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8 Polar Coordinates, Parametric Equations, and Vectors

In this chapter we study different ways of describing points and curves in the plane. We are already familiar with *rectangular coordinates* (Section 1.9). Using rectangular coordinates is like describing a location in a city by saying that it's at the corner of 2nd Street and 4th Avenue: such directions would be useful to a person who is driving on the city streets. But we may also describe this same location "as the crow flies;" we can say, for example, that it is 1.5 miles northeast of City Hall. These directions would help a drone or a hot-air balloon pilot find the location. That's what we do using *polar coordinates*—we specify the location of a point in the plane by giving its distance and direction from a fixed reference point. We also study *parametric equations*: such equations allow us to track the location of a moving point in the coordinate plane. For example, parametric equations can model the shape of a winding road as well as the location of a car on the road at any time. *Vectors* allow us to describe two quantities simultaneously, such as the magnitude and direction of a force at a given point in the coordinate plane.

8.1 Polar Coordinates

■ Definition of Polar Coordinates ■ Relationship Between Polar and Rectangular Coordinates ■ Polar Equations

In this section we define polar coordinates, and we learn how polar coordinates are related to rectangular coordinates.

■ Definition of Polar Coordinates

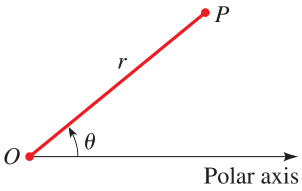

Figure 1

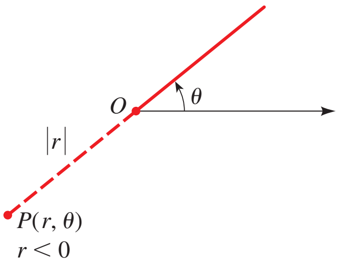

Figure 2

The **polar coordinate system** uses distances and directions to specify the location of a point in the plane. To set up this system, we choose a fixed point O in the plane called the **pole** (or **origin**) and draw from O a ray (half-line) called the **polar axis** as in Figure 1. Then each point P can be assigned polar coordinates $P(r, \theta)$, where

r is the *distance* from O to P

θ is the *angle* between the polar axis and the segment $\overline{OP}$

We use the convention that θ is positive if measured in a counterclockwise direction from the polar axis or negative if measured in a clockwise direction. If r is negative, then $P(r, \theta)$ is defined to be the point that lies $|r|$ units from the pole in the direction opposite to that given by θ (see Figure 2).

Example 1 ■ Plotting Points in Polar Coordinates

Plot the points whose polar coordinates are given.

(a) $(1, 3\pi/4)$ **(b)** $(3, -\pi/6)$ **(c)** $(3, 3\pi)$ **(d)** $(-4, \pi/4)$

Solution The points are plotted in Figure 3. The point in part (d) lies 4 units from the origin along the angle $5\pi/4$ because the given value of r is negative.

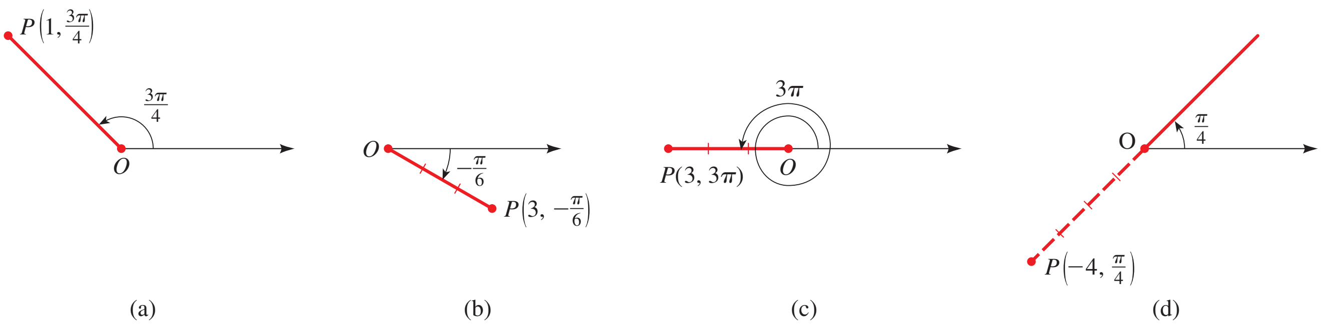

Figure 3

Now Try Exercises 5 and 7 ■

Note that the coordinates (r, θ) and $(-r, \theta + \pi)$ represent the same point, as shown in Figure 4. Moreover, because the angles $\theta + 2n\pi$ (where n is any integer) all have the

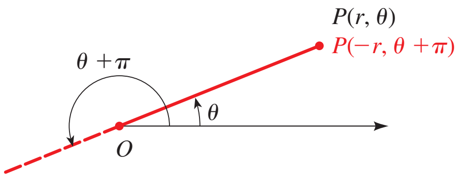

Figure 4

same terminal side as the angle θ, each point in the plane has infinitely many representations in polar coordinates. In fact, any point $P(r, \theta)$ can also be represented by

$$P(r, \theta + 2n\pi) \qquad \text{and} \qquad P(-r, \theta + (2n + 1)\pi)$$

for any integer n.

Example 2 ■ Different Polar Coordinates for the Same Point

(a) Graph the point with polar coordinates $P(2, \pi/3)$.

(b) Find two other polar coordinate representations of P with $r > 0$ and two with $r < 0$.

Solution

(a) The graph is shown in Figure 5(a).

(b) Other representations with $r > 0$ are

$$\left(2, \frac{\pi}{3} + 2\pi\right) = \left(2, \frac{7\pi}{3}\right) \qquad \text{Add } 2\pi \text{ to } \theta$$

$$\left(2, \frac{\pi}{3} - 2\pi\right) = \left(2, -\frac{5\pi}{3}\right) \qquad \text{Add } -2\pi \text{ to } \theta$$

Other representations with $r < 0$ are

$$\left(-2, \frac{\pi}{3} + \pi\right) = \left(-2, \frac{4\pi}{3}\right) \qquad \text{Replace } r \text{ by } -r \text{ and add } \pi \text{ to } \theta$$

$$\left(-2, \frac{\pi}{3} - \pi\right) = \left(-2, -\frac{2\pi}{3}\right) \qquad \text{Replace } r \text{ by } -r \text{ and add } -\pi \text{ to } \theta$$

The graphs in Figure 5 explain why these coordinates represent the same point.

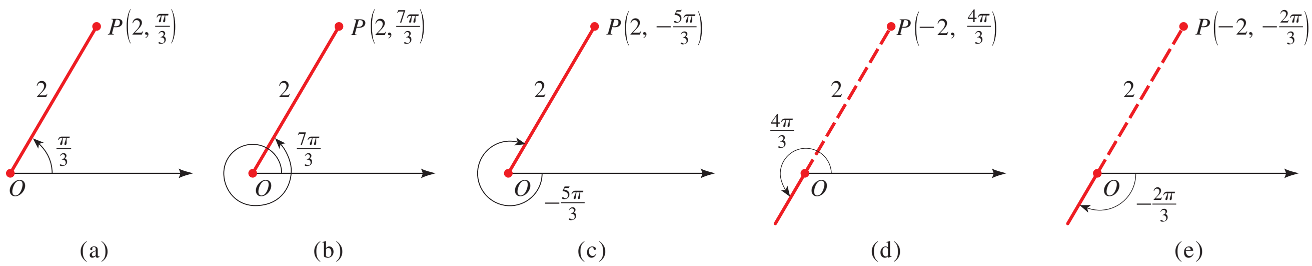

Figure 5

Now Try Exercise 11

■ Relationship Between Polar and Rectangular Coordinates

Situations often arise in which we need to consider polar and rectangular coordinates simultaneously. The connection between the two systems is illustrated in Figure 6 (on the next page), where the polar axis coincides with the positive x-axis. The formulas in the box are obtained from the figure using the definitions of the trigonometric functions and the Pythagorean Theorem. (Although we have pictured the case in which $r > 0$ and θ is acute, the formulas hold for any angle θ and for any value of r.)

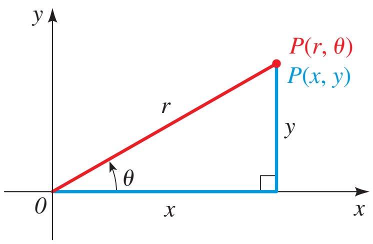

Figure 6

Relationship Between Polar and Rectangular Coordinates

1. To change from polar to rectangular coordinates, use the formulas

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta$$

2. To change from rectangular to polar coordinates, use the formulas

$$r^2 = x^2 + y^2 \qquad \text{and} \qquad \tan\theta = \frac{y}{x} \quad (x \neq 0)$$

Example 3 ■ Converting Polar Coordinates to Rectangular Coordinates

Find rectangular coordinates for the point that has polar coordinates $(4, 2\pi/3)$.

Solution Since $r = 4$ and $\theta = 2\pi/3$, we have

$$x = r\cos\theta = 4\cos\frac{2\pi}{3} = 4\cdot\left(-\frac{1}{2}\right) = -2$$

$$y = r\sin\theta = 4\sin\frac{2\pi}{3} = 4\cdot\frac{\sqrt{3}}{2} = 2\sqrt{3}$$

Thus the point has rectangular coordinates $(-2, 2\sqrt{3})$.

Now Try Exercise 29 ■

Example 4 ■ Converting Rectangular Coordinates to Polar Coordinates

Find polar coordinates for the point that has rectangular coordinates $(2, -2)$.

Solution Using $x = 2$, $y = -2$, we get

$$r^2 = x^2 + y^2 = 2^2 + (-2)^2 = 8$$

so $r = 2\sqrt{2}$ or $-2\sqrt{2}$. Also

$$\tan\theta = \frac{y}{x} = \frac{-2}{2} = -1$$

so $\theta = 3\pi/4$ or $-\pi/4$. Since the point $(2, -2)$ lies in Quadrant IV (see Figure 7), we can represent it in polar coordinates as $(2\sqrt{2}, -\pi/4)$ or $(-2\sqrt{2}, 3\pi/4)$.

Now Try Exercise 37 ■

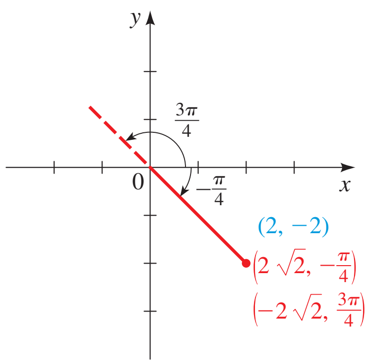

Figure 7

Discovery Project ■ Mapping the World

In the *Focus on Modeling* following Chapter 6 we learned how surveyors can make a map of a city or town. But mapping the whole world introduces a new difficulty. How is it possible to represent a *spherical* world on a *flat* map? This challenge was faced by Renaissance explorers and their mapmakers, who developed several ingenious solutions. In this project we see how polar coordinates and trigonometry can help us make a map of the whole world on a flat sheet of paper. You can find the project at **www.stewartmath.com**.

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Note that the equations relating polar and rectangular coordinates do not uniquely determine r or θ. When we use these equations to find the polar coordinates of a point, we must be careful that the values we choose for r and θ give us a point in the correct quadrant, as we did in Example 4.

Polar Equations

In Examples 3 and 4 we converted points from one coordinate system to the other. Now we consider the same problem for equations. A **polar equation** is an equation in the polar coordinates r and θ; similarly, a **rectangular equation** is an equation in the rectangular coordinates x and y.

Example 5 ■ Converting an Equation from Rectangular Coordinates to Polar Coordinates

Express the equation $x^2 = 4y$ in polar coordinates.

Solution We use the formulas $x = r\cos\theta$ and $y = r\sin\theta$.

$$\begin{aligned} x^2 &= 4y && \text{Rectangular equation} \\ (r\cos\theta)^2 &= 4(r\sin\theta) && \text{Substitute } x = r\cos\theta,\ y = r\sin\theta \\ r^2\cos^2\theta &= 4r\sin\theta && \text{Expand} \\ r &= 4\frac{\sin\theta}{\cos^2\theta} && \text{Divide by } r\cos^2\theta \\ r &= 4\sec\theta\tan\theta && \text{Simplify} \end{aligned}$$

Now Try Exercise 47 ■

As Example 5 shows, converting an equation from rectangular coordinates to polar coordinates is straightforward: Just replace x by $r\cos\theta$ and y by $r\sin\theta$, and then simplify. But converting an equation from polar to rectangular form often requires more thought.

Example 6 ■ Converting Equations from Polar Coordinates to Rectangular Coordinates

Express each polar equation in rectangular coordinates. If possible, determine the graph of the equation from its rectangular form.

(a) $r = 5\sec\theta$ **(b)** $r = 2\sin\theta$ **(c)** $r = 2 + 2\cos\theta$

Solution

(a) Since $\sec\theta = 1/\cos\theta$, we multiply both sides by $\cos\theta$.

$$\begin{aligned} r &= 5\sec\theta && \text{Polar equation} \\ r\cos\theta &= 5 && \text{Multiply by } \cos\theta \\ x &= 5 && \text{Substitute } x = r\cos\theta \end{aligned}$$

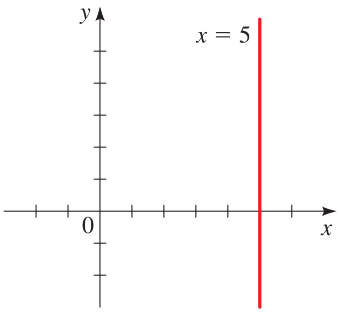


Figure 8

The graph of $x = 5$ is the vertical line in Figure 8.

(b) We multiply both sides of the equation by r, because then we can use the formulas $r^2 = x^2 + y^2$ and $r\sin\theta = y$.

$$\begin{aligned} r &= 2\sin\theta && \text{Polar equation} \\ r^2 &= 2r\sin\theta && \text{Multiply by } r \\ x^2 + y^2 &= 2y && r^2 = x^2 + y^2 \text{ and } r\sin\theta = y \\ x^2 + y^2 - 2y &= 0 && \text{Subtract } 2y \\ x^2 + (y-1)^2 &= 1 && \text{Complete the square in } y \end{aligned}$$

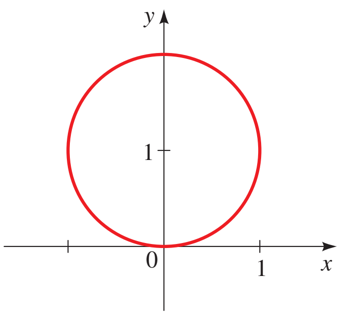


Figure 9

This is the equation of a circle of radius 1 centered at the point $(0, 1)$. It is graphed in Figure 9.

(c) We first multiply both sides of the equation by r:

$$r = 2 + 2\cos\theta$$
$$r^2 = 2r + 2r\cos\theta$$

Using $r^2 = x^2 + y^2$ and $x = r\cos\theta$, we can convert two terms in the equation into rectangular coordinates, but eliminating the remaining r requires more work.

$$x^2 + y^2 = 2r + 2x \qquad r^2 = x^2 + y^2 \text{ and } r\cos\theta = x$$
$$x^2 + y^2 - 2x = 2r \qquad \text{Subtract } 2x$$
$$(x^2 + y^2 - 2x)^2 = 4r^2 \qquad \text{Square both sides}$$
$$(x^2 + y^2 - 2x)^2 = 4(x^2 + y^2) \qquad r^2 = x^2 + y^2$$

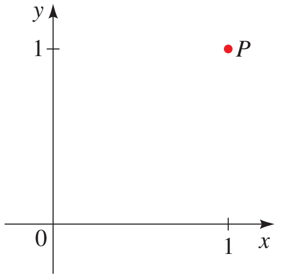
In this case the rectangular equation looks more complicated than the polar equation. Although we cannot easily determine the graph of the equation from its rectangular form, we will see in the next section how to graph it using the polar equation.

Now Try Exercises 57, 59, and 61 ■

8.1 Exercises

Concepts

1. We can describe the location of a point in the plane using different ________ systems. The point P shown in the figure has rectangular coordinates (,) and polar coordinates (,).



2. Let P be a point in the plane.

(a) If P has polar coordinates (r, θ), then it has rectangular coordinates (x, y) where $x =$ ________ and $y =$ ________.

(b) If P has rectangular coordinates (x, y), then it has polar coordinates (r, θ) where $r^2 =$ ________ and $\tan\theta =$ ________.

3–4 ■ *Yes or No*? If *No*, give a reason.

3. Do the polar coordinates $(2, \pi/6)$ and $(-2, 7\pi/6)$ represent the same point?

4. Do the equations relating polar and rectangular coordinates uniquely determine r and θ?

Skills

5–10 ■ **Plotting Points in Polar Coordinates** Plot the point that has the given polar coordinates.

5. $\left(2, \frac{\pi}{2}\right)$ **6.** $(1, 0)$ **7.** $\left(3, -\frac{\pi}{4}\right)$

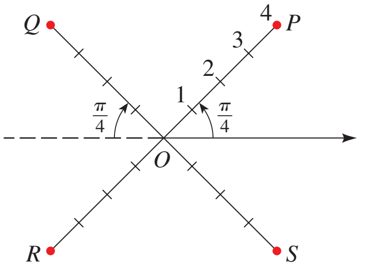
8. $\left(4, -\frac{5\pi}{6}\right)$ **9.** $(-2, 4\pi/3)$ **10.** $\left(-3, \frac{7\pi}{3}\right)$

11–16 ■ **Different Polar Coordinates for the Same Point** Plot the point that has the given polar coordinates. Then give two other polar coordinate representations of the point, one with $r < 0$ and the other with $r > 0$.

11. $(3, \pi/2)$ **12.** $(2, 3\pi/4)$ **13.** $(-1, 7\pi/6)$

14. $(-2, -\pi/3)$ **15.** $(-5, 0)$ **16.** $(3, 1)$

17–24 ■ **Points in Polar Coordinates** Determine which point in the figure—P, Q, R, or S—has the given polar coordinates.



17. $(4, 3\pi/4)$ **18.** $(4, -3\pi/4)$

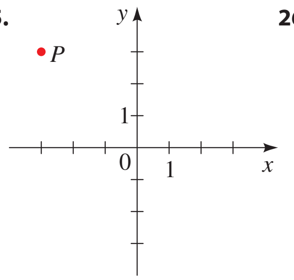
19. $(-4, -\pi/4)$ **20.** $(-4, 13\pi/4)$

21. $(4, -23\pi/4)$ **22.** $(-4, 23\pi/4)$

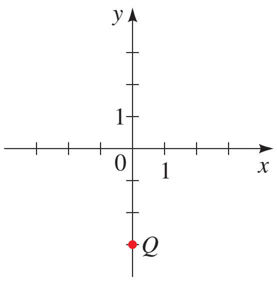
23. $(-4, 101\pi/4)$ **24.** $(4, 103\pi/4)$

25–26 ■ Rectangular Coordinates to Polar Coordinates A point is graphed in rectangular form. Find polar coordinates for the point, with $r > 0$ and $0 < \theta < 2\pi$.

25.

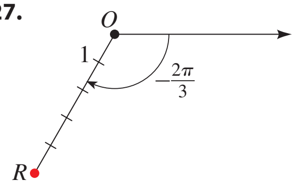


26.

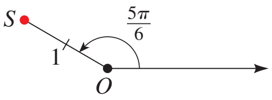


27–28 ■ Polar Coordinates to Rectangular Coordinates A point is graphed in polar form. Find its rectangular coordinates.

27.



28.



29–36 ■ Polar Coordinates to Rectangular Coordinates Find the rectangular coordinates for the point whose polar coordinates are given.

29. $(3, \pi/2)$ **30.** $(6, 2\pi/3)$

31. $(\sqrt{2}, -\pi/4)$ **32.** $(-1, 5\pi/2)$

33. $(5, 5\pi)$ **34.** $(0, 13\pi)$

35. $(\sqrt{3}, -\pi/6)$ **36.** $(2\sqrt{2}, -3\pi/4)$

37–44 ■ Rectangular Coordinates to Polar Coordinates Convert the rectangular coordinates to polar coordinates with $r > 0$ and $0 \le \theta < 2\pi$.

37. $(-1, 1)$ **38.** $(3\sqrt{3}, -3)$

39. $(\sqrt{8}, \sqrt{8})$ **40.** $(-\sqrt{6}, -\sqrt{2})$

41. $(3, 4)$ **42.** $(1, -2)$

43. $(-6, 0)$ **44.** $(0, -\sqrt{3})$

45–52 ■ Rectangular Equations to Polar Equations Convert the equation to polar form.

45. $x = y$ **46.** $x^2 + y^2 = 9$

47. $x = y^2$ **48.** $y = 5$

49. $x = 4$ **50.** $x^2 - y^2 = 1$

51. $x^2 + y^2 = y$ **52.** $(x^2 + y^2)^{3/2} = 6xy$

53–72 ■ Polar Equations to Rectangular Equations Convert the polar equation to rectangular coordinates.

53. $r = 7$ **54.** $r = -3$

55. $\theta = -\dfrac{\pi}{2}$ **56.** $\theta = \pi$

57. $r\cos\theta = 6$ **58.** $r = 2\csc\theta$

59. $r = 4\sin\theta$ **60.** $r = 6\cos\theta$

61. $r = 1 + \cos\theta$ **62.** $r = 3(1 - \sin\theta)$

63. $r = 1 + 2\sin\theta$ **64.** $r = 2 - \cos\theta$

65. $r = \dfrac{1}{\sin\theta - \cos\theta}$ **66.** $r = \dfrac{1}{1 + \sin\theta}$

67. $r = \dfrac{4}{1 + 2\sin\theta}$ **68.** $r = \dfrac{2}{1 - \cos\theta}$

69. $r^2 = \tan\theta$ **70.** $r^2 = \sin 2\theta$

71. $\sec\theta = 2$ **72.** $\cos 2\theta = 1$

Discuss ■ Discover ■ Prove ■ Write

73. Discuss ■ Prove: The Distance Formula in Polar Coordinates

(a) Use the Law of Cosines to prove that the distance between the polar points (r_1, θ_1) and (r_2, θ_2) is

$$d = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2 - \theta_1)}$$

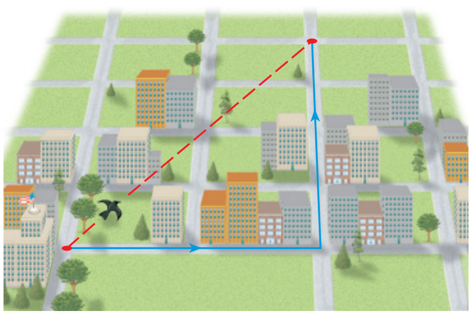
(b) Find the distance between the points whose polar coordinates are $(3, 3\pi/4)$ and $(-1, 7\pi/6)$, using the formula from part (a).

(c) Convert the points in part (b) to rectangular coordinates. Find the distance between them, using the usual Distance Formula. Do you get the same answer?

74. Discuss: Different Coordinate Systems Certain curves are more naturally described in one coordinate system than in another. In each of the following situations, which coordinate system would be appropriate: rectangular or polar? Give reasons to support your answer.

(a) You need to give driving directions to your home to a friend.

(b) You need to give directions to your home to a homing pigeon.



8.2 Graphs of Polar Equations

■ Graphing Polar Equations ■ Symmetry ■ Graphing Polar Equations with Graphing Devices

The **graph of a polar equation** $r = f(\theta)$ consists of all points P that have at least one polar representation (r, θ) whose coordinates satisfy the equation. Many curves that arise in mathematics and its applications are more easily and naturally represented by polar equations than by rectangular equations.

■ Graphing Polar Equations

A rectangular grid is helpful for plotting points in rectangular coordinates (see Figure 1(a)). To plot points in polar coordinates, it is convenient to use a grid consisting of circles centered at the pole and rays emanating from the pole, as shown in Figure 1(b). We will use such grids to help us sketch polar graphs.

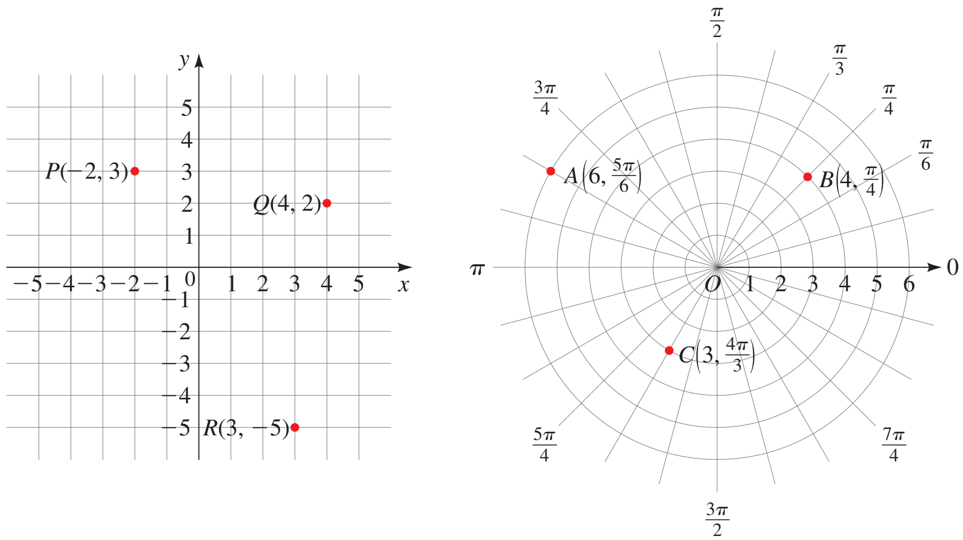


Figure 1 (a) Grid for rectangular coordinates (b) Grid for polar coordinates

In Examples 1 and 2 we see that circles centered at the origin and lines that pass through the origin have particularly simple equations in polar coordinates.

Example 1 ■ Sketching the Graph of a Polar Equation

Sketch a graph of the equation $r = 3$, and express the equation in rectangular coordinates.

Solution The graph consists of all points whose r-coordinate is 3, that is, all points that are 3 units away from the origin. So the graph is a circle of radius 3 centered at the origin, as shown in Figure 2.

Squaring both sides of the equation, we get

$$r^2 = 3^2 \qquad \text{Square both sides}$$

$$x^2 + y^2 = 9 \qquad \text{Substitute } r^2 = x^2 + y^2$$

So the equivalent equation in rectangular coordinates is $x^2 + y^2 = 9$.

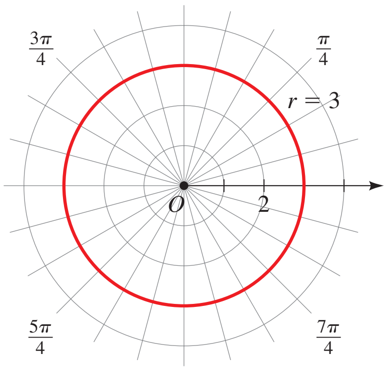


Figure 2

Now Try Exercise 17 ■

In general, the graph of the equation $r = a$ is a circle of radius $|a|$ centered at the origin. Squaring both sides of this equation, we see that the equivalent equation in rectangular coordinates is $x^2 + y^2 = a^2$.

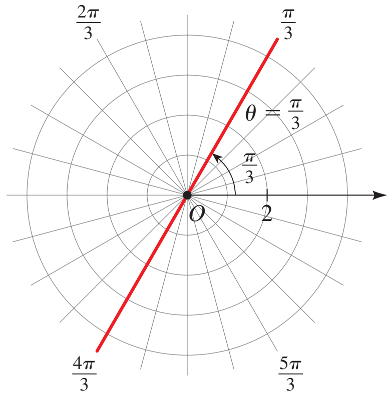


Figure 3

Example 2 ■ Sketching the Graph of a Polar Equation

Sketch a graph of the equation $\theta = \pi/3$, and express the equation in rectangular coordinates.

Solution The graph consists of all points whose θ-coordinate is $\pi/3$. This is the straight line that passes through the origin and makes an angle of $\pi/3$ with the polar axis (see Figure 3). Note that the points $(r, \pi/3)$ on the line with $r > 0$ lie in Quadrant I, whereas those with $r < 0$ lie in Quadrant III. If the point (x, y) lies on this line, then

$$\frac{y}{x} = \tan\theta = \tan\frac{\pi}{3} = \sqrt{3}$$

Thus the rectangular equation of this line is $y = \sqrt{3}x$.

Now Try Exercise 19 ■

To sketch a polar curve whose graph isn't as obvious as the ones in the preceding examples, we plot points calculated for sufficiently many values of θ and then join them in a continuous curve. (This is what we did when we first learned to graph equations in rectangular coordinates.)

Example 3 ■ Sketching the Graph of a Polar Equation

Sketch a graph of the polar equation $r = 2\sin\theta$.

Recall that the functions sine and cosine are periodic with period 2π.

Solution We first use the equation to determine the polar coordinates of several points on the curve. The results are shown in the following table.

$\boldsymbol{\theta}$	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
$\boldsymbol{r = 2\sin\theta}$	0	1	$\sqrt{2}$	$\sqrt{3}$	2	$\sqrt{3}$	$\sqrt{2}$	1	0

We plot these points in Figure 4 and then join them to sketch the curve. The graph appears to be a circle. We have used values of θ only between 0 and π because the same points (this time expressed with negative r-coordinates) would be obtained if we allowed θ to range from π to 2π.

The polar equation $r = 2\sin\theta$ in rectangular coordinates is

$$x^2 + (y - 1)^2 = 1$$

[see Example 8.1.6(b)]. From the rectangular form of the equation we see that the graph is a circle of radius 1 centered at $(0, 1)$.

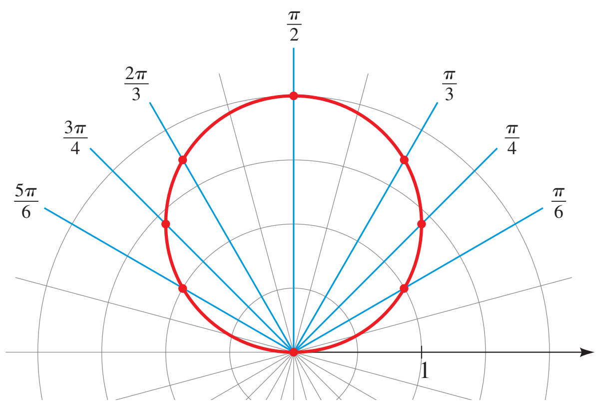


Figure 4 | $r = 2\sin\theta$

Now Try Exercise 21 ■

In general, the graph of an equation of the form

$$r = 2a \sin \theta \qquad \text{or} \qquad r = 2a \cos \theta$$

is a **circle** with radius $|a|$ centered at the points with polar coordinates $(a, \pi/2)$ and $(a, 0)$, respectively.

