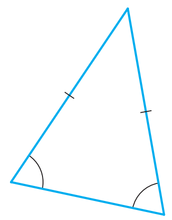
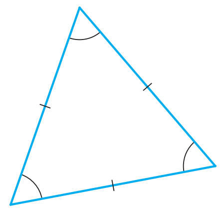
Appendix A | Geometry Review

■ Congruent Triangles ■ Similar Triangles ■ The Pythagorean Theorem ■ Parallel Lines ■ Circles

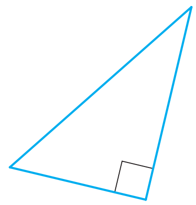
In this appendix we review some concepts from geometry. Several of these involve triangles, so let's recall the names of some special types of triangles.



Isosceles triangle
Two sides equal



Equilateral triangle
All three sides equal



Right triangle
Has one right angle

An angle is **acute** if its measure is between 0 and 90 degrees.

In an isosceles triangle the angles opposite the equal sides are equal. In an equilateral triangle all three angles are equal to each other. In a right triangle one angle is a right angle and the other two angles are acute.

■ Congruent Triangles

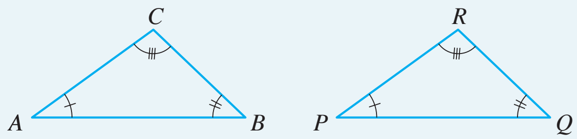
Two geometric figures are congruent if they have the same shape and size. In particular, two line segments are congruent if they have the same length, and two angles are congruent if they have the same measure. For triangles we have the following definition.

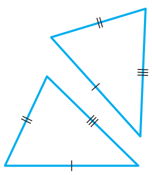
Congruent Triangles

Two triangles are **congruent** if their vertices can be matched up so that corresponding sides and angles are congruent.

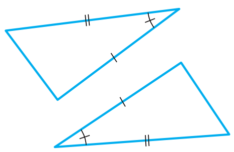
We write $\triangle ABC \cong \triangle PQR$ to mean that triangle ABC is congruent to triangle PQR and that the sides and angles correspond as follows.

$$AB = PQ \qquad \angle A = \angle P$$
$$BC = QR \qquad \angle B = \angle Q$$
$$AC = PR \qquad \angle C = \angle R$$

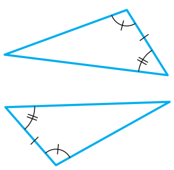




(a) SSS



(b) SAS



(c) ASA

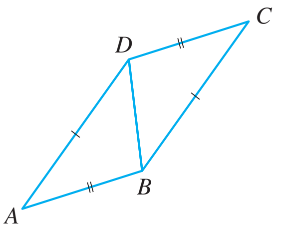
Figure 1

To prove that two triangles are congruent, we don't need to show that all six corresponding parts (side and angles) are congruent. For instance, if all three sides are congruent, then all three angles must also be congruent. You can verify that the following properties lead to congruent triangles.

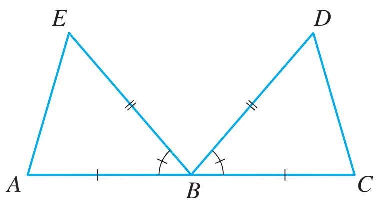
- **Side-Side-Side (SSS).** If each side of one triangle is congruent to the corresponding side of another triangle, then the two triangles are congruent. See Figure 1(a).
- **Side-Angle-Side (SAS).** If two sides and the included angle in one triangle are congruent to the corresponding sides and angle in another triangle, then the two triangles are congruent. See Figure 1(b).
- **Angle-Side-Angle (ASA).** If two angles and the included side in one triangle are congruent to the corresponding angles and side in another triangle, then the triangles are congruent. See Figure 1(c).

Example 1 ■ Congruent Triangles

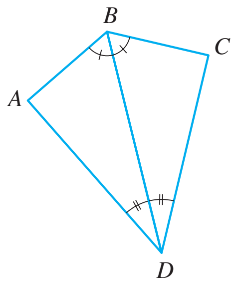
(a) $\triangle ADB \cong \triangle CBD$ by SSS.



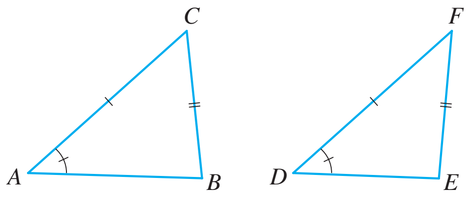
(b) $\triangle ABE \cong \triangle CBD$ by SAS.



(c) $\triangle ABD \cong \triangle CBD$ by ASA.



(d) These triangles are not necessarily congruent. "Side-side-angle" does *not* determine congruence.



