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11

Sequences and Series

Throughout this book we have used functions to model real-world situations. The functions we've used have always had real numbers as inputs. But many situations occur in stages: stage 1, 2, 3, ... To model such situations, we need functions whose inputs are the natural numbers 1, 2, 3, . . . (representing the stages). For example, the peaks of a bouncing ball are represented by the natural numbers 1, 2, 3, . . . (representing peak 1, 2, 3, . . .). A function f that models the height of the ball at each peak has natural numbers 1, 2, 3, . . . as inputs and gives the heights as $f(1)$, $f(2)$, $f(3)$, ... In general a function whose inputs are the natural numbers is called a *sequence*. We can think of a sequence as simply a list of numbers written in a specific order.

The applications of sequences are varied. For instance, the amount in a bank account at the end of each month and the number of ancestors for each successive generation are both sequences. Many patterns in nature can also be modeled by sequences. For example, the Fibonacci sequence describes the growth of a rabbit population, the arrangements of leaves on a plant, or the spiral patterns of seeds in a sunflower (pictured above).

11.1 Sequences and Summation Notation

■ Sequences ■ Recursively Defined Sequences ■ The Partial Sums of a Sequence
■ Sigma Notation

Generally speaking, a sequence is an infinite list of numbers. The numbers in the sequence are often written as $a_1, a_2, a_3, \ldots$. The ellipsis (three dots) means that the list continues indefinitely. A simple example is the sequence

$$\begin{array}{ccccc} 5, & 10, & 15, & 20, & 25, \ldots \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ a_1 & a_2 & a_3 & a_4 & a_5 \ldots \end{array}$$

We can describe the pattern of the sequence displayed above by the following *formula*:

$$a_n = 5n$$

You may have already thought of a different way to describe the pattern—namely, "you go from one number to the next by adding 5." This natural way of describing the sequence is expressed by the *recursive formula*:

$$a_n = a_{n-1} + 5 \qquad (n = 2, 3, \ldots)$$

starting with $a_1 = 5$. Try substituting $n = 1, 2, 3, \ldots$ in each of these formulas to see how they produce the numbers in the sequence. In this section we see how these different ways are used to describe specific sequences.

■ Sequences

Any ordered list of numbers can be viewed as a function whose input values are 1, 2, 3, . . . and whose output values are the numbers in the list. So we define a sequence as follows.

Definition of a Sequence

A **sequence** is a function a whose domain is the set of natural numbers. The **terms of the sequence** are the function values

$$a(1), a(2), a(3), \ldots, a(n), \ldots$$

We usually write a_n instead of the function notation $a(n)$. So the terms of the sequence are written as

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

The number a_1 is called the **first term**, a_2 is called the **second term**, and in general, a_n is called the ***n*th term**.

Here is a simple example of a sequence:

$$2, 4, 6, 8, 10, \ldots$$

Another way to write this sequence is to use function notation:

$$a(n) = 2n$$

so $a(1) = 2, a(2) = 4, a(3) = 6, \ldots$

We can write a sequence in this way when it's clear what the subsequent terms of the sequence are. This sequence consists of even numbers. To be more accurate, however, we need to specify a procedure for finding *all* the terms of the sequence. This can be done by giving a formula for the nth term a_n of the sequence. In this case,

$$a_n = 2n$$

and the sequence can be written as

$2,$	$4,$	$6,$	$8,$	$\ldots,$	$2n,$	$\ldots$
1st term	2nd term	3rd term	4th term		nth term	

Notice how the formula $a_n = 2n$ gives all the terms of the sequence. For instance, substituting 1, 2, 3, and 4 for n gives the first four terms:

$$a_1 = 2 \cdot 1 = 2 \qquad a_2 = 2 \cdot 2 = 4$$

$$a_3 = 2 \cdot 3 = 6 \qquad a_4 = 2 \cdot 4 = 8$$

To find the 103rd term of this sequence, we use $n = 103$ to get

$$a_{103} = 2 \cdot 103 = 206$$

Example 1 ■ Finding the Terms of a Sequence

Find the first five terms and the 100th term of the sequence defined by each formula.

(a) $a_n = 2n - 1$ **(b)** $c_n = n^2 - 1$

(c) $t_n = \dfrac{n}{n + 1}$ **(d)** $r_n = \dfrac{(-1)^n}{2^n}$

Solution To find the first five terms, we substitute $n = 1, 2, 3, 4$, and 5 in the formula for the nth term. To find the 100th term, we substitute $n = 100$. This gives the following.

nth Term	First Five Terms	100th Term
(a) $2n - 1$	$1, 3, 5, 7, 9$	199
(b) $n^2 - 1$	$0, 3, 8, 15, 24$	9999
(c) $\dfrac{n}{n+1}$	$\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}, \dfrac{5}{6}$	$\dfrac{100}{101}$
(d) $\dfrac{(-1)^n}{2^n}$	$-\dfrac{1}{2}, \dfrac{1}{4}, -\dfrac{1}{8}, \dfrac{1}{16}, -\dfrac{1}{32}$	$\dfrac{1}{2^{100}}$

Now Try Exercises 3, 5, 7, and 9 ■

In Example 1(d) the presence of $(-1)^n$ in the sequence has the effect of making successive terms alternately negative and positive.

It is often useful to picture a sequence by sketching its graph. Since a sequence is a function whose domain is the natural numbers, we can draw its graph in the Cartesian plane. For instance, the graph of the sequence

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots, \frac{1}{n}, \ldots$$

is shown in Figure 1.

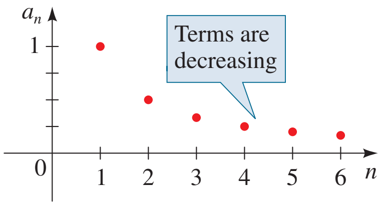


Figure 1

Compare the graph of the sequence shown in Figure 1 to the graph of

$$1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{6}, \ldots, \frac{(-1)^{n+1}}{n}, \ldots$$

shown in Figure 2. The graph of every sequence consists of isolated points that are *not* connected.

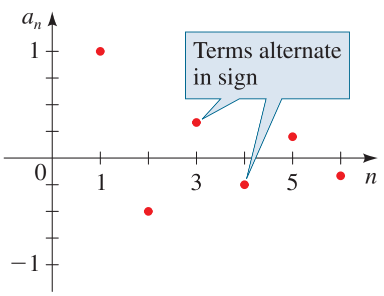


Figure 2

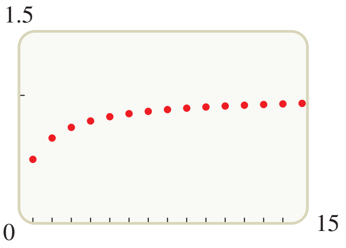


Figure 3 | $t(n) = n/(n+1)$

Not all sequences can be defined by a formula. For example, there is no known formula for the sequence of prime numbers:*

2, 3, 5, 7, 11, 13, 17, 19, 23, . . .

We can use a graphing device to graph a sequence. The graph of the sequence in Example 1(c) is shown in Figure 3.

Finding patterns is an important part of mathematics. Consider a sequence that begins

$$1, 4, 9, 16, \ldots$$

Can you detect a pattern in these numbers? In other words, can you define a sequence whose first four terms are these numbers? The answer to this question seems straightforward; these numbers are the squares of the numbers 1, 2, 3, 4. Thus the sequence we are looking for is defined by $a_n = n^2$. However, this is not the *only* sequence whose first four terms are 1, 4, 9, 16. In other words, the answer to our problem is not unique (see Exercise 86). In the next example we are interested in finding an *obvious* sequence whose first few terms agree with the given ones.

Example 2 ■ Finding the *n*th Term of a Sequence

Find the nth term of a sequence whose first several terms are given.

(a) $\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \ldots$ **(b)** $-2, 4, -8, 16, -32, \ldots$

Solution

(a) We notice that the numerators of these fractions are the odd numbers and the denominators are the even numbers. Even numbers are of the form $2n$, and odd numbers are of the form $2n - 1$ (an odd number differs from an even number by 1). So a sequence that has these numbers for its first four terms is given by

$$a_n = \frac{2n-1}{2n}$$

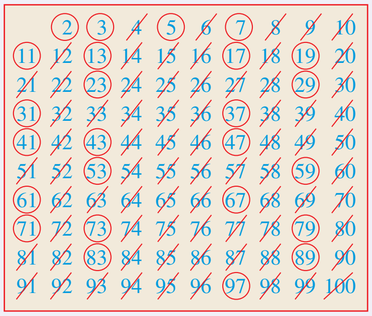
(b) These numbers are powers of 2, and they alternate in sign, so a sequence that agrees with these terms is given by

$$a_n = (-1)^n 2^n$$

You should check that these formulas do indeed generate the given terms.

Now Try Exercises 29 and 35 ■

ERATOSTHENES (circa 276–195 B.C.) was a renowned Greek geographer, mathematician, and astronomer. He accurately calculated the circumference of the earth by an ingenious method (see Exercise 6.1.80). He is most famous, however, for his method for finding primes, now called the *sieve of Eratosthenes*. The method consists of listing the integers, beginning with 2 (the first prime), and then crossing out all the multiples of 2, which are not prime. The next number remaining on the list is 3 (the second prime), so we again cross out all multiples of it. The next remaining number is 5 (the third prime number), and we cross out all multiples of it, and so on. In this way all numbers that are not prime are crossed out, and the remaining numbers are the primes.



■ Recursively Defined Sequences

Some sequences do not have simple defining formulas like those shown in the preceding example. The nth term of a sequence may depend on some or all of the terms preceding it. A sequence defined in this way is called **recursive**. Here are two examples.

Example 3 ■ Finding the Terms of a Recursively Defined Sequence

A sequence is defined recursively by $a_1 = 1$ and

$$a_n = 3(a_{n-1} + 2)$$

(a) Find the first five terms of the sequence.

(b) Use a graphing device to find the 20th term of the sequence.

Solution

(a) The defining formula for this sequence is recursive. It allows us to find the nth term a_n if we know the preceding term a_{n-1}. Thus we can find the second term from the first term, the third term from the second term, the fourth term from

* A *prime number* is a whole number p whose only divisors are p and 1. (By convention the number 1 is not considered prime.)

Large Prime Numbers
The search for large primes fascinates many people. As of this writing, the largest known prime number is

$$2^{82,589,933} - 1$$

It was discovered by a computer volunteered by Patrick Laroche of Ocala, Florida, to the distributed computing project known as GIMPS (the Great Internet Mersenne Prime Search). In decimal notation this number contains 24,862,048 digits. If it were written in full, it would occupy more than six times as many pages as this book contains. Numbers of the form $2^p - 1$, where p is prime, are called Mersenne numbers and are named for the French monk Marin Mersenne who first studied them in the 1600s. Such numbers are more easily checked for primality than others. That is why the largest known primes are of this form.

You can find online calculators for computing recursive sequences.

the third term, and so on. Since we are given the first term $a_1 = 1$, we can proceed as follows.

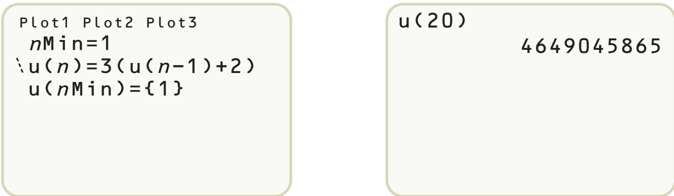
$$a_2 = 3(a_1 + 2) = 3(1 + 2) = 9$$
$$a_3 = 3(a_2 + 2) = 3(9 + 2) = 33$$
$$a_4 = 3(a_3 + 2) = 3(33 + 2) = 105$$
$$a_5 = 3(a_4 + 2) = 3(105 + 2) = 321$$

Thus the first five terms of this sequence are

$$1, 9, 33, 105, 321, \ldots$$

(b) Note that to find the 20th term of the recursive sequence, we must first find all 19 preceding terms. This is most easily done by using a graphing device. Figure 4(a) shows how to enter this sequence on the TI-83 calculator. From Figure 4(b) we see that the 20th term of the sequence is

$$a_{20} = 4{,}649{,}045{,}865$$



(a) (b)

Figure 4 | $u(n) = 3(u(n-1) + 2), u(1) = 1$

Now Try Exercises 15 and 25 ■

Example 4 ■ The Fibonacci Sequence

Find the first 11 terms of the sequence defined recursively by $F_1 = 1$, $F_2 = 1$, and

$$F_n = F_{n-1} + F_{n-2}$$

Solution To find F_n, we need to find the two preceding terms, F_{n-1} and F_{n-2}. Since we are given F_1 and F_2, we proceed as follows.

$$F_3 = F_2 + F_1 = 1 + 1 = 2$$
$$F_4 = F_3 + F_2 = 2 + 1 = 3$$
$$F_5 = F_4 + F_3 = 3 + 2 = 5$$

It's clear what is happening here. Each term is the sum of the two terms that precede it, so we can write down as many terms as we please. Here are the first 11 terms.

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

Now Try Exercise 19 ■

The sequence in Example 4 is called the **Fibonacci sequence**, named after the 13th century Italian mathematician who used it to solve a problem about the breeding of rabbits (see Exercise 85). The sequence also occurs in numerous other applications in nature. (See Figures 5 and 6.) In fact, so many phenomena behave like the Fibonacci sequence that one mathematical journal, the *Fibonacci Quarterly*, is devoted entirely to its properties.

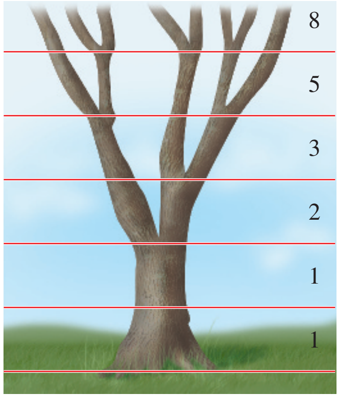


Figure 5 | The Fibonacci sequence in the branching of a tree

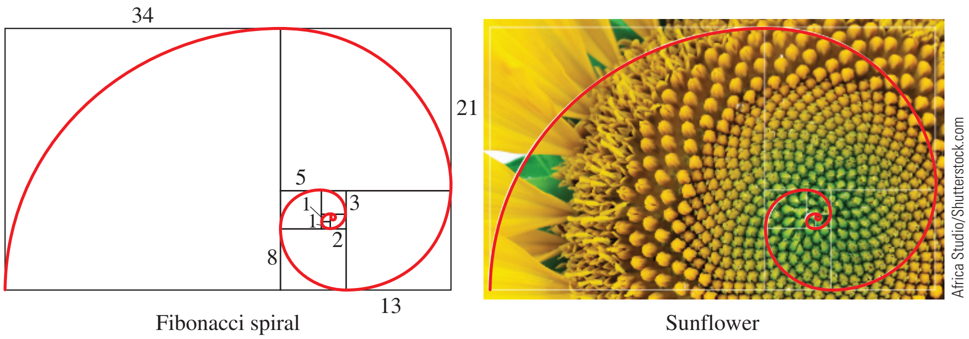


Figure 6 | The Fibonacci sequence in the pattern of seeds in a sunflower

The Partial Sums of a Sequence

In calculus we are often interested in adding the terms of a sequence. This leads to the following definition.

The Partial Sums of a Sequence

For the sequence

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

the **partial sums** are

$$\begin{aligned} S_1 &= a_1 \\ S_2 &= a_1 + a_2 \\ S_3 &= a_1 + a_2 + a_3 \\ S_4 &= a_1 + a_2 + a_3 + a_4 \\ &\vdots \\ S_n &= a_1 + a_2 + a_3 + \cdots + a_n \\ &\vdots \end{aligned}$$

S_1 is called the **first partial sum**, S_2 is the **second partial sum**, and so on. S_n is called the ***n*th partial sum**. The sequence $S_1, S_2, S_3, \ldots, S_n, \ldots$ is called the **sequence of partial sums**.

Example 5 ■ Finding the Partial Sums of a Sequence

Find the first four partial sums and the nth partial sum of the sequence given by $a_n = 1/2^n$.

Solution The terms of the sequence are

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$$

The first four partial sums are

$$\begin{aligned}
S_1 &= \frac{1}{2} && = \frac{1}{2}\\
S_2 &= \frac{1}{2} + \frac{1}{4} && = \frac{3}{4}\\
S_3 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} && = \frac{7}{8}\\
S_4 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} && = \frac{15}{16}
\end{aligned}$$

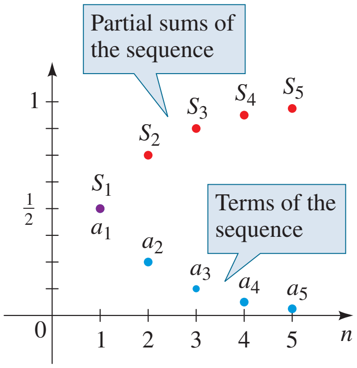


Figure 7 | Graph of the sequence a_n and the sequence of partial sums S_n

Notice that in the value of each partial sum, the denominator is a power of 2 and the numerator is one less than the denominator. In general, the nth partial sum is

$$S_n = \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n}$$

The first five terms of a_n and S_n are graphed in Figure 7.

Now Try Exercise 43 ■

Example 6 ■ Finding the Partial Sums of a Sequence

Find the first four partial sums and the nth partial sum of the sequence given by

$$a_n = \frac{1}{n} - \frac{1}{n+1}$$

Solution The first four partial sums are

$$\begin{aligned}
S_1 &= \left(1 - \frac{1}{2}\right) && = 1 - \frac{1}{2}\\
S_2 &= \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \frac{1}{3}\right) && = 1 - \frac{1}{3}\\
S_3 &= \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{4}\right) && = 1 - \frac{1}{4}\\
S_4 &= \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{4}} - \frac{1}{5}\right) && = 1 - \frac{1}{5}
\end{aligned}$$

Do you detect a pattern here? Of course. The nth partial sum is

$$S_n = 1 - \frac{1}{n+1}$$

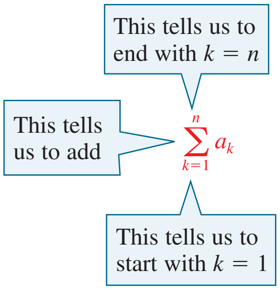
Now Try Exercise 45 ■

Sigma Notation

Given a sequence

$$a_1, a_2, a_3, a_4, \ldots$$

we can write the sum of the first n terms using **summation notation**, or **sigma notation**. This notation derives its name from the Greek letter Σ (capital sigma, corresponding to our S for "sum"). Sigma notation is used as follows:



$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + a_4 + \cdots + a_n$$

The left side of this expression is read, "The sum of a_k from $k = 1$ to $k = n$." The letter k is called the **index of summation**, or the **summation variable**, and the idea is to replace k in the expression after the sigma by the integers $1, 2, 3, \ldots, n$, and add the resulting expressions, arriving at the right-hand side of the equation.

Example 7 ■ Sigma Notation

Find each sum.

(a) $\sum_{k=1}^{5} k^2$ **(b)** $\sum_{j=3}^{5} \frac{1}{j}$ **(c)** $\sum_{k=5}^{10} k$ **(d)** $\sum_{i=1}^{6} 2$

Solution

(a) $\sum_{k=1}^{5} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55$

(b) $\sum_{j=3}^{5} \frac{1}{j} = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{47}{60}$

(c) $\sum_{k=5}^{10} k = 5 + 6 + 7 + 8 + 9 + 10 = 45$

(d) $\sum_{i=1}^{6} 2 = 2 + 2 + 2 + 2 + 2 + 2 = 12$

Now Try Exercises 47 and 49 ■

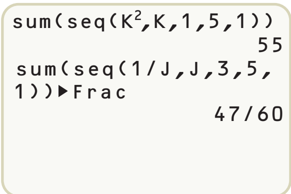


Figure 8

We can use a graphing device to evaluate sums. For instance, Figure 8 shows how the TI-83 can be used to evaluate the sums in parts (a) and (b) of Example 7. You can also find online calculators for computing partial sums of sequences.

The Golden Ratio

The ancient Greeks considered a line segment to be divided into the **golden ratio** if the ratio of the shorter part to the longer part is the same as the ratio of the longer part to the whole segment.

Thus the segment shown is divided into the golden ratio if

$$\frac{1}{x} = \frac{x}{1 + x}$$

This leads to a quadratic equation whose positive solution is

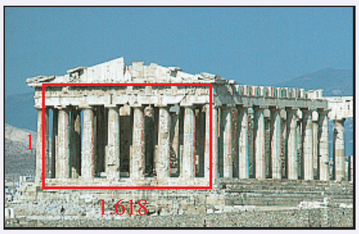
$$x = \frac{1 + \sqrt{5}}{2} \approx 1.618$$

This ratio occurs naturally in many places. For instance, psychology experiments show that the most pleasing shape of rectangle is one whose sides are in golden ratio. The ancient Greeks agreed with this and built their temples in this ratio.

