8.1. We sample a periodic continuous-time signal with a sampling rate:

$$F_s = \frac{\Omega s}{2\pi} = \frac{1}{T} = \frac{6}{10^{-3}}\text{Hz}$$

(a) The sampled signal is given by:

$$x[n] = x_c(nT)$$

Expressed as a Discrete Fourier Series:

$$x[n] = \sum_{k=-9}^{9} a_k e^{j\frac{2\pi}{6}kn}$$

We note that, in accordance with the discussion of Section 8.1, the sampled signal is represented by the summation of harmonically-related complex exponentials. The fundamental frequency of this set of exponentials is $2\pi/N$, where $N = 6$.
Therefore, the sequence $x[n]$ is periodic with period 6.

(b) For any bandlimited continuous-time signal, the Nyquist Criterion may be stated from Eq. (4.14b) as:

$$F_s \geq 2F_N,$$

where F_s is the sampling rate (Hz), and F_N corresponds to the highest frequency component in the signal (also Hz).
As evident by the finite Fourier series representation of $x_c(t)$, this continuous-time signal is, indeed, bandlimited with a maximum frequency of $F_n = \frac{9}{10^{-3}}$ Hz.
Therfore, by sampling at a rate of $F_s = \frac{6}{10^{-3}}$ Hz, the Nyquist Criterion is violated, and aliasing results.

(c) We use the analysis equation of Eq. (8.11):

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n]e^{-j\frac{2\pi}{N}kn}$$

From part (a), $\tilde{x}[n]$ is periodic with $N = 6$.
Substitution yields:

$$\begin{aligned}\tilde{X}[k] &= \sum_{n=0}^{5}\left(\sum_{m=-9}^{9} a_m e^{j\frac{2\pi}{6}mn}\right)e^{-j\frac{2\pi}{6}kn} \\ &= \sum_{n=0}^{5}\sum_{m=-9}^{9} a_m e^{j(2\pi/6)(m-k)n}\end{aligned}$$

We reverse the order of the summations, and use the orthogonality relationship from Example 8.1:

$$\tilde{X}[k] = 6\sum_{m=-9}^{9} a_m \sum_{r=-\infty}^{\infty} \delta[m-k+rN]$$

Taking the infinite summation to the outside, we recognize the convolution between a_m and shifted impulses (Recall $a_m = 0$ for $|m| > 9$). Thus,

$$\tilde{X}[k] = 6\sum_{r=-\infty}^{\infty} a_{k-6r}$$

Note that from $\tilde{X}[k]$, the aliasing is apparent.

8.2. (a) Using the analysis equation of Eq. (8.11)

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn}$$

Since $\tilde{x}[n]$ is also periodic with period $3N$,

$$\begin{aligned}\tilde{X}_3[k] &= \sum_{n=0}^{3N-1} \tilde{x}[n] W_{3N}^{kn} \\ &= \sum_{n=0}^{N-1} \tilde{x}[n] W_{3N}^{kn} + \sum_{n=N}^{2N-1} \tilde{x}[n] W_{3N}^{kn} + \sum_{n=2N}^{3N-1} \tilde{x}[n] W_{3N}^{kn}\end{aligned}$$

Performing a change of variables in the second and third summations of $\bar{X}_3[k]$,

$$\tilde{X}_3[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_{3N}^{kn} + W_{3N}^{kN} \sum_{n=0}^{N-1} \tilde{x}[n+N] W_{3N}^{kn} + W_{3N}^{2kN} \sum_{n=0}^{N-1} \tilde{x}[n+2N] W_{3N}^{kn}$$

Since $\tilde{x}[n]$ is periodic with period N, and $W_{3N}^{kn} = W_N^{(\frac{k}{3})n}$,

$$\begin{aligned}\tilde{X}_3[k] &= \left(1 + e^{-j2\pi(\frac{k}{3})} + e^{-j2\pi(\frac{2k}{3})}\right) \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{(\frac{k}{3}n)} \\ &= \left(1 + e^{-j2\pi(\frac{k}{3})} + e^{-j2\pi(\frac{2k}{3})}\right) \tilde{X}[k] \\ &= \begin{cases} 3\tilde{X}[k/3], & k = 3\ell \\ 0, & \text{otherwise} \end{cases}\end{aligned}$$

(b) Using $N = 2$ and $\tilde{x}[n]$ as in Fig P8.2-1:

$$\begin{aligned}\tilde{X}[k] &= \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn} \\ &= \sum_{n=0}^{1} \tilde{x}[n] e^{-j\frac{2\pi}{2}kn} \\ &= \tilde{x}[0] + \tilde{x}[1] e^{-j\pi k} \\ &= 1 + 2(-1)^k \\ &= \begin{cases} 3, & k = 0 \\ -1, & k = 1 \end{cases}\end{aligned}$$

Observe, from Fig. P8.2-1, that $\tilde{x}[n]$ is also periodic with period $3N = 6$:

$$\begin{aligned}\tilde{X}_3[k] &= \sum_{n=0}^{3N-1} \tilde{x}[n] W_{3N}^{kn} \\ &= \sum_{n=0}^{5} \tilde{x}[n] e^{-j\frac{\pi}{3}kn} \\ &= (1 + e^{-j\frac{2\pi}{3}k} + e^{-j\frac{4\pi}{3}k})(1 + 2(-1)^{\frac{k}{3}}) \\ &= (1 + e^{-j\frac{2\pi}{3}k} + e^{-j\frac{4\pi}{3}k})\tilde{X}[k/3] \\ &= \begin{cases} 9, & k = 0 \\ -3, & k = 3 \\ 0, & k = 1, 2, 4, 5. \end{cases}\end{aligned}$$

8.3. (a) The DFS coefficients will be real if $\tilde{x}[n]$ is even. Only signal B can be even (i.e., $\tilde{x}_B[n] = \tilde{x}_B[-n]$; if the origin is selected as the midpoint of either the nonzero block, or the zero block).

(b) The DFS coefficients will be imaginary if $\tilde{x}[n]$ is even. None of the sequences in Fig P8.3-1 can be odd.

(c) We use the analysis equation, Eq. (8.11) and the closed form expression for a geometric series. Assuming unit amplitudes and discarding DFS points which are zero:

$$
\begin{aligned}
\tilde{X}_A[k] &= \sum_{n=0}^{3} e^{j\frac{2\pi}{8}kn} \\
&= \frac{1-e^{j\frac{\pi}{4}4k}}{1-e^{j\frac{\pi}{4}k}} \\
&= \frac{1-(-1)^k}{1-e^{j\frac{\pi}{4}k}} = 0, k = \pm 2, \pm 4, \ldots \\
\tilde{X}_B[k] &= \sum_{n=0}^{2} e^{j\frac{2\pi}{8}kn} \\
&= \frac{1-e^{j\frac{\pi}{4}3k}}{1-e^{j\frac{\pi}{4}k}} \\
\tilde{X}_C[k] &= \sum_{n=0}^{3} e^{j\frac{2\pi}{8}kn} - \sum_{n=4} 7e^{j\frac{2\pi}{8}kn} \\
&= \sum_{n=0}^{3} \left(e^{j\frac{\pi}{4}kn} - e^{j\frac{\pi}{4}k(n+4)}\right) \\
&= \left(1-e^{j\pi k}\right)\frac{1-e^{j\pi k}}{1-e^{j\frac{\pi}{4}k}} \\
&= 0, \quad k = \pm 2, \pm 4, \ldots
\end{aligned}
$$

8.4. A periodic sequence is constructed from the sequence:

$$x[n] = \alpha^n u[n] \ , \ |\alpha| < 1$$

as follows:

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n+rN] \ , \ |\alpha| < 1$$

(a) The Fourier transform of $x[n]$:

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\ &= \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} \\ &= \frac{1}{1-\alpha e^{-j\omega}}, \qquad |\alpha| < 1 \end{aligned}$$

(b) The DFS of $\tilde{x}[n]$:

$$\begin{aligned} \tilde{X}[k] &= \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn} \\ &= \sum_{n=0}^{N-1} \sum_{r=-\infty}^{\infty} x[n+rN] W_N^{kn} \\ &= \sum_{n=0}^{N-1} \sum_{r=-\infty}^{\infty} \alpha^{n+rN} u[n+rN] W_N^{kn} \\ &= \sum_{n=0}^{N-1} \sum_{r=0}^{\infty} \alpha^{n+rN} W_N^{kn} \end{aligned}$$

Rearranging the summations gives:

$$\begin{aligned} \tilde{X}[k] &= \sum_{r=0}^{\infty} \alpha^{rN} \sum_{n=0}^{N-1} \alpha^n W_N^{kn} \\ &= \sum_{r=0}^{\infty} \alpha^{rN} \left(\frac{1-\alpha^N e^{-j2\pi k}}{1-\alpha e^{-j\frac{2\pi k}{N}}} \right) \ , \ |\alpha| < 1 \\ &= \frac{1}{1-\alpha^N} \left(\frac{1-\alpha^N e^{-j2\pi k}}{1-\alpha e^{-j\frac{2\pi k}{N}}} \right) \ , \ |\alpha| < 1 \\ \tilde{X}[k] &= \frac{1}{1-\alpha e^{-j(2\pi k/N)}} \ , \ |\alpha| < 1 \end{aligned}$$

(c) Comparing the results of part (a) and part (b):

$$\tilde{X}[k] = X(e^{j\omega})\big|_{\omega = 2\pi k/N} \, .$$

8.5. (a)

$$\begin{aligned} x[n] &= \delta[n] \\ X[k] &= \sum_{n=0}^{N-1} \delta[n] W_N^{kn}, \quad 0 \le k \le (N-1) \\ &= 1 \end{aligned}$$

(b)

$$\begin{aligned} x[n] &= \delta[n-n_0], \quad 0 \le n_0 \le (N-1) \\ X[k] &= \sum_{n=0}^{N-1} \delta[n-n_0] W_N^{kn}, \quad 0 \le k \le (N-1) \\ &= W_N^{kn_0} \end{aligned}$$

(c)

$$\begin{aligned} x[n] &= \begin{cases} 1, & n \text{ even} \\ 0, & n \text{ odd} \end{cases} \\ X[k] &= \sum_{n=0}^{N-1} x[n] W_N^{kn}, \quad 0 \le k \le (N-1) \\ &= \sum_{n=0}^{(N/2)-1} W_N^{2kn} \\ &= \frac{1-e^{-j2\pi k}}{1-e^{-j(\pi k/N)}} \\ X[k] &= \begin{cases} N/2, & k = 0, N/2 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

(d)

$$\begin{aligned} x[n] &= \begin{cases} 1, & 0 \le n \le ((N/2)-1) \\ 0, & N/2 \le n \le (N-1) \end{cases} \\ X[k] &= \sum_{n=0}^{N-1} x[n] W_N^{kn}, \quad 0 \le k \le (N-1) \\ &= \sum_{n=0}^{(N/2)-1} W_N^{kn} \\ &= \frac{1-e^{-j\pi k}}{1-e^{-j(2\pi k)/N}} \\ X[k] &= \begin{cases} N/2, & k = 0 \\ \dfrac{2}{1-e^{-j(2\pi k/N)}}, & k \text{ odd} \\ 0, & k \text{ even}, 0 \le k \le (N-1) \end{cases} \end{aligned}$$

(e)

$$\begin{aligned} x[n] &= \begin{cases} a^n, & 0 \le n \le (N-1) \\ 0, & \text{otherwise} \end{cases} \\ X[k] &= \sum_{n=0}^{N-1} a^n W_N^{kn}, \quad 0 \le k \le (N-1) \\ &= \frac{1-a^N e^{-j2\pi k}}{1-ae^{-j(2\pi k)/N}} \\ X[k] &= \frac{1-a^N}{1-ae^{-j(2\pi k)/N}} \end{aligned}$$

8.6. Consider the finite-length sequence

$$x[n] = \begin{cases} e^{j\omega_0 n}, & 0 \le n \le (N-1) \\ 0, & \text{otherwise} \end{cases}$$

(a) The Fourier transform of $x[n]$:

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\ &= \sum_{n=0}^{N-1} e^{j\omega_0 n}e^{-j\omega n} \\ X(e^{j\omega}) &= \frac{1-e^{-j(\omega-\omega_0)N}}{1-e^{-j(\omega-\omega_0)}} \\ &= \frac{e^{-j(\omega-\omega_0)(N/2)}}{e^{-j(\omega-\omega_0)/2}} \left(\frac{\sin[(\omega-\omega_0)(N/2)]}{\sin[(\omega-\omega_0)/2]}\right) \\ X(e^{j\omega}) &= e^{-j(\omega-\omega_0)((N-1)/2)} \left(\frac{\sin[(\omega-\omega_0)(N/2)]}{\sin[(\omega-\omega_0)/2]}\right) \end{aligned}$$

(b) N-point DFT:

$$\begin{aligned} X[k] &= \sum_{n=0}^{N-1} x[n]W_N^{kn}, \qquad 0 \le k \le (N-1) \\ &= \sum_{n=0}^{N-1} e^{j\omega_0 n}W_N^{kn} \\ &= \frac{1-e^{-j((2\pi k/N)-\omega_0)N}}{1-e^{-j((2\pi k/N)-\omega_0)}} \\ &= e^{-j\left(\frac{2\pi k}{N}-\omega_0\right)\left(\frac{N-1}{2}\right)} \frac{\sin\left[\left(\frac{2\pi k}{N}-\omega_0\right)\frac{N}{2}\right]}{\sin\left[\left(\frac{2\pi k}{N}-\omega_0\right)/2\right]} \end{aligned}$$

Note that $X[k] = X(e^{j\omega})|_{\omega=(2\pi k)/N}$

(c) Suppose $\omega_0 = (2\pi k_0)/N$, where k_0 is an integer:

$$\begin{aligned} X[k] &= \frac{1-e^{-j(k-k_0)2\pi}}{1-e^{-j(k-k_0)(2\pi)/N}} \\ &= e^{-j(2\pi/N)(k-k_0)((N-1)/2)} \frac{\sin\pi(k-k_0)}{\sin(\pi(k-k_0)/N)} \end{aligned}$$

8.7. We have a six-point uniform sequence, $x[n]$, which is nonzero for $0 \leq n \leq 5$. We sample the Z-transform of $x[n]$ at four equally-spaced points on the unit circle.

$$X[k] = X(z)|_{z=e^{(2\pi k/4)}}$$

We seek the sequence $x_1[n]$ which is the inverse DFT of $X[k]$. Recall the definition of the Z-transform:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Since $x[n]$ is zero for all n outside $0 \leq n \leq 5$, we may replace the infinite summation with a finite summation. Furthermore, after substituting $z = e^{j(2\pi k/4)}$, we obtain

$$X[k] = \sum_{n=0}^{5} x[n]W_4^{kn}, \qquad 0 \leq k \leq 4$$

Note that we have taken a 4-point DFT, as specified by the sampling of the Z-transform; however, the original sequence was of length 6. As a result, we can expect some aliasing when we return to the time domain via the inverse DFT.

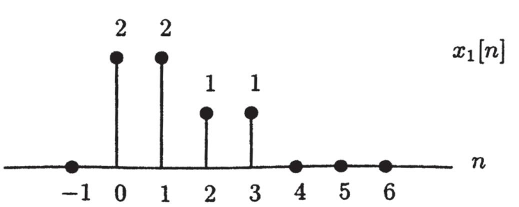
Performing the DFT,

$$X[k] = W_4^{0k} + W_4^{k} + W_4^{2k} + W_4^{3k} + W_4^{4k} + W_4^{5k}, \qquad 0 \leq k \leq 4$$

Taking the inverse DFT by inspection, we note that there are six impulses (one for each value of n above). However,

$$W_4^{4k} = W_4^{0k} \text{ and } W_4^{5k} = W_4^{k},$$

so two points are aliased. The resulting time-domain signal is



8.8. Fourier transform of $x[n] = (1/2)^n u[n]$:

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n e^{-j\omega n} \\ &= \frac{1}{1-\frac{1}{2}e^{-j\omega}} \end{aligned}$$

Now, sample the frequency spectra of $x[n]$:

$$Y[k] = X(e^{j\omega})|_{\omega=2\pi k/10}, \quad 0 \le k \le 9$$

We have the 10-pt DFT:

$$\begin{aligned} Y[k] &= \frac{1}{1-\frac{1}{2}e^{-j(2\pi k/10)}}, \quad 0 \le k \le 9 \\ &= \sum_{n=0}^{9} y[n]W_{10}^{kn} \end{aligned}$$

Recall:

$$\left(\frac{1}{2}\right)^n \xrightarrow[\text{N-pt}]{\text{DFT}} \frac{1-(\frac{1}{2})^N}{1-\frac{1}{2}e^{-j(2\pi k/N)}}$$

So, we may infer:

$$y[n] = \frac{(\frac{1}{2})^n}{1-(\frac{1}{2})^{10}}, \quad 0 \le n \le 9$$

8.9. Given a 20-pt finite-duration sequence $x[n]$:

(a) We wish to obtain $X(e^{j\omega})|_{\omega=4\pi/5}$ using the smallest DFT possible. A possible size of the DFT is evident by the periodicity of $e^{j\omega}|_{\omega=4\pi/5}$. Suppose we choose the size of the DFT to be $M = 5$. The data sequence is 20 points long, so we use the time-aliasing technique derived in the previous problem. Specifically, we alias $x[n]$ as:

$$x_1[n] = \sum_{r=-\infty}^{\infty} x[n+5r]$$

This aliased version of $x[n]$ is periodic with period 5 now. The 5-pt DFT is computed. The desired value occurs at a frequency corresponding to:

$$\frac{2\pi k}{N} = \frac{4\pi}{5}$$

For $N = 5$, $k = 2$, so the desired value may be obtained as $X[k]|_{k=2}$.

(b) Next, we wish to obtain $X(e^{j\omega})|_{\omega=10\pi/27}$.
The smallest DFT is of size $L = 27$. Since the DFT is larger than the data block size, we pad $x[n]$ with 7 zeros as follows:

$$x_2[n] = \begin{cases} x[n], & 0 \le n \le 19 \\ 0, & 20 \le n \le 26 \end{cases}$$

We take the 27-pt DFT, and the desired value corresponds to $X[k]$ evaluated at $k = 5$.

8.10. From Fig P8.10-1, the two 8-pt sequences are related through a circular shift. Specifically,

$$x_2[n] = x_1[((n-4))_8]$$

From property 5 in Table 8.2,

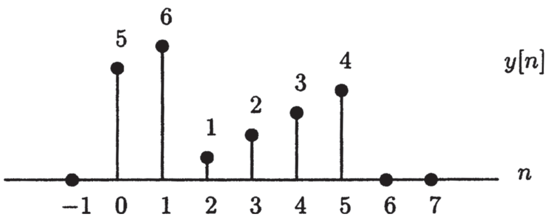
$$\text{DFT}\{x_1[((n-4))_8]\} = W_8^{4k} X_1[k]$$

Thus,

$$\begin{aligned} X_2[k] &= W_8^{4k} X_1[k] \\ &= e^{-j\pi k} X_1[k] \\ X_2[k] &= (-1)^k X_1[k] \end{aligned}$$

8.11. We wish to perform the circular convolution between two 6-pt sequences. Since $x_2[n]$ is just a shifted impulse, the circular-convolution coincides with a circular shift of $x_1[n]$ by two points.

$$\begin{aligned} y[n] &= x_1[n] \, ⑥ \, x_2[n] \\ &= x_1[n] \, ⑥ \, \delta[n-2] \\ &= x_1[((n-2))_6] \end{aligned}$$



8.12. (a)

$$x[n] = \cos(\frac{\pi n}{2}), \quad 0 \le n \le 3$$

transforms to

$$X[k] = \sum_{n=0}^{3} \cos(\frac{\pi n}{2}) W_4^{kn}, \quad 0 \le k \le 3$$

The cosine term contributes only two non-zero values to the summation, giving:

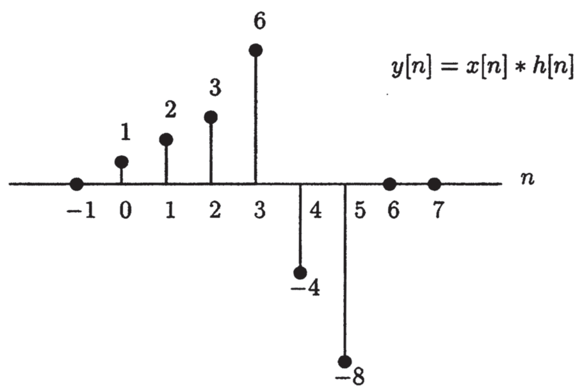
$$\begin{aligned} X[k] &= 1 - e^{-j\pi k}, \quad 0 \le k \le 3 \\ &= 1 - W_4^{2k} \end{aligned}$$

(b)

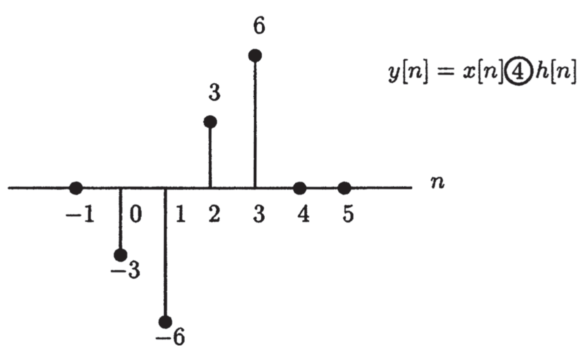
$$h[n] = 2^n, \quad 0 \le n \le 3$$

$$\begin{aligned} H[k] &= \sum_{n=0}^{3} 2^n W_4^{kn}, \quad 0 \le k \le 3 \\ &= 1 + 2W_4^k + 4W_4^{2k} + 8W_4^{3k} \end{aligned}$$

(c) Remember, circular convolution equals linear convolution plus aliasing. We need $N \ge 3+4-1 = 6$ to avoid aliasing. Since $N = 4$,we expect to get aliasing here. First, find $y[n] = x[n] * h[n]$:



For this problem, aliasing means the last three points ($n = 4, 5, 6$) will wrap-around on top of the first three points, giving $y[n] = x[n] ④ h[n]$:



(d) Using the DFT values we calculated in parts (a) and (b):

$$\begin{aligned} Y[k] &= X[k]H[k] \\ &= 1 + 2W_4^k + 4W_4^{2k} + 8W_4^{3k} - W_4^{2k} - 2W_4^{3k} - 4W_4^{4k} - 8W_4^{5k} \end{aligned}$$

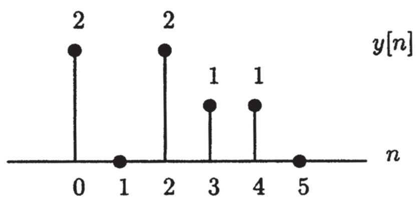
Since $W_4^{4k} = W_4^{0k}$ and $W_4^{5k} = W_4^k$

$$Y[k] = -3 - 6W_4^k + 3W_4^{2k} + 6W_4^{3k}, \quad 0 \le k \le 3$$

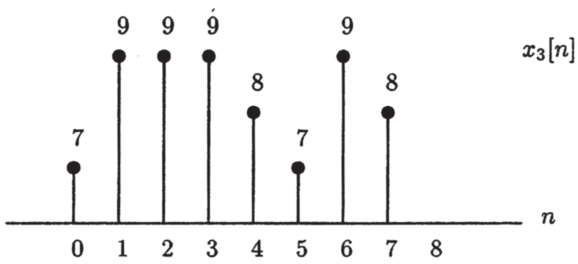
Taking the inverse DFT:

$$y[n] = -3\delta[n] - 6\delta[n-1] + 3\delta[n-2] + 6\delta[n-3], \quad 0 \le n \le 3$$

8.13. Using the properties of the DFT, we get $y[n] = x[((n-2))_5]$, that is $y[n]$ is equal to $x[n]$ circularly shifted by 2. We get:

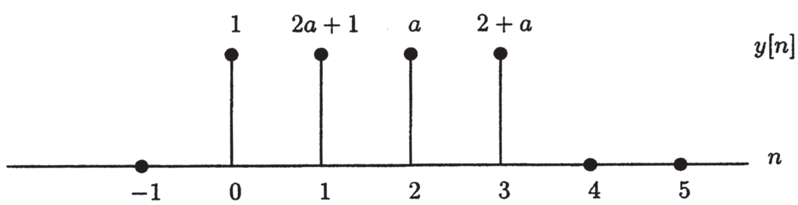


8.14. $x_3[n]$ is the linear convolution of $x_1[n]$ and $x_2[n]$ time-aliased to $N = 8$. Carrying out the 8-point circular convolution, we get:



We thus conclude $x_3[2] = 9$.