■ Similar Triangles

Two geometric figures are similar if they have the same shape, but not necessarily the same size. (See *Discovery Project: Similarity* referenced in Section 6.2.) In the case of triangles we can define similarity as follows.

Similar Triangles

Two triangles are **similar** if their vertices can be matched up so that corresponding angles are congruent. In this case corresponding sides are proportional.

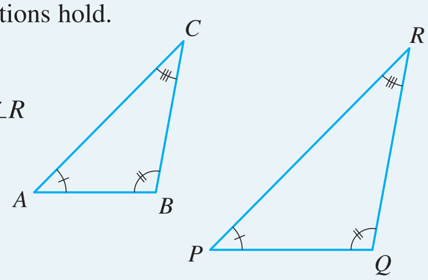
We write $\triangle ABC \sim \triangle PQR$ to mean that triangle ABC is similar to triangle PQR and that the following conditions hold.

The angles correspond as follows:

$$\angle A = \angle P, \quad \angle B = \angle Q, \quad \angle C = \angle R$$

The sides are proportional as follows:

$$\frac{AB}{PQ} = \frac{BC}{QR} = \frac{AC}{PR}$$

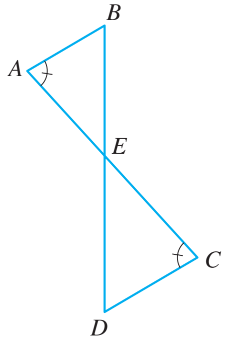


The sum of the angles in any triangle is 180°. [See Example 7(a).] So if we know two angles in a triangle, the third is determined. Thus to prove that two triangles are similar, we need only show that two angles in one triangle are congruent to two angles in the other.

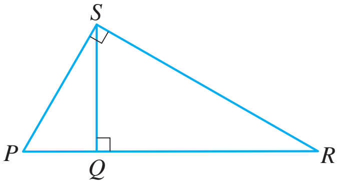
Example 2 ■ Similar Triangles

Find all pairs of similar triangles in the figures.

(a)



(b)



Solution

(a) Since $\angle AEB$ and $\angle CED$ are opposite angles, they are equal. Thus

$$\triangle AEB \sim \triangle CED$$

(b) Since all three triangles in the figure are right triangles, we have

$$\angle QSR + \angle QRS = 90°$$

$$\angle QSR + \angle QSP = 90°$$

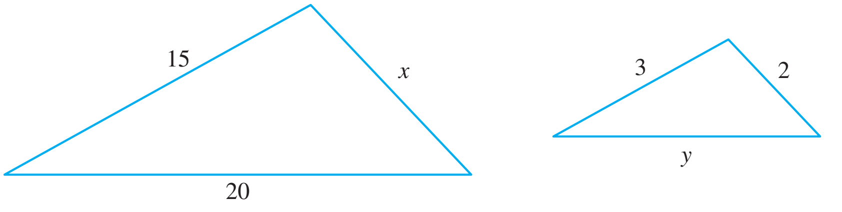
Subtracting these equations, we find that $\angle QSP = \angle QRS$. Thus

$$\triangle PQS \sim \triangle SQR \sim \triangle PSR$$

■

Example 3 ■ Proportional Sides in Similar Triangles

Given that the triangles in the figure are similar, find the lengths x and y.



Solution By similarity, we know that the lengths of corresponding sides in the triangles are proportional. First we find x.

$$\frac{x}{2} = \frac{15}{3} \qquad \text{Corresponding sides are proportional}$$

$$x = \frac{2 \cdot 15}{3} = 10 \qquad \text{Solve for } x$$

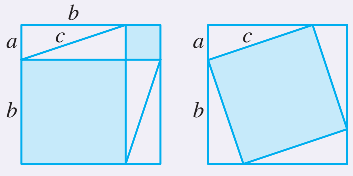
Now we find y.

$$\frac{15}{3} = \frac{20}{y} \qquad \text{Corresponding sides are proportional}$$

$$y = \frac{20 \cdot 3}{15} = 4 \qquad \text{Solve for } y$$

■

BHASKARA (1114–1185 A.D.) was an Indian mathematician, astronomer, and astrologer. Among his many accomplishments was an ingenious proof of the Pythagorean Theorem. He simply drew the two figures shown below and wrote "Behold!" Do you see how the two figures together show that $a^2 + b^2 = c^2$ for the right triangle with sides a, b, and c? Bhaskara's mathematical book *Līlāvatī* (The Beautiful) consists of algebra problems posed in the form of stories to his daughter Lilavati. Many of the problems begin "Oh beautiful maiden, suppose . . . ". The story is told that, using astrology, Bhaskara had determined that great misfortune would befall his daughter if she married at any time other than at a certain hour on a certain day. On her wedding day, as she was anxiously watching the water clock—unbeknownst to her—a pearl fell from her headdress, stopping the flow of water in the clock and causing her to miss the opportune moment for marriage. Bhaskara's *Līlāvatī* was written to console her.



