfer function from $x[n]$ to $y_x[n]$ is

$$H_{xy}(z) = \frac{\frac{z^{-1}}{1-z^{-1}}}{1 + \frac{z^{-1}}{1-z^{-1}}} = z^{-1}$$

Hence $y_x[n] = x[n-1]$.

(ii) The transfer function from $e[n]$ to $y_e[n]$ is

$$H_{ey}(z) = \frac{1}{1 + \frac{z^{-1}}{1-z^{-1}}} = 1 - z^{-1}$$

So

$$\begin{aligned} P_{y_e}(\omega) &= P_e(\omega) H_{ey}(e^{j\omega}) H_{ey} e^{-j\omega} \\ &= \sigma_e^2 (1 - e^{-j\omega})(1 - e^{j\omega}) \\ &= \sigma_e^2 (2 - 2\cos(\omega)) \end{aligned}$$

(b) (i) $x[n]$ contributes only to $y_1[n]$, but not $y_2[n]$. Therefore

$$\begin{aligned} y_{1x}[n] &= x[n-1] \\ r_x[n] &= x[n-2] \end{aligned}$$

(ii) In part(a), the difference equation describing the sigma-delta noise-shaper is

$$y[n] = x[n-1] + e[n] - e[n-1].$$

So here we apply the difference equation to both sigma-delta modulators:

$$\begin{aligned} y_{1e}[n] &= e_1[n] - e_1[n-1] \\ y_{2e}[n] &= e_1[n-1] + e_2[n] - e_2[n-1] \\ r_e[n] &= y_{1e}[n-1] - (y_{2e}[n] - y_{2e}[n-1]) \\ &= -e_2[n] + 2e_2[n-1] - e_w[n-2] \\ H_{e_2r}(z) &= -(1 - z^{-1})^2 \\ P_{r_e}(\omega) &= \sigma_e^2 (2 - 2\cos\omega)^2 \end{aligned}$$