Example 4 ■ Sketching the Graph of a Cardioid

Sketch a graph of $r = 2 + 2 \cos \theta$.

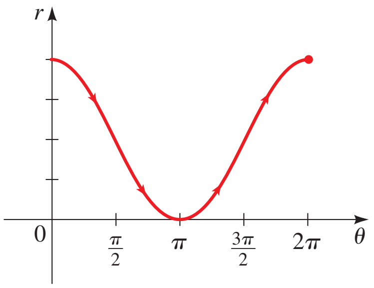


Figure 5 | $r = 2 + 2 \cos \theta$ sketched in rectangular coordinates

Solution Instead of plotting points as in Example 3, we first sketch the graph of $r = 2 + 2 \cos \theta$ in *rectangular* coordinates in Figure 5. We can think of this graph as a table of values that enables us to read at a glance the values of r that correspond to increasing values of θ. For instance, we see that as θ increases from 0 to $\pi/2$, r (the distance from O) decreases from 4 to 2, so we sketch the corresponding part of the polar graph in Figure 6(a). As θ increases from $\pi/2$ to π, Figure 5 shows that r decreases from 2 to 0, so we sketch the next part of the graph as in Figure 6(b). As θ increases from π to $3\pi/2$, r increases from 0 to 2, as shown in part (c). Finally, as θ increases from $3\pi/2$ to 2π, r increases from 2 to 4, as shown in part (d). If we let θ increase beyond 2π or decrease beyond 0, we would simply retrace our path. Combining the portions of the graph from parts (a) through (d) of Figure 6, we sketch the complete graph in part (e).

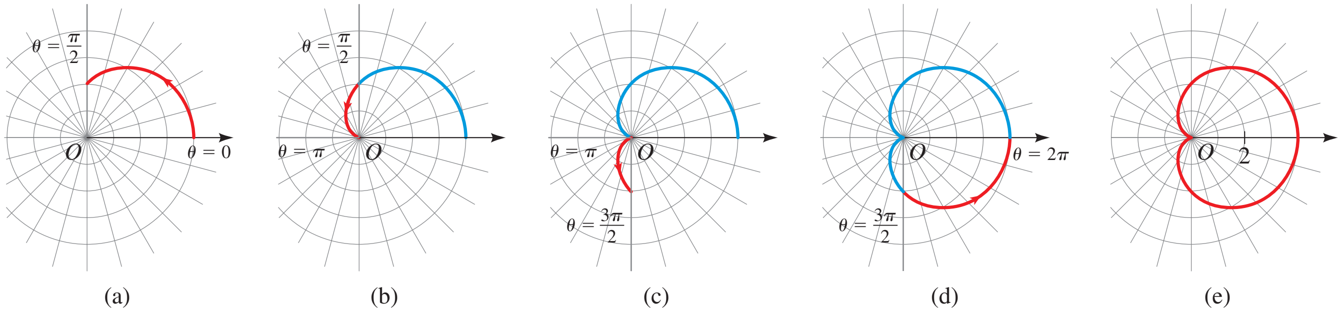


Figure 6 | Steps in sketching $r = 2 + 2 \cos \theta$ in polar coordinates

Now Try Exercise 25 ■

The polar equation $r = 2 + 2 \cos \theta$ in rectangular coordinates is

$$(x^2 + y^2 - 2x)^2 = 4(x^2 + y^2)$$

[see Example 8.1.6(c)]. The simpler form of the polar equation shows that it is more natural to describe cardioids using polar coordinates.

The curve in Figure 6 is called a **cardioid** because it is heart-shaped. In general, the graph of any equation of the form

$$r = a(1 \pm \cos \theta) \qquad \text{or} \qquad r = a(1 \pm \sin \theta)$$

is a cardioid.

Example 5 ■ Sketching the Graph of a Four-Leaved Rose

Sketch the curve $r = \cos 2\theta$.

Solution As in Example 4, we first sketch the graph of $r = \cos 2\theta$ in *rectangular* coordinates, as shown in Figure 7. As θ increases from 0 to $\pi/4$, Figure 7 shows that r decreases from 1 to 0, so we draw the corresponding portion of the polar curve in Figure 8 (indicated by 1). As θ increases from $\pi/4$ to $\pi/2$, the value of r goes from 0 to -1. This means that the distance from the origin increases from 0 to 1, but instead of being in Quadrant I, this portion of the polar curve (indicated by 2) lies on the opposite side of the origin in Quadrant III. The remainder of the curve is drawn in a similar fashion, with the arrows and numbers indicating the order in which the portions are traced out. The resulting curve has four petals and is called a **four-leaved rose**.

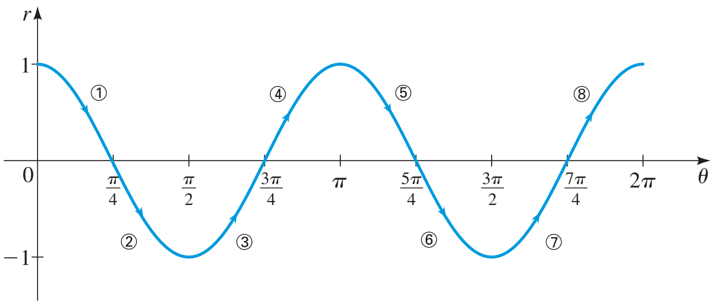


Figure 7 | Graph of $r = \cos 2\theta$ sketched in rectangular coordinates

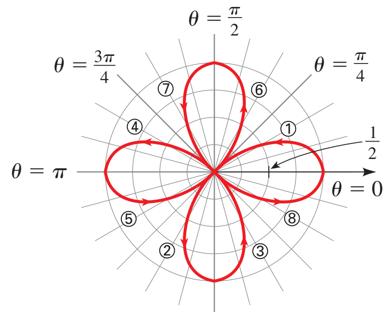


Figure 8 | Four-leaved rose $r = \cos 2\theta$ sketched in polar coordinates

Now Try Exercise 29

In general, the graph of an equation of the form

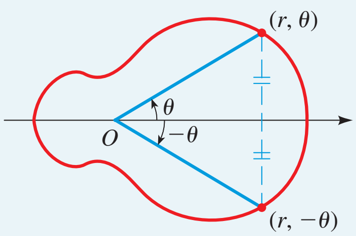
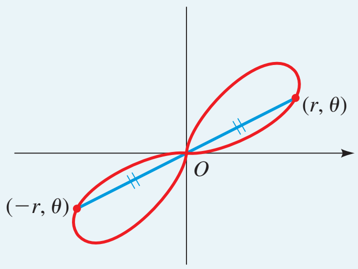
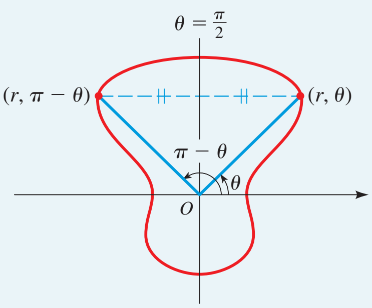
$$r = a \cos n\theta \qquad \text{or} \qquad r = a \sin n\theta$$

is an ***n*-leaved rose** if n is odd or a $2n$-leaved rose if n is even (as in Example 5).

Symmetry

In graphing a polar equation, it's often helpful to take advantage of symmetry. We list three tests for symmetry.

Types of Symmetry

Symmetry	Test	Graph	Property of Graph
With respect to the polar axis	The polar equation is unchanged if we replace θ by $-\theta$.		Graph is unchanged when reflected about the polar axis. See Figures 2, 6(e), and 8.
With respect to the pole	The polar equation is unchanged if we replace r by $-r$ or θ by $\theta + \pi$.		Graph is unchanged when rotated π radians about the pole. See Figure 8.
With respect to the line $\theta = \pi/2$	The polar equation is unchanged if we replace θ by $\pi - \theta$.		Graph is unchanged when reflected about the vertical line $\theta = \pi/2$. See Figures 4 and 8.

In rectangular coordinates the zeros of the function $y = f(x)$ correspond to the x-intercepts of the graph. In polar coordinates the zeros of the function $r = f(\theta)$ are the angles θ at which the curve crosses the pole. The zeros help us sketch the graph, as is illustrated in the next example.

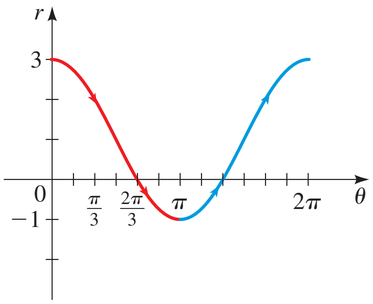


Figure 9

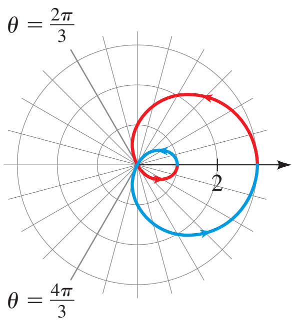


Figure 10 | $r = 1 + 2\cos\theta$

Example 6 ■ Using Symmetry to Sketch a Limaçon

Sketch a graph of the equation $r = 1 + 2\cos\theta$.

Solution We use the following steps as aids in sketching the graph.

Symmetry. Since the equation is unchanged when θ is replaced by $-\theta$, the graph is symmetric about the polar axis.

Zeros. To find the zeros, we solve

$$0 = 1 + 2\cos\theta$$

$$\cos\theta = -\frac{1}{2}$$

$$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$$

Table of values. As in Example 4, we sketch the graph of $r = 1 + 2\cos\theta$ in *rectangular* coordinates to serve as a table of values (Figure 9).

Now we sketch the polar graph of $r = 1 + 2\cos\theta$ from $\theta = 0$ to $\theta = \pi$ and then use symmetry to complete the graph, as in Figure 10.

Now Try Exercise 37 ■

The curve in Figure 10 is called a **limaçon**, after the Middle French word for snail. In general, the graph of an equation of the form

$$r = a \pm b\cos\theta \qquad \text{or} \qquad r = a \pm b\sin\theta$$

is a limaçon. The shape of the limaçon depends on the relative size of a and b.

■ Graphing Polar Equations with Graphing Devices

Although it's useful to be able to sketch simple polar graphs, we need to use a graphing calculator or computer when the graph is as complicated as the one in Figure 11. Fortunately, most graphing devices are capable of graphing polar equations directly.

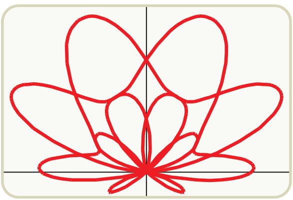


Figure 11 | $r = \sin\theta + \sin^3(5\theta/2)$

Example 7 ■ Drawing the Graph of a Polar Equation

Graph the equation $r = \cos(2\theta/3)$.

Solution We need to determine the domain for θ. So we ask ourselves: How many times must θ go through a complete rotation (2π radians) before the graph starts to repeat itself? The graph repeats itself when the same value of r is obtained at θ and $\theta + 2n\pi$. Thus we need to find an integer n so that

$$\cos\frac{2(\theta + 2n\pi)}{3} = \cos\frac{2\theta}{3}$$

For this equality to hold, $4n\pi/3$ must be a multiple of 2π, and this first happens when $n = 3$. Therefore we obtain the entire graph if we choose values of θ between $\theta = 0$ and $\theta = 0 + 2(3)\pi = 6\pi$. The graph is shown in Figure 12.

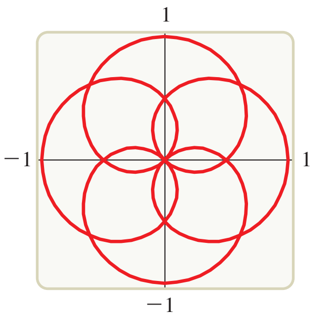


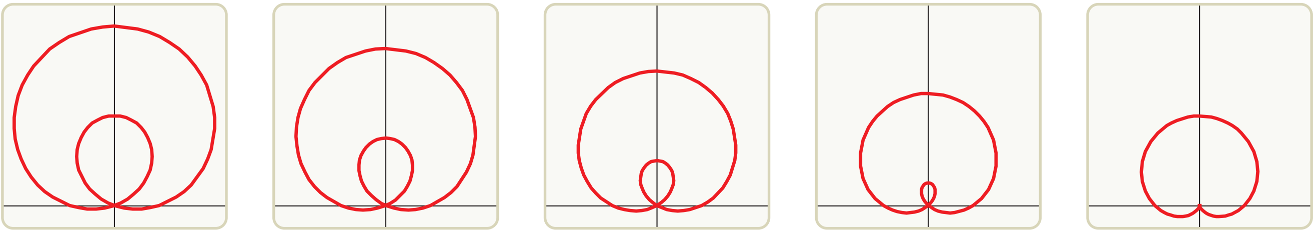
Figure 12 | $r = \cos(2\theta/3)$

Now Try Exercise 47 ■

Example 8 ■ A Family of Polar Equations

Graph the family of polar equations $r = 1 + c \sin \theta$ for $c = 3, 2.5, 2, 1.5, 1$. How does the shape of the graph change as c changes?

Solution Figure 13 shows computer-drawn graphs for the given values of c. When $c > 1$, the graph has an inner loop; the loop decreases in size as c decreases. When $c = 1$, the loop disappears, and the graph becomes a cardioid (see Example 4).



$c = 3.0$ $c = 2.5$ $c = 2.0$ $c = 1.5$ $c = 1.0$

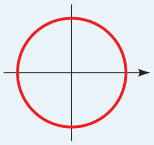
Figure 13 | A family of limaçons $r = 1 + c \sin \theta$ in the viewing rectangle $[-2.5, 2.5]$ by $[-0.5, 4.5]$

Now Try Exercise 51 ■

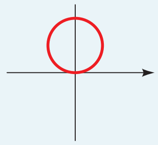
The box below gives a summary of some of the basic polar graphs used in calculus.

Some Common Polar Curves

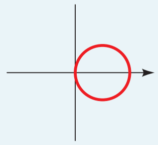
Circles and Spiral



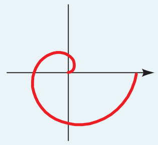
$r = a$
circle



$r = a \sin \theta$
circle



$r = a \cos \theta$
circle



$r = a\theta$
spiral

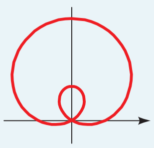
Limaçons

$r = a \pm b \sin \theta$

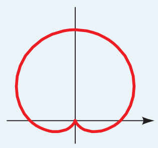
$r = a \pm b \cos \theta$

$(a > 0, b > 0)$

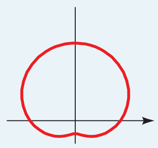
Orientation depends on the trigonometric function (sine or cosine) and the sign of b.



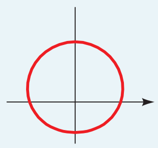
$a < b$
limaçon with inner loop



$a = b$
cardioid



$b < a < 2b$
dimpled limaçon



$2b \le a$
convex limaçon

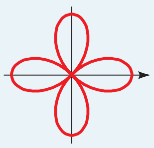
Roses

$r = a \sin n\theta$

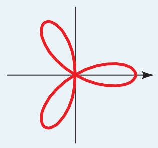
$r = a \cos n\theta$

n-leaved if n is odd

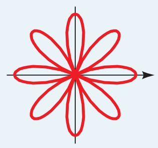
$2n$-leaved if n is even



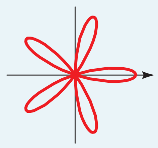
$r = a \cos 2\theta$
4-leaved rose



$r = a \cos 3\theta$
3-leaved rose



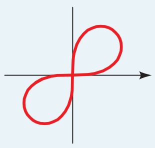
$r = a \cos 4\theta$
8-leaved rose



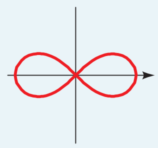
$r = a \cos 5\theta$
5-leaved rose

Lemniscates

Figure-eight-shaped curves



$r^2 = a^2 \sin 2\theta$
lemniscate



$r^2 = a^2 \cos 2\theta$
lemniscate

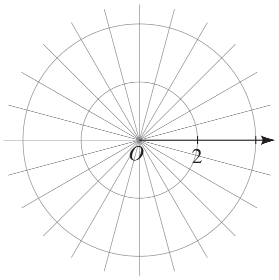
8.2 Exercises

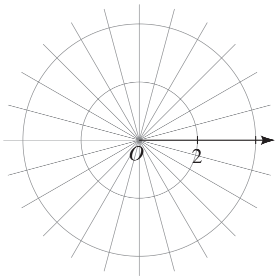
Concepts

1. To plot points in polar coordinates, we use a grid consisting of ________ centered at the pole and ________ emanating from the pole.

2. (a) To graph a polar equation $r = f(\theta)$, we plot all the points (r, θ) that ________ the equation.

(b) The simplest polar equations are obtained by setting r or θ equal to a constant. The graph of the polar equation $r = 3$ is a ________ with radius ________ centered at the ________. The graph of the polar equation $\theta = \pi/4$ is a ________ passing through the ________ with slope ________. Graph these polar equations below.





Skills

3–8 ■ Graphs of Polar Equations Match the polar equation with the graphs labeled I–VI. Use the table of common polar curves to help you.

3. $r = 3\cos\theta$

4. $r = 3$

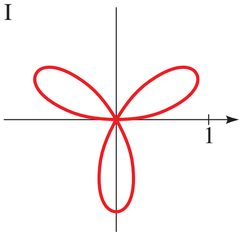
5. $r = 2 + 2\sin\theta$

6. $r = 1 + 2\cos\theta$

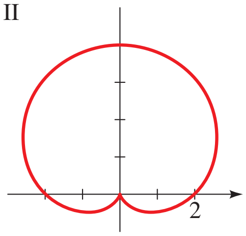
7. $r = \sin 3\theta$

8. $r = \sin 4\theta$

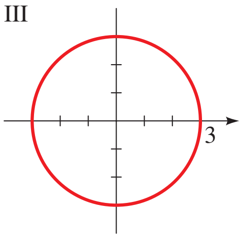
I



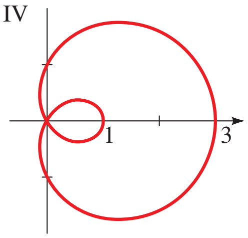
II



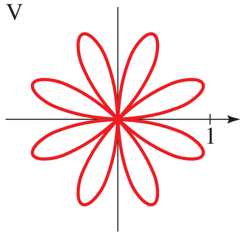
III



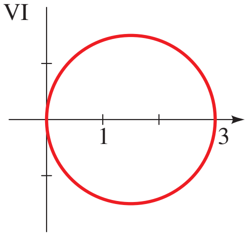
IV



V



VI



9–16 ■ Testing for Symmetry Test the polar equation for symmetry with respect to the polar axis, the pole, and the line $\theta = \pi/2$.

9. $r = 2 - \sin\theta$

10. $r = 4 + 8\cos\theta$

11. $r = 3\sec\theta$

12. $r = 5\cos\theta\csc\theta$

13. $r = \dfrac{4}{3 - 2\sin\theta}$

14. $r = \dfrac{5}{1 + 3\cos\theta}$

15. $r^2 = 4\cos 2\theta$

16. $r^2 = 9\sin\theta$

17–22 ■ Polar to Rectangular Sketch a graph of the polar equation, and express the equation in rectangular coordinates.

17. $r = 2$

18. $r = -1$

19. $\theta = -\pi/2$

20. $\theta = 5\pi/6$

21. $r = 6\sin\theta$

22. $r = \cos\theta$

23–46 ■ Graphing Polar Equations Sketch a graph of the polar equation.

23. $r = -2\cos\theta$

24. $r = 3\sin\theta$

25. $r = 2 - 2\cos\theta$

26. $r = 1 + \sin\theta$

27. $r = -3(1 + \sin\theta)$

28. $r = \cos\theta - 1$

29. $r = \sin 2\theta$

30. $r = 2\cos 3\theta$

31. $r = -\cos 5\theta$

32. $r = \sin 4\theta$

33. $r = 2\sin 5\theta$

34. $r = -3\cos 4\theta$

35. $r = \sqrt{3} - 2\sin\theta$

36. $r = 2 + \sin\theta$

37. $r = \sqrt{3} + \cos\theta$

38. $r = 1 - 2\cos\theta$

39. $r = 2 - 2\sqrt{2}\cos\theta$

40. $r = 3 + 6\sin\theta$

41. $r^2 = \cos 2\theta$

42. $r^2 = 4\sin 2\theta$

43. $r = \theta, \quad \theta \geq 0$ (spiral)

44. $r\theta = 1, \quad \theta > 0$ (reciprocal spiral)

45. $r = 2 + \sec\theta$ (conchoid)

46. $r = \sin\theta\tan\theta$ (cissoid)

47–50 ■ Graphing Polar Equations Use a graphing device to graph the polar equation. Choose the domain of θ to make sure you produce the entire graph.

47. $r = \cos(\theta/2)$

48. $r = \sin(8\theta/5)$

49. $r = 1 + 2\sin(\theta/2)$ (nephroid)

50. $r = \sqrt{1 - 0.8\sin^2\theta}$ (hippopede)

51–52 ■ Families of Polar Equations These exercises involve families of polar equations.

51. Graph the family of polar equations $r = 1 + \sin n\theta$ for $n = 1, 2, 3, 4$, and 5. How is the number of loops related to n?

52. Graph the family of polar equations $r = 1 + c \sin 2\theta$ for $c = 0.3, 0.6, 1, 1.5$, and 2. How does the graph change as c increases?

53–56 ■ Special Polar Equations Match the polar equation with the graphs labeled I–IV. Give reasons for your answers.

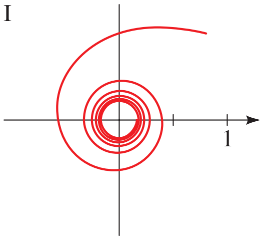
53. $r = \sin(\theta/2)$

54. $r = 1/\sqrt{\theta}$

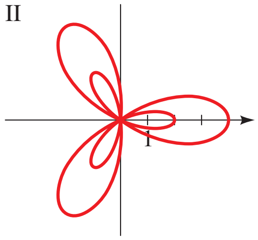
55. $r = \theta \sin \theta$

56. $r = 1 + 3\cos(3\theta)$

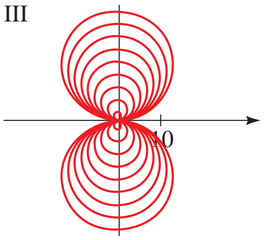
I



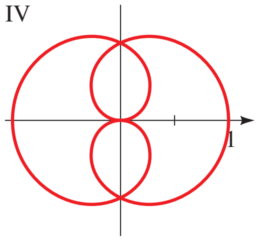
II



III



IV



Skills Plus

57–60 ■ Rectangular to Polar Sketch a graph of the rectangular equation. [*Hint:* First convert the equation to polar coordinates.]

57. $(x^2 + y^2)^3 = 4x^2y^2$

58. $(x^2 + y^2)^3 = (x^2 - y^2)^2$

59. $(x^2 + y^2)^2 = x^2 - y^2$

60. $x^2 + y^2 = (x^2 + y^2 - x)^2$

61. A Circle in Polar Coordinates Consider the polar equation $r = a\cos\theta + b\sin\theta$.

(a) Express the equation in rectangular coordinates, and use this to show that the graph of the equation is a circle. What are the center and radius?

(b) Use your answer to part (a) to graph the equation $r = 2\sin\theta + 2\cos\theta$.

62. A Parabola in Polar Coordinates

(a) Graph the polar equation $r = \tan\theta \sec\theta$ in the viewing rectangle $[-3, 3]$ by $[-1, 9]$.

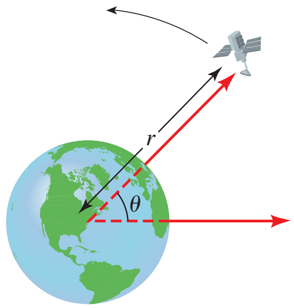
(b) Note that your graph in part (a) looks like a parabola (see Section 3.1). Confirm this by converting the equation to rectangular coordinates.

Applications

63. Orbit of a Satellite Scientists and engineers often use polar equations to model the motion of satellites in earth orbit. Let's consider a satellite whose orbit is modeled by the equation $r = 22500/(4 - \cos\theta)$, where r is the distance in miles between the satellite and the center of the earth and θ is the angle shown in the figure.

(a) On the same viewing screen, graph the circle $r = 3960$ (to represent the earth, which we will assume to be a sphere of radius 3960 mi) and the polar equation of the satellite's orbit. Describe the motion of the satellite as θ increases from 0 to 2π.

(b) For what angle θ is the satellite closest to the earth? Find the height of the satellite above the earth's surface for this value of θ.



64. An Unstable Orbit The orbit described in Exercise 63 is stable because the satellite traverses the same path over and over as θ increases. Suppose that a meteor strikes the satellite and changes its orbit to

$$r = \frac{22500\left(1 - \dfrac{\theta}{40}\right)}{4 - \cos\theta}$$

(a) On the same viewing screen, graph the circle $r = 3960$ and the new orbit equation, with θ increasing from 0 to 3π. Describe the new motion of the satellite.

(b) Estimate graphically the value of θ at the moment the satellite crashes into the earth.

Discuss ■ Discover ■ Prove ■ Write

65. Discuss ■ Discover: A Transformation of Polar Graphs How are the graphs of

$$r = 1 + \sin\left(\theta - \frac{\pi}{6}\right)$$

and

$$r = 1 + \sin\left(\theta - \frac{\pi}{3}\right)$$

related to the graph of $r = 1 + \sin\theta$? In general, how is the graph of $r = f(\theta - \alpha)$ related to the graph of $r = f(\theta)$?

66. Discuss: Choosing a Convenient Coordinate System Compare the polar equation of the circle $r = 2$ with its equation in rectangular coordinates. In which coordinate system is the equation simpler? Do the same for the equation of the four-leaved rose $r = \sin 2\theta$. Which coordinate system would you choose to study these curves?

67. Discuss: Choosing a Convenient Coordinate System Compare the rectangular equation of the line $y = 2$ with its polar equation. In which coordinate system is the equation simpler? Which coordinate system would you choose to study lines?

8.3 Polar Form of Complex Numbers; De Moivre's Theorem

■ Graphing Complex Numbers ■ Polar Form of Complex Numbers
■ De Moivre's Theorem ■ *n*th Roots of Complex Numbers

Complex numbers are discussed in Section 1.6.

In this section we represent complex numbers in polar (or trigonometric) form. This enables us to find the nth roots of complex numbers. To describe the polar form of complex numbers, we must first learn to work with complex numbers graphically.

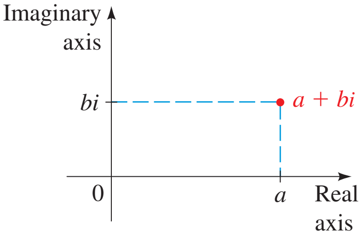


Figure 1

Graphing Complex Numbers

To graph real numbers or sets of real numbers, we have been using the number line, which has just one dimension. Complex numbers, however, have two components: a real part and an imaginary part. This suggests that we need two axes to graph complex numbers: one for the real part and one for the imaginary part. We call these the **real axis** and the **imaginary axis**, respectively. The plane determined by these two axes is called the **complex plane**. To graph the complex number $a + bi$, we plot the ordered pair of numbers (a, b) in this plane, as indicated in Figure 1.

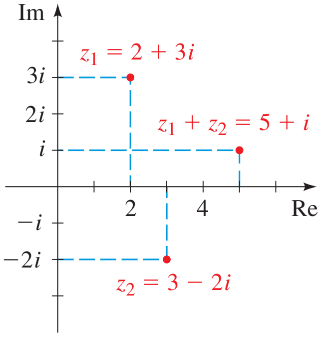


Figure 2

Example 1 ■ Graphing Complex Numbers

Graph the complex numbers $z_1 = 2 + 3i$, $z_2 = 3 - 2i$, and $z_1 + z_2$.

Solution We have $z_1 + z_2 = (2 + 3i) + (3 - 2i) = 5 + i$. The graph is shown in Figure 2.

Now Try Exercise 19 ■

Example 2 ■ Graphing Sets of Complex Numbers

Graph each set of complex numbers.

(a) $S = \{a + bi \mid a \geq 0\}$

(b) $T = \{a + bi \mid a < 1, b \geq 0\}$

Solution

(a) S is the set of complex numbers whose real part is nonnegative. The graph is shown in Figure 3(a).

(b) T is the set of complex numbers for which the real part is less than 1 and the imaginary part is nonnegative. The graph is shown in Figure 3(b).

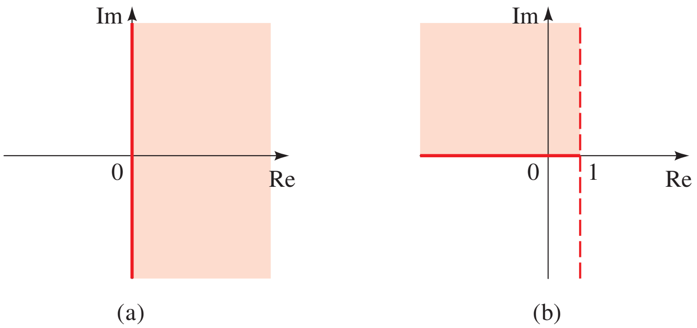


Figure 3

Now Try Exercise 21 ■

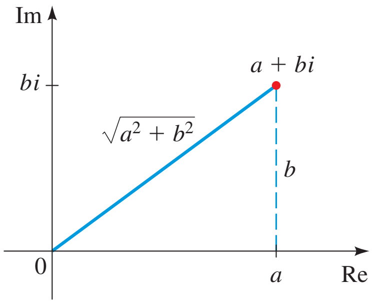


Figure 4

Recall that the absolute value of a real number can be thought of as its distance from the origin on the real number line (see Section 1.1). We define absolute value for complex numbers in a similar fashion. Using the Pythagorean Theorem, we can see from Figure 4 that the distance between $a + bi$ and the origin in the complex plane is $\sqrt{a^2 + b^2}$. This leads to the following definition.