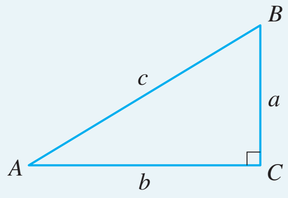
■ The Pythagorean Theorem

In a right triangle the side opposite the right angle is called the **hypotenuse**, and the other two sides are called the **legs**.

The Pythagorean Theorem

In a right triangle the square of the hypotenuse is equal to the sum of the squares of the legs. That is, in $\triangle ABC$ in the figure

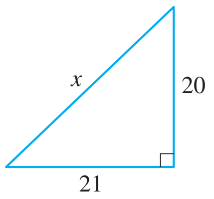
$$a^2 + b^2 = c^2$$



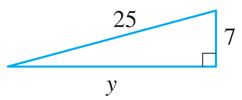
Example 4 ■ Using the Pythagorean Theorem

Find the lengths x and y in the right triangles shown.

(a)



(b)



Solution

(a) We use the Pythagorean Theorem with $a = 20$, $b = 21$, and $c = x$. Then $x^2 = 20^2 + 21^2 = 841$. So $x = \sqrt{841} = 29$.

(b) We use the Pythagorean Theorem with $c = 25$, $a = 7$, and $b = y$. Then $25^2 = 7^2 + y^2$, so $y^2 = 25^2 - 7^2 = 576$. Thus $y = \sqrt{576} = 24$. ■

The converse of the Pythagorean Theorem is also true.

Converse of the Pythagorean Theorem

If the square of one side of a triangle is equal to the sum of the squares of the other two sides, then the triangle is a right triangle.

Example 5 ■ Proving That a Triangle Is a Right Triangle

Prove that the triangle with sides of length 8, 15, and 17 is a right triangle.

Solution You can check that $8^2 + 15^2 = 17^2$. So the triangle must be a right triangle by the converse of the Pythagorean Theorem. ■

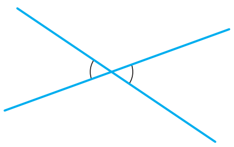


Figure 2 | Opposite angles are equal

■ Parallel Lines

Recall that if two lines intersect, then **opposite angles** (or **vertex angles**) formed by the lines are equal (see Figure 2). Two lines that never intersect are called *parallel*. To determine whether two lines are parallel we first draw a **transversal**—that is, a line that intersects both lines. We identify pairs of angles formed by the lines and the transversal, as shown in Figure 3.

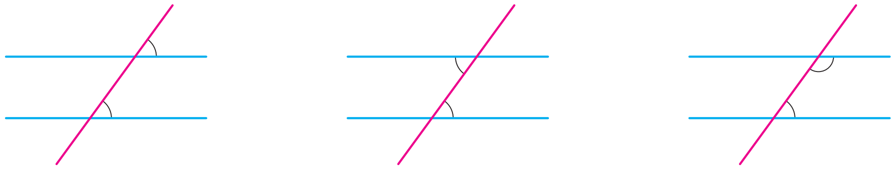
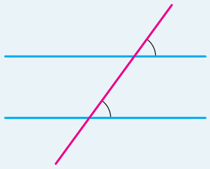


Figure 3 **Corresponding angles** **Alternate interior angles** **Co-interior angles**

Parallel Lines

Two lines in the plane are called **parallel** if they do not intersect. We can determine if two lines are parallel by using the following theorem:

> Two lines are parallel if and only if corresponding angles formed by the lines and a transversal are congruent (equal).



There are several equivalent ways of showing that two lines are parallel. For instance, by showing that alternate interior angles are equal (Example 6), by showing that alternate exterior angles are equal [Exercise 41(a)], or by showing that the sum of co-interior angles is 180° [Exercise 41(b)].

Example 6 ■ Parallel Lines

A transversal intersects two lines. Prove that if alternate interior angles are equal, then the lines are parallel. Also prove the converse: if the lines are parallel, then alternate interior angles are equal.

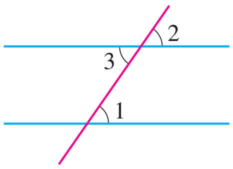


Figure 4

Solution We use the theorem about parallel lines stated in the preceding box. In Figure 4, $\angle 1$ and $\angle 2$ are corresponding angles and $\angle 1$ and $\angle 3$ are alternate interior angles.

($\Rightarrow$) If alternate interior angles are equal, then $\angle 1 = \angle 3$. But $\angle 2 = \angle 3$ because they are opposite angles. It follows that $\angle 1 = \angle 2$. Thus corresponding angles are equal and so the lines are parallel by the preceding theorem.

