Graiki/Moment/Getty Images

2 Functions

A *function* is a rule that describes how one quantity depends on another. Many real-world situations follow precise rules, so they can be modeled by functions. For example, there is a rule that relates the distance skydivers fall to the time they have been falling. So the distance traveled by a skydiver is a *function* of time. Knowing this function allows skydivers to determine when to open their parachutes. In this chapter we study functions and their graphs, as well as many real-world applications of functions. In the *Focus on Modeling* at the end of the chapter we explore different real-world situations that can be modeled by functions.

2.1 Functions

■ Functions All Around Us ■ Definition of Function ■ Evaluating a Function
■ The Domain of a Function ■ Four Ways to Represent a Function

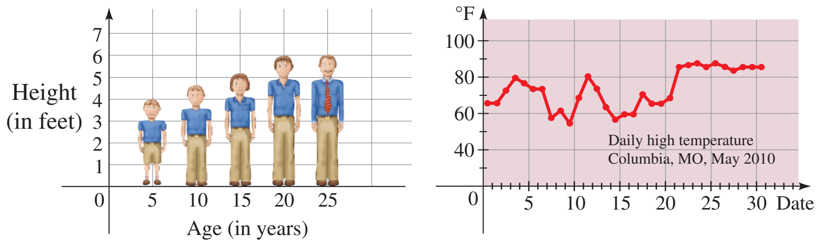
In this section we introduce the concept of a *function* and explore four different ways of describing a function—verbally, numerically, graphically, and algebraically.

■ Functions All Around Us

In nearly every physical phenomenon we observe that one quantity depends on another. For example, your height depends on your age, the temperature depends on the date, the cost of mailing a package depends on its weight (see Figure 1). We use the term *function* to describe this dependence of one quantity on another. That is, we say the following:

- Height is a function of age.
- Temperature is a function of date.
- Cost of mailing a package is a function of weight.

The US Post Office uses a simple rule to determine the cost of mailing a first-class parcel on the basis of its weight. But it's not so easy to describe the rule that relates height to age or the rule that relates temperature to date.



w (ounces)	2022 Postage (dollars)
$0 < w \le 1$	1.20
$1 < w \le 2$	1.44
$2 < w \le 3$	1.68
$3 < w \le 4$	1.92
$4 < w \le 5$	2.16
$5 < w \le 6$	2.40

Figure 1 Height is a function of age. Temperature is a function of date. Postage is a function of weight.

Can you think of other functions? Here are some more examples:

- The area of a circle is a function of its radius.
- The number of bacteria in a culture is a function of time.
- The weight of an astronaut is a function of elevation.
- The price of a commodity is a function of the demand for that commodity.

The rule that describes how the area A of a circle depends on its radius r is given by the formula $A = \pi r^2$. Even when a precise rule or formula describing a function is not available, we can still describe the function by a graph, as in the next example.

Example 1 ■ Describing a Function by a Graph

When you turn on a hot-water faucet that is connected to a hot-water tank, the temperature T of the water depends on how long the water has been running. So we can say:

- The temperature of water from the faucet is a function of time.

We can sketch a rough graph of the temperature T of the water as a function of the time t that has elapsed since the faucet was turned on, as shown in Figure 2. The graph shows that the initial temperature of the water is close to room temperature. When the water from the hot-water tank reaches the faucet, the water's temperature T increases quickly.

In the next phase, T is constant at the temperature of the water in the tank. When the tank is drained, T decreases to the temperature of the cold-water supply.

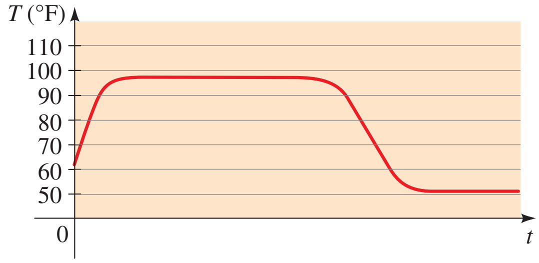


Figure 2 | Graph of water temperature T as a function of time t

Now Try Exercise 95 ■

■ Definition of Function

We have previously used letters to stand for numbers. Here we do something quite different: We use letters to represent *rules*.

A function is a rule. To talk about a function, we need to give it a name. We will use letters such as $f, g, h, \ldots$ to represent functions. For example, we can use the letter f to represent a rule as follows:

$$\text{“}f\text{”} \quad \text{is the rule} \quad \text{“square the number”}$$

When we write $f(2)$, we mean "apply the rule f to the number 2." Applying the rule gives $f(2) = 2^2 = 4$. Similarly, $f(3) = 3^2 = 9$, $f(4) = 4^2 = 16$, and in general $f(x) = x^2$.

Definition of a Function

A **function** f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B.

The square root key $\boxed{\sqrt{\ }}$ on your calculator is a good example of a function as a machine. First you input x into the display. Then you press the key labeled $\boxed{\sqrt{\ }}$. If $x < 0$, then x is not in the domain of this function; that is, x is not an acceptable input, and the calculator will indicate an error. If $x \geq 0$, then an *approximation* to $\sqrt{x}$ appears in the display, correct to a certain number of decimal places. (Thus the $\boxed{\sqrt{\ }}$ key on your calculator is not quite the same as the exact mathematical function f defined by $f(x) = \sqrt{x}$.)