Modulus of a Complex Number

The **modulus** (or **absolute value**) of the complex number $z = a + bi$ is

$$|z| = \sqrt{a^2 + b^2}$$

Example 3 ■ Calculating the Modulus

The plural of *modulus* is *moduli*.

Find the moduli of the complex numbers $3 + 4i$ and $8 - 5i$.

Solution

$$|3 + 4i| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

$$|8 - 5i| = \sqrt{8^2 + (-5)^2} = \sqrt{89}$$

Now Try Exercise 9 ■

Example 4 ■ Graphing a Set of Complex Numbers

Graph each set of complex numbers.

(a) $C = \{z \mid |z| = 1\}$ **(b)** $D = \{z \mid |z| \le 1\}$

Solution

(a) C is the set of complex numbers whose distance from the origin is 1. Thus C is a circle of radius 1 with center at the origin, as shown in Figure 5.

(b) D is the set of complex numbers whose distance from the origin is less than or equal to 1. Thus D is the disk that consists of all complex numbers on and inside the circle C of part (a), as shown in Figure 6.

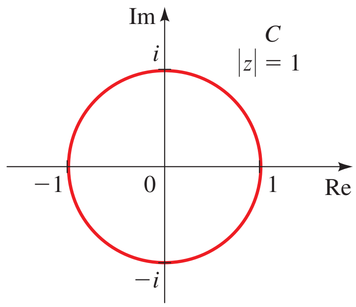


Figure 5

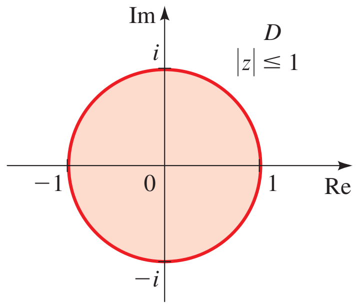


Figure 6

Now Try Exercises 23 and 25 ■

■ Polar Form of Complex Numbers

Let $z = a + bi$ be a complex number, and in the complex plane let's draw the line segment joining the origin to the point $a + bi$. (See Figure 7 on the next page.) The length of this line segment is $r = |z| = \sqrt{a^2 + b^2}$. If θ is an angle in standard position whose terminal side coincides with this line segment, then by the definitions of sine and cosine (see Section 6.3)

$$a = r\cos\theta \qquad \text{and} \qquad b = r\sin\theta$$

so $z = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$. We have shown the following.

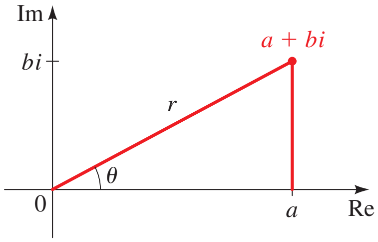


Figure 7

Polar Form of Complex Numbers

A complex number $z = a + bi$ has the **polar form** (or **trigonometric form**)

$$z = r(\cos\theta + i\sin\theta)$$

where $r = |z| = \sqrt{a^2 + b^2}$ and $\tan\theta = b/a$. The number r is the **modulus** of z, and θ is called an **argument** of z.

Note The argument of z is not unique, but any two arguments of z differ by a multiple of 2π. When determining the argument, we must consider the quadrant in which z lies, as we see in the next example.

Example 5 ■ Writing Complex Numbers in Polar Form

Write each complex number in polar form.

(a) $1 + i$ **(b)** $-1 + \sqrt{3}i$ **(c)** $-4\sqrt{3} - 4i$ **(d)** $3 + 4i$

Solution These complex numbers are graphed in Figure 8, in order to help us find their arguments.

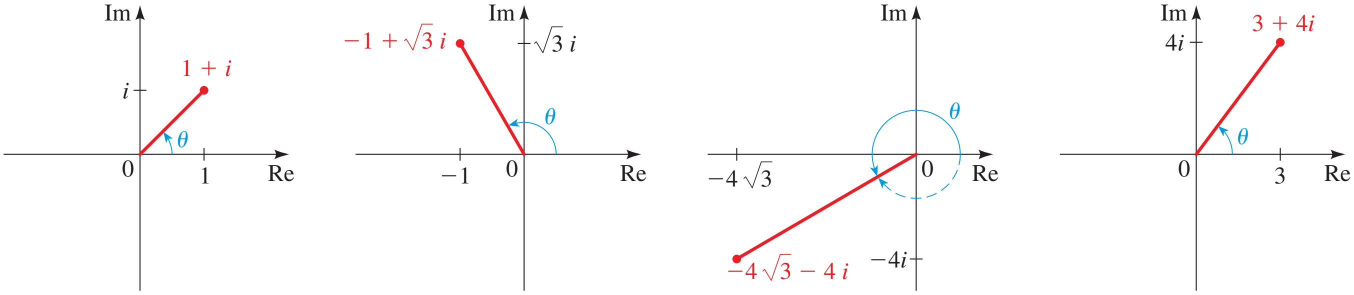


Figure 8 (a) (b) (c) (d)

$\tan\theta = \frac{1}{1} = 1$ and θ in Quadrant I, so $\theta = \frac{\pi}{4}$

(a) An argument is $\theta = \pi/4$ and $r = \sqrt{1 + 1} = \sqrt{2}$. Thus

$$1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$\tan\theta = \dfrac{\sqrt{3}}{-1} = -\sqrt{3}$ and θ in Quadrant II, so $\theta = \frac{2\pi}{3}$

(b) An argument is $\theta = 2\pi/3$ and $r = \sqrt{1 + 3} = 2$. Thus

$$-1 + \sqrt{3}i = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

$\tan\theta = \dfrac{-4}{-4\sqrt{3}} = \dfrac{1}{\sqrt{3}}$ and θ in Quadrant III, so $\theta = \frac{7\pi}{6}$

(c) An argument is $\theta = 7\pi/6$ (or we could use $\theta = -5\pi/6$), and $r = \sqrt{48 + 16} = 8$. Thus

$$-4\sqrt{3} - 4i = 8\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$$

$\tan\theta = \frac{4}{3}$ and θ in Quadrant I, so $\theta = \tan^{-1}\left(\frac{4}{3}\right)$

(d) An argument is $\theta = \tan^{-1}\left(\frac{4}{3}\right) \approx 0.93$ and $r = \sqrt{3^2 + 4^2} = 5$. So

$$3 + 4i = 5\left[\cos\left(\tan^{-1}\left(\tfrac{4}{3}\right)\right) + i\sin\left(\tan^{-1}\left(\tfrac{4}{3}\right)\right)\right] \approx 5[\cos 0.93 + i\sin 0.93]$$

Now Try Exercises 29, 31, 33, and 43 ■

The Addition Formulas for Sine and Cosine that we discussed in Section 7.2 simplify the multiplication and division of complex numbers in polar form. The following theorem shows how.

Multiplication and Division of Complex Numbers

If the two complex numbers z_1 and z_2 have the polar forms

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \qquad \text{and} \qquad z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

then z_1z_2 and z_1/z_2 have the following polar forms:

Multiplication

$$z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

Division

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)] \qquad (z_2 \neq 0)$$

This theorem says:

To multiply two complex numbers, multiply the moduli and add the arguments.

To divide two complex numbers, divide the moduli and subtract the arguments.

Proof To prove the Multiplication Formula, we multiply the two complex numbers:

$$\begin{aligned} z_1z_2 &= r_1r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= r_1r_2[\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)] \\ &= r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \end{aligned}$$

In the last step we used the Addition Formulas for Sine and Cosine.

The proof of the Division Formula is left as an exercise. (See Exercise 102.) ■

Example 6 ■ Multiplying and Dividing Complex Numbers

Let

$$z_1 = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \qquad \text{and} \qquad z_2 = 5\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

Find **(a)** z_1z_2 and **(b)** z_1/z_2.

Solution

(a) By the Multiplication Formula

$$\begin{aligned} z_1z_2 &= (2)(5)\left[\cos\left(\frac{\pi}{4} + \frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{4} + \frac{\pi}{3}\right)\right] \\ &= 10\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right) \end{aligned}$$

To approximate the answer, we use a calculator in radian mode and get

$$\begin{aligned} z_1z_2 &\approx 10(-0.2588 + 0.9659i) \\ &= -2.588 + 9.659i \end{aligned}$$

(b) By the Division Formula

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{2}{5}\left[\cos\left(\frac{\pi}{4} - \frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{4} - \frac{\pi}{3}\right)\right] \\ &= \frac{2}{5}\left[\cos\left(-\frac{\pi}{12}\right) + i\sin\left(-\frac{\pi}{12}\right)\right] \\ &= \frac{2}{5}\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right)\end{aligned}$$

Using a calculator in radian mode, we get the approximate answer:

$$\frac{z_1}{z_2} \approx \frac{2}{5}(0.9659 - 0.2588i) = 0.3864 - 0.1035i$$

Now Try Exercise 49 ■

■ De Moivre's Theorem

Repeated use of the Multiplication Formula gives the following useful formula for raising a complex number to a power n for any positive integer n.

De Moivre's Theorem

If $z = r(\cos\theta + i\sin\theta)$, then for any integer n

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

This theorem says: *To take the nth power of a complex number, take the nth power of the modulus and multiply the argument by n.*

Proof By the Multiplication Formula

$$\begin{aligned}z^2 = zz &= r^2[\cos(\theta + \theta) + i\sin(\theta + \theta)] \\ &= r^2(\cos 2\theta + i\sin 2\theta)\end{aligned}$$

Now we multiply z^2 by z to get

$$\begin{aligned}z^3 = z^2z &= r^3[\cos(2\theta + \theta) + i\sin(2\theta + \theta)] \\ &= r^3(\cos 3\theta + i\sin 3\theta)\end{aligned}$$

Repeating this argument, we see that for any positive integer n

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

A similar argument using the Division Formula shows that this also holds for negative integers. ■

Example 7 ■ Finding a Power Using De Moivre's Theorem

Find $\left(\frac{1}{2} + \frac{1}{2}i\right)^{10}$.

Solution Since $\frac{1}{2} + \frac{1}{2}i = \frac{1}{2}(1 + i)$, it follows from Example 5(a) that

$$\frac{1}{2} + \frac{1}{2}i = \frac{\sqrt{2}}{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

So by de Moivre's Theorem

$$\left(\frac{1}{2}+\frac{1}{2}i\right)^{10} = \left(\frac{\sqrt{2}}{2}\right)^{10}\left(\cos\frac{10\pi}{4} + i\sin\frac{10\pi}{4}\right)$$

$$= \frac{2^5}{2^{10}}\left(\cos\frac{5\pi}{2} + i\sin\frac{5\pi}{2}\right) = \frac{1}{32}i$$

Now Try Exercise 65

nth Roots of Complex Numbers

An ***n*th root** of a complex number z is any complex number w such that $w^n = z$. De Moivre's Theorem gives us a method for calculating the nth roots of any complex number.

nth Roots of Complex Numbers

If $z = r(\cos\theta + i\sin\theta)$ and n is a positive integer, then z has the n distinct nth roots

$$w_k = r^{1/n}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$

for $k = 0, 1, 2, \ldots, n-1$.

Proof To find the nth roots of z, we need to find a complex number w such that

$$w^n = z$$

Let's write z in polar form:

$$z = r(\cos\theta + i\sin\theta)$$

One nth root of z is

$$w = r^{1/n}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)$$

because by de Moivre's Theorem, $w^n = z$. But the argument θ of z can be replaced by $\theta + 2k\pi$ for any integer k. Since this expression gives a different value of w for $k = 0, 1, 2, \ldots, n-1$, we have proved the formula in the theorem.

The following observations help us use the preceding formula.

Finding the nth Roots of $z = r(\cos\theta + i\sin\theta)$

1. **Modulus.** The modulus of each nth root is $r^{1/n}$.
2. **Argument.** The argument of the first root is θ/n.
3. **Roots.** Repeatedly add $2\pi/n$ to get the argument of each successive root.

These observations show that, when graphed, the nth roots of z are spaced equally on the circle of radius $r^{1/n}$.

Example 8 ■ Finding Roots of a Complex Number

Find the six sixth roots of $z = -64$, and graph these roots in the complex plane.

Solution In polar form, $z = 64(\cos \pi + i \sin \pi)$. Applying the formula for nth roots with $n = 6$, we get

$$w_k = 64^{1/6}\left[\cos\left(\frac{\pi + 2k\pi}{6}\right) + i \sin\left(\frac{\pi + 2k\pi}{6}\right)\right]$$

for $k = 0, 1, 2, 3, 4, 5$. Since $64^{1/6} = 2$, we find that the six sixth roots of -64 are

We add $2\pi/6 = \pi/3$ to each argument to get the argument of the next root.

$$w_0 = 2\left(\cos\frac{\pi}{6} + i \sin\frac{\pi}{6}\right) = \sqrt{3} + i \qquad k = 0$$

$$w_1 = 2\left(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}\right) = 2i \qquad k = 1$$

$$w_2 = 2\left(\cos\frac{5\pi}{6} + i \sin\frac{5\pi}{6}\right) = -\sqrt{3} + i \qquad k = 2$$

$$w_3 = 2\left(\cos\frac{7\pi}{6} + i \sin\frac{7\pi}{6}\right) = -\sqrt{3} - i \qquad k = 3$$

$$w_4 = 2\left(\cos\frac{3\pi}{2} + i \sin\frac{3\pi}{2}\right) = -2i \qquad k = 4$$

$$w_5 = 2\left(\cos\frac{11\pi}{6} + i \sin\frac{11\pi}{6}\right) = \sqrt{3} - i \qquad k = 5$$

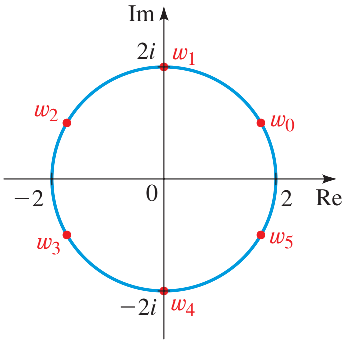


Figure 9 | The six sixth roots of $z = -64$

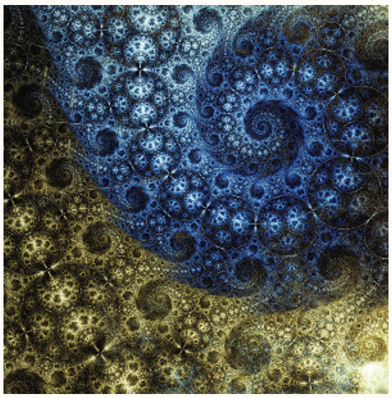
All these points lie on a circle of radius 2, as shown in Figure 9.

Now Try Exercise 81 ■

When finding roots of complex numbers, we sometimes write the argument θ of the complex number in degrees. In this case the nth roots are obtained from the formula

$$w_k = r^{1/n}\left[\cos\left(\frac{\theta + 360°k}{n}\right) + i \sin\left(\frac{\theta + 360°k}{n}\right)\right]$$

for $k = 0, 1, 2, \ldots, n - 1$.



Discovery Project ■ Fractals

Most of the things we model in this book follow regular, predictable patterns. But many real-world phenomena—such as a cloud, a jagged coastline, or a flickering flame—appear to have random or even chaotic shapes. Fractals allow us to model these sorts of shapes. Surprisingly, the extremely complex shapes of fractals and their infinite detail are produced by exceedingly simple rules and endless repetitions that involve iterating simple functions whose inputs and outputs are complex numbers. You can find the project at the book companion website **www.stewartmath.com**.

Example 9 ■ Finding Cube Roots of a Complex Number

Find the three cube roots of $z = 2 + 2i$, and graph these roots in the complex plane.

Solution First we write z in polar form using degrees. We have $r = \sqrt{2^2 + 2^2} = 2\sqrt{2}$ and $\theta = 45°$. Thus

$$z = 2\sqrt{2}(\cos 45° + i \sin 45°)$$

Applying the formula for nth roots (in degrees) with $n = 3$, we find that the cube roots of z are of the form

We add $360°/3 = 120°$ to each argument to get the argument of the next root.

$$w_k = (2\sqrt{2})^{1/3}\left[\cos\left(\frac{45° + 360°k}{3}\right) + i \sin\left(\frac{45° + 360°k}{3}\right)\right]$$

where $k = 0, 1, 2$. Since $(2\sqrt{2})^{1/3} = (2^{3/2})^{1/3} = 2^{1/2} = \sqrt{2}$, the three cube roots are

$$w_0 = \sqrt{2}(\cos 15° + i \sin 15°) \approx 1.366 + 0.366i \qquad k = 0$$

$$w_1 = \sqrt{2}(\cos 135° + i \sin 135°) = -1 + i \qquad k = 1$$

$$w_2 = \sqrt{2}(\cos 255° + i \sin 255°) \approx -0.366 - 1.366i \qquad k = 2$$

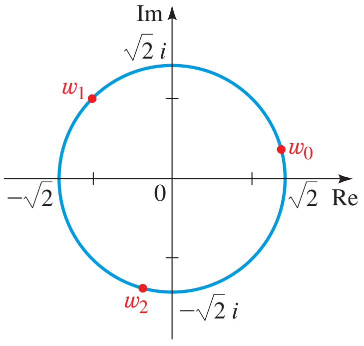


Figure 10 | The three cube roots of $z = 2 + 2i$

The three cube roots of z are graphed in Figure 10. These roots are spaced equally on a circle of radius $\sqrt{2}$.

Now Try Exercise 77 ■

Example 10 ■ Solving an Equation Using the nth Roots Formula

Solve the equation $z^6 + 64 = 0$.

Solution This equation can be written as $z^6 = -64$. Thus the solutions are the six sixth roots of -64, which we found in Example 8.

Now Try Exercise 87 ■

8.3 Exercises

Concepts

1. A complex number $z = a + bi$ has two parts: a is the ________ part, and b is the ________ part. To graph $a + bi$, we graph the ordered pair (___ , ___) in the complex plane.

2. Let $z = a + bi$.

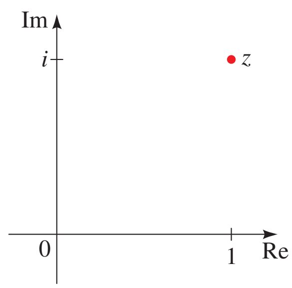
(a) The modulus of z is $r =$ ________, and an argument of z is an angle θ satisfying $\tan\theta =$ ________.

(b) We can express z in polar form as $z =$ ____________, where r is the modulus of z and θ is the argument of z.

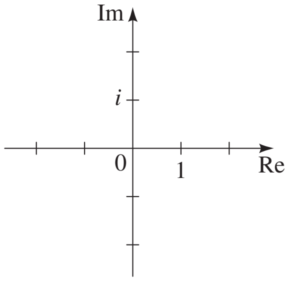
3. (a) The complex number $z = -1 + i$ in polar form is $z =$ ________.

(b) The complex number $z = 2\left(\cos\frac{\pi}{6} + i \sin\frac{\pi}{6}\right)$ in rectangular form is $z =$ ________.

(c) The complex number graphed below can be expressed in rectangular form as ________ or in polar form as ________.



4. How many different nth roots does a nonzero complex number have? ________. The number 16 has ________ fourth roots. These roots are ________, ________, ________, and ________. In the complex plane these roots all lie on a circle of radius ________. Graph the roots on the following graph.



Skills

5–14 ■ A Complex Number and Its Modulus Graph the complex number, and find its modulus.

5. $4i$ **6.** $-3i$

7. -2 **8.** 6

9. $5 + 2i$ **10.** $7 - 3i$

11. $\sqrt{3} + i$ **12.** $-1 - \dfrac{\sqrt{3}}{3}i$

13. $\dfrac{3 + 4i}{5}$ **14.** $\dfrac{-\sqrt{2} + \sqrt{2}i}{2}$

15–16 ■ Graphing Complex Numbers Sketch the complex numbers z, $2z$, $-z$, and $\frac{1}{2}z$ on the same complex plane.

15. $z = 1 + i$ **16.** $z = -1 + \sqrt{3}i$

17–18 ■ Graphing a Complex Number and Its Complex Conjugate Sketch the complex number z and its complex conjugate $\bar{z}$ on the same complex plane.

17. $z = 8 + 2i$ **18.** $z = -5 + 6i$

19–20 ■ Graphing Complex Numbers Sketch z_1, z_2, $z_1 + z_2$, and $z_1 z_2$ on the same complex plane.

19. $z_1 = 2 - i, \quad z_2 = 2 + i$

20. $z_1 = -1 + i, \quad z_2 = 2 - 3i$

21–28 ■ Graphing Sets of Complex Numbers Sketch the set in the complex plane.

21. $\{z = a + bi \mid a \le 0, b \ge 0\}$

22. $\{z = a + bi \mid a > 1, b > 1\}$

23. $\{z \mid |z| = 3\}$ **24.** $\{z \mid |z| \ge 1\}$

25. $\{z \mid |z| < 2\}$ **26.** $\{z \mid 2 \le |z| \le 5\}$

27. $\{z = a + bi \mid a + b < 2\}$

28. $\{z = a + bi \mid a \ge b\}$

29–48 ■ Polar Form of Complex Numbers Write the complex number in polar form with argument θ between 0 and 2π.

29. $1 + i$ **30.** $1 - i$ **31.** $-2 + 2i$

32. $-\sqrt{2} - \sqrt{2}i$ **33.** $-\sqrt{3} - i$ **34.** $-5 + 5\sqrt{3}i$

35. $2\sqrt{3} - 2i$ **36.** $3 + 3\sqrt{3}i$ **37.** $2i$

38. $-5i$ **39.** -3 **40.** $\sqrt{2}$

41. $-\sqrt{6} + \sqrt{2}i$ **42.** $-\sqrt{5} - \sqrt{15}i$ **43.** $4 + 3i$

44. $3 + 2i$ **45.** $4(\sqrt{3} - i)$ **46.** $i(\sqrt{2} - \sqrt{6}i)$

47. $-3(1 - i)$ **48.** $2i(1 + i)$

49–56 ■ Products and Quotients of Complex Numbers Find the product $z_1 z_2$ and the quotient z_1/z_2. Express your answer in polar form.

49. $z_1 = 3\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right), \quad z_2 = 2\left(\cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6}\right)$

50. $z_1 = \sqrt{3}\left(\cos\dfrac{5\pi}{4} + i\sin\dfrac{5\pi}{4}\right), \quad z_2 = 2(\cos\pi + i\sin\pi)$

51. $z_1 = \sqrt{2}\left(\cos\dfrac{5\pi}{3} + i\sin\dfrac{5\pi}{3}\right),$

$z_2 = 2\sqrt{2}\left(\cos\dfrac{3\pi}{2} + i\sin\dfrac{3\pi}{2}\right)$

52. $z_1 = \cos\dfrac{3\pi}{4} + i\sin\dfrac{3\pi}{4}, \quad z_2 = \cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}$

53. $z_1 = 4(\cos 120° + i\sin 120°),$

$z_2 = 2(\cos 30° + i\sin 30°)$

54. $z_1 = \sqrt{2}(\cos 75° + i\sin 75°),$

$z_2 = 3\sqrt{2}(\cos 60° + i\sin 60°)$

55. $z_1 = 4(\cos 200° + i\sin 200°),$

$z_2 = 25(\cos 150° + i\sin 150°)$

56. $z_1 = \frac{4}{5}(\cos 25° + i\sin 25°),$

$z_2 = \frac{1}{5}(\cos 155° + i\sin 155°)$

57–64 ■ Products and Quotients of Complex Numbers Write z_1 and z_2 in polar form, and then find the product $z_1 z_2$ and the quotients z_1/z_2 and $1/z_1$. Express your answers in polar form.

57. $z_1 = \sqrt{3} + i, \quad z_2 = 1 + \sqrt{3}i$

58. $z_1 = \sqrt{2} - \sqrt{2}i, \quad z_2 = 1 - i$

59. $z_1 = 2\sqrt{3} - 2i, \quad z_2 = -1 + i$

60. $z_1 = -\sqrt{2}i, \quad z_2 = -3 - 3\sqrt{3}i$

61. $z_1 = 5 + 5i, \quad z_2 = 4$

62. $z_1 = 4\sqrt{3} - 4i, \quad z_2 = 8i$

63. $z_1 = -20, \quad z_2 = \sqrt{3} + i$

64. $z_1 = 3 + 4i, \quad z_2 = 2 - 2i$

65–76 ■ Powers Using De Moivre's Theorem Find the indicated power using de Moivre's Theorem.

65. $(-\sqrt{3} + i)^6$

66. $(1 - i)^{10}$

67. $(-\sqrt{2} - \sqrt{2}i)^5$

68. $(1 + i)^7$

69. $\left(\dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2}i\right)^{12}$

70. $(\sqrt{3} - i)^{-10}$

71. $(2 - 2i)^8$

72. $\left(-\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i\right)^{15}$

73. $(-1 - i)^7$

74. $(3 + \sqrt{3}i)^4$

75. $(2\sqrt{3} + 2i)^{-5}$

76. $(1 - i)^{-8}$

77–86 ■ Roots of Complex Numbers Find the indicated roots, and graph the roots in the complex plane.

77. The square roots of $4\sqrt{3} + 4i$

78. The cube roots of $4\sqrt{3} + 4i$

79. The fourth roots of $-81i$

80. The fifth roots of 32

81. The eighth roots of 1

82. The cube roots of $1 + i$

83. The cube roots of i

84. The fifth roots of i

85. The fourth roots of -1

86. The fifth roots of $-16 - 16\sqrt{3}i$

87–92 ■ Solving Equations Using *n*th Roots Solve the equation.

87. $z^4 + 1 = 0$

88. $z^8 - i = 0$

89. $z^3 - 4\sqrt{3} - 4i = 0$

90. $z^6 - 1 = 0$

91. $z^3 + 1 = -i$

92. $z^3 - 1 = 0$

Skills Plus

93–96 ■ Complex Coefficients and the Quadratic Formula The quadratic formula works whether the coefficients of the equation are real or complex. Solve the following equations using the quadratic formula and, if necessary, de Moivre's Theorem.

93. $z^2 - iz + 1 = 0$

94. $z^2 + iz + 2 = 0$

95. $z^2 - 2iz - 2 = 0$

96. $z^2 + (1 + i)z + i = 0$

97–98 ■ Finding *n*th Roots of a Complex Number Let $w = \cos(2\pi/n) + i\sin(2\pi/n)$, where n is a positive integer.

97. Show that the n distinct roots of 1 are

$$1, w, w^2, w^3, \ldots, w^{n-1}$$

98. If $z \neq 0$ and s is any nth root of z, show that the n distinct roots of z are

$$s, sw, sw^2, sw^3, \ldots, sw^{n-1}$$

99. Properties of the Modulus Verify the property for the complex numbers w and z.

(a) $z\bar{z} = |z|^2$

(b) $|wz| = |w||z|$

(c) $\left|\dfrac{1}{z}\right| = \dfrac{1}{|z|}$

(d) $\left|\dfrac{w}{z}\right| = \dfrac{|w|}{|z|}$

Discuss ■ Discover ■ Prove ■ Write

100. Discuss: Sums of Roots of Unity Find the exact values of all three cube roots of 1 (see Exercise 97), and then add them. Do the same for the fourth, fifth, sixth, and eighth roots of 1. What do you think is the sum of the nth roots of 1 for any n?

101. Discuss: Products of Roots of Unity Find the product of the three cube roots of 1 (see Exercise 97). Do the same for the fourth, fifth, sixth, and eighth roots of 1. What do you think is the product of the nth roots of 1 for any n?

102. Prove: Division in Polar Form If the two complex numbers z_1 and z_2 have the polar forms

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$$
$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

show that

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

[*Hint:* Multiply numerator and denominator by the complex conjugate of z_2 and simplify.]

103. Prove: Sums of Squares of Integers Use complex numbers to show that the following statement is true: For any integers a and b there exist integers c and d such that

$$c^2 + d^2 = 2(a^2 + b^2)$$

PS *Introduce something extra.* Although the statement involves only integers, one way to prove it is to introduce complex numbers. Let $w = 1 + i$ and $z = a + bi$. Evaluate $|wz|^2$ in two ways using Exercise 99(b).

8.4 Plane Curves and Parametric Equations

■ Plane Curves and Parametric Equations ■ Eliminating the Parameter ■ Finding Parametric Equations for a Curve ■ Using Graphing Devices to Graph Parametric Curves

So far, we have described a curve by giving an equation (in rectangular or polar coordinates) that the coordinates of all the points on the curve must satisfy. But not all curves in the plane can be described in this way. In this section we study parametric equations, which are a general method for describing any curve.

■ Plane Curves and Parametric Equations

We can think of a curve as the path of a point moving in the plane; the x- and y-coordinates of the point are then functions of time. This idea leads to the following definition.

Plane Curves and Parametric Equations

If f and g are functions defined on an interval I, then the set of points $(f(t), g(t))$ is a **plane curve**. The equations

$$x = f(t) \qquad y = g(t)$$

where $t \in I$, are **parametric equations** for the curve, with **parameter** t.

Example 1 ■ Sketching a Plane Curve

Sketch the curve defined by the parametric equations

$$x = t^2 - 3t \qquad y = t - 1$$

Solution For every value of t we get a point on the curve. For example, if $t = 0$, then $x = 0$ and $y = -1$, so the corresponding point is $(0, -1)$. In Figure 1 we plot the points (x, y) determined by the values of t shown in the following table.

The arrows on the curve indicate the direction of the curve for increasing values of t.

t	x	y
-2	10	-3
-1	4	-2
0	0	-1
1	-2	0
2	-2	1
3	0	2
4	4	3
5	10	4

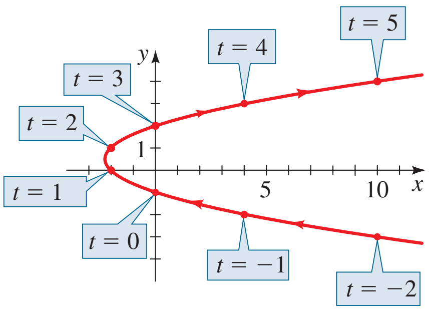


Figure 1

As t increases, a particle whose position is given by the parametric equations moves along the curve in the direction of the arrows.