The golden ratio is related to the Fibonacci sequence (see Exercise 12.4.43). The ratio of two successive Fibonacci numbers

$$\frac{F_{n+1}}{F_n}$$

gets closer to the golden ratio the larger the value of n.



Clark Dunbar/Flame/Corbis


Stefano Bianchetti/Getty Images

FIBONACCI (circa 1170–1250) was born in Pisa, Italy, and was educated in North Africa. He traveled widely in the Mediterranean area and learned the various methods then in use for writing numbers. On returning to Pisa in 1202, Fibonacci advocated the use of the Hindu-Arabic decimal system, the one we use today, over the Roman numeral system that was used in Europe in his time. His most famous book, *Liber Abaci*, expounds on the advantages of the Hindu-Arabic numerals. In fact, multiplication and division were so complicated using Roman numerals that the equivalent of a college degree was necessary to master these skills. Interestingly, in 1299 the city of Florence outlawed the use of the decimal system for merchants and businesses, requiring numbers to be written in Roman numerals or words. One can only speculate about the reasons for this law.

Example 8 ■ Writing Sums in Sigma Notation

Write each sum using sigma notation.

(a) $1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3$ **(b)** $\sqrt{3} + \sqrt{4} + \sqrt{5} + \cdots + \sqrt{77}$

Solution

(a) We can write

$$1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 = \sum_{k=1}^{7} k^3$$

(b) A natural way to write this sum is

$$\sqrt{3} + \sqrt{4} + \sqrt{5} + \cdots + \sqrt{77} = \sum_{k=3}^{77} \sqrt{k}$$

However, there is no unique way of writing a sum in sigma notation. We could also write this sum as

$$\sqrt{3} + \sqrt{4} + \sqrt{5} + \cdots + \sqrt{77} = \sum_{k=0}^{74} \sqrt{k+3}$$

or
$$\sqrt{3} + \sqrt{4} + \sqrt{5} + \cdots + \sqrt{77} = \sum_{k=1}^{75} \sqrt{k+2}$$

Now Try Exercises 67 and 69 ■

The following properties of sums are natural consequences of properties of the real numbers.

Properties of Sums

Let $a_1, a_2, a_3, a_4, \ldots$ and $b_1, b_2, b_3, b_4, \ldots$ be sequences. Then for every positive integer n and any real number c the following properties hold.

1. $\displaystyle\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$ **2.** $\displaystyle\sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$

3. $\displaystyle\sum_{k=1}^{n} ca_k = c\left(\sum_{k=1}^{n} a_k\right)$

Proof To prove Property 1, we write out the left-hand side of the equation to get

$$\sum_{k=1}^{n} (a_k + b_k) = (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) + \cdots + (a_n + b_n)$$

Because addition is commutative and associative, we can rearrange the terms on the right-hand side to read

$$\sum_{k=1}^{n} (a_k + b_k) = (a_1 + a_2 + a_3 + \cdots + a_n) + (b_1 + b_2 + b_3 + \cdots + b_n)$$

Rewriting the right side using sigma notation gives Property 1. Property 2 is proved in a similar manner. To prove Property 3, we use the Distributive Property:

$$\begin{aligned}\sum_{k=1}^{n} ca_k &= ca_1 + ca_2 + ca_3 + \cdots + ca_n \\ &= c(a_1 + a_2 + a_3 + \cdots + a_n) = c\left(\sum_{k=1}^{n} a_k\right)\end{aligned}$$ ■

11.1 Exercises

Concepts

1. A sequence is a function whose domain is ________.

2. The nth partial sum of a sequence is the sum of the first ________ terms of the sequence. So for the sequence $a_n = n^2$ the fourth partial sum is $S_4 =$ ____ + ____ + ____ + ____ = ____.

Skills

3–14 ■ Terms of a Sequence Find the first four terms and the 100th term of the sequence whose nth term is given.

3. $a_n = n - 3$

4. $a_n = 2n - 1$

5. $a_n = \dfrac{1}{3n - 4}$

6. $a_n = n^3 + 2$

7. $a_n = 3^n$

8. $a_n = \left(\dfrac{-1}{5}\right)^{n-1}$

9. $a_n = \dfrac{(-1)^n}{n^2}$

10. $a_n = \dfrac{1}{n^2}$

11. $a_n = 1 + (-1)^n$

12. $a_n = (-1)^{n+1}\dfrac{n}{n+1}$

13. $a_n = n^n$

14. $a_n = 3$

15–20 ■ Recursive Sequences A sequence is defined recursively by the given formulas. Find the first five terms of the sequence.

15. $a_n = 2(a_{n-1} + 3)$ and $a_1 = 4$

16. $a_n = \dfrac{a_{n-1}}{6}$ and $a_1 = -24$

17. $a_n = 2a_{n-1} + 1$ and $a_1 = 1$

18. $a_n = \dfrac{1}{1 + a_{n-1}}$ and $a_1 = 1$

19. $a_n = a_{n-1} + a_{n-2}$ and $a_1 = 1, a_2 = 2$

20. $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ and $a_1 = a_2 = a_3 = 1$

21–26 ■ Terms of a Sequence Use a graphing device to do the following. **(a)** Find the first ten terms of the sequence. **(b)** Graph the first ten terms of the sequence.

21. $a_n = 4n + 3$

22. $a_n = n^2 + n$

23. $a_n = \dfrac{12}{n}$

24. $a_n = 4 - 2(-1)^n$

25. $a_n = \dfrac{1}{a_{n-1}}$ and $a_1 = 2$

26. $a_n = a_{n-1} - a_{n-2}$ and $a_1 = 1, a_2 = 3$

27–38 ■ *n*th term of a Sequence Find the nth term of a sequence whose first several terms are given.

27. $2, 4, 6, 8, \ldots$

28. $1, 3, 5, 7, \ldots$

29. $-3, 9, -27, 81, \ldots$

30. $-\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \frac{1}{81}, \ldots$

31. $4, 9, 14, 19, \ldots$

32. $10, 3, -4, -11, \ldots$

33. $5, -25, 125, -625, \ldots$

34. $3, 0.3, 0.03, 0.003, \ldots$

35. $1, \frac{3}{4}, \frac{5}{9}, \frac{7}{16}, \frac{9}{25}, \ldots$

36. $\frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \ldots$

37. $0, 2, 0, 2, 0, 2, \ldots$

38. $1, \frac{1}{2}, 3, \frac{1}{4}, 5, \frac{1}{6}, \ldots$

39–42 ■ Partial Sums Find the first six partial sums $S_1, S_2, S_3, S_4, S_5, S_6$ of the sequence whose first several terms are given.

39. $2, 4, 6, 8, \ldots$

40. $1^2, 2^2, 3^2, 4^2, \ldots$

41. $\dfrac{1}{3}, \dfrac{1}{3^2}, \dfrac{1}{3^3}, \dfrac{1}{3^4}, \ldots$

42. $4, -4, 4, -4, \ldots$

43–46 ■ *n*th Partial Sum Find the first four partial sums and the nth partial sum of the sequence a_n.

43. $a_n = \dfrac{2}{3^n}$

44. $a_n = \dfrac{1}{n+1} - \dfrac{1}{n+2}$

45. $a_n = \sqrt{n} - \sqrt{n+1}$

46. $a_n = \log\left(\dfrac{n}{n+1}\right)$ [*Hint:* Use a property of logarithms to write the nth term as a difference.]

47–54 ■ Evaluating a Sum Find the sum.

47. $\displaystyle\sum_{k=1}^{4} k$

48. $\displaystyle\sum_{k=1}^{4} k^2$

49. $\displaystyle\sum_{k=1}^{4} \frac{1}{3k}$

50. $\displaystyle\sum_{j=1}^{51} (-1)^j$

51. $\displaystyle\sum_{i=1}^{8} [1 + (-1)^i]$

52. $\displaystyle\sum_{i=4}^{12} 10$

53. $\displaystyle\sum_{k=1}^{5} 2^{k-1}$

54. $\displaystyle\sum_{i=1}^{3} i2^i$

55–60 ■ Evaluating a Sum Use a graphing device to evaluate the sum.

55. $\displaystyle\sum_{k=1}^{10} k^2$

56. $\displaystyle\sum_{k=1}^{100} (3k + 4)$

57. $\displaystyle\sum_{j=7}^{20} j^2(1 + j)$

58. $\displaystyle\sum_{j=5}^{15} \frac{1}{j^2 + 1}$

59. $\displaystyle\sum_{n=0}^{22} (-1)^n 2n$

60. $\displaystyle\sum_{n=1}^{100} \frac{(-1)^n}{n}$

61–66 ■ Sigma Notation Write the sum without using sigma notation.

61. $\displaystyle\sum_{k=1}^{4} k^3$

62. $\displaystyle\sum_{j=1}^{4} \sqrt{\frac{j-1}{j+1}}$

63. $\displaystyle\sum_{k=0}^{6} \sqrt{k+4}$

64. $\displaystyle\sum_{k=6}^{9} k(k+3)$

65. $\displaystyle\sum_{k=3}^{100} x^k$

66. $\displaystyle\sum_{j=1}^{n} (-1)^{j+1} x^j$

67–74 ■ Sigma Notation Write the sum using sigma notation.

67. $4 + 8 + 12 + 16 + \cdots + 48$

68. $2 + 5 + 8 + \cdots + 29$

69. $1^2 + 2^2 + 3^2 + \cdots + 10^2$

70. $\dfrac{1}{2\ln 2} - \dfrac{1}{3\ln 3} + \dfrac{1}{4\ln 4} - \dfrac{1}{5\ln 5} + \cdots + \dfrac{1}{100\ln 100}$

71. $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4} + \cdots + \dfrac{1}{999 \cdot 1000}$

72. $\dfrac{\sqrt{1}}{1^2} + \dfrac{\sqrt{2}}{2^2} + \dfrac{\sqrt{3}}{3^2} + \cdots + \dfrac{\sqrt{n}}{n^2}$

73. $1 + x + x^2 + x^3 + \cdots + x^{100}$

74. $1 - 2x + 3x^2 - 4x^3 + 5x^4 + \cdots - 100x^{99}$

Skills Plus

75. *n*th Term of a Sequence Find a formula for the nth term of the sequence

$$\sqrt{2}, \quad \sqrt{2\sqrt{2}}, \quad \sqrt{2\sqrt{2\sqrt{2}}}, \quad \sqrt{2\sqrt{2\sqrt{2\sqrt{2}}}}, \ldots$$

[*Hint:* Write each term as a power of 2.]

76. A Formula for the Fibonacci Sequence It is known that the nth term of the Fibonacci sequence is given by the formula

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n}\right)$$

Use a calculator to find the first ten terms of the Fibonacci sequence using this formula.

Applications

77. Compound Interest An amount of \$2000 is deposited in a savings account that pays 2.4% interest per year compounded monthly. The amount in the account after n months is given by

$$A_n = 2000\left(1 + \frac{0.024}{12}\right)^n$$

(a) Find the first six terms of the sequence.

(b) Find the amount in the account after 3 years.

78. Compound Interest An amount of \$100 is deposited at the end of each month into an account that pays 6% interest per year compounded monthly. The amount of interest accumulated after n months is given by

$$I_n = 100\left(\frac{1.005^n - 1}{0.005} - n\right)$$

(a) Find the first six terms of the sequence.

(b) Find the interest accumulated after 5 years.

79. Population of a City A city was incorporated in 2004 with a population of 35,000. It is expected that the population will increase at a rate of 2% per year. The population n years after 2004 is given by

$$P_n = 35{,}000(1.02)^n$$

(a) Find the first five terms of the sequence.

(b) Find the population in 2014.

80. Paying off a Debt A loan of \$10,000 is to be repaid in monthly installments of \$200. Interest is charged on the balance at a rate of 0.5% per month.

(a) Show that the balance A_n in the nth month is given recursively by $A_0 = 10{,}000$ and

$$A_n = 1.005A_{n-1} - 200$$

(b) Find the balance after 6 months.

81. Fish Farming A fish farm has 5000 catfish in a pond. The number of catfish increases by 8% per month, and 300 catfish are harvested per month.

(a) Show that the catfish population P_n after n months is given recursively by $P_0 = 5000$ and

$$P_n = 1.08P_{n-1} - 300$$

(b) How many fish are there in the pond after 12 months?

82. Price of a House The median price of a house in a certain county increases by about 6% per year. In 2022 the median price was \$240,000. Let P_n be the median price n years after 2022.

(a) Find a formula for the sequence P_n.

(b) Find the expected median price in 2030.

83. Salary Increases A management position provides a salary of \$45,000 a year with a \$2000 raise every year. Let A_n be the salary in the nth year of employment.

(a) Find a recursive definition of A_n.

(b) Find the salary in the fifth year of employment.

84. Concentration of a Solution An expirement is set up to find the optimal salt concentration for the growth of a certain species of mollusk. The experiment begins with a brine solution that has a salt concentration of 4 g/L. The concentration of salt is increased by 10% every day. Let C_0 denote the initial concentration, and let C_n be the concentration after n days.

(a) Find a recursive definition of C_n.

(b) Find the salt concentration after 8 days.

85. Fibonacci's Rabbits Fibonacci posed the following problem: Suppose that rabbits live forever and that every month each pair produces a new pair that becomes productive at age 2 months. If we start with one newborn pair, how many pairs of rabbits will we have in the nth month? Show that the answer is F_n, where F_n is the nth term of the Fibonacci sequence.

Discuss ■ Discover ■ Prove ■ Write

86. Discover ■ Prove: Different Sequences That Start the Same

(a) Show that the first four terms of the sequence defined by $a_n = n^2$ are

$$1, 4, 9, 16, \ldots$$

(b) Show that the first four terms of the sequence defined by $a_n = n^2 + (n-1)(n-2)(n-3)(n-4)$ are also

$$1, 4, 9, 16, \ldots$$

(c) Find a sequence whose first six terms are the same as those of $a_n = n^2$ but whose succeeding terms differ from this sequence.

(d) Find two different sequences that begin

$$2, 4, 8, 16, \ldots$$

87. Discuss: A Recursively Defined Sequence Find the first 40 terms of the sequence defined by

$$a_{n+1} = \begin{cases} \dfrac{a_n}{2} & \text{if } a_n \text{ is an even number} \\ 3a_n + 1 & \text{if } a_n \text{ is an odd number} \end{cases}$$

and $a_1 = 11$. Do the same if $a_1 = 25$. Make a conjecture about this type of sequence. Try several other values for a_1 to test your conjecture.

88. Discuss: A Different Type of Recursion Find the first 10 terms of the sequence defined by

$$a_n = a_{n-a_{n-1}} + a_{n-a_{n-2}}$$

with

$$a_1 = 1 \qquad \text{and} \qquad a_2 = 1$$

How is this recursive sequence different from the others in this section?

11.2 Arithmetic Sequences

■ Arithmetic Sequences ■ Partial Sums of Arithmetic Sequences

In this section we study a special type of sequence, called an arithmetic sequence.

■ Arithmetic Sequences

Perhaps the simplest way to generate a sequence is to start with a number a and add to it a fixed constant d, over and over again.

Definition of an Arithmetic Sequence

An **arithmetic sequence** is a sequence of the form

$$a, a + d, a + 2d, a + 3d, a + 4d, \ldots$$

The number a is the **first term**, and d is the **common difference** of the sequence. The ***n*th term** of an arithmetic sequence is given by

$$a_n = a + (n - 1)d$$

The number d is called the common difference because any two consecutive terms of an arithmetic sequence differ by d.

Example 1 ■ Arithmetic Sequences

(a) If $a = 2$ and $d = 3$, then we have the arithmetic sequence

$$2, 2 + 3, 2 + 6, 2 + 9, \ldots$$

or

$$2, 5, 8, 11, \ldots$$

Any two consecutive terms of this sequence differ by $d = 3$. The nth term is $a_n = 2 + 3(n - 1)$.

(b) Consider the arithmetic sequence

$$9, 4, -1, -6, -11, \ldots$$

Here the common difference is $d = -5$. The terms of an arithmetic sequence decrease if the common difference is negative. The nth term is $a_n = 9 - 5(n - 1)$.

(c) The graph of the arithmetic sequence $a_n = 1 + 2(n - 1)$ is shown in Figure 1. Notice that the points in the graph lie on the straight line $y = 2x - 1$, which has slope $d = 2$.

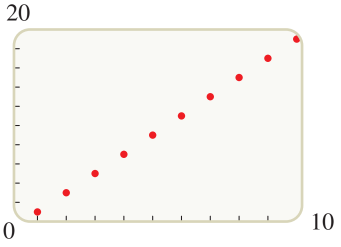


Figure 1

Now Try Exercises 5, 11, and 17 ■

An arithmetic sequence is determined completely by the first term a and the common difference d. Thus if we know the first two terms of an arithmetic sequence, then we can find a formula for the nth term, as the next example shows.

Mathematics in the Modern World

Fair Division of Assets

Dividing an asset fairly among a number of people is of great interest to mathematicians. Problems of this nature include dividing the national budget, disputed land, or assets in divorce cases. In 1994 Brams and Taylor found a mathematical way of dividing things fairly. Their solution has been applied to division problems in political science, legal proceedings, and other areas. To understand the problem, consider the following example. Suppose persons A and B want to divide a property fairly between them. To divide it *fairly* means that both A and B must be satisfied with the outcome of the division. Solution: A gets to divide the property into two pieces, then B gets to choose the piece they want. Since both A and B had a part in the division process, each should be satisfied. The situation becomes much more complicated if three or more people are involved (and that's where mathematics comes in).

Dividing things fairly involves much more than simply cutting things in half; it must take into account the *relative worth* each person attaches to the thing being divided. A story from the Bible illustrates this clearly. Two women appear before King Solomon, each claiming to be the mother of the same newborn baby. To discover which of these two women is the real mother, King Solomon ordered his swordsman to cut the baby in half! The real mother, who attaches far more worth to the baby than anyone else does, immediately gives up her claim to the baby to save the baby's life.

Mathematical solutions to fair-division problems have been applied in an international treaty, the Convention on the Law of the Sea (1982). If a country wants to develop a portion of the sea floor, it is required to divide the portion into two parts, one part to be used by itself and the other by a consortium that will preserve it for later use by a less developed country. The consortium gets first choice.

Example 2 ■ Finding Terms of an Arithmetic Sequence

Find the common difference, the first six terms, the nth term, and the 300th term of the arithmetic sequence

$$13, 7, 1, -5, \ldots$$

Solution Since the first term is 13, we have $a = 13$. The common difference is $d = 7 - 13 = -6$. Thus the nth term of this sequence is

$$a_n = 13 - 6(n - 1)$$

From this we find the first six terms:

$$13, 7, 1, -5, -11, -17, \ldots$$

The 300th term is $a_{300} = 13 - 6(300 - 1) = -1781$.

Now Try Exercise 33 ■

The next example shows that an arithmetic sequence is determined completely by *any* two of its terms.

Example 3 ■ Finding Terms of an Arithmetic Sequence

The 11th term of an arithmetic sequence is 52, and the 19th term is 92. Find the 1000th term.

Solution To find the nth term of this sequence, we need to find a and d in the formula

$$a_n = a + (n - 1)d$$

From this formula we get

$$a_{11} = a + (11 - 1)d = a + 10d$$

$$a_{19} = a + (19 - 1)d = a + 18d$$

Since $a_{11} = 52$ and $a_{19} = 92$, we get the following two equations:

$$\begin{cases} 52 = a + 10d \\ 92 = a + 18d \end{cases}$$

Solving this system for a and d, we get $a = 2$ and $d = 5$. (Verify this.) Thus the nth term of this sequence is

$$a_n = 2 + 5(n - 1)$$

The 1000th term is $a_{1000} = 2 + 5(1000 - 1) = 4997$.

Now Try Exercise 47 ■

■ Partial Sums of Arithmetic Sequences

Suppose we want to find the sum of the numbers 1, 2, 3, 4, . . . , 100, that is,

$$\sum_{k=1}^{100} k$$

When the now famous mathematician C. F. Gauss (see Section 3.5) was a schoolboy, his teacher posed this problem to the class and expected that it would keep the students busy for a long time. But Gauss answered the question almost immediately. His idea was this: Since we are adding numbers produced according to a fixed pattern, there must also be a pattern (or formula) for finding the sum. He started by writing the numbers

Mathematics in the Modern World

Fair Voting Methods

Mathematics has been applied to voting systems. You may ask, What is the problem with how we vote in elections now? Well, suppose candidates A, B, and C are running for office. The final vote tally is as follows: A got 40%, B got 39%, and C got 21%. So, candidate A wins. But 60% of voters *did not* want A. Moreover, most of the voters who voted for C prefer B over A and would have been willing to change their vote to B so that A would not win. So we have a situation where most of the voters prefer B over A, but A won. Is this fair?