We usually consider functions for which the sets A and B are sets of real numbers. The symbol $f(x)$ is read "f of x" or "f at x" and is called the **value of f at x**, or the **image of x under f**. The set A is called the **domain** of the function. The **range** of f is the subset of B that consists of all possible values of $f(x)$ as x varies throughout the domain, that is,

$$\text{range of } f = \{f(x) \mid x \in A\}$$

The symbol that represents an arbitrary number in the domain of a function f is called an **independent variable**. The symbol that represents a number in the range of f is called a **dependent variable**. So if we write $y = f(x)$, then x is the independent variable and y is the dependent variable.

It is helpful to think of a function as a **machine** (see Figure 3). If x is in the domain of the function f, then when x enters the machine, it is accepted as an **input** and the machine produces an **output** $f(x)$ according to the rule of the function. Thus we can think of the domain as the set of all possible inputs and the range as the set of all possible outputs.

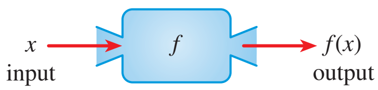


Figure 3 | Machine diagram of f

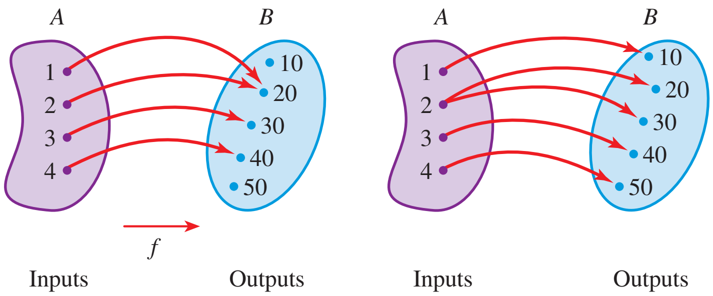
Function Input Output

$$f(x) = y$$

Note The notation $y = f(x)$ is called **function notation**. The letter f is the name of the function (or rule), x is the input, and y is the corresponding output. See the figure in the margin.

Another way to picture a function f is by an **arrow diagram** as in Figure 4(a). Each arrow associates an input from A to the corresponding output in B. Since a function associates *exactly* one output to each input, the diagram in Figure 4(a) represents a function but the diagram in Figure 4(b) does *not* represent a function.

The correspondence illustrated in the diagram in part (b) is not a function, but it is a *relation*. Relations are defined in Section 2.2.



(a) Function (b) Not a function

Figure 4 | Arrow diagrams

Example 2 ■ Analyzing a Function

A function f is defined by the formula

$$f(x) = x^2 + 4$$

(a) Express in words how f acts on the input x to produce the output $f(x)$.

(b) Evaluate $f(3)$, $f(-2)$, and $f(\sqrt{5})$.

(c) Find the domain and range of f.

(d) Draw a machine diagram for f.

Solution

(a) The formula tells us that f first squares the input x and then adds 4 to the result. So f is the function

"square, then add 4"

(b) The values of f are found by substituting for x in the formula $f(x) = x^2 + 4$.

$$f(3) = 3^2 + 4 = 13 \qquad \text{Replace } x \text{ by } 3$$

$$f(-2) = (-2)^2 + 4 = 8 \qquad \text{Replace } x \text{ by } -2$$

$$f(\sqrt{5}) = (\sqrt{5})^2 + 4 = 9 \qquad \text{Replace } x \text{ by } \sqrt{5}$$

(c) The domain of f consists of all possible inputs for f. Since we can evaluate the formula $f(x) = x^2 + 4$ for every real number x, the domain of f is the set $\mathbb{R}$ of all real numbers.

The range of f consists of all possible outputs of f. Because $x^2 \geq 0$ for all real numbers x, we have $x^2 + 4 \geq 4$, so for every output of f we have $f(x) \geq 4$. Thus the range of f is $\{y \mid y \geq 4\} = [4, \infty)$.

(d) A machine diagram for f is shown in Figure 5.

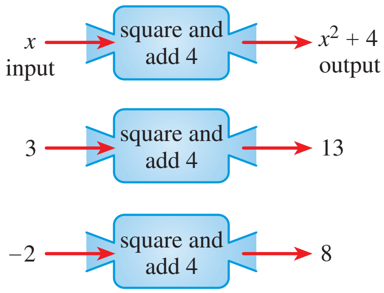


Figure 5 | Machine diagrams

Now Try Exercises 13, 17, 21, and 53 ■

■ Evaluating a Function

In the definition of a function the independent variable x plays the role of a placeholder. For example, the function $f(x) = 3x^2 + x - 5$ can be thought of as

$$f(\square) = 3 \cdot \square^2 + \square - 5$$

To evaluate f at a number, we substitute the number for the placeholder.

Example 3 ■ Evaluating a Function

Let $f(x) = 3x^2 + x - 5$. Evaluate each function value.

(a) $f(-2)$ **(b)** $f(0)$ **(c)** $f(4)$ **(d)** $f(\frac{1}{2})$

Solution To evaluate f at a number, we substitute the number for x in the definition of f.

