

Linear Algebra: Matrices

A matrix is simply a rectangular array of numbers and the *dimension* of a matrix is written as *rows* \times *columns*. For the example below, the dimension of the matrix would be 2×3 . Often times you will see the matrix dimensionality depicted as $\mathbb{R}^{2 \times 3}$ where \mathbb{R} simply denotes a *real number*.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

To reference a specific element in a matrix we would specify the row and column. For the example below, matrix A , A_{12} would give us the number in the 1st row and 2nd column which would be the number 2. The indexes of matrices can be *1-indexed* or *0-indexed*, meaning, the first element (row or column) starts at either 1 or 0. Matrices are also, by convention, referenced with *uppercase* letters and numbers and vectors with *lowercase* letters.

$$A = \begin{bmatrix} 10 & 24 & 35 \\ 34 & 15 & 76 \end{bmatrix} \therefore A_{12} = 24$$

Matrix Addition & Subtraction

When adding or subtracting one matrix from another we simply add or subtract each number in each column with it's corresponding number in the other matrix. Also, both matrices must be of the same dimension. For example:

$$\begin{bmatrix} 1 & 3 & 5 \\ 7 & 9 & 2 \\ 3 & 4 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 2 & 0 \\ 3 & 1 & 4 \\ 6 & 9 & 8 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 5 \\ 10 & 10 & 6 \\ 9 & 13 & 9 \end{bmatrix}$$

Matrix Multiplication and Division

To multiply or divide two matrices the number of columns in one matrix must be equal to the number of rows in the other matrix.

To multiply matrices you will need to multiply the first number of the *multiplier* vector by the first number of in the first column of the *multiplicand* vector. Then, multiply the second number in the multiplier vector by the second number in the multiplicand vector and add the the products which becomes the first value in the product vector. Do this for each number in the multiplicand vector. For example:

$$\begin{bmatrix} 1 & 3 \\ 4 & 0 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} (1 \cdot 1) + (3 \cdot 5) \\ (4 \cdot 1) + (0 \cdot 5) \\ (2 \cdot 1) + (1 \cdot 5) \end{bmatrix} = \begin{bmatrix} 16 \\ 4 \\ 7 \end{bmatrix}$$

Here is an example with a larger multiplier matrix:

$$\begin{bmatrix} 2 & 3 \\ 7 & 9 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 2 & 5 & 1 \\ 3 & 6 & 2 \end{bmatrix} = \begin{bmatrix} (2 \cdot 2) + (3 \cdot 3) \\ (5 \cdot 7) + (6 \cdot 9) \\ (1 \cdot 3) + (2 \cdot 4) \end{bmatrix} = \begin{bmatrix} 13 \\ 89 \\ 11 \end{bmatrix}$$

Example With Hypothesis Function

Below is an example of using matrices to compute predictions for a hypothesis function. This is the preferred way to solve using code (such as Python) as it is more computationally efficient.

Given a set of features:

2014, 1416, 1534, 852

And a hypothesis:

$$h_{\theta}(x) = -40 + 0.25x$$

Then, using matrices:

$$\begin{bmatrix} 1 & 2014 \\ 1 & 1416 \\ 1 & 1534 \\ 1 & 852 \end{bmatrix} \cdot \begin{bmatrix} -40 \\ 0.25 \end{bmatrix} = \begin{bmatrix} 463.50 \\ 314.00 \\ 343.50 \\ 173.00 \end{bmatrix}$$

Scalar Multiplication and Division

Scalar multiplication and division are performed the same way. Below is an example of multiplying a matrix by a scalar, in this case, the number 2:

$$2 \cdot \begin{bmatrix} 2 & 3 & 5 \\ 7 & 9 & 2 \\ 3 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 6 & 10 \\ 14 & 18 & 4 \\ 6 & 4 & 2 \end{bmatrix}$$

Combination of Operands

Just as with any mathematical equation you must follow the order of operations. For example, with the below:

$$\begin{aligned} & 3 \cdot \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} - \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} \div 3 \\ &= \begin{bmatrix} 3 \\ 12 \\ 6 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ \frac{2}{3} \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 12 \\ 10\frac{1}{3} \end{bmatrix} \end{aligned}$$

Identity Matrix

Just as the number 1 in multiplication is the identity property:

$$3 \cdot 1 = 1 \cdot 3 = 3$$

Matrices also have an identity which has all zeros and number 1 in the diagonal. These matrices are denoted by the capital letter I. For example, the below is an identity matrix for a 2x2 matrix:

$$I_{2 \times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

When we apply this to another 2x2 matrix the result is the same as the original matrix:

$$\begin{aligned} & \text{If } A = \begin{bmatrix} 3 & 4 \\ 2 & 16 \end{bmatrix} \\ & \text{And } I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Then $A(I) = A$

$$\begin{bmatrix} 3 & 4 \\ 2 & 16 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 16 \end{bmatrix}$$

Special Matrix Operations

Matrix Inverse

Just as many numbers have an inverse such as:

$$3(3^{-1}) = 1 \text{ or } 34(34^{-1}) = 1$$

Matrices can also have an inverse:

$$A(A^{-1}) = A^{-1}(A) = I$$

For example:

$$\begin{bmatrix} 3 & 4 \\ 2 & 16 \end{bmatrix} \cdot \begin{bmatrix} 0.4 & -0.1 \\ -0.05 & 0.075 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

However, some numbers, such as 0 do not have an inverse:

$$0(0^{-1}) = \text{undefined}$$

In the same way the inverse of a matrix with all zeros is undefined. Such matrices are called *singular* or *degenerate*:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Matrix Transposition

This is simply converting matrix rows to columns, for example:

$$A = \begin{bmatrix} 3 & 4 \\ 2 & 16 \end{bmatrix}$$

Transposed, it would be:

$$A^T = \begin{bmatrix} 3 & 2 \\ 4 & 16 \end{bmatrix}$$

Therefore:

$$A_{m \times n} = A_{n \times m}^T$$

For example:

$$A_{2,1} = A_{1,2}^T = 4$$