In the 1950s, Kenneth Arrow showed mathematically that no democratic method of voting can be completely fair, and he later won a Nobel Prize for his work. Mathematicians continue to work on finding voting systems that are more fair for the voter. The system most often used in federal, state, and local elections is called *plurality voting* (the candidate with the most votes wins). Other systems include *majority voting* (if no candidate gets a majority, a runoff is held between the top two votegetters), *approval voting* (each voter can vote for as many candidates as they approve of), *ranked-choice voting* (each voter orders the candidates according to his or her preference). In this last system, it is advantageous for candidates to appeal to a wider audience in order to also receive second-choice (or third-choice) votes and thus increase their chances of winning. It is thought that such a system would result in a less polarized electorate.

from 1 to 100 and then below them wrote the same numbers in reverse order. Writing S for the sum and adding corresponding terms give

$$\begin{array}{rcccccccccccc} S &=& 1 &+& 2 &+& 3 &+\cdots+& 98 &+& 99 &+& 100 \\ S &=& 100 &+& 99 &+& 98 &+\cdots+& 3 &+& 2 &+& 1 \\ \hline 2S &=& 101 &+& 101 &+& 101 &+\cdots+& 101 &+& 101 &+& 101 \end{array}$$

It follows that $2S = 100(101) = 10{,}100$, so $S = 5050$.

Of course, the sequence of natural numbers 1, 2, 3, . . . is an arithmetic sequence (with $a = 1$ and $d = 1$), and the method for summing the first 100 terms of this sequence can be used to find a formula for the nth partial sum of any arithmetic sequence. We want to find the sum of the first n terms of the arithmetic sequence whose terms are $a_k = a + (k-1)d$; that is, we want to find

$$\begin{aligned} S_n &= \sum_{k=1}^{n} [a + (k-1)d] \\ &= a + (a+d) + (a+2d) + (a+3d) + \cdots + [a + (n-1)d] \end{aligned}$$

Using Gauss's method, we write

$$\begin{array}{rcccccccc} S_n &=& a &+& (a+d) &+\cdots+& [a+(n-2)d] &+& [a+(n-1)d] \\ S_n &=& [a+(n-1)d] &+& [a+(n-2)d] &+\cdots+& (a+d) &+& a \\ \hline 2S_n &=& [2a+(n-1)d] &+& [2a+(n-1)d] &+\cdots+& [2a+(n-1)d] &+& [2a+(n-1)d] \end{array}$$

There are n identical terms on the right side of this equation, so

$$2S_n = n[2a + (n-1)d]$$

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

Notice that $a_n = a + (n-1)d$ is the nth term of this sequence. So we can write

$$S_n = \frac{n}{2}[a + a + (n-1)d] = n\left(\frac{a + a_n}{2}\right)$$

This last formula says that the sum of the first n terms of an arithmetic sequence is the average of the first and nth terms multiplied by n, the number of terms in the sum. We now summarize this result.

Partial Sums of an Arithmetic Sequence

For the arithmetic sequence given by $a_n = a + (n-1)d$, the ***n*th partial sum**

$$S_n = a + (a+d) + (a+2d) + (a+3d) + \cdots + [a + (n-1)d]$$

is given by either of the following formulas.

1. $S_n = \dfrac{n}{2}[2a + (n-1)d]$ **2.** $S_n = n\left(\dfrac{a + a_n}{2}\right)$

Example 4 ■ Finding a Partial Sum of an Arithmetic Sequence

Find the sum of the first 50 odd numbers.

Solution The odd numbers form an arithmetic sequence with $a = 1$ and $d = 2$. The nth term is $a_n = 1 + 2(n-1) = 2n - 1$, so the 50th odd number is $a_{50} = 2(50) - 1 = 99$. Substituting in Formula 2 for the partial sum of an arithmetic sequence, we get

$$S_{50} = 50\left(\frac{a + a_{50}}{2}\right) = 50\left(\frac{1 + 99}{2}\right) = 50 \cdot 50 = 2500$$

Now Try Exercise 51 ■

Example 5 ■ Finding a Partial Sum of an Arithmetic Sequence

Find the following partial sum of an arithmetic sequence.

$$3 + 7 + 11 + 15 + \cdots + 159$$

Solution For this sequence $a = 3$ and $d = 4$, so $a_n = 3 + 4(n - 1)$. To find which term of the sequence is the last term 159, we use the formula for the nth term and solve for n.

$$159 = 3 + 4(n - 1) \qquad \text{Set } a_n = 159$$

$$39 = n - 1 \qquad \text{Subtract 3; divide by 4}$$

$$n = 40 \qquad \text{Add 1}$$

To find the partial sum of the first 40 terms, we use Formula 1 for the nth partial sum of an arithmetic sequence:

$$S_{40} = \tfrac{40}{2}[2(3) + 4(40 - 1)] = 3240$$

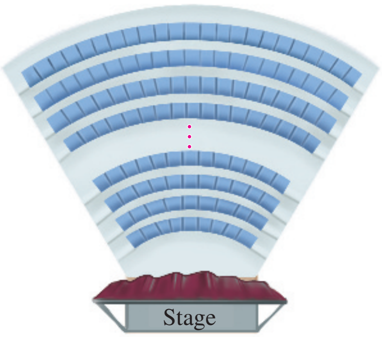
We can also use Formula 2:

$$S_{40} = 40\left(\frac{3 + 159}{2}\right) = 3240$$

Now Try Exercise 57 ■

Example 6 ■ Finding the Seating Capacity of an Amphitheater

An amphitheater has 50 rows of seats with 30 seats in the first row, 32 in the second, 34 in the third, and so on. Find the total number of seats.



Solution The numbers of seats in the rows form an arithmetic sequence with $a = 30$ and $d = 2$. Since there are 50 rows, the total number of seats is the sum

$$S_{50} = \tfrac{50}{2}[2(30) + 49(2)] \qquad S_n = \frac{n}{2}[2a + (n - 1)d]$$

$$= 3950$$

Thus the amphitheater has 3950 seats.

Now Try Exercise 75 ■

Example 7 ■ Finding the Number of Terms in a Partial Sum

How many terms of the arithmetic sequence 5, 7, 9, . . . must be added to get 572?

Solution We are asked to find n when $S_n = 572$. Substituting $a = 5$, $d = 2$, and $S_n = 572$ in Formula 1 for the partial sum of an arithmetic sequence, we get

$$572 = \frac{n}{2}[2 \cdot 5 + (n - 1)2] \qquad S_n = \frac{n}{2}[2a + (n - 1)d]$$

$$572 = 5n + n(n - 1) \qquad \text{Distributive Property}$$

$$0 = n^2 + 4n - 572 \qquad \text{Expand}$$

$$0 = (n - 22)(n + 26) \qquad \text{Factor}$$

This gives $n = 22$ or $n = -26$. But since n is the *number* of terms in this partial sum, we must have $n = 22$.

Now Try Exercise 65 ■

11.2 Exercises

Concepts

1. An arithmetic sequence is a sequence in which the ________ between successive terms is constant.

2. The sequence given by $a_n = a + (n - 1)d$ is an arithmetic sequence in which a is the first term and d is the ________ ________. So for the arithmetic sequence $a_n = 2 + 5(n - 1)$ the first term is ________, and the common difference is ________.

3–4 ■ *True or False*? If *False*, give a reason.

3. The nth partial sum of an arithmetic sequence is the average of the first and last terms times n.

4. If we know the first and second terms of an arithmetic sequence, then we can find any other term.

Skills

5–10 ■ Terms of an Arithmetic Sequence The nth term of an arithmetic sequence is given. **(a)** Find the first five terms of the sequence. **(b)** What is the common difference d? **(c)** Graph the terms you found in part (a).

5. $a_n = 7 + 3(n - 1)$
6. $a_n = -10 + 20(n - 1)$
7. $a_n = -3 - 5(n - 1)$
8. $a_n = 7 - 3(n - 1)$
9. $a_n = 1.5 + 0.5(n - 1)$
10. $a_n = \frac{1}{2}(n - 1)$

11–16 ■ nth Term of an Arithmetic Sequence Find the nth term of the arithmetic sequence with given first term a and common difference d. What is the 10th term?

11. $a = -10, \quad d = 6$
12. $a = 5, \quad d = -2$
13. $a = 0.6, \quad d = -1$
14. $a = 1.8, \quad d = -0.2$
15. $a = \frac{5}{2}, \quad d = -\frac{1}{2}$
16. $a = \sqrt{3}, \quad d = \sqrt{3}$

17–26 ■ Arithmetic Sequence? The first four terms of a sequence are given. Can these terms be the terms of an arithmetic sequence? If so, find the common difference.

17. $11, 17, 23, 29, \ldots$
18. $-31, -19, -7, 5, \ldots$
19. $16, 9, 2, -4, \ldots$
20. $100, 68, 36, 4, \ldots$
21. $2, 4, 8, 16, \ldots$
22. $2, 4, 6, 8, \ldots$
23. $3, \frac{3}{2}, 0, -\frac{3}{2}, \ldots$
24. $\ln 2, \ln 4, \ln 8, \ln 16, \ldots$
25. $2.6, 4.3, 6.0, 7.7, \ldots$
26. $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$

27–32 ■ Arithmetic Sequence? Find the first five terms of the sequence, and determine whether it is arithmetic. If it is arithmetic, find the common difference, and express the nth term of the sequence in the standard form $a_n = a + (n - 1)d$.

27. $a_n = 4 + 7n$
28. $a_n = 4 + 2^n$
29. $a_n = \dfrac{1}{1 + 2n}$
30. $a_n = 1 + \dfrac{n}{2}$
31. $a_n = 6n - 10$
32. $a_n = 3 + (-1)^n n$

33–44 ■ Terms of an Arithmetic Sequence Determine the common difference, the fifth term, the nth term, and the 100th term of the arithmetic sequence.

33. $6, 8, 10, 12, \ldots$
34. $-5, 0, 5, 10, \ldots$
35. $29, 11, -7, -25, \ldots$
36. $64, 49, 34, 19, \ldots$
37. $4, 9, 14, 19, \ldots$
38. $11, 8, 5, 2, \ldots$
39. $-12, -8, -4, 0, \ldots$
40. $\frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}, \ldots$
41. $25, 26.5, 28, 29.5, \ldots$
42. $15, 12.3, 9.6, 6.9, \ldots$
43. $2, 2 + s, 2 + 2s, 2 + 3s, \ldots$
44. $-t, -t + 3, -t + 6, -t + 9, \ldots$

45–50 ■ Finding Terms of an Arithmetic Sequence Find the indicated term of the arithmetic sequence with the given description.

45. The 50th term is 1000, and the common difference is 6. Find the first and second terms.

46. The 100th term is -750, and the common difference is -20. Find the fifth term.

47. The fourteenth term is $\frac{2}{3}$, and the ninth term is $\frac{1}{4}$. Find the first term and the nth term.

48. The twelfth term is 118, and the eighth term is 146. Find the first term and the nth term.

49. The first term is 25, and the common difference is 18. Which term of the sequence is 601?

50. The first term is 3500, and the common difference is -15. Which term of the sequence is 2795?

51–56 ■ Partial Sums of an Arithmetic Sequence Find the partial sum S_n of the arithmetic sequence that satisfies the given conditions.

51. $a = 3, \quad d = 5, \quad n = 20$
52. $a = 10, \quad d = -8, \quad n = 30$
53. $a = -40, \quad d = 14, \quad n = 15$
54. $a = -2, \quad d = 23, \quad n = 25$
55. $a_1 = 4, \quad a_3 = -2, \quad n = 15$
56. $a_3 = 45, \quad a_7 = 55, \quad n = 49$

57–64 ■ Partial Sums of an Arithmetic Sequence A partial sum of an arithmetic sequence is given. Find the sum.

57. $1 + 5 + 9 + \cdots + 401$
58. $-5 - 2.5 + 0 + 2.5 + \cdots + 60$

59. $250 + 233 + 216 + \cdots + 97$

60. $89 + 85 + 81 + \cdots + 13$

61. $0.7 + 2.7 + 4.7 + \cdots + 56.7$

62. $-10 - 9.9 - 9.8 - \cdots - 0.1$

63. $\sum_{k=0}^{10} (3 + 0.25k)$

64. $\sum_{n=0}^{20} (1 - 2n)$

65–66 ■ Adding Terms of an Arithmetic Sequence Find the number of terms of the arithmetic sequence with the given description that must be added to get a value of 2700.

65. The first term is 5, and the common difference is 2.

66. The first term is 12, and the common difference is 8.

Skills Plus

67. Special Triangle Show that a right triangle whose sides are in arithmetic progression is similar to a 3–4–5 triangle.

68. Product of Numbers Find the product of the numbers

$$10^{1/10}, 10^{2/10}, 10^{3/10}, 10^{4/10}, \ldots, 10^{19/10}$$

69. Harmonic Sequence A sequence is **harmonic** if the reciprocals of the terms of the sequence form an arithmetic sequence. Determine whether the following sequence is harmonic.

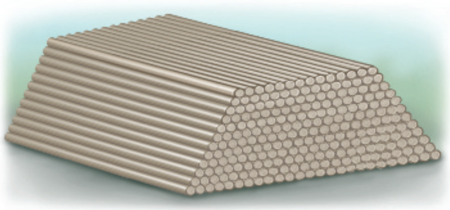
$$1, \tfrac{3}{5}, \tfrac{3}{7}, \tfrac{1}{3}, \ldots$$

70. Harmonic Mean The **harmonic mean** of two numbers is the reciprocal of the average of the reciprocals of the two numbers. Find the harmonic mean of 3 and 5.

Applications

71. Depreciation The purchase value of an office computer client-server is \$12,500. Its annual depreciation is \$1875. Find the value of the computer after 6 years.

72. Poles in a Pile Utility poles are being stored in a pile with 25 poles in the first layer, 24 in the second, and so on. If there are 12 layers, how many utility poles does the pile contain?



73. Salary Increases A management position provides a salary of \$45,000 a year with a \$2000 raise every year. Find the total earnings for the first 10 years.

74. Drive-In Theater A drive-in theater has spaces for 20 cars in the first parking row, 22 in the second, 24 in the third, and so on. If there are 21 rows in the theater, find the number of cars that can be parked.

75. Theater Seating An architect designs a theater with 15 seats in the first row, 18 in the second, 21 in the third, and so on. If the theater is to have a seating capacity of 870, how many rows must the architect use in the design?

76. Falling Ball When an object is allowed to fall freely near the surface of the earth, the gravitational pull is such that the object falls 16 ft in the first second, 48 ft in the next second, 80 ft in the next second, and so on.

(a) Find the total distance a ball falls in 6 s.

(b) Find a formula for the total distance a ball falls in n seconds.

77. The Twelve Days of Christmas In the well-known song "The Twelve Days of Christmas," a sweetheart receives k gifts on the kth day for each of the 12 days of Christmas. The sweetheart also receives every previous gift identically on each subsequent day. Thus on the 12th day the sweetheart receives a gift for the first day, 2 gifts for the second, 3 gifts for the third, and so on. Show that the number of gifts received on the 12th day is a partial sum of an arithmetic sequence. Find this sum.

Discuss ■ Discover ■ Prove ■ Write

78. Discuss: Arithmetic Means The **arithmetic mean** (or average) of two numbers a and b is

$$m = \frac{a + b}{2}$$

Note that m is the same distance from a as from b, so a, m, b is an arithmetic sequence. In general, if $m_1, m_2, \ldots, m_k$ are equally spaced between a and b so that

$$a, m_1, m_2, \ldots, m_k, b$$

is an arithmetic sequence, then $m_1, m_2, \ldots, m_k$ are called k arithmetic means between a and b.

(a) Insert two arithmetic means between 10 and 18.

(b) Insert three arithmetic means between 10 and 18.

(c) Suppose a doctor needs to increase a patient's dosage of a certain medicine from 100 mg to 300 mg per day in five equal steps. How many arithmetic means must be inserted between 100 and 300 to give the progression of daily doses, and what are these means?

11.3 Geometric Sequences

■ Geometric Sequences ■ Partial Sums of Geometric Sequences ■ What Is an Infinite Series?
■ Infinite Geometric Series

In this section we study geometric sequences. This type of sequence occurs frequently in applications to finance, population growth, and other fields.

■ Geometric Sequences

Recall that an arithmetic sequence is generated when we repeatedly add a number d to an initial term a. A *geometric* sequence is generated when we start with a number a and repeatedly *multiply* by a fixed nonzero constant r.

Definition of a Geometric Sequence

A **geometric sequence** is a sequence of the form

$$a, ar, ar^2, ar^3, ar^4, \ldots$$

The number a is the **first term**, and r is the **common ratio** of the sequence. The ***n*th term** of a geometric sequence is given by

$$a_n = ar^{n-1}$$

The number r is called the common ratio because the ratio of any two consecutive terms of the sequence is r.

Example 1 ■ Geometric Sequences

(a) If $a = 3$ and $r = 2$, then we have the geometric sequence

$$3, \quad 3 \cdot 2, \quad 3 \cdot 2^2, \quad 3 \cdot 2^3, \quad 3 \cdot 2^4, \quad \ldots$$

or

$$3, 6, 12, 24, 48, \ldots$$

Notice that the ratio of any two consecutive terms is $r = 2$. The nth term is $a_n = 3(2)^{n-1}$.

(b) The sequence

$$2, -10, 50, -250, 1250, \ldots$$

is a geometric sequence with $a = 2$ and $r = -5$. When r is negative, the terms of the sequence alternate in sign. The nth term is $a_n = 2(-5)^{n-1}$.

(c) The sequence

$$1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots$$

is a geometric sequence with $a = 1$ and $r = \frac{1}{3}$. The nth term is $a_n = 1\left(\frac{1}{3}\right)^{n-1}$.

(d) The graph of the geometric sequence defined by $a_n = \frac{1}{5} \cdot 2^{n-1}$ is shown in Figure 1. Notice that the points in the graph coincide with the graph of the exponential function $y = \frac{1}{5} \cdot 2^{x-1}$.

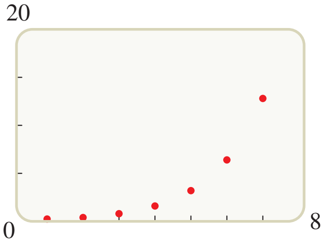


Figure 1

If $0 < r < 1$, then the terms of the geometric sequence ar^{n-1} decrease, whereas if $r > 1$, then the terms increase. (What happens if $r = 1$?)

Now Try Exercises 5, 9, and 13 ■

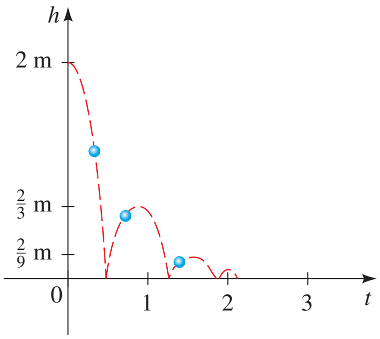


Figure 2

Geometric sequences occur naturally. Here is an example. Suppose a ball has elasticity such that when it is dropped, it bounces up one-third of the distance it has fallen. If this ball is dropped from a height of 2 meters, then it bounces up to a height of $2\left(\frac{1}{3}\right) = \frac{2}{3}$ meters. On its second bounce, it returns to a height of $\left(\frac{2}{3}\right)\left(\frac{1}{3}\right) = \frac{2}{9}$ meters, and so on (see Figure 2). Thus the height h_n that the ball reaches on its nth bounce is given by the geometric sequence

$$h_n = \tfrac{2}{3}\left(\tfrac{1}{3}\right)^{n-1} = 2\left(\tfrac{1}{3}\right)^n$$

We can find the nth term of a geometric sequence if we know any two terms, as the following examples show.

Example 2 ■ Finding Terms of a Geometric Sequence

Find the common ratio, the first term, the nth term, and the eighth term of the geometric sequence

$$5, 15, 45, 135, \ldots$$

Solution To find a formula for the nth term of this sequence, we need to find the first term a and the common ratio r. Clearly, $a = 5$. To find r, we find the ratio of any two consecutive terms. For instance, $r = \frac{45}{15} = 3$. Thus

$$a_n = 5(3)^{n-1} \qquad a_n = ar^{n-1}$$

The eighth term is $a_8 = 5(3)^{8-1} = 5(3)^7 = 10{,}935$.