(a) $f(-2) = 3 \cdot (-2)^2 + (-2) - 5 = 5$

(b) $f(0) = 3 \cdot 0^2 + 0 - 5 = -5$

(c) $f(4) = 3 \cdot (4)^2 + 4 - 5 = 47$

(d) $f(\frac{1}{2}) = 3 \cdot (\frac{1}{2})^2 + \frac{1}{2} - 5 = -\frac{15}{4}$

Now Try Exercise 23 ■

A **piecewise-defined function** (or a **piecewise function**) is a function that is defined by different formulas on different parts of its domain, as in the next example.

Example 4 ■ A Piecewise-Defined Function

A cell phone plan costs \$39 a month. The plan includes 5 gigabytes (GB) of free high-speed data and charges \$15 per gigabyte for any additional high-speed data used. The monthly charges are a function of the number of gigabytes of data used, given by

$$C(x) = \begin{cases} 39 & \text{if } 0 \le x \le 5 \\ 39 + 15(x - 5) & \text{if } x > 5 \end{cases}$$

Find $C(0.5)$, $C(5)$, and $C(8)$.

Solution Remember that a function is a rule. Here is how we apply the rule for this function. First we look at the value of the input, x. If $0 \le x \le 5$, then the value of $C(x)$ is 39. On the other hand, if $x > 5$, then the value of $C(x)$ is $39 + 15(x - 5)$.

Since $0.5 \le 5$, we have $C(0.5) = 39$.

Since $5 \le 5$, we have $C(5) = 39$.

Since $8 > 5$, we have $C(8) = 39 + 15(8 - 5) = 84$.

Thus the plan charges \$39 for 0.5 GB, \$39 for 5 GB, and \$84 for 8 GB.

Now Try Exercises 33 and 91 ■

From Examples 3 and 4 we see that the values of a function can change from one input to another. The **net change** in the value of a function f as the input changes from a to b (where $a \le b$) is given by

$$\boxed{f(b) - f(a)}$$

The next example illustrates this concept.

Example 5 ■ Finding Net Change

You can check that the values of the function in Example 5 decrease and then increase between -2 and 2, but the net change from -2 to 2 is 0 because $f(-2)$ and $f(2)$ have the same value.

Let $f(x) = x^2$. Find the net change in the value of f between the given inputs.

(a) From 1 to 3 **(b)** From -2 to 2

Solution

(a) The net change is $f(3) - f(1) = 9 - 1 = 8$.

(b) The net change is $f(2) - f(-2) = 4 - 4 = 0$.

Now Try Exercise 41 ■

Example 6 ■ Evaluating a Function

If $f(x) = 2x^2 + 3x - 1$, evaluate the following.

(a) $f(a)$ **(b)** $f(-a)$ **(c)** $f(a + h)$ **(d)** $\dfrac{f(a + h) - f(a)}{h}$, $h \neq 0$

Expressions like the one in part (d) of Example 6 occur frequently in calculus; they are called *difference quotients*, and they represent the average change in the value of f between $x = a$ and $x = a + h$. (See Section 2.4.)

Solution

(a) $f(a) = 2a^2 + 3a - 1$

(b) $f(-a) = 2(-a)^2 + 3(-a) - 1 = 2a^2 - 3a - 1$

(c)
$$\begin{aligned} f(a + h) &= 2(a + h)^2 + 3(a + h) - 1 \\ &= 2(a^2 + 2ah + h^2) + 3(a + h) - 1 \\ &= 2a^2 + 4ah + 2h^2 + 3a + 3h - 1 \end{aligned}$$

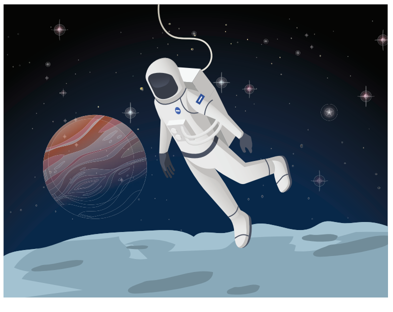
(d) Using the results from parts (c) and (a), we have

$$\begin{aligned} \frac{f(a + h) - f(a)}{h} &= \frac{(2a^2 + 4ah + 2h^2 + 3a + 3h - 1) - (2a^2 + 3a - 1)}{h} \\ &= \frac{4ah + 2h^2 + 3h}{h} = 4a + 2h + 3 \end{aligned}$$

Now Try Exercise 45 ■

A **table of values** for a function is a table with two headings, one for inputs and one for the corresponding outputs. A table of values helps us to analyze a function numerically, as shown in the next example.

Example 7 ■ The Weight of an Astronaut



If an astronaut weighs 130 lb on the earth, then the astronaut's weight h miles above the surface of the earth is given by the function

$$w(h) = 130\left(\frac{3960}{3960 + h}\right)^2$$

(a) What is the astronaut's weight 100 mi above the earth?

(b) Construct a table of values for the function w that gives the astronaut's weight at heights from 0 to 500 mi. What do you conclude from the table?

(c) Find the net change in the astronaut's weight from ground level to a height of 500 mi.

The weight of an object on or near the earth is the gravitational force that the earth exerts on it. When in orbit around the earth, an astronaut experiences the sensation of "weightlessness" because the centripetal force that keeps the astronaut in orbit is exactly the same as the gravitational pull of the earth.

Solution

(a) We want the value of the function w when $h = 100$; that is, we must calculate $w(100)$:

$$w(100) = 130\left(\frac{3960}{3960 + 100}\right)^2 \approx 123.67$$

So at a height of 100 mi the astronaut weighs about 124 lb.