8.15. $y[n]$ is the linear convolution of $x_1[n]$ and $x_2[n]$ time-aliased to $N = 4$. Carrying out the 4-point circular convolution, we get:



Matching the above sequence to the one given, we get $a = -1$, which is unique.

8.16. $X_1[k]$ is the 4-point DFT of $x[n]$ and $x_1[n]$ is the 4-point inverse DFT of $X_1[k]$, therefore $x_1[n]$ is $x[n]$ time aliased to $N = 4$. In other words, $x_1[n]$ is one period of $\tilde{x}[n] = x[((n))_4]$. We thus have:

$$4 = b + 1.$$

Therefore, $b = 3$. This is clearly unique.

8.17. Looking at the sequences, we see that $x_1[n] * x_2[n]$ is non-zero for $1 \le n \le 8$. The smallest N such that $x_1[n] \textcircled{N} x_2[n] = x_1[n] * x_2[n]$ is therefore $N = 9$.

8.18. Taking the inverse DFT of $X_1[k]$ and using the properties of the DFT, we get:

$$x_1[n] = x[((n+3))_5].$$

Therefore:

$$x_1[0] = x[3] = c.$$

We thus conclude that $c = 2$.

8.19. $x_1[n]$ and $x[n]$ are related by a circular shift as can be seen from the plots. Using the properties of the DFT and the relationship between $X_1[k]$ and $X[k]$, we have:

$$x_1[n] = x[((n-m))_6].$$

$m = 2$ works, clearly this choice is not unique, any $m = 2 + 6l$, where l is an integer, would work.

8.20.

$$X_1[k] = X[k]e^{+j(2\pi k2/N)}.$$

Using the properties of the DFT, we get:

$$x_1[n] = x[((n+2))_N].$$

From the figures, we conclude that:

$$N = 5.$$

This choice of N is unique.

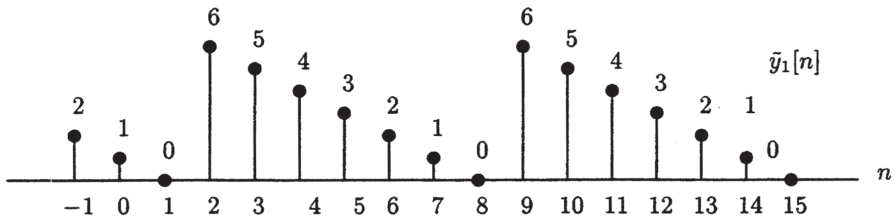
8.21. (a) We seek a sequence $\tilde{y}_1[n]$ such that

$$\tilde{Y}_1[k] = \tilde{X}_1[k]\tilde{X}_2[k]$$

From the discussion of Section 8.2.5, $\tilde{y}[n]$ is the result of the periodic convolution between $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$.

$$\tilde{y}_1[n] = \sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m]$$

Since $\tilde{x}_2[n]$ is a periodic impulse, shifted by two, the resultant sequence will be a shifted (by two) replica of $\tilde{x}_1[n]$.



Using the analysis equation of Eq. (8.11), we may rigorously derive $\tilde{y}_1[n]$:

$$\begin{aligned}
\tilde{X}_1[k] &= \sum_{n=0}^{6} \tilde{x}_1[n]W_7^{kn} \\
&= 6 + 5W_7^{k} + 4W_7^{2k} + 3W_7^{3k} + 2W_7^{4k} + W_7^{5k} \\
\tilde{X}_2[k] &= \sum_{n=0}^{6} \tilde{x}_2[n]W_7^{kn} \\
&= W_7^{2k} \\
\tilde{Y}_1[k] &= \tilde{X}_1[k]\tilde{X}_2[k] \\
&= 6W_7^{2k} + 5W_7^{3k} + 4W_7^{4k} + 3W_7^{5k} + 2W_7^{6k} + W_7^{7k}
\end{aligned}$$

Noting that $W_7^{7k} = e^{j\frac{2\pi}{7}(7k)} = 1 = W_7^{0k}$, we use the synthesis equation of Eq. (8.12) to construct $\tilde{y}_1[n]$. The result is identical to the sequence depicted above.

(b) The DFS of the signal illustrated in Fig. P8.21-2.is given by:

$$\begin{aligned}
\tilde{X}_3[k] &= \sum_{n=0}^{6} \tilde{x}_3[n]W_7^{kn} \\
&= 1 + W_7^{4k}
\end{aligned}$$

Therefore:

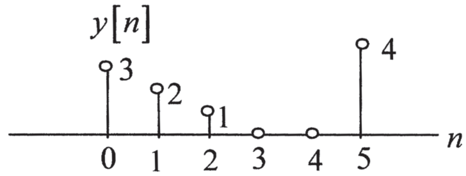
$$\begin{aligned}
\tilde{Y}_2[k] &= \tilde{X}_1[k]\tilde{X}_3[k] \\
&= \tilde{X}_1[k] + W_7^{4k}\tilde{X}_1[k]
\end{aligned}$$

Since the DFS is linear, the inverse DFS of $\tilde{Y}_2[k]$ is given by:

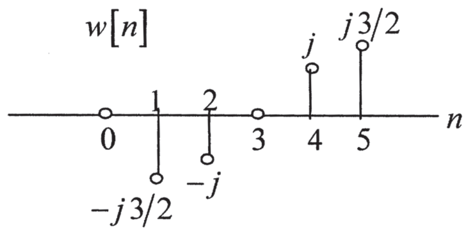
$$\tilde{y}_2[n] = \tilde{x}_1[n] + \tilde{x}_1[n-4].$$

8.22. We cannot conclude $\mathrm{Re}\{X(e^{j\omega})\}=0$ for $-\pi \leq \omega \leq \pi$. A counterexample is $x[n]=\delta[n-1]-\delta[n-2]$ with $N=3$. When we replicate $x[n]$ to get $\tilde{x}[n]$, we find that $\tilde{x}[n]$ has odd symmetry. Then for real-valued $\tilde{x}[n]$, the discrete Fourier series is imaginary, implying a purely imaginary DFT for $x[n]$. However, for real-valued $x[n]$, the DTFT is not purely imaginary.

8.23. **(a)** If $Y[k]=W_6^{5k}X[k]$, then $y[n]$ is a five-sample circular shift of $x[n]$, where $x[n]$ is taken as a six-point sequence. That is,



(b) If $W[k]=\text{Im}\{X[k]\}$, then $jW[k]=j\,\text{Im}\{X[k]\}$. Since $x_{op}[n]\xleftrightarrow{DFT} j\,\text{Im}\{X[k]\}$, we have $w[n]=-jx_{op}[n]\xleftrightarrow{DFT}\text{Im}\{X[k]\}$. That is,



(c) We can proceed by direct calculation:

$$X[k]=4+3W_6^k+2W_6^{2k}+W_6^{3k}.$$

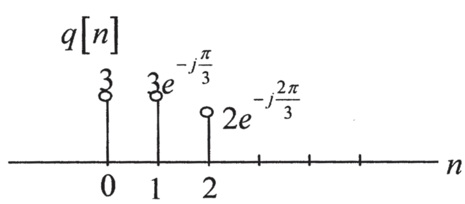
Let $Q[k]=X[2k+1]$,

$$\begin{aligned} Q[k]&=4+3W_6^{(2k+1)}+2W_6^{2(2k+1)}+W_6^{3(2k+1)}\\ &=4+3W_6^1W_6^{2k}+2W_6^2W_6^{4k}+W_6^3W_6^{6k}\\ &=4+e^{-j\frac{\pi}{3}}W_3^k+2e^{-j\frac{2\pi}{3}}W_3^{2k}+e^{-j\pi}W_3^{3k}. \end{aligned}$$

Noting that $W_3^{3k}=W_3^{0k}=1$,

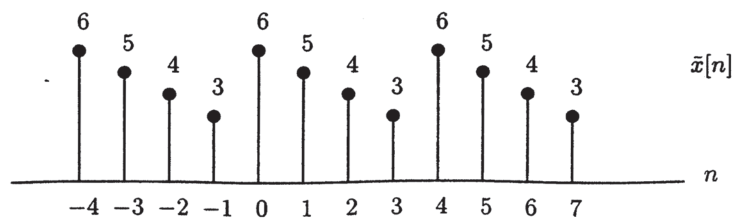
$$Q[k]=3+3e^{-j\frac{\pi}{3}}W_3^k+2e^{-j\frac{2\pi}{3}}W_3^{2k}.$$

The sequence $q[n]$ is then

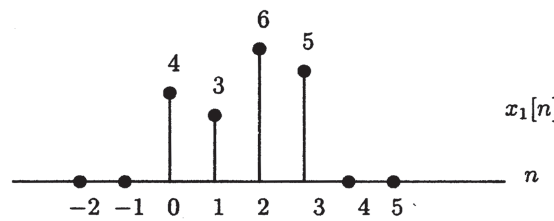


The result shows the effects of both frequency-domain shifting and time-domain aliasing.

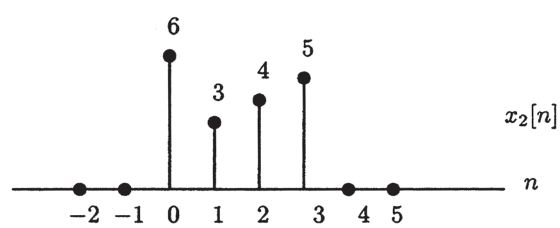
8.24. We may approach this problem in two ways. First, the notion of modulo arithmetic may be simplified if we utilize the implied periodic extension. That is, we redraw the original signal as if it were periodic with period $N = 4$. A few periods are sufficient:



To obtain $x_1[n] = x[((n-2))_4]$, we shift by two (to the right) and only keep those points which lie in the original domain of the signal (i.e. $0 \leq n \leq 3$):



To obtain $x_2[n] = x[((-n))_4]$, we fold the pseudo-periodic version of $x[n]$ over the origin (time-reversal), and again we set all points outside $0 \leq n \leq 3$ equal to zero. Hence,



Note that $x[((0))_4] = x[0]$, etc.

In the second approach, we work with the given signal. The signal is confined to
$0 \leq n \leq 3$; therefore, the circular nature must be maintained by picturing the signal on the circumference of a cylinder.

8.25. Problem 1 in Spring 2003 Final exam.
Appears in: Spring05 PS8, Spring04 PS7.

Problem

Consider the signal $x[n] = \delta[n-4] + 2\delta[n-5] + \delta[n-6]$.

(a) Find $X(e^{j\omega})$ the discrete-time Fourier transform of $x[n]$. Write expressions for the magnitude and phase of $X(e^{j\omega})$, and sketch these functions.

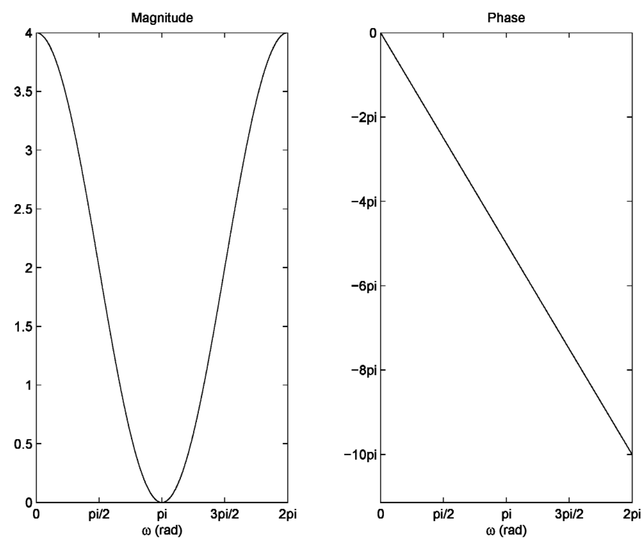
(b) Find all values of N for which the N-point DFT is a set of real numbers.

(c) Can you find a 3-point causal signal $x_1[n]$ (i.e., $x_1[n] = 0$ for $n < 0$) for which the 3-point DFT of $x_1[n]$ is:

$$X_1[k] = |X[k]| \qquad k = 0, 1, 2$$

Solution from Spring05 PS8

(a) $X(e^{j\omega}) = e^{-j4\omega} + 2e^{-j5\omega} + e^{-j6\omega} = e^{-j5\omega}2(1 + cos(\omega))$.
$|X(e^{j\omega})| = |2(1 + cos(\omega))| = 2(1 + cos(\omega))$, and $\angle(X(e^{j\omega})) = -5\omega$.

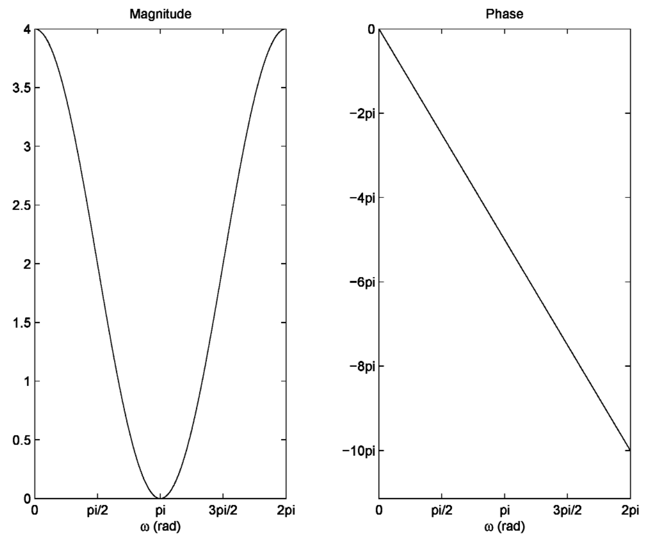


(b) The N-point DFT of $x[n]$ is the set of N samples of $X(e^{j\omega})$, at $\omega_k = 2\pi k/N$, $k = 0, 1, .., N-1$. The set of frequencies where $X(e^{j\omega})$ is purely real is $\omega = 2\pi k/10$, $k = 0, 1, .., 9$ (we need $\angle X(e^{j\omega}) = 5\omega = \pi M$, for some integer M). That means that the DFT will be a real sequence for $N = 1, 2, 5, 10$.

(c) The three-point time-aliased version of $x[n]$ is the sequence $\delta[n] + \delta[n-1] + 2\delta[n-2]$. The sequence of absolute values of its 3-point DFT is $|X_3[k]| = 4\delta[k] + \delta[k-1] + \delta[k-2]$. The inverse 3-point DFT of this sequence is the desired sequence, $x_1[n] = 2\delta[n] + \delta[n-1] + \delta[n-2]$. Note that $x_1[n]$ replicated is a real and even signal so that the frequency domain is also real and even.

Solution from Spring04 PS7

(a) $X(e^{j\omega}) = e^{-j4\omega} + 2e^{-j5\omega} + e^{-j6\omega} = e^{-j5\omega}2(1 + cos(\omega))$.
$|X(e^{j\omega})| = |2(1 + cos(\omega))| = 2(1 + cos(\omega))$, and $\angle(X(e^{j\omega})) = -5\omega$.

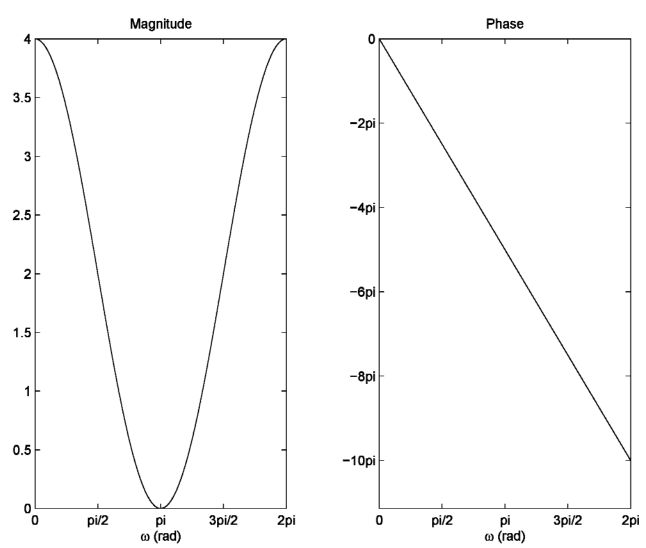


(b) The N-point DFT of $x[n]$ is the set of N samples of $X(e^{j\omega})$, at $\omega_k = 2\pi k/N$, $k = 0, 1, .., N-1$. The set of frequencies where $X(e^{j\omega})$ is purely real is $\omega = 2\pi k/10$, $k = 0, 1, .., 9$ (we need $\angle X(e^{j\omega}) = 5\omega = \pi M$, for some integer M). That means that the DFT will be a real sequence for $N = 1, 2, 5, 10$.

(c) The three-point time-aliased version of $x[n]$ is the sequence $\delta[n] + \delta[n-1] + 2\delta[n-2]$. The sequence of absolute values of its 3-point DFT is $|X_3[k]| = 4\delta[k] + \delta[k-1] + \delta[k-2]$. The inverse 3-point DFT of this sequence is the desired sequence, $x_1[n] = 2\delta[n] + \delta[n-1] + \delta[n-2]$.

Solution from Spring03 Final

(a) $X(e^{j\omega}) = e^{-j4\omega} + 2e^{-j5\omega} + e^{-j6\omega}$



(b) $N = 5, 10$

(c) $x_1[n] = 2\delta[n] + \delta[n-1] + \delta[n-2]$

8.26. A. We know

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$$

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{2\pi}{N}kn}$$

$$= \sum_{n=0}^{N-1} x[n] e^{-j\frac{\pi}{N}n} e^{-j\frac{2\pi}{N}kn}$$

$$= \sum_{n=0}^{N-1} x[n] e^{-j\frac{\pi}{N}n(1+2k)}$$

Let $\omega_k = \dfrac{\pi(1+2k)}{N}$. Then

$$\tilde{X}[k] = \sum_{n=0}^{N-1} x[n] e^{-j\omega_k n} = X\left(e^{j\omega_k}\right)$$

$$= X\left(e^{j\left(\frac{\pi+2\pi k}{N}\right)}\right), \quad k = 0,1,\ldots,N-1.$$

B. The frequencies of sampling are given by

$$\omega_k = \frac{\pi(1+2k)}{N}, \quad k = 0,1,\ldots,N-1.$$

C. Given the modified $\tilde{X}[k]$, we can use the inverse transform to find $\tilde{x}[n]$. To get $x[n]$ from $\tilde{x}[n]$ it is a simple point-by-point multiplication given by

$$x[n] = e^{j\frac{\pi}{N}n} \tilde{x}[n].$$

8.27. I. If $G[k]=10\delta[k]$, then $g[n]=1, \quad n=0,\ldots,9$. We can now find $G(e^{j\omega})$ as

$$\begin{aligned}
G(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} g[n]e^{-j\omega n} \\
&= \sum_{n=0}^{9} e^{-j\omega n} \\
&= \frac{1-e^{-j10\omega}}{1-e^{-j\omega}} \\
&= e^{-j\frac{9}{2}\omega}\,\frac{\sin(5\omega)}{\sin(\omega/2)}.
\end{aligned}$$

8.28. **(a)** Using the analysis equation

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$
$$= \sum_{n=0}^{5} x[n] W_6^{kn}$$
$$= 6W_6^0 + 5W_6^k + 4W_6^{2k} + 3W_6^{3k} + 2W_6^{4k} + W_6^{5k}.$$

(b)

$$W[k] = W_6^{-2k} X[k]$$
$$= 6W_6^{-2k} + 5W_6^{-k} + 4 + 3W_6^k + 2W_6^{2k} + W_6^{3k}.$$

Using the fact that $W_6^k = e^{-j\frac{2\pi k}{6}}$,

$$W_6^{-2k} = e^{j\frac{4\pi k}{6}} = e^{j\frac{4\pi k}{6}} \times e^{-j2\pi k} \text{ (since } e^{-j2\pi k} = 1)$$
$$= e^{-j\frac{8\pi k}{6}} = W_6^{4k},$$

and similarly

$$W_6^{-k} = W_6^{5k}.$$

Then

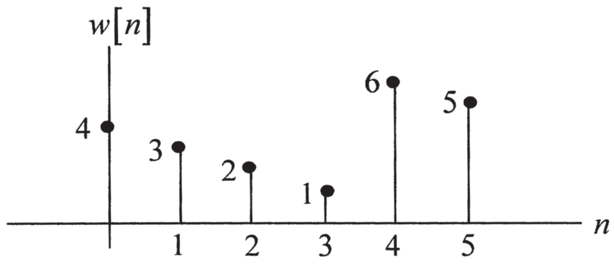
$$W[k] = 4 + 3W_6^k + 2W_6^{2k} + W_6^{3k} + 6W_6^{4k} + 5W_6^{5k}.$$

Using the synthesis equation,

$$w[n] = \frac{1}{6}\sum_{k=0}^{5} W[k] W_6^{-kn}.$$

We could go ahead and solve the problem in this "brute force" method, but notice that each $\delta[n-k] \xrightarrow{DFT} W_N^k$. Then,

$$w[n] = 4\delta[n] + 3\delta[n-1] + 2\delta[n-2] + \delta[n-3] + 6\delta[n-4] + 5\delta[n-5].$$



Notice that multiplying by W_6^{-2k} in frequency has the effect of a shift of 2 in time, but modulo 6.