Now Try Exercise 29 ■

Example 3 ■ Finding Terms of a Geometric Sequence

The third term of a geometric sequence is $\frac{63}{4}$, and the sixth term is $\frac{1701}{32}$. Find the fifth term.

Solution Since this sequence is geometric, its nth term is given by the formula $a_n = ar^{n-1}$. Thus

$$a_3 = ar^{3-1} = ar^2$$
$$a_6 = ar^{6-1} = ar^5$$

From the values we are given for these two terms, we get the following system of equations:

$$\begin{cases} \frac{63}{4} = ar^2 \\ \frac{1701}{32} = ar^5 \end{cases}$$

We solve this system by dividing.

$$\frac{ar^5}{ar^2} = \frac{\frac{1701}{32}}{\frac{63}{4}}$$

$$r^3 = \frac{27}{8} \qquad \text{Simplify}$$

$$r = \frac{3}{2} \qquad \text{Take cube root of each side}$$

Substituting for r in the first equation gives

$$\frac{63}{4} = a\left(\frac{3}{2}\right)^2 \qquad \text{Substitute } r = \tfrac{3}{2} \text{ in } \tfrac{63}{4} = ar^2$$

$$a = 7 \qquad \text{Solve for } a$$

Science Source

SRINIVASA RAMANUJAN (1887–1920) was born into a poor family in the small town of Kumbakonam in India. Self-taught in mathematics, he worked in virtual isolation from other mathematicians. At the age of 25 he wrote a letter to G. H. Hardy, the leading British mathematician at the time, listing some of his discoveries. His discoveries included the following series for calculating π:

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801}\sum_{k=0}^{\infty}\frac{(4k)!(1103 + 26390k)}{(k!)^4 396^{4k}}$$

Hardy immediately recognized Ramanujan's genius, and for the next six years the two worked together in London until Ramanujan fell ill and returned to his hometown in India, where he died a year later. Ramanujan was a genius with a phenomenal ability to see hidden patterns in the properties of numbers. Most of his discoveries were written as complicated infinite series, the importance of which was not recognized until many years after his death. In the last year of his life he wrote 130 pages of mysterious formulas, many of which still defy proof. Hardy tells the story that when he visited Ramanujan in a hospital and arrived in a taxi, he remarked to Ramanujan that the cab's number, 1729, was uninteresting. Ramanujan replied "No, it is a very interesting number. It is the smallest number expressible as the sum of two cubes in two different ways." The 2015 movie *The Man Who Knew Infinity* is a biographical drama about Ramanujan.

It follows that the nth term of this sequence is

$$a_n = 7\left(\tfrac{3}{2}\right)^{n-1}$$

Thus the fifth term is

$$a_5 = 7\left(\tfrac{3}{2}\right)^{5-1} = 7\left(\tfrac{3}{2}\right)^4 = \tfrac{567}{16}$$

Now Try Exercise 41 ■

■ Partial Sums of Geometric Sequences

For the geometric sequence $a, ar, ar^2, ar^3, ar^4, \ldots, ar^{n-1}, \ldots$, the nth partial sum is

$$S_n = \sum_{k=1}^{n} ar^{k-1} = a + ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1}$$

To find a formula for S_n, we multiply S_n by r and subtract from S_n.

$$\begin{aligned} S_n &= a + ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1} \\ rS_n &= \phantom{a + {}} ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1} + ar^n \\ \hline S_n - rS_n &= a - ar^n \end{aligned}$$

So

$$\begin{aligned} S_n(1 - r) &= a(1 - r^n) \\ S_n &= \frac{a(1 - r^n)}{1 - r} \qquad (r \neq 1) \end{aligned}$$

We summarize this result.

Partial Sums of a Geometric Sequence

For the geometric sequence defined by $a_n = ar^{n-1}$, the ***n*th partial sum**

$$S_n = a + ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1} \qquad (r \neq 1)$$

is given by

$$S_n = a\frac{1 - r^n}{1 - r}$$

Example 4 ■ Finding a Partial Sum of a Geometric Sequence

Find the following partial sum of a geometric sequence.

$$1 + 4 + 16 + \cdots + 4096$$

Solution For this sequence $a = 1$ and $r = 4$, so $a_n = 4^{n-1}$. Since $4^6 = 4096$, we use the formula for S_n with $n = 7$, and we have

$$S_7 = 1 \cdot \frac{1 - 4^7}{1 - 4} = 5461$$

Thus this partial sum is 5461.

Now Try Exercises 49 and 53 ■

Example 5 ■ Finding a Partial Sum of a Geometric Sequence

Find the sum $\displaystyle\sum_{k=1}^{6} 7\left(-\tfrac{2}{3}\right)^{k-1}$.

Solution The given sum is the sixth partial sum of a geometric sequence with first term $a = 7\left(-\frac{2}{3}\right)^0 = 7$ and $r = -\frac{2}{3}$. Thus by the formula for S_n with $n = 6$ we have

$$S_6 = 7 \cdot \frac{1 - \left(-\frac{2}{3}\right)^6}{1 - \left(-\frac{2}{3}\right)} = 7 \cdot \frac{1 - \frac{64}{729}}{\frac{5}{3}} = \frac{931}{243} \approx 3.83$$

Now Try Exercise 59 ■

■ What Is an Infinite Series?

An expression of the form

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + a_4 + \cdots$$

is called an **infinite series**. The dots mean that we are to continue the addition indefinitely. What meaning can we attach to the sum of infinitely many numbers? It seems at first that it is not possible to add infinitely many numbers and arrive at a finite number. But consider the following problem. You want to eat a cake by first eating half the cake, then eating half of what remains, then again eating half of what remains. This process can continue indefinitely because at each stage, some of the cake remains (see Figure 3).

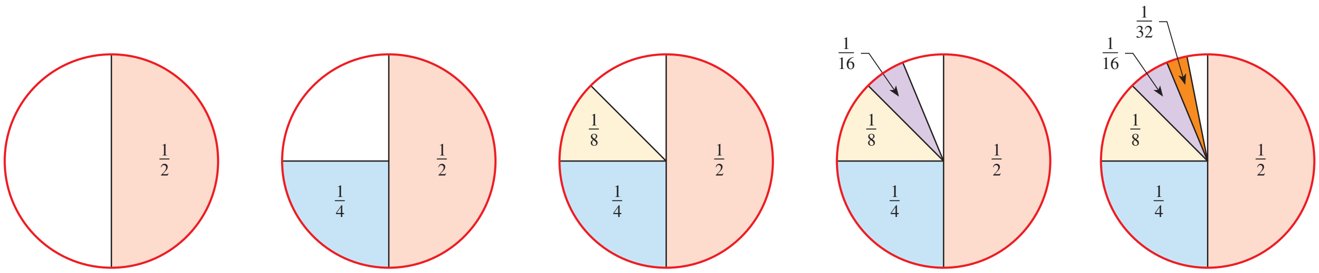


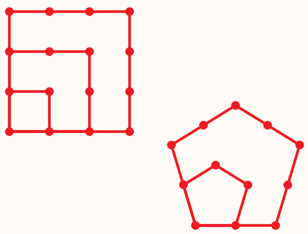
Figure 3

Does this mean that it's impossible to eat all of the cake? Of course not. Let's write down what you have eaten from this cake:

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

This is an infinite series, and we note two things about it: First, from Figure 3 it's clear that no matter how many terms of this series we add, the total will never exceed 1. Second, the more terms of this series we add, the closer the sum is to 1. (See Figure 3.) This suggests that the number 1 can be written as the sum of infinitely many smaller numbers:

$$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots$$



Discovery Project ■ Finding Patterns

Finding patterns in nature is an important part of mathematical modeling. If we can find a pattern (or a formula) that describes the terms of a sequence, then we can use the pattern to predict subsequent terms of the sequence. In this project we investigate difference sequences and how they help us find patterns in triangular, square, pentagonal, and other polygonal numbers. You can find the project at the book companion website **www.stewartmath.com**.

To make this more precise, let's look at the partial sums of this series:

$$S_1 = \frac{1}{2} \qquad\qquad\qquad\qquad = \frac{1}{2}$$

$$S_2 = \frac{1}{2} + \frac{1}{4} \qquad\qquad\quad = \frac{3}{4}$$

$$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \qquad = \frac{7}{8}$$

$$S_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$$

and, in general (see Example 11.1.5),

$$S_n = 1 - \frac{1}{2^n}$$

As n gets larger and larger, we are adding more and more terms of this series. Intuitively, as n gets larger, S_n gets closer to the sum of the series. Now notice that as n gets large, $1/2^n$ gets closer and closer to 0. Thus S_n gets close to $1 - 0 = 1$. Using the notation of Section 3.6, we can write

$$S_n \to 1 \quad \text{as} \quad n \to \infty$$

In general, if S_n gets close to a finite number S as n gets large, we say that the infinite series **converges** (or is **convergent**). The number S is called the **sum of the infinite series**. If an infinite series does not converge, we say that the series **diverges** (or is **divergent**).

Infinite Geometric Series

Here is another way to arrive at the formula for the sum of an infinite geometric series:

$$\begin{aligned} S &= a + ar + ar^2 + ar^3 + \cdots \\ &= a + r(a + ar + ar^2 + \cdots) \\ &= a + rS \end{aligned}$$

Solve the equation $S = a + rS$ for S to get

$$\begin{aligned} S - rS &= a \\ (1 - r)S &= a \\ S &= \frac{a}{1 - r} \end{aligned}$$

An **infinite geometric series** is a series of the form

$$a + ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1} + \cdots$$

We can apply the reasoning that we used earlier to find the sum of an infinite geometric series. The nth partial sum of such a series is given by the formula

$$S_n = a\frac{1 - r^n}{1 - r} \qquad (r \neq 1)$$

It can be shown that if $|r| < 1$, then r^n gets close to 0 as n gets large (you can easily convince yourself of this using a calculator). It follows that S_n gets close to $a/(1 - r)$ as n gets large, or

$$S_n \to \frac{a}{1 - r} \quad \text{as} \quad n \to \infty$$

Thus the sum of this infinite geometric series is $a/(1 - r)$.

Sum of an Infinite Geometric Series

If $|r| < 1$, then the infinite geometric series

$$\sum_{k=1}^{\infty} ar^{k-1} = a + ar + ar^2 + ar^3 + \cdots$$

converges and has the sum

$$S = \frac{a}{1 - r}$$

If $|r| \geq 1$, the series diverges.

Example 6 ■ Infinite Series

Determine whether the infinite geometric series is convergent or divergent. If it is convergent, find its sum.

(a) $2 + \frac{2}{5} + \frac{2}{25} + \frac{2}{125} + \cdots$ **(b)** $1 + \frac{7}{5} + \left(\frac{7}{5}\right)^2 + \left(\frac{7}{5}\right)^3 + \cdots$

Solution

(a) This is an infinite geometric series with $a = 2$ and $r = \frac{1}{5}$. Since $|r| = \left|\frac{1}{5}\right| < 1$, the series converges. By the formula for the sum of an infinite geometric series, we have

$$S = \frac{2}{1 - \frac{1}{5}} = \frac{5}{2}$$

(b) This is an infinite geometric series with $a = 1$ and $r = \frac{7}{5}$. Since $|r| = \left|\frac{7}{5}\right| > 1$, the series diverges.

Now Try Exercises 65 and 69

Example 7 ■ Writing a Repeated Decimal as a Fraction

Find the fraction that represents the rational number $2.3\overline{51}$.

Solution This repeating decimal can be written as a series:

$$\frac{23}{10} + \frac{51}{1000} + \frac{51}{100{,}000} + \frac{51}{10{,}000{,}000} + \frac{51}{1{,}000{,}000{,}000} + \cdots$$

After the first term, the terms of this series form an infinite geometric series with

$$a = \frac{51}{1000} \quad \text{and} \quad r = \frac{1}{100}$$

Thus the sum of this part of the series is

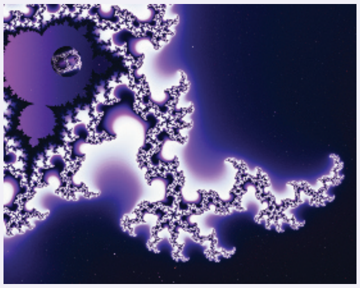
$$S = \frac{\frac{51}{1000}}{1 - \frac{1}{100}} = \frac{\frac{51}{1000}}{\frac{99}{100}} = \frac{51}{1000} \cdot \frac{100}{99} = \frac{51}{990}$$

It follows that

$$2.3\overline{51} = \frac{23}{10} + \frac{51}{990} = \frac{2328}{990} = \frac{388}{165}$$

Now Try Exercise 77

Mathematics in the Modern World



Bill Ross/Getty Images

Fractals

Many of the things we model in this book have regular predictable shapes. But recent advances in mathematics have made it possible to model such seemingly random or even chaotic shapes as those of a cloud, a flickering flame, a mountain, or a jagged coastline. The basic tools in this type of modeling are the fractals invented by the mathematician Benoit Mandelbrot. A *fractal* is a geometric shape built up from a simple basic shape by scaling and repeating the shape indefinitely according to a given rule. Fractals have infinite detail; this means the closer you look, the more you see. They are also *self-similar*; that is, zooming in on a portion of the fractal yields the same detail as the original shape. Because of their beautiful shapes, fractals are used by moviemakers to create fictional landscapes and exotic backgrounds.

Although a fractal is a complex shape, it is produced according to very simple rules. This property of fractals is exploited in a process of storing pictures on a computer called *fractal image compression*.

11.3 Exercises

Concepts

1. A geometric sequence is a sequence in which the ________ of successive terms is constant.

2. The sequence given by $a_n = ar^{n-1}$ is a geometric sequence in which a is the first term and r is the ________ ________. So for the geometric sequence $a_n = 2(5)^{n-1}$ the first term is ________, and the common ratio is ________.

3. *True or False*? If we know the first and second terms of a geometric sequence, then we can find any other term.

4. (a) The nth partial sum of a geometric sequence $a_n = ar^{n-1}$ is given by $S_n =$ ________.

(b) The series $\sum_{k=1}^{\infty} ar^{k-1} = a + ar + ar^2 + ar^3 + \cdots$ is an infinite ________ series. If $|r| < 1$, then this series ________, and its sum is $S =$ ________. If $|r| \geq 1$, the series ________.

Skills

5–8 ■ *n*th Term of a Geometric Sequence The nth term of a sequence is given. **(a)** Find the first five terms of the sequence. **(b)** What is the common ratio r? **(c)** Graph the terms you found in (a).

5. $a_n = 7(3)^{n-1}$ **6.** $a_n = 6(-0.5)^{n-1}$

7. $a_n = 8\left(-\frac{1}{4}\right)^{n-1}$ **8.** $a_n = -\frac{1}{9}(3)^{n-1}$

9–12 ■ *n*th Term of a Geometric Sequence Find the nth term of the geometric sequence with given first term a and common ratio r. What is the fourth term?

9. $a = 7, \quad r = 4$ **10.** $a = -\frac{3}{2}, \quad r = 3$

11. $a = 5, r = -3$ **12.** $a = \sqrt{3}, \quad r = \sqrt{3}$

13–22 ■ Geometric Sequence? The first four terms of a sequence are given. Determine whether these terms can be the terms of a geometric sequence. If the sequence is geometric, find the common ratio.

13. $3, 6, 12, 24, \ldots$ **14.** $3, 48, 93, 138, \ldots$

15. $\frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, \ldots$ **16.** $432, -144, 48, -16, \ldots$

17. $3, \frac{3}{2}, \frac{3}{4}, \frac{3}{8}, \ldots$ **18.** $10, \frac{10}{3}, \frac{10}{9}, \frac{10}{27}, \ldots$

19. $-\frac{1}{2}, \frac{1}{2}, -\frac{1}{4}, \frac{1}{4}, \ldots$ **20.** $e^2, e^4, e^6, e^8, \ldots$

21. $1.0, 1.1, 1.21, 1.331, \ldots$ **22.** $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots$

23–28 ■ Geometric Sequence? Find the first five terms of the sequence, and determine whether it is geometric. If it is geometric, find the common ratio, and express the nth term of the sequence in the standard form $a_n = ar^{n-1}$.

23. $a_n = 2(3)^n$ **24.** $a_n = 4 + 3^n$

25. $a_n = \dfrac{1}{4^n}$ **26.** $a_n = (-1)^n 2^n$

27. $a_n = \ln(5^{n-1})$ **28.** $a_n = n^n$

29–38 ■ Terms of a Geometric Sequence Determine the common ratio, the fifth term, and the nth term of the geometric sequence.

29. $2, 6, 18, 54, \ldots$ **30.** $7, \frac{14}{3}, \frac{28}{9}, \frac{56}{27}, \ldots$

31. $0.3, -0.09, 0.027, -0.0081, \ldots$

32. $1, \sqrt{2}, 2, 2\sqrt{2}, \ldots$

33. $144, -12, 1, -\frac{1}{12}, \ldots$ **34.** $-8, -2, -\frac{1}{2}, -\frac{1}{8}, \ldots$

35. $3, 3^{5/3}, 3^{7/3}, 27, \ldots$ **36.** $t, \dfrac{t^2}{2}, \dfrac{t^3}{4}, \dfrac{t^4}{8}, \ldots$

37. $1, s^{2/7}, s^{4/7}, s^{6/7}, \ldots$ **38.** $5, 5^{c+1}, 5^{2c+1}, 5^{3c+1}, \ldots$

39–46 ■ Finding Terms of a Geometric Sequence Find the indicated term(s) of the geometric sequence with the given description.

39. The first term is 14 and the second term is 4. Find the fourth term.

40. The first term is 8 and the second term is 6. Find the fifth term.

41. The third term is $-\frac{1}{3}$ and the sixth term is 9. Find the first and second terms.

42. The fourth term is 12 and the seventh term is $\frac{32}{9}$. Find the first and nth terms.

43. The third term is -18 and the sixth term is 9216. Find the first and nth terms.

44. The third term is -54 and the sixth term is $\frac{729}{256}$. Find the first and second terms.

45. The common ratio is 0.75 and the fourth term is 729. Find the first three terms.

46. The common ratio is $\frac{1}{6}$ and the third term is 18. Find the first and seventh terms.

47. Which Term? The first term of a geometric sequence is 1536 and the common ratio is $\frac{1}{2}$. Which term of the sequence is 6?

48. Which Term? The second and fifth terms of a geometric sequence are 30 and 3750, respectively. Which term of the sequence is 468,750?

49–52 ■ Partial Sums of a Geometric Sequence Find the partial sum S_n of the geometric sequence that satisfies the given conditions.

49. $a = 5, \quad r = 2, \quad n = 6$ **50.** $a = \frac{2}{3}, \quad r = \frac{1}{3}, \quad n = 4$

51. $a_3 = 28, \quad a_6 = 224, \quad n = 6$

52. $a_2 = 0.12, \quad a_5 = 0.00096, \quad n = 4$

53–58 ■ Partial Sums of a Geometric Sequence Find the sum.

53. $1 + 3 + 9 + \cdots + 2187$

54. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots - \frac{1}{512}$

55. $-15 + 30 - 60 + \cdots - 960$

56. $5120 + 2560 + 1280 + \cdots + 20$

57. $1.25 + 12.5 + 125 + \cdots + 12{,}500{,}000$

58. $10800 + 1080 + 108 + \cdots + 0.000108$

59–64 ■ Partial Sums of a Geometric Sequence Find the sum.

59. $\sum_{k=1}^{5} 3\left(\frac{1}{2}\right)^{k-1}$ **60.** $\sum_{k=1}^{5} 8\left(-\frac{3}{2}\right)^{k-1}$

61. $\sum_{k=1}^{6} 5(-2)^{k-1}$ **62.** $\sum_{k=1}^{6} 10(5)^{k-1}$

63. $\sum_{k=1}^{5} 3\left(\frac{2}{3}\right)^{k-1}$ **64.** $\sum_{k=1}^{6} 64\left(\frac{3}{2}\right)^{k-1}$

65–76 ■ Infinite Geometric Series Determine whether the infinite geometric series is convergent or divergent. If it is convergent, find its sum.

65. $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots$ **66.** $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$

67. $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \cdots$ **68.** $\frac{2}{5} + \frac{4}{25} + \frac{8}{125} + \cdots$

69. $1 + \frac{3}{2} + \left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^3 + \cdots$

70. $\frac{1}{3^6} + \frac{1}{3^8} + \frac{1}{3^{10}} + \frac{1}{3^{12}} + \cdots$

71. $3 - \frac{3}{2} + \frac{3}{4} - \frac{3}{8} + \cdots$

72. $1 - 1 + 1 - 1 + \cdots$

73. $3 - 3(1.1) + 3(1.1)^2 - 3(1.1)^3 + \cdots$

74. $-\frac{100}{9} + \frac{10}{3} - 1 + \frac{3}{10} - \cdots$

75. $\frac{1}{\sqrt{2}} + \frac{1}{2} + \frac{1}{2\sqrt{2}} + \frac{1}{4} + \cdots$

76. $1 - \sqrt{2} + 2 - 2\sqrt{2} + 4 - \cdots$

77–82 ■ Repeated Decimal Express the repeating decimal as a fraction.