(b) The table gives the astronaut's weight, rounded to the nearest pound, at 100-mile increments. The values in the table are calculated like the one in part (a).

h	$w(h)$
0	130
100	124
200	118
300	112
400	107
500	102

The table indicates that the astronaut's weight decreases as the height above the surface of the earth increases.

(c) The net change in the astronaut's weight from $h = 0$ to $h = 500$ is

$$w(500) - w(0) = 102 - 130 = -28$$

The negative sign indicates that the astronaut's weight *decreased* by about 28 lb.

Now Try Exercise 83 ■

The Domain of a Function

Recall that the *domain* of a function is the set of all inputs for the function. The domain of a function may be stated explicitly. For example, if we write

$$f(x) = x^2 \qquad 0 \le x \le 5$$

then the domain is the set of all real numbers x for which $0 \le x \le 5$. If the function is given by an algebraic expression and the domain is not stated explicitly, then by convention *the domain of the function is the domain of the algebraic expression—that is, the set of all real numbers for which the expression is defined as a real number*. For example, consider the functions

Domains of algebraic expressions are discussed in Section 1.4.

$$f(x) = \frac{1}{x - 4} \qquad g(x) = \sqrt{x}$$

The function f is not defined at $x = 4$, so its domain is $\{x \mid x \ne 4\}$. The function g is not defined for negative x, so its domain is $\{x \mid x \ge 0\}$.

Example 8 ■ Finding Domains of Functions

Find the domain of each function.

(a) $f(x) = \dfrac{1}{x^2 - x}$ **(b)** $g(x) = \sqrt{9 - x^2}$ **(c)** $h(t) = \dfrac{t}{\sqrt{t + 1}}$

Solution

(a) A rational expression is not defined when the denominator is 0. Since

$$f(x) = \frac{1}{x^2 - x} = \frac{1}{x(x - 1)}$$

we see that $f(x)$ is not defined when $x = 0$ or $x = 1$. Thus the domain of f is

$$\{x \mid x \ne 0, x \ne 1\}$$

The domain may also be written in interval notation as

$$(-\infty, 0) \cup (0, 1) \cup (1, \infty)$$

(b) We can't take the square root of a negative number, so we must have $9 - x^2 \ge 0$. Using the methods of Section 1.8, we can solve this inequality to find that $-3 \le x \le 3$. Thus the domain of g is

$$\{x \mid -3 \le x \le 3\} = [-3, 3]$$

(c) We can't take the square root of a negative number, and we can't divide by 0, so we must have $t + 1 > 0$, that is, $t > -1$. So the domain of h is

$$\{t \mid t > -1\} = (-1, \infty)$$

Now Try Exercises 59, 67, and 73 ■

Four Ways to Represent a Function

We have used machine and arrow diagrams to help us understand what a function is. We can describe a specific function in the following four ways:

- verbally (by a description in words)
- algebraically (by an explicit formula)
- visually (by a graph)
- numerically (by a table of values)

A single function may be represented in all four ways, and it is often useful to go from one representation to another to gain insight into the function. However, certain functions are described more naturally by one method than by the others. We summarize all four methods in the following box.

Four Ways to Represent a Function

Verbal Using words:

"To convert from Celsius to Fahrenheit, multiply the Celsius temperature by $\frac{9}{5}$, then add 32."

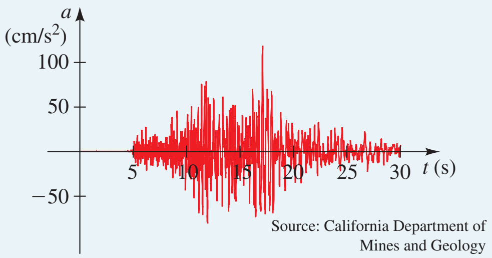
Relation between Celsius and Fahrenheit temperature scales

Algebraic Using a formula:

$$A(r) = \pi r^2$$

Area of a circle

Visual Using a graph:



Vertical acceleration during an earthquake

Numerical Using a table of values:

w (ounces)	$C(w)$ (dollars)
$0 < w \le 1$	\$1.20
$1 < w \le 2$	\$1.44
$2 < w \le 3$	\$1.68
$3 < w \le 4$	\$1.92
$4 < w \le 5$	\$2.16
$\vdots$	$\vdots$

Cost of mailing a large first-class envelope

Example 9 ■ Representing a Function Verbally, Algebraically, Numerically, and Graphically

Let $F(C)$ be the Fahrenheit temperature corresponding to the Celsius temperature C. (Thus F is the function that converts Celsius inputs to Fahrenheit outputs.) This function is described verbally by "multiply the Celsius temperature by $\frac{9}{5}$, then add 32". Find ways to represent this function in the other three ways:

(a) Algebraically (using a formula)

(b) Numerically (using a table of values)

(c) Visually (using a graph)

Solution

For the function F, the inputs are the temperatures in Celsius and the outputs are the corresponding temperatures in Fahrenheit.