(c) One way to do this is to compute the linear convolution and then add copies of it shifted by N (6 in this case). Another method is to use the DFT, find the product $H[k]X[k]$, and then take an inverse DFT. We know

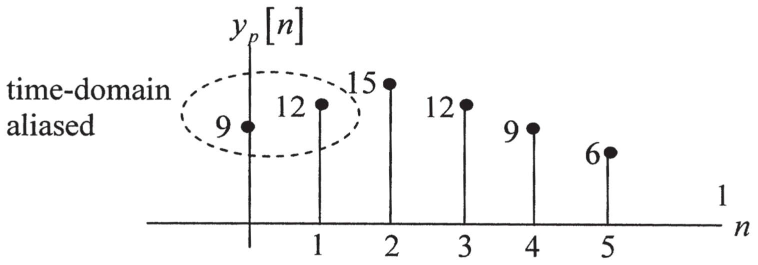
$$X[k] = 6W_6^0 + 5W_6^k + 4W_6^{2k} + 3W_6^{3k} + 2W_6^{4k} + W_6^{5k}$$
$$H[k] = 1 + W_6^k + W_6^{2k}$$

Then

$$\begin{aligned}Y_p[k] = 6 &+ 5W_6^k + 4W_6^{2k} + 3W_6^{3k} + 2W_6^{4k} + W_6^{5k} \\ &+ 6W_6^k + 5W_6^{2k} + 4W_6^{3k} + 3W_6^{4k} + 2W_6^{5k} + W_6^{6k} \\ &+ 6W_6^{2k} + 5W_6^{3k} + 4W_6^{4k} + 3W_6^{5k} + 2W_6^{6k} + W_6^{7k} \\ &= 9 + 12W_6^k + 15W_6^{2k} + 12W_6^{3k} + 9W_6^{4k} + 6W_6^{5k},\end{aligned}$$

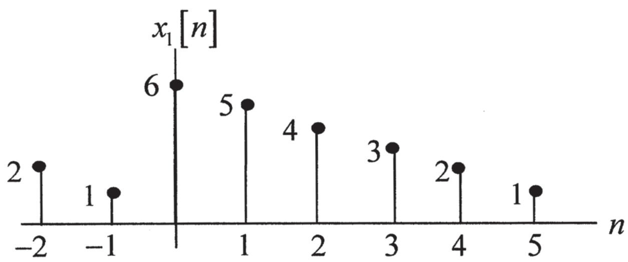
where we have used $W_6^{6k} = 1$ and $W_6^{7k} = W_6^k$. Now we have

$$y_p[n] = 9\delta[n] + 12\delta[n-1] + 15\delta[n-2] + 12\delta[n-3] + 9\delta[n-4] + 6\delta[n-5].$$



(d) To ensure that no time-domain aliasing occurs in the output, N should be large enough to accommodate the length of the linear convolution. That is, $N \geq 6+3-1=8$

(e)

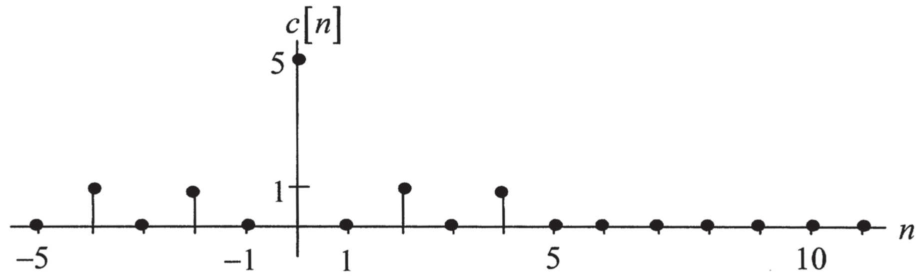


This new input when convolved with $h[n]$ will give the circular convolution found in (c)

We merely extend $x[n]$ as a periodic signal with period 6 samples.

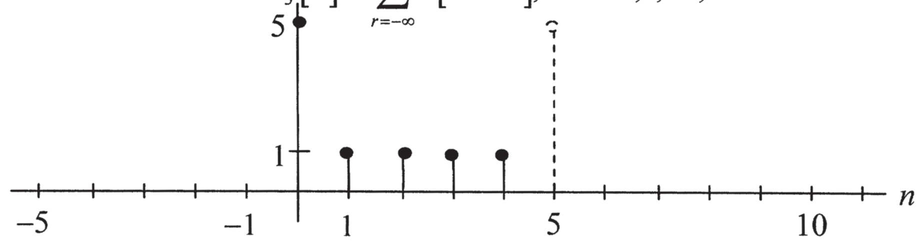
(f) In general $x_1[n]$ is constructed by extending $x[n]$ periodically for $n = -1, K, -M$.

8.29. (a)



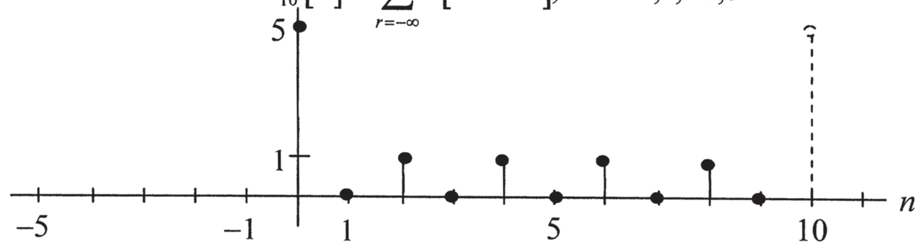
(b)

$$c_5[n] = \sum_{r=-\infty}^{\infty} c[n-r5], \quad n = 0,1,K\ ,4$$

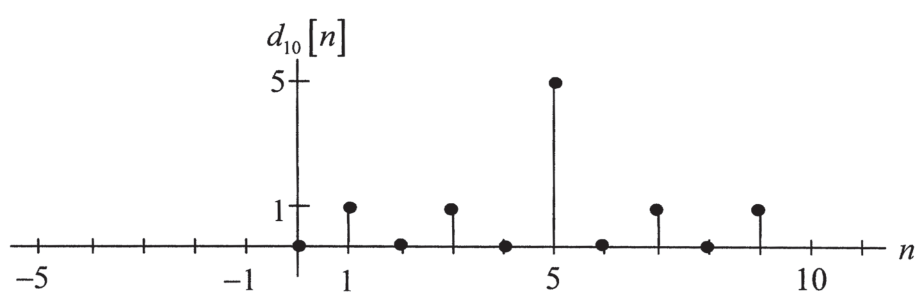


(c)

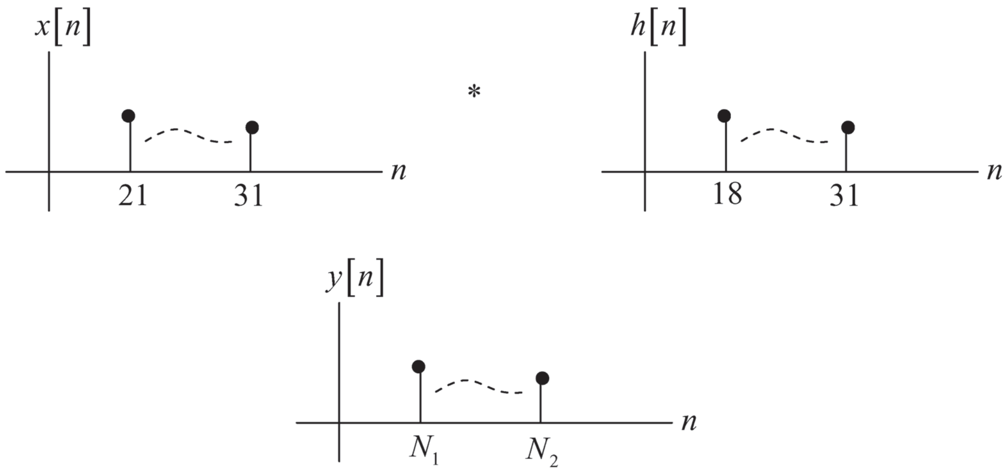
$$c_{10}[n] = \sum_{r=-\infty}^{\infty} c[n-r10], \quad n = 0,1,K\ ,9$$



(d) $W_{10}^{5k} \Rightarrow$ rotate right by 5 .



8.30. A.



$$N_1 = 21 + 18 = 39$$
$$N_2 = 31 + 31 = 62.$$

B. The sequence $y_1[n]$ is the 32-point circular convolution of $x_1[n]$ with $h_1[n]$. That is,

$$\begin{aligned} y_1[n] &= \sum_{r=-\infty}^{\infty} y[n + r32] \\ &= y[n+32], \quad n = 0, 1, \ldots 31, \end{aligned}$$

since $y[n+32]$ is the only one that fits in $0 \le n \le 31$.

C. If we add zeros at the ends too, we can get $y_1[n] = y[n]$ if $N > 62$.

8.31. Given $x[n] = 2\delta[n] + \delta[n-1] - \delta[n-2]$,

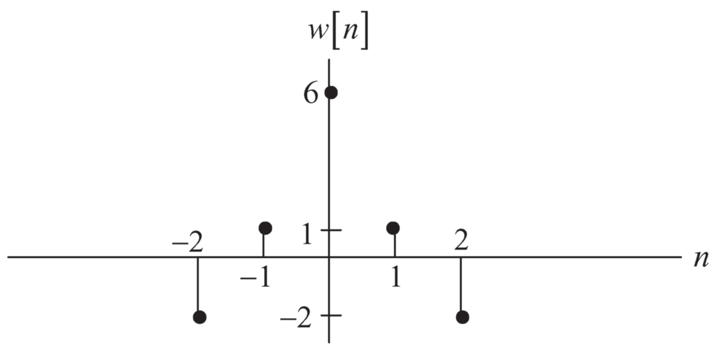
A. $X(e^{j\omega}) = 2 + e^{-j\omega} - e^{-j\omega 2}$.

$Y(e^{j\omega}) = 2 + e^{j\omega} - e^{j\omega 2}$ for $y[n] = x[-n]$.

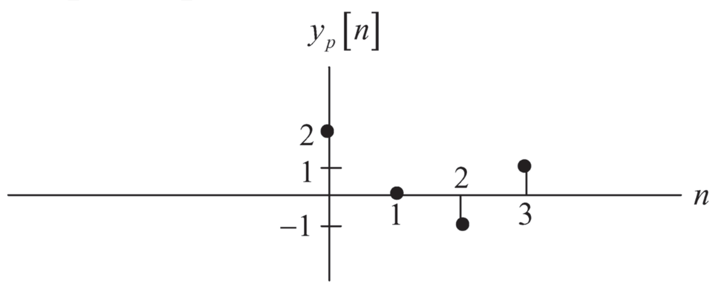
B. If $W(e^{j\omega}) = X(e^{j\omega})Y(e^{j\omega})$, then

$$W(e^{j\omega}) = (2 + e^{-j\omega} - e^{-j\omega 2})(2 + e^{j\omega} - e^{j\omega 2})$$
$$= -2e^{j\omega 2} + e^{j\omega} + 6 + e^{-j\omega} - 2e^{-j\omega 2}.$$

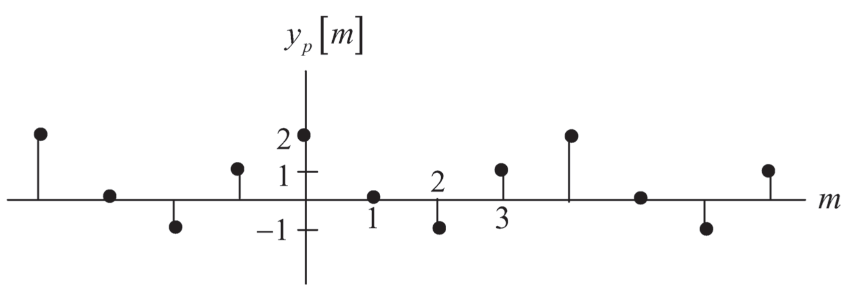
C.

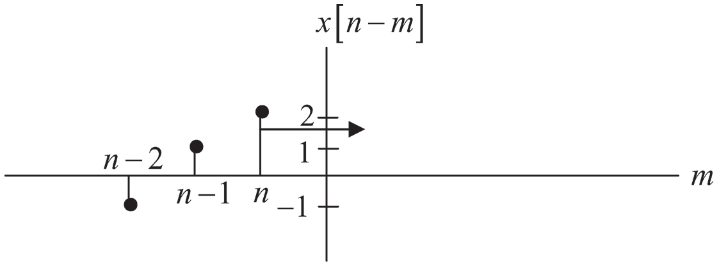


D. Define $y_p[n] = x[((-n))_4]$.

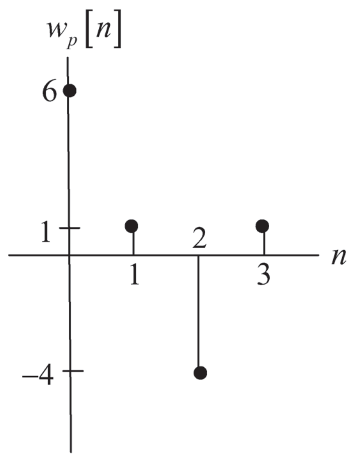


E. To circularly convolve $x[n]$ with $y_p[n]$, consider the construction shown below.





The result, $w_p[n]$ is plotted below.



Note that $w_p[n]$ is periodic; only one period is shown in the plot.

F. Since $x[n]$ has three contiguous non-zero samples, there will be no time-domain aliasing if $N \geq 5$. The circular convolution of $x[n]$ with $x\left[((-n))_5\right]$ will be identical to $w[n]$ plotted in C above.

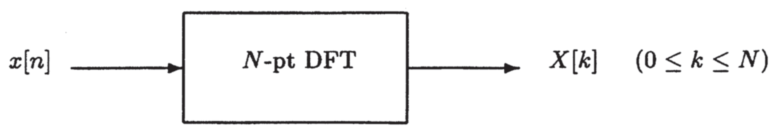
8.32. We have $x[n]$ for $0 \le n \le P$.

We desire to compute $X(z)|_{z=e^{-j(2\pi k/N)}}$ using one N-pt DFT.

(a) Suppose $N > P$ (the DFT size is larger than the data segment). The technique used in this case is often referred to as zero-padding. By appending zeros to a small data block, a larger DFT may be used. Thus the frequency spectra may be more finely sampled. It is a common misconception to believe that zero-padding enhances spectral resolution. The addition of a larger block of data to a larger DFT would enhance this quality.

So, we append $N_z = N - P$ zeros to the end of the sequence as follows:

$$x'[n] = \begin{cases} x[n], & 0 \le n \le (P-1) \\ 0, & P \le n \le N \end{cases}$$



(b) Suppose $N > P$, consider taking a DFT which is smaller than the data block. Of course, some aliasing is expected. Perhaps we could introduce time aliasing to offset the effects.

Consider the N-pt inverse DFT of $X[k]$,

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{-kn}, \qquad 0 \le n \le (N-1)$$

Suppose $X[k]$ was obtained as the result of an infinite summation of complex exponents:

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1}\left(\sum_{m=0}^{\infty} x[m]e^{-j(2\pi k/N)m}\right) W_N^{-kn}$$

Rearrange to get:

$$x[n] = \sum_{m=-\infty}^{\infty} x[m]\left(\frac{1}{N}\sum_{k=0}^{N-1} e^{-j(2\pi/N)(m-n)k}\right)$$

Using the orthogonality relationship of Example 8.1:

$$\begin{aligned} x[n] &= \sum_{m=-\infty}^{\infty} x[m] \sum_{r=-\infty}^{\infty} \delta[m-n+rN] \\ x[n] &= \sum_{r=-\infty}^{\infty} x[n-rN] \end{aligned}$$

So, we should alias $x[n]$ as above. Then we take the N-pt DFT to get $X[k]$.

8.33. Problem 4 in Spring2005 final exam.

Problem

A filter $h[n]$ is a 10-point FIR filter, *i.e.*,

$$h[n] = 0 \qquad \text{for } n < 0 \text{ and for } n > 9.$$

Given that the 10-point DFT of $h[n]$ is given by

$$H[k] = \frac{1}{5}\delta[k-1] + \frac{1}{3}\delta[k-7],$$

find $H(e^{j\omega})$, the DTFT of $h[n]$.

Solution from Spring2005 final

By computing the IDFT we find

$$h[n] = \frac{1}{10}\left(\frac{1}{5}e^{j(2\pi/10)n} + \frac{1}{3}e^{j(2\pi/10)7n}\right) \qquad \text{for } n = 0, 1, \ldots, 9.$$

Now

$$\begin{aligned}
H(e^{j\omega}) &= \sum_{n=0}^{9} h[n]e^{j\omega n} = \frac{1}{10}\sum_{n=0}^{9}\left(\frac{1}{5}e^{j(2\pi/10)n} + \frac{1}{3}e^{j(2\pi/10)7n}\right)e^{-j\omega n} \\
&= \frac{1}{10}\left(\frac{1}{5}\sum_{n=0}^{9} e^{j((2\pi/10)-\omega)n} + \frac{1}{3}\sum_{n=0}^{9} e^{j((2\pi/10)7-\omega)n}\right) \\
&= \frac{1}{10}\left(\frac{1}{5}\cdot\frac{1-(e^{j((2\pi/10)-\omega)})^{10}}{1-e^{j((2\pi/10)-\omega)}} + \frac{1}{3}\cdot\frac{1-(e^{j((2\pi/10)7-\omega)})^{10}}{1-e^{j((2\pi/10)7-\omega)}}\right) \\
&= \frac{1}{10}\left(\frac{1}{5}\cdot\frac{1-e^{-j10\omega}}{1-e^{j((2\pi/10)-\omega)}} + \frac{1}{3}\cdot\frac{1-e^{-j10\omega}}{1-e^{j((2\pi/10)7-\omega)}}\right).
\end{aligned}$$

(Sampling $H(e^{j\omega})$ at $\omega_k = \frac{2\pi k}{10}$ gives $H[k]$.)

8.34. II. The z-transform $X_1(z)$ of $x_1[n]$ is given by

$$X_1(z) = \sum_{n=0}^{N-1} x_1[n] z^{-n}.$$

At $z = \frac{1}{2} e^{-j\frac{2\pi k}{N}}$ we have

$$X_1(z)\Big|_{z=\frac{1}{2}e^{-j\frac{2\pi k}{N}}} = \sum_{n=0}^{N-1} x_1[n]\left(\tfrac{1}{2}\right)^{-n} e^{j\frac{2\pi kn}{N}}, \quad k = 0,\ldots,N-1.$$

Now $X_2[k]$ is given by

$$X_2[k] = \sum_{n=0}^{N-1} x_2[n] e^{-j\frac{2\pi kn}{N}}$$
$$= \sum_{n=0}^{N-1} x_2\left[((-n))_N\right] e^{j\frac{2\pi kn}{N}}, \quad k = 0,\ldots,N-1.$$

Then if $X_2[k] = X_1(z)\Big|_{z=\frac{1}{2}e^{-j\frac{2\pi k}{N}}}$, $\quad k = 0,\ldots,N$ we have

$$x_1[n]\left(\tfrac{1}{2}\right)^{-n} = x_2\left[((-n))_N\right], \quad n = 0,\ldots,N-1.$$

8.35. Given

$$x[0]=1, x[1]=0, x[2]=2, x[3]=2, x[4]=b, x[5]=1,$$

we have

$$X\left(e^{j\omega}\right)=1+2e^{-j2\omega}+2e^{-j3\omega}+be^{-j4\omega}+e^{-j5\omega}.$$

Define

$$X_1[k]=X\left(e^{j\omega}\right)\Big|_{\omega=\frac{\pi}{2}k}, \quad k=0,1,2,3.$$

Sampling $X\left(e^{j\omega}\right)$ with $N=4$ causes time-domain aliasing of $x[n]$. That is,

$$x_1[0]=x[0]+x[4], x_1[1]=x[1]+x[5], x_1[2]=x[2], x_1[3]=x[3],$$

or

$$x_1[0]=1+b, x_1[1]=1, x_1[2]=2, x_1[3]=2.$$

We are given $x_1[0]=4$, so $b=3$.

Note that this can also be solved by direct calculation.

8.36. A. To obtain five samples of $X\left(e^{j\omega}\right)$ we need to time-alias $x[n]$ to $0 \le n < 5$ and take a DFT. Sampling at five points in frequency corresponds to periodically replicating $x[n]$ with a period of five, summing the replicas, and extracting the first five points by multiplying with a window.

$$\begin{aligned} g[n] &= \sum_{m=-\infty}^{\infty} x[n+5m] \quad \text{for } 0 \le n < 5 \\ &= \sum_{m=0}^{\infty} x[n+5m] \quad \text{for } 0 \le n < 5 \\ &= \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^{n+5m} \quad \text{for } 0 \le n < 5 \\ &= \left(\frac{1}{2}\right)^{n} \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^{5m} \quad \text{for } 0 \le n < 5 \\ &= \left(\frac{1}{2}\right)^{n} \left(\frac{1}{1-(1/2)^5}\right) \quad \text{for } 0 \le n < 5. \end{aligned}$$

B. To get the samples of $W\left(e^{j\omega}\right)$ at five frequency points, we need to alias $w[n]$ to five points and take the DFT. If

$$w_a[n] = \begin{cases} w[n] + w[n+5], & 0 \le n < 5 \\ 0, & \text{otherwise,} \end{cases}$$

then $w_a[n]$ must be equal to $g[n]$ to have the same DFT. So $w[n]$ must satisfy $w[n] + w[n+5] = g[n]$ for $0 \le n < 5$.

We can find one answer that works by constraining the DTFT of $w[n]$ to be equal to the DTFT of $X\left(e^{j\omega}\right)$ at ten points in frequency, which would include the five required by the problem. Then we time-alias $x[n]$ to the $0 \le n < 10$ range to obtain

$$w[n] = \left(\frac{1}{2}\right)^{n} \left(\frac{1}{1-(1/2)^{10}}\right) \quad \text{for } 0 \le n < 10$$

as a possible answer, which satisfies the constraint above.