Now Try Exercise 3 ■

If we replace t by $-t$ in Example 1, we obtain the parametric equations

$$x = t^2 + 3t \qquad y = -t - 1$$

The graph of these parametric equations (see Figure 2) is the same as the curve in Figure 1 but traced out in the opposite direction. On the other hand, if we replace t by

Bettmann/Getty Images

MARIA GAETANA AGNESI (1718–1799) is famous for having written *Instituzioni Analitiche*, one of the first calculus textbooks.

Agnesi was born into a wealthy family in Milan, Italy, the oldest of 21 children. She was a child prodigy, mastering many languages at an early age, including Latin, Greek, and Hebrew. At the age of 20 she published a series of essays on philosophy and natural science. After her mother died, Agnesi took on the task of educating her brothers. In 1748 Agnesi published her famous textbook, which she originally wrote as a text for tutoring her brothers. The book compiled and explained the mathematical knowledge of the day. It contains many carefully chosen examples, one of which is the curve now known as the "witch of Agnesi" (see Exercise 66). One review calls her book an "exposition by examples rather than by theory." The book gained Agnesi immediate recognition. Pope Benedict XIV appointed her to a position at the University of Bologna, writing, "we have had the idea that you should be awarded the well-known chair of mathematics, by which it comes of itself that you should not thank us but we you." This appointment was an extremely high honor for a woman, since very few women then were even allowed to attend university. Just two years later, Agnesi's father died, and she left mathematics completely. She became a nun and devoted the rest of her life and her wealth to caring for sick and dying women, herself dying in poverty at a poorhouse of which she had once been director.

$2t$ in Example 1, we obtain the parametric equations

$$x = 4t^2 - 6t \qquad y = 2t - 1$$

The graph of these parametric equations (see Figure 3) is again the same curve but is traced out "twice as fast." *Thus a parametrization contains more information than just the shape of the curve; it also indicates* how *the curve is being traced out*.

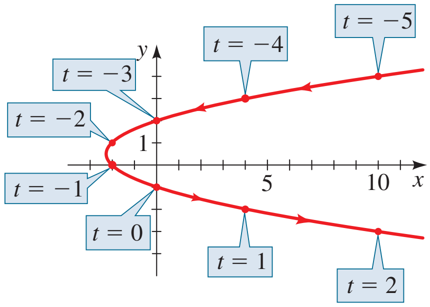


Figure 2 | $x = t^2 + 3t,\ y = -t - 1$

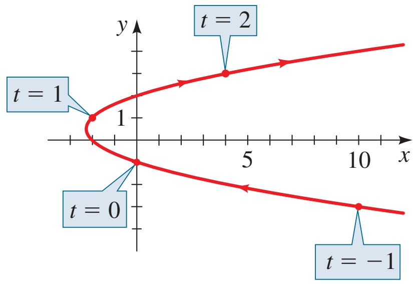


Figure 3 | $x = 4t^2 - 6t,\ y = 2t - 1$

■ Eliminating the Parameter

Often a curve given by parametric equations can also be represented by a single rectangular equation in x and y. The process of finding this equation is called *eliminating the parameter*. One way to do this is to solve for t in one equation, then substitute into the other.

Example 2 ■ Eliminating the Parameter

Eliminate the parameter in the parametric equations of Example 1.

Solution First we solve for t in the simpler equation, then we substitute into the other equation. From the equation $y = t - 1$ we get $t = y + 1$. Substituting into the equation for x, we get

$$x = t^2 - 3t = (y + 1)^2 - 3(y + 1) = y^2 - y - 2$$

Thus the curve in Example 1 has the rectangular equation $x = y^2 - y - 2$, so it is a parabola.

Now Try Exercise 5 ■

Eliminating the parameter often helps us identify the shape of a curve, as we see in the next two examples.

Example 3 ■ Modeling Circular Motion

The following parametric equations model the position of a moving object at time t (in seconds):

$$x = \cos t \qquad y = \sin t \qquad t \geq 0$$

Describe and graph the path of the object.

Solution To identify the curve, we eliminate the parameter. Since $\cos^2 t + \sin^2 t = 1$ and since $x = \cos t$ and $y = \sin t$ for every point (x, y) on the curve, we have

$$x^2 + y^2 = (\cos t)^2 + (\sin t)^2 = 1$$

This means that all points on the curve satisfy the equation $x^2 + y^2 = 1$, so the graph is a circle of radius 1 centered at the origin. As t increases from 0 to 2π, the point given by the parametric equations starts at $(1, 0)$ and moves counterclockwise once around the circle, as shown in Figure 4. So the object completes one revolution around the circle in 2π seconds. Notice that the parameter t can be interpreted as the angle shown in the figure.

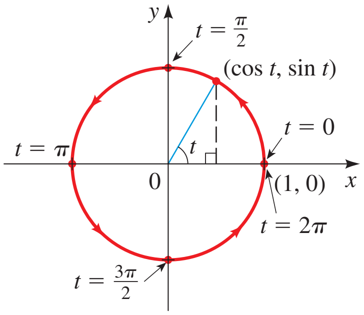


Figure 4

Now Try Exercise 27 ■

Example 4 ■ Sketching a Parametric Curve

Eliminate the parameter, and sketch the graph of the parametric equations

$$x = \sin t \qquad y = 2 - \cos^2 t$$

Solution To eliminate the parameter, we first use the trigonometric identity $\cos^2 t = 1 - \sin^2 t$ to change the second equation:

$$y = 2 - \cos^2 t = 2 - (1 - \sin^2 t) = 1 + \sin^2 t$$

Now we can substitute $\sin t = x$ from the first equation to get

$$y = 1 + x^2$$

so the point (x, y) moves along the parabola $y = 1 + x^2$. However, since $-1 \le \sin t \le 1$, we have $-1 \le x \le 1$, so the parametric equations represent only the part of the parabola between $x = -1$ and $x = 1$. Since $\sin t$ is periodic, the point $(x, y) = (\sin t, 2 - \cos^2 t)$ moves back and forth infinitely often along the parabola between the points $(-1, 2)$ and $(1, 2)$, as shown in Figure 5.

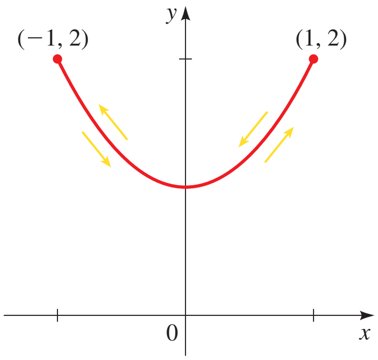


Figure 5

Now Try Exercise 15 ■

■ Finding Parametric Equations for a Curve

It is often possible to find parametric equations for a curve by using some geometric properties that define the curve, as in the next two examples.

Example 5 ■ Finding Parametric Equations for a Graph

Find parametric equations for the line of slope 3 that passes through the point $(2, 6)$.

Solution Let's start at the point $(2, 6)$ and move up and to the right along this line. Because the line has slope 3, for every 1 unit we move to the right, we must move upward 3 units. In other words, if we increase the x-coordinate by t units, we must correspondingly increase the y-coordinate by $3t$ units. This leads to the parametric equations

$$x = 2 + t \qquad y = 6 + 3t$$

To confirm that these equations give the desired line, we eliminate the parameter. We solve for t in the first equation and substitute into the second to get

$$y = 6 + 3(x - 2) = 3x$$

Thus the slope-intercept form of the equation of this line is $y = 3x$, which is a line of slope 3 that does pass through $(2, 6)$ as required. The graph is shown in Figure 6.

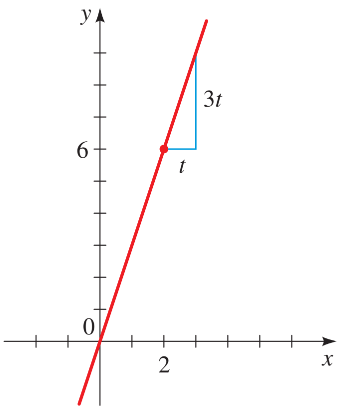


Figure 6

Now Try Exercise 33 ■

Example 6 ■ Parametric Equations for the Cycloid

As a circle rolls along a straight line, the curve traced out by a fixed point P on the circumference of the circle is called a **cycloid** (see Figure 7). If the circle has radius a and rolls along the x-axis, with one position of the point P being at the origin, find parametric equations for the cycloid.

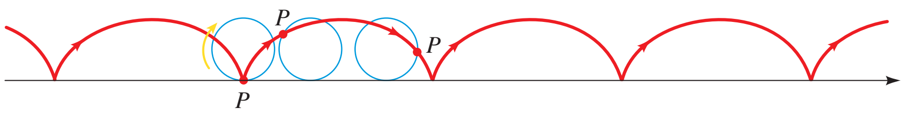


Figure 7

Solution Figure 8 shows the circle and the point P after the circle has rolled through an angle θ (in radians). The distance $d(O, T)$ that the circle has rolled must be the same as the length of the arc PT, which, by the arc length formula, is $a\theta$. (See Section 6.1.) This means that the center of the circle is $C(a\theta, a)$.

Let the coordinates of P be (x, y). Then from Figure 8 (which illustrates the case $0 < \theta < \pi/2$), we see that

$$x = d(O, T) - d(P, Q) = a\theta - a\sin\theta = a(\theta - \sin\theta)$$

$$y = d(T, C) - d(Q, C) = a - a\cos\theta = a(1 - \cos\theta)$$

so parametric equations for the cycloid are

$$x = a(\theta - \sin\theta) \qquad y = a(1 - \cos\theta)$$

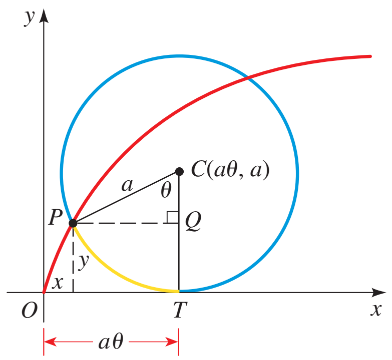


Figure 8

Now Try Exercise 53 ■

The cycloid has a number of interesting physical properties. It is the "curve of quickest descent" in the following sense. Let's choose two points P and Q that are not directly above each other and join them with a wire. Suppose we allow a bead to slide down the wire under the influence of gravity (ignoring friction). Of all possible shapes into which the wire can be bent, the bead will slide from P to Q the fastest when the shape is half of an arch of an inverted cycloid (see Figure 9). The cycloid is also the "curve of equal descent" in the sense that no matter where we place a bead B on a cycloid-shaped wire, it takes the same time to slide to the bottom (see Figure 10). These rather surprising properties of the cycloid were proved (using calculus) in the 17th century by several mathematicians and physicists, including Johann Bernoulli, Blaise Pascal, and Christiaan Huygens.

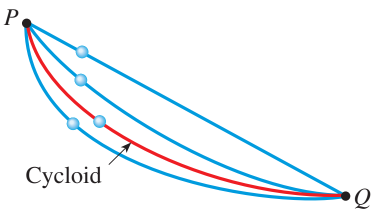


Figure 9

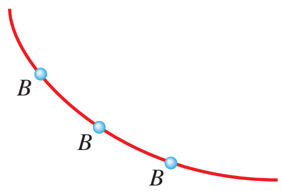
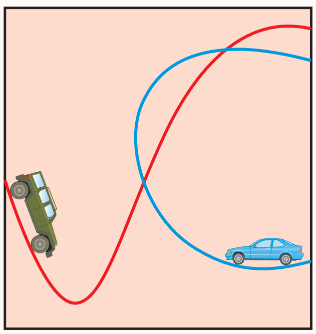


Figure 10



Discovery Project ■ Collision

Two cars are travelling on two different roads. The roads intersect, but will the cars collide? In this project we model the location of each car by a pair of parametric equations, with time t as the parameter. The equations will tell us if the cars collide, that is, if they will be at the same location at the same time. We will also use graphing devices to "animate" the parametric equations, so we can visually confirm whether the cars collide. You can find the project at **www.stewartmath.com**.

■ Using Graphing Devices to Graph Parametric Curves

Most graphing calculators and computer graphing programs can be used to graph parametric equations. Such devices are particularly useful in sketching complicated curves like the one shown in Figure 11.

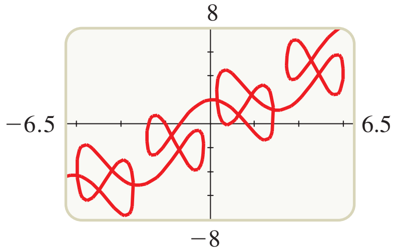
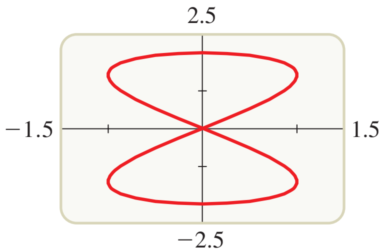
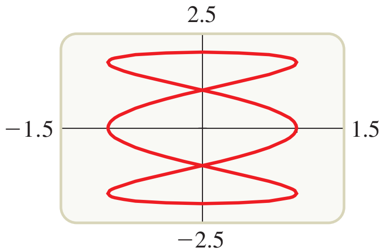


Figure 11 | $x = t + 2\sin 2t$, $y = t + 2\cos 5t$



(a) $x = \sin 2t$, $y = 2\cos t$



(b) $x = \sin 3t$, $y = 2\cos t$

Figure 12

Example 7 ■ Graphing Parametric Curves

Use a graphing device to draw the following parametric curves. Discuss their similarities and differences.

(a) $x = \sin 2t$
$y = 2\cos t$

(b) $x = \sin 3t$
$y = 2\cos t$

Solution In both parts (a) and (b) the graph will lie inside the rectangle given by $-1 \le x \le 1$, $-2 \le y \le 2$, because both the sine and the cosine of any number will be between -1 and 1. Thus we may use the viewing rectangle $[-1.5, 1.5]$ by $[-2.5, 2.5]$.

(a) Since $2\cos t$ is periodic with period 2π (see Section 5.3) and since $\sin 2t$ has period π, letting t vary over the interval $0 \le t \le 2\pi$ gives us the complete graph, which is shown in Figure 12(a).

(b) Again, letting t take on values between 0 and 2π gives the complete graph shown in Figure 12(b).

Both graphs are *closed curves*, which means that they form loops with the same starting and ending point; also, both graphs cross over themselves. However, the graph in Figure 12(a) has two loops, like a figure eight, whereas the graph in Figure 12(b) has three loops.

Now Try Exercise 39 ■

The curves graphed in Example 7 are called Lissajous figures. A **Lissajous figure** is the graph of a pair of parametric equations of the form

$$x = A\sin \omega_1 t \qquad y = B\cos \omega_2 t$$

where A, B, ω_1, and ω_2 are positive real constants. Since both $\sin \omega_1 t$ and $\cos \omega_2 t$ are between -1 and 1, a Lissajous figure will lie inside the rectangle determined by $-A \le x \le A$, $-B \le y \le B$. This fact can be used to choose a viewing rectangle when graphing a Lissajous figure, as we did in Example 7.

Recall from Section 8.1 that rectangular coordinates (x, y) and polar coordinates (r, θ) are related by the equations $x = r\cos\theta$, $y = r\sin\theta$. Thus we can graph the polar equation $r = f(\theta)$ by changing it to parametric form as follows.

$$x = r\cos\theta = f(\theta)\cos\theta \qquad \text{Since } r = f(\theta)$$
$$y = r\sin\theta = f(\theta)\sin\theta$$

Replacing θ by the standard parametric variable t, we have the following result.

Polar Equations in Parametric Form

The graph of the polar equation $r = f(\theta)$ is the same as the graph of the parametric equations

$$x = f(t)\cos t \qquad y = f(t)\sin t$$

Example 8 ■ Parametric Form of a Polar Equation

Consider the polar equation $r = \theta$, $1 \le \theta \le 10\pi$.

(a) Express the equation in parametric form.

(b) Draw a graph of the parametric equations from part (a).

Solution Here, $r = f(\theta) = \theta$, so $f(t) = t$.

(a) The given polar equation is equivalent to the parametric equations

$$x = t\cos t \qquad y = t\sin t$$

(b) Since $10\pi \approx 31.42$, we use the viewing rectangle $[-32, 32]$ by $[-32, 32]$, and we let t vary from 1 to 10π. The resulting graph shown in Figure 13 is a *spiral*.

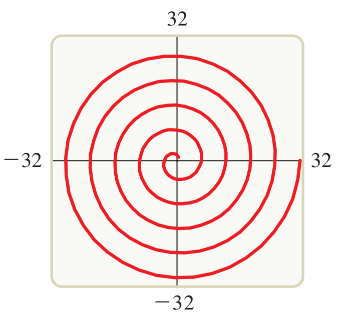


Figure 13 | $x = t\cos t$, $y = t\sin t$

Now Try Exercise 47 ■

8.4 Exercises

Concepts

1. (a) The parametric equations

$$x = f(t) \qquad y = g(t)$$

give the coordinates of a point $(x, y) = (f(t), g(t))$ for appropriate values of t. The variable t is called a ________.

(b) Suppose that the parametric equations

$$x = t \qquad y = t^2 \qquad t \ge 0$$

model the position of a moving object at time t. When $t = 0$, the object is at (___ , ___), and when $t = 1$, the object is at (___ , ___).

(c) If we eliminate the parameter in part (b), we get the equation $y =$ ________. We see from this equation that the path of the moving object is a ________.

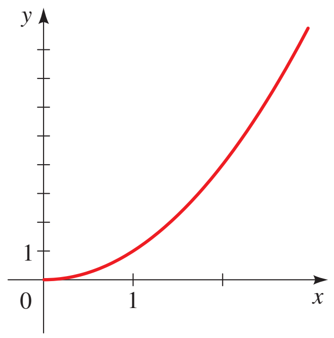
2. (a) *True or False?* The same curve can be described by parametric equations in many different ways.

(b) The parametric equations

$$x = 2t \qquad y = (2t)^2$$

model the position of a moving object at time t. When $t = 0$, the object is at (___ , ___), and when $t = 1$, the object is at (___ , ___).

(c) If we eliminate the parameter, we get the equation $y =$ ________, which is the same equation as in Exercise 1(c). So the objects in Exercises 1(b) and 2(b) move along the same ________ but traverse the path differently. Indicate the position of each object when $t = 0$ and when $t = 1$ on the following graph.



Skills

3–26 ■ Sketching a Curve by Eliminating the Parameter A pair of parametric equations is given. **(a)** Sketch the curve represented by the parametric equations. Use arrows to indicate the direction of the curve as t increases. **(b)** Find an equation in rectangular coordinates for the curve by eliminating the parameter.

3. $x = 2t, \quad y = t + 6$

4. $x = 6t - 4, \quad y = 3t, \quad t \ge 0$

5. $x = t^2, \quad y = t - 2, \quad 2 \le t \le 4$

6. $x = 2t + 1, \quad y = \left(t + \frac{1}{2}\right)^2$

7. $x = \sqrt{t}, \quad y = 1 - t$

8. $x = t^2, \quad y = t^4 + 1$

9. $x = \dfrac{1}{t}, \quad y = t + 1$

10. $x = t + 1, \quad y = \dfrac{t}{t+1}$

11. $x = 4t^2, \quad y = 8t^3$

12. $x = |t|, \quad y = |1 - |t||$

13. $x = 2\sin t, \quad y = 2\cos t, \quad 0 \le t \le \pi$

14. $x = 2\cos t, \quad y = 3\sin t, \quad 0 \le t \le 2\pi$

15. $x = \sin^2 t, \quad y = \sin^4 t$

16. $x = \sin^2 t, \quad y = \cos t$

17. $x = \cos t, \quad y = \cos 2t$

18. $x = \cos 2t, \quad y = \sin 2t$

19. $x = \sec t, \quad y = \tan t, \quad 0 \le t < \pi/2$

20. $x = \cot t, \quad y = \csc t, \quad 0 < t < \pi$

21. $x = \tan t, \quad y = \cot t, \quad 0 < t < \pi/2$

22. $x = e^{-t}, \quad y = e^t$

23. $x = e^{2t}, \quad y = e^t$

24. $x = \sec t, \quad y = \tan^2 t, \quad 0 \le t < \pi/2$

25. $x = \cos^2 t, \quad y = \sin^2 t$

26. $x = \cos^3 t, \quad y = \sin^3 t, \quad 0 \le t \le 2\pi$

27–30 ■ Circular Motion The position of an object in circular motion is modeled by the given parametric equations. Describe the path of the object by stating the radius of the circle, the position at time $t = 0$, the orientation of the motion (clockwise or counterclockwise), and the time t that it takes to complete one revolution around the circle.

27. $x = 3\cos t, \quad y = 3\sin t$

28. $x = 2\sin t, \quad y = 2\cos t$

29. $x = \sin 2t, \quad y = \cos 2t$

30. $x = 4\cos 3t, \quad y = 4\sin 3t$

31–32 ■ Parametric Equations for Circular Motion Find parametric equations for the position of a particle moving along a circle centered at the origin, as described.

31. The particle travels clockwise around a circle with radius 5 and completes a revolution in 4π seconds.

32. The particle travels counterclockwise around a circle with radius 1 and completes a revolution in 2 seconds.

33–36 ■ Parametric Equations for Curves Find parametric equations for the curve with the given properties.

33. The line with slope $\frac{1}{2}$, passing through $(4, -1)$

34. The line passing through $(6, 7)$ and $(7, 8)$

35. The circle $x^2 + y^2 = a^2$.

36. The ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

37. Path of a Projectile If a projectile is launched from a cannon with an initial speed of v_0 ft/s at an angle α above the horizontal, then its position after t seconds is given by the parametric equations

$$x = (v_0 \cos \alpha)t \qquad y = (v_0 \sin \alpha)t - 16t^2$$

(where x and y are measured in feet). Show that the path of the projectile is a parabola by eliminating the parameter t.

38. Path of a Projectile Referring to Exercise 37, suppose the projectile is fired into the air with an initial speed of 2048 ft/s at an angle of 30° to the horizontal.

(a) After how many seconds will the projectile hit the ground?

(b) How far from the cannon will the projectile hit the ground?

(c) What is the maximum height attained by the projectile?

39–44 ■ Graphs of Parametric Equations Use a graphing device to draw the curve represented by the parametric equations.

39. $x = \sin t, \quad y = 2\cos 3t$

40. $x = 2\sin t, \quad y = \cos 4t$

41. $x = 3\sin 5t, \quad y = 5\cos 3t$

42. $x = \sin 4t, \quad y = \cos 3t$

43. $x = \sin(\cos t), \quad y = \cos t^{3/2}, \quad 0 \le t \le 2\pi$

44. $x = 2\cos t + \cos 2t, \quad y = 2\sin t - \sin 2t$

45–48 ■ Parametric Form of a Polar Equation A polar equation is given. **(a)** Express the polar equation in parametric form. **(b)** Use a graphing device to graph the parametric equations you found in part (a).

45. $r = 2^{\theta/12}, \quad 0 \le \theta \le 4\pi$

46. $r = \sin\theta + 2\cos\theta$

47. $r = \dfrac{4}{2 - \cos\theta}$

48. $r = 2^{\sin\theta}$

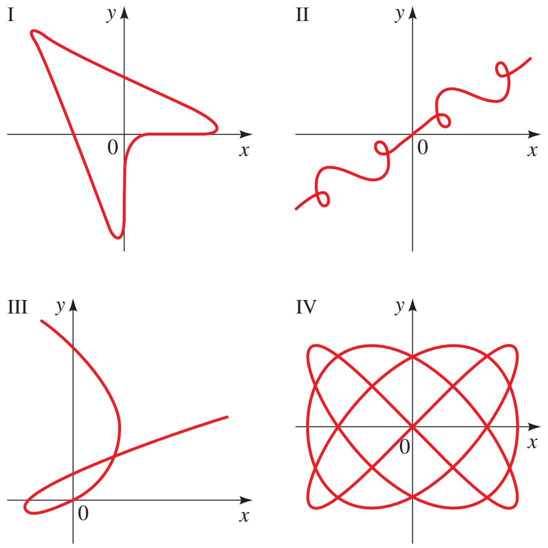
49–52 ■ Graphs of Parametric Equations Match the parametric equations with the graphs labeled I–IV. Give reasons for your answers.

49. $x = t^3 - 2t, \quad y = t^2 - t$

50. $x = \sin 3t, \quad y = \sin 4t$

51. $x = t + \sin 2t, \quad y = t + \sin 3t$

52. $x = \sin(t + \sin t), \quad y = \cos(t + \cos t)$

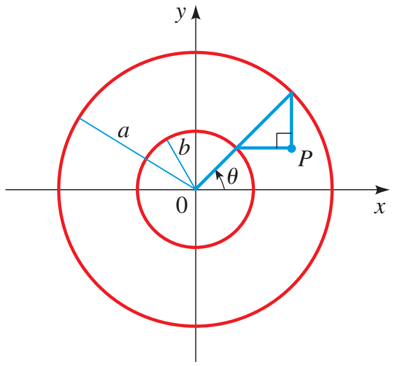


53. Finding Parametric Equations for a Curve Two circles of radius a and b are centered at the origin, as shown in the following figure. As the angle θ increases, the point P traces out a curve that lies between the circles.

(a) Find parametric equations for the curve, using θ as the parameter.

(b) Graph the curve using a graphing device, with $a = 3$ and $b = 2$.

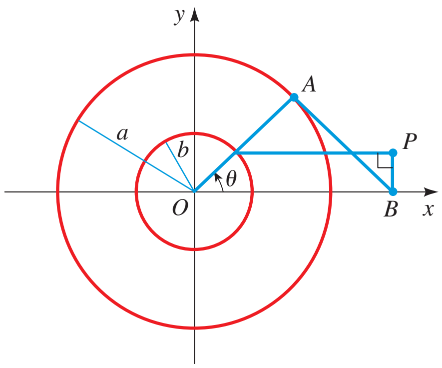
(c) Eliminate the parameter, and identify the curve.



54. Finding Parametric Equations for a Curve Two circles of radius a and b are centered at the origin, as shown in the figure. The line segment AB is tangent to the larger circle so that angle OAB is a right angle. (See Appendix A *Geometry Review*.)

(a) Find parametric equations for the curve traced out by the point P, using the angle θ as the parameter.

(b) Graph the curve using a graphing device, with $a = 3$ and $b = 2$.



55. Curtate Cycloid

(a) In Example 6, suppose the point P that traces out the curve lies not on the edge of the circle but rather at a fixed point inside the rim, at a distance b from the center (with $b < a$). The curve traced out by P is called a **curtate cycloid** (or **trochoid**). Show that parametric equations for the curtate cycloid are

$$x = a\theta - b\sin\theta \qquad y = a - b\cos\theta$$

(b) Sketch the graph using $a = 3$ and $b = 2$.

56. Prolate Cycloid

(a) In Exercise 55, if the point P lies *outside* the circle at a distance b from the center (with $b > a$), then the curve traced out by P is called a **prolate cycloid**. Show that parametric equations for the prolate cycloid are the same as the equations for the curtate cycloid.

(b) Sketch the graph for the case in which $a = 1$ and $b = 2$.

Skills Plus

57. Parametric Equations of a Hyperbola Eliminate the parameter θ in the following parametric equations. (This curve is called a **hyperbola**; see Section 10.3.)

$$x = a\tan\theta \qquad y = b\sec\theta$$

58. Parametric Equations of a Hyperbola Show that the following parametric equations represent a part of the hyperbola of Exercise 57.

$$x = a\sqrt{t} \qquad y = b\sqrt{t+1}$$

59–62 ■ Graphs of Parametric Equations Sketch the curve given by the parametric equations.

59. $x = t\cos t, \quad y = t\sin t, \quad t \ge 0$

60. $x = \sin t, \quad y = \sin 2t$

61. $x = \dfrac{3t}{1+t^3}, \quad y = \dfrac{3t^2}{1+t^3}$

62. $x = \cot t, \quad y = 2\sin^2 t, \quad 0 < t < \pi$

63. Hypocycloid A circle C of radius b rolls on the inside of a larger circle of radius a centered at the origin. Let P be a fixed point on the smaller circle, with initial position at the point $(a, 0)$ as shown in the figure. The curve traced out by P is called a **hypocycloid**.

(a) Show that parametric equations for the hypocycloid are

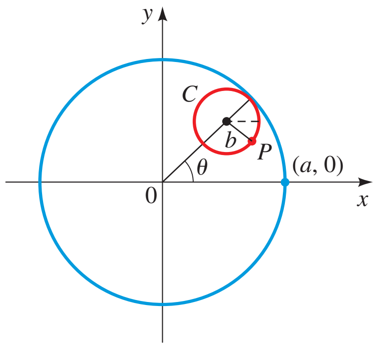
$$x = (a-b)\cos\theta + b\cos\left(\frac{a-b}{b}\theta\right)$$

$$y = (a-b)\sin\theta - b\sin\left(\frac{a-b}{b}\theta\right)$$

(b) If $a = 4b$, the hypocycloid is called an **astroid**. Show that in this case the parametric equations can be reduced to

$$x = a\cos^3\theta \qquad y = a\sin^3\theta$$

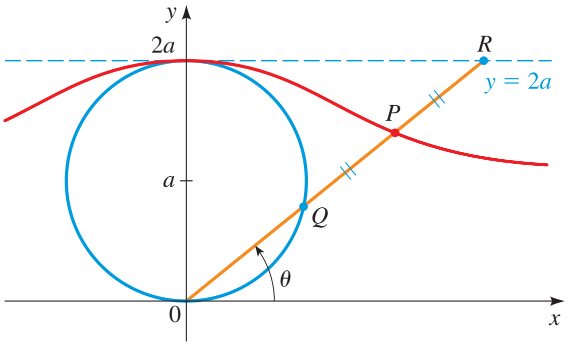
Sketch the curve. Eliminate the parameter to obtain an equation for the astroid in rectangular coordinates.



64. Epicycloid If the circle C of Exercise 63 rolls on the *outside* of the larger circle, the curve traced out by P is called an epicycloid. Find parametric equations for the epicycloid.

65. Longbow Curve In the following figure, the circle of radius a is stationary, and for every θ, the point P is the midpoint of the segment QR. The curve traced out by P for $0 < \theta < \pi$

is called the **longbow curve**. Find parametric equations for this curve.