(a) The verbal description tells us that we should first multiply the input C by $\frac{9}{5}$ and then add 32 to the result. So we get

$$F(C) = \tfrac{9}{5}C + 32$$

(b) We use the algebraic formula for F that we found in part (a) to construct a table of values:

C (Celsius)	F (Fahrenheit)	(C, F)
-10	14	$(-10, 14)$
0	32	$(0, 32)$
10	50	$(10, 50)$
20	68	$(20, 68)$
30	86	$(30, 86)$
40	104	$(40, 104)$

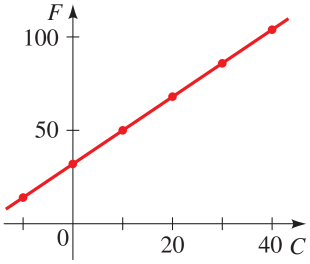


Figure 6 | Celsius and Fahrenheit

(c) We use the points (ordered pairs) tabulated in part (b) to help us draw the graph of this function in Figure 6.

Now Try Exercise 77 ■

2.1 Exercises

Concepts

1. If $f(x) = x^3 + 1$, then

(a) the value of f at $x = -1$ is $f(\_\_\_) = \_\_\_\_\_$.

(b) the value of f at $x = 2$ is $f(\_\_\_) = \_\_\_\_\_$.

(c) the net change in the value of f between $x = -1$ and $x = 2$ is $f(\_\_\_) - f(\_\_\_) = \_\_\_\_\_$.

2. For a function f, the set of all possible inputs is called the _____ of f, and the set of all possible outputs is called the _____ of f.

3. (a) Which of the following functions have 5 in their domain?

$$f(x) = x^2 - 3x \qquad g(x) = \frac{x-5}{x} \qquad h(x) = \sqrt{x - 10}$$

(b) For the functions from part (a) that *do* have 5 in their domain, find the value of the function at 5.

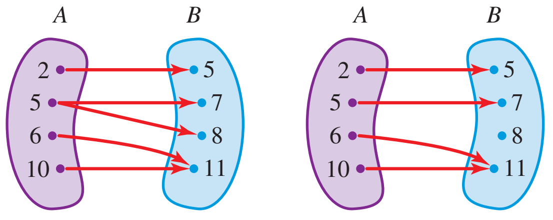
4. A function is given algebraically by the formula $f(x) = (x - 4)^2 + 3$. Complete these other ways to represent f:

(a) *Verbal:* "Subtract 4, then _____ and _____."

(b) *Numerical:*

x	$f(x)$
0	19
2	
4	
6	

5. A function f is a rule that assigns to each element x in a set A exactly _____ element(s) called $f(x)$ in a set B. Which of the following diagrams represents a function?



(a) (b)

6. A function f is given by a table of values.

(a) From the table, $f(-1) = \_\_\_\_\_$ and $f(2) = \_\_\_\_\_$

(b) Can a function have the same output for two different inputs?

x	-2	-1	0	1	2	3
$f(x)$	5	4	-3	2	4	0

7–8 ■ *Yes or No*? If *No*, give a reason. Let f be a function.

7. Is it possible that $f(1) = 5$ and $f(2) = 5$?

8. Is it possible that $f(1) = 5$ and $f(1) = 6$?

Skills

9–12 ■ **Function Notation** Express the rule in function notation. (For example, the rule "square, then subtract 5" is expressed as the function $f(x) = x^2 - 5$.)

9. Multiply by 3, then subtract 5

10. Add 2, then multiply by 5

11. Square, add 1, then take the square root

12. Add 1, take the square root, then divide by 6

13–16 ■ Functions in Words Express the function (or rule) in words.

13. $f(x) = 5x + 1$ **14.** $g(x) = 4(x^2 - 2)$

15. $h(x) = \dfrac{\sqrt{x} - 4}{3}$ **16.** $k(x) = \dfrac{\sqrt{x^2 + 9}}{2}$

17–18 ■ Machine Diagram Draw a machine diagram for the function.

17. $f(x) = \sqrt{x - 1}$ **18.** $f(x) = \dfrac{3}{x - 2}$

19–20 ■ Table of Values Complete the table.

19. $f(x) = 2(x - 1)^2$

x	$f(x)$
-1	
0	
1	
2	
3	

20. $g(x) = |2x + 3|$

x	$g(x)$
-3	
-2	
0	
1	
3	

21–32 ■ Evaluating Functions Evaluate the function at the indicated values.

21. $f(x) = 3x^2 + 1$; $f(-2), f(2), f(0), f(\frac{1}{3}), f(\sqrt{5})$

22. $f(x) = 4x - x^3$; $f(-2), f(0), f(2), f(1), f(\frac{1}{2})$

23. $g(x) = \dfrac{1 - x}{5}$;

$g(-2), g(0), g(2), g(-a), g(x^2), g(a - 2)$

24. $h(x) = \dfrac{\sqrt{x + 3}}{2}$;

$h(-1), h(0), h(1), h(a), h(x - 2), h(a^2 - 2)$

25. $f(x) = x^2 + 2x$;

$f(0), f(3), f(-3), f(a), f(-x), f\left(\dfrac{1}{a}\right)$

26. $h(t) = t + \dfrac{1}{t}$;

$h(-1), h(2), h(\frac{1}{2}), h(x - 1), h\left(\dfrac{1}{x}\right)$

27. $g(x) = \dfrac{1 - x}{1 + x}$;

$g(2), g(-1), g(\frac{1}{2}), g(a), g(a - 1), g(x^2 - 1)$

28. $g(t) = \dfrac{t + 2}{t - 2}$;

$g(-2), g(2), g(0), g(a), g(a^2 - 2), g(a + 1)$

29. $k(x) = 3x^2 - x + 1$;

$k(-1), k(0), k(2), k(\sqrt{5}), k(a - 1), k(x^2)$

30. $k(x) = x^4 - x^3$;

$k(-2), k(-1), k(1), k\left(\dfrac{a}{3}\right), k(a^2), k\left(\dfrac{1}{t}\right)$

31. $f(x) = 2|x - 1|$;

$f(-2), f(0), f(\frac{1}{2}), f(2), f(x + 1), f(x^2 + 2)$

32. $f(x) = \dfrac{|x|}{x}$;

$f(-2), f(-1), f(0), f(5), f(x^2), f\left(\dfrac{1}{x}\right)$

33–36 ■ Piecewise-Defined Functions Evaluate the piecewise defined function at the indicated values.