77. $0.999\ldots$ **78.** $0.2\overline{53}$ **79.** $0.030303\ldots$

80. $2.1\overline{125}$ **81.** $0.\overline{112}$ **82.** $0.123123123\ldots$

Skills Plus

83. Geometric Means If the numbers $a_1, a_2, \ldots, a_n$ form a geometric sequence, then $a_2, a_3, \ldots, a_{n-1}$ are **geometric means** between a_1 and a_n. Insert three geometric means between 5 and 80.

84. Partial Sum of a Geometric Sequence Find the sum of the first ten terms of the sequence

$$a + b, a^2 + 2b, a^3 + 3b, a^4 + 4b, \ldots$$

85–86 ■ Arithmetic or Geometric? The first four terms of a sequence are given. Determine whether these terms can be the terms of an arithmetic sequence, a geometric sequence, or neither. If the sequence is arithmetic or geometric, find the next term.

85. (a) $5, -3, 5, -3, \ldots$ **(b)** $\frac{1}{3}, 1, \frac{5}{3}, \frac{7}{3}, \ldots$

(c) $\sqrt{3}, 3, 3\sqrt{3}, 9, \ldots$ **(d)** $-3, -\frac{3}{2}, 0, \frac{3}{2}, \ldots$

86. (a) $1, -1, 1, -1, \ldots$ **(b)** $\sqrt{5}, \sqrt[3]{5}, \sqrt[6]{5}, 1, \ldots$

(c) $2, -1, \frac{1}{2}, 2, \ldots$ **(d)** $x - 1, x, x + 1, x + 2, \ldots$

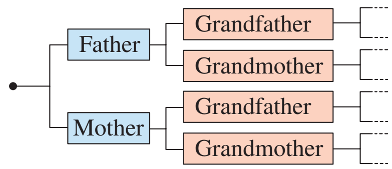
Applications

87. Depreciation A construction company purchases a bulldozer for \$160,000. Each year the value of the bulldozer depreciates by 20% of its value in the preceding year. Let V_n be the value of the bulldozer in the nth year. (Let $n = 1$ be the year the bulldozer is purchased.)

(a) Find a formula for V_n.

(b) In what year will the value of the bulldozer be less than \$100,000?

88. Ancestors A person has two parents, four grandparents, eight great-grandparents, and so on. How many ancestors does a person have 15 generations back?

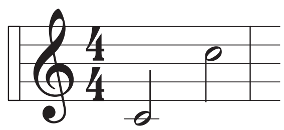


89. Bouncing Ball A ball is dropped from a height of 80 feet. The elasticity of this ball is such that it rebounds three-fourths the distance it has fallen. How high does the ball rebound on the fifth bounce? Find a formula for how high the ball rebounds on the nth bounce.

90. Bacteria Culture A culture initially has 5000 bacteria, and its size increases by 8% every hour. How many bacteria are present at the end of 5 hours? Find a formula for the number of bacteria present after n hours.

91. Mixing Coolant A truck radiator holds 5 gallons and is filled with water. A gallon of water is removed from the radiator and replaced with a gallon of antifreeze; then a gallon of the mixture is removed from the radiator and again replaced by a gallon of antifreeze. This process is repeated indefinitely. How much water remains in the tank after this process is repeated 3 times? 5 times? n times?

92. Musical Frequencies The frequencies of musical notes (measured in cycles per second) form a geometric sequence. Middle C has a frequency of 256, and the C that is an octave higher has a frequency of 512. Find the frequency of C two octaves below middle C.



93. Bouncing Ball A ball is dropped from a height of 9 feet. The elasticity of the ball is such that it always bounces up one-third the distance it has fallen.

(a) Find the total distance the ball has traveled at the instant it hits the ground the fifth time.

(b) Find a formula for the total distance the ball has traveled at the instant it hits the ground the nth time.

94. Salary Increases A management position provides a salary of \$45,000 in the first year. Each year the salary increases by 5% of the preceding year's salary. Find the total earnings in the first 10 years.

95. St. Ives The following is a well-known children's rhyme:

As I was going to St. Ives,
I met a man with seven wives;
Every wife had seven sacks;
Every sack had seven cats;
Every cat had seven kits;
Kits, cats, sacks, and wives,
How many were going to St. Ives?

Assuming that the entire group is actually going to St. Ives, show that the answer to the question in the rhyme is a partial sum of a geometric sequence, and find the sum.

96. Drug Concentration A certain drug is administered once a day. The concentration of the drug in the patient's bloodstream increases rapidly at first, but each successive dose has less effect than the preceding one. The total amount of the drug (in mg) in the bloodstream after the nth dose is given by

$$\sum_{k=1}^{n} 50\left(\tfrac{1}{2}\right)^{k-1}$$

(a) Find the amount of the drug in the bloodstream after $n = 10$ days.

(b) If the drug is taken on a long-term basis, the amount in the bloodstream is approximated by the infinite series $\sum_{k=1}^{\infty} 50\left(\tfrac{1}{2}\right)^{k-1}$. Find the sum of this series.

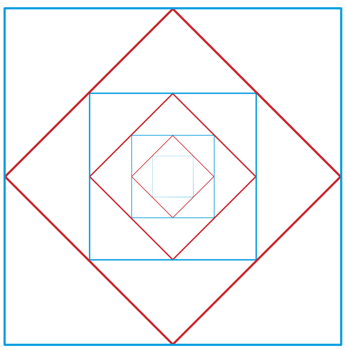
97. Bouncing Ball A certain ball rebounds to half the height from which it is dropped. Use an infinite geometric series to approximate the total distance the ball travels after being dropped from 1 meter above the ground until it comes to rest.

98. Bouncing Ball If the ball in Exercise 97 is dropped from a height of 8 feet, then 1 second is required for its first complete bounce—from the instant it first touches the ground until it next touches the ground. Each subsequent complete bounce requires $1/\sqrt{2}$ as long as the preceding complete bounce. Use an infinite geometric series to estimate the time interval from the instant the ball first touches the ground until it stops bouncing.

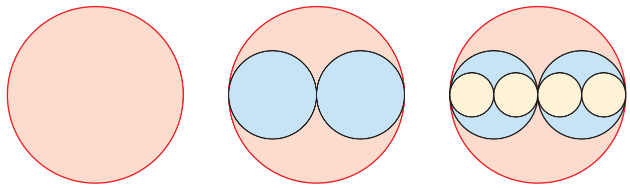
99. Geometry The midpoints of the sides of a square of side 1 are joined to form a new square. This procedure is repeated for each new square (see the figure).

(a) Find the sum of the areas of all the squares.

(b) Find the sum of the perimeters of all the squares.

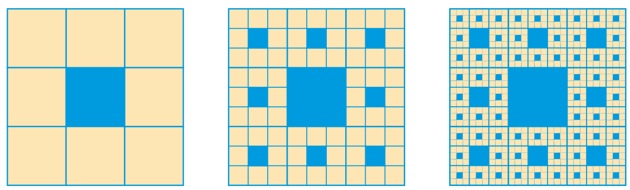


100. Geometry A circular disk of radius R is cut out of paper, as shown in figure (a). Two disks of radius $\frac{1}{2}R$ are cut out of paper and placed on top of the first disk, as in figure (b), and then four disks of radius $\frac{1}{4}R$ are placed on these two disks, as in figure (c). Assuming that this process can be repeated indefinitely, find the total area of all the disks.



(a) (b) (c)

101. Geometry A yellow square of side 1 is divided into nine smaller squares, and the middle square is colored blue as shown in the figure. Each of the smaller yellow squares is in turn divided into nine squares, and each middle square is colored blue. If this process is continued indefinitely, what is the total area that is colored blue?



■ Discuss ■ Discover ■ Prove ■ Write

102. Prove: Reciprocals of a Geometric Sequence If $a_1, a_2, a_3, \ldots$ is a geometric sequence with common ratio r, show that the sequence

$$\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \ldots$$

is also a geometric sequence, and find the common ratio.

103. Prove: Logarithms of a Geometric Sequence If $a_1, a_2, a_3, \ldots$ is a geometric sequence with a common ratio $r > 0$ and $a_1 > 0$, show that the sequence

$$\log a_1, \log a_2, \log a_3, \ldots$$

is an arithmetic sequence, and find the common difference.

104. Prove: Exponentials of an Arithmetic Sequence If $a_1, a_2, a_3, \ldots$ is an arithmetic sequence with common difference d, show that the sequence

$$10^{a_1}, 10^{a_2}, 10^{a_3}, \ldots$$

is a geometric sequence, and find the common ratio.

105. Prove: A Factoring Formula Show that for $r \neq 1$ and k a natural number

$$(1 + r + r^2 + \cdots + r^{2^k-1}) = (1 + r + r^2 + \cdots + r^{2^{k-1}-1})(r^{2^{k-1}} + 1)$$

PS *Look for something familiar.* Use the formula for the sum of a geometric sequence.

11.4 Mathematical Induction

■ Conjecture and Proof ■ Mathematical Induction

There are two aspects to mathematics—discovery and proof—and they are of equal importance. We must discover something before we can attempt to prove it, and we cannot be certain of its truth until it has been proved. In this section we examine the relationship between these two key components of mathematics more closely.

■ Conjecture and Proof

Let's try an experiment. We add more and more of the odd numbers as follows:

$$1 = 1$$
$$1 + 3 = 4$$
$$1 + 3 + 5 = 9$$
$$1 + 3 + 5 + 7 = 16$$
$$1 + 3 + 5 + 7 + 9 = 25$$

What do you notice about the numbers on the right-hand side of these equations? They are, in fact, all perfect squares. These equations say the following:

The sum of the first 1 odd number is 1^2.

The sum of the first 2 odd numbers is 2^2.

The sum of the first 3 odd numbers is 3^2.

The sum of the first 4 odd numbers is 4^2.

The sum of the first 5 odd numbers is 5^2.

This leads naturally to the following question: Is it true that for every natural number n, the sum of the first n odd numbers is n^2? Could this remarkable property be true? We could try a few more numbers and find that the pattern persists for the first 6, 7, 8, 9, and 10 odd numbers. At this point we feel fairly confident that this is always true, so we make a *conjecture*:

The sum of the first n odd numbers is n^2.

Since we know that the nth odd number is $2n - 1$, we can write this statement more precisely as

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

It is important to realize that this is still a conjecture. We cannot conclude by checking a finite number of cases that a property is true for all numbers (there are infinitely many). To see this more clearly, suppose some mathematicians tell us that they have added up the first trillion odd numbers and found that they do *not* add up to 1 trillion squared. What would you tell these mathematicians? It would be silly to say that you're sure it's true because you have already checked the first five cases. You could, however, take out paper and pencil and start checking it yourself, but this task would probably take the rest of your life. The tragedy would be that after completing this task, you would still not be sure of the truth of the conjecture! Do you see why?

Herein lies the power of mathematical proof. A **proof** is a clear argument that demonstrates the truth of a statement beyond doubt.

Consider the polynomial

$$p(n) = n^2 - n + 41$$

Here are some values of $p(n)$:

$$p(1) = 41 \qquad p(2) = 43$$
$$p(3) = 47 \qquad p(4) = 53$$
$$p(5) = 61 \qquad p(6) = 71$$
$$p(7) = 83 \qquad p(8) = 97$$

All the values so far are prime numbers. In fact, if you keep going, you will find that $p(n)$ is prime for all natural numbers up to $n = 40$. It might seem reasonable at this point to conjecture that $p(n)$ is prime for *every* natural number n. But that conjecture would be too hasty because it is easily seen that $p(41)$ is *not* prime. This illustrates that we cannot be certain of the truth of a statement no matter how many special cases we check. We need a convincing argument—a *proof*—to determine the truth of a statement.

BLAISE PASCAL (1623–1662) is considered one of the most versatile minds in modern history. He was a writer and philosopher as well as a gifted mathematician and physicist. Among his contributions that appear in this book are Pascal's triangle and the Principle of Mathematical Induction.

Pascal's father, himself a mathematician, believed that his son should not study mathematics until he was 15 or 16. But at age 12, Blaise insisted on learning geometry and proved most of its elementary theorems himself. At 19 he invented the first mechanical adding machine. In 1647, after writing a major treatise on the conic sections, he abruptly abandoned mathematics because he felt that his intense studies were contributing to his ill health. He devoted himself instead to frivolous recreations such as gambling, but this only served to pique his interest in probability. In 1654 he miraculously survived a carriage accident in which his horses ran off a bridge. Taking this to be a sign from God, Pascal entered a monastery, where he pursued theology and philosophy, writing his famous *Pensées*. He also continued his mathematical research. He valued faith and intuition more than reason as the source of truth, declaring that "the heart has its own reasons, which reason cannot know."

■ Mathematical Induction

Let's consider a special kind of proof called **mathematical induction**. Here is how it works: Suppose we have a statement that says something about all natural numbers n. For example, for any natural number n, let $P(n)$ be the following statement:

$$P(n): \quad \text{The sum of the first } n \text{ odd numbers is } n^2.$$

Since this statement is about all natural numbers, it contains infinitely many statements; we will call them $P(1)$, $P(2)$, ...

$$\begin{aligned} P(1)&: \quad \text{The sum of the first 1 odd number is } 1^2. \\ P(2)&: \quad \text{The sum of the first 2 odd numbers is } 2^2. \\ P(3)&: \quad \text{The sum of the first 3 odd numbers is } 3^2. \\ &\vdots \qquad\qquad\qquad \vdots \end{aligned}$$

How can we prove all of these statements at once? Mathematical induction is a clever way of doing just that.

The crux of the idea is this: Suppose we can prove that whenever one of these statements is true, then the one following it in the list is also true. In other words,

$$\text{For every } k, \text{ if } P(k) \text{ is true, then } P(k+1) \text{ is true.}$$

This is called the **induction step** because it leads us from the truth of one statement to the truth of the next. Now suppose that we can also prove that

$$P(1) \text{ is true.}$$

The induction step now leads us through the following chain of statements:

$$\begin{aligned} &P(1) \text{ is true, so } P(2) \text{ is true.} \\ &P(2) \text{ is true, so } P(3) \text{ is true.} \\ &P(3) \text{ is true, so } P(4) \text{ is true.} \\ &\quad\vdots \qquad\qquad\quad \vdots \end{aligned}$$

So we see that if both the induction step and $P(1)$ are proved, then statement $P(n)$ is proved for all n. Here is a summary of this important method of proof.

Principle of Mathematical Induction

For each natural number n, let $P(n)$ be a statement depending on n. Suppose that the following two conditions are satisfied.

1. $P(1)$ is true.
2. For every natural number k, if $P(k)$ is true, then $P(k+1)$ is true.

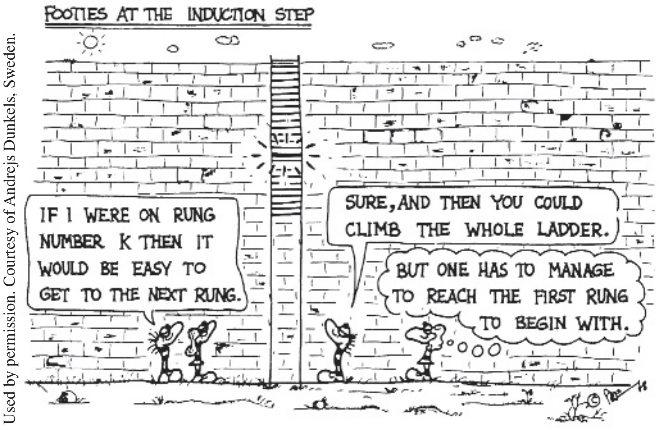
Then $P(n)$ is true for all natural numbers n.

To apply this principle, there are two steps:

Step 1 Prove that $P(1)$ is true.

Step 2 Assume that $P(k)$ is true, and use this assumption to prove that $P(k+1)$ is true.

Notice that in Step 2 we do not prove that $P(k)$ is true. We only show that *if* $P(k)$ is true, *then* $P(k+1)$ is also true. The assumption that $P(k)$ is true is called the **induction hypothesis**.



We now use mathematical induction to prove that the conjecture that we made at the beginning of this section is true.

Example 1 ■ A Proof by Mathematical Induction

Prove that for all natural numbers n,

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

Solution Let $P(n)$ denote the statement $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

Step 1 We need to show that $P(1)$ is true. But $P(1)$ is simply the statement that $1 = 1^2$, which is of course true.

Step 2 We assume that $P(k)$ is true. Thus our induction hypothesis is

$$1 + 3 + 5 + \cdots + (2k - 1) = k^2$$

We want to use this to show that $P(k+1)$ is true, that is,

$$1 + 3 + 5 + \cdots + (2k - 1) + [2(k + 1) - 1] = (k + 1)^2$$

[Note that we get $P(k+1)$ by substituting $k + 1$ for each n in the statement $P(n)$.] We start with the left-hand side and use the induction hypothesis to obtain the right-hand side of the equation.

$$\begin{aligned}
&1 + 3 + 5 + \cdots + (2k - 1) + [2(k + 1) - 1] \\
&= [1 + 3 + 5 + \cdots + (2k - 1)] + [2(k + 1) - 1] && \text{Group the first } k \text{ terms} \\
&= k^2 + [2(k + 1) - 1] && \text{Induction hypothesis} \\
&= k^2 + [2k + 2 - 1] && \text{Distributive Property} \\
&= k^2 + 2k + 1 && \text{Simplify} \\
&= (k + 1)^2 && \text{Factor}
\end{aligned}$$

This equals k^2 by the induction hypothesis

Thus $P(k+1)$ follows from $P(k)$, and this completes the induction step.

Having proved Steps 1 and 2, we conclude by the Principle of Mathematical Induction that $P(n)$ is true for all natural numbers n.

Now Try Exercise 3 ■

Example 2 ■ A Proof by Mathematical Induction

Prove that for every natural number n,

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Solution Let $P(n)$ be the statement $1 + 2 + 3 + \cdots + n = n(n+1)/2$. We want to show that $P(n)$ is true for all natural numbers n.

Step 1 We need to show that $P(1)$ is true. But $P(1)$ says that

$$1 = \frac{1(1+1)}{2}$$

and this statement is clearly true.

Step 2 Assume that $P(k)$ is true. Thus our induction hypothesis is

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$$

We want to use this to show that $P(k+1)$ is true, that is,

$$1 + 2 + 3 + \cdots + k + (k+1) = \frac{(k+1)[(k+1)+1]}{2}$$

So we start with the left-hand side and use the induction hypothesis to obtain the right-hand side.

This equals $\dfrac{k(k+1)}{2}$ by the induction hypothesis

$$\begin{aligned}
&1 + 2 + 3 + \cdots + k + (k+1) \\
&\quad = [1 + 2 + 3 + \cdots + k] + (k+1) && \text{Group the first } k \text{ terms} \\
&\quad = \frac{k(k+1)}{2} + (k+1) && \text{Induction hypothesis} \\
&\quad = (k+1)\left(\frac{k}{2} + 1\right) && \text{Factor } k+1 \\
&\quad = (k+1)\left(\frac{k+2}{2}\right) && \text{Common denominator} \\
&\quad = \frac{(k+1)[(k+1)+1]}{2} && \text{Write } k+2 \text{ as } k+1+1
\end{aligned}$$

Thus $P(k+1)$ follows from $P(k)$, and this completes the induction step.

Having proved Steps 1 and 2, we conclude by the Principle of Mathematical Induction that $P(n)$ is true for all natural numbers n.

Now Try Exercise 5 ■

The following box gives formulas for the sums of powers of the first n natural numbers. These formulas are important in calculus. Formula 1 is proved in Example 2. The other formulas are also proved by using mathematical induction (see Exercises 6 and 9).

Note that we have numbered each equation with the number that corresponds to the power being summed up.

Sums of Powers

0. $\displaystyle\sum_{k=1}^{n} 1 = n$

1. $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

2. $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

3. $\displaystyle\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$

It might happen that a statement $P(n)$ is false for the first few natural numbers but true from some number on. For example, we might want to prove that $P(n)$ is true for $n \geq 5$. Notice that if we prove that $P(5)$ is true, then this fact, together with the induction step, would imply the truth of $P(5), P(6), P(7), \ldots$. The next example illustrates this point.

Example 3 ■ Proving an Inequality by Mathematical Induction

Prove that $4n < 2^n$ for all $n \geq 5$.