8.37. Appears in: Fall05 PS7, Fall04 PS7, Fall02 PS7, Spring01 PS6.

Problem

A discrete-time LTI filter S is to be implemented using the overlap-save method. Recall that in the overlap-save method the input is divided into *overlapping* blocks, as opposed to the overlap-add method where the input blocks are non-overlapping. For this implementation, the input signal $x[n]$ will be divided into overlapping 256-point blocks $x_r[n]$. Adjacent blocks will overlap by 255 points so that they differ by only one sample. Mathematically, we have the following relation between $x_r[n]$ and $x[n]$, where r ranges over all integers and we obtain a different block $x_r[n]$ for each value of r:

$$x_r[n] = \begin{cases} x[n+r] & 0 \le n \le 255 \\ 0 & \text{otherwise} \end{cases}$$

Each block is processed by taking the 256-point DFT of $x_r[n]$, multiplying the result with $H[k]$ given below, and taking the 256-point inverse DFT of the product. The unaliased samples from each block computation (in this case only a single sample per block) are then "saved" as part of the overall output.

$$H[k] = \begin{cases} 1 & 0 \le k \le 31 \\ 0 & 32 \le k \le 224 \\ 1 & 225 \le k \le 255 \end{cases}$$

In this problem you may assume that the overlap-save method has been correctly implemented.

(a) Is S an ideal frequency-selective filter? Justify your answer.

(b) Is the impulse response of S real valued? Justify your answer.

(c) Determine the impulse response of S.

Solution from Fall05 PS7

(a) Assuming that the overlap-save method is correctly implemented, the output $y[n]$ of S can be represented as the *linear* convolution $y[n] = x[n] * h[n]$. The impulse response $h[n]$ corresponding to $H[k]$ is a finite sequence of length 256. However, an ideal frequency-selective filter has an infinite impulse response. Therefore, S cannot be an ideal frequency-selective filter.

(b) The impulse response $h[n]$ of S is the IDFT of $H[k]$. Since $H[k]$ is real and even in the circular sense ($H[k] = H[((-k))_{256}]$), $h[n]$ is real.

(c)

$$\begin{aligned}
h[n] &= \frac{1}{256}\sum_{k=0}^{255} H[k]W_{256}^{-kn} \qquad 0 \le n \le 255 \\
&= \frac{1}{256}\sum_{k=0}^{31} W_{256}^{-kn} + \frac{1}{256}\sum_{k=225}^{255} W_{256}^{-n(k-256)} \\
&= \frac{1}{256}\sum_{k=0}^{31} W_{256}^{-kn} + \frac{1}{256}\sum_{k=-31}^{-1} W_{256}^{-kn} \\
&= \frac{1}{256}\sum_{k=-31}^{31} W_{256}^{-kn} \\
&= \frac{1}{256}\frac{W_{256}^{31n} - W_{256}^{-32n}}{1 - W_{256}^{-n}} \\
&= \frac{1}{256}\frac{W_{256}^{-0.5n}(W_{256}^{31.5n} - W_{256}^{-31.5n})}{W_{256}^{-0.5n}\left(W_{256}^{0.5n} - W_{256}^{-0.5n}\right)} \\
&= \frac{\sin\frac{63\pi n}{256}}{256\sin\frac{\pi n}{256}}
\end{aligned}$$

In sum,

$$h[n] = \begin{cases} \dfrac{\sin\dfrac{63\pi n}{256}}{256\sin\dfrac{\pi n}{256}} & 0 \le n \le 255 \\ 0 & \text{otherwise} \end{cases}$$

Solution from Fall04 PS7

(a) S can be represented according to the equation $y[n] = x[n] * h[n]$. An ideal frequency selective filter has an infinite impulse response, but $h[n]$ is a 256 point (i.e. finite) sequence. Therefore, S is NOT an ideal frequency selective filter.

(b) The impulse response of S is $IDFT\{H[k]\} = h[n]$. Since $\tilde{H}[k]$ is real and even, so is $\tilde{h}[n]$ and therefore $h[n]$ is real.

(c)

$$\begin{aligned}
h[n] &= \frac{1}{256}\sum_{k=0}^{255} H[k]W_{256}^{-kn} \qquad 0 \le n \le 255 \\
&= \frac{1}{256}\sum_{k=0}^{31} H[k]W_{256}^{-kn} + \frac{1}{256}\sum_{k=225}^{255} H[k]W_{256}^{-n(k-256)} \\
&= \frac{1}{256}\sum_{k=0}^{31} H[k]W_{256}^{-kn} + \frac{1}{256}\sum_{k=-31}^{-1} H[k]W_{256}^{-kn} \\
&= \frac{1}{256}\sum_{k=-31}^{31} W_{256}^{-kn}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{256}\frac{W_{256}^{31n} - W_{256}^{-32n}}{1 - W_{256}^{-n}} \\
&= \frac{1}{256}\frac{W_{256}^{-0.5n}(W_{256}^{31.5n} - W_{256}^{-31.5n})}{W_{256}^{-0.5n}\left(W_{256}^{0.5n} - W_{256}^{-0.5n}\right)} \\
&= \frac{\sin\frac{63n\pi}{256}}{256\sin\frac{n\pi}{256}}
\end{aligned}
$$

In sum,

$$
h[n] = \begin{cases} \dfrac{\sin\dfrac{63n\pi}{256}}{256\sin\dfrac{n\pi}{256}} & 0 \le n \le 255 \\ 0 & \text{otherwise} \end{cases}
$$

Solution from Fall02 PS7

(a) S can be represented according to the equation $y[n] = x[n] * h[n]$. An ideal frequency selective filter has an infinite impulse response, but $h[n]$ is a 256 point (i.e. finite) sequence. Therefore, S is NOT an ideal frequency selective filter.

(b) The impulse response of S is $IDFT\{H[k]\} = h[n]$. Since $\tilde{H}[k]$ is real and even, so is $\tilde{h}[n]$ and therefore $h[n]$ is real.

(c)

$$
\begin{aligned}
h[n] &= \frac{1}{256}\sum_{k=0}^{255} H[k]W_{256}^{-kn} \qquad 0 \le n \le 255 \\
&= \frac{1}{256}\sum_{k=0}^{31} H[k]W_{256}^{-kn} + \frac{1}{256}\sum_{k=225}^{255} H[k]W_{256}^{-n(k-256)} \\
&= \frac{1}{256}\sum_{k=0}^{31} H[k]W_{256}^{-kn} + \frac{1}{256}\sum_{k=-31}^{-1} H[k]W_{256}^{-kn} \\
&= \frac{1}{256}\sum_{k=-31}^{31} W_{256}^{-kn} \\
&= \frac{1}{256}\frac{W_{256}^{31n} - W_{256}^{-32n}}{1 - W_{256}^{-n}} \\
&= \frac{1}{256}\frac{W_{256}^{-0.5n}(W_{256}^{31.5n} - W_{256}^{-31.5n})}{W_{256}^{-0.5n}\left(W_{256}^{0.5n} - W_{256}^{-0.5n}\right)} \\
&= \frac{\sin\frac{63n\pi}{256}}{256\sin\frac{n\pi}{256}}
\end{aligned}
$$

In sum,

$$
h[n] = \begin{cases} \dfrac{\sin\dfrac{63n\pi}{256}}{256\sin\dfrac{n\pi}{256}} & 0 \le n \le 255 \\ 0 & \text{otherwise} \end{cases}
$$

Solution from Spring01 PS6

N/A

8.38. Problem 5 in Fall 2002 final exam.
Appears in: Spring05 PS8, Fall04 PS7, Spring04 PS7.

Problem

$x[n]$ is a real-valued finite length sequence of length 512, i.e.,

$$x[n] = 0 \qquad n < 0, n \geq 512$$

and has been stored in a 512 point data memory. It is known that $X[k]$ the 512-point DFT of $x[n]$ is bandlimited, specifically

$$X[k] = 0 \qquad 250 \leq k \leq 262$$

In storing the data, **one** data point at most may have been corrupted. Specifically, if $s[n]$ denotes the stored data, $s[n] = x[n]$ except possibly at one unknown memory location n_0.

To test and possibly correct the data, you are able to examine $S[k]$, the 512 point DFT of $s[n]$.

(a) Specify whether, by examining $S[k]$, it is possible and if so, how, to detect whether an error has been made in one data point, i.e., whether or not $s[n] = x[n]$.

In parts (b) and (c), assume that you know for sure that one data point has been corrupted, i.e., that $s[n] = x[n]$ except at $n = n_0$.

(b) In this part, assume the value of n_0 is unknown. Specify a procedure for determining from $S[k]$ the value of n_0.

(c) In this part, assume that you know the value of n_0. Specify a procedure for determining $x[n_0]$ from $S[k]$.

Solution from Spring05 PS8

(a) Yes. By linearity and the definition of the DFT,

$$S[k] = X[k] + (s[n_0] - x[n_0])W^{kn_0}, \text{ where } W = e^{-j\frac{2\pi}{512}}$$

But $X[k] = 0$ for $250 \leq k \leq 262$. Therefore, in that interval,

$$S[k] = (s[n_0] - x[n_0])W^{kn_0},$$

which is nonzero if $s[n_0] \neq x[n_0]$.

(b) Pick two sequential non-zero $S[k]$ and $S[k+1]$, where $250 \leq k \leq 261$:

$$\frac{S[k+1]}{S[k]} = W^{n_0} \Rightarrow n_0 = \frac{\ln\left(\frac{S[k+1]}{S[k]}\right)}{\ln(W)} = \frac{j512}{2\pi}\ln\left(\frac{S[k+1]}{S[k]}\right)$$

(c) We know that for $250 \le k \le 262$, $S[k] = (s[n_0] - x[n_0])W^{kn_0}$, where $s[n_0]$ can be determined by the IDFT:

$$s[n_0] = \sum_{i=0}^{N-1} S[i]W^{-in_0}.$$

Thus,

$$\begin{aligned} x[n_0] &= s[n_0] - S[k]W^{-kn_0} \\ &= \sum_{i=0}^{N-1} S[i]W^{-in_0} - S[k]W^{-kn_0} \end{aligned}$$

Solution from Fall04 PS7

(a) Yes. By linearity and the definition of the DFT,

$$S[k] = X[k] + (s[n_0] - x[n_0])W^{kn_0}, \text{ where } W = e^{-j\frac{2\pi}{512}}$$

But $X[k] = 0$ for $250 \le k \le 262$. Therefore, in that interval,

$$S[k] = (s[n_0] - x[n_0])W^{kn_0},$$

which is nonzero if $s[n_0] \ne x[n_0]$.

(b) Pick two sequential non-zero $S[k]$ and $S[k+1]$, where $250 \le k \le 261$:

$$\frac{S[k+1]}{S[k]} = W^{n_0} \Rightarrow n_0 = \frac{\ln\left(\frac{S[k+1]}{S[k]}\right)}{\ln(W)} = \frac{j512}{2\pi}\ln\left(\frac{S[k+1]}{S[k]}\right)$$

(c) We know that for $250 \le k \le 262$, $S[k] = (s[n_0] - x[n_0])W^{kn_0}$, where $s[n_0]$ can be determined by the IDFT:

$$s[n_0] = \sum_{i=0}^{N-1} S[i]W^{-in_0}.$$

Thus,

$$\begin{aligned} x[n_0] &= s[n_0] - S[k]W^{-kn_0} \\ &= \sum_{i=0}^{N-1} S[i]W^{-in_0} - S[k]W^{-kn_0} \end{aligned}$$

Solution from Spring04 PS7

Suppose that the value $x[n_0]$, $0 \le n_0 < 512$ has been corrupted. Then we have

$$s[n] = x[n] + (s[n_0] - x[n_0])\delta[n - n_0] = x[n] + a\delta[n - n_0] \tag{1}$$

where we define $a = s[n_0] - x[n_0]$. If no corruption transpired then $a = 0$. The DFT of $s[n]$ is:

$$S[k] = X[k] + DFT\{a\delta[n - n_0]\} = X[k] + ae^{-j2\pi kn_0/N}, \quad N = 512 \tag{2}$$

For $250 \le k \le 262$, $X[k] = 0$, and $S[k] = ae^{-j2\pi kn_0/N}$.

(a) It is possible to detect corruption of one sample by looking at $S[k]$ for $250 \le k \le 262$ - in case of corruption these samples will have non-zero values.

(b) For $250 \le k \le 262$ $S[k] = ae^{-j2\pi k n_0/N}$, and we can find n_0 from the phase of $S[k]$. However, this requires some care, because we observe the phase of $S[k]$ modulo 2π:

$$\angle(S[250]) \bmod 2\pi = -\frac{2\pi 250 n_0}{512} \bmod 2\pi$$
$$\angle(S[251]) \bmod 2\pi = -\frac{2\pi 251 n_0}{512} \bmod 2\pi = \left(-\frac{2\pi 250 n_0}{512} - \frac{2\pi n_0}{512}\right) \bmod 2\pi$$

It follows that

$$(\angle(S[250]) - \angle(S[251])) \bmod 2\pi = \frac{2\pi n_0}{512} \bmod 2\pi = \frac{2\pi n_0}{512}$$

The last equality is true because we know that $0 \le n_0 < 512$, and hence $0 \le \frac{2\pi n_0}{512} < 2\pi$. Hence, $n_0 = 512\frac{(\angle(S[250]) - \angle(S[251])) \bmod 2\pi}{2\pi}$.

(c) To determine $a = s[n_0] - x[n_0]$, (including its sign), we can use the value of n_0 that we found in part (b): $a = S[250]/e^{-j2\pi 250 n_0/N}$. Then $x[n_0] = s[n_0] - a$.

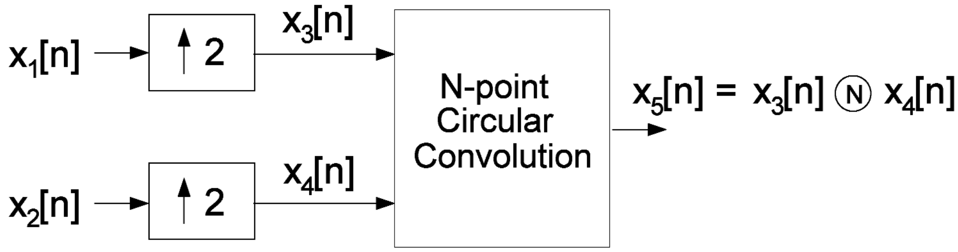
Solution from Fall02 Final

N/A

8.39. Problem 2 from sp 2005 final exam Appears in: Fall05 PS2.

Problem

In the system shown in the figure below, $x_1[n]$ and $x_2[n]$ are both causal, 32-point sequences, *i.e.*, they are both zero outside the interval $0 \leq n \leq 31$. $y[n]$ denotes the linear convolution of $x_1[n]$ and $x_2[n]$, *i.e.*, $y[n] = x_1[n] * x_2[n]$.



1. Determine the values of N for which all the values of $y[n]$ can be completely recovered from $x_5[n]$.

2. Specify explicitly how to recover $y[n]$ from $x_5[n]$ for the *smallest* value of N which you determined in part (1).

Solution from Fall05 PS2

1. The signals $x_3[n]$ and $x_4[n]$ are each length 63. In that sense, the linear convolution $x_3[n] * x_4[n]$ has length $63 + 63 - 1 = 125$, and can be obtained using N-point circular convolution with $N \geq 125$. Every other sample of that linear convolution is 0, as can be seen using the flip-and-slide graphical interpretation of linear convolution of $x_3[n]$ and $x_4[n]$, which both have values of 0 for every other sample. The nonzero values correspond to values of $y[n]$.

   However, since $y[n]$ is only length $32 + 32 - 1 = 63$, it seems like we may be able to use smaller values of N in the circular convolution, though we need $N \geq 63$. Suppose we make N an odd number such that $63 \leq N < 125$. Then first, the aliased samples from circular convolution will fall in the positions of the 0 values in $x_5[n]$. Second, when these values (of aliased samples) are nonzero, they correspond to values that are actually in the linear convolution $x_1[n] * x_2[n]$. And third, since we have chosen a large enough value for N, we obtain all values of $y[n]$.

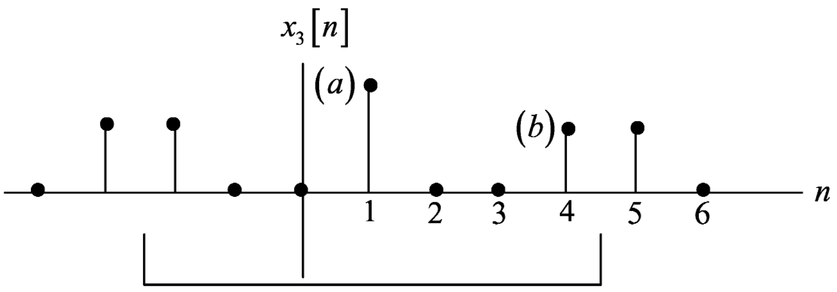
   Therefore, we can use any odd number for N such that $63 \leq N < 125$, and any value such that $N \geq 125$.

2. For $N = 63$, start with $x_5[0]$ and select every other sample. These are $y[0]$ through $y[31]$. Then start with $x_5[1]$ and select every other sample. These are $y[32]$ through $y[62]$. This interleaving is shown below:

   $$\begin{array}{cccccccc} x_5[0] & x_5[1] & x_5[2] & x_5[3] & \ldots & x_5[60] & x_5[61] & x_5[62] \\ y[0] & y[32] & y[1] & y[33] & \ldots & y[30] & y[62] & y[31] \end{array}$$

Solution from Spring05 Final

8.40. A DFT $X[k]$ will exhibit generalized linear phase if the periodic extension of the signal $x[n]$ has even or odd symmetry. Of the three given signals, only $x_3[n]$ has the required symmetry.



The figure shows even symmetry about $n=1$. Therefore for this signal, $\alpha=1$.

For a direct demonstration, we can calculate $X_3[k]$. That is,

$$
\begin{aligned}
X_3[k] &= \sum_{n=0}^{6} x[n] e^{-j2\pi\frac{nk}{7}} \\
&= ae^{-j2\pi\frac{k}{7}} + be^{-j2\pi\frac{4k}{7}} + be^{-j2\pi\frac{5k}{7}} \\
&= ae^{-j2\pi\frac{k}{7}} + be^{-j2\pi\frac{4k}{7}} + be^{j2\pi\frac{2k}{7}} \\
&= \left(a + be^{-j2\pi\frac{3k}{7}} + be^{j2\pi\frac{3k}{7}}\right) e^{-j2\pi\frac{k}{7}} \\
&= \left(a + 2b\cos(3k/7)\right) e^{-j2\pi\frac{k}{7}},
\end{aligned}
$$

which is in the required form with $\alpha=1$.

8.41. Initially we have

$$x[0]=1+j3, x[1]=0, x[2]=2-j, x[3]=0,$$
$$x[4]=-1-j3, x[5]=0, x[6]=-1+j3, x[7]=0.$$

Let $X[k]$ represent the DFT of $x[n]$. Then if $\hat{X}[k]=X[k+1]$ we have $\hat{x}[n]=x[n]e^{-j\frac{\pi}{4}n}$. That is,

$$\hat{x}[0]=1+j3, \hat{x}[1]=0, \hat{x}[2]=-1-j2, \hat{x}[3]=0,$$
$$\hat{x}[4]=1+j3, \hat{x}[5]=0, \hat{x}[6]=-3-j, \hat{x}[7]=0.$$

If we compress $\hat{X}[k]$ by a factor of $M=2$ we obtain $Y[k]=\hat{X}[2k], \quad k=0,1,2,3$. That is, $Y[k]=X[2k+1], \quad k=0,1,2,3$. The inverse DFT $y[n]$ of $Y[k]$ is the quantity we seek.

Compressing $\hat{X}[k]$ in the frequency domain will cause aliasing of $\hat{x}[n]$ in the time domain. We have

$$\begin{aligned}
y[n] &= \tfrac{1}{4}\sum_{k=0}^{3} Y[k]e^{j2\pi\frac{kn}{4}} \\
&= \tfrac{1}{4}\sum_{k=0}^{3} \hat{X}[2k]e^{j2\pi\frac{kn}{4}} \\
&= \tfrac{1}{4}\sum_{k=0}^{3}\left(\sum_{m=0}^{7}\hat{x}[m]e^{-j2\pi\frac{(2k)m}{8}}\right)e^{j2\pi\frac{kn}{4}} \\
&= \sum_{m=0}^{7}\hat{x}[m]\tfrac{1}{4}\sum_{k=0}^{3} e^{j2\pi\frac{(n-m)k}{4}} \\
&= \sum_{m=0}^{7}\hat{x}[m]\sum_{r=-\infty}^{\infty}\delta[n-m-4r], \quad n=0,1,2,3.
\end{aligned}$$

The last expression is the convolution of $\hat{x}[n]$ with a train of impulses spaced every four samples. Taking into account that $\hat{x}[n]$ has a length of eight samples, we have

$$y[n]=\hat{x}[n]+\hat{x}[n+1], \quad n=0,1,2,3.$$

That is,

$$y[0]=2+j6, y[1]=0, y[2]=-4-j3, y[3]=0.$$

Then

$$y_r[0]=2, y_r[1]=0, y_r[2]=-4, y_r[3]=0$$

and

$$y_i[0]=6, y_i[1]=0, y_i[2]=-3, y_i[3]=0.$$

Note that this problem can also be solved by direct calculation.