($\Leftarrow$) Conversely, if the lines are parallel, then by the theorem, corresponding angles are equal; that is, $\angle 1 = \angle 2$. But since $\angle 2 = \angle 3$ (opposite angles), it follows that $\angle 1 = \angle 3$. That is, alternate interior angles are equal. ■

An **exterior angle** of a triangle is an angle between a side of the triangle and an outward extended adjacent side. Part (b) in the next example is called the **Exterior Angle Theorem**.

Example 7 ■ Exterior Angle Theorem

Prove each of the following.

(a) The sum of the angles of a triangle is 180°.

(b) An exterior angle of a triangle is equal to the sum of the two opposite interior angles.

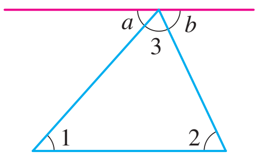


Figure 5

Solution

(a) Let's introduce a line parallel to one side of the triangle and passing through the opposite vertex, as shown in Figure 5. By Example 6, $\angle 1 = \angle a$ (alternate interior angles). Similarly, $\angle 2 = \angle b$. So, the sum of the angles of the triangle is

$$\angle 1 + \angle 2 + \angle 3 = \angle a + \angle b + \angle 3 = 180°$$

because $\angle a$, $\angle 3$, and $\angle b$ together form a straight angle (180° angle).

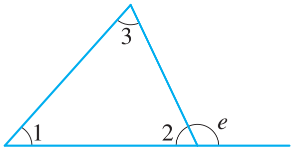


Figure 6

(b) In Figure 6, $\angle e$ is an exterior angle. By part (a), $\angle 1 + \angle 2 + \angle 3 = 180°$. Also, $\angle 2 + \angle e = 180°$ (because they form a straight angle). It follows that

$$\angle 1 + \angle 2 + \angle 3 = \angle e + \angle 2$$

So, $\angle e = \angle 1 + \angle 3$. ■

A **quadrilateral** is a four-sided figure. Certain quadrilaterals have special names, as shown in Figure 7.

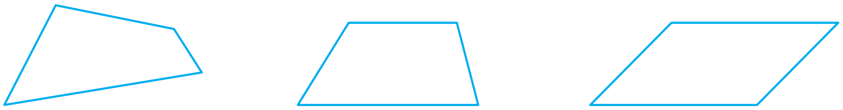


Figure 7

Quadrilateral	**Trapezoid**	**Parallelogram**
Four-sided figure	One pair of parallel sides	Two pairs of parallel sides

Example 8 ■ Opposite Sides of a Parallelogram Are Congruent

Prove that opposite sides of a parallelogram are congruent.

Solution Let $ABCD$ be a parallelogram, as sketched in Figure 8(a). Let's introduce the diagonal AC as shown in Figure 8(b). Now, since opposite sides are parallel and the diagonal is a transversal, it follows that $\angle 1 = \angle 3$ (alternate interior angles) and similarly, $\angle 2 = \angle 4$. So $\triangle ABC \cong \triangle CDA$ by ASA because the side AC is common to both triangles. Thus $AB = CD$ and $DA = BC$.

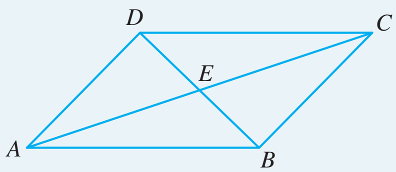
Figure 8 (a) Parallelogram (b) Parallelogram with diagonal

■

Diagonals of a Parallelogram

We can use the following theorem to test whether a quadrilateral is a parallelogram.

A quadrilateral is a parallelogram if and only if the diagonals bisect each other.



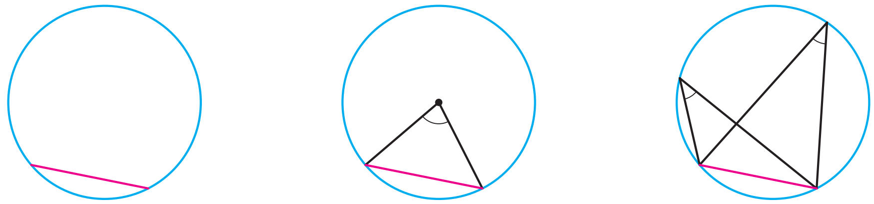
Proof In the figure, the diagonals of quadrilateral $ABCD$ intersect at E.

($\Rightarrow$) If $ABCD$ is a parallelogram, then AB is parallel to DC and the diagonals are transversals. We see that $\triangle AEB \cong \triangle CED$ by ASA because $\angle ABE = \angle CDE$ and $\angle BAE = \angle DCE$ (alternate interior angles), and $AB = CD$ by Example 8. So $AE = CE$ and $BE = DE$ (corresponding sides of congruent triangles). Thus the diagonals bisect each other.