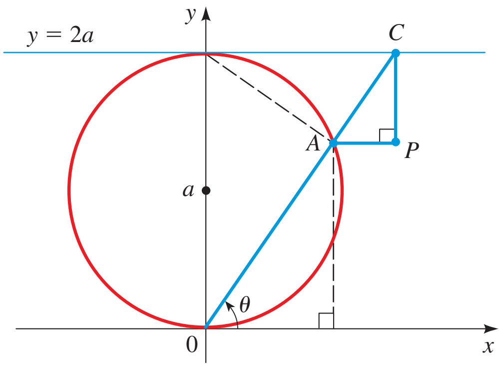
66. The Witch of Agnesi A curve, called a **witch of Agnesi**, consists of all points P determined as shown in the figure.

(a) Show that parametric equations for this curve can be written as

$$x = 2a \cot \theta \qquad y = 2a \sin^2 \theta$$

[*Hint:* A triangle inscribed in a semicircle is a right triangle. See Appendix A *Geometry Review*.]

(b) Graph the curve using a graphing device, with $a = 3$.



67. Eliminating the Parameter Eliminate the parameter θ in the parametric equations for the cycloid (Example 6) to obtain an equation in rectangular coordinates for the section of the curve given by $0 \le \theta \le \pi$.

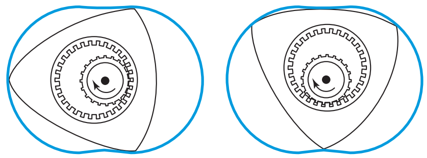
Applications

68. The Rotary Engine The Mazda MX-30 uses an unconventional engine (invented by Felix Wankel in 1954) in which the pistons are replaced by a rotor in the shape of a Reuleaux triangle (Exercise 6.1.95). The rotor turns in a special housing as shown in the figure. The vertices of the rotor maintain contact with the housing at all times, while the center of the rotor traces out a circle of radius r, turning the drive shaft. (For an animation go to www.wikipedia.org/wiki/Wankel_engine.) The shape of the housing is given by the following parametric equations (where R is the distance between the vertices and center of the rotor):

$$x = r \cos 3\theta + R \cos \theta \qquad y = r \sin 3\theta + R \sin \theta$$

(a) Suppose that the drive shaft has radius $r = 1$. Graph the curve given by the parametric equations for the following values of R: 0.5, 1, 3, 5.

(b) Which of the four values of R given in part (a) seems to best model the engine housing illustrated in the figure?



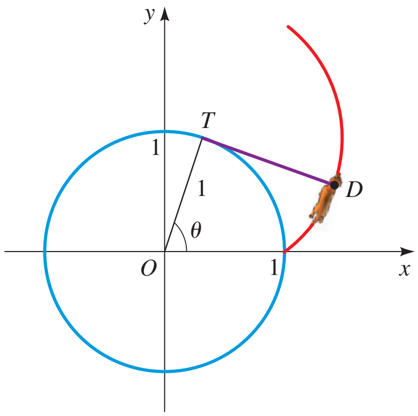
69. Spiral Path of a Dog A dog is tied by a long leash to a cylindrical tree trunk of radius 1 ft. He has managed to wrap the entire leash around the tree while playing in the yard, and the dog finds himself at the point $(1, 0)$ in the figure. Seeing a squirrel, he runs around the tree counterclockwise, unwinding the leash and keeping it taut while chasing the intruder.

(a) Show that parametric equations for the dog's path (called an **involute of a circle**) are

$$x = \cos \theta + \theta \sin \theta \qquad y = \sin \theta - \theta \cos \theta$$

[*Hint:* Note that the leash is always tangent to the tree, so OT is perpendicular to TD.]

(b) Graph the path of the dog for $0 \le \theta \le 4\pi$.



Discuss ■ Discover ■ Prove ■ Write

70. Discover ■ Write: More Information in Parametric Equations In this section we stated that parametric equations contain more information than just the shape of a curve. Write a short paragraph explaining this statement. Use the following example and your answers to parts (a) and (b) below in your explanation.

The position of a particle is given by the parametric equations

$$x = \sin t \qquad y = \cos t$$

where t represents time. We know that the shape of the path of the particle is a circle.

(a) How long does it take the particle to travel once around the circle? Find parametric equations for the case when the particle moves twice as fast around the circle.

(b) Does the particle travel clockwise or counterclockwise around the circle? Find parametric equations for the case when the particle moves in the opposite direction around the circle.

71. Discuss: Different Ways of Tracing Out a Curve The curves C, D, E, and F are defined parametrically as follows, where the parameter t takes on all real values unless otherwise stated:

$$C: \quad x = t, \quad y = t^2$$

$$D: \quad x = \sqrt{t}, \quad y = t, \quad t \geq 0$$

$$E: \quad x = \sin t, \quad y = \sin^2 t$$

$$F: \quad x = 3^t, \quad y = 3^{2t}$$

(a) Show that the points on all four of these curves satisfy the same equation in rectangular coordinates.

(b) Draw the graph of each curve and explain how the curves differ from one another.

8.5 Vectors

■ Geometric Description of Vectors ■ Vectors in the Coordinate Plane ■ Using Vectors to Model Velocity and Force

In applications of mathematics, certain quantities are determined completely by their magnitude—for example, length, mass, area, temperature, and energy. We speak of a length of 5 m or a mass of 3 kg; only one number is needed to describe each of these quantities. Such a quantity is called a **scalar**.

On the other hand, to describe the displacement of an object, two numbers are required: the *magnitude* and the *direction* of the displacement. To describe the velocity of a moving object, we must specify both the *speed* and the *direction* of travel. Quantities such as displacement, velocity, acceleration, and force that involve magnitude as well as direction are called *directed quantities*. One way to represent such quantities mathematically is through the use of *vectors*.

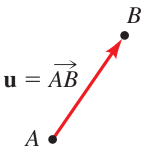


Figure 1

■ Geometric Description of Vectors

A **vector** in the plane is a line segment with an assigned direction. We sketch a vector as shown in Figure 1 with an arrow to specify the direction. We denote this vector by $\overrightarrow{AB}$. Point A is the **initial point**, and B is the **terminal point** of the vector $\overrightarrow{AB}$. The length of the line segment AB is called the **magnitude** or **length** of the vector and is denoted by $|\overrightarrow{AB}|$. We use boldface letters to denote vectors. Thus we write $\mathbf{u} = \overrightarrow{AB}$.

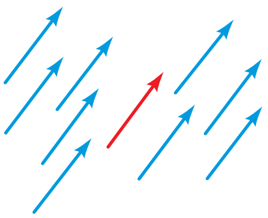


Figure 2

Two vectors are considered **equal** if they have equal magnitude and the same direction. Thus all the vectors in Figure 2 are equal. This definition of equality makes sense if we think of a vector as representing a displacement. Two such displacements are the same if they have equal magnitudes and the same direction. So the vectors in Figure 2 can be thought of as the *same* displacement applied to objects in different locations in the plane.

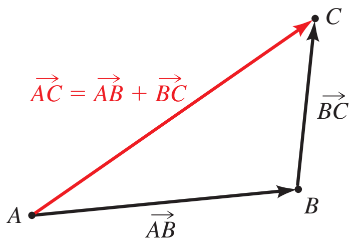


Figure 3

If the displacement $\mathbf{u} = \overrightarrow{AB}$ is followed by the displacement $\mathbf{v} = \overrightarrow{BC}$, then the resulting displacement is $\overrightarrow{AC}$ as shown in Figure 3. In other words, the single displacement represented by the vector $\overrightarrow{AC}$ has the same effect as the other two displacements together. We call the vector $\overrightarrow{AC}$ the **sum** of the vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$, and we write $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$. (The **zero vector**, denoted by $\mathbf{0}$, represents no displacement.) Thus to find the sum of any two vectors $\mathbf{u}$ and $\mathbf{v}$, we sketch vectors equal to $\mathbf{u}$ and $\mathbf{v}$ with the initial point of one at the terminal point of the other [see Figure 4(a)]. If we draw $\mathbf{u}$ and $\mathbf{v}$ starting at the same point, then $\mathbf{u} + \mathbf{v}$ is the vector that is the diagonal of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$ shown in Figure 4(b).

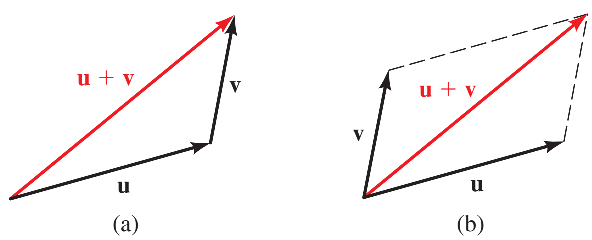


Figure 4 | Addition of vectors

If c is a real number and $\mathbf{v}$ is a vector, we define a new vector $c\mathbf{v}$ as follows: The vector $c\mathbf{v}$ has magnitude $|c||\mathbf{v}|$ and has the same direction as $\mathbf{v}$ if $c > 0$ and the opposite direction if $c < 0$. If $c = 0$, then $c\mathbf{v} = \mathbf{0}$, the zero vector. This process is called **multiplication of a vector by a scalar**. Multiplying a vector by a scalar has the effect of stretching or shrinking the vector. Figure 5 shows graphs of the vector $c\mathbf{v}$ for different values of c. We write the vector $(-1)\mathbf{v}$ as $-\mathbf{v}$. Thus $-\mathbf{v}$ is the vector with the same length as $\mathbf{v}$ but with opposite direction.

The **difference** of two vectors $\mathbf{u}$ and $\mathbf{v}$ is defined by $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$. Figure 6 shows that the vector $\mathbf{u} - \mathbf{v}$ is the other diagonal of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$.

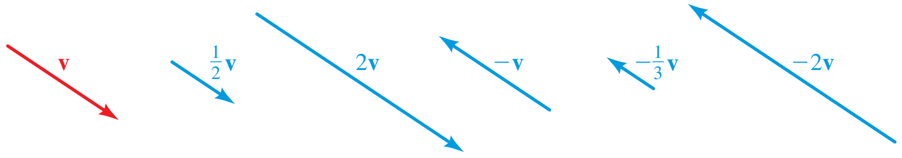


Figure 5 | Multiplication of a vector by a scalar

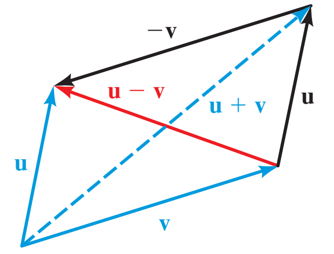


Figure 6 | Subtraction of vectors

Vectors in the Coordinate Plane

So far, we've discussed vectors geometrically. By placing a vector in a coordinate plane, we can describe it analytically (that is, by using components). In Figure 7(a), to go from the initial point of the vector $\mathbf{v}$ to the terminal point, we move a_1 units to the right and a_2 units upward. We represent $\mathbf{v}$ as an ordered pair of real numbers.

Note the distinction between the *vector* $\langle a_1, a_2\rangle$ and the *point* (a_1, a_2).

$$\mathbf{v} = \langle a_1, a_2\rangle$$

where a_1 is the **horizontal component** of $\mathbf{v}$ and a_2 is the **vertical component** of $\mathbf{v}$. Remember that a vector represents a magnitude and a direction, not a particular arrow in the plane. Thus the vector $\langle a_1, a_2\rangle$ has many different representations, depending on its initial point [see Figure 7(b)].

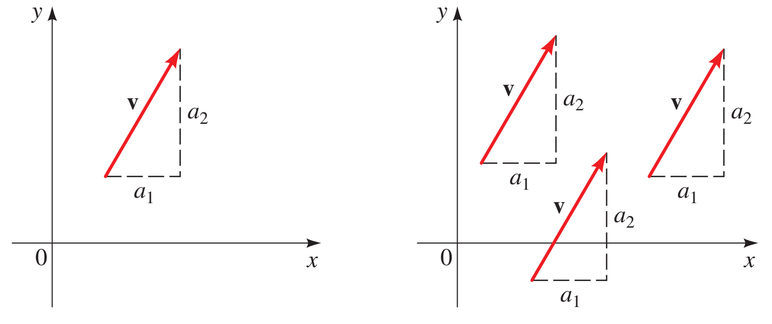


Figure 7

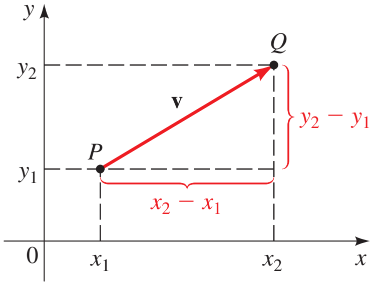


Figure 8

Using Figure 8, we can state the relationship between a geometric representation of a vector and the analytic one as follows.

Component Form of a Vector

If a vector $\mathbf{v}$ is represented in the plane with initial point $P(x_1, y_1)$ and terminal point $Q(x_2, y_2)$, then

$$\mathbf{v} = \langle x_2 - x_1, y_2 - y_1\rangle$$

Example 1 ■ Describing Vectors in Component Form

(a) Find the component form of the vector $\mathbf{u}$ with initial point $(-2, 5)$ and terminal point $(3, 7)$.

(b) If the vector $\mathbf{v} = \langle 3, 7 \rangle$ is sketched with initial point $(2, 4)$, what is its terminal point?

(c) Sketch representations of the vector $\mathbf{w} = \langle 2, 3 \rangle$ with initial points at $(0, 0)$, $(2, 2)$, $(-2, -1)$, and $(1, 4)$.

Solution

(a) The desired vector is

$$\mathbf{u} = \langle 3 - (-2), 7 - 5 \rangle = \langle 5, 2 \rangle$$

(b) Let the terminal point of $\mathbf{v}$ be (x, y). Then

$$\langle x - 2, y - 4 \rangle = \langle 3, 7 \rangle$$

So $x - 2 = 3$ and $y - 4 = 7$, or $x = 5$ and $y = 11$. The terminal point is $(5, 11)$.

(c) Representations of the vector $\mathbf{w}$ are sketched in Figure 9.

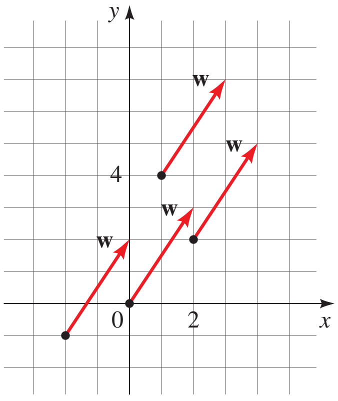


Figure 9

Now Try Exercises 11, 19, and 23 ■

We now give analytic definitions of the various operations on vectors that we have described geometrically. Let's start with equality of vectors. We've said that two vectors are equal if they have equal magnitude and the same direction. For the vectors $\mathbf{u} = \langle a_1, a_2 \rangle$ and $\mathbf{v} = \langle b_1, b_2 \rangle$ this means that $a_1 = b_1$ and $a_2 = b_2$. In other words, two vectors are **equal** if and only if their corresponding components are equal. Thus all the arrows in Figure 7(b) represent the same vector, as do all the arrows in Figure 9.

Applying the Pythagorean Theorem to the triangle in Figure 10, we obtain the following formula for the magnitude of a vector.

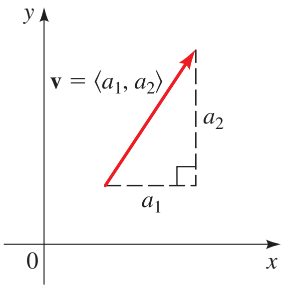


Figure 10

Magnitude of a Vector

The **magnitude** or **length** of a vector $\mathbf{v} = \langle a_1, a_2 \rangle$ is

$$|\mathbf{v}| = \sqrt{a_1^2 + a_2^2}$$

Example 2 ■ Magnitudes of Vectors

Find the magnitude of each vector.

(a) $\mathbf{u} = \langle 2, -3 \rangle$ **(b)** $\mathbf{v} = \langle 5, 0 \rangle$ **(c)** $\mathbf{w} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$

Solution

(a) $|\mathbf{u}| = \sqrt{2^2 + (-3)^2} = \sqrt{13}$

(b) $|\mathbf{v}| = \sqrt{5^2 + 0^2} = \sqrt{25} = 5$

(c) $|\mathbf{w}| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = \sqrt{\frac{9}{25} + \frac{16}{25}} = 1$

Now Try Exercise 37 ■

The following definitions of addition, subtraction, and scalar multiplication of vectors correspond to the geometric descriptions given earlier. Figure 11 (on the next page) shows how the analytic definition of addition corresponds to the geometric one.

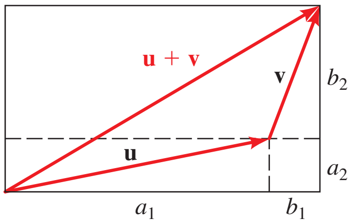


Figure 11

Algebraic Operations on Vectors

If $\mathbf{u} = \langle a_1, a_2 \rangle$ and $\mathbf{v} = \langle b_1, b_2 \rangle$, then

$$\mathbf{u} + \mathbf{v} = \langle a_1 + b_1, a_2 + b_2 \rangle$$
$$\mathbf{u} - \mathbf{v} = \langle a_1 - b_1, a_2 - b_2 \rangle$$
$$c\mathbf{u} = \langle ca_1, ca_2 \rangle \qquad (c \in \mathbb{R})$$

Example 3 ■ Operations with Vectors

If $\mathbf{u} = \langle 2, -3 \rangle$ and $\mathbf{v} = \langle -1, 2 \rangle$, find $\mathbf{u} + \mathbf{v}$, $\mathbf{u} - \mathbf{v}$, $2\mathbf{u}$, $-3\mathbf{v}$, and $2\mathbf{u} + 3\mathbf{v}$.

Solution By the definitions of the vector operations we have

$$\mathbf{u} + \mathbf{v} = \langle 2, -3 \rangle + \langle -1, 2 \rangle = \langle 1, -1 \rangle$$
$$\mathbf{u} - \mathbf{v} = \langle 2, -3 \rangle - \langle -1, 2 \rangle = \langle 3, -5 \rangle$$
$$2\mathbf{u} = 2\langle 2, -3 \rangle = \langle 4, -6 \rangle$$
$$-3\mathbf{v} = -3\langle -1, 2 \rangle = \langle 3, -6 \rangle$$
$$2\mathbf{u} + 3\mathbf{v} = 2\langle 2, -3 \rangle + 3\langle -1, 2 \rangle = \langle 4, -6 \rangle + \langle -3, 6 \rangle = \langle 1, 0 \rangle$$

Now Try Exercise 31 ■

The following properties for vector operations can be proved from the definitions. The **zero vector** is the vector $\mathbf{0} = \langle 0, 0 \rangle$. It plays the same role for addition of vectors as the number 0 does for addition of real numbers.

Properties of Vectors

Vector Addition	Multiplication by a Scalar
$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	$c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$	$(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
$\mathbf{u} + \mathbf{0} = \mathbf{u}$	$(cd)\mathbf{u} = c(d\mathbf{u}) = d(c\mathbf{u})$
$\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$	$1\mathbf{u} = \mathbf{u}$
Length of a Vector	$0\mathbf{u} = \mathbf{0}$
$\lvert c\mathbf{u} \rvert = \lvert c \rvert \lvert \mathbf{u} \rvert$	$c\mathbf{0} = \mathbf{0}$

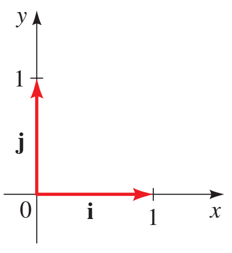


Figure 12

A vector of length 1 is called a **unit vector**. For instance, in Example 2(c) the vector $\mathbf{w} = \langle \frac{3}{5}, \frac{4}{5} \rangle$ is a unit vector. Two useful unit vectors are $\mathbf{i}$ and $\mathbf{j}$, defined by

$$\mathbf{i} = \langle 1, 0 \rangle \qquad \mathbf{j} = \langle 0, 1 \rangle$$

(See Figure 12.) These vectors are special because any vector can be expressed in terms of them. (See Figure 13.)

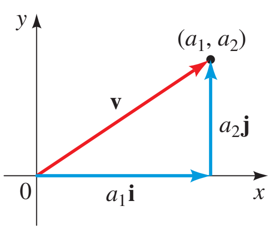


Figure 13

Vectors in Terms of i and j

The vector $\mathbf{v} = \langle a_1, a_2 \rangle$ can be expressed in terms of $\mathbf{i}$ and $\mathbf{j}$ by

$$\mathbf{v} = \langle a_1, a_2 \rangle = a_1\mathbf{i} + a_2\mathbf{j}$$

Example 4 ■ Vectors in Terms of i and j

(a) Write the vector $\mathbf{u} = \langle 5, -8 \rangle$ in terms of $\mathbf{i}$ and $\mathbf{j}$.

(b) If $\mathbf{u} = 3\mathbf{i} + 2\mathbf{j}$ and $\mathbf{v} = -\mathbf{i} + 6\mathbf{j}$, write $2\mathbf{u} + 5\mathbf{v}$ in terms of $\mathbf{i}$ and $\mathbf{j}$.

Solution

(a) $\mathbf{u} = 5\mathbf{i} + (-8)\mathbf{j} = 5\mathbf{i} - 8\mathbf{j}$

(b) The properties of addition and scalar multiplication of vectors show that we can manipulate vectors in the same way as algebraic expressions. Thus

$$\begin{aligned} 2\mathbf{u} + 5\mathbf{v} &= 2(3\mathbf{i} + 2\mathbf{j}) + 5(-\mathbf{i} + 6\mathbf{j}) \\ &= (6\mathbf{i} + 4\mathbf{j}) + (-5\mathbf{i} + 30\mathbf{j}) \\ &= \mathbf{i} + 34\mathbf{j} \end{aligned}$$

Now Try Exercises 27 and 35 ■

Let **v** be a vector in the plane with its initial point at the origin. The **direction** of **v** is θ, the smallest positive angle in standard position formed by the positive x-axis and **v**. (See Figure 14.) If we know the magnitude and direction of a vector, then Figure 14 shows that we can find the horizontal and vertical components of the vector.

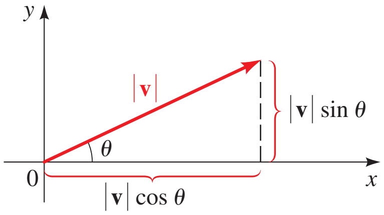


Figure 14

Horizontal and Vertical Components of a Vector

Let **v** be a vector with magnitude $|\mathbf{v}|$ and direction θ.

Then $\mathbf{v} = \langle a_1, a_2 \rangle = a_1\mathbf{i} + a_2\mathbf{j}$, where

$$a_1 = |\mathbf{v}|\cos\theta \qquad \text{and} \qquad a_2 = |\mathbf{v}|\sin\theta$$

Thus we can express **v** as

$$\mathbf{v} = |\mathbf{v}|\cos\theta\,\mathbf{i} + |\mathbf{v}|\sin\theta\,\mathbf{j}$$

Example 5 ■ Components and Direction of a Vector

(a) A vector **v** has length 8 and direction $\pi/3$. Find the horizontal and vertical components, and write **v** in terms of **i** and **j**.

(b) Find the direction of the vector $\mathbf{u} = -\sqrt{3}\,\mathbf{i} + \mathbf{j}$.

Solution

(a) We have $\mathbf{v} = \langle a, b \rangle$, where the components are given by

$$a = 8\cos\frac{\pi}{3} = 4 \qquad \text{and} \qquad b = 8\sin\frac{\pi}{3} = 4\sqrt{3}$$

Thus $\mathbf{v} = \langle 4, 4\sqrt{3} \rangle = 4\mathbf{i} + 4\sqrt{3}\,\mathbf{j}$.

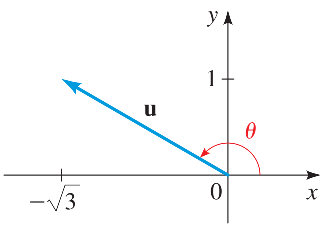


Figure 15

(b) From Figure 15 we see that the direction θ has the property that

$$\tan\theta = \frac{1}{-\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

Thus the reference angle for θ is $\pi/6$. Since the terminal point of the vector **u** is in Quadrant II, it follows that $\theta = 5\pi/6$.

Now Try Exercises 41 and 51 ■

■ Using Vectors to Model Velocity and Force

The use of bearings (such as N 30° E) to describe directions is explained in Section 6.6.

The **velocity** of a moving object is modeled by a vector whose direction is the direction of motion and whose magnitude is the speed. Figure 16 on the next page shows some vectors **u**, representing the velocity of wind flowing in the direction N 30° E, and a vector **v**, representing the velocity of an airplane flying through this wind at the point P. From our experience we know that wind affects both the speed and the direction of an airplane. The true velocity of the plane (relative to the ground) is given by the vector $\mathbf{w} = \mathbf{u} + \mathbf{v}$, as shown in Figure 17.

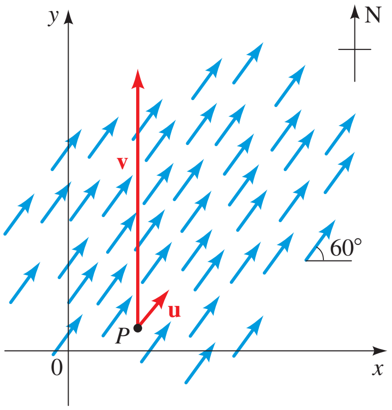


Figure 16

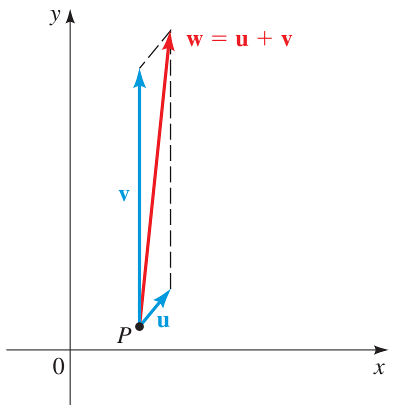


Figure 17

Example 6 ■ The True Speed and Direction of an Airplane

An airplane heads due north at 300 mi/h. It experiences a 40 mi/h crosswind flowing in the direction N 30° E, as shown in Figure 16.

(a) Express the velocity $\mathbf{v}$ of the airplane relative to the air and the velocity $\mathbf{u}$ of the wind, in component form.

(b) Find the true velocity of the airplane as a vector.

(c) Find the true speed and direction of the airplane.

Solution

(a) The velocity of the airplane relative to the air is $\mathbf{v} = 0\,\mathbf{i} + 300\,\mathbf{j} = 300\,\mathbf{j}$. By the formulas for the components of a vector we find that the velocity of the wind is

$$\begin{aligned} \mathbf{u} &= (40\cos 60°)\mathbf{i} + (40\sin 60°)\mathbf{j} \\ &= 20\,\mathbf{i} + 20\sqrt{3}\,\mathbf{j} \\ &\approx 20\,\mathbf{i} + 34.64\,\mathbf{j} \end{aligned}$$

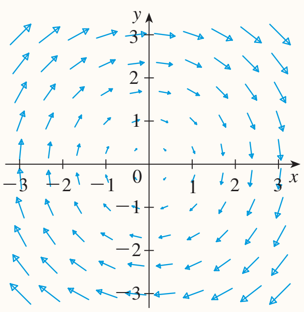
The **true velocity** is the velocity relative to the ground.

(b) The true velocity of the airplane is given by the vector $\mathbf{w} = \mathbf{u} + \mathbf{v}$:

$$\begin{aligned} \mathbf{w} = \mathbf{u} + \mathbf{v} &= (20\,\mathbf{i} + 20\sqrt{3}\,\mathbf{j}) + (300\,\mathbf{j}) \\ &= 20\,\mathbf{i} + (20\sqrt{3} + 300)\mathbf{j} \\ &\approx 20\,\mathbf{i} + 334.64\,\mathbf{j} \end{aligned}$$

(c) The true speed of the airplane is given by the magnitude of $\mathbf{w}$:

$$|\mathbf{w}| \approx \sqrt{(20)^2 + (334.64)^2} \approx 335.2 \text{ mi/h}$$



Discovery Project ■ Vector Fields

A *vector field* is a collection of vectors, like the vectors that model wind velocity at each point in some region. In this project we use a graphing device to graph vector fields that are given by a rule. The graph shown here displays at each point (x, y) the vector $y\,\mathbf{i} - x\,\mathbf{j}$. You can see that the vectors appear to rotate about the origin. We'll graph different vector fields and then visually determine the path a particle would take when it is put in the field. You can find the project at **www.stewartmath.com**.

The direction of the airplane is the direction θ of the vector $\mathbf{w}$. The angle θ has the property that $\tan\theta \approx 334.64/20 = 16.732$, so $\theta \approx 86.6°$. Thus the airplane is heading in the direction N 3.4° E.

Now Try Exercise 59 ■

Example 7 ■ Calculating a Heading

A boater launches a boat from one shore of a straight river and wants to land at the point directly on the opposite shore. If the speed of the boat (relative to the water) is 10 mi/h and the river is flowing east at the rate of 5 mi/h, in what direction should the boat be headed in order to arrive at the desired landing point?

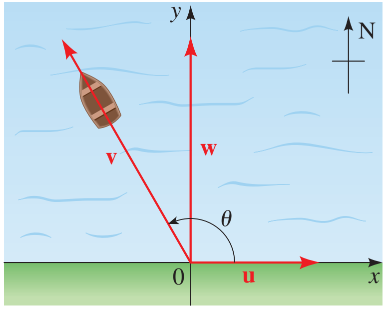


Figure 18

Solution We choose a coordinate system with the origin at the initial position of the boat, as shown in Figure 18. Let $\mathbf{u}$ and $\mathbf{v}$ represent the velocities of the river and the boat, respectively. Thus $\mathbf{u} = 5\,\mathbf{i}$ and, since the speed of the boat is 10 mi/h, we have $|\mathbf{v}| = 10$, so

$$\mathbf{v} = (10\cos\theta)\mathbf{i} + (10\sin\theta)\mathbf{j}$$

where the angle θ is as shown in Figure 18. The true course of the boat is given by the vector $\mathbf{w} = \mathbf{u} + \mathbf{v}$. We have

$$\begin{aligned}\mathbf{w} = \mathbf{u} + \mathbf{v} &= 5\,\mathbf{i} + (10\cos\theta)\mathbf{i} + (10\sin\theta)\mathbf{j}\\ &= (5 + 10\cos\theta)\mathbf{i} + (10\sin\theta)\mathbf{j}\end{aligned}$$

Since the boater wants to land at a point directly across the river, the direction of the boat should have horizontal component 0. In other words, the boat should be pointed at an angle θ in such a way that

$$\begin{aligned}5 + 10\cos\theta &= 0\\ \cos\theta &= -\tfrac{1}{2}\\ \theta &= 120°\end{aligned}$$

Thus the boat should be headed in the direction $\theta = 120°$ (or N 30° W).