33. $f(x) = \begin{cases} 3x + 1 & \text{if } x < 5 \\ x^2 - 1 & \text{if } x \geq 5 \end{cases}$

$f(-5), f(0), f(\frac{1}{3}), f(5), f(6)$

34. $f(x) = \begin{cases} 5 & \text{if } x \leq 2 \\ 2x - 3 & \text{if } x > 2 \end{cases}$

$f(-3), f(0), f(2), f(3), f(5)$

35. $f(x) = \begin{cases} x^2 + 2x & \text{if } x \leq -1 \\ x & \text{if } -1 < x \leq 1 \\ -1 & \text{if } x > 1 \end{cases}$

$f(-4), f(-\frac{3}{2}), f(-1), f(0), f(25)$

36. $f(x) = \begin{cases} 3x & \text{if } x < 0 \\ x + 1 & \text{if } 0 \leq x \leq 2 \\ (x - 2)^2 & \text{if } x > 2 \end{cases}$

$f(-5), f(0), f(1), f(2), f(5)$

37–40 ■ Evaluating Functions Use the function to evaluate the indicated expressions and simplify.

37. $f(x) = x^2 + 1$; $f(x + 2), f(x) + f(2)$

38. $f(x) = 3x - 1$; $f(2x), 2f(x)$

39. $f(x) = x + 4$; $f(x^2), (f(x))^2$

40. $f(x) = 6x - 18$; $f\left(\dfrac{x}{3}\right), \dfrac{f(x)}{3}$

41–44 ■ Net Change Find the net change in the value of the function between the given inputs.

41. $f(x) = 3x - 2$; from 1 to 5

42. $f(x) = 4 - 5x$; from 3 to 5

43. $g(t) = 1 - t^2$; from -2 to 5

44. $h(t) = t^2 + 5$; from -3 to 6

45–52 ■ Difference Quotient Find $f(a)$, $f(a + h)$, and the difference quotient $\dfrac{f(a + h) - f(a)}{h}$, where $h \neq 0$.

45. $f(x) = 3 - x$ **46.** $f(x) = x^2 - 4x$

47. $f(x) = 5$ **48.** $f(x) = \dfrac{1}{x + 1}$

49. $f(x) = \dfrac{x}{x + 1}$ **50.** $f(x) = \dfrac{x - 1}{x}$

51. $f(x) = 3 - 5x + 4x^2$ **52.** $f(x) = x^3$

53–58 ■ Domain and Range Find the domain and range of the function.

53. $f(x) = 3x$

54. $f(x) = 5x^2 + 4$

55. $f(x) = |x| + 3$

56. $f(x) = 2 + \sqrt{x - 1}$

57. $f(x) = 3x, \quad -2 \le x \le 6$

58. $f(x) = 5x^2 + 4, \quad 0 \le x \le 2$

59–76 ■ Domain Find the domain of the function.

59. $f(x) = \dfrac{2}{3 + x}$

60. $f(x) = \dfrac{x}{4 - x}$

61. $f(x) = \dfrac{x + 2}{x^2 - 1}$

62. $f(x) = \dfrac{x^4}{x^2 + x - 6}$

63. $f(t) = \sqrt{2 - t}$

64. $g(t) = \sqrt{t^2 + 9}$

65. $f(t) = \sqrt[3]{2t + 5}$

66. $g(x) = \sqrt{7 - 3x}$

67. $f(t) = \sqrt{t^2 - 25}$

68. $g(t) = \sqrt{36 - t^2}$

69. $g(x) = \dfrac{\sqrt{2 + x}}{3 - x}$

70. $g(x) = \dfrac{\sqrt{x}}{2x^2 + x - 1}$

71. $g(x) = \sqrt[4]{x^2 - 6x}$

72. $g(x) = \sqrt{x^2 - 2x - 8}$

73. $f(x) = \dfrac{4}{\sqrt{2 - x}}$

74. $g(x) = \dfrac{3x}{\sqrt{x + 2}}$

75. $f(x) = \dfrac{(x + 1)^2}{\sqrt{2x - 1}}$

76. $f(x) = \dfrac{x}{\sqrt[4]{9 - x^2}}$

77–80 ■ Four Ways to Represent a Function A verbal description of a function is given. Find **(a)** algebraic, **(b)** numerical, and **(c)** graphical representations for the function.

77. To evaluate $f(x)$, square the input and add 1 to the result.

78. To evaluate $f(x)$, add 2 to the input and square the result.

79. Let $T(x)$ be the amount of sales tax charged in Lemon County on a purchase of x dollars. To find the tax, take 8% of the purchase price.