Solution Let $P(n)$ denote the statement $4n < 2^n$.

Step 1 $P(5)$ is the statement that $4 \cdot 5 < 2^5$, or $20 < 32$, which is true.

Step 2 Assume that $P(k)$ is true. Thus our induction hypothesis is

$$4k < 2^k$$

We get $P(k+1)$ by replacing n by $k+1$ in the statement $P(n)$.

We want to use this to show that $P(k+1)$ is true, that is,

$$4(k+1) < 2^{k+1}$$

So we start with the left-hand side of the inequality and use the induction hypothesis to show that it is less than the right-hand side. For $k \geq 5$ we have

$$\begin{aligned} 4(k+1) &= 4k + 4 && \text{Distributive Property} \\ &< 2^k + 4 && \text{Induction hypothesis} \\ &< 2^k + 4k && \text{Because } 4 < 4k \\ &< 2^k + 2^k && \text{Induction hypothesis} \\ &= 2 \cdot 2^k \\ &= 2^{k+1} && \text{Property of exponents} \end{aligned}$$

This is less than 2^k by the induction hypothesis

Thus $P(k+1)$ follows from $P(k)$, and this completes the induction step.

Having proved Steps 1 and 2, we conclude by the Principle of Mathematical Induction that $P(n)$ is true for all natural numbers $n \geq 5$.

Now Try Exercise 21 ■

11.4 Exercises

Concepts

1. Mathematical induction is a method of proving that a statement $P(n)$ is true for all ________ numbers n. In Step 1 we prove that ________ is true.

2. Which of the following is true about Step 2 in a proof by mathematical induction?

(i) We prove "$P(k+1)$ is true."

(ii) We prove "If $P(k)$ is true, then $P(k+1)$ is true."

Skills

3–14 ■ Proving a Formula Use mathematical induction to prove that the formula is true for all natural numbers n.

3. $2 + 4 + 6 + \cdots + 2n = n(n+1)$

4. $1 + 4 + 7 + \cdots + (3n-2) = \dfrac{n(3n-1)}{2}$

5. $5 + 8 + 11 + \cdots + (3n+2) = \dfrac{n(3n+7)}{2}$

6. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

7. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \dfrac{n(n+1)(n+2)}{3}$

8. $1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + n(n+2) = \dfrac{n(n+1)(2n+7)}{6}$

9. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$

10. $1^3 + 3^3 + 5^3 + \cdots + (2n-1)^3 = n^2(2n^2-1)$

11. $2^3 + 4^3 + 6^3 + \cdots + (2n)^3 = 2n^2(n+1)^2$

12. $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4} + \cdots + \dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$

13. $1 \cdot 2 + 2 \cdot 2^2 + 3 \cdot 2^3 + 4 \cdot 2^4 + \cdots + n \cdot 2^n$
$= 2[1 + (n - 1)2^n]$

14. $1 + 2 + 2^2 + \cdots + 2^{n-1} = 2^n - 1$

15–24 ■ Proving a Statement Use mathematical induction to show that the given statement is true.

15. $n^2 + n$ is divisible by 2 for all natural numbers n.

16. $5^n - 1$ is divisible by 4 for all natural numbers n.

17. $n^2 - n + 41$ is odd for all natural numbers n.

18. $n^3 - n + 3$ is divisible by 3 for all natural numbers n.

19. $8^n - 3^n$ is divisible by 5 for all natural numbers n.

20. $3^{2n} - 1$ is divisible by 8 for all natural numbers n.

21. $n < 2^n$ for all natural numbers n.

22. $(n + 1)^2 < 2n^2$ for all natural numbers $n \geq 3$.

23. If $x > -1$, then $(1 + x)^n \geq 1 + nx$ for all natural numbers n.

24. $100n \leq n^2$ for all $n \geq 100$.

25. Formula for a Recursive Sequence A sequence is defined recursively by $a_{n+1} = 3a_n$ and $a_1 = 5$. Show that $a_n = 5 \cdot 3^{n-1}$ for all natural numbers n.

26. Formula for a Recursive Sequence A sequence is defined recursively by $a_{n+1} = 3a_n - 8$ and $a_1 = 4$. Find an explicit formula for a_n, and then use mathematical induction to prove that the formula you found is true.

27. Proving a Factorization Show that $x - y$ is a factor of $x^n - y^n$ for all natural numbers n.
[*Hint:* $x^{k+1} - y^{k+1} = x^k(x - y) + (x^k - y^k)y$.]

28. Proving a Factorization Show that $x + y$ is a factor of $x^{2n-1} + y^{2n-1}$ for all natural numbers n.

Skills Plus

29–33 ■ Fibonacci Sequence F_n denotes the nth term of the Fibonacci sequence discussed in Section 11.1. Use mathematical induction to prove the statement.

29. F_{3n} is even for all natural numbers n.

30. $F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1$

31. $F_1^2 + F_2^2 + F_3^2 + \cdots + F_n^2 = F_n F_{n+1}$

32. $F_1 + F_3 + \cdots + F_{2n-1} = F_{2n}$

33. For all $n \geq 2$,

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$$

34. Formula Using Fibonacci Numbers Let a_n be the nth term of the sequence defined recursively by

$$a_{n+1} = \frac{1}{1 + a_n}$$

and let $a_1 = 1$. Find a formula for a_n in terms of the Fibonacci numbers F_n. Prove that the formula you found is valid for all natural numbers n.

35. Discover and Prove an Inequality Let F_n be the nth term of the Fibonacci sequence. Find and prove an inequality relating n and F_n for all natural numbers $n \geq 5$.

36. Discover and Prove an Inequality Find and prove an inequality relating $100n$ and n^3.

Discuss ■ Discover ■ Prove ■ Write

37. Discover ■ Prove: True or False? Determine whether each statement is true or false. If you think the statement is true, prove it. If you think it is false, give an example for which it fails.

(a) $p(n) = n^2 - n + 11$ is prime for all n.

(b) $n^2 > n$ for all $n \geq 2$.

(c) $2^{2n+1} + 1$ is divisible by 3 for all $n \geq 1$.

(d) $n^3 \geq (n + 1)^2$ for all $n \geq 2$.

(e) $n^3 - n$ is divisible by 3 for all $n \geq 2$.

(f) $n^3 - 6n^2 + 11n$ is divisible by 6 for all $n \geq 1$.

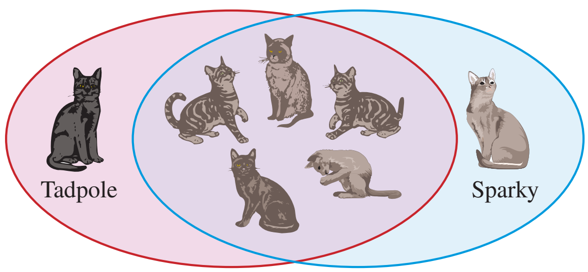
38. Discuss: All Cats Are Black? What is wrong with the following "proof" by mathematical induction that all cats are black? Let $P(n)$ denote the statement "In any group of n cats, if one cat is black, then they are all black."

Step 1 The statement is clearly true for $n = 1$.

Step 2 Suppose that $P(k)$ is true. We show that $P(k + 1)$ is true.

Suppose we have a group of $k + 1$ cats, one of whom is black; call this cat "Tadpole." Remove some other cat (call it "Sparky") from the group. We are left with k cats, one of whom (Tadpole) is black, so by the induction hypothesis, all k of these are black. Now put Sparky back in the group and take out Tadpole. We again have a group of k cats, all of whom—except possibly Sparky—are black. Then by the induction hypothesis, Sparky must be black too. So all $k + 1$ cats in the original group are black.

Thus by induction $P(n)$ is true for all n. Since everyone has seen at least one black cat, it follows that all cats are black.



11.5 The Binomial Theorem

■ Expanding $(a + b)^n$ ■ The Binomial Coefficients ■ The Binomial Theorem
■ Proof of the Binomial Theorem

An expression of the form $a + b$ is called a **binomial**. Although in principle we can raise $a + b$ to any power, raising it to a very high power would be tedious. In this section we find a formula that gives the expansion of $(a + b)^n$ for any natural number n and then prove the formula using mathematical induction.

■ Expanding $(a + b)^n$

To find a pattern in the expansion of $(a + b)^n$, we first look at some special cases.

$$
\begin{aligned}
(a+b)^1 &= a + b \\
(a+b)^2 &= a^2 + 2ab + b^2 \\
(a+b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\
(a+b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \\
(a+b)^5 &= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 \\
&\vdots
\end{aligned}
$$

The following patterns emerge for the expansion of $(a + b)^n$.

1. There are $n + 1$ terms, the first being a^n and the last being b^n.
2. The exponents of a decrease by 1 from term to term, while the exponents of b increase by 1.
3. The sum of the exponents of a and b in each term is n.

For instance, notice how the exponents of a and b behave in the expansion of $(a + b)^5$.

The exponents of *a* decrease:

$$(a+b)^5 = a^{5} + 5a^{4}b^1 + 10a^{3}b^2 + 10a^{2}b^3 + 5a^{1}b^4 + b^5$$

The exponents of *b* increase:

$$(a+b)^5 = a^5 + 5a^4b^{1} + 10a^3b^{2} + 10a^2b^{3} + 5a^1b^{4} + b^{5}$$

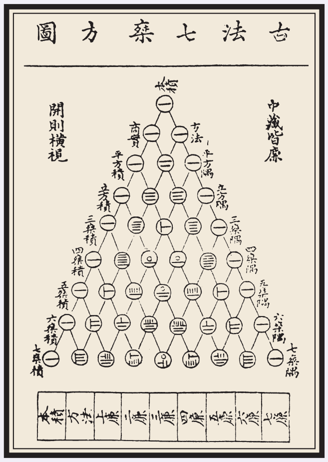
With these observations we can write the form of the expansion of $(a + b)^n$ for any natural number n. For example, writing a question mark for the missing coefficients, we have

$$(a+b)^8 = a^8 + ?a^7b + ?a^6b^2 + ?a^5b^3 + ?a^4b^4 + ?a^3b^5 + ?a^2b^6 + ?ab^7 + b^8$$

To complete the expansion, we need to determine these coefficients. To find a pattern, let's write the coefficients in the expansion of $(a + b)^n$ for the first few values of n in a triangular array as shown in the following array, which is called **Pascal's triangle**.

$$
\begin{array}{lccccccccccc}
(a+b)^0 & & & & & & 1 & & & & & \\
(a+b)^1 & & & & & 1 & & 1 & & & & \\
(a+b)^2 & & & & 1 & & 2 & & 1 & & & \\
(a+b)^3 & & & 1 & & 3 & & 3 & & 1 & & \\
(a+b)^4 & & 1 & & 4 & & 6 & & 4 & & 1 & \\
(a+b)^5 & 1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

What we now call **Pascal's triangle** appears in this Chinese document by Chu Shih-Chieh, dated 1303. The title reads "The Old Method Chart of the Seven Multiplying Squares." The triangle was rediscovered by Pascal (see Section 11.4).



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The row corresponding to $(a + b)^0$ is called the zeroth row and is included to show the symmetry of the array. The key observation about Pascal's triangle is the following property.

Key Property of Pascal's Triangle

Every entry (other than a 1) is the sum of the two entries diagonally above it.

From this property we can find any row of Pascal's triangle from the row above it. For instance, we find the sixth and seventh rows, starting with the fifth row:

$$
\begin{array}{lccccccccccccccc}
(a+b)^5 & & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
(a+b)^6 & & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
(a+b)^7 & 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}
$$

To see why this property holds, let's consider the following expansions:

$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

$$(a + b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

We arrive at the expansion of $(a + b)^6$ by multiplying $(a + b)^5$ by $(a + b)$. Notice, for instance, that the circled term in the expansion of $(a + b)^6$ is obtained via this multiplication from the two circled terms above it. We get this term when the two terms above it are multiplied by b and a, respectively. Thus its coefficient is the sum of the coefficients of these two terms. We will use this observation at the end of this section when we prove the Binomial Theorem.

Having found these patterns, we can now easily obtain the expansion of any binomial, at least to relatively small powers.

Example 1 ■ Expanding a Binomial Using Pascal's Triangle

Find the expansion of $(a + b)^7$ using Pascal's triangle.

Solution The first term in the expansion is a^7, and the last term is b^7. Using the fact that the exponent of a decreases by 1 from term to term and that of b increases by 1 from term to term, we have

$$(a + b)^7 = a^7 + ?a^6b + ?a^5b^2 + ?a^4b^3 + ?a^3b^4 + ?a^2b^5 + ?ab^6 + b^7$$

The appropriate coefficients appear in the seventh row of Pascal's triangle. Thus

$$(a + b)^7 = a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$$

Now Try Exercise 5 ■

Example 2 ■ Expanding a Binomial Using Pascal's Triangle

Use Pascal's triangle to expand $(2 - 3x)^5$.

Solution We find the expansion of $(a + b)^5$ and then substitute 2 for a and $-3x$ for b. Using Pascal's triangle for the coefficients, we get

$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

Substituting $a = 2$ and $b = -3x$ gives

$$
\begin{aligned}
(2 - 3x)^5 &= (2)^5 + 5(2)^4(-3x) + 10(2)^3(-3x)^2 + 10(2)^2(-3x)^3 + 5(2)(-3x)^4 + (-3x)^5 \\
&= 32 - 240x + 720x^2 - 1080x^3 + 810x^4 - 243x^5
\end{aligned}
$$

Now Try Exercise 13 ■

The Binomial Coefficients

Although Pascal's triangle is useful in finding the binomial expansion for reasonably small values of n, it isn't practical for finding $(a + b)^n$ for large values of n. The reason is that the method we use for finding the successive rows of Pascal's triangle is recursive. Thus to find the 100th row of this triangle, we must first find the preceding 99 rows.

We need to examine the pattern in the coefficients more carefully to develop a formula that allows us to calculate directly any coefficient in the binomial expansion. Such a formula exists, and the rest of this section is devoted to finding and proving it. However, to state this formula, we need some notation.

The product of the first n natural numbers is denoted by $\boldsymbol{n!}$ and is called $\boldsymbol{n}$ **factorial**.

$4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$

$7! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 = 5040$

$10! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 = 3{,}628{,}800$

$$\boxed{n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-1) \cdot n}$$

We also define 0! as follows:

$$\boxed{0! = 1}$$

This definition of 0! makes many formulas involving factorials shorter and easier to write.

The Binomial Coefficient

Let n and r be nonnegative integers with $r \le n$. The **binomial coefficient** is denoted by $\binom{n}{r}$ and is defined by

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Example 3 ■ Calculating Binomial Coefficients

(a) $$\binom{9}{4} = \frac{9!}{4!(9-4)!} = \frac{9!}{4!\,5!} = \frac{\cancel{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \cdot 6 \cdot 7 \cdot 8 \cdot 9}{(1 \cdot 2 \cdot 3 \cdot 4)\cancel{(1 \cdot 2 \cdot 3 \cdot 4 \cdot 5)}}$$
$$= \frac{6 \cdot 7 \cdot 8 \cdot 9}{1 \cdot 2 \cdot 3 \cdot 4} = 126$$

(b) $$\binom{100}{3} = \frac{100!}{3!(100-3)!} = \frac{\cancel{1 \cdot 2 \cdot 3 \cdot \cdots \cdot 97} \cdot 98 \cdot 99 \cdot 100}{(1 \cdot 2 \cdot 3)\cancel{(1 \cdot 2 \cdot 3 \cdot \cdots \cdot 97)}}$$
$$= \frac{98 \cdot 99 \cdot 100}{1 \cdot 2 \cdot 3} = 161{,}700$$

(c) $$\binom{100}{97} = \frac{100!}{97!(100-97)!} = \frac{\cancel{1 \cdot 2 \cdot 3 \cdot \cdots \cdot 97} \cdot 98 \cdot 99 \cdot 100}{\cancel{(1 \cdot 2 \cdot 3 \cdot \cdots \cdot 97)}(1 \cdot 2 \cdot 3)}$$
$$= \frac{98 \cdot 99 \cdot 100}{1 \cdot 2 \cdot 3} = 161{,}700$$

Now Try Exercises 17 and 19 ■

Although the binomial coefficient $\binom{n}{r}$ is defined in terms of a fraction, all the results of Example 3 are natural numbers. In fact, $\binom{n}{r}$ is always a natural number (see Exercise 54).

Notice that the binomial coefficients in parts (b) and (c) of Example 3 are equal. This is a special case of the following relation, which you are asked to prove in Exercise 52.

$$\boxed{\binom{n}{r} = \binom{n}{n-r}}$$

To see the connection between the binomial coefficients and the binomial expansion of $(a + b)^n$, let's calculate the following binomial coefficients:

$$\binom{5}{2} = \frac{5!}{2!(5-2)!} = 10$$

$$\binom{5}{0} = 1 \qquad \binom{5}{1} = 5 \qquad \binom{5}{2} = 10 \qquad \binom{5}{3} = 10 \qquad \binom{5}{4} = 5 \qquad \binom{5}{5} = 1$$

These are precisely the entries in the fifth row of Pascal's triangle. In fact, we can write Pascal's triangle as follows.

$$\begin{array}{ccccccccccccc}
 & & & & & & \binom{0}{0} & & & & & & \\
 & & & & & \binom{1}{0} & & \binom{1}{1} & & & & & \\
 & & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & & \\
 & & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & & \\
 & & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & & \\
 & \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} & \\
 & \cdot & & \cdot & & \cdot & \cdot & \cdot & & \cdot & & \cdot & \\
\binom{n}{0} & & \binom{n}{1} & & \binom{n}{2} & & \cdots & & & \binom{n}{n-1} & & \binom{n}{n} &
\end{array}$$

To demonstrate that this pattern holds, we need to show that any entry in this version of Pascal's triangle is the sum of the two entries diagonally above it. In other words, we must show that each entry satisfies the key property of Pascal's triangle. We now state this property in terms of the binomial coefficients.

Key Property of the Binomial Coefficients

For any nonnegative integers r and k with $r \le k$,

$$\binom{k}{r-1} + \binom{k}{r} = \binom{k+1}{r}$$

Notice that the two terms on the left-hand side of this equation are adjacent entries in the kth row of Pascal's triangle and the term on the right-hand side is the entry diagonally below them, in the $(k + 1)$st row. Thus this equation is a restatement of the key property of Pascal's triangle in terms of the binomial coefficients. A proof of this formula is outlined in Exercise 53.

The Binomial Theorem

We are now ready to state the Binomial Theorem.

The Binomial Theorem

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$$

We prove this theorem at the end of this section. First, let's look at some applications of the Binomial Theorem.

Example 4 ■ Expanding a Binomial Using the Binomial Theorem

Use the Binomial Theorem to expand $(x+y)^4$.

Solution By the Binomial Theorem,

$$(x+y)^4 = \binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4$$

Verify that

$$\binom{4}{0} = 1 \qquad \binom{4}{1} = 4 \qquad \binom{4}{2} = 6 \qquad \binom{4}{3} = 4 \qquad \binom{4}{4} = 1$$

It follows that

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

Now Try Exercise 25 ■

Example 5 ■ Expanding a Binomial Using the Binomial Theorem

Use the Binomial Theorem to expand $\left(\sqrt{x} - 1\right)^8$.

Solution We first find the expansion of $(a+b)^8$ and then substitute $\sqrt{x}$ for a and -1 for b. Using the Binomial Theorem, we have

$$\begin{aligned}(a+b)^8 = {} & \binom{8}{0}a^8 + \binom{8}{1}a^7b + \binom{8}{2}a^6b^2 + \binom{8}{3}a^5b^3 + \binom{8}{4}a^4b^4 \\ & + \binom{8}{5}a^3b^5 + \binom{8}{6}a^2b^6 + \binom{8}{7}ab^7 + \binom{8}{8}b^8\end{aligned}$$

Verify that

$$\binom{8}{0} = 1 \qquad \binom{8}{1} = 8 \qquad \binom{8}{2} = 28 \qquad \binom{8}{3} = 56 \qquad \binom{8}{4} = 70$$

$$\binom{8}{5} = 56 \qquad \binom{8}{6} = 28 \qquad \binom{8}{7} = 8 \qquad \binom{8}{8} = 1$$

So

$$\begin{aligned}(a+b)^8 = {} & a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4 + 56a^3b^5 \\ & + 28a^2b^6 + 8ab^7 + b^8\end{aligned}$$

Performing the substitutions $a = x^{1/2}$ and $b = -1$ gives

$$\begin{aligned}(\sqrt{x} - 1)^8 = (x^{1/2})^8 &+ 8(x^{1/2})^7(-1) + 28(x^{1/2})^6(-1)^2 + 56(x^{1/2})^5(-1)^3 \\ &+ 70(x^{1/2})^4(-1)^4 + 56(x^{1/2})^3(-1)^5 + 28(x^{1/2})^2(-1)^6 \\ &+ 8(x^{1/2})(-1)^7 + (-1)^8\end{aligned}$$

This simplifies to

$$(\sqrt{x} - 1)^8 = x^4 - 8x^{7/2} + 28x^3 - 56x^{5/2} + 70x^2 - 56x^{3/2} + 28x - 8x^{1/2} + 1$$

Now Try Exercise 27 ■

The Binomial Theorem can be used to find a particular term of a binomial expansion without having to find the entire expansion.