Now Try Exercise 61 ■

Force is also represented by a vector. Intuitively, we can think of force as describing a push or a pull on an object, for example, a horizontal push of a book across a table or the downward pull of the earth's gravity on a ball. Force is measured in pounds (or in newtons, in the metric system). For instance, a man weighing 200 lb exerts a force of 200 lb downward on the ground. If several forces are acting on an object, the **resultant force** experienced by the object is the vector sum of these forces.

Example 8 ■ Resultant Force

Two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ with magnitudes 10 lb and 20 lb, respectively, act on an object at a point P, as shown in Figure 19. Find the resultant force acting at P.

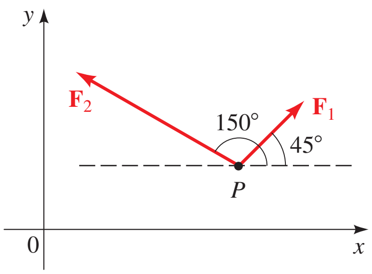


Figure 19

Solution We write $\mathbf{F}_1$ and $\mathbf{F}_2$ in component form:

$$\begin{aligned}\mathbf{F}_1 &= (10\cos 45°)\mathbf{i} + (10\sin 45°)\mathbf{j} = 10\frac{\sqrt{2}}{2}\mathbf{i} + 10\frac{\sqrt{2}}{2}\mathbf{j}\\ &= 5\sqrt{2}\,\mathbf{i} + 5\sqrt{2}\,\mathbf{j}\\ \mathbf{F}_2 &= (20\cos 150°)\mathbf{i} + (20\sin 150°)\mathbf{j} = -20\frac{\sqrt{3}}{2}\mathbf{i} + 20\left(\frac{1}{2}\right)\mathbf{j}\\ &= -10\sqrt{3}\,\mathbf{i} + 10\mathbf{j}\end{aligned}$$

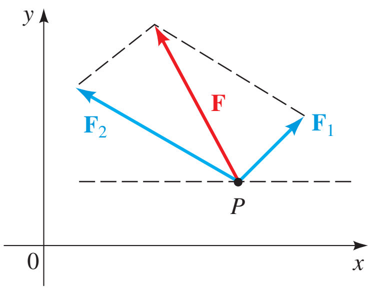


Figure 20

So the resultant force **F** is

$$\begin{aligned}\mathbf{F} &= \mathbf{F}_1 + \mathbf{F}_2\\ &= (5\sqrt{2}\,\mathbf{i} + 5\sqrt{2}\,\mathbf{j}) + (-10\sqrt{3}\,\mathbf{i} + 10\,\mathbf{j})\\ &= (5\sqrt{2} - 10\sqrt{3})\mathbf{i} + (5\sqrt{2} + 10)\mathbf{j}\\ &\approx -10\,\mathbf{i} + 17\,\mathbf{j}\end{aligned}$$

The resultant force **F** is shown in Figure 20.

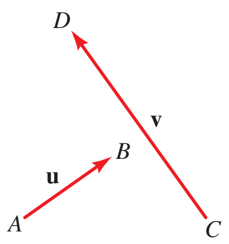
Now Try Exercise 67

8.5 Exercises

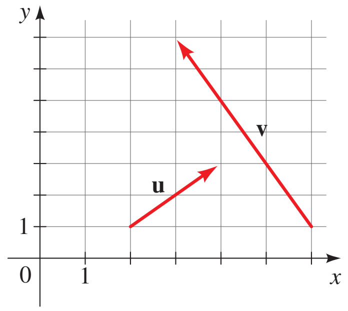
Concepts

1. (a) A vector in the plane is a line segment with an assigned direction. In Figure I below, the vector **u** has initial point ________ and terminal point ________. Sketch the vectors $2\mathbf{u}$ and $\mathbf{u} + \mathbf{v}$.

(b) A vector in a coordinate plane is expressed by using components. In Figure II below, the vector **u** has initial point (____, ____) and terminal point (____, ____). In component form we write $\mathbf{u} = \langle$ ____, ____ $\rangle$, and $\mathbf{v} = \langle$ ____, ____ $\rangle$. Then $2\mathbf{u} = \langle$ ____, ____ $\rangle$ and $\mathbf{u} + \mathbf{v} = \langle$ ____, ____ $\rangle$.



I



II

2. (a) The length of a vector $\mathbf{w} = \langle a_1, a_2\rangle$ is $|\mathbf{w}| =$ ________, so the length of the vector **u** in Figure II above is $|\mathbf{u}| =$ ________.

(b) If we know the length $|\mathbf{w}|$ and direction θ of a vector **w**, then we can express the vector in component form as $\mathbf{w} = \langle$ ________, ________ $\rangle$.

Skills

3–8 ■ Sketching Vectors Sketch the vector indicated. (The vectors **u** and **v** are shown in the figure.)

3. $2\mathbf{u}$

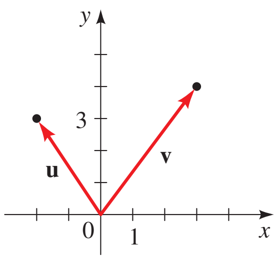
4. $-\mathbf{v}$

5. $\mathbf{u} + \mathbf{v}$

6. $\mathbf{u} - \mathbf{v}$

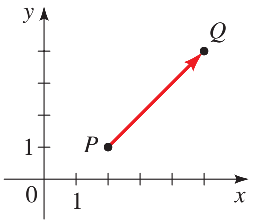
7. $\mathbf{v} - 2\mathbf{u}$

8. $2\mathbf{u} + \mathbf{v}$

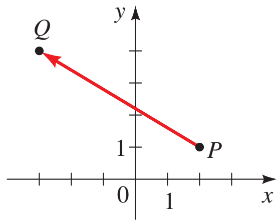


9–18 ■ Component Form of Vectors Express the vector with initial point P and terminal point Q in component form.

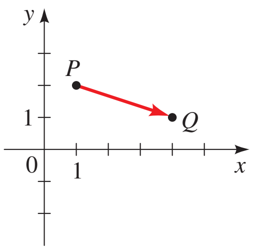
9.



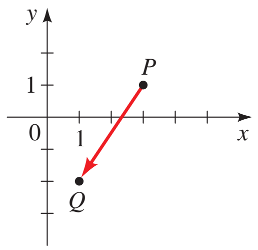
10.



11.



12.



13. $P(1, 3)$, $Q(4, 5)$

14. $P(2, 5)$, $Q(3, 1)$

15. $P(5, 3)$, $Q(1, 0)$

16. $P(-1, 3)$, $Q(-6, -1)$

17. $P(-1, -1)$, $Q(-1, 1)$

18. $P(-8, -6)$, $Q(-1, -1)$

19–22 ■ Sketching Vectors Sketch the given vector with initial point (4, 3), and find the terminal point.

19. $\mathbf{u} = \langle 2, 4\rangle$

20. $\mathbf{u} = \langle -1, 2\rangle$

21. $\mathbf{u} = \langle 4, -3\rangle$

22. $\mathbf{u} = \langle -8, -1\rangle$

23–26 ■ Sketching Vectors Sketch representations of the given vector with initial points at $(0, 0)$, $(2, 3)$, and $(-3, 5)$.

23. $\mathbf{u} = \langle 3, 5\rangle$

24. $\mathbf{u} = \langle 4, -6\rangle$

25. $\mathbf{u} = \langle -7, 2\rangle$

26. $\mathbf{u} = \langle 0, -9\rangle$

27–30 ■ Writing Vectors in Terms of i and j Write the given vector in terms of **i** and **j**.

27. $\mathbf{u} = \langle 2, 3\rangle$

28. $\mathbf{u} = \langle -1, 0\rangle$

29. $\mathbf{u} = \langle 0, -2\rangle$

30. $\mathbf{u} = \langle -4, -5\rangle$

31–36 ■ Operations with Vectors Find $2\mathbf{u}$, $-3\mathbf{v}$, $\mathbf{u}+\mathbf{v}$, and $3\mathbf{u}-4\mathbf{v}$ for the given vectors $\mathbf{u}$ and $\mathbf{v}$.

31. $\mathbf{u} = \langle 1, 4\rangle$, $\mathbf{v} = \langle -1, 2\rangle$ **32.** $\mathbf{u} = \langle -2, 5\rangle$, $\mathbf{v} = \langle 2, -8\rangle$

33. $\mathbf{u} = \langle 0, -1\rangle$, $\mathbf{v} = \langle -2, 0\rangle$ **34.** $\mathbf{u} = \mathbf{i}$, $\mathbf{v} = -2\mathbf{j}$

35. $\mathbf{u} = 2\mathbf{i} - \mathbf{j}$, $\mathbf{v} = \mathbf{j}$ **36.** $\mathbf{u} = \mathbf{i} + \mathbf{j}$, $\mathbf{v} = \mathbf{i} - \mathbf{j}$

37–40 ■ Magnitude of Vectors Find $|\mathbf{u}|$, $|\mathbf{v}|$, $|2\mathbf{u}|$, $|\frac{1}{2}\mathbf{v}|$, $|\mathbf{u}+\mathbf{v}|$, $|\mathbf{u}-\mathbf{v}|$, and $|\mathbf{u}| - |\mathbf{v}|$.

37. $\mathbf{u} = 3\mathbf{i} - \mathbf{j}$, $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}$

38. $\mathbf{u} = -2\mathbf{i} + 3\mathbf{j}$, $\mathbf{v} = \mathbf{i} - 2\mathbf{j}$

39. $\mathbf{u} = \langle 10, -1\rangle$, $\mathbf{v} = \langle -2, -2\rangle$

40. $\mathbf{u} = \langle -6, 6\rangle$, $\mathbf{v} = \langle -2, -1\rangle$

41–46 ■ Components of a Vector Find the horizontal and vertical components of the vector with given length and direction, and write the vector in terms of the vectors $\mathbf{i}$ and $\mathbf{j}$.

41. $|\mathbf{v}| = 10$, $\theta = 60°$ **42.** $|\mathbf{v}| = 20$, $\theta = 150°$

43. $|\mathbf{v}| = 1$, $\theta = 225°$ **44.** $|\mathbf{v}| = 800$, $\theta = 125°$

45. $|\mathbf{v}| = 4$, $\theta = 10°$ **46.** $|\mathbf{v}| = \sqrt{3}$, $\theta = 300°$

47–52 ■ Magnitude and Direction of a Vector Find the magnitude and direction (in degrees) of the vector.

47. $\mathbf{v} = \langle 3, 4\rangle$ **48.** $\mathbf{v} = \left\langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right\rangle$

49. $\mathbf{v} = \langle -12, 5\rangle$ **50.** $\mathbf{v} = \langle 40, 9\rangle$

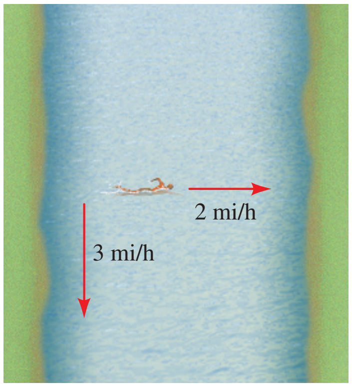
51. $\mathbf{v} = \mathbf{i} + \sqrt{3}\,\mathbf{j}$ **52.** $\mathbf{v} = \mathbf{i} + \mathbf{j}$

Applications

53. Components of a Force A landscaper pushes a lawn mower with a force of 30 lb exerted at an angle of 30° to the ground. Find the horizontal and vertical components of the force.

54. Components of a Velocity A jet is flying in a direction N 20° E with a speed of 500 mi/h. Find the north and east components of the velocity.

55. Velocity A river flows due south at 3 mi/h. A swimmer attempting to cross the river heads due east swimming at 2 mi/h relative to the water. Find the true velocity of the swimmer as a vector.



56. Velocity Suppose that in Exercise 55 the current is flowing at 1.2 mi/h due south. In what direction should the swimmer head in order to arrive at a landing point due east of the starting point?

57. Velocity A migrating salmon heads in the direction N 45° E, swimming at 5 mi/h relative to the water. The prevailing ocean currents flow due east at 3 mi/h. Find the true velocity of the fish as a vector.

58. A jet is flying through a wind that is blowing 40 mi/h due west. The jet has a speed of 585 mi/h relative to the air, and the pilot heads the jet in the direction N 45° W. Find the true speed and direction of the jet.

59. True Velocity of a Jet A pilot heads a jet due east. The jet has a speed of 425 mi/h relative to the air. The wind is blowing due north with a speed of 40 mi/h.

(a) Express the velocity of the wind as a vector in component form.

(b) Express the velocity of the jet relative to the air as a vector in component form.

(c) Find the true velocity of the jet as a vector.

(d) Find the true speed and direction of the jet.

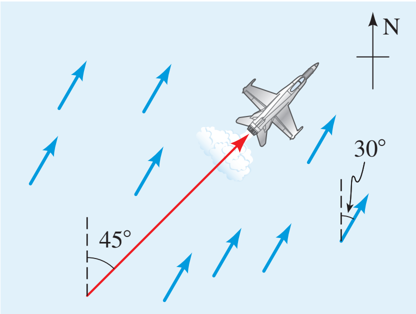
60. True Velocity of a Jet A jet is flying through a wind that is blowing with a speed of 55 mi/h in the direction N 30° E (see the figure). The jet has a speed of 765 mi/h relative to the air, and the pilot heads the jet in the direction N 45° E.

(a) Express the velocity of the wind as a vector in component form.

(b) Express the velocity of the jet relative to the air as a vector in component form.

(c) Find the true velocity of the jet as a vector.

(d) Find the true speed and direction of the jet.



61. True Velocity of a Jet In what direction should the pilot in Exercise 60 head the jet for the true course to be due north?

62. Velocity The speed of an airplane is 300 mi/h relative to the air. The wind is blowing due north with a speed of 30 mi/h. In what direction should the airplane head in order to arrive at a point due west of its location?

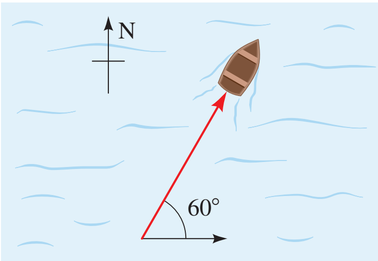
63. Velocity of a Boat A straight river flows east at a speed of 10 mi/h. A boater starts at the south shore of the river and heads in a direction 60° from the shore (see the figure). The boat has a speed of 20 mi/h relative to the water.

(a) Express the velocity of the river as a vector in component form.

(b) Express the velocity of the boat relative to the water as a vector in component form.

(c) Find the true velocity of the boat.

(d) Find the true speed and direction of the boat.



64. Velocity of a Boat The boater in Exercise 63 wants to arrive at a point on the north shore of the river directly opposite the starting point. In what direction should the boat be headed?

65. Velocity of a Boat A boat heads in the direction N 72° E. The speed of the boat relative to the water is 24 mi/h. The water is flowing directly south. It is observed that the true direction of the boat is directly east.

(a) Express the velocity of the boat relative to the water as a vector in component form.

(b) Find the speed of the water and the true speed of the boat.

66. Velocity A sailor walks due west on the deck of an ocean liner at 2 mi/h. The ocean liner is moving due north at a speed of 25 mi/h. Find the speed and direction of the sailor relative to the surface of the water.

67–72 ■ Equilibrium of Forces The forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$ acting at the same point P are said to be in equilibrium if the resultant force is zero, that is, if $\mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_n = \mathbf{0}$. Find **(a)** the resultant forces acting at P, and **(b)** the additional force required (if any) for the forces to be in equilibrium.

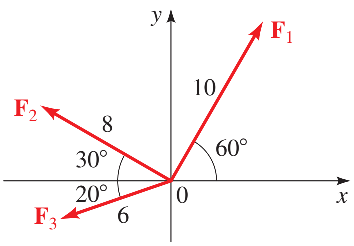
67. $\mathbf{F}_1 = \langle 2, 5 \rangle$, $\mathbf{F}_2 = \langle 3, -8 \rangle$

68. $\mathbf{F}_1 = \langle 3, -7 \rangle$, $\mathbf{F}_2 = \langle 4, -2 \rangle$, $\mathbf{F}_3 = \langle -7, 9 \rangle$

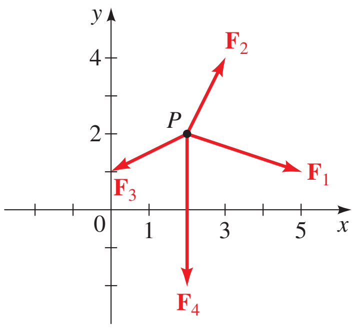
69. $\mathbf{F}_1 = 4\mathbf{i} - \mathbf{j}$, $\mathbf{F}_2 = 3\mathbf{i} - 7\mathbf{j}$, $\mathbf{F}_3 = -8\mathbf{i} + 3\mathbf{j}$, $\mathbf{F}_4 = \mathbf{i} + \mathbf{j}$

70. $\mathbf{F}_1 = \mathbf{i} - \mathbf{j}$, $\mathbf{F}_2 = \mathbf{i} + \mathbf{j}$, $\mathbf{F}_3 = -2\mathbf{i} + \mathbf{j}$

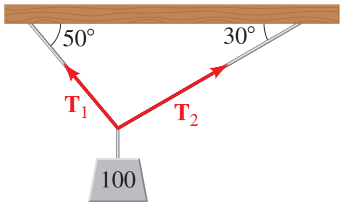
71.



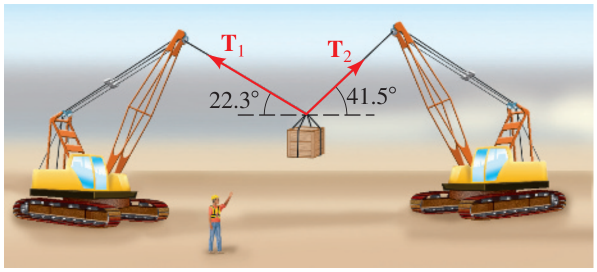
72.



73. Equilibrium of Tensions A 100-lb weight hangs from a string as shown in the figure. Find the tensions $\mathbf{T}_1$ and $\mathbf{T}_2$ in the string.

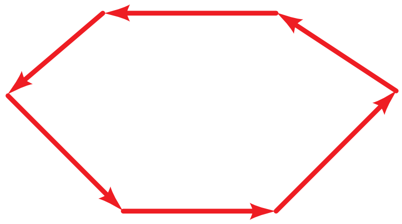


74. Equilibrium of Tensions The cranes in the figure are lifting an object that weighs 18,278 lb. Find the tensions $\mathbf{T}_1$ and $\mathbf{T}_2$.



Discuss ■ Discover ■ Prove ■ Write

75. Discuss: Vectors That Form a Polygon Suppose that n vectors can be placed head to tail in the plane so that they form a polygon. (The figure shows the case of a hexagon.) Explain why the sum of these vectors is $\mathbf{0}$.



PS *Try to recognize something familiar.* Think about the geometric definition for vector addition.

8.6 The Dot Product

■ The Dot Product of Vectors ■ The Component of **u** along **v**
■ The Projection of **u** onto **v** ■ Work

In this section we define an operation on vectors called the dot product. This concept is especially useful in calculus and in applications of vectors to physics and engineering.

■ The Dot Product of Vectors

We begin by defining the dot product of two vectors.

Definition of the Dot Product

If $\mathbf{u} = \langle a_1, a_2 \rangle$ and $\mathbf{v} = \langle b_1, b_2 \rangle$ are vectors, then their **dot product**, denoted by $\mathbf{u} \cdot \mathbf{v}$, is defined by

$$\mathbf{u} \cdot \mathbf{v} = a_1 b_1 + a_2 b_2$$

Thus to find the dot product of **u** and **v**, we multiply corresponding components and add. The dot product of vectors is *not* a vector; it is a real number, or scalar.

Example 1 ■ Calculating Dot Products

(a) If $\mathbf{u} = \langle 3, -2 \rangle$ and $\mathbf{v} = \langle 4, 5 \rangle$ then

$$\mathbf{u} \cdot \mathbf{v} = (3)(4) + (-2)(5) = 2$$

(b) If $\mathbf{u} = 2\mathbf{i} + \mathbf{j}$ and $\mathbf{v} = 5\mathbf{i} - 6\mathbf{j}$, then

$$\mathbf{u} \cdot \mathbf{v} = (2)(5) + (1)(-6) = 4$$

Now Try Exercises 5(a) and 11(a) ■

The proof of each of the following properties of the dot product follows from the definition.

Properties of the Dot Product

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$
3. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
4. $|\mathbf{u}|^2 = \mathbf{u} \cdot \mathbf{u}$

Proof We prove only the last property. The proofs of the others are left as exercises. Let $\mathbf{u} = \langle a_1, a_2 \rangle$. Then

$$\mathbf{u} \cdot \mathbf{u} = a_1 a_1 + a_2 a_2 = a_1^2 + a_2^2 = |\mathbf{u}|^2$$ ■

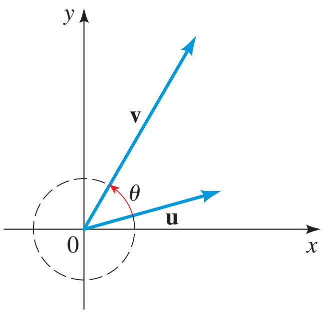


Figure 1

Let **u** and **v** be vectors, and sketch them with initial points at the origin. We define the **angle** θ **between u and v** to be the smaller of the angles formed by these representations of **u** and **v**. (See Figure 1.) Thus $0 \le \theta \le \pi$. The next theorem relates the angle between two vectors to their dot product.

The Dot Product Theorem

If θ is the angle between two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$, then

$$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$$

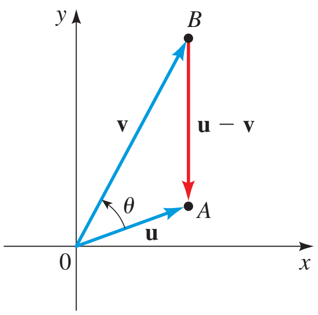


Figure 2

Proof Applying the Law of Cosines to triangle AOB in Figure 2 gives

$$|\mathbf{u}-\mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}||\mathbf{v}|\cos\theta$$

Using the properties of the dot product, we write the left-hand side as follows:

$$\begin{aligned} |\mathbf{u}-\mathbf{v}|^2 &= (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) \\ &= \mathbf{u}\cdot\mathbf{u} - \mathbf{u}\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} \\ &= |\mathbf{u}|^2 - 2(\mathbf{u}\cdot\mathbf{v}) + |\mathbf{v}|^2 \end{aligned}$$

Equating the right-hand sides of the displayed equations, we get

$$\begin{aligned} |\mathbf{u}|^2 - 2(\mathbf{u}\cdot\mathbf{v}) + |\mathbf{v}|^2 &= |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}||\mathbf{v}|\cos\theta \\ -2(\mathbf{u}\cdot\mathbf{v}) &= -2|\mathbf{u}||\mathbf{v}|\cos\theta \\ \mathbf{u}\cdot\mathbf{v} &= |\mathbf{u}||\mathbf{v}|\cos\theta \end{aligned}$$

This proves the theorem. ■

The Dot Product Theorem is useful because it allows us to find the angle between two vectors if we know the components of the vectors. The angle is obtained by solving the equation in the Dot Product Theorem for $\cos\theta$. We state this important result explicitly.

Angle Between two Vectors

If θ is the angle between two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$, then

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$

Example 2 ■ Finding the Angle between two Vectors

Find the angle between the vectors $\mathbf{u} = \langle 2, 5\rangle$ and $\mathbf{v} = \langle 4, -3\rangle$.

Solution By the formula for the angle between two vectors we have

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{(2)(4) + (5)(-3)}{\sqrt{4+25}\sqrt{16+9}} = \frac{-7}{5\sqrt{29}}$$

Thus the angle between $\mathbf{u}$ and $\mathbf{v}$ is

$$\theta = \cos^{-1}\left(\frac{-7}{5\sqrt{29}}\right) \approx 105.1°$$

Now Try Exercises 5(b) and 11(b) ■

Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are called **perpendicular**, or **orthogonal**, if the angle between them is $\pi/2$. The following theorem shows that we can determine whether two vectors are perpendicular by finding their dot product.

Orthogonal Vectors

Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular if and only if $\mathbf{u}\cdot\mathbf{v} = 0$.

Proof If **u** and **v** are perpendicular, then the angle between them is $\pi/2$, so

$$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\frac{\pi}{2} = 0$$

Conversely, if $\mathbf{u}\cdot\mathbf{v} = 0$, then

$$|\mathbf{u}||\mathbf{v}|\cos\theta = 0$$

Since **u** and **v** are nonzero vectors, we conclude that $\cos\theta = 0$, so $\theta = \pi/2$. Thus **u** and **v** are orthogonal. ■

Example 3 ■ Checking Whether Two Vectors Are Perpendicular

Determine whether the vectors in each pair are perpendicular.

(a) $\mathbf{u} = \langle 3, 5\rangle$ and $\mathbf{v} = \langle 2, -8\rangle$ **(b)** $\mathbf{u} = \langle 2, 1\rangle$ and $\mathbf{v} = \langle -1, 2\rangle$

Solution

(a) $\mathbf{u}\cdot\mathbf{v} = (3)(2) + (5)(-8) = -34 \neq 0$, so **u** and **v** are not perpendicular.

(b) $\mathbf{u}\cdot\mathbf{v} = (2)(-1) + (1)(2) = 0$, so **u** and **v** are perpendicular.

Now Try Exercises 17 and 19 ■

■ The Component of u along v

The **component of u along v** (also called the **component of u in the direction of v** or the **scalar projection of u onto v**) is defined to be

$$|\mathbf{u}|\cos\theta$$

Note that the component of **u** along **v** is a scalar, not a vector.

where θ is the angle between **u** and **v**. Figure 3 gives a geometric interpretation of this concept. Intuitively, the component of **u** along **v** is the magnitude of the portion of **u** that points in the direction of **v**. Notice that the component of **u** along **v** is negative if $\pi/2 < \theta \leq \pi$.

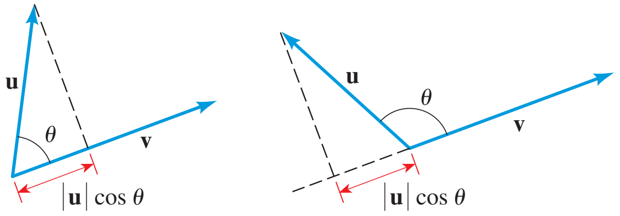


Figure 3

In analyzing forces in physics and engineering, it's often helpful to express a vector as a sum of two vectors lying in perpendicular directions. For example, suppose a car is parked on an inclined driveway as shown in Figure 4. The weight of the car is a vector **w** that points directly downward. We can write

$$\mathbf{w} = \mathbf{u} + \mathbf{v}$$

where **u** is parallel to the driveway and **v** is perpendicular to the driveway. The vector **u** is the force that tends to roll the car down the driveway, and **v** is the force experienced by the surface of the driveway. The magnitudes of these forces are the components of **w** along **u** and **v**, respectively.

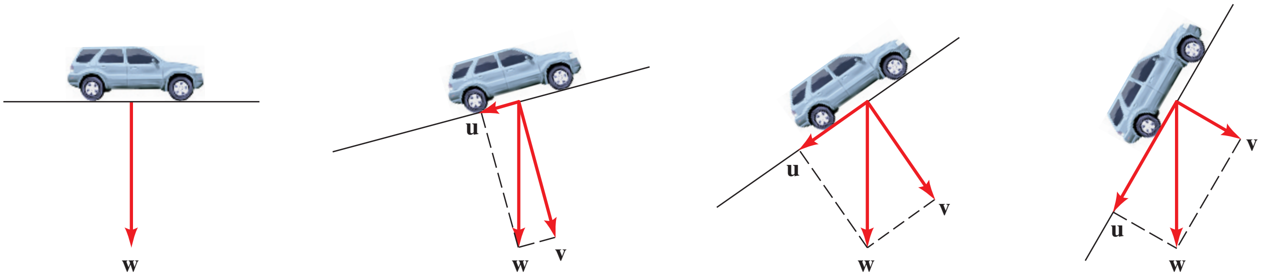


Figure 4

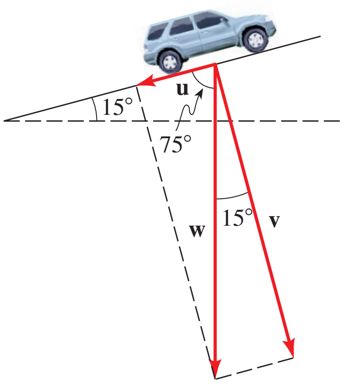


Figure 5

Example 4 ■ Resolving a Force into Components

A car weighing 3000 lb is parked on a driveway that is inclined 15° to the horizontal, as shown in Figure 5.

(a) Find the magnitude of the force required to prevent the car from rolling down the driveway.

(b) Find the magnitude of the force experienced by the driveway due to the weight of the car.

Solution The car exerts a force $\mathbf{w}$ of 3000 lb directly downward. We resolve $\mathbf{w}$ into the sum of two vectors $\mathbf{u}$ and $\mathbf{v}$, one parallel to the surface of the driveway and the other perpendicular to it, as shown in Figure 5.

(a) The magnitude of the part of the force $\mathbf{w}$ that causes the car to roll down the driveway is

$$|\mathbf{u}| = \text{component of } \mathbf{w} \text{ along } \mathbf{u} = 3000 \cos 75° \approx 776$$

Thus the force needed to prevent the car from rolling down the driveway is about 776 lb.

(b) The magnitude of the force exerted by the car on the driveway is

$$|\mathbf{v}| = \text{component of } \mathbf{w} \text{ along } \mathbf{v} = 3000 \cos 15° \approx 2898$$

The force experienced by the driveway is about 2898 lb.

Now Try Exercise 51 ■

The component of $\mathbf{u}$ along $\mathbf{v}$ can be computed by using dot products:

$$|\mathbf{u}| \cos \theta = \frac{|\mathbf{v}||\mathbf{u}| \cos \theta}{|\mathbf{v}|} = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|}$$

We have shown the following.