80. Let $V(d)$ be the volume of a sphere of diameter d. To find the volume, take the cube of the diameter, then multiply by π and divide by 6.

Skills Plus

81–82 ■ Domain and Range Find the domain and range of f.

81. $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 5 & \text{if } x \text{ is irrational} \end{cases}$

82. $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 5x & \text{if } x \text{ is irrational} \end{cases}$

Applications

83. Torricelli's Law A tank holds 50 gallons of water, which drains from a leak at the bottom, causing the tank to empty in 20 minutes. The tank drains faster when it is nearly full because the pressure on the leak is greater. **Torricelli's Law** gives the volume of water remaining in the tank after t minutes as

$$V(t) = 50\left(1 - \frac{t}{20}\right)^2 \qquad 0 \le t \le 20$$

(a) Find $V(0)$ and $V(20)$.

(b) What do your answers to part (a) represent?

(c) Make a table of values of $V(t)$ for $t = 0, 5, 10, 15, 20$.

(d) Find the net change in the volume V as t changes from 0 min to 20 min.

84. Area of a Sphere The surface area S of a sphere is a function of its radius r given by

$$S(r) = 4\pi r^2$$

(a) Find $S(2)$ and $S(3)$.

(b) What do your answers in part (a) represent?

85. Relativity According to the Theory of Relativity, the length L of an object is a function of its velocity v with respect to an observer. For an object whose length at rest is 10 meters, the function is given by

$$L(v) = 10\sqrt{1 - \frac{v^2}{c^2}}$$

where c is the speed of light (300,000 km/s).

(a) Find $L(0.5c)$, $L(0.75c)$, and $L(0.9c)$.

(b) How does the length of an object change as its velocity increases?

86. Blackbody Radiation A *blackbody* is an ideal object that absorbs all electromagnetic radiation. A blackbody with temperature above 0 K radiates heat. The radiance L emitted by a blackbody is a function of the temperature T (in kelvins) given by

$$L(T) = \frac{a}{\pi}T^4$$

where $a = 5.67 \times 10^{-8}\,\mathrm{W \cdot m^{-2} \cdot K^{-4}}$.

(a) Find $L(300)$, $L(350)$, and $L(1000)$.

(b) How does the radiance change as temperature increases?

(c) The sun is well approximated by a blackbody. Find the radiance L of the sun given that its surface temperature is about 5778 K.

87. Pupil Size When the brightness x of a light source is increased, the eye reacts by decreasing the radius R of the pupil. The dependence of R on x is given by the function

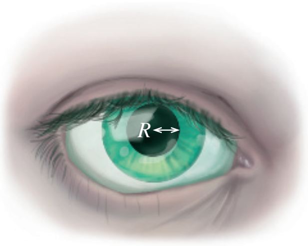
$$R(x) = \sqrt{\frac{13 + 7x^{0.4}}{1 + 4x^{0.4}}}$$

where R is measured in millimeters and x is measured in appropriate units of brightness.

(a) Find $R(1)$, $R(10)$, and $R(100)$.

(b) Make a table of values of $R(x)$.

(c) Find the net change in the radius R as x changes from 10 to 100.



88. Blood Flow As blood moves through a vein or an artery, its velocity v is greatest along the central axis and decreases as the distance r from the central axis increases (see the figure). The formula that gives v as a function of r is called the **law of laminar flow**. For an artery with radius 0.5 cm, the relationship between v (in cm/s) and r (in cm) is given by the function

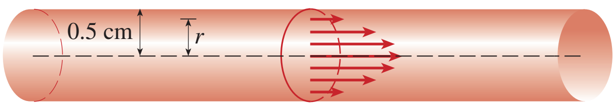
$$v(r) = 18{,}500(0.25 - r^2) \qquad 0 \le r \le 0.5$$

(a) Find $v(0.1)$ and $v(0.4)$.

(b) What do your answers to part (a) tell you about the flow of blood in this artery?

(c) Make a table of values of $v(r)$ for $r = 0, 0.1, 0.2, 0.3, 0.4, 0.5$.

(d) Find the net change in the velocity v as r changes from 0.1 cm to 0.5 cm.



89. How Far Can You See? Because of the curvature of the earth, the maximum distance D that you can see from the top of a tall building or from an airplane at height h is given by the function

$$D(h) = \sqrt{2rh + h^2}$$

where $r = 3960$ mi is the radius of the earth and D and h are measured in miles.

(a) Find $D(0.1)$ and $D(0.2)$.

(b) How far can you see from the observation deck of Toronto's CN Tower, 1135 ft above the ground?

(c) Commercial aircraft fly at an altitude of about 7 mi. How far can the pilot see?

(d) Find the net change in the value of distance D as h changes from 1135 ft to 7 mi.

90. Population Growth The population P (in millions) of the state of Arizona is a function of the year t. The table gives the population at ten-year intervals from 1960 to 2020.

(a) Find $P(1960)$, $P(1980)$, and $P(2020)$.

(b) Find the net change in the population from 1960 to 1980 and from 1980 to 2020.

t	1960	1970	1980	1990	2000	2010	2020
P	1.3	1.8	2.7	3.7	5.1	6.4	7.2

Source: US Census Bureau

91. Income Tax In a certain country, income tax T is assessed according to the following function of income x:

$$T(x) = \begin{cases} 0 & \text{if } 0 \le x \le 10{,}000 \\ 0.08(x - 10{,}000) & \text{if } 10{,}000 < x \le 20{,}000 \\ 800 + 0.15(x - 20{,}000) & \text{if } 20{,}000 < x \end{cases}$$

(a) Find $T(5{,}000)$, $T(12{,}000)$, and $T(25{,}000)$.