8.42. (a) $c_{xx}[m] = x[n] * x[-n]\big|_{n=m} = 0$ for $m < -1023$, $m > 1023$.

$$|X_N[k]|^2 = X_N^*[k] X_N[k] = N\text{-point DFT}\left\{x\left[((-n))_N\right] * x[n]\right\}$$

$$g_N[m] = N\text{-point IDFT}\left\{N\text{-point DFT}\left\{x\left[((-n))_N\right] * x[n]\right\}\right\}$$

$$= \begin{cases} x\left[((-n))_N\right] * x[n]\big|_{n=m}, & 0 \le m \le N-1 \\ 0, & \text{otherwise.} \end{cases}$$

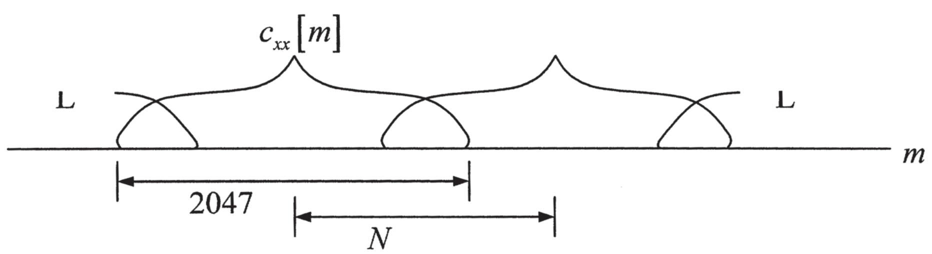
For $N = 2047$ and $0 \le m \le N-1$, $g_N[m] = x\left[((-n))_{2047}\right] * x[n]\big|_{n=m}$. For smaller N, circular convolution introduces time aliasing. To obtain $c_{xx}[m]$ use

$$c_{xx}[m] = \begin{cases} g_{2047}[m], & 0 \le m < 1024 \\ g_{2047}[2047+m], & -1024 < m \le -1. \end{cases}$$

(b) For $0 \le m \le N-1$, $g_N[m] = x\left[((-n))_N\right] * x[n]\big|_{n=m}$. We would now like to use a variant of the earlier technique but for smaller N. For general even N our "post-processing" step is

$$\hat{c}_{xx}[m] = \begin{cases} g_N[m], & 0 \le m < \dfrac{N+1}{2} \\ g_N[N+m], & -\dfrac{(N+1)}{2} < m \le -1. \end{cases}$$

If we want $\hat{c}_{xx}[m] = c_{xx}[m]$ for $|m| \le 10$, we need to ensure that the time aliasing from circular convolution does not affect $g_N[m]$ for $0 \le m \le 10$ and for $N-11 \le m \le N-1$.



For the lowest possible N, $N = 1024$, we have only $g_N[0]$ unaffected by aliasing. For $N = 1025$, $g_N[0]$, $g_N[1]$, and $g_N[1024]$ are unaffected, etc. Keeping this trent in mind we pick $N = 1034$. Our post-processing step becomes

$$c_{xx}[m] = \begin{cases} g_{1034}[m], & 0 \le m \le 10 \\ g_{1034}[1034+m], & -10 \le m \le -1. \end{cases}$$

8.43. Problem 5 in Fall2005 midterm exam.

Problem

In Figure 1, $x[n]$ is a finite sequence of length 1024. The sequence $R[k]$ is obtained by taking the 1024-point DFT of $x[n]$ and compressing the result by 2.

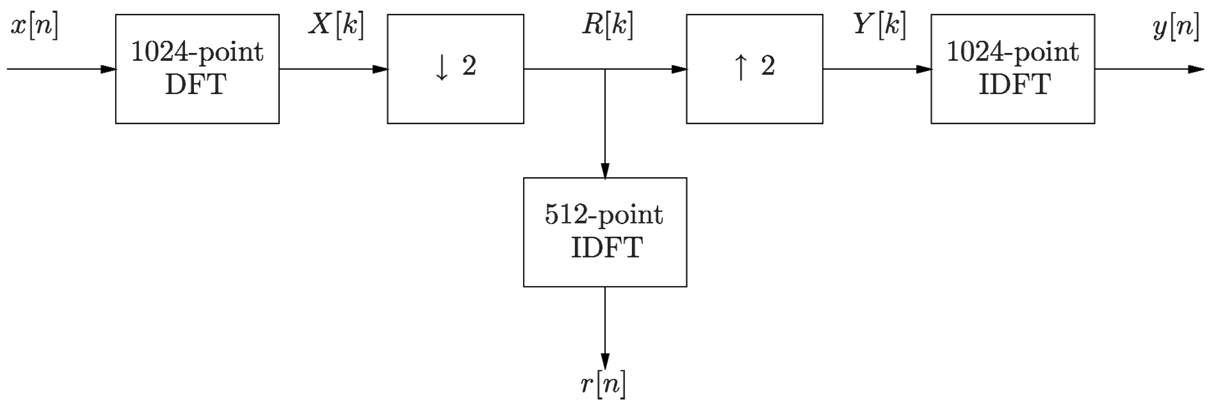


Figure 1:

(a) Choose the most accurate statement for $r[n]$, the 512-point inverse DFT of $R[k]$. Justify your choice in a few concise sentences.

A. $r[n] = x[n],\ 0 \le n \le 511$

B. $r[n] = x[2n],\ 0 \le n \le 511$

C. $r[n] = x[n] + x[n+512],\ 0 \le n \le 511$

D. $r[n] = x[n] + x[-n+512],\ 0 \le n \le 511$

E. $r[n] = x[n] + x[1023-n],\ 0 \le n \le 511$

In all cases $r[n] = 0$ outside $0 \le n \le 511$.

(b) The sequence $Y[k]$ is obtained by expanding $R[k]$ by 2. Choose the most accurate statement for $y[n]$, the 1024-point inverse DFT of $Y[k]$. Justify your choice in a few concise sentences.

A. $y[n] = \begin{cases} \frac{1}{2}\left(x[n] + x[n+512]\right), & 0 \le n \le 511 \\ \frac{1}{2}\left(x[n] + x[n-512]\right), & 512 \le n \le 1023 \end{cases}$

B. $y[n] = \begin{cases} x[n], & 0 \le n \le 511 \\ x[n-512], & 512 \le n \le 1023 \end{cases}$

C. $y[n] = \begin{cases} x[n], & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$

D. $y[n] = \begin{cases} x[2n], & 0 \le n \le 511 \\ x[2(n-512)], & 512 \le n \le 1023 \end{cases}$

E. $y[n] = \frac{1}{2}\left(x[n] + x[1023-n]\right),\ 0 \le n \le 1023$

In all cases $y[n] = 0$ outside $0 \le n \le 1023$.

Solution from Fall05 Midterm

Answer: C

Compressing the 1024-point DFT $X[k]$ by 2 undersamples the DTFT $X(e^{j\omega})$. Undersampling in the frequency domain corresponds to aliasing in the time domain. In this specific case, the second half of $x[n]$ is folded onto the first half, as described by statement C.

Answer: A

We can first think about how expanding by 2 in the time domain affects the DFT. Expanding a time sequence $x[n]$ by 2 compresses the DTFT $X(e^{j\omega})$ by 2 in frequency. As a result, the $2N$-point DFT of the expanded sequence samples *two* periods of $X(e^{j\omega})$ and equals two copies of the N-point DFT $X[k]$.

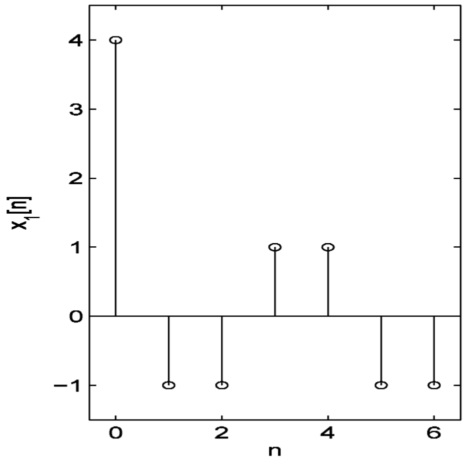
By duality, expanding the DFT $R[k]$ by 2 corresponds to repeating $r[n]$ back-to-back, with an additional scaling by $\frac{1}{2}$. Thus statement A is correct.

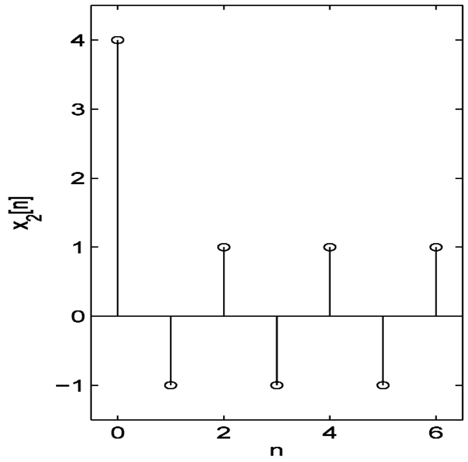
Alternatively, $Y[k] = \frac{1}{2}\left(1 + (-1)^k\right) X[k]$. Modulating the DFT by $(-1)^k$ corresponds to a circular time shift of $N/2 = 512$.

8.44. Problem 3 in Spring2005 final exam.

Problem

Below are two finite-length sequences $x_1[n]$ and $x_2[n]$ of length 7. $X_i(e^{j\omega})$ denotes the DTFT of $x_i[n]$, and $X_i[k]$ denotes the seven-point DFT of $x_i[n]$.





For each of the sequences $x_1[n]$ and $x_2[n]$, indicate whether each one of the following properties holds:

(a) $X_i(e^{j\omega})$ can be written in the form

$$X_i(e^{j\omega}) = A_i(\omega)e^{j\alpha_i\omega}, \quad \text{for } \omega \in (-\pi, \pi),$$

where $A_i(\omega)$ is real and α_i is a constant.

(b) $X_i[k]$ can be written in the form

$$X_i[k] = B_i[k]e^{j\beta_i k},$$

where $B_i[k]$ is real and β_i is a constant.

Solution from Spring2005

For the case of a length-7 sequence, we have generalized linear phase if and only if there is even or odd symmetry about the $n = 3$ sample. Neither sequence has this property.

Linear phase in the DFT arises if the *periodic extension* of the signal has even or odd symmetry. The periodic extension of $x_1[n]$ has the required symmetry but the periodic extension of $x_2[n]$ does not.

8.45. Problem 8 in Spring2005 final exam.

Problem

The sequence $x[n]$ is a 128-point sequence (*i.e.*, $x[n] = 0$ for $n < 0$ and for $n > 127$), and $x[n]$ has at least one non-zero sample. The DTFT of $x[n]$ is denoted $X(e^{j\omega})$.

What is the largest integer M such that it is possible for $X(e^{j\frac{2\pi}{M}k})$ to be zero for all integer values of k? Construct an example for the maximal M that you have found.

Solution from Spring05 final

Let $X_M[k]$ denote the M-pt DFT of $x[n]$. Then, $X_M[k] = X(e^{j\omega})\big|_{\omega=\frac{2\pi}{M}k} = X(e^{j\frac{2\pi}{M}k})$. In order for the desired quantities to be zero, we need $X_M[k]$ to be zero. When this DFT is zero for all k, the corresponding periodic signal is zero for all time. Since the M-pt DFT of a finite-length signal is the Fourier series representation of the signal replicated with period M, the question can then be reformulated as: What is the largest integer M such that when $x[n]$ is replicated with period M to form $\tilde{x}[n]$, $\tilde{x}[n] = 0$ for all n?

Since at least one sample of $x[n]$ is non-zero, then any M that is equal to or greater than 128 (no time-aliasing) does not produce the desired result. We claim that the largest possible M is 127. An example that demonstrates this suffices as proof.

Let $x[n] = \delta[n] - \delta[n-127]$. Then

$$\begin{aligned}
\tilde{x}[n] &= \sum_{m=-\infty}^{\infty} x[n+127m] \\
&= \sum_{m=-\infty}^{\infty} (\delta[n+127m] - \delta[n+127m-127]) \\
&= \sum_{m=-\infty}^{\infty} (\delta[n+127m] - \delta[n+127(m-1)]) \\
&= \sum_{m=-\infty}^{\infty} \delta[n+127m] - \sum_{m=-\infty}^{\infty} \delta[n+127(m-1)] \\
&= \sum_{m=-\infty}^{\infty} \delta[n+127m] - \sum_{k=-\infty}^{\infty} \delta[n+127k], \qquad \text{where } k = m-1 \text{ in the second sum} \\
&= 0.
\end{aligned}$$

Alternatively, we can think of constructing the example by placing 127 equally-spaced zeros on the unit circle (beginning at $z = 1$), which would make the DTFT zero at those locations. In this case, the z-transform of $x[n]$ is

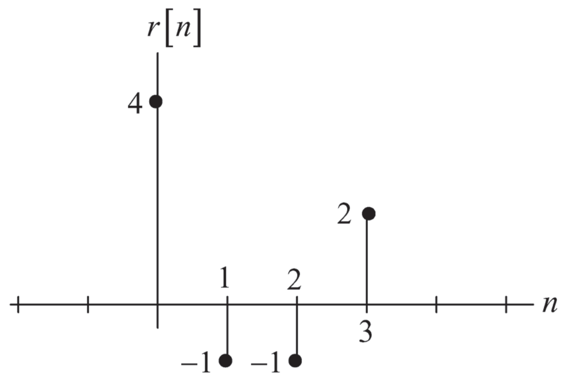
$$X(z) = (1 - z^{-1})(1 - e^{j\frac{2\pi}{127}}z^{-1})(1 - e^{j\frac{2\pi}{127}2}z^{-1}) \cdots (1 - e^{j\frac{2\pi}{127}126}z^{-1}).$$

Simplifying this expression produces

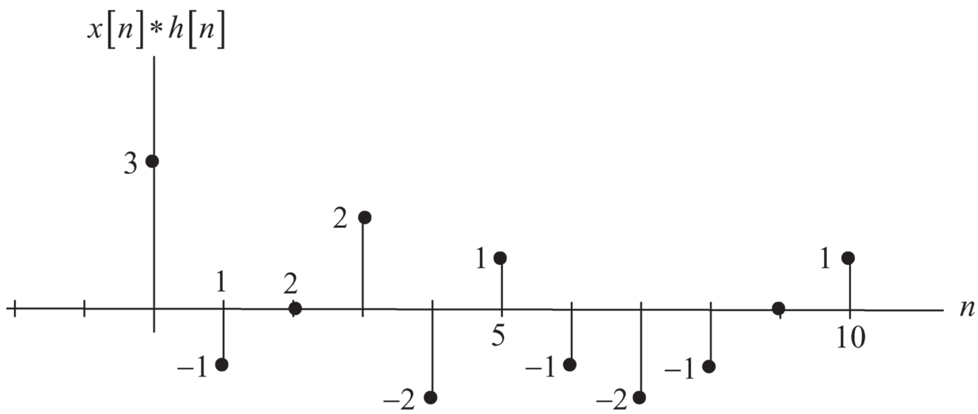
$$X(z) = 1 - z^{-127},$$

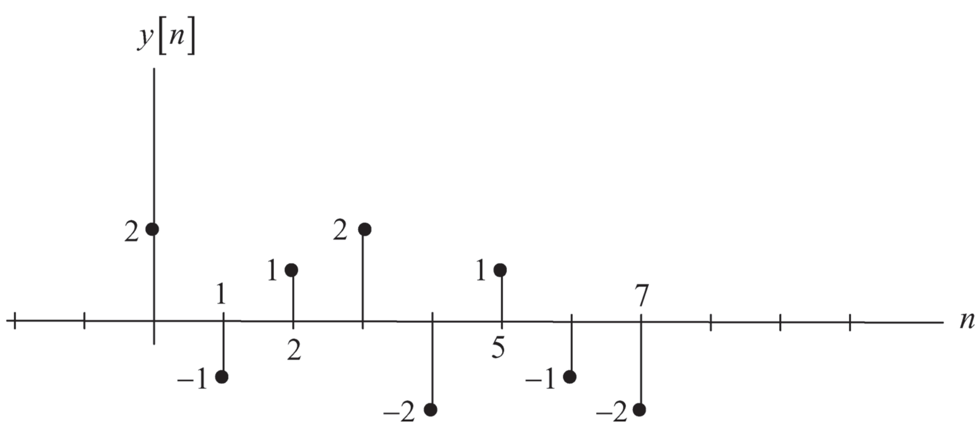
which in the time domain is $x[n] = \delta[n] - \delta[n-127]$, just as before.

8.46. A. $R[k]$ is the four-point DFT of $x[n]$. Inverting $R[k]$ creates time aliasing.

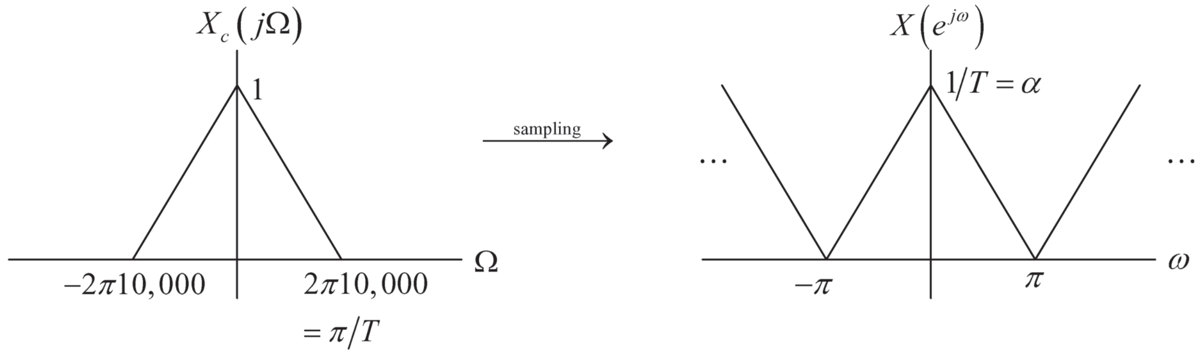


B. The product of the DFTs corresponds to circular convolution, i.e., linear convolution followed by time aliasing.

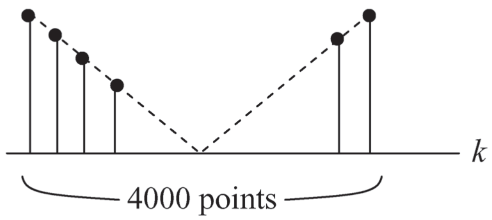




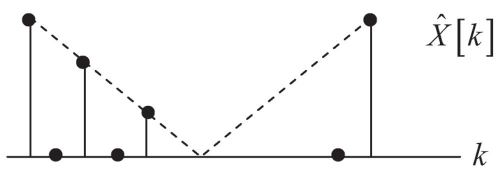
8.47.



Desired DFT ($X[k]$):



But Eq. (2) is equivalent to the expansion by two of the 2000-point DFT of $x_c(nT)$. That is,



$$\hat{X}[k] = \begin{cases} X[k], & k \text{ even} \\ 0, & k \text{ odd.} \end{cases}$$

8.48. C. Since we know $x[n]$ for $n = 64,\ldots,1023$, we can form the sums $y[m], \quad m = 0,\ldots,63$ given by

$$y[n] = \sum_{i=1}^{15} x[n+64i], \quad n = 0,\ldots,63.$$

Next we can form the function $\hat{X}[k], \quad k = 0,\ldots,1023$, given by

$$\hat{X}[k] = \begin{cases} X[k], & k = 0,16,32,\ldots,1008 \\ 0, & \text{otherwise.} \end{cases}$$

The 1024-point IDFT $\hat{x}[n]$ of $\hat{X}[k]$ is a time-aliased verson of $x[n]$. That is

$$\hat{x}[n] = \tfrac{1}{16} \sum_{i=0}^{15} x[n+64i], \quad n = 0,\ldots,63.$$

We have

$$x[n] = 16\hat{x}[n] - y[n], \quad n = 0,\ldots,63.$$

8.49. A. Given

$$c_{xy}[n] = y[-n] * x[n],$$

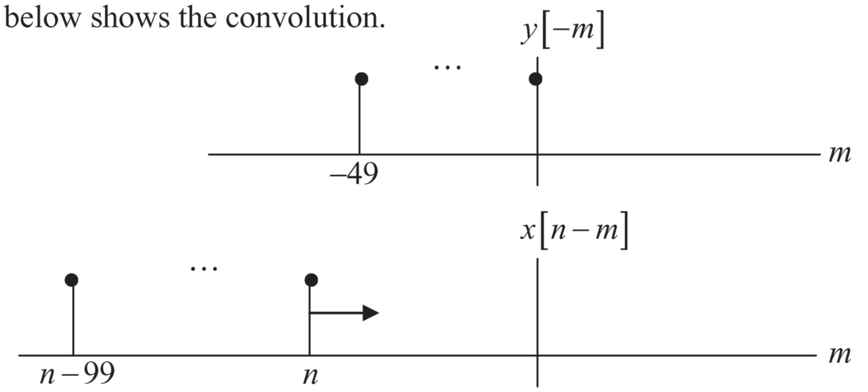
we have

$$C_{xy}(e^{j\omega}) = F[y[-n]]X(e^{j\omega}).$$

Now $F[y[-n]] = Y(e^{-j\omega})$, and $Y(e^{-j\omega}) = Y^*(e^{j\omega})$ since $y[n]$ is real. Substituting gives

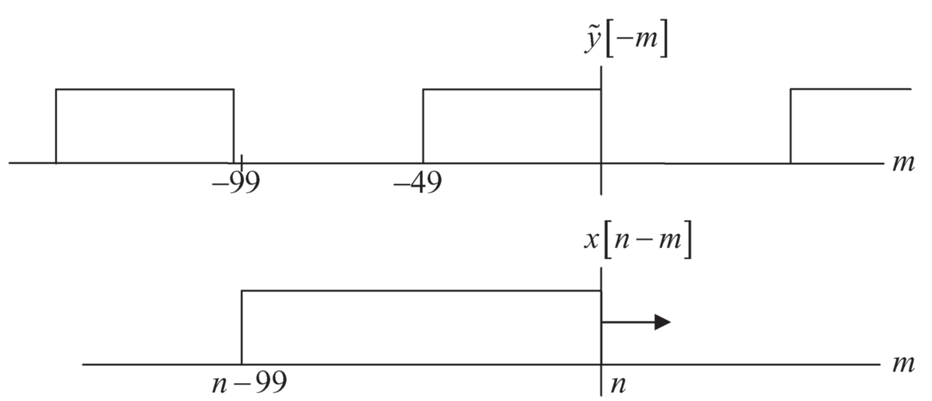
$$C_{xy}(e^{j\omega}) = Y^*(e^{j\omega})X(e^{j\omega}).$$

B. Given that $x[n]$ has a length of $N = 100$ samples and $y[n]$ has a length of $L = 50$ samples, then $y[-n]$ will also have a length of $L = 50$ samples and $c_{xy}[n] = y[-n] * x[n]$ will have a length of $N + L - 1 = 149$ samples. The construction shown below shows the convolution.



The construction shows that $c_{xy}[n]$ can be nonzero only over the range $N_1 \le n \le N_2$, where $N_1 = -49$ and $N_2 = 99$.

C. The prescribed procedure calculates the N-point circular convolution of $x[n]$ with $y[-n]$. The construction below shows the convolution.



Correct values of $c_{xy}[n]$ will be obtained for $0 \le n \le 20$ if $N \ge 100$.