Recall that

$$\binom{n}{r} = \binom{n}{n-r}$$

General Term of the Binomial Expansion

The term that contains a^r in the expansion of $(a + b)^n$ is

$$\binom{n}{r} a^r b^{n-r}$$

Example 6 ■ Finding a Particular Term in a Binomial Expansion

Find the term that contains x^5 in the expansion of $(2x + y)^{20}$.

Solution The term that contains x^5 is given by the formula for the general term with $a = 2x$, $b = y$, $n = 20$, and $r = 5$. So this term is

$$\binom{20}{5} a^5 b^{15} = \frac{20!}{5!(20-5)!}(2x)^5 y^{15} = \frac{20!}{5!\,15!} 32x^5 y^{15} = 496{,}128x^5y^{15}$$

Now Try Exercise 39 ■

Example 7 ■ Finding a Particular Term in a Binomial Expansion

Find the coefficient of x^8 in the expansion of $\left(x^2 + \dfrac{1}{x}\right)^{10}$.

Solution Both x^2 and $1/x$ are powers of x, so the power of x in each term of the expansion is determined by both terms of the binomial. To find the required coefficient, we first find the general term in the expansion. By the formula we have $a = x^2$, $b = 1/x$, and $n = 10$, so the general term is

$$\binom{10}{r}(x^2)^r\left(\frac{1}{x}\right)^{10-r} = \binom{10}{r} x^{2r}(x^{-1})^{10-r} = \binom{10}{r} x^{3r-10}$$

Thus the term that contains x^8 is the term in which

$$\begin{aligned} 3r - 10 &= 8 \\ r &= 6 \end{aligned}$$

So the required coefficient is

$$\binom{10}{6} = 210$$

Now Try Exercise 41 ■

Proof of the Binomial Theorem

We now give a proof of the Binomial Theorem using mathematical induction.

Proof Let $P(n)$ denote the statement

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$$

Step 1 We show that $P(1)$ is true. But $P(1)$ is just the statement

$$(a+b)^1 = \binom{1}{0}a^1 + \binom{1}{1}b^1 = 1a + 1b = a + b$$

which is certainly true.

Step 2 We assume that $P(k)$ is true. Thus our induction hypothesis is

$$(a+b)^k = \binom{k}{0}a^k + \binom{k}{1}a^{k-1}b + \binom{k}{2}a^{k-2}b^2 + \cdots + \binom{k}{k-1}ab^{k-1} + \binom{k}{k}b^k$$

We use this to show that $P(k+1)$ is true.

$$\begin{aligned}
(a+b)^{k+1} &= (a+b)[(a+b)^k] \\
&= (a+b)\left[\binom{k}{0}a^k + \binom{k}{1}a^{k-1}b + \binom{k}{2}a^{k-2}b^2 + \cdots + \binom{k}{k-1}ab^{k-1} + \binom{k}{k}b^k\right] && \text{Induction hypothesis} \\
&= a\left[\binom{k}{0}a^k + \binom{k}{1}a^{k-1}b + \binom{k}{2}a^{k-2}b^2 + \cdots + \binom{k}{k-1}ab^{k-1} + \binom{k}{k}b^k\right] \\
&\quad + b\left[\binom{k}{0}a^k + \binom{k}{1}a^{k-1}b + \binom{k}{2}a^{k-2}b^2 + \cdots + \binom{k}{k-1}ab^{k-1} + \binom{k}{k}b^k\right] && \text{Distributive Property} \\
&= \binom{k}{0}a^{k+1} + \binom{k}{1}a^kb + \binom{k}{2}a^{k-1}b^2 + \cdots + \binom{k}{k-1}a^2b^{k-1} + \binom{k}{k}ab^k \\
&\quad + \binom{k}{0}a^kb + \binom{k}{1}a^{k-1}b^2 + \binom{k}{2}a^{k-2}b^3 + \cdots + \binom{k}{k-1}ab^k + \binom{k}{k}b^{k+1} && \text{Distributive Property} \\
&= \binom{k}{0}a^{k+1} + \left[\binom{k}{0} + \binom{k}{1}\right]a^kb + \left[\binom{k}{1} + \binom{k}{2}\right]a^{k-1}b^2 \\
&\quad + \cdots + \left[\binom{k}{k-1} + \binom{k}{k}\right]ab^k + \binom{k}{k}b^{k+1} && \text{Group like terms}
\end{aligned}$$

Using the key property of the binomial coefficients, we can write each of the expressions in square brackets as a single binomial coefficient. Also, writing the first and last coefficients as $\binom{k+1}{0}$ and $\binom{k+1}{k+1}$ (these are equal to 1 by Exercise 50) gives

$$(a+b)^{k+1} = \binom{k+1}{0}a^{k+1} + \binom{k+1}{1}a^kb + \binom{k+1}{2}a^{k-1}b^2 + \cdots + \binom{k+1}{k}ab^k + \binom{k+1}{k+1}b^{k+1}$$

But this last equation is precisely $P(k+1)$, and this completes the induction step.

Having proved Steps 1 and 2, we conclude by the Principle of Mathematical Induction that the theorem is true for all natural numbers n. ■

11.5 Exercises

Concepts

1. An algebraic expression of the form $a + b$, which consists of a sum of two terms, is called a ________.

2. We can find the coefficients in the expansion of $(a + b)^n$ from the nth row of ________ triangle. So

$$(a + b)^4 = \square a^4 + \square a^3b + \square a^2b^2 + \square ab^3 + \square b^4$$

3. The binomial coefficients can be calculated directly by using the formula $\binom{n}{k} =$ ________. So $\binom{4}{3} =$ ________.

4. To expand $(a + b)^n$, we can use the ________ Theorem. Using this theorem, we find the expansion $(a + b)^4 =$

$$\binom{\square}{\square}a^4 + \binom{\square}{\square}a^3b + \binom{\square}{\square}a^2b^2 + \binom{\square}{\square}ab^3 + \binom{\square}{\square}b^4$$

Skills

5–16 ■ Pascal's Triangle Use Pascal's triangle to expand the expression.

5. $(x + y)^6$

6. $(2x + 1)^4$

7. $\left(x + \dfrac{1}{x}\right)^4$

8. $(x - y)^5$

9. $(x - 1)^5$

10. $(\sqrt{a} + \sqrt{b})^6$

11. $(x^2y - 1)^5$

12. $(1 + \sqrt{2})^6$

13. $(2x - 3y)^3$

14. $(1 + x^3)^3$

15. $\left(\dfrac{1}{x} - \sqrt{x}\right)^5$

16. $\left(2 + \dfrac{x}{2}\right)^5$

17–24 ■ Calculating Binomial Coefficients Evaluate the expression.

17. $\binom{6}{4}$

18. $\binom{8}{3}$

19. $\binom{100}{98}$

20. $\binom{10}{5}$

21. $\binom{3}{1}\binom{4}{2}$

22. $\binom{5}{2}\binom{5}{3}$

23. $\binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5}$

24. $\binom{5}{0} - \binom{5}{1} + \binom{5}{2} - \binom{5}{3} + \binom{5}{4} - \binom{5}{5}$

25–28 ■ Binomial Theorem Use the Binomial Theorem to expand the expression.

25. $(x + 2y)^4$

26. $(1 - x)^5$

27. $\left(1 + \dfrac{1}{x}\right)^6$

28. $(2A + B^2)^4$

29–42 ■ Terms of a Binomial Expansion Find the indicated terms in the expansion of the given binomial.

29. The first three terms in the expansion of $(x + 2y)^{20}$

30. The first four terms in the expansion of $(x^{1/2} + 1)^{30}$

31. The last two terms in the expansion of $(a^{2/3} + a^{1/3})^{25}$

32. The first three terms in the expansion of

$$\left(x + \frac{1}{x}\right)^{40}$$

33. The middle term in the expansion of $(x^2 + 1)^{18}$

34. The fifth term in the expansion of $(ab - 1)^{20}$

35. The 24th term in the expansion of $(a + b)^{25}$

36. The 28th term in the expansion of $(A - B)^{30}$

37. The 100th term in the expansion of $(1 + y)^{100}$

38. The second term in the expansion of

$$\left(x^2 - \frac{1}{x}\right)^{25}$$

39. The term containing x^4 in the expansion of $(x + 2y)^{10}$

40. The term containing y^3 in the expansion of $(\sqrt{2} + y)^{12}$

41. The term containing b^8 in the expansion of $(a + b^2)^{12}$

42. The term that does not contain x in the expansion of

$$\left(8x + \frac{1}{2x}\right)^8$$

43–46 ■ Factoring Factor using the Binomial Theorem.

43. $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

44. $(x - 1)^5 + 5(x - 1)^4 + 10(x - 1)^3 + 10(x - 1)^2 + 5(x - 1) + 1$

45. $8a^3 + 12a^2b + 6ab^2 + b^3$

46. $x^8 + 4x^6y + 6x^4y^2 + 4x^2y^3 + y^4$

47–48 ■ Simplifying a Difference Quotient Simplify using the Binomial Theorem.

47. $\dfrac{(x + h)^3 - x^3}{h}$

48. $\dfrac{(x + h)^4 - x^4}{h}$

Skills Plus

49–52 ■ Proving a Statement Show that the given statement is true.

49. $(1.01)^{100} > 2$. [*Hint:* Note that $(1.01)^{100} = (1 + 0.01)^{100}$, and use the Binomial Theorem to show that the sum of the first three terms of the expansion is greater than 2.]

50. $\binom{n}{0} = 1$ and $\binom{n}{n} = 1$

51. $\binom{n}{1} = \binom{n}{n-1} = n$

52. $\binom{n}{r} = \binom{n}{n-r}$ for $0 \le r \le n$

53. Proving an Identity In this exercise we prove the identity

$$\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$$

(a) Write the left-hand side of this equation as the sum of two fractions.

(b) Show that a common denominator of the expression that you found in part (a) is $r!(n-r+1)!$.

(c) Add the two fractions using the common denominator in part (b), simplify the numerator, and note that the resulting expression is equal to the right-hand side of the equation.

54. Proof Using Induction Prove that $\binom{n}{r}$ is an integer for all n and for $0 \le r \le n$. [*Suggestion:* Use induction to show that the statement is true for all n, and use Exercise 53 for the induction step.]

Applications

55. Difference in Volumes of Cubes The volume of a cube of side x inches is given by $V(x) = x^3$, so the volume of a cube of side $x + 2$ inches is given by $V(x+2) = (x+2)^3$. Use the Binomial Theorem to show that the difference in volume between the larger and smaller cubes is $6x^2 + 12x + 8$ cubic inches.

56. Probability of Hitting a Target The probability that an archer hits the target is $p = 0.9$, so the probability that the archer misses the target is $q = 0.1$. It is known that in this situation the probability that the archer hits the target exactly r times in n attempts is given by the term containing p^r in the binomial expansion of $(p+q)^n$. Find the probability that the archer hits the target exactly three times in five attempts.

Discuss ■ Discover ■ Prove ■ Write

57. Discuss: Powers of Factorials Which is larger, $(100!)^{101}$ or $(101!)^{100}$?

PS *Look for something familiar.* Try factoring the expressions. Do they have any common factors?

58. Discover ■ Prove: Sums of Binomial Coefficients Add each of the first five rows of Pascal's triangle, as indicated. Do you see a pattern?

$$1 + 1 = \boxed{?}$$
$$1 + 2 + 1 = \boxed{?}$$
$$1 + 3 + 3 + 1 = \boxed{?}$$
$$1 + 4 + 6 + 4 + 1 = \boxed{?}$$
$$1 + 5 + 10 + 10 + 5 + 1 = \boxed{?}$$

On the basis of the pattern you have found, find the sum of the nth row:

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$$

Prove your result by expanding $(1+1)^n$ using the Binomial Theorem.

59. Discover ■ Prove: Alternating Sums of Binomial Coefficients Find the sum

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n\binom{n}{n}$$

by finding a pattern as in Exercise 58. Prove your result by expanding $(1-1)^n$ using the Binomial Theorem.

Chapter 11 Review

Properties and Formulas

Sequences | Section 11.1

A **sequence** is a function whose domain is the set of natural numbers. Instead of writing $a(n)$ for the value of the sequence at n, we generally write a_n, and we refer to this value as the ***n*th term** of the sequence. Sequences are often described in list form:

$$a_1, a_2, a_3, \ldots$$

Partial Sums of a Sequence | Section 11.1

For the sequence $a_1, a_2, a_3, \ldots$ the ***n*th partial sum** S_n is the sum of the first n terms of the sequence:

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n$$

The nth partial sum of a sequence can also be expressed by using **sigma notation**:

$$S_n = \sum_{k=1}^{n} a_k$$

Arithmetic Sequences | Section 11.2

An **arithmetic sequence** is a sequence whose terms are obtained by adding the same fixed constant d to each term to get the next term. Thus an arithmetic sequence has the form

$$a, a+d, a+2d, a+3d, \ldots$$

The number a is the **first term** of the sequence, and the number d is the **common difference**. The nth term of the sequence is

$$a_n = a + (n-1)d$$

Partial Sums of an Arithmetic Sequence | Section 11.2

For the arithmetic sequence $a_n = a + (n-1)d$ the nth partial sum $S_n = \sum_{k=1}^{n} [a + (k-1)d]$ is given by either of the following equivalent formulas:

1. $S_n = \frac{n}{2}[2a + (n-1)d]$ **2.** $S_n = n\left(\frac{a + a_n}{2}\right)$

Geometric Sequences | Section 11.3

A **geometric sequence** is a sequence whose terms are obtained by multiplying each term by the same fixed constant r to get the next term. Thus a geometric sequence has the form

$$a, ar, ar^2, ar^3, \ldots$$

The number a is the **first term** of the sequence, and the number r is the **common ratio**. The nth term of the sequence is

$$a_n = ar^{n-1}$$

Partial Sums of a Geometric Sequence | Section 11.3

For the geometric sequence $a_n = ar^{n-1}$ the nth partial sum $S_n = \sum_{k=1}^{n} ar^{k-1}$ (where $r \neq 1$) is given by

$$S_n = a\frac{1 - r^n}{1 - r}$$

Infinite Geometric Series | Section 11.3

An **infinite geometric series** is a series of the form

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots$$

An infinite geometric series for which $|r| < 1$ has the sum

$$S = \frac{a}{1 - r}$$

Principle of Mathematical Induction | Section 11.4

For each natural number n, let $P(n)$ be a statement that depends on n. Suppose that each of the following conditions is satisfied.

1. $P(1)$ is true.

2. For every natural number k, if $P(k)$ is true, then $P(k + 1)$ is true.

Then $P(n)$ is true for all natural numbers n.

Sums of Powers | Section 11.4

0. $\displaystyle\sum_{k=1}^{n} 1 = n$

1. $\displaystyle\sum_{k=1}^{n} k = \frac{n(n + 1)}{2}$

2. $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n + 1)(2n + 1)}{6}$

3. $\displaystyle\sum_{k=1}^{n} k^3 = \frac{n^2(n + 1)^2}{4}$

Binomial Coefficients | Section 11.5

If n and r are positive integers with $n \geq r$, then the **binomial coefficient** $\binom{n}{r}$ is defined by

$$\binom{n}{r} = \frac{n!}{r!(n - r)!}$$

Binomial coefficients satisfy the following properties:

$$\binom{n}{r} = \binom{n}{n - r}$$

$$\binom{k}{r - 1} + \binom{k}{r} = \binom{k + 1}{r}$$

The Binomial Theorem | Section 11.5

$$(a + b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n}b^n$$

The term that contains a^r in the expansion of $(a + b)^n$ is $\binom{n}{r}a^r b^{n-r}$.

Concept Check

1. (a) What is a sequence? What notation do we use to denote the terms of a sequence?

(b) Find a formula for the sequence of even numbers and a formula for the sequence of odd numbers.

(c) Find the first three terms and the 10th term of the sequence given by $a_n = n/(n + 1)$.

2. (a) What is a recursively defined sequence?

(b) Find the first four terms of the sequence recursively defined by $a_1 = 3$ and $a_n = n + 2a_{n-1}$.

3. (a) What is meant by the partial sums of a sequence?

(b) Find the first three partial sums of the sequence given by $a_n = 1/n$.

4. (a) What is an arithmetic sequence? Write a formula for the nth term of an arithmetic sequence.

(b) Write a formula for the arithmetic sequence that starts as follows: 3, 8, . . . Write the first five terms of this sequence.

(c) Write two different formulas for the sum of the first n terms of an arithmetic sequence.

(d) Find the sum of the first 20 terms of the sequence in part (b).

5. (a) What is a geometric sequence? Write an expression for the nth term of a geometric sequence that has first term a and common ratio r.

(b) Write an expression for the geometric sequence with first term $a = 3$ and common ratio $r = \frac{1}{2}$. Give the first five terms of this sequence.

(c) Write an expression for the sum of the first n terms of a geometric sequence.

(d) Find the sum of the first five terms of the sequence in part (b).

6. (a) What is an infinite geometric series?

(b) What does it mean for an infinite series to converge? For what values of r does an infinite geometric series converge? If an infinite geometric series converges, then what is its sum?

(c) Write the first four terms of the infinite geometric series with first term $a = 5$ and common ratio $r = 0.4$. Does the series converge? If so, find its sum.

7. (a) Write $1^3 + 2^3 + 3^3 + 4^3 + 5^3$ using sigma notation.

(b) Write $\displaystyle\sum_{k=3}^{5} 2k^2$ without using sigma notation.

8. **(a)** State the Principle of Mathematical Induction.

(b) Use mathematical induction to prove that for all natural numbers n, $3^n - 1$ is an even number.

9. **(a)** Write Pascal's triangle. How are the entries in the triangle related to each other?

Row 0

Row 1

Row 2

Row 3

(b) Use Pascal's triangle to expand$(x + c)^3$.

10. **(a)** What does the symbol $n!$ mean? Find $5!$.

(b) Define $\binom{n}{r}$, and find $\binom{5}{2}$.

11. **(a)** State the Binomial Theorem.

(b) Use the Binomial Theorem to expand $(x + 2)^3$.

(c) Use the Binomial Theorem to find the term containing x^4 in the expansion of $(x + 2)^{10}$.

Answers to the Concept Check can be found at the book companion website stewartmath.com.

Exercises

1–6 ■ Terms of a Sequence Find the first four terms as well as the tenth term of the sequence with the given nth term.

1. $a_n = \dfrac{n^2}{n + 1}$ **2.** $a_n = (-1)^n \dfrac{2^n}{n}$

3. $a_n = \dfrac{(-1)^n + 1}{n^3}$ **4.** $a_n = \dfrac{n(n + 1)}{2}$

5. $a_n = \dfrac{(2n)!}{2^n n!}$ **6.** $a_n = \dbinom{n + 1}{2}$

7–10 ■ Recursive Sequences A sequence is defined recursively. Find the first seven terms of the sequence.

7. $a_n = a_{n-1} + 2n - 1, \quad a_1 = 1$

8. $a_n = \dfrac{a_{n-1}}{n}, \quad a_1 = 1$

9. $a_n = a_{n-1} + 2a_{n-2}, \quad a_1 = 1, a_2 = 3$

10. $a_n = \sqrt{3a_{n-1}}, \quad a_1 = \sqrt{3}$

11–14 ■ Arithmetic or Geometric Sequence? The nth term of a sequence is given. **(a)** Find the first five terms of the sequence. **(b)** Graph the terms you found in part (a). **(c)** Find the fifth partial sum of the sequence. **(d)** Determine whether the sequence is arithmetic or geometric. Find the common difference or the common ratio.

11. $a_n = 2n + 5$ **12.** $a_n = \dfrac{5}{2^n}$

13. $a_n = \dfrac{3^n}{2^{n+1}}$ **14.** $a_n = 4 - \dfrac{n}{2}$

15–22 ■ Arithmetic or Geometric Sequence? The first four terms of a sequence are given. Determine whether they can be the terms of an arithmetic sequence, a geometric sequence, or neither. If the sequence is arithmetic or geometric, find the fifth term.