($\Leftarrow$) Conversely, suppose that the diagonals bisect each other. Then $AE = CE$ and $BE = DE$. Also, $\angle AEB = \angle CED$ because they are opposite angles. So $\triangle AEB \cong \triangle CED$ by SAS. But then $\angle ABE = \angle CDE$ and these are alternate interior angles formed by the lines AB and CD and the transversal AC. It follows that AB is parallel to CD. By an analogous argument using triangles AED and CEB we can show that AD is parallel to CB. Thus $ABCD$ is a parallelogram. ■

Circles

A **chord** of a circle is a line segment whose endpoints lie on the circle, as shown in Figure 9(a). The angles shown in Figure 9(b) and 9(c) are said to be **subtended by the chord** (shown in red). The angle in Figure 9(b) is said to be subtended by the chord *at the center of the circle*. The angles in Figure 9(c), are subtended by the chord *on the circle* (or *on the circumference of the circle*).



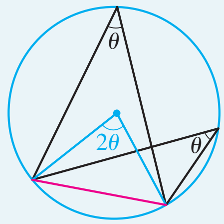
(a) Chord
(b) Angle subtended at center of circle
(c) Angles subtended on the circle

Figure 9

Circles and Chords

The angle at the center of a circle subtended by a chord is twice any angle on the circle subtended by the same chord.

Chords of equal length subtend equal angles.



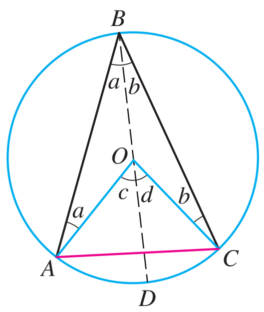


Figure 10

Proof We prove the case illustrated in Figure 10, where the center of the circle is inside $\angle ABC$. The other cases are proved similarly. Let's introduce the diameter BD to the figure. Then triangle AOB is isosceles because two sides are radii of the circle, so the base angles are equal; each is labeled $\angle a$ in the figure. By the Exterior Angle Theorem [Example 7(b)], it follows that $\angle c = \angle a + \angle a = 2\angle a$. Similarly, $\angle d = 2\angle b$. From the figure we see that

$$
\begin{aligned}
\angle AOC &= \angle c + \angle d \\
&= 2\angle a + 2\angle b && \text{Exterior Angle Thorem} \\
&= 2\,(\angle a + \angle b) \\
&= 2\angle ABC
\end{aligned}
$$

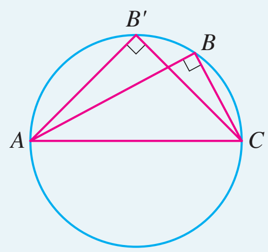
Thus $\angle AOC = 2\angle ABC$ and this completes the proof. ■

When a chord is a diameter of a circle, we get the following important special case of the preceding result. A proof is outlined in Exercise 48.

Triangle Inscribed in a Semicircle

An angle subtended by a diameter of a circle is a right angle. So a triangle inscribed in a semicircle is a right triangle.

In the figure, AC is a diameter of the circle, so angles B and B' are right angles and $\triangle ABC$ and $\triangle AB'C$ are right triangles.



To illustrate the preceding theorems, consider the circle and chords in Figure 11. In the figure, $\angle a$ is subtended by the chord BD on the circle, whereas $\angle b$ is subtended by the same chord BD at the center of the circle. So by the theorem on Circles and Chords, $\angle b = 2\angle a$. Also, both $\angle a$ and $\angle c$ are subtended by the chord BC on the circle, so $\angle a = \angle c$. By the theorem for a Triangle Inscribed in a Semicircle, $\angle BDC$ is a right angle because it is subtended by a diameter of the circle.

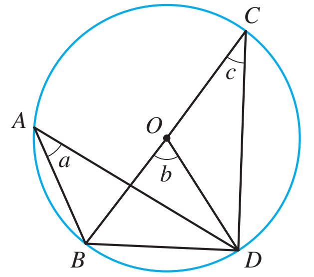


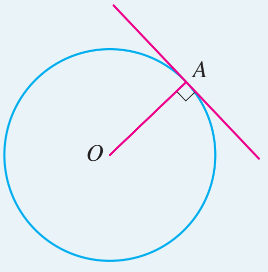
Figure 11

Intuitively, a tangent line to a curve at a given point is a line that just touches (but does not cross) the curve at that point.

Tangent Lines to Circles

A line that intersects a circle at exactly one point is said to be **tangent** to the circle at that point. The following theorem gives a key property of tangents.