The component of $\mathbf{u}$ along $\mathbf{v}$ is a scalar, not a vector.

The Component of u Along v

The component of $\mathbf{u}$ along $\mathbf{v}$ (or the scalar projection of $\mathbf{u}$ onto $\mathbf{v}$) is

$$\text{comp}_{\mathbf{v}}\, \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|}$$

Example 5 ■ Finding Components

Let $\mathbf{u} = \langle 1, 4 \rangle$ and $\mathbf{v} = \langle -2, 1 \rangle$. Find the component of $\mathbf{u}$ along $\mathbf{v}$.

Solution From the formula for the component of $\mathbf{u}$ along $\mathbf{v}$ we have

$$\text{comp}_{\mathbf{v}}\, \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|} = \frac{(1)(-2) + (4)(1)}{\sqrt{4+1}} = \frac{2}{\sqrt{5}}$$

Now Try Exercise 27 ■

■ The Projection of u onto v

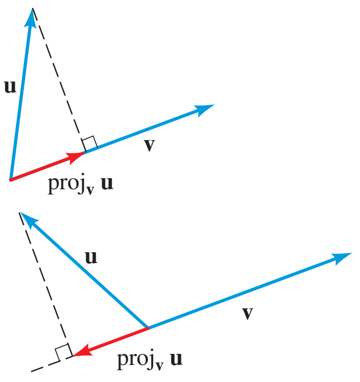


Figure 6

The projection of $\mathbf{u}$ onto $\mathbf{v}$, denoted by $\text{proj}_{\mathbf{v}}\, \mathbf{u}$, is the vector *parallel* to $\mathbf{v}$ and whose *length* is the component of $\mathbf{u}$ along $\mathbf{v}$ as shown in Figure 6. To find an expression for $\text{proj}_{\mathbf{v}}\, \mathbf{u}$, we first find a unit vector in the direction of $\mathbf{v}$ and then multiply it by the component of $\mathbf{u}$ along $\mathbf{v}$, as follows.

$$\text{proj}_{\mathbf{v}}\,\mathbf{u} = (\text{component of } \mathbf{u} \text{ along } \mathbf{v})(\text{unit vector in direction of } \mathbf{v})$$

$$= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}\right)\frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}$$

We often need to **resolve** a vector $\mathbf{u}$ into the sum of two vectors, one parallel to $\mathbf{v}$ and one orthogonal to $\mathbf{v}$. That is, we want to write $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$, where $\mathbf{u}_1$ is parallel to $\mathbf{v}$ and $\mathbf{u}_2$ is orthogonal to $\mathbf{v}$. In this case, $\mathbf{u}_1 = \text{proj}_{\mathbf{v}}\,\mathbf{u}$ and $\mathbf{u}_2 = \mathbf{u} - \text{proj}_{\mathbf{v}}\,\mathbf{u}$. (see Exercise 45.)

The Vector Projection of u Onto v

The **projection of u onto v** is the vector $\text{proj}_{\mathbf{v}}\,\mathbf{u}$ given by

$$\text{proj}_{\mathbf{v}}\,\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}$$

If the vector $\mathbf{u}$ is **resolved** into $\mathbf{u}_1$ and $\mathbf{u}_2$, where $\mathbf{u}_1$ is parallel to $\mathbf{v}$ and $\mathbf{u}_2$ is orthogonal to $\mathbf{v}$, then

$$\mathbf{u}_1 = \text{proj}_{\mathbf{v}}\,\mathbf{u} \qquad \text{and} \qquad \mathbf{u}_2 = \mathbf{u} - \text{proj}_{\mathbf{v}}\,\mathbf{u}$$

The projection of **u** onto **v** is a vector, not a scalar.

Example 6 ■ Resolving a Vector into Orthogonal Vectors

Let $\mathbf{u} = \langle -2, 9\rangle$ and $\mathbf{v} = \langle -1, 2\rangle$.

(a) Find $\text{proj}_{\mathbf{v}}\,\mathbf{u}$.

(b) Resolve $\mathbf{u}$ into $\mathbf{u}_1$ and $\mathbf{u}_2$, where $\mathbf{u}_1$ is parallel to $\mathbf{v}$ and $\mathbf{u}_2$ is orthogonal to $\mathbf{v}$.

Solution

(a) By the formula for the projection of one vector onto another we have

$$\text{proj}_{\mathbf{v}}\,\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v} \qquad \text{Formula for projection}$$

$$= \left(\frac{\langle -2, 9\rangle\cdot\langle -1, 2\rangle}{(-1)^2 + 2^2}\right)\langle -1, 2\rangle \qquad \text{Definition of } \mathbf{u} \text{ and } \mathbf{v}$$

$$= 4\langle -1, 2\rangle = \langle -4, 8\rangle$$

(b) By the formula in the preceding box we have $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$, where

$$\mathbf{u}_1 = \text{proj}_{\mathbf{v}}\,\mathbf{u} = \langle -4, 8\rangle \qquad \text{From part (a)}$$

$$\mathbf{u}_2 = \mathbf{u} - \text{proj}_{\mathbf{v}}\,\mathbf{u} = \langle -2, 9\rangle - \langle -4, 8\rangle = \langle 2, 1\rangle$$

Now Try Exercise 31 ■

■ Work

One use of the dot product occurs in calculating work. In everyday use, the term *work* means the total amount of effort required to perform a task. In physics, *work* has a technical meaning that conforms to this intuitive meaning. If a constant force of magnitude F moves an object through a distance d along a straight line, then the **work** done is

$$W = Fd \qquad \text{or} \qquad \text{work} = \text{force} \times \text{distance}$$

If F is measured in pounds and d in feet, then the unit of work is a foot-pound (ft-lb). For example, how much work is done in lifting a 20-lb weight 6 ft off the ground? Since a force of 20 lb is required to lift this weight and since the weight moves through a distance of 6 ft, the amount of work done is

$$W = Fd = (20)(6) = 120 \text{ ft-lb}$$

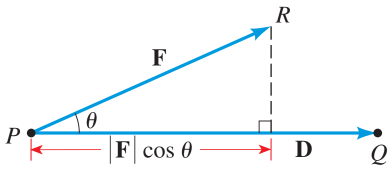


Figure 7

This formula applies only when the force is directed along the direction of motion. In the general case, if the force **F** moves an object from P to Q, as illustrated in Figure 7, then only the component of the force in the direction of $\mathbf{D} = \overrightarrow{PQ}$ affects the object. Thus the effective magnitude of the force on the object is

$$\text{comp}_{\mathbf{D}}\,\mathbf{F} = |\mathbf{F}|\cos\theta$$

So the work done is

$$W = \text{force} \times \text{distance} = (|\mathbf{F}|\cos\theta)|\mathbf{D}| = |\mathbf{F}||\mathbf{D}|\cos\theta = \mathbf{F}\cdot\mathbf{D}$$

We have derived the following formula for calculating work.

Work

The **work** W done by a force **F** in moving along a displacement vector **D** is

$$W = \mathbf{F}\cdot\mathbf{D}$$

Example 7 ■ Calculating Work

A force is given by the vector $\mathbf{F} = \langle 2, 3\rangle$ and moves an object from the point $(1, 3)$ to the point $(5, 9)$. Find the work done.

Solution The displacement vector is

$$\mathbf{D} = \langle 5 - 1, 9 - 3\rangle = \langle 4, 6\rangle$$

So the work done is

$$W = \mathbf{F}\cdot\mathbf{D} = \langle 2, 3\rangle\cdot\langle 4, 6\rangle = 26$$

If the unit of force is pounds and the distance is measured in feet, then the work done is 26 ft-lb.

Now Try Exercise 37 ■

Example 8 ■ Calculating Work

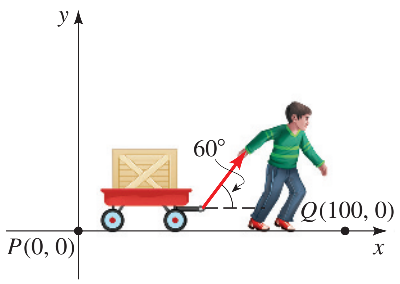


Figure 8

A child pulls a wagon horizontally by exerting a force of 20 lb on the handle. If the handle makes an angle of 60° with the horizontal, find the work done in moving the wagon 100 ft.

Solution We choose a coordinate system with the origin at the initial position of the wagon (see Figure 8). That is, the wagon moves from the point $P(0, 0)$ to the point $Q(100, 0)$. The vector that represents this displacement is

$$\mathbf{D} = 100\,\mathbf{i}$$

James L. Amos/SuperStock

Discovery Project ■ Sailing Against the Wind

Sailors depend on the wind to propel their boats. But what if the wind is blowing in a direction opposite to the direction in which they want to travel? Although it is impossible to sail directly against the wind, it is possible to sail at an angle into the wind so that the sailboat can make headway against the wind. In this project we discover how vectors that model the sail, the keel, and the wind can be combined to find the direction in which the boat will move. You can find the project at **www.stewartmath.com**.

The force on the handle can be written in terms of components (see Section 8.5) as

$$\mathbf{F} = (20\cos 60°)\mathbf{i} + (20\sin 60°)\mathbf{j} = 10\mathbf{i} + 10\sqrt{3}\,\mathbf{j}$$

Thus the work done is

$$W = \mathbf{F}\cdot\mathbf{D} = (10\mathbf{i} + 10\sqrt{3}\,\mathbf{j})\cdot(100\mathbf{i}) = 1000 \text{ ft-lb}$$

Now Try Exercise 49

8.6 Exercises

Concepts

1–2 ■ Let $\mathbf{u} = \langle a_1, a_2\rangle$ and $\mathbf{v} = \langle b_1, b_2\rangle$ be nonzero vectors in the plane, and let θ be the angle between them.

1. The dot product of $\mathbf{u}$ and $\mathbf{v}$ is defined by

$$\mathbf{u}\cdot\mathbf{v} = \underline{\qquad\qquad\qquad}$$

The dot product of two vectors is a ________, not a vector.

2. The angle θ satisfies

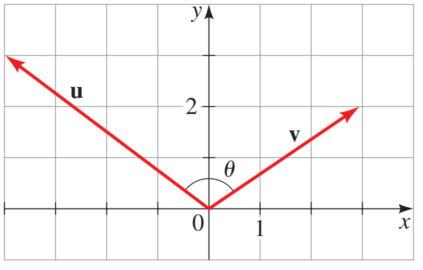
$$\cos\theta = \frac{\square}{\square}$$

So if $\mathbf{u}\cdot\mathbf{v} = 0$, the vectors are ________.

To find the angle θ between the vectors $\mathbf{u}$ and $\mathbf{v}$ in the figure, we first find

$$\cos\theta = \frac{\langle \square, \square\rangle\cdot\langle\square, \square\rangle}{|\langle\square, \square\rangle||\langle\square, \square\rangle|} = \underline{\qquad\qquad}$$

and so $\theta \approx$ ________, rounded to the nearest degree.

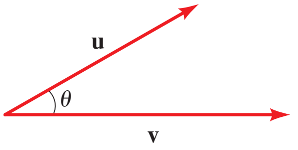


3. (a) The component of $\mathbf{u}$ along $\mathbf{v}$ is the scalar $|\mathbf{u}|\cos\theta$ and can be expressed in terms of the dot product as

$\text{comp}_{\mathbf{v}}\,\mathbf{u} =$ ________. Sketch this component in the figure below.

(b) The projection of $\mathbf{u}$ onto $\mathbf{v}$ is the vector

$\text{proj}_{\mathbf{v}}\,\mathbf{u} =$ ________. Sketch this projection in the figure below.



4. The work done by a force $\mathbf{F}$ in moving an object along a vector $\mathbf{D}$ is $W =$ ________.

Skills

5–16 ■ **Dot Products and Angles Between Vectors** Find **(a)** $\mathbf{u}\cdot\mathbf{v}$ and **(b)** the angle between $\mathbf{u}$ and $\mathbf{v}$ to the nearest degree.

5. $\mathbf{u} = \langle 2, 0\rangle$, $\mathbf{v} = \langle 1, 1\rangle$

6. $\mathbf{u} = \mathbf{i} + \sqrt{3}\,\mathbf{j}$, $\mathbf{v} = -\sqrt{3}\,\mathbf{i} + \mathbf{j}$

7. $\mathbf{u} = \langle 1, 0\rangle$, $\mathbf{w} = \langle 1, \sqrt{3}\rangle$

8. $\mathbf{u} = \langle -6, 6\rangle$, $\mathbf{v} = \langle 1, -1\rangle$

9. $\mathbf{u} = \langle 3, -2\rangle$, $\mathbf{v} = \langle 1, 2\rangle$

10. $\mathbf{u} = \langle 3, 4\rangle$, $\mathbf{w} = \langle 4, 3\rangle$

11. $\mathbf{u} = -5\mathbf{j}$, $\mathbf{v} = -\mathbf{i} - \sqrt{3}\,\mathbf{j}$

12. $\mathbf{u} = \mathbf{i} + \mathbf{j}$, $\mathbf{v} = \mathbf{i} - \mathbf{j}$

13. $\mathbf{u} = \mathbf{i} + 3\mathbf{j}$, $\mathbf{v} = 4\mathbf{i} - \mathbf{j}$

14. $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j}$, $\mathbf{v} = -2\mathbf{i} - \mathbf{j}$

15. $\mathbf{u} = \langle 1, \sqrt{3}\rangle$, $\mathbf{w} = \langle 1, -\sqrt{3}\rangle$

16. $\mathbf{u} = \langle 6, 8\rangle$, $\mathbf{w} = \langle 3, 4\rangle$

17–22 ■ **Perpendicular Vectors?** Determine whether the given vectors are perpendicular.

17. $\mathbf{u} = \langle 6, 4\rangle$, $\mathbf{v} = \langle -2, 3\rangle$

18. $\mathbf{u} = \langle 0, -5\rangle$, $\mathbf{v} = \langle 4, 0\rangle$

19. $\mathbf{u} = \langle -2, 6\rangle$, $\mathbf{v} = \langle 4, 2\rangle$

20. $\mathbf{u} = 2\mathbf{i}$, $\mathbf{v} = -7\mathbf{j}$

21. $\mathbf{u} = 2\mathbf{i} - 8\mathbf{j}$, $\mathbf{v} = -12\mathbf{i} - 3\mathbf{j}$

22. $\mathbf{u} = 4\mathbf{i}$, $\mathbf{v} = -\mathbf{i} + 3\mathbf{j}$

23–26 ■ **Dot Products** Find the indicated quantity, given that $\mathbf{u} = 2\mathbf{i} + \mathbf{j}$, $\mathbf{v} = \mathbf{i} - 3\mathbf{j}$, and $\mathbf{w} = 3\mathbf{i} + 4\mathbf{j}$.

23. $\mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w}$

24. $\mathbf{u}\cdot(\mathbf{v} + \mathbf{w})$

25. $(\mathbf{u} + \mathbf{v})\cdot(\mathbf{u} - \mathbf{v})$

26. $(\mathbf{u}\cdot\mathbf{v})(\mathbf{u}\cdot\mathbf{w})$

27–30 ■ **The Component of u along v** Find the component of $\mathbf{u}$ along $\mathbf{v}$.

27. $\mathbf{u} = \langle 4, 6\rangle$, $\mathbf{v} = \langle 3, -4\rangle$

28. $\mathbf{u} = \langle -3, 5\rangle$, $\mathbf{v} = \langle 1/\sqrt{2}, 1/\sqrt{2}\rangle$

29. $\mathbf{u} = 7\mathbf{i} - 24\mathbf{j}$, $\mathbf{v} = \mathbf{j}$

30. $\mathbf{u} = 7\mathbf{i}$, $\mathbf{v} = 8\mathbf{i} + 6\mathbf{j}$

31–36 ■ Vector Projection of u onto v **(a)** Calculate $\text{proj}_{\mathbf{v}}\,\mathbf{u}$. **(b)** Resolve $\mathbf{u}$ into $\mathbf{u}_1$ and $\mathbf{u}_2$, where $\mathbf{u}_1$ is parallel to $\mathbf{v}$ and $\mathbf{u}_2$ is orthogonal to $\mathbf{v}$.

31. $\mathbf{u} = \langle -2, 4\rangle, \quad \mathbf{v} = \langle 1, 1\rangle$

32. $\mathbf{u} = \langle 7, -4\rangle, \quad \mathbf{v} = \langle 2, 1\rangle$

33. $\mathbf{u} = \langle 1, 2\rangle, \quad \mathbf{v} = \langle 1, -3\rangle$

34. $\mathbf{u} = \langle 11, 3\rangle, \quad \mathbf{v} = \langle -3, -2\rangle$

35. $\mathbf{u} = \langle 2, 9\rangle, \quad \mathbf{v} = \langle -3, 4\rangle$

36. $\mathbf{u} = \langle 1, 1\rangle, \quad \mathbf{v} = \langle 2, -1\rangle$

37–40 ■ Calculating Work Find the work done by the force $\mathbf{F}$ in moving an object from P to Q.

37. $\mathbf{F} = 4\mathbf{i} - 5\mathbf{j}; \quad P(0, 0), Q(3, 8)$

38. $\mathbf{F} = 400\mathbf{i} + 50\mathbf{j}; \quad P(-1, 1), Q(200, 1)$

39. $\mathbf{F} = 10\mathbf{i} + 3\mathbf{j}; \quad P(2, 3), Q(6, -2)$

40. $\mathbf{F} = -4\mathbf{i} + 20\mathbf{j}; \quad P(0, 10), Q(5, 25)$

Skills Plus

41–44 ■ Properties of Vectors Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors, and let c be a scalar. Prove the given property.

41. $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$

42. $(c\mathbf{u})\cdot\mathbf{v} = c(\mathbf{u}\cdot\mathbf{v}) = \mathbf{u}\cdot(c\mathbf{v})$

43. $(\mathbf{u} + \mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot\mathbf{w} + \mathbf{v}\cdot\mathbf{w}$

44. $(\mathbf{u} - \mathbf{v})\cdot(\mathbf{u} + \mathbf{v}) = |\mathbf{u}|^2 - |\mathbf{v}|^2$

45. Projection Show that $\text{proj}_{\mathbf{v}}\,\mathbf{u}$ and $\mathbf{u} - \text{proj}_{\mathbf{v}}\,\mathbf{u}$ are orthogonal.

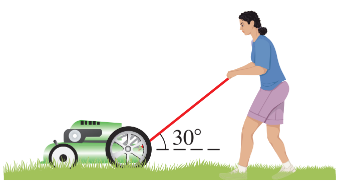
46. Projection Show that $\mathbf{v}\cdot\text{proj}_{\mathbf{v}}\,\mathbf{u} = \mathbf{u}\cdot\mathbf{v}$.

Applications

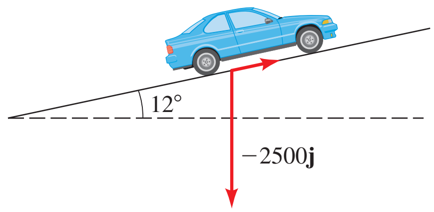
47. Work The force $\mathbf{F} = 4\mathbf{i} - 7\mathbf{j}$ moves an object 4 ft along the x-axis in the positive direction. Find the work done if the unit of force is the pound.

48. Work A constant force $\mathbf{F} = \langle 2, 8\rangle$ moves an object along a straight line from the point $(2, 5)$ to the point $(11, 13)$. Find the work done if the distance is measured in feet and the force is measured in pounds.

49. Work A lawn mower is pushed a distance of 200 ft along a horizontal path by a constant force of 50 lb. The handle of the lawn mower is held at an angle of 30° from the horizontal (see the figure). Find the work done.



50. Work A car drives 500 ft on a road that is inclined 12° to the horizontal, as shown in the following figure. The car weighs 2500 lb. Thus gravity acts straight down on the car with a constant force $\mathbf{F} = -2500\mathbf{j}$. Find the work done by the car in overcoming gravity.



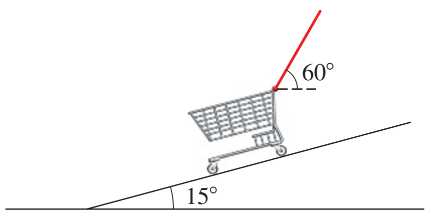
51. Force A car is on a driveway that is inclined 10° to the horizontal. A force of 490 lb is required to keep the car from rolling down the driveway.

(a) Find the weight of the car.
(b) Find the force the car exerts against the driveway.

52. Force A car is on a driveway that is inclined 25° to the horizontal. If the car weighs 2755 lb, find the force required to keep it from rolling down the driveway.

53. Force A package that weighs 200 lb is placed on an inclined plane. If a force of 80 lb is just sufficient to keep the package from sliding, find the angle of inclination of the plane. (Ignore the effects of friction.)

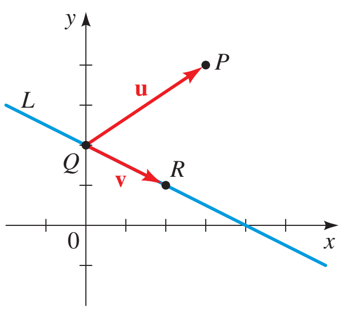
54. Force A cart weighing 40 lb is placed on a ramp inclined at 15° to the horizontal. The cart is held in place by a rope inclined at 60° to the horizontal, as shown in the figure. Find the force that the rope must exert on the cart to keep it from rolling down the ramp.



Discuss ■ Discover ■ Prove ■ Write

55. Discuss ■ Discover ■ Write: Distance from a Point to a Line Let L be the line $2x + 4y = 8$, and let P be the point $(3, 4)$.

(a) Show that the points $Q(0, 2)$ and $R(2, 1)$ lie on L.
(b) Let $\mathbf{u} = \overrightarrow{QP}$ and $\mathbf{v} = \overrightarrow{QR}$, as shown in the figure. Find $\mathbf{w} = \text{proj}_{\mathbf{v}}\,\mathbf{u}$.
(c) Sketch a graph that explains why $|\mathbf{u} - \mathbf{w}|$ is the distance from P to L. Find this distance.
(d) Write a short paragraph describing the steps you would take to find the distance from a given point to a given line.

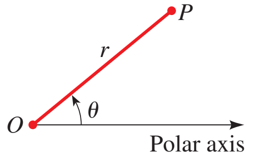


Chapter 8 Review

Properties and Formulas

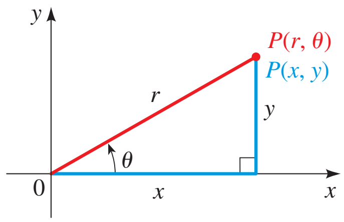
Polar Coordinates | Section 8.1

In the **polar coordinate** system the location of a point P in the plane is determined by an ordered pair (r, θ), where r is the distance from the pole O to P and θ is the angle formed by the polar axis and the segment $\overrightarrow{OP}$, as shown in the figure.



Polar and Rectangular Coordinates | Section 8.1

Any point P in the plane has polar coordinates $P(r, \theta)$ and rectangular coordinates $P(x, y)$, as shown.



- **To change from polar to rectangular coordinates**, we use the equations

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta$$

- **To change from rectangular to polar coordinates**, we use the equations

$$r^2 = x^2 + y^2 \qquad \text{and} \qquad \tan\theta = \frac{y}{x}$$

Polar Equations and Graphs | Section 8.2

A **polar equation** is an equation in the variables r and θ. The **graph of a polar equation** $r = f(\theta)$ consists of all points (r, θ) whose coordinates satisfy the equation.

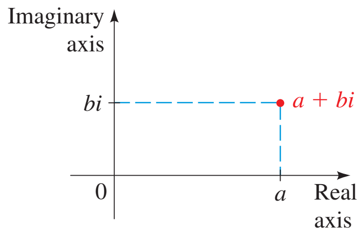
Symmetry in Graphs of Polar Equations | Section 8.2

The graph of a polar equation is

- **symmetric about the polar axis** if the equation is unchanged when we replace θ by $-\theta$;
- **symmetric about the pole** if the equation is unchanged when we replace r by $-r$, or θ by $\theta + \pi$.
- **symmetric about the vertical line** $\theta = \pi/2$ if the equation is unchanged when we replace θ by $\pi - \theta$.

Complex Numbers | Section 8.3

A **complex number** is a number of the form $a + bi$, where $i^2 = -1$ and where a and b are real numbers. For the complex number $z = a + bi$, a is called the **real part** and b is called the **imaginary part**. A complex number $a + bi$ is graphed in the complex plane as shown.



The **modulus** (or **absolute value**) of a complex number $z = a + bi$ is

$$|z| = \sqrt{a^2 + b^2}$$

Polar Form of Complex Numbers | Section 8.3

A complex number $z = a + bi$ has the **polar form** (or **trigonometric form**)

$$z = r(\cos\theta + i\sin\theta)$$

where $r = |z|$ and $\tan\theta = b/a$. The number r is the modulus of z and θ is the argument of z.

Multiplication and Division of Complex Numbers in Polar Form | Section 8.3

Suppose the complex numbers z_1 and z_2 have the following polar form:

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$$

$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

Then

$$z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

De Moivre's Theorem | Section 8.3

If $z = r(\cos\theta + i\sin\theta)$ is a complex number in polar form and n is a positive integer, then

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

*n*th Roots of Complex Numbers | Section 8.3

If $z = r(\cos\theta + i\sin\theta)$ is a complex number in polar form and n is a positive integer, then z has the n distinct nth roots $w_0, w_1, \ldots, w_{n-1}$, where

$$w_k = r^{1/n}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$

where $k = 0, 1, 2, \ldots, n - 1$.

Finding the *n*th Roots of *z* | Section 8.3

To find the nth roots of $z = r(\cos\theta + i\sin\theta)$, we use the following observations:

1. The modulus of each nth root is $r^{1/n}$.

2. The argument of the first root w_0 is θ/n.

3. Repeatedly add $2\pi/n$ to get the argument of each successive root.

Parametric Equations | Section 8.4

If f and g are functions defined on an interval I, then the set of points $(f(t), g(t))$ is a **plane curve**. The equations

$$x = f(t) \qquad y = g(t)$$

where $t \in I$, are **parametric equations** for the curve, with **parameter** t.

Polar Equations in Parametric Form | Section 8.4

The graph of the polar equation $r = f(\theta)$ is the same as the graph of the parametric equations

$$x = f(t)\cos t \qquad y = f(t)\sin t$$

Vectors | Section 8.5

A **vector** is a quantity with both magnitude and direction.
A vector in the coordinate plane is expressed in terms of two coordinates or components

$$\mathbf{v} = \langle a_1, a_2\rangle$$

If a vector $\mathbf{v}$ has its initial point at $P(x_1, y_1)$ and its terminal point at $Q(x_2, y_2)$, then

$$\mathbf{v} = \langle x_2 - x_1, y_2 - y_1\rangle$$

Let $\mathbf{u} = \langle a_1, a_2\rangle$, $\mathbf{v} = \langle b_1, b_2\rangle$, and $c \in \mathbb{R}$. The operations on vectors are defined as follows.

$$\mathbf{u} + \mathbf{v} = \langle a_1 + b_1, a_2 + b_2\rangle \qquad \text{Addition}$$

$$\mathbf{u} - \mathbf{v} = \langle a_1 - b_1, a_2 - b_2\rangle \qquad \text{Subtraction}$$

$$c\mathbf{u} = \langle ca_1, ca_2\rangle \qquad \text{Scalar multiplication}$$

The unit vectors **i** and **j** are defined by

$$\mathbf{i} = \langle 1, 0\rangle \qquad \mathbf{j} = \langle 0, 1\rangle$$

Any vector $\mathbf{v} = \langle a_1, a_2\rangle$ can be expressed as

$$\mathbf{v} = a_1\mathbf{i} + a_2\mathbf{j}$$

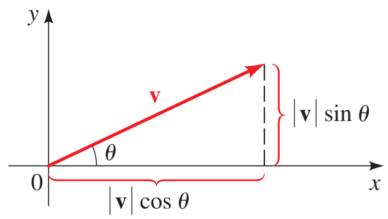
Let $\mathbf{v} = \langle a_1, a_2\rangle$. The **magnitude** (or **length**) of **v** is

$$|\mathbf{v}| = \sqrt{a_1^2 + a_2^2}$$

The **direction** of **v** is the smallest positive angle θ in standard position formed by the positive x-axis and **v** (see the figure below).

If $\mathbf{v} = \langle a_1, a_2\rangle$, then the components of **v** satisfy

$$a_1 = |\mathbf{v}|\cos\theta \qquad a_2 = |\mathbf{v}|\sin\theta$$



The Dot Product of Vectors | Section 8.6

If $\mathbf{u} = \langle a_1, a_2\rangle$ and $\mathbf{v} = \langle b_1, b_2\rangle$, then their **dot product** is

$$\mathbf{u}\cdot\mathbf{v} = a_1b_1 + a_2b_2$$

If θ is the angle between **u** and **v**, then

$$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$$

The angle θ between **u** and **v** satisfies

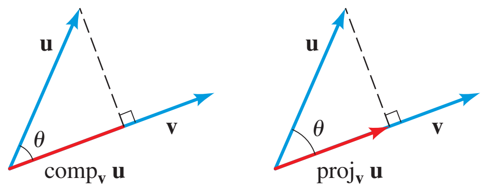
$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$

The vectors **u** and **v** are perpendicular if and only if

$$\mathbf{u}\cdot\mathbf{v} = 0$$

The **component of u along v** (a scalar) and the **projection of u onto v** (a vector) are given by

$$\text{comp}_{\mathbf{v}}\,\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|} \qquad \text{proj}_{\mathbf{v}}\,\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}$$



The **work** W done by a force **F** in moving along a vector **D** is

$$W = \mathbf{F}\cdot\mathbf{D}$$

Concept Check

1. (a) Explain the polar coordinate system.

(b) Graph the points with polar coordinates $(2, \pi/3)$ and $(-1, 3\pi/4)$.

(c) State the equations that relate the rectangular coordinates of a point to its polar coordinates.

(d) Find rectangular coordinates for $(2, \pi/3)$.

(e) Find polar coordinates for $P(-2, 2)$.

2. (a) What is a polar equation?

(b) Convert the polar equation $r = \sin\theta$ to an equivalent rectangular equation.