15. $5, 5.5, 6, 6.5, \ldots$ **16.** $\sqrt{2}, 2\sqrt{2}, 3\sqrt{2}, 4\sqrt{2}, \ldots$

17. $t - 3, t - 2, t - 1, t, \ldots$ **18.** $\sqrt{2}, 2, 2\sqrt{2}, 4, \ldots$

19. $t^3, t^2, t, 1, \ldots$ **20.** $1, -\frac{3}{2}, 2, -\frac{5}{2}, \ldots$

21. $\frac{3}{4}, \frac{1}{2}, \frac{1}{3}, \frac{2}{9}, \ldots$ **22.** $a, 1, \dfrac{1}{a}, \dfrac{1}{a^2}, \ldots$

23. Proving a Sequence Is Geometric Show that $3, 6i, -12, -24i, \ldots$ is a geometric sequence, and find the common ratio. (Here $i = \sqrt{-1}$.)

24. nth Term of a Geometric Sequence Find the nth term of the geometric sequence $2, 2 + 2i, 4i, -4 + 4i, -8, \ldots$. (Here $i = \sqrt{-1}$.)

25–28 ■ Finding Terms of Arithmetic and Geometric Sequences Find the indicated term of the arithmetic or geometric sequence with the given description.

25. The fourth term of an arithmetic sequence is 11, and the sixth term is 17. Find the second term.

26. The 20th term of an arithmetic sequence is 96, and the common difference is 5. Find the nth term.

27. The third term of a geometric sequence is 9, and the common ratio is $\frac{3}{2}$. Find the fifth term.

28. The second term of a geometric sequence is 10, and the fifth term is $\frac{1250}{27}$. Find the nth term.

29–30 ■ Salary Increases A school advertises two teaching positions.

Position I: Starting salary \$52,000 and each year the salary increases by 4% of the preceding year

Position II: Starting salary \$55,000 and each year the salary increases by \$1600

29. For Position I, find a formula for the salary A_n in the nth year of employment and list the salaries for the first six years of employment.

30. For Position II, find a formula for the salary A_n in the nth year of employment and list the salaries for the first six years of employment. Which teaching position has the larger salary in the sixth year?

31. Bacteria Culture A certain type of bacteria divides every 5 seconds. If three of these bacteria are put into a petri dish, how many bacteria are in the dish at the end of 1 minute?

32. Arithmetic Sequences If $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are arithmetic sequences, show that $a_1 + b_1, a_2 + b_2, a_3 + b_3, \ldots$ is also an arithmetic sequence.

33. Geometric Sequences If $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are geometric sequences, show that $a_1b_1, a_2b_2, a_3b_3, \ldots$ is also a geometric sequence.

34. Arithmetic or Geometric Sequence?

(a) If $a_1, a_2, a_3, \ldots$ is an arithmetic sequence, is the sequence $a_1 + 2, a_2 + 2, a_3 + 2, \ldots$ arithmetic?

(b) If $a_1, a_2, a_3, \ldots$ is a geometric sequence, is the sequence $5a_1, 5a_2, 5a_3, \ldots$ geometric?

35. Arithmetic and Geometric Sequences Find the values of x for which the sequence $6, x, 12, \ldots$ is

(a) arithmetic **(b)** geometric

36. Arithmetic and Geometric Sequences Find the values of x and y for which the sequence $2, x, y, 17, \ldots$ is

(a) arithmetic **(b)** geometric

37–40 ■ Partial Sums Find the sum.

37. $\sum_{k=3}^{6} (k+1)^2$ **38.** $\sum_{i=1}^{4} \frac{2i}{2i-1}$

39. $\sum_{k=1}^{6} (k+1)2^{k-1}$ **40.** $\sum_{m=1}^{5} 3^{m-2}$

41–44 ■ Sigma Notation Write the sum without using sigma notation. Do not evaluate.

41. $\sum_{k=1}^{10} (k-1)^2$ **42.** $\sum_{j=2}^{100} \frac{1}{j-1}$

43. $\sum_{k=1}^{50} \frac{3^k}{2^{k+1}}$ **44.** $\sum_{n=1}^{10} n^2 2^n$

45–48 ■ Sigma Notation Write the sum using sigma notation. Do not evaluate.

45. $3 + 6 + 9 + 12 + \cdots + 99$

46. $1^2 + 2^2 + 3^2 + \cdots + 100^2$

47. $1 \cdot 2^3 + 2 \cdot 2^4 + 3 \cdot 2^5 + 4 \cdot 2^6 + \cdots + 100 \cdot 2^{102}$

48. $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{999 \cdot 1000}$

49–54 ■ Sums of Arithmetic and Geometric Sequences Determine whether the expression is a partial sum of an arithmetic or geometric sequence. Then find the sum.

49. $1 + 0.9 + (0.9)^2 + \cdots + (0.9)^5$

50. $3 + 3.7 + 4.4 + \cdots + 10$

51. $\sqrt{5} + 2\sqrt{5} + 3\sqrt{5} + \cdots + 100\sqrt{5}$

52. $\frac{1}{3} + \frac{2}{3} + 1 + \frac{4}{3} + \cdots + 33$

53. $\sum_{n=0}^{6} 3(-4)^n$ **54.** $\sum_{k=0}^{8} 7(5)^{k/2}$

55–60 ■ Infinite Geometric Series Determine whether the infinite geometric series is convergent or divergent. If it is convergent, find its sum.

55. $1 - \frac{2}{5} + \frac{4}{25} - \frac{8}{125} + \cdots$

56. $0.1 + 0.01 + 0.001 + 0.0001 + \cdots$

57. $5 - 5(1.01) + 5(1.01)^2 - 5(1.01)^3 + \cdots$

58. $1 + \frac{1}{3^{1/2}} + \frac{1}{3} + \frac{1}{3^{3/2}} + \cdots$

59. $-1 + \frac{9}{8} - \left(\frac{9}{8}\right)^2 + \left(\frac{9}{8}\right)^3 - \cdots$

60. $a + ab^2 + ab^4 + ab^6 + \cdots, \quad |b| < 1$

61. Terms of an Arithmetic Sequence The first term of an arithmetic sequence is $a = 7$, and the common difference is $d = 3$. How many terms of this sequence must be added to obtain 325?

62. Terms of an Geometric Sequence The sum of the first three terms of a geometric series is 52, and the common ratio is $r = 3$. Find the first term.

63. Ancestors A person has two parents, four grandparents, eight great-grandparents, and so on. What is the total number of a person's ancestors in 15 generations?

64–66 ■ Mathematical Induction Use mathematical induction to prove that the formula is true for all natural numbers n.

64. $1 + 4 + 7 + \cdots + (3n - 2) = \frac{n(3n-1)}{2}$

65. $\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$

66. $\left(1 + \frac{1}{1}\right)\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right)\cdots\left(1 + \frac{1}{n}\right) = n + 1$

67–69 ■ Proof by Induction Use mathematical induction to show that the given statement is true.

67. $7^n - 1$ is divisible by 6 for all natural numbers n.

68. The Fibonacci number F_{4n} is divisible by 3 for all natural numbers n.

69. Formula for a Recursive Sequence A sequence is defined recursively by $a_{n+1} = 3a_n + 4$ and $a_1 = 4$. Show that $a_n = 2 \cdot 3^n - 2$ for all natural numbers n.

70–73 ■ Binomial Coefficients Evaluate the expression.

70. $\binom{5}{2}\binom{5}{3}$ **71.** $\binom{10}{2} + \binom{10}{6}$

72. $\sum_{k=0}^{5} \binom{5}{k}$ **73.** $\sum_{k=0}^{8} \binom{8}{k}\binom{8}{8-k}$

74–77 ■ Binomial Expansion Expand the expression.

74. $(A - B)^3$ **75.** $(x + 2)^5$

76. $(1 - x^2)^6$ **77.** $(2x + y)^4$

78–80 ■ Terms in a Binomial Expansion Find the indicated terms in the given binomial expansion.

78. Find the 20th term in the expansion of $(a + b)^{22}$.

79. Find the first three terms in the expansion of $(b^{-2/3} + b^{1/3})^{20}$.

80. Find the term containing A^6 in the expansion of $(A + 3B)^{10}$.

Chapter 11 | Test

1. Find the first six terms and the sixth partial sum of the sequence whose nth term is $a_n = 2n^2 - n$.

2. A sequence is defined recursively by $a_{n+1} = 3a_n - n$, $a_1 = 2$. Find the first six terms of the sequence.

3. An arithmetic sequence begins 2, 5, 8, 11, 14, ...

(a) Find the common difference d for this sequence.

(b) Find a formula for the nth term a_n of the sequence.

(c) Find the 35th term of the sequence.

4. A geometric sequence begins 12, 3, $\frac{3}{4}$, $\frac{3}{16}$, $\frac{3}{64}$, ...

(a) Find the common ratio r for this sequence.

(b) Find a formula for the nth term a_n of the sequence.

(c) Find the tenth term of the sequence.

5. The first term of a geometric sequence is 25, and the fourth term is $\frac{1}{5}$.

(a) Find the common ratio r and the fifth term.

(b) Find the sum of the first eight terms.

6. The first term of an arithmetic sequence is 10, and the tenth term is 2.

(a) Find the common difference and the 100th term of the sequence.

(b) Find the sum of the first ten terms.

7. Let $a_1, a_2, a_3, \ldots$ be a geometric sequence with initial term a and common ratio r. Show that $a_1^2, a_2^2, a_3^2, \ldots$ is also a geometric sequence by finding its common ratio.

8. Write the expression without using sigma notation, and then find the sum.

(a) $\displaystyle\sum_{n=1}^{5}(1 - n^2)$ **(b)** $\displaystyle\sum_{n=3}^{6}(-1)^n 2^{n-2}$

9. Find the sum.

(a) $\dfrac{1}{3} + \dfrac{2}{3^2} + \dfrac{2^2}{3^3} + \dfrac{2^3}{3^4} + \cdots + \dfrac{2^9}{3^{10}}$

(b) $1 + \dfrac{1}{2^{1/2}} + \dfrac{1}{2} + \dfrac{1}{2^{3/2}} + \cdots$

10. Use mathematical induction to prove that for all natural numbers n,

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

11. Expand $(2x + y^2)^5$.

12. Find the term containing x^3 in the binomial expansion of $(3x - 2)^{10}$.

13. A puppy weighs 0.85 lb at birth, and each week he gains 24% in weight. Let a_n be his weight in pounds at the end of his nth week of life.

(a) Find a formula for a_n.

(b) How much does the puppy weigh when he is 6 weeks old?

(c) Is the sequence $a_1, a_2, a_3, \ldots$ arithmetic, geometric, or neither?

Many real-world processes occur in stages. Population growth can be viewed in stages—each new generation represents a new stage. Compound interest is paid in stages—each interest payment creates a new account balance. Many things that change continuously are more easily measured in discrete stages. For example, we can measure the temperature of a continuously cooling object in one-hour intervals. In this *Focus on Modeling* we learn how recursive sequences are used to model such situations. In some cases we can get an explicit formula for a sequence from the recursion relation that defines it by finding a pattern in the terms of the sequence.

Recursive Sequences as Models

Suppose you deposit some money in an account that pays 6% interest compounded monthly. The bank has a definite rule for paying interest: At the end of each month the bank adds to your account $\frac{1}{2}\%$ (or 0.005) of the amount in your account at that time. Let's express this rule as follows:

$$\boxed{\text{amount at the end of this month}} = \boxed{\text{amount at the end of last month}} + 0.005 \times \boxed{\text{amount at the end of last month}}$$

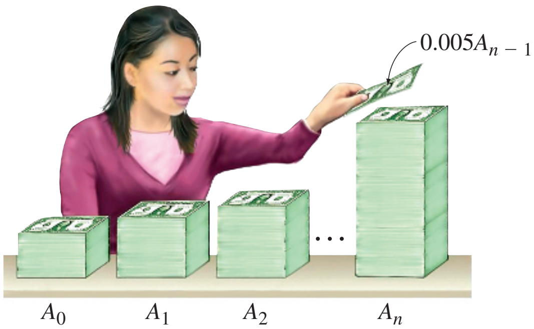
Using the Distributive Property, we can write this as

$$\boxed{\text{amount at the end of this month}} = 1.005 \times \boxed{\text{amount at the end of last month}}$$

To model this statement using algebra, let A_0 be the amount of the original deposit, let A_1 be the amount at the end of the first month, let A_2 be the amount at the end of the second month, and so on. So A_n is the amount at the end of the nth month. Thus

$$A_n = 1.005A_{n-1}$$

We recognize this as a recursively defined sequence—it gives us the amount at each stage in terms of the amount at the preceding stage.



To find a formula for A_n, let's find the first few terms of the sequence and look for a pattern.

$$A_1 = 1.005A_0$$
$$A_2 = 1.005A_1 = (1.005)^2A_0$$
$$A_3 = 1.005A_2 = (1.005)^3A_0$$
$$A_4 = 1.005A_3 = (1.005)^4A_0$$

We can use mathematical induction to prove that the formula we found for A_n is valid for all natural numbers n.

We see that in general, $A_n = (1.005)^nA_0$.

Example 1 ■ Population Growth

A certain animal population grows by 2% each year. The initial population is 5000.

(a) Find a recursive sequence that models the population P_n at the end of the nth year.

(b) Find the first five terms of the sequence P_n.

(c) Find a formula for P_n.

Solution

(a) We can model the population using the following rule:

$$\boxed{\text{population at the end of this year}} = 1.02 \times \boxed{\text{population at the end of last year}}$$

Algebraically, we can write this as the recursion relation

$$P_n = 1.02P_{n-1}$$

(b) Since the initial population is 5000, we have

$$\begin{aligned}
P_0 &= 5000 \\
P_1 &= 1.02P_0 = (1.02)5000 \\
P_2 &= 1.02P_1 = (1.02)^2 5000 \\
P_3 &= 1.02P_2 = (1.02)^3 5000 \\
P_4 &= 1.02P_3 = (1.02)^4 5000
\end{aligned}$$

(c) We see from the pattern exhibited in part (b) that $P_n = (1.02)^n 5000$. (Note that P_n is a geometric sequence, with common ratio $r = 1.02$.) ■

Example 2 ■ Daily Drug Dose

A patient is instructed to take a 50-mg pill of a certain drug every morning. It is known that the body eliminates 40% of the drug every 24 hours.

(a) Find a recursive sequence that models the amount A_n of the drug in the patient's body after each pill is taken.

(b) Find the first four terms of the sequence A_n.

(c) Find a formula for A_n.

(d) How much of the drug remains in the patient's body after 5 days? How much will accumulate in the patient's system after prolonged use?

Solution

(a) Each morning, 60% of the drug remains, plus the patient takes an additional 50 mg (the daily dose).

$$\boxed{\begin{array}{c}\text{amount of drug}\\ \text{this morning}\end{array}} = 0.6 \times \boxed{\begin{array}{c}\text{amount of drug}\\ \text{yesterday morning}\end{array}} + 50 \text{ mg}$$

We can express this as a recursion relation

$$A_n = 0.6A_{n-1} + 50$$

(b) Since the initial dose is 50 mg, we have

$$A_0 = 50$$

$$A_1 = 0.6A_0 + 50 = 0.6(50) + 50$$

$$\begin{aligned} A_2 &= 0.6A_1 + 50 = 0.6[0.6(50) + 50] + 50 \\ &= 0.6^2(50) + 0.6(50) + 50 \\ &= 50(0.6^2 + 0.6 + 1) \end{aligned}$$

$$\begin{aligned} A_3 &= 0.6A_2 + 50 = 0.6[0.6^2(50) + 0.6(50) + 50] + 50 \\ &= 0.6^3(50) + 0.6^2(50) + 0.6(50) + 50 \\ &= 50(0.6^3 + 0.6^2 + 0.6 + 1) \end{aligned}$$

(c) From the pattern in part (b) we see that

$$\begin{aligned} A_n &= 50(1 + 0.6 + 0.6^2 + \cdots + 0.6^n) \\ &= 50\left(\frac{1 - 0.6^{n+1}}{1 - 0.6}\right) && \text{Partial sum of a geometric sequence} \\ &= 125(1 - 0.6^{n+1}) && \text{Simplify} \end{aligned}$$

(d) To find the amount remaining after 5 days, we substitute $n = 5$ and get

$$A_5 = 125(1 - 0.6^{5+1}) \approx 119 \text{ mg}$$

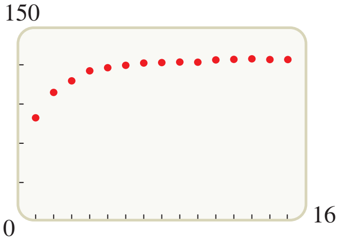


Figure 1

To find the amount remaining after prolonged use, we let n become large. As n gets large, 0.6^n approaches 0. That is, $0.6^n \to 0$ as $n \to \infty$ (see Section 4.1). So as $n \to \infty$,

$$A_n = 125(1 - 0.6^{n+1}) \to 125(1 - 0) = 125$$

Thus after prolonged use, the amount of drug in the patient's system approaches 125 mg (see Figure 1, where we have used a graphing device to graph the sequence).

Problems

1. **Retirement Accounts** Many college professors keep retirement savings with TIAA, the largest annuity program in the world. Interest on these accounts is compounded and credited *daily*. A professor has $275,000 on deposit with TIAA at the start of 2022 and earns 3.65% interest per year.
   - **(a)** Find a recursive sequence that models the amount A_n in the account at the end of the nth day of 2022.
   - **(b)** Find the first eight terms of the sequence A_n, rounded to the nearest cent.
   - **(c)** Find a formula for A_n.

2. **Fitness Program** A student decides to embark on a swimming program as the best way to maintain cardiovascular health. The student begins by swimming 5 minutes on the first day, then adds $1\frac{1}{2}$ minutes every day after that.
   - **(a)** Find a recursive formula for the number of minutes T_n spent swimming on the nth day of the program.
   - **(b)** Find the first six terms of the sequence T_n.
   - **(c)** Find a formula for T_n. What kind of sequence is this?
   - **(d)** On what day does the student attain the goal of swimming at least 65 minutes a day?
   - **(e)** What is the total amount of time the student will have swum after 30 days?

3. Monthly Savings Program A student begins a monthly savings program by depositing \$100 on January 1 in a savings account that pays 3% interest compounded monthly. An amount of \$100 is added to the account at the end of each month, when the interest is credited.

(a) Find a recursive formula for the amount A_n in the account at the end of the nth month. (Include the interest credited for that month and the monthly deposit.)

(b) Find the first five terms of the sequence A_n.

(c) Use the pattern you observed in part (b) to find a formula for A_n. [*Hint:* To find the pattern most easily, it's best *not* to simplify the terms *too* much.]

(d) How much has been saved after five years?

4. Pollution A chemical plant discharges 2400 tons of pollutants every year into an adjacent lake. Through natural runoff, 70% of the pollutants contained in the lake at the beginning of the year are expelled by the end of the year.

(a) Explain why the following sequence models the amount A_n of the pollutant in the lake at the end of the nth year that the plant is operating.

$$A_n = 0.30A_{n-1} + 2400$$

(b) Find the first five terms of the sequence A_n.

(c) Find a formula for A_n.

(d) How much of the pollutant remains in the lake after six years? How much will remain after the plant has been operating for many years?

(e) Verify your answer to part (d) by graphing A_n with a graphing device for $n = 1$ to $n = 20$.

5. Comparing Annual Saving Plans An amount of \$5000 is invested in a one-year Certificate of Deposit (CD) that yields 5% interest per year. At the end of each year, when the CD matures, the total amount (principal and interest) is reinvested, together with an additional amount according to one of the following plans.

Plan I: Add 10% of the total amount at the end of each year.

Plan II: Add \$500n to the total amount at the end of year n.

(a) Explain why the following recursion formulas give the amounts U_n and V_n that is reinvested in the nth year for Plans I and II, respectively.

$$U_n = 1.05U_{n-1} + 0.1(1.05U_{n-1}) \qquad V_n = 1.05V_{n-1} + 500n$$

(b) Calculate several values of U_n and V_n. These are most conveniently calculated using a graphing calculator, as shown in the figure. Observe that Plan II seems to accumulate more savings, but Plan I eventually pulls ahead in this savings race. In what year does this occur?

```
Plot1 Plot2 Plot3
\u(n) = 1.05 u(n - 1)
 +0.1(1.05 u(n - 1))
 u(nMin) = {5000}
\v(n) = 1.05 v(n - 1)
 +500 n
 v(nMin) = {5000}
```

Entering the sequences

n	u(n)	v(n)
0	5000	5000
1	5775	5750
2	6670.1	7037.5
3	7704	8889.4
4	8898.1	11334
5	10277	14401
6	11870	18121
n=0		

Table of values of the sequences