> A line is tangent to a circle at a point P if and only if the line is perpendicular to the radius of the circle drawn to the point P.



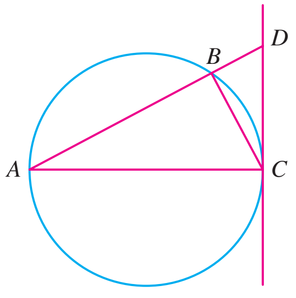


Figure 12

Example 10 ■ Tangent to a Circle

In Figure 12 the line CD is tangent to the circle at C and AC is a diameter of the circle. Show that $\triangle ACD$ is similar to $\triangle ABC$.

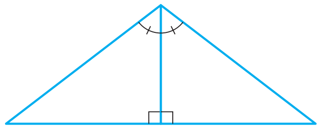
Solution Note that $\angle ABC$ is a right angle because it is subtended by a diameter of the circle. Also, $\angle ACD$ is a right angle because the line CD is tangent to the circle. Both triangles share $\angle BAC$. So two angles in $\triangle ACD$ are congruent to two angles in $\triangle ABC$. It follows that the triangles are similar.

■

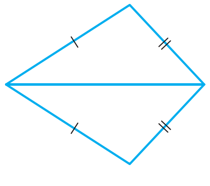
Appendix A Exercises

1–4 ■ Congruent Triangles? Determine whether the pair of triangles is congruent. If so, state the congruence principle you are using.

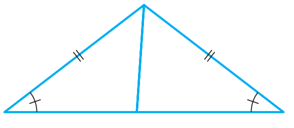
1.



2.



3.



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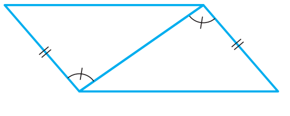
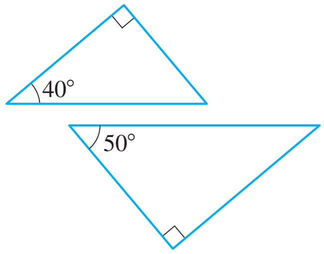


5–8 ■ Similar Triangles? Determine whether the pair of triangles is similar.

5.



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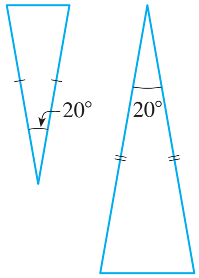

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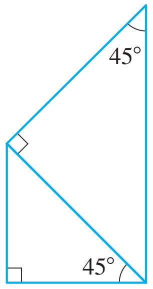

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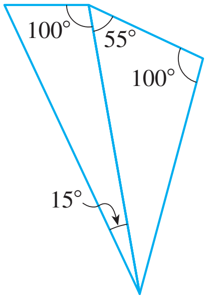
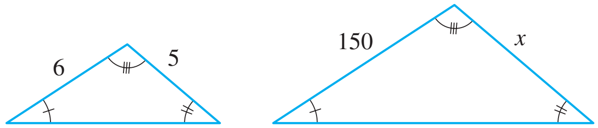


9–12 ■ Similar Triangles Given that the pair of triangles is similar, find the length(s) x and/or y.

9.



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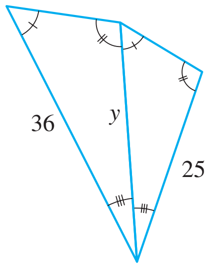

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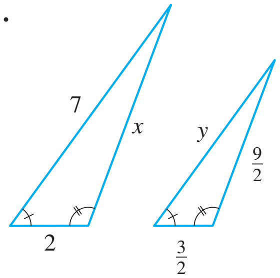

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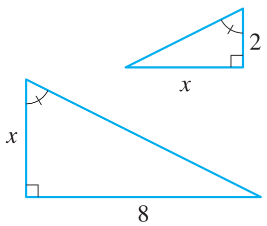
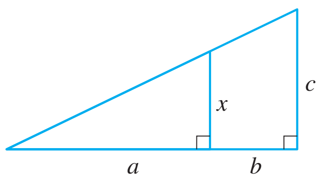


13–14 ■ Using Similarity Express x in terms of a, b, and c.

13.



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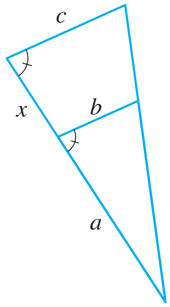
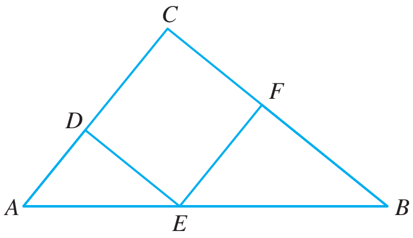


15. Proving Similarity In the figure $CDEF$ is a rectangle. Prove that $\triangle ABC \sim \triangle AED \sim \triangle EBF$.