3. (a) How do we graph a polar equation?

(b) Sketch a graph of the polar equation $r = 4 + 4\cos\theta$. What is the graph called?

4. (a) What is the complex plane? How do we graph a complex number $z = a + bi$ in the complex plane?

(b) What are the modulus and argument of the complex number $z = a + bi$?

(c) Graph the point $z = \sqrt{3} - i$, and find the modulus and argument of z.

5. **(a)** How do we express the complex number z in polar form?

(b) Express $z = \sqrt{3} - i$ in polar form.

6. Let $$z_1 = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

and $$z_2 = 5\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

(a) Find the product $z_1 z_2$.

(b) Find the quotient z_1/z_2.

7. **(a)** State de Moivre's Theorem.

(b) Use de Moivre's Theorem to find the fifth power of $z = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$.

8. **(a)** State the formula for the nth roots of a complex number $z = r(\cos\theta + i\sin\theta)$.

(b) How do we find the nth roots of a complex number?

(c) Find the three third roots of $z = -8$.

9. **(a)** What are parametric equations?

(b) Sketch a graph of the following parametric equations, using arrows to indicate the direction of the curve.

$$x = t + 1 \qquad y = t^2 \qquad -2 \le t \le 2$$

(c) Eliminate the parameter to obtain an equation in x and y.

10. **(a)** What is a vector in the plane? How do we represent a vector in the coordinate plane?

(b) Find the vector with initial point $(2, 3)$ and terminal point $(4, 10)$.

(c) Let $\mathbf{v} = \langle 2, 1\rangle$. If the initial point of $\mathbf{v}$ is placed at $P(1, 1)$, where is its terminal point? Sketch several representations of $\mathbf{v}$.

(d) How is the magnitude of $\mathbf{v} = \langle a_1, a_2\rangle$ defined? Find the magnitude of $\mathbf{w} = \langle 3, 4\rangle$.

(e) What are the vectors $\mathbf{i}$ and $\mathbf{j}$? Express the vector $\mathbf{v} = \langle 5, 9\rangle$ in terms of $\mathbf{i}$ and $\mathbf{j}$.

(f) Let $\mathbf{v} = \langle a_1, a_2\rangle$ be a vector in the coordinate plane. What is meant by the direction θ of $\mathbf{v}$? What are the components of $\mathbf{v}$ in terms of its length and direction? Sketch a figure to illustrate your answer.

(g) Suppose that $\mathbf{v}$ has length $|\mathbf{v}| = 5$ and direction $\theta = \pi/6$. Express $\mathbf{v}$ in terms of its coordinates.

11. **(a)** Define addition and scalar multiplication for vectors.

(b) If $\mathbf{u} = \langle 2, 3\rangle$ and $\mathbf{v} = \langle 5, 9\rangle$, find $\mathbf{u} + \mathbf{v}$ and $4\mathbf{u}$.

12. **(a)** Define the dot product of the vectors $\mathbf{u} = \langle a_1, a_2\rangle$ and $\mathbf{v} = \langle b_1, b_2\rangle$, and state the formula for the angle θ between $\mathbf{u}$ and $\mathbf{v}$.

(b) If $\mathbf{u} = \langle 2, 3\rangle$ and $\mathbf{v} = \langle 1, 4\rangle$, find $\mathbf{u}\cdot\mathbf{v}$ and find the angle between $\mathbf{u}$ and $\mathbf{v}$.

13. How much work is done by the force $\mathbf{F}$ in moving an object along a displacement vector $\mathbf{D}$?

Answers to the Concept Check can be found at the book companion website stewartmath.com.

Exercises

1–4 ■ Polar Coordinates to Rectangular Coordinates A point $P(r, \theta)$ is given in polar coordinates. **(a)** Plot the point P. **(b)** Find rectangular coordinates for P.

1. $(12, \pi/6)$ **2.** $(8, -3\pi/4)$

3. $(-3, 7\pi/4)$ **4.** $(-\sqrt{3}, 2\pi/3)$

5–8 ■ Rectangular Coordinates to Polar Coordinates A point $P(x, y)$ is given in rectangular coordinates. **(a)** Plot the point P. **(b)** Find polar coordinates for P with $r \ge 0$. **(c)** Find polar coordinates for P with $r \le 0$.

5. $(8, 8)$ **6.** $(-\sqrt{2}, \sqrt{6})$

7. $(-6\sqrt{2}, -6\sqrt{2})$ **8.** $(4, -4)$

9–12 ■ Rectangular Equations to Polar Equations **(a)** Convert the equation to polar coordinates and simplify. **(b)** Graph the equation. [*Hint:* Use the form of the equation that you find easier to graph.]

9. $x + y = 4$ **10.** $xy = 1$

11. $x^2 + y^2 = 4x + 4y$ **12.** $(x^2 + y^2)^2 = 2xy$

13–20 ■ Polar Equations to Rectangular Equations **(a)** Sketch the graph of the polar equation. **(b)** Express the equation in rectangular coordinates.

13. $r = 3 + 3\cos\theta$ **14.** $r = 3\sin\theta$

15. $r = 2\sin 2\theta$ **16.** $r = 4\cos 3\theta$

17. $r^2 = \sec 2\theta$ **18.** $r^2 = 4\sin 2\theta$

19. $r = \sin\theta + \cos\theta$ **20.** $r = \dfrac{4}{2 + \cos\theta}$

21–24 ■ Graphing Polar Equations Use a graphing device to graph the polar equation. Choose the domain of θ to produce the entire graph.

21. $r = \cos(\theta/3)$ **22.** $r = \sin(9\theta/4)$

23. $r = 1 + 4\cos(\theta/3)$ **24.** $r = \sin(2^\theta) - 2, \quad 0 \le \theta \le 2\pi$

25–30 ■ Complex Numbers A complex number is given. **(a)** Graph the complex number in the complex plane. **(b)** Find the modulus and argument. **(c)** Write the number in polar form.

25. $4 + 4i$ **26.** $-10i$

27. $5 + 3i$ **28.** $1 + \sqrt{3}i$

29. $-1 + i$ **30.** -20

31–34 ■ Powers Using de Moivre's Theorem Use de Moivre's Theorem to find the indicated power.

31. $(1 - \sqrt{3}i)^4$ **32.** $(1 + i)^8$

33. $(\sqrt{3} + i)^{-4}$ **34.** $\left(\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i\right)^{20}$

35–38 ■ Roots of Complex Numbers Find the indicated roots.

35. The square roots of $-16i$

36. The cube roots of $4 + 4\sqrt{3}i$

37. The sixth roots of 1 **38.** The eighth roots of i

39–42 ■ Parametric Curves A pair of parametric equations is given. **(a)** Sketch the curve represented by the parametric equations. Use arrows to indicate the direction of the curve as t increases. **(b)** Find an equation in rectangular coordinates for the curve by eliminating the parameter.

39. $x = 1 - t^2, \quad y = 1 + t$ **40.** $x = t^2 - 1, \quad y = t^2 + 1$

41. $x = 1 + \cos t, \quad y = 1 - \sin t, \quad 0 \le t \le \pi/2$

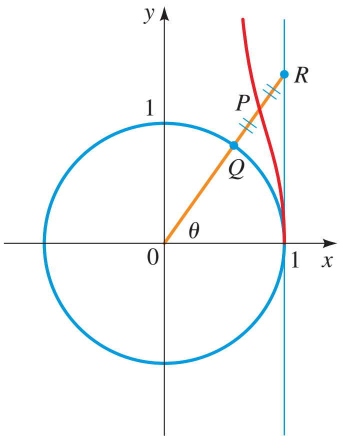
42. $x = \dfrac{1}{t} + 2, \quad y = \dfrac{2}{t^2}, \quad 0 < t \le 2$

43–44 ■ Graphs of Parametric Equations Use a graphing device to draw the parametric curve.

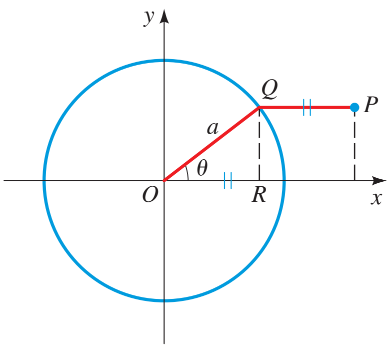
43. $x = \cos 2t, \quad y = \sin 3t$

44. $x = \sin(t + \cos 2t), \quad y = \cos(t + \sin 3t)$

45. Finding Parametric Equations for a Curve In the figure, the point P is the midpoint of the segment QR and $0 \le \theta < \pi/2$. Using θ as the parameter, find a parametric representation for the curve traced out by P.



46. Finding Parametric Equations for a Curve Find parametric equations for the curve traced out by the points P shown in the figure, using the angle θ as the parameter. In the figure, $|OR| = |QP|$.



47–48 ■ Operations with Vectors Find $|\mathbf{u}|$, $\mathbf{u} + \mathbf{v}$, $\mathbf{u} - \mathbf{v}$, $2\mathbf{u}$, and $3\mathbf{u} - 2\mathbf{v}$.

47. $\mathbf{u} = \langle -2, 3\rangle, \quad \mathbf{v} = \langle 8, 1\rangle$ **48.** $\mathbf{u} = 2\mathbf{i} + \mathbf{j}, \quad \mathbf{v} = \mathbf{i} - 2\mathbf{j}$

49. Component Form of a Vector A vector has initial point $P(0, 3)$ and terminal point $Q(3, -1)$. Express the vector in component form.

50. Terminal Point of a Vector If the vector $5\mathbf{i} - 8\mathbf{j}$ is placed in the plane with its initial point at $P(5, 6)$, find its terminal point.

51–52 ■ Length and Direction of Vectors Find the length and direction of the given vector.

51. $\mathbf{u} = \langle -2, 2\sqrt{3}\rangle$ **52.** $\mathbf{v} = 2\mathbf{i} - 5\mathbf{j}$

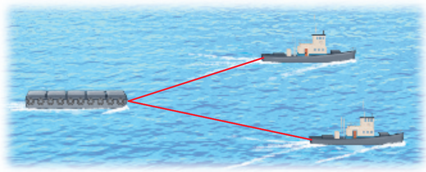
53–54 ■ Component Form of a Vector The length $|\mathbf{u}|$ and direction θ of a vector $\mathbf{u}$ are given. Express $\mathbf{u}$ in component form.

53. $|\mathbf{u}| = 20, \quad \theta = 60°$ **54.** $|\mathbf{u}| = 13.5, \quad \theta = 125°$

55. Resultant Force Two tugboats are pulling a barge as shown in the figure. One pulls with a force of 2.0×10^4 lb in the direction N 50° E, and the other pulls with a force of 3.4×10^4 lb in the direction S 75° E.

(a) Find the resultant force on the barge as a vector.

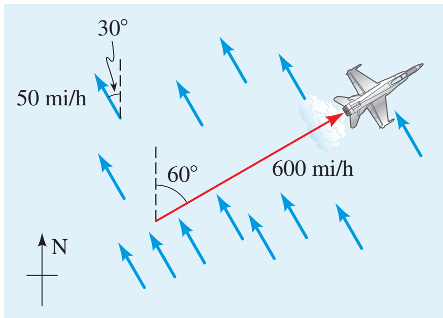
(b) Find the magnitude and direction of the resultant force.



56. True Velocity of a Plane An airplane heads N 60° E at a speed of 600 mi/h relative to the air. A wind begins to blow in the direction N 30° W at 50 mi/h. (See the figure.)

(a) Find the velocity of the airplane as a vector.

(b) Find the true speed and direction of the airplane.



57–60 ■ Dot Products Find the vectors $|\mathbf{u}|$, $\mathbf{u} \cdot \mathbf{u}$, and $\mathbf{u} \cdot \mathbf{v}$.

57. $\mathbf{u} = \langle 4, -3\rangle, \quad \mathbf{v} = \langle 9, -8\rangle$ **58.** $\mathbf{u} = \langle 5, 12\rangle, \quad \mathbf{v} = \langle 10, -4\rangle$

59. $\mathbf{u} = -2\mathbf{i} + 2\mathbf{j}, \quad \mathbf{v} = \mathbf{i} + \mathbf{j}$ **60.** $\mathbf{u} = 10\mathbf{j}, \quad \mathbf{v} = 5\mathbf{i} - 3\mathbf{j}$

61–64 ■ Orthogonal Vectors Are $\mathbf{u}$ and $\mathbf{v}$ orthogonal? If not, find the angle between them.

61. $\mathbf{u} = \langle -4, 2\rangle, \quad \mathbf{v} = \langle 3, 6\rangle$ **62.** $\mathbf{u} = \langle 5, 3\rangle, \quad \mathbf{v} = \langle -2, 6\rangle$

63. $\mathbf{u} = 2\mathbf{i} + \mathbf{j}, \quad \mathbf{v} = \mathbf{i} + 3\mathbf{j}$ **64.** $\mathbf{u} = \mathbf{i} - \mathbf{j}, \quad \mathbf{v} = \mathbf{i} + \mathbf{j}$

65–66 ■ Scalar and Vector Projections Two vectors $\mathbf{u}$ and $\mathbf{v}$ are given. **(a)** Find the component of $\mathbf{u}$ along $\mathbf{v}$. **(b)** Find $\text{proj}_{\mathbf{v}}\mathbf{u}$. **(c)** Resolve $\mathbf{u}$ into the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$, where $\mathbf{u}_1$ is parallel to $\mathbf{v}$ and $\mathbf{u}_2$ is perpendicular to $\mathbf{v}$.

65. $\mathbf{u} = \langle 3, 1\rangle, \quad \mathbf{v} = \langle 6, -1\rangle$ **66.** $\mathbf{u} = \mathbf{i} + 2\mathbf{j}, \quad \mathbf{v} = 4\mathbf{i} - 9\mathbf{j}$

67. Work A force $\mathbf{F} = 2\mathbf{i} + 9\mathbf{j}$ moves an object from the point $(7, -1)$ to the point $(1, 1)$. Find the work done if the distance is measured in feet and the force is measured in pounds.

68. Work A force $\mathbf{F}$ with magnitude 250 lb moves an object in the direction of a vector $\mathbf{D}$ a distance of 20 ft. If the work done is 3800 ft-lb, find the angle between $\mathbf{F}$ and $\mathbf{D}$.

Chapter 8 | Test

1. (a) Convert the point whose polar coordinates are $(8, 5\pi/4)$ to rectangular coordinates.

   (b) Find two polar coordinate representations for the rectangular coordinate point $(-6, 2\sqrt{3})$, one with $r > 0$ and one with $r < 0$ and both with $0 \le \theta < 2\pi$.

2. (a) Graph the polar equation $r = 8 \cos \theta$. What type of curve is this?

   (b) Convert the equation to rectangular coordinates.

3. Graph the polar equation $r = 3 + 6 \sin \theta$. What type of curve is this?

4. Let $z = 1 + \sqrt{3}i$.

   (a) Graph z in the complex plane.

   (b) Write z in polar form.

   (c) Find the complex number z^9.

5. Let $z_1 = 4\left(\cos \dfrac{7\pi}{12} + i \sin \dfrac{7\pi}{12}\right)$ and $z_2 = 2\left(\cos \dfrac{5\pi}{12} + i \sin \dfrac{5\pi}{12}\right)$.

   Find $z_1 z_2$ and $\dfrac{z_1}{z_2}$.

6. Find the cube roots of $27i$, and sketch these roots in the complex plane.

7. (a) Sketch the curve represented by the parametric equations below. Use arrows to indicate the direction of the curve as t increases.

   $$x = 3 \sin t + 3 \qquad y = 2 \cos t \qquad 0 \le t \le \pi$$

   (b) Eliminate the parameter t in part (a) to obtain an equation for this curve in rectangular coordinates.

8. Find parametric equations for the line of slope 2 that passes through the point $(3, 5)$.

9. The position of an object in circular motion is modeled by the parametric equations

   $$x = 3 \sin 2t \qquad y = 3 \cos 2t$$

   where t is measured in seconds.

   (a) Describe the path of the object by stating the radius of the circle, the position at time $t = 0$, the orientation of motion (clockwise or counterclockwise), and the time t it takes to complete one revolution around the circle.

   (b) Suppose the speed of the object is doubled. Find new parametric equations that model the motion of the object.

   (c) Find an equation in rectangular coordinates for the same curve by eliminating the parameter.

   (d) Find a polar equation for the same curve.

10. Let $\mathbf{u}$ be the vector with initial point $P(3, -1)$ and terminal point $Q(-3, 9)$.

    (a) Graph $\mathbf{u}$ in the coordinate plane.

    (b) Express $\mathbf{u}$ in terms of $\mathbf{i}$ and $\mathbf{j}$.

    (c) Find the length of $\mathbf{u}$.

11. Let $\mathbf{u} = \langle 1, 3 \rangle$, and let $\mathbf{v} = \langle -6, 2 \rangle$.

    (a) Find $\mathbf{u} - 3\mathbf{v}$.

    (b) Find $|\mathbf{u} + \mathbf{v}|$.

    (c) Find $\mathbf{u} \cdot \mathbf{v}$.

    (d) Are $\mathbf{u}$ and $\mathbf{v}$ perpendicular?

12. Let $\mathbf{u} = \langle -4\sqrt{3}, 4 \rangle$.

(a) Graph $\mathbf{u}$ in the coordinate plane, with initial point $(0, 0)$.

(b) Find the length and direction of $\mathbf{u}$.

13. A river is flowing due east at 8 mi/h. A motorboat heads in the direction N 30° E in the river. The speed of the motorboat relative to the water is 12 mi/h.

(a) Express the true velocity of the motorboat as a vector.

(b) Find the true speed and direction of the motorboat.

14. Let $\mathbf{u} = 3\,\mathbf{i} + 2\,\mathbf{j}$ and $\mathbf{v} = 5\,\mathbf{i} - \mathbf{j}$.

(a) Find the angle between $\mathbf{u}$ and $\mathbf{v}$.

(b) Find the component of $\mathbf{u}$ along $\mathbf{v}$.

(c) Find $\text{proj}_{\mathbf{v}}\,\mathbf{u}$.

15. A force $\mathbf{F} = 3\,\mathbf{i} - 5\,\mathbf{j}$ moves an object from the point $(2, 2)$ to the point $(7, -13)$. Find the work done if the distance is measured in feet and the force is measured in pounds.

In this section we use parametric equations and vectors to model the motion of a projectile, such as a ball thrown upward, an object launched from a catapult, or a cannonball fired from a cannon.

Parametric Equations for the Path of a Projectile

Suppose that we fire a projectile into the air from ground level, with an initial speed v_0 and at an angle θ upward from the ground. If there were no gravity (and no air resistance), the projectile would just keep moving indefinitely at the same speed and in the same direction. Since distance = speed × time, at time t the projectile would have traveled a distance $v_0 t$, so its position at time t would be given by the following parametric equations (assuming that the origin of our coordinate system is placed at the initial location of the projectile; see Figure 1):

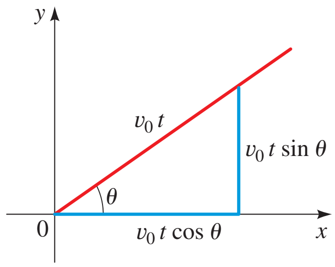


Figure 1 | Path of a projectile with no gravity

$$x = (v_0 \cos\theta)t \qquad y = (v_0 \sin\theta)t \qquad \text{No gravity}$$

But, of course, we know that gravity will pull the projectile back to ground level. It can be shown that the effect of gravity after t seconds can be accounted for by subtracting $\frac{1}{2}gt^2$ from the vertical position of the projectile. In this expression, g is the gravitational acceleration: $g \approx 32$ ft/s^2 ≈ 9.8 m/s^2. Thus we have the following parametric equations for the path of the projectile:

$$x = (v_0 \cos\theta)t \qquad y = (v_0 \sin\theta)t - \tfrac{1}{2}gt^2 \qquad \text{Position at time } t$$

Example ■ The Path of a Cannonball

Find parametric equations that model the path of a cannonball fired into the air with an initial speed of 150 m/s at a 30° angle of elevation. Sketch the path of the cannonball.

The path of a projectile can be "animated" on a graphing device. See the Discovery Project *Collsion* referenced in Section 8.4.

Solution Substituting the given initial speed and angle into the general parametric equations of the path of a projectile, we get

$$x = (150 \cos 30°)t \qquad y = (150 \sin 30°)t - \tfrac{1}{2}(9.8)t^2 \qquad \text{Substitute } v_0 = 150,\ \theta = 30°,\ g = 9.8$$

$$x \approx 129.9t \qquad y = 75t - 4.9t^2 \qquad \text{Simplify}$$

This path is graphed in Figure 2.

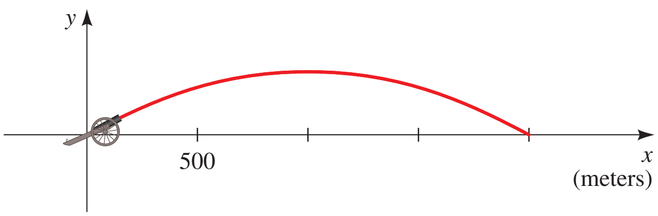


Figure 2 | Path of a cannonball

■

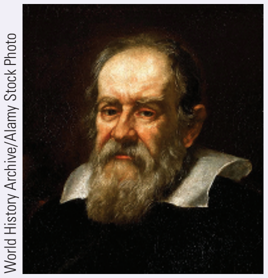
Range of a Projectile

How can we tell where and when the cannonball of the first example hits the ground? Since ground level corresponds to $y = 0$, we substitute this value for y and solve for t.

$$0 = 75t - 4.9t^2 \qquad \text{Set } y = 0$$

$$0 = t(75 - 4.9t) \qquad \text{Factor}$$

$$t = 0 \quad \text{or} \quad t = \frac{75}{4.9} \approx 15.3 \qquad \text{Solve for } t$$



GALILEO GALILEI (1564–1642) was born in Pisa, Italy. He studied medicine but later abandoned this in favor of science and mathematics. At the age of 25, by dropping cannonballs of various sizes from the Leaning Tower of Pisa, he demonstrated that light objects fall at the same rate as heavier ones. This contradicted the then-accepted view of Aristotle that heavier objects fall more quickly. Galileo also showed that the distance an object falls is proportional to the square of the time it has been falling, and from this he was able to prove that the path of a projectile is a parabola.

Galileo constructed the first telescope and, using it, discovered the moons of Jupiter. His advocacy of the Copernican view that the earth revolves around the sun (rather than being stationary) led to his being called before the Inquisition. By then an old man, he was forced to recant his views, but he is said to have muttered under his breath, "Nevertheless, it does move." Galileo revolutionized science by expressing scientific principles in the language of mathematics. He said, "The great book of nature is written in mathematical symbols."

The first solution, $t = 0$, is the time when the cannon was fired; the second solution means that the cannonball hits the ground after 15.3 s of flight. To see *where* this happens, we substitute this value of t into the equation for x, the horizontal location of the cannonball.

$$x \approx 129.9t = 129.9(15.3) \approx 1987.5 \text{ m}$$

The cannonball travels almost 2 km before hitting the ground—that is the *range* is 2 km. In general, for a projectile launched from ground level, the **range of the projectile** is the horizontal distance from the point it is launched to the point it returns to the ground.

Figure 3 shows the paths of several projectiles, all fired with the same initial speed but at different angles. From the graphs we see that if the firing angle is too high or too low, the projectile doesn't travel very far.

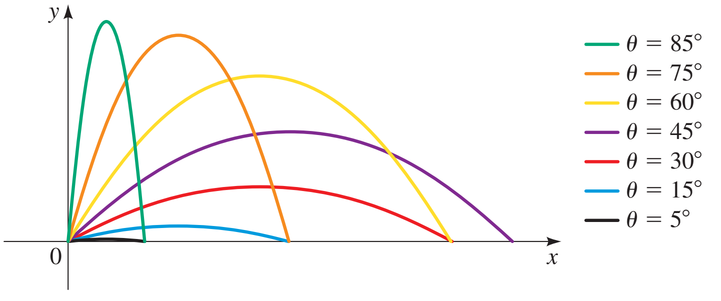


Figure 3 | Paths of projectiles

Let's try to find the optimal firing angle—the angle that shoots the projectile as far as possible. We'll go through the same steps as we did when calculating the range, but we'll use the general parametric equations instead. First, we solve for the time when the projectile hits the ground by substituting $y = 0$.

$$\begin{aligned} 0 &= (v_0 \sin\theta)t - \tfrac{1}{2}gt^2 && \text{Substitute } y = 0 \\ 0 &= t(v_0 \sin\theta - \tfrac{1}{2}gt) && \text{Factor} \\ 0 &= v_0 \sin\theta - \tfrac{1}{2}gt && \text{Set second factor equal to 0} \\ t &= \frac{2v_0 \sin\theta}{g} && \text{Solve for } t \end{aligned}$$

Now we substitute this into the equation for x to see how far the projectile has traveled horizontally when it hits the ground.

$$\begin{aligned} x &= (v_0 \cos\theta)t && \text{Parametric equation for } x \\ &= (v_0 \cos\theta)\left(\frac{2v_0 \sin\theta}{g}\right) && \text{Substitute } t = (2v_0 \sin\theta)/g \\ &= \frac{2v_0^2 \sin\theta \cos\theta}{g} && \text{Simplify} \\ &= \frac{v_0^2 \sin 2\theta}{g} && \text{Use identity } \sin 2\theta = 2\sin\theta\cos\theta \end{aligned}$$

We want to choose θ so that x is as large as possible. The largest value that the sine of an angle can have is 1, the sine of 90°. Thus we want $2\theta = 90°$, or $\theta = 45°$. So to send the projectile as far as possible, it should be launched at an angle of 45°. From the last equation in the preceding display, we see that it will travel a distance $x = v_0^2/g$.

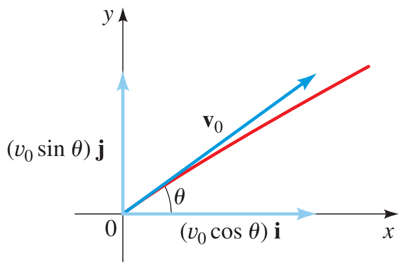


Figure 4 | Initial velocity vector

Vector Equation for the Velocity of a Projectile

We found parametric equations for the path of a projectile. Now we find a vector equation for the velocity of the projectile.

A projectile is fired with initial speed v_0 m/s at an angle θ upward from the ground. Figure 4 shows that the initial velocity vector is

$$\mathbf{v}_0 = (v_0 \cos\theta)\mathbf{i} + (v_0 \sin\theta)\mathbf{j} \qquad \text{Initial velocity}$$

The horizontal component of the velocity vector does not change with time, but the vertical component decreases as gravity tugs downward on the projectile. Gravity reduces the upward velocity by g m/s every second. Thus, after t seconds the vertical component is reduced by gt m/s. So the velocity $\mathbf{v}_t$ at any time t is given by

$$\mathbf{v}_t = (v_0 \cos\theta)\mathbf{i} + (v_0 \sin\theta - gt)\mathbf{j} \qquad \text{Velocity at time } t$$

For the cannonball in the example, the velocity at time t is

$$\begin{aligned}\mathbf{v}_t &= (150 \cos 30°)\mathbf{i} + (150 \sin 30° - 9.8t)\mathbf{j} \\ &\approx 129.9\mathbf{i} + (75 - 9.8\text{t})\mathbf{j}\end{aligned}$$

Figure 5 shows a graph of the path of the projectile and velocity vectors $\mathbf{v}_t$ (and their component vectors) at four points on the path.

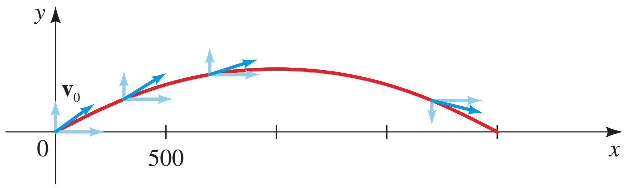


Figure 5 | Path and velocity of a projectile

Problems

1. **Trajectories Are Parabolas** From the graphs in Figure 3 the paths of projectiles appear to be parabolas that open downward. Eliminate the parameter t from the general parametric equations to verify that these are indeed parabolas.

2. **Path of a Baseball** Suppose a baseball is thrown at 30 ft/s at a 60° angle to the horizontal from a height of 4 ft above the ground.
   (a) Find parametric equations for the path of the baseball, and sketch its graph.
   (b) How far does the baseball travel horizontally, and when does it hit the ground?

3. **Path of a Rocket** Suppose that a rocket is fired at an angle of 5° from the vertical with an initial speed of 1000 ft/s.
   (a) Find the length of time the rocket is in the air.
   (b) Find the greatest height it reaches.
   (c) Find the horizontal distance it has traveled when it hits the ground.
   (d) Graph the rocket's path.

4. **Firing a Projectile** The initial speed of a projectile is 330 m/s.
   (a) At what angle should the projectile be fired so that it hits a target 10 km away? (You should find that there are two possible angles.) Graph the projectile paths for both angles.
   (b) For which angle is the target hit sooner?

5. Maximum Height Show that the maximum height reached by a projectile as a function of its initial speed v_0 and its firing angle θ is

$$y = \frac{v_0^2 \sin^2\theta}{2g}$$

6. Shooting into the Wind Suppose that a projectile is fired into a headwind that pushes it back so as to reduce its horizontal speed by a constant amount w. Find parametric equations for the path of the projectile.

7. Shooting into the Wind Using the parametric equations you derived in Problem 6, draw graphs of the path of a projectile with initial speed $v_0 = 32$ ft/s, fired into a headwind of $w = 24$ ft/s, for the angles $\theta = 5°, 15°, 30°, 40°, 45°, 55°, 60°$, and $75°$. Is it still true that the greatest range is attained when firing at 45°? Draw some more graphs for different angles, and use these graphs to estimate the optimal firing angle.

8. Path and Velocity of a Projectile A cannonball is fired from ground level with initial speed 200 m/s at an angle of elevation of 60°.

(a) Find parametric equations for the position of the cannonball t seconds after it is fired.

(b) Find the velocity vectors $\mathbf{v}_t$ of the cannonball at time t. What is the velocity at its highest point? What is the speed when it hits the ground?

(c) Draw a graph of the path of the cannonball and sketch velocity vectors on the graph at the points corresponding to $t = 6, 10, 18, 30$, as shown in Figure 4.