8.50. We have a 10-point sequence, $x[n]$. We want a modified sequence, $x_1[n]$, such that the 10-pt. DFT of $x_1[n]$ corresponds to

$$X_1[k] = X(z)|_{z=\frac{1}{2}e^{j[(2\pi k/10)+(\pi/10)]}}$$

Recall the definition of the Z-transform of $x[n]$:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Since $x[n]$ is of finite duration ($N = 10$), we assume:

$$x[n] = \begin{cases} \text{nonzero}, & 0 \le n \le 9 \\ 0, & \text{otherwise} \end{cases}$$

Therefore,

$$X(z) = \sum_{n=0}^{9} x[n]z^{-n}$$

Substituting in $z = \frac{1}{2}e^{j[(2\pi k/10)+(\pi/10)]}$:

$$X(z)|_{z=\frac{1}{2}e^{j[(2\pi k/10)+(\pi/10)]}} = \sum_{n=0}^{9} x[n]\left(\frac{1}{2}e^{j[(2\pi k/10)+(\pi/10)]}\right)^{-n}$$

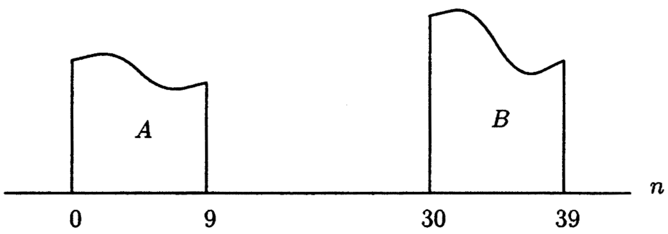
We seek the signal $x_1[n]$, whose 10-pt. DFT is equivalent to the above expression. Recall the analysis equation for the DFT:

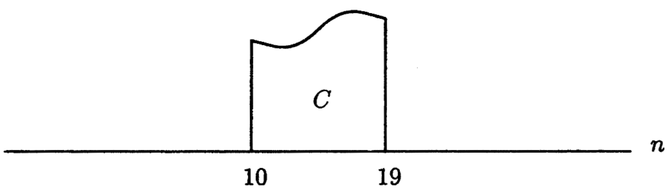
$$X_1[k] = \sum_{n=0}^{9} x_1[n]W_{10}^{kn}, \qquad 0 \le k \le 9$$

Since $W_{10}^{kn} = e^{-j(2\pi/10)kn}$, by comparison

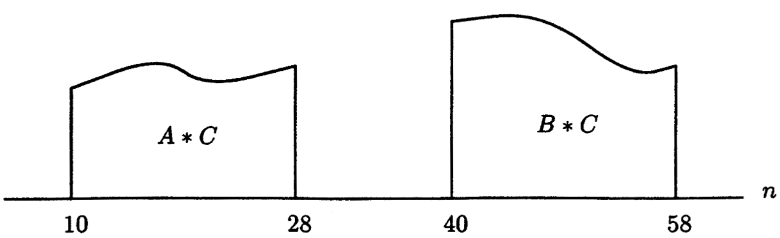
$$x_1[n] = x[n]\left(\frac{1}{2}e^{j(\pi/10)}\right)^{-n}$$

8.51. We have





(a) The linear convolution $x[n] * y[n]$ is a $40 + 20 - 1 = 59$ point sequence:



Thus, $x[n] * y[n] = w[n]$ is nonzero for $10 \leq n \leq 28$ and $40 \leq n \leq 58$.

(b) The 40-pt circluar convolution can be obtained by aliasing the linear convolution. Specifically, we alias the points in the range $40 \leq n \leq 58$ to the range $0 \leq n \leq 18$.
Since $w[n] = x[n] * y[n]$ is zero for $0 \leq n \leq 9$, the circular convolution $g[n] = x[n] (40) y[n]$ consists of only the (aliased) values:

$$w[n] = x[n] * y[n], \qquad 40 \leq n \leq 49$$

Also, the points of $g[n]$ for $18 \leq n \leq 39$ will be equivalent to the points of $w[n]$ in this range.
To conclude,

$$\begin{aligned} w[n] &= g[n], \quad 18 \leq n \leq 39 \\ w[n+40] &= g[n], \quad 0 \leq n \leq 9 \end{aligned}$$

8.52. (a)

$$\frac{1}{8}\sum_{k=0}^{7} X[k]e^{j\frac{2\pi}{8}k9} = \frac{1}{8}\sum_{k=0}^{7} X[k]e^{j\frac{2\pi}{8}k} = x[1].$$

(b)

$$\begin{aligned}
V[k] &= X(z)\big|_{z=2e^{j(\frac{2\pi k+\pi}{8})}} \\
&= \sum_{n=-\infty}^{n=\infty} x[n]z^{-n}\big|_{z=2e^{j(\frac{2\pi k+\pi}{8})}} \\
&= \sum_{n=0}^{n=8} x[n]z^{-n}\big|_{z=2e^{j(\frac{2\pi k+\pi}{8})}} \\
&= \sum_{n=0}^{n=8} x[n](2e^{j\frac{\pi}{8}})^{-n}e^{-j\frac{2\pi k}{8}n} \\
&= \sum_{n=0}^{n=8} v[n]e^{-j\frac{2\pi k}{8}n}.
\end{aligned}$$

We thus conclude that

$$v[n] = x[n](2e^{j\frac{\pi}{8}})^{-n}.$$

(c)

$$\begin{aligned}
w[n] &= \frac{1}{4}\sum_{k=0}^{3} W[k]W_4^{-kn} \\
&= \frac{1}{4}\sum_{k=0}^{3}(X[k] + X[k+4])e^{+j\frac{2\pi}{4}kn} \\
&= \frac{1}{4}\sum_{k=0}^{3} X[k]e^{+j\frac{2\pi}{4}kn} + \frac{1}{4}\sum_{k=0}^{3} X[k+4]e^{+j\frac{2\pi}{4}kn} \\
&= \frac{1}{4}\sum_{k=0}^{3} X[k]e^{+j\frac{2\pi}{4}kn} + \frac{1}{4}\sum_{k=4}^{7} X[k]e^{+j\frac{2\pi}{4}kn} \\
&= \frac{1}{4}\sum_{k=0}^{7} X[k]e^{+j\frac{2\pi}{8}k2n} \\
&= 2x[2n].
\end{aligned}$$

We thus conclude that

$$w[n] = 2x[2n].$$

(d) Note that $Y[k]$ can be written as:

$$\begin{aligned}
Y[k] &= X[k] + (-1)^k X[k] \\
&= X[k] + W_8^{4k}X[k].
\end{aligned}$$

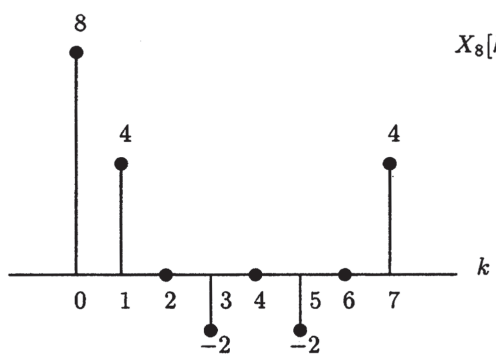
Using the DFT properties, we thus conclude that

$$y[n] = x[n] + x[((n-4))_8].$$

8.53. (a) Let $n = 0, ..., 7$, we can write $x[n]$ as:

$$\begin{aligned}
x[n] &= 1 + \frac{1}{2}(e^{j\frac{\pi}{4}n} + e^{-j\frac{\pi}{4}n}) - \frac{1}{4}(e^{j\frac{3\pi}{4}n} + e^{-j\frac{3\pi}{4}n}) \\
&= 1 + \frac{1}{2}e^{j\frac{2\pi}{8}n} + \frac{1}{2}e^{j\frac{2\pi}{8}n7} - \frac{1}{4}e^{j\frac{2\pi}{8}n3} - \frac{1}{4}e^{j\frac{2\pi}{8}n5} \\
&= \frac{1}{8}(8 + 4e^{j\frac{2\pi}{8}n} + 4e^{j\frac{2\pi}{8}n7} - 2e^{j\frac{2\pi}{8}n3} - 2e^{j\frac{2\pi}{8}n5}) \\
&= \frac{1}{8}\sum_{k=0}^{7} X_8[k]e^{j\frac{2\pi k}{8}n}
\end{aligned}$$

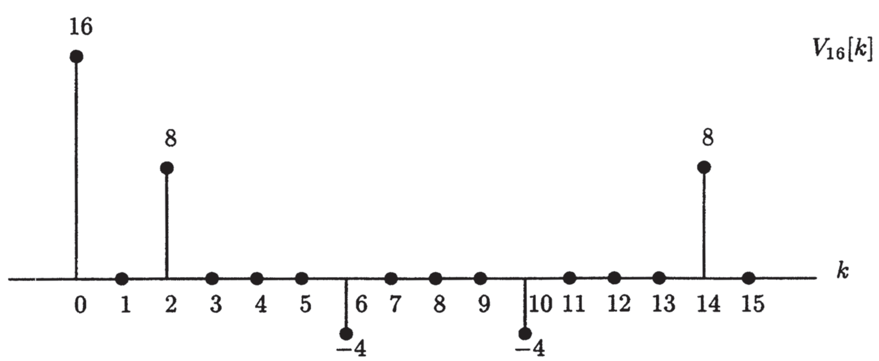
We thus get the following plot for $X_8[k]$:



(b) Now let $n = 0, ..., 15$, we can write $v[n]$ as:

$$\begin{aligned}
v[n] &= 1 + \frac{1}{2}(e^{j\frac{\pi}{4}n} + e^{-j\frac{\pi}{4}n}) - \frac{1}{4}(e^{j\frac{3\pi}{4}n} + e^{-j\frac{3\pi}{4}n}) \\
&= 1 + \frac{1}{2}e^{j\frac{2\pi}{16}n2} + \frac{1}{2}e^{j\frac{2\pi}{16}n14} - \frac{1}{4}e^{j\frac{2\pi}{16}n6} - \frac{1}{4}e^{j\frac{2\pi}{16}n10} \\
&= \frac{1}{16}(16 + 8e^{j\frac{2\pi}{16}n2} + 8e^{j\frac{2\pi}{16}n14} - 4e^{j\frac{2\pi}{16}n6} - 4e^{j\frac{2\pi}{16}n10}) \\
&= \frac{1}{16}\sum_{k=0}^{15} V_{16}[k]e^{j\frac{2\pi k}{16}n}
\end{aligned}$$

We thus get the following plot for $V_{16}[k]$:



(c)

$$|X_{16}[k]| = X(e^{j\omega})|_{\omega=\frac{2\pi}{16}k} \qquad 0 \le k \le 15$$

where $X(e^{j\omega})$ is the Fourier transform of $x[n]$.
Note that $x[n]$ can be expressed as:

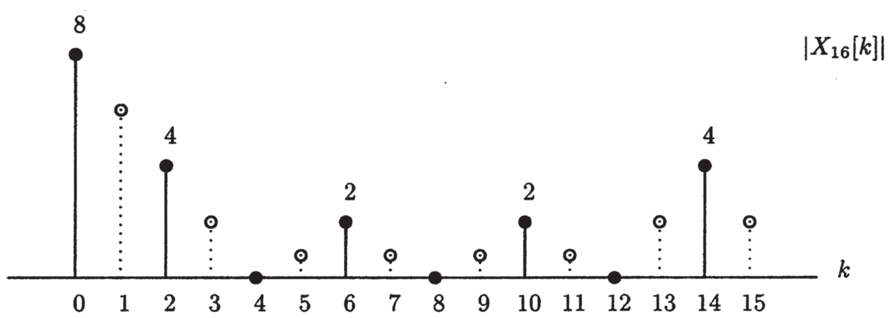
$$x[n] = y[n]w[n]$$

where:

$$y[n] = 1 + \cos(\frac{\pi n}{4}) - \frac{1}{2}\cos(\frac{3\pi n}{4})$$

and $w[n]$ is an eight-point rectangular window.
$|X_{16}[k]|$ will therefore have as its even points the sequence $|X_8[k]|$ .The odd points will correspond to the bandlimited interpolation between the even-point samples. The values that we can find exactly by inspection are thus:

$$|X_{16}[k]| = |X_8[k/2]| \qquad k = 0, 2, 4, ..., 14.$$



8.54. We wish to verify the identity of Eq. (8.7):

$$\frac{1}{N}\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-r)n} = \begin{cases} 1, & k-r = mN,\ m: \text{ integer} \\ 0, & \text{otherwise} \end{cases}$$

(a) For $k - r = mN$,

$$\begin{aligned} e^{j\frac{2\pi}{N}(k-r)n} &= e^{j\frac{2\pi}{N}(mN)n} \\ & e^{j2\pi mn} \\ & (1)^{mn} \end{aligned}$$

Since m and n are integers;

$$e^{j\frac{2\pi}{N}(k-r)n} = 1 \text{ , for } k-r = mN$$

So,

$$\begin{aligned} \frac{1}{N}\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-r)n} &= \frac{1}{N}\sum_{n=0}^{N-1} 1 \\ &= 1 \text{ , for } k-r = mN \end{aligned}$$

(b)

$$\frac{1}{N}\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}ln} = \frac{1 - e^{j\frac{2\pi}{N}lN}}{1 - e^{j\frac{2\pi}{N}l}}$$

This closed form solution is indeterminate for $l = mN$ only.
For the case when $l = mN$, we use L'Hôpital's Rule to find:

$$\begin{aligned} \lim_{l \to mN} \frac{1 - e^{j2\pi l}}{1 - e^{j\frac{2\pi}{N}l}} &= \left[\frac{-j2\pi e^{j2\pi l}}{-j\frac{2\pi}{N}e^{j\frac{2\pi}{N}l}}\right]_{l=mN} \\ &= N \end{aligned}$$

(c) For the case when $k - r \neq mN$:

$$\begin{aligned} \frac{1}{N}\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-r)n} &= \frac{1 - e^{j2\pi(k-r)}}{1 - e^{j\frac{2\pi}{N}(k-r)}} \\ &= 0 \end{aligned}$$

Note that the denominator is nonzero, while the numerator will always be zero for $k - r \neq mN$.

8.55. (a) We know from Eq. (8.11) that if $\tilde{x}_1[n] = \tilde{x}[n-m]$, we have:

$$\tilde{X}_1[k] = \sum_{n=0}^{N-1} \tilde{x}[n-m] W_N^{kn}$$

If we substitute $r = n - m$ into this equation, we get:

$$\begin{aligned} \tilde{X}_1[k] &= \sum_{r=-m}^{N-1-m} \tilde{x}[r] W_N^{k(r+m)} \\ &= W_N^{km} \sum_{r=-m}^{N-1-m} \tilde{x}[r] W_N^{kr} \end{aligned}$$

(b) We can decompose the summation from part (a) into

$$\tilde{X}_1[k] = W_N^{km} \left[\sum_{r=-m}^{-1} \tilde{x}[r] W_N^{kr} + \sum_{r=0}^{N-1-m} \tilde{x}[r] W_N^{kr} \right]$$

Using the fact that $\tilde{x}[r]$ and W_N^{kr} are periodic with period N:

$$\sum_{r=-m}^{-1} \tilde{x}[r] W_N^{kr} = \sum_{r=-m}^{-1} \tilde{x}[r+N] W_N^{k(r+N)}$$

Substituting $\ell = r + N$

$$\sum_{r=-m}^{-1} \tilde{x}[r] W_N^{kr} = \sum_{\ell=N-m}^{N-1} \tilde{x}[\ell] W_N^{k\ell}$$

(c) Using the result from part (b):

$$\begin{aligned} \tilde{X}_1[k] &= W_N^{km} \left[\sum_{r=N-m}^{N-1} \tilde{x}[r] W_N^{kr} + \sum_{r=0}^{N-m-1} \tilde{x}[r] W_N^{kr} \right] \\ &= W_N^{km} \sum_{r=0}^{N-1} \tilde{x}[r] W_N^{kr} \\ &= W_N^{km} \tilde{X}[k] \end{aligned}$$

Hence, if $\tilde{x}_1[n] = \tilde{x}[n-m]$, then $\tilde{X}_1[k] = W_N^{km} \tilde{X}[k]$.

8.56. (a) 1. The DFS of $\tilde{x}^*[n]$ is given by:

$$\begin{aligned}\sum_{n=0}^{N-1} \tilde{x}^*[n] W_N^{kn} &= \left(\sum_{n=0}^{N-1} \tilde{x}[n] W_N^{-kn}\right)^* \\ &= \tilde{X}^*[-k]\end{aligned}$$

2. The DFS of $\tilde{x}^*[-n]$:

$$\begin{aligned}\sum_{n=0}^{N-1} \tilde{x}^*[-n] W_N^{kn} &= \left(\sum_{l=-N+1}^{0} \tilde{x}[l] W_N^{kl}\right)^* \\ &= X^*[k]\end{aligned}$$

3. The DFS of $Re\{\tilde{x}[n]\}$:

$$\begin{aligned}\sum_{n=0}^{N-1} \frac{\tilde{x}[n] + \tilde{x}^*[n]}{2} W_N^{kn} &= \frac{1}{2}\left(\tilde{X}[k] + \tilde{X}^*[-k]\right) \\ &= \tilde{X}_e[k]\end{aligned}$$

4. The DFS of $jIm\{\tilde{x}[n]\}$:

$$\begin{aligned}\sum_{n=0}^{N-1} \frac{\tilde{x}[n] - \tilde{x}^*[n]}{2} W_N^{kn} &= \frac{1}{2}\left(\tilde{X}[k] - \tilde{X}^*[-k]\right) \\ &= \tilde{X}_0[k]\end{aligned}$$

(b) Consider $\tilde{x}[n]$ real:
1.

$$Re\{\tilde{X}[k]\} = \frac{\tilde{X}[k] + \tilde{X}^*[k]}{2}$$

From part (a), if $\tilde{x}[n]$ is real,

$$\begin{aligned}\text{DFS }\{\tilde{x}[n]\} &= \text{DFS }\{\tilde{x}^*[n]\} \\ \text{DFS }\{\tilde{x}[-n]\} &= \text{DFS }\{\tilde{x}^*[-n]\}\end{aligned}$$

So,

$$\begin{aligned}\tilde{X}[k] &= \tilde{X}^*[-k] \\ \tilde{X}[-k] &= \tilde{X}^*[k] \\ Re\{\tilde{X}[k]\} &= \frac{X^*[k] + X[-k]}{2} \\ &= Re\{\tilde{X}[-k]\}\end{aligned}$$

(i.e. the real part of $\tilde{X}[k]$ is even.)
2.

$$\begin{aligned}Im\{\tilde{X}[k]\} &= \frac{\tilde{X}[k] - \tilde{X}^*[k]}{2} \\ &= \frac{\tilde{X}^*[-k] - \tilde{X}[-k]}{2} \\ &= Im\{\tilde{X}[-k]\}\end{aligned}$$

(i.e., the imaginary part of $\tilde{X}[k]$ is odd.)
3.

$$\begin{aligned}|\tilde{X}[k]| &= \sqrt{\tilde{X}[k]\tilde{X}^*[k]} \\ &= \sqrt{\tilde{X}^*[-k]\tilde{X}[-k]} \\ &= |\tilde{X}[-k]|\end{aligned}$$

(i.e., the magnitude of $\tilde{X}[k]$ is even)

4.

$$\begin{aligned}
\angle\tilde{X}[k] &= \arctan\left(\frac{\mathcal{I}m\{\tilde{X}[k]\}}{\mathcal{R}e\{\tilde{X}[k]\}}\right) \\
&= \arctan\left(\frac{\mathcal{I}m\{\tilde{X}[-k]\}}{\mathcal{R}e\{\tilde{X}[-k]\}}\right) \\
&= -\angle\tilde{X}[-k]
\end{aligned}$$

(i.e., the angle of $\tilde{X}[k]$ is odd.)

8.57. 1. Let $x[n]$ $(0 \le n \le N-1)$ be one period of the periodic sequence $\tilde{x}[n]$. The Fourier transform of this periodic sequence can be expressed as:

$$\tilde{X}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \tilde{x}[n]e^{-j\omega n}$$

Recall the synthesis equation, Eq. (8.12):

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k]W_N^{-kn}$$

Substitution yields:

$$\tilde{X}(e^{j\omega}) = \sum_{n=-\infty}^{\infty}\left(\frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]W_N^{-kn}\right)e^{-j\omega n}$$

Rearranging the summations and combining terms:

$$\tilde{X}(e^{j\omega}) = \sum_{k=0}^{N-1}\tilde{X}[k]\left(\frac{1}{N}\sum_{n=-\infty}^{\infty} e^{j(\frac{2\pi k}{N}-\omega)n}\right)$$

The infinite summation is recognized as an impulse at $\omega = (2\pi k/N)$:

$$\tilde{X}(e^{j\omega}) = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]\delta\left(\omega - \frac{2\pi k}{N}\right)$$

2. Since $x[n]$ corresponds to one period of $\tilde{x}[n]$, we must apply a rectangular window (unit amplitude and length N) to the periodic sequence. Thus, to extract one period from $\tilde{x}[n]$:

$$x[n] = \tilde{x}[n]w[n]$$

where,

$$w[n] = \begin{cases} 1, & 0 \le n \le (N-1) \\ 0, & \text{otherwise} \end{cases}$$

The window has a Fourier transform:

$$\begin{aligned} W(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} w[n]e^{-j\omega n} \\ &= \sum_{n=0}^{N-1} e^{-j\omega n} \\ &= \frac{1-e^{-j\omega N}}{1-e^{-j\omega}} \quad = \frac{e^{-j\omega\frac{N}{2}}}{e^{-j\frac{\omega}{2}}}\frac{e^{j\frac{\omega N}{2}} - e^{-j\frac{\omega N}{2}}}{e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}} \\ &\quad = e^{-j\omega(\frac{N-1}{2})}\frac{\sin\left(\omega\frac{N}{2}\right)}{\sin\left(\frac{\omega}{2}\right)} \end{aligned}$$

3. Since $x[n] = \tilde{x}[n]w[n]$, the Fourier transform of $x[n]$ can be represented by the periodic convolution (see Eq. (8.28)).

$$X(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]\delta\left(\theta - \frac{2\pi k}{N}\right)\frac{\sin\left[\frac{N}{2}(w-\theta)\right]}{\sin\left(\frac{\omega-\theta}{2}\right)}e^{-j(\frac{N-1}{2})(\omega-\theta)}$$

Integration over $-\pi \le \theta \le \pi$ reduces to the summation (note the impulse train):

$$\tilde{X}(e^{j\omega}) = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]\frac{\sin\left[(N\omega - 2\pi k)/2\right]}{\sin\left[\left(\omega - \frac{2\pi k}{N}\right)/2\right]}e^{-j(\frac{N-1}{2})(\omega - \frac{2\pi k}{N})}$$

Hence, the Fourier transform is obtained from the DFS via an interpolation formula.