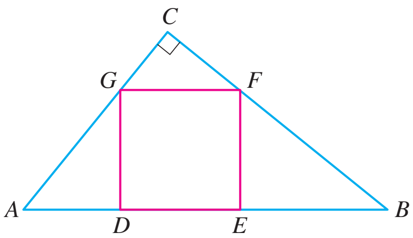
16. Proving Similarity In the figure $DEFG$ is a square. Prove the following:

(a) $\triangle ADG \sim \triangle GCF$

(b) $\triangle ADG \sim \triangle FEB$

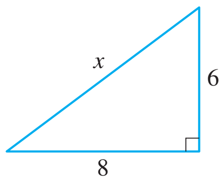
(c) $AD \cdot EB = DG \cdot FE$

(d) $DE = \sqrt{AD \cdot EB}$



17–22 ■ Pythagorean Theorem In the given right triangle, find the side labeled x.

17.



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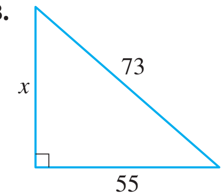

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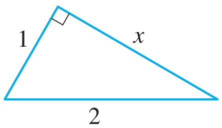

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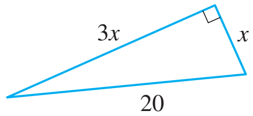

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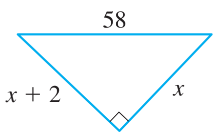

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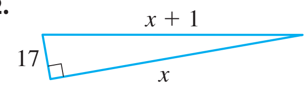


23–28 ■ Right Triangle? The lengths of the sides of a triangle are given. Determine whether the triangle is a right triangle.

23. 5, 12, 13

24. 15, 20, 25

25. 8, 10, 12

26. 6, 17, 18

27. 48, 55, 73

28. 13, 84, 85

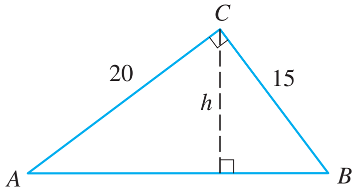
29–32 ■ Pythagorean Theorem These exercises require the use of the Pythagorean Theorem.

29. One leg of a right triangle measures 11 cm. The hypotenuse is 1 cm longer than the other leg. Find the length of the hypotenuse.

30. The length of a rectangle is 1 ft greater than its width. Each diagonal is 169 ft long. Find the dimensions of the rectangle.

31. Each of the diagonals of a quadrilateral is 27 cm long. Two adjacent sides measure 17 cm and 21 cm. Is the quadrilateral a rectangle?

32. Find the height h of the right triangle ABC shown in the figure. [*Hint:* Find the area of triangle ABC in two different ways.]



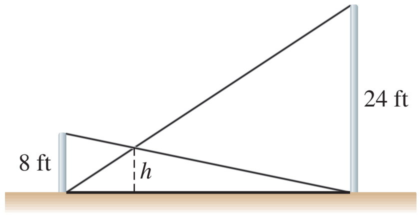
33. Pythagorean Triples If a, b, c are positive integers such that $a^2 + b^2 = c^2$, then (a, b, c) is called a **Pythagorean triple**.

(a) Let m and n be positive integers with $m > n$. Let $a = m^2 - n^2$, $b = 2mn$, and $c = m^2 + n^2$. Show that (a, b, c) is a Pythagorean triple.

(b) Use part (a) to find the rest of the Pythagorean triples given in the table.

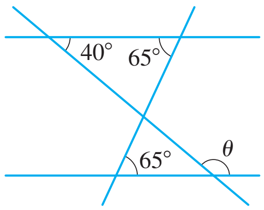
m	n	(a, b, c)
2	1	(3, 4, 5)
3	1	(8, 6, 10)
3	2	
4	1	
4	2	
4	3	
5	1	
5	2	
5	3	
5	4	

34. Finding a Length Two vertical poles, one 8 ft tall and the other 24 ft tall, have ropes stretched from the top of each to the base of the other (see the figure). How high above the ground is the point where the ropes cross? [*Hint:* Use similarity.]



35–40 ■ Angle Measure Find the measure of the angle labeled θ. Give reasons for your answer.

35.