8.58. The N-point DFT of the N-pt sequence, $x[n]$ is given by

$$\begin{aligned} X[k] &= \sum_{n=0}^{N-1} x[n]W_N^{kn}, \qquad 0 \le k \le (N-1) \\ X[0] &= \sum_{n=0}^{N-1} x[n] \end{aligned}$$

(a) Suppose $x[n] = -x[N-1-n]$. For N even, all elements of $x[n]$ will cancel with an antisymmetric component. For N odd, all elements have a counterpart with opposite sign. However, $x[(N-1)/2]$ must also be zero.

Therefore, for $x[n] = -x[N-1-n]$, $X[0] = 0$.

(b) Suppose $x[n] = x[N-1-n]$ and N even.

$$\begin{aligned} X[N/2] &= \sum_{n=0}^{N-1} x[n]W_N^{(N/2)n} \\ &= \sum_{n=0}^{N-1} x[n](-1)^n \\ &= x[0] - x[1] + x[2] - x[3] + \cdots + x[N-2] - x[N-1] \end{aligned}$$

Since $x[n] = x[N-1-n]$, then

$$\begin{aligned} x[0] &= x[N-1] \\ x[1] &= x[N-2] \\ &\vdots \end{aligned}$$

Therefore, $X[N/2] = 0$.

8.59. (a) The conjugate-symmetric part of a sequence:

$$x_e[n] = \frac{1}{2}\Big(x[n] + x^*[-n]\Big)$$

The periodic conjugate-symmetric part:

$$x_{ep}[n] = \frac{1}{2}\Big(x[((n))_N] + x^*[((-n))_N]\Big), \qquad 0 \le n \le (N-1)$$

Note that:

$$\begin{aligned} x[((n))_N] &= x[n], \quad 0 \le n \le (N-1) \\ x^*[((-n))_N] &= x^*[-n+N] + x^*[0]\delta[n] - x^*[0]\delta[n-N] \end{aligned}$$

Substituting into $x_{ep}[n]$:

$$x_{ep}[n] = \frac{1}{2}\Big[x[n] + x^*[-n+N] + x^*[0]\delta[n] - x^*[0]\delta[n-N]\Big], \qquad 0 \le n \le (N-1)$$

Since,

$$\begin{aligned} x_e[n] &= \frac{1}{2}\Big(x[n] + x^*[-n]\Big) \\ &= \frac{1}{2}\Big(x[n] + x^*[0]\delta[n]\Big), \quad 0 \le n \le (N-1) \\ \text{and } x_e[n-N] &= \frac{1}{2}\Big(x[n-N] + x^*[N-n]\Big) \\ &= \frac{1}{2}\Big(-x^*[0]\delta[n-N] + x^*[N-n]\Big) \quad 0 \le n \le (N-1) \end{aligned}$$

We can combine to get:

$$x_{ep}[n] = x_e[n] + x_e[n-N] \qquad 0 \le n \le (N-1)$$

The periodic conjugate-antisymmetric part is given as

$$x_{op}[n] = (x_o[n] + x_o[n-N]), \qquad 0 \le n \le (N-1)$$

Recall that the odd part can be expressed as

$$x_o[n] = \frac{1}{2}\Big(x[n] - x^*[-n]\Big)$$

So,

$$x_o[n-N] = \frac{1}{2}\Big(x[n-N] - x^*[N-n]\Big)$$

For $0 \le n \le (N-1)$:

$$\begin{aligned} x_o[n] &= \frac{1}{2}\Big(x[n] - x^*[0]\delta[n]\Big), \quad 0 \le n \le N-1 \\ x_o[n-N] &= \frac{1}{2}\Big(x[0]\delta[n-N] - x^*[N-n]\Big) \end{aligned}$$

From the definition of $x_{op}[n]$:

$$\begin{aligned} x_{op}[n] &= \frac{1}{2}\Big(x[((n))_N] - x^*[((-n))_N]\Big), \quad 0 \le n \le (N-1) \\ &= \frac{1}{2}\Big(x[n] - x^*[0]\delta[n] + x^*[0]\delta[n-N] - x^*[N-n]\Big), \quad 0 \le n \le (N-1) \end{aligned}$$

Recognizing the expressions for $x_o[n]$ and $x_o[n-N]$ in $x_{op}[n]$, we have

$$x_{op}[n] = x_o[n] + x_o[n-N], \qquad 0 \le n \le (N-1)$$

(b) $x[n]$ **is a sequence of length** N**; however,**

$$x[n] = \begin{cases} x_1[n], & 0 \le n \le N/2 \\ 0, & N/2 \le n \le N-1 \end{cases}$$

The even part: (assume N is even)

$$x_e[n] = \begin{cases} \dfrac{x[n]}{2} + \dfrac{x^*[0]\delta[n]}{2}, & 0 \le n \le N/2 \\ \dfrac{x^*[-n]}{2}, & -N/2 \le n \le -1 \\ 0, & \text{otherwise} \end{cases}$$

From part (a):

$$x_{ep}[n] = x_e[n] + x_e[n-N], \qquad 0 \le n \le (N-1)$$

Because $x[n] = 0$ for $|n| \ge N/2$,

$$x_{ep}[n] = x_e[n], \qquad 0 \le n \le (N/2-1)$$

Also, since $x_e[n] = x_e^*[-n]$,

$$x_e[n] = x_{ep}^*[-n], \qquad -N/2 < n \le -1$$

To conclude:

$$x_e[n] = \begin{cases} x_{ep}[n], & 0 \le n \le N/2 \\ \dfrac{x_{ep}[n]}{2}, & n = N/2 \\ x_{ep}^*[-n], & -N/2 < n \le -1 \\ \dfrac{x_{ep}^*[-n]}{2}, & n = -N/2 \end{cases}$$

8.60.

$$\sum_{n=0}^{N-1} |x[n]|^2 = \sum_{n=0}^{N-1} x[n]x^*[n]$$

From the synthesis equation:

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{-kn}$$

Hence,

$$x^*[n] = \frac{1}{N}\sum_{k=0}^{N-1} X^*[k]W_N^{kn}$$

substituting:

$$\begin{aligned}
\sum_{n=0}^{N-1} |x[n]|^2 &= \sum_{n=0}^{N-1} x[n]\left(\frac{1}{N}\sum_{k=0}^{N-1} X^*[k]W_N^{kn}\right) \\
&= \frac{1}{N}\sum_{k=0}^{N-1} X^*[k]\left(\sum_{n=0}^{N-1} x[n]W_N^{kn}\right) \\
&= \frac{1}{N}\sum_{k=0}^{N-1} X^*[k]X[k] \\
\sum_{n=0}^{N-1} |x[n]|^2 &= \frac{1}{N}\sum_{k=0}^{N-1} |X[k]|^2
\end{aligned}$$

8.61. (a) This statement is TRUE:

$$\begin{aligned}
X[k] &= X(e^{j\omega})|_{\omega=2\pi k/N} \\
&= B\left(\frac{2\pi k}{N}\right) e^{j(2\pi/N)k\alpha} \\
A[k] &= B\left(\frac{2\pi k}{N}\right) \\
\gamma &= \frac{2\pi\alpha}{N}
\end{aligned}$$

(b) This statement is FALSE:
Suppose $x[n] = \delta[n] + \frac{1}{2}\delta[n-1]$,

$$\begin{aligned}
X[k] &= 1 + \frac{1}{2}e^{-j\pi k} \\
&= 1 + \frac{1}{2}(-1)^k
\end{aligned}$$

Expressed in the form

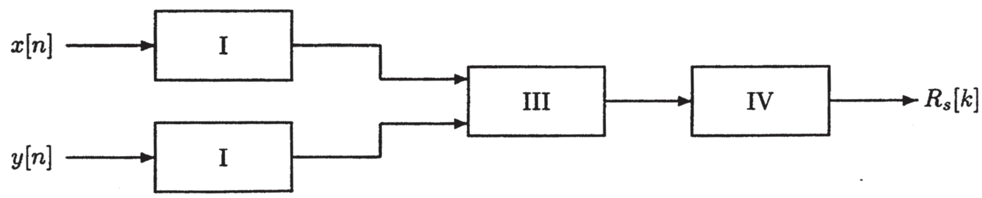
$$\begin{aligned}
X[k] &= A[k]e^{j\gamma k}, \\
A[k] &= 1 + \frac{1}{2}(-1)^k \\
\text{and } \gamma &= 0
\end{aligned}$$

The Fourier transform of $x[n]$ is $X(e^{j\omega}) = 1 + \frac{1}{2}e^{j\omega}$, which cannot be expressed in the form $X(e^{-j\omega}) = B(\omega)e^{j\alpha\omega}$.

8.62. We desire 128 samples of $X(e^{j\omega})Y(e^{j\omega})$.

Since $x[n]$ and $y[n]$ are 256 points long, the linear convolution, $x[n] * y[n]$, will be 512 points long.

We are given a 128-pt DFT only. Therefore, we must time-alias to get 128 samples. The most efficient implementation is:



Total cost = 110.

8.63. Ideally, the inverse system would be:

$$H_i(z) = z - bz^{-1}$$

Hence,

$$X(z) = (z - bz^{-1})Y(z)$$

and

$$x[n] = y[n+1] - by[n-1], \qquad -\infty \leq n \leq \infty$$

If we use an N-pt block of $y[n]$:

$$y[n]: \qquad 0 \leq n \leq N,$$

then

$$V[k] = (W_N^{-k} - bW_N^{k})Y[k]$$

and

$$v[n] = y[((n+1))_N] - by[((n-1))_N]$$

Because the shift is circular, the points at $n = 0$ and $n = (N-1)$ will not be correct. Therefore, only the points in the range $1 \leq n \leq (N-2)$ are valid.

8.64. (a)

$$\begin{aligned} X[k] &= \sum_{n=0}^{N-1} x[n]e^{-j(2\pi/N)kn}, \quad 0 \le k \le (N-1) \\ X_M[k] &= \sum_{n=0}^{N-1} x[n]e^{-j(2\pi k/N+\pi/N)n} \\ &= \sum_{n=0}^{N-1} x[n]e^{-j(\pi n/N)}e^{-j(2\pi/N)kn} \end{aligned}$$

So, $x_M[n] = x[n]e^{-j(\pi n/N)}$.

(b)

$$\begin{aligned} X_M[N-k] &= \sum_{n=0}^{N-1} x[n]e^{-j(\pi n/N)}e^{-j(2\pi/N)(N-k)n}, \quad 0 \le k \le (N-1) \\ &= \sum_{n=0}^{N-1} x[n]e^{-j(\pi n/N)}e^{j(2\pi/N)k} \\ X_M[N-(k+1)] &= \sum_{n=0}^{N-1} x[n]e^{-j(\pi n/N)}e^{-j(2\pi/N)(N-k-1)n} \\ &= \sum_{n=0}^{N-1} x[n]e^{-j(\pi n/N)}e^{-j(2\pi/N)(N-1)n}e^{j(2\pi/N)kn} \\ &= \sum_{n=0}^{N-1} x[n]e^{j(\pi n/N)}e^{j(2\pi/N)kn} \\ &= X_M^*[k] \end{aligned}$$

So,

$$X_M[k] = X_M^*[N-(k+1)] \text{ , for } 0 \le k \le (N-1) \text{ and } x[n] \text{ real.}$$

(c) (i) $N-k-1$ is odd when k is even. If $R[k] = X_M[2k]$, we may obtain $X_M[k]$ from $R[k]$ as follows:

$$G[k] = \begin{cases} R[k/2], & \text{k even} \\ R^*[(N-(k+1))/2], & \text{k odd} \end{cases}$$

where we note that

$$R^*[(N-(k+1))/2] = X_M^*[N-(k+1)]$$

for k odd.

(ii)

$$\begin{aligned} R[k] &= X_M[2k] \\ &= \sum_{n=0}^{(N/2)-1} x[n]e^{-j(4\pi k/N+\pi/N)n} \\ &= \sum_{n=0}^{N-1} x[n]e^{-j(\pi n/N)}e^{-j\left(\frac{2\pi kn}{N/2}\right)} \\ &= \sum_{n=0}^{(N/2)-1} x[n]e^{-j(\pi n/N)}e^{-j\left(\frac{2\pi kn}{N/2}\right)} + \sum_{n=N/2}^{N-1} x[n]e^{-j(\pi n/N)}e^{-j\left(\frac{2\pi kn}{N/2}\right)} \\ &= \sum_{n=0}^{(N/2)-1} \left(x[n]e^{-j(\pi n/N)} + x[n+N/2]e^{-j(\pi n/N)}e^{-j(\pi/2)}\right)e^{-j\left(\frac{2\pi kn}{N/2}\right)} \end{aligned}$$

So,

$$r[n] = (x[n] - jx[n+N/2])e^{-j(\pi n/N)}, \quad 0 \le n \le \left(\frac{N}{2}-1\right)$$

(d)

$$\begin{aligned} X_{3M}[k] &= X_{1M}[k]X_{2M}[k] \\ X_{3M}[n] &= \sum_{r=0}^{N-1} x_{1M}[r]x_{2M}[((n-r))_N] \end{aligned}$$

From part (a):

$$\begin{aligned} x_{1M}[n] &= x_1[n]e^{-j(\pi n/N)} \\ x_{2M}[n] &= x_2[n]e^{-j(\pi n/N)} \\ x_{3M}[n] &= x_3[n]e^{-j(\pi n/N)} \end{aligned}$$

So,

$$\begin{aligned} x_3[n] &= e^{j(\pi n/N)} \sum_{r=0}^{N-1} x_1[r]x_2[((n-r))_N]e^{-j(\pi/N)[((n-r))_N+r]} \\ &= \sum_{r=0}^{N-1} x_1[r]x_2[((n-r))_N]e^{-j(\pi/N)[((n-r))_N-(n-r)]} \end{aligned}$$

Since,

$$((n-r))_N = \begin{cases} n-r, & n \geq r \\ N+n-r, & n < r \end{cases}$$

$$((n-r))_N - (n-r) = \begin{cases} 0, & n \geq r \\ N, & n < r \end{cases}$$

then

$$e^{-j(\pi/N)[((n-r))_N-(n-r)]} = \text{sgn}\,[n-r] = \begin{cases} 1, & n \geq r \\ -1, & n < r \end{cases}$$

and

$$x_3 = \sum_{r=0}^{N-1} x_1[r]x_2[((n-r))_N]\text{sgn}\,[n-r]$$

(e) Suppose, that for $n \geq N/2$:

$$\begin{aligned} x_{1M}[n] &= x_1[n]e^{-j(\pi n/N)} = 0 \\ x_{2M}[n] &= x_2[n]e^{-j(\pi n/N)} = 0 \end{aligned}$$

then the modified circular convolution is equivalent to the modified linear convolution:

$$x_{1M}[n] \, \textcircled{N} \, x_{2M}[n] = x_{1M}[n] * x_{2M}[n]$$

(i.e. no aliasing occurs.)

$$\begin{aligned} x_{3M}[n] &= x_{1M}[n] * x_{2M}[n] \\ &= \sum_{r=0}^{N-1} x_{1M}[r]x_{2M}[n-r] \end{aligned}$$

Thus,

$$\begin{aligned} x_3[n] &= e^{j(\pi n/N)} \sum_{r=0}^{N-1} x_1[n]x_2[n-r]e^{-j(\pi n/N)}e^{-j(\pi n/N)(n-r)} \\ &= \sum_{r=0}^{N-1} x_1[r]x_2[n-r]e^{-j(\pi/N)(n-r)} \end{aligned}$$

So,

$$x_3[n] = \sum_{r=0}^{N-1} x_1[r]x_2[n-r]e^{-j(\pi/N)(n-r)} = x_1[n] * x_2[n]$$

8.65. (a) We wish to compute $x[n] \, 63 \, h[n]$:

$$\begin{aligned}
\text{let } x_1[n] &= x[n], && 0 \leq n \leq 31 \\
x_2[n] &= x[n+32], && 0 \leq n \leq 30 \\
h_1[n] &= h[n], && 0 \leq n \leq 31 \\
h_2[n] &= h[n+32], && 0 \leq n \leq 30
\end{aligned}$$

$$\begin{aligned}
x[n] * h[n] &= x_1[n] * h_1[n] + x_1[n] * h_2[n] * \delta[n-32] + x_2[n] * h_1[n] * \delta[n-32] \\
&\quad + x_2[n] * h_2[n] * \delta[n-32] * \delta[n-32]
\end{aligned}$$

Let

$$\begin{aligned}
y_1[n] &= x_1[n] * h_1[n] &&= x_1[n] \, 64 \, h_1[n] \\
y_2[n] &= x_1[n] * h_2[n] &&= x_1[n] \, 64 \, h_2[h] \\
y_3[n] &= x_2[n] * h_1[n] &&= x_2[n] \, 64 \, h_1[n] \\
y_4[n] &= x_2[n] * h_2[n] &&= x_2[n] \, 64 \, h_2[n]
\end{aligned}$$

We can compute each of the above circular convolutions with two 64-pt DFTs and one 64-pt inverse DFT.

$$\begin{aligned}
y[n] &= x[n] * h[n] \\
&= y_1[n] + y_2[n-32] + y_3[n-32] + y_4[n-64]
\end{aligned}$$

So

$$x[n] \, 63 \, h[n] = y[n] + y[n+63], \qquad 0 \leq n \leq 62$$

The total computational cost is 12 DFTs of size $N = 64$.

(b) Using two 128-pt DFTs and one 128-pt inverse DFT:

$$y[n] = x[n] \, 128 \, h[n] = x[n] * h[n]$$

The 63-pt circular convolution:

$$x[n] \, 63 \, h[n] = y[n] + y[n+63], \qquad 0 \leq n \leq 62$$

(c) Using the 64-pt DFT method of part (a):

$$\#\text{mult} = 4(12)(64 \log_2(64)) = 18432$$

Using 128-pt DFTs:

$$\#\text{mult} = 4(3)(128 \log_2(128)) = 10752$$

Direct convolution:

$$\#\text{mult} = 2 \sum_{n=1}^{63} n - 63 = 3969$$

8.66. From each circular convolution, the first 49 points will be incorrect. Therefore, we get 51 good points and the input must be overlapped by $100 - 51 = 49$ points.

(a) $V = 49$

(b) $M = 51$

(c) The points extracted correspond to the range $49 \leq n \leq 99$.

Distorting filter: $h[n] = \delta[n] - \frac{1}{2}\delta[n - n_0]$

8.67. (a) The Z-transform of $h[n]$

$$\begin{aligned} H(z) &= \sum_{n=-\infty}^{\infty} h[n]z^{-n} \\ H(z) &= 1 - \frac{1}{2}z^{-n_0} \end{aligned}$$

The N-pt DFT of $h[n]$: $(N = 4n_0)$

$$\begin{aligned} H[k] &= \sum_{n=0}^{4n_0-1} h[n]W_{4n}^{kn_0}, \quad 0 \le k \le (4n_0 - 1) \\ &= 1 - \frac{1}{2}W_{4n_0}^{kn_0} \\ H[k] &= 1 - \frac{1}{2}e^{-j(\pi/2)k} \end{aligned}$$

(b)

$$\begin{aligned} H_i(z) &= \frac{1}{1 + 1/2z^{-n_0}}, \quad |z| > \left(\frac{1}{2}\right)^{-n_0} \text{ for causality} \\ h_i[n] &= \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{n/n_0} \delta[n - kn_0] \end{aligned}$$

The filter is IIR.

(c)

$$G[k] = \frac{1}{H[k]} = \frac{1}{1 - e^{-j(\pi/2)k}}, \quad 0 \le k \le (4n_0 - 1)$$

The impulse response, $g[n]$, is just $h_i[n]$ time-aliased by $4n_0$ points:

$$\begin{aligned} g[n] &= \left(1 + \frac{1}{16} + \frac{1}{256} + \cdots\right)\delta[n] + \left(\frac{1}{2} + \frac{1}{32} + \frac{1}{512} + \cdots\right)\delta[n - n_0] \\ &+ \left(\frac{1}{4} + \frac{1}{64} + \frac{1}{1024} + \cdots\right)\delta[n - 2n_0] + \left(\frac{1}{8} + \frac{1}{128} + \frac{1}{2048} + \cdots\right)\delta[n - 3n_0] \\ g[n] &= \frac{16}{15}\delta[n] + \frac{8}{15}\delta[n - n_0] + \frac{4}{15}\delta[n - 2n_0] + \frac{2}{15}\delta[n - 3n_0] \end{aligned}$$

(d) Indeed,

$$G[k]H[k] = 1, \quad 0 \le k \le (4n_0 - 1)$$

However, this relationship is only true at $4n_0$ distinct frequencies. This fact does not imply that for all ω:

$$G(e^{j\omega})H(e^{j\omega}) = 1$$

(e)

$$\begin{aligned} y[n] &= g[n] * h[n] \\ &= \frac{16}{15}\delta[n] + \frac{8}{15}\delta[n - n_0] + \frac{4}{15}\delta[n - 2n_0] + \frac{2}{15}\delta[n - 3n_0] - \frac{8}{15}\delta[n - n_0] \\ &\quad - \frac{4}{15}\delta[n - 2n_9] - \frac{2}{15}\delta[n - 3n_0] - \frac{1}{15}\delta[n - 4n_0] \\ y[n] &= \frac{16}{15}\delta[n] - \frac{1}{15}\delta[n - 4n_0] \end{aligned}$$

8.68. (a) We start by computing $\tilde{X}_H[k+N]$:

$$\begin{aligned}
\tilde{X}_H[k+N] &= \sum_{n=0}^{N-1} \tilde{x}[n] H_N[n(k+N)] \\
&= \sum_{n=0}^{N-1} \tilde{x}[n](\cos(\frac{2\pi(nk+nN)}{N}) + \sin(\frac{2\pi(nk+nN)}{N}) \\
&= \sum_{n=0}^{N-1} \tilde{x}[n](\cos(\frac{2\pi nk}{N}) + \sin(\frac{2\pi nk}{N})) \\
&= \sum_{n=0}^{N-1} \tilde{x}[n] H_N[nk] \\
&= \tilde{X}_H[k].
\end{aligned}$$

We thus conclude that the DHS coefficients form a sequence that is also periodic with period N.

(b) We have:

$$\begin{aligned}
\frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}_H[k] H_N[nk] &= \frac{1}{N}\sum_{k=0}^{N-1}(\sum_{m=0}^{N-1} \tilde{x}[m] H_N[mk]) H_N[nk] \\
&= \frac{1}{N}\sum_{m=0}^{N-1} \tilde{x}[m] \sum_{k=0}^{N-1} H_N[mk] H_N[nk] \\
&= \frac{1}{N}\tilde{x}[n] N \\
&= \tilde{x}[n].
\end{aligned}$$

Where we have used the fact that $\sum_{k=0}^{N-1} H_N[mk]H_N[nk] = N$ only if $((m))_N = ((n))_N$, otherwise it's 0.
This completes the derivation of the DHS synthesis formula.

(c) We have:

$$\begin{aligned}
H_N[a+N] &= \cos(\frac{2\pi(a+N)}{N}) + \sin(\frac{2\pi(a+N)}{N}) \\
&= \cos(\frac{2\pi a}{N} + 2\pi) + \sin(\frac{2\pi a}{N} + 2\pi) \\
&= \cos(\frac{2\pi a}{N}) + \sin(\frac{2\pi a}{N}) \\
&= H_N[a].
\end{aligned}$$

And:

$$\begin{aligned}
H_N[a+b] &= \cos(\frac{2\pi(a+b)}{N}) + \sin(\frac{2\pi(a+b)}{N}) \\
&= (C_N[a]C_N[b] - S_N[a]S_N[b]) + (S_N[a]C_N[b] + C_N[a]S_N[b]) \\
&= C_N[b](C_N[a] + S_N[a]) + S_N[b](-S_N[a] + C_N[a]) \\
&= C_N[b](C_N[a] + S_N[a]) + S_N[b](S_N[-a] + C_N[-a]) \\
&= C_N[b]H_N[a] + S_N[b]H_N[-a] \\
&= C_N[a]H_N[b] + S_N[a]H_N[-b] \text{ (since } H_N[a+b] = H_N[b+a] \text{)}
\end{aligned}$$

Where we have used trigonometric properties.

(d) We have:

$$\begin{aligned}
DHS(\tilde{x}[n-n_0]) &= \sum_{n=0}^{N-1} \tilde{x}[n-n_0]H_N[nk] \\
&= \sum_{n=-n_0}^{N-1-n_0} \tilde{x}[n]H_N[(n+n_0)k] \\
&= \sum_{n=-n_0}^{N-1-n_0} \tilde{x}[n](H_N[nk]C_N[n_0k] + H_N[-nk]S_N[n_0k]) \\
&= C_N[n_0k]\sum_{n=-n_0}^{N-1-n_0} \tilde{x}[n]H_N[nk] + S_N[n_0k]\sum_{n=-n_0}^{N-1-n_0} \tilde{x}[n]H_N[-nk] \\
&= C_N[n_0k]\sum_{n=0}^{N-1} \tilde{x}[n]H_N[nk] + S_N[n_0k]\sum_{n=0}^{N-1} \tilde{x}[n]H_N[-nk] \\
&= C_N[n_0k]\tilde{X}_H[k] + S_N[n_0k]\tilde{X}_H[-k]
\end{aligned}$$

Where we have used the periodicity of $H_N[nk]$ and $\tilde{x}[n]$.

(e) We have:

$$\begin{aligned}
DHT\{\tilde{x}_3[n]\} &= DHT\{\sum_{m=0}^{N-1} x_1[m]x_2[((n-m))_N]\} \\
X_{H3}[k] &= \sum_{n=0}^{N-1}(\sum_{m=0}^{N-1} x_1[m]x_2[((n-m))_N])H_N[nk] \\
&= \sum_{m=0}^{N-1} x_1[m]\sum_{n=0}^{N-1} x_2[((n-m))_N]H_N[nk] \\
&= \sum_{m=0}^{N-1} x_1[m]DHT\{x_2[((n-m))_N]\} \\
&= \sum_{m=0}^{N-1} x_1[m](X_{H2}[k]C_N[mk] + X_{H2}[((-k))_N]S_N[mk]) \text{ (using P8.65-7)} \\
&= \sum_{m=0}^{N-1} x_1[m]X_{H2}[k]C_N[mk] + \sum_{m=0}^{N-1} x_1[m]X_{H2}[((-k))_N]S_N[mk] \\
&= \sum_{m=0}^{N-1} x_1[m]X_{H2}[k](\frac{H_N[mk]+H_N[-mk]}{2}) \\
&\quad + \sum_{m=0}^{N-1} x_1[m]X_{H2}[((-k))_N](\frac{H_N[mk]-H_N[-mk]}{2}) \\
&= \frac{1}{2}X_{H2}[k](X_{H1}[k] + X_{H1}[((-k))_N]) + \frac{1}{2}X_{H2}[((-k))_N](X_{H1}[k] - X_{H1}[((-k))_N]) \\
&= \frac{1}{2}X_{H1}[k](X_{H2}[k] + X_{H2}[((-k))_N]) + \frac{1}{2}X_{H1}[((-k))_N](X_{H2}[k] - X_{H2}[((-k))_N])
\end{aligned}$$

This is the desired convolution property.

(f) Since the DFT of $x[n]$ is given by:

$$\begin{aligned}
X[k] &= \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi kn}{N}} \\
&= \sum_{n=0}^{N-1} x[n](\cos(-\frac{2\pi kn}{N}) + j\sin(-\frac{2\pi kn}{N}))
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{N-1} x[n](\cos(\frac{2\pi kn}{N}) - j\sin(\frac{2\pi kn}{N})) \\
&= \sum_{n=0}^{N-1} x[n](C_N[kn] - jS_N[kn])
\end{aligned}
$$

then:

$$
\begin{aligned}
\sum_{n=0}^{N-1} x[n]C_N[kn] &= \frac{1}{2}(X[k] + X[((-k))_N]) \\
\sum_{n=0}^{N-1} x[n]S_N[kn] &= -\frac{1}{2j}(X[k] - X[((-k))_N])
\end{aligned}
$$

We thus get:

$$
\begin{aligned}
X_H[k] &= \sum_{n=0}^{N-1} x[n](C_N[kn] + S_N[kn]) \\
&= \frac{1}{2}(X[k] + X[((-k))_N]) - \frac{1}{2j}(X[k] - X[((-k))_N]) \\
&= (\frac{1}{2} - \frac{1}{2j})X[k] + (\frac{1}{2} + \frac{1}{2j})X[((-k))_N]
\end{aligned}
$$

This allows us to obtain $X_H[k]$ from $X[k]$.

(g) We have:

$$
X_H[k] = \sum_{n=0}^{N-1} x[n](C_N[kn] + S_N[kn])
$$

Therefore:

$$
\begin{aligned}
\sum_{n=0}^{N-1} x[n]C_N[kn] &= \frac{1}{2}(X_H[k] + X_H[((-k))_N]) \\
\sum_{n=0}^{N-1} x[n]S_N[kn] &= \frac{1}{2}(X_H[k] - X_H[((-k))_N])
\end{aligned}
$$

We thus get:

$$
\begin{aligned}
X[k] &= \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi kn}{N}} \\
&= \sum_{n=0}^{N-1} x[n](C_N[kn] - jS_N[kn]) \\
&= \frac{1}{2}(X_H[k] + X_H[((-k))_N]) - j\frac{1}{2}(X_H[k] - X_H[((-k))_N]) \\
&= (\frac{1}{2} - \frac{j}{2})X_H[k] + (\frac{1}{2} + \frac{j}{2})X_H[((-k))_N]
\end{aligned}
$$

This allows us to obtain $X[k]$ from $X_H[k]$.

8.69. (a) The DTFT is given by:

$$\begin{aligned}\tilde{X}(e^{jw}) &= X(e^{jw}) + X(e^{jw})e^{-jwN} \\ &= X(e^{jw})(1+e^{-jwN})\end{aligned}$$

The DFT is just samples of the DTFT:

$$\begin{aligned}\tilde{X}[k] &= \tilde{X}(e^{jw})|_{w=\frac{2\pi k}{2N}} \\ &= X(e^{j2\pi k2N})(1+(-1)^k)\end{aligned}$$

Therefore:

$$\tilde{X}[k] = \begin{cases} 2X[\frac{k}{2}] &, \ k \text{ even} \\ 0 &, \ k \text{ odd} \end{cases}$$

(b) The original system computes the following:

$$\tilde{X}[k]H[k] = \begin{cases} 2X[\frac{k}{2}]H[k] &, \ k \text{ even} \\ 0 &, \ k \text{ odd} \end{cases}$$

We thus want:

$$\begin{aligned}X[k]G[k] &= 2X[k]H[2k] & k = 0, ..., N-1 \\ G[k] &= 2H[2k] \\ &= 2\sum_{n=0}^{2N-1} h[n]e^{-j\frac{2\pi 2kn}{2N}} & k = 0, ..., N-1 \\ g[n] &= 2(h[n] + h[n+N])\end{aligned}$$

System A time aliases and multiplies by 2.

For system B, we need:

$$Y[k] = \begin{cases} W[\frac{k}{2}] &, \ k \text{ even} \\ 0 &, \ k \text{ odd} \end{cases}$$

Thus:

$$y[n] = \begin{cases} w[n] &, \ 0 \le n \le N-1 \\ w[n-N] &, \ N \le n \le 2N-1 \\ 0 &, \ \text{otherwise} \end{cases}$$

System B regenerates the $2N$−point sequence by repeating w[n].

8.70. (a) We have:

$$|H(e^{jw})| = \begin{cases} 1 & , \quad |w| \leq \frac{\pi}{4} \\ 0 & , \quad \text{otherwise} \end{cases}$$

Since $h[n]$ is FIR, we assume it is non-zero over $0 \leq n \leq N$. The phase of $H(e^{jw})$ should be set such that $h[n]$ is symmetric about the center of its range, i.e. $\frac{N}{2}$. Therefore, the phase of $H(e^{jw})$ should be $e^{j\frac{wN}{2}}$. So one possible $H[k]$ may be:

$$H[k] = \begin{cases} e^{j\frac{2\pi}{4N}\frac{N}{2}k} & , \quad 0 \leq k \leq \frac{1}{8}4N \\ 0 & , \quad \text{otherwise} \\ e^{j\frac{2\pi}{4N}\frac{N}{2}k} & , \quad 4N - \frac{N}{2} \leq k \leq 4N \end{cases}$$

that is:

$$H[k] = \begin{cases} e^{j\frac{\pi}{4}k} & , \quad 0 \leq k \leq \frac{N}{2} \\ 0 & , \quad \text{otherwise} \\ e^{j\frac{\pi}{4}k} & , \quad 4N - \frac{N}{2} \leq k \leq 4N \end{cases}$$

(b) System A needs to perform the following operations:

$$Y_2[k] = \begin{cases} X[k]H'[k] & , \quad 0 \leq k \leq \frac{N}{2} \\ 0 & , \quad \text{otherwise} \\ X[k-3N]H'[k-3N] & , \quad 4N - \frac{N}{2} \leq k \leq 4N \end{cases}$$

Where $H'[k]$ is the N-point DFT of $h[n]$.

(c) It is cheaper to implement N-point DFTs than $4N$-point DFTs, therefore the implementation in Figure P8.67-2 is usually preferable to the one in Figure P8.67-1.

8.71. Substituting the expression for $X_1[k]$ from equation (8.164) into equation (8.165), we get:

$$\begin{aligned} x_1[n] &= \frac{1}{2N-2}\sum_{k=0}^{2N-3} X_1[k]e^{j2\pi kn/(2N-2)} \\ &= \frac{1}{2N-2}\left(\sum_{k=0}^{N-1} X^{c1}[k]e^{j2\pi kn/(2N-2)} + \sum_{k=N}^{2N-3} X^{c1}[2N-2-k]e^{j2\pi kn/(2N-2)}\right) \end{aligned}$$

Note that:

$$\begin{aligned} \sum_{k=N}^{2N-3} X^{c1}[2N-2-k]e^{j2\pi kn/(2N-2)} &= \sum_{r=1}^{N-2} X^{c1}[r]e^{j2\pi(2N-2-r)n/(2N-2)} \\ &= \sum_{k=1}^{N-2} X^{c1}[k]e^{-j2\pi kn/(2N-2)} \end{aligned}$$

therefore:

$$\begin{aligned} x_1[n] &= \frac{1}{2N-2}\left(\sum_{k=0}^{N-1} X^{c1}[k]e^{j2\pi kn/(2N-2)} + \sum_{k=1}^{N-2} X^{c1}[k]e^{-j2\pi kn/(2N-2)}\right) \\ &= \frac{1}{2N-2}\left(X^{c1}[0] + X^{c1}[N-1]e^{j2\pi n} + \sum_{k=1}^{N-2} X^{c1}[k]\left(e^{j2\pi kn/(2N-2)} + e^{-j2\pi kn/(2N-2)}\right)\right) \\ &= \frac{1}{2N-2}\left(X^{c1}[0] + X^{c1}[N-1]e^{j\pi n} + \sum_{k=1}^{N-2} X^{c1}[k]2\cos\left(\frac{\pi kn}{N-1}\right)\right) \end{aligned}$$

and:

$$\begin{aligned} x[n] &= x_1[n] && \text{for } n = 0, 1, ..., N-1 \\ &= \frac{1}{N-1}\left(\sum_{k=0}^{N-1} \alpha[k]X^{c1}[k]\cos\left(\frac{\pi kn}{N-1}\right)\right) && 0 \le n \le N-1 \end{aligned}$$

where $\alpha[k]$ is given by:

$$\alpha[k] = \begin{cases} \frac{1}{2} & , \quad k = 0 \text{ and } N-1 \\ 1 & , \quad 1 \le k \le N-2. \end{cases}$$

This completes the derivation.

8.72.

$$v[n] = x_2[2n]$$

therefore, for $k = 0, 1, ..., N-1$:

$$V[k] = \frac{1}{2}(X_2[k] + X_2[k+N]).$$

Using equation (8.168), we have:

$$\begin{aligned}
V[k] &= \frac{1}{2}(X_2[k] + X_2[k+N]) \\
&= e^{j\frac{\pi k}{2N}} Re\{X[k]e^{-j\frac{\pi k}{2N}}\} + e^{j\frac{\pi(k+N)}{2N}} Re\{X[k+N]e^{-j\frac{\pi(k+N)}{2N}}\} \\
&= e^{j\frac{\pi k}{2N}} Re\{X[k]e^{-j\frac{\pi k}{2N}}\} + e^{j\frac{\pi}{2}} e^{j\frac{\pi k}{2N}} Re\{X[k+N]e^{-j\frac{\pi k}{2N}} e^{-j\frac{\pi}{2}}\} \\
&= e^{j\frac{\pi k}{2N}} Re\{X[k]e^{-j\frac{\pi k}{2N}}\} + je^{j\frac{\pi k}{2N}} Re\{-jX[k+N]e^{-j\frac{\pi k}{2N}}\} \\
&= e^{j\frac{\pi k}{2N}} (Re\{X[k]e^{-j\frac{\pi k}{2N}}\} + jIm\{X[k+N]e^{-j\frac{\pi k}{2N}}\}.
\end{aligned}$$

Using the above expression, we get:

$$\begin{aligned}
2Re\{e^{-j\frac{2\pi k}{4N}} V[k]\} &= 2Re\{e^{-j\frac{\pi k}{2N}} e^{j\frac{\pi k}{2N}} (Re\{X[k]e^{-j\frac{\pi k}{2N}}\} + jIm\{X[k+N]e^{-j\frac{\pi k}{2N}}\})\} \\
&= 2Re\{X[k]e^{-j\frac{\pi k}{2N}}\} \\
&= X^{c2}[k] \qquad \text{where we used equation (8.170).}
\end{aligned}$$

Furthermore, we have:

$$\begin{aligned}
2Re\{e^{-j\frac{2\pi k}{4N}} V[k]\} &= 2Re\{e^{-j\frac{2\pi k}{4N}} \sum_{n=0}^{N-1} v[n]e^{-j\frac{2\pi kn}{N}}\} \\
&= 2Re\{\sum_{n=0}^{N-1} v[n]e^{-j(\frac{2\pi k}{N}(n+\frac{1}{4}))}\} \\
&= 2\sum_{n=0}^{N-1} Re\{v[n]e^{-j(\frac{2\pi k}{N}(n+\frac{1}{4}))}\} \\
&= 2\sum_{n=0}^{N-1} v[n]\cos(\frac{2\pi k}{N}(n+\frac{1}{4})) \\
&= 2\sum_{n=0}^{N-1} v[n]\cos(\frac{\pi k(4n+1)}{2N}).
\end{aligned}$$

and:

$$\begin{aligned}
2Re\{e^{-j\frac{2\pi k}{4N}} V[k]\} &= 2Re\{X[k]e^{-j\frac{\pi k}{2N}}\} \\
&= 2Re\{\sum_{n=0}^{2N-1} x[n]e^{-j\frac{2\pi kn}{2N}} e^{-j\frac{\pi k}{2N}}\} \\
&= 2Re\{\sum_{n=0}^{N-1} x[n]e^{-j\frac{\pi k}{2N}(2n+1)}\} \\
&= 2\sum_{n=0}^{N-1} Re\{x[n]e^{-j\frac{\pi k}{2N}(2n+1)}\} \\
&= 2\sum_{n=0}^{N-1} x[n]\cos(\frac{\pi k(2n+1)}{2N}).
\end{aligned}$$

From the results above, we conclude, for $k = 0, 1, ..., N-1$:

$$\begin{aligned}
X^{c2}[k] &= 2Re\{e^{-j\frac{2\pi k}{4N}}V[k]\} \\
&= 2\sum_{n=0}^{N-1} v[n]\cos(\frac{\pi k(4n+1)}{2N}) \\
&= 2\sum_{n=0}^{N-1} x[n]\cos(\frac{\pi k(2n+1)}{2N}).
\end{aligned}$$

8.73. Substituting the expression for $X_2[k]$ from equation (8.174) into equation (8.175), we get:

$$\begin{aligned}
x_2[n] &= \frac{1}{2N}\sum_{k=0}^{2N-1} X_2[k]e^{j2\pi kn/(2N)} \\
&= \frac{1}{2N}(X^{c2}[0] + \sum_{k=1}^{N-1} X^{c2}[k]e^{j\pi k/(2N)}e^{j2\pi kn/(2N)} - \sum_{k=N+1}^{2N-1} X^{c2}[2N-k]e^{j\pi k/(2N)}e^{j2\pi kn/(2N)}) \\
&= \frac{1}{2N}(X^{c2}[0] + \sum_{k=1}^{N-1} X^{c2}[k]e^{j\pi k(2n+1)/(2N)} - \sum_{k=N+1}^{2N-1} X^{c2}[2N-k]e^{j\pi k(2n+1)/(2N)}) \\
&= \frac{1}{2N}(X^{c2}[0] + \sum_{k=1}^{N-1} X^{c2}[k]e^{j\pi k(2n+1)/(2N)} - \sum_{k=1}^{N-1} X^{c2}[k]e^{j\pi(2N-k)(2n+1)/(2N)}) \\
&= \frac{1}{2N}(X^{c2}[0] + \sum_{k=1}^{N-1} X^{c2}[k]e^{j\pi k(2n+1)/(2N)} + \sum_{k=1}^{N-1} X^{c2}[k]e^{-j\pi k(2n+1)/(2N)}) \\
&= \frac{1}{2N}(X^{c2}[0] + \sum_{k=1}^{N-1} X^{c2}[k](e^{j\pi k(2n+1)/(2N)} + e^{-j\pi k(2n+1)/(2N)})) \\
&= \frac{1}{2N}(X^{c2}[0] + \sum_{k=1}^{N-1} X^{c2}[k]\cos(\frac{\pi k(2n+1)}{2N})).
\end{aligned}$$

Furthermore:

$$\begin{aligned}
x[n] &= x_2[n] && \text{for } n = 0, 1, ..., N-1 \\
&= \frac{1}{N}\sum_{k=0}^{N-1} \beta[k]X^{c2}[k]\cos(\frac{\pi k(2n+1)}{2N}) && 0 \le n \le N-1
\end{aligned}$$

where $\beta[k]$ is given by:

$$\beta[k] = \begin{cases} \frac{1}{2} &, \quad k = 0 \\ 1 &, \quad 1 \le k \le N-1. \end{cases}$$

This completes the derivation.

8.74. First we derive Parseval's theorem for the DFT.
Let $x[n]$ be an N point sequence and define $y[n]$ as follows:

$$y[n] = x[n] \textcircled{N} x^*[((-n))_N].$$

Using the properties of the DFT, we have:

$$Y[k] = X[k]X^*[k] = |X[k]|^2.$$

Note that:

$$y[0] = \sum_n |x[n]|^2$$

and using the DFT synthesis equation, we get:

$$y[0] = \frac{1}{N}\sum_{k=0}^{N-1} Y[k].$$

Parseval's Theorem for the DFT is therefore:

$$\sum_n |x[n]|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X[k]|^2.$$

(a) Note that:

$$\sum_{n=0}^{N-1} |X^{c1}[k]|^2 = \sum_{n=0}^{N-1} |X_1[k]|^2$$

and, using equation (8.164):

$$\sum_{n=0}^{2N-3} |X_1[k]|^2 = 2\sum_{n=0}^{N-1} |X^{c1}[k]|^2 - |X^{c1}[0]|^2 - |X^{c1}[N-1]|^2.$$

Using the DFT properties:

$$\sum_n |x_1[n]|^2 = \frac{1}{2N-2}\sum_{k=0}^{2N-3} |X_1[k]|^2$$

and, using equation (8.161):

$$\sum_{n=0}^{2N-3} |x_1[n]|^2 = 2\sum_{n=0}^{N-1} |x[n]|^2 - |x[0]|^2 - |x[N-1]|^2.$$

We thus conclude:

$$\frac{1}{2N-2}(2\sum_{n=0}^{N-1} |X^{c1}[k]|^2 - |X^{c1}[0]|^2 - |X^{c1}[N-1]|^2) = 2\sum_{n=0}^{N-1} |x[n]|^2 - |x[0]|^2 - |x[N-1]|^2.$$

(b) Using equation (8.171),

$$\sum_{n=0}^{N-1} |X^{c2}[k]|^2 = \sum_{n=0}^{N-1} |X_2[k]|^2.$$

Note that, using equation (8.167):

$$\sum_{k=0}^{2N-1} |X_2[k]|^2 = 2\sum_{k=0}^{N-1} |X[k]|^2 - |X[0]|^2,$$

and, using equation (8.166):

$$\sum_{n=0}^{2N-1} |x_2[n]|^2 = 2\sum_{n=0}^{N-1} |x[n]|^2.$$

Using the DFT properties:

$$\sum_{n=0}^{2N-1} |x_2[n]|^2 = \frac{1}{2N}\sum_{k=0}^{2N-1} |X_2[k]|^2.$$

We thus conclude:

$$\frac{1}{2N}(2\sum_{k=0}^{N-1} |X[k]|^2 - |X[0]|^2) = 2\sum_{n=0}^{N-1} |x[n]|^2.$$