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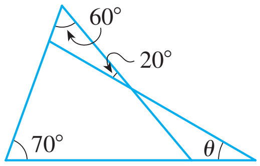

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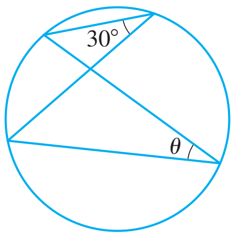

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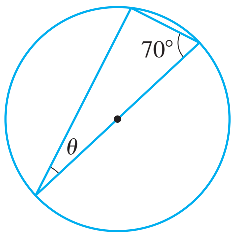

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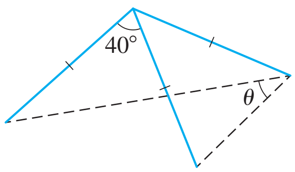

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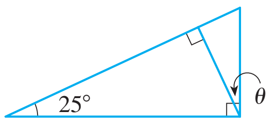
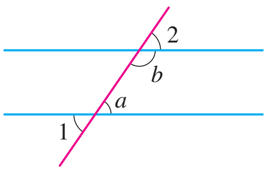


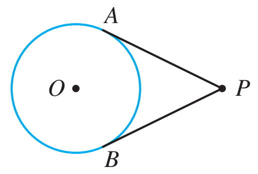
41. Showing Two Lines Are Parallel In the figure, $\angle 1$ and $\angle 2$ are called *alternate exterior angles*. Recall that $\angle a$ and $\angle b$ are co-interior angles. Prove the following.

(a) Two lines are parallel if and only if alternate exterior angles are equal.

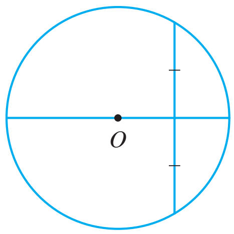
(b) Two lines are parallel if and only if the sum of co-interior angles is 180°.



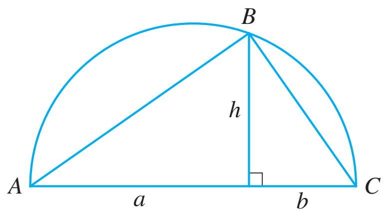
42. Tangents to a Circle Prove that tangents to a circle from a point P outside the circle have the same length. [*Hint*: Draw radii of the circle to the points of tangency and draw the line segment OP.]



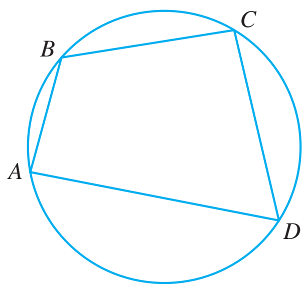
43. Diameter and Chord Prove that if a diameter of a circle bisects a chord, then it is perpendicular to the chord.



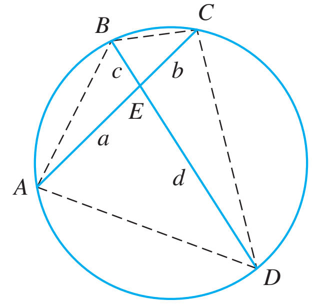
44. Altitude In the figure, triangle ABC is inscribed in a semicircle. Show that the altitude is $h = \sqrt{ab}$.



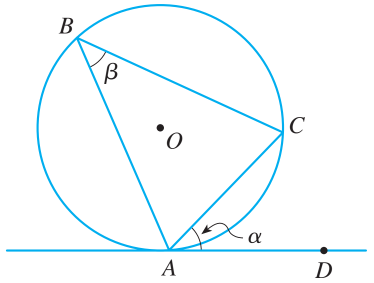
45. Cyclic Quadrilateral A quadrilateral is called *cyclic* if it can be inscribed in a circle (see the figure). Prove that if a quadrilateral is cyclic, then the sum of each pair of opposite angles is 180°. [*Hint*: Draw the diagonals and use the fact that the sum of the angles of a quadrilateral is 360°.]



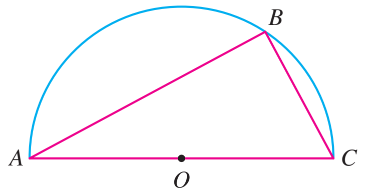
46. Chords Theorem The figure shows a circle and two intersecting chords. Show that $ab = cd$. [*Hint*: First show that $\triangle ABE \sim \triangle CDE$.]



47. Tangent and Chord In the figure, the line is tangent to the circle at A. Prove that $\angle\alpha = \angle\beta$. [*Hint*: Add radii of the circle to points A and C.]



48. Triangle Inscribed in a Semicircle In the figure, $\triangle ABC$ is inscribed in a semicircle. Prove that $\angle ABC$ is a right angle. [*Hint:* Apply the Exterior Angle Theorem to the isosceles triangles formed by introducing the radius OB.]



The following appendixes can be found at **www.stewartmath.com.**

APPENDIX B: *Calculations and Significant Figures*

APPENDIX C: *Graphing with a Graphing Calculator*

APPENDIX D: *Using the TI-83/84 Graphing Calculator*

APPENDIX E: *Three-Dimensional Coordinate Geometry*

APPENDIX F: *Mathematics of Finance*

APPENDIX G: *Probability and Statistics*