(b) What do your answers in part (a) represent?

92. Internet Purchases An online bookstore charges \$9 shipping for orders under \$50 but provides free shipping for orders of \$50 or more. The cost C of an order is a function of the total price x of the books purchased, given by

$$C(x) = \begin{cases} x + 9 & \text{if } x < 50 \\ x & \text{if } x \ge 50 \end{cases}$$

(a) Find $C(25)$, $C(45)$, $C(50)$, and $C(65)$.

(b) What do your answers in part (a) represent?

93. Cost of a Hotel Stay A hotel chain charges \$114 each night for the first two nights and \$99 for each additional night's stay. The total cost T is a function of the number of nights x that a guest stays.

(a) Complete the expressions in the following piecewise-defined function.

$$T(x) = \begin{cases} \square & \text{if } 0 \le x \le 2 \\ \square & \text{if } x > 2 \end{cases}$$

(b) Find $T(2)$, $T(3)$, and $T(5)$.

(c) What do your answers in part (b) represent?

94. Speeding Tickets In a certain state the maximum speed permitted on freeways is 65 mi/h, and the minimum is 40 mi/h. The fine F for violating these limits is \$15 for every mile above the maximum or below the minimum.

(a) Complete the expressions in the following piecewise-defined function, where x is the speed at which you are driving.

$$F(x) = \begin{cases} \square & \text{if } 0 < x < 40 \\ \square & \text{if } 40 \le x \le 65 \\ \square & \text{if } x > 65 \end{cases}$$

(b) Find $F(30)$, $F(50)$, and $F(75)$.

(c) What do your answers in part (b) represent?

95. Height of Grass A homeowner mows the lawn every Wednesday afternoon. Sketch a rough graph of the height of the grass as a function of time over the course of a four-week period beginning on a Sunday.

96. Temperature Change You place a frozen pie in an oven and bake it for an hour. Then you take the pie out and let it cool before eating it. Sketch a rough graph of the temperature of the pie as a function of time.

97. Outdoor Temperature Sketch a rough graph of the outdoor temperature as a function of time during a typical spring day.

98. Price of a Car Sketch a rough graph of the market value of a car as a function of the number of years since the car was purchased, over a period of 40 years. Assume the car is well maintained. How would your graph change if the car becomes a collectible antique?

■ Discuss ■ Discover ■ Prove ■ Write

99. Discuss: Examples of Functions At the beginning of this section we discussed three examples of everyday, ordinary functions: Height is a function of age, temperature is a function of date, and postage cost is a function of weight. Give three other examples of functions from everyday life.

100. Discuss: Four Ways to Represent a Function Think of a function that can be represented in all four ways described in this section, and give the four representations.

101. Discuss: Piecewise Defined Functions In Exercises 91–94 we worked with real-world situations modeled by piecewise defined functions. Find other examples of real-world situations that can be modeled by piecewise defined functions, and express the models in function notation.

2.2 Graphs of Functions

■ Graphing Functions by Plotting Points ■ Graphing Functions with Graphing Devices ■ Graphing Piecewise-Defined Functions ■ Which Graphs Represent Functions? The Vertical Line Test ■ Which Equations Represent Functions? ■ Which Relations Represent Functions?

In Section 2.1 we explored how a function can be represented by a graph. In this section we investigate in more detail the concept of the graph of a function.

■ Graphing Functions by Plotting Points

To graph a function f, we plot the points $(x, f(x))$ in a coordinate plane. In other words, we plot the points (x, y) whose x-coordinate is an input and whose y-coordinate is the corresponding output of the function.

The Graph of a Function

If f is a function with domain A, then the **graph** of f is the set of ordered pairs

$$\{(x, f(x)) \mid x \in A\}$$

plotted in a coordinate plane. In other words, the graph of f is the set of all points (x, y) such that $y = f(x)$; that is, the graph of f is the graph of the equation $y = f(x)$.

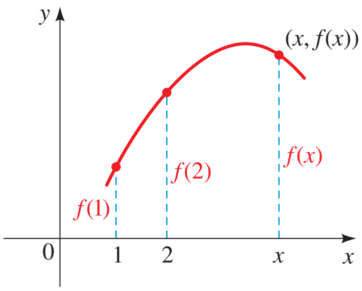


Figure 1 | The height of the graph above x is the value of $f(x)$.

The graph of a function f gives a picture of the behavior or "life history" of the function. We can read the value of $f(x)$ from the graph as being the height of the graph above x (see Figure 1).

Equations of lines are studied in Section 1.10.

A function f of the form $f(x) = mx + b$ is called a **linear function** because its graph is the graph of the equation $y = mx + b$, which represents a line with slope m and y-intercept b. A special case of a linear function occurs when the slope is $m = 0$. The function $f(x) = b$, where b is a given number, is called a **constant function** because all its values are the same number, namely, b. Its graph is the horizontal line $y = b$. Figure 2 shows the graphs of the constant function $f(x) = 3$ and the linear function $f(x) = 2x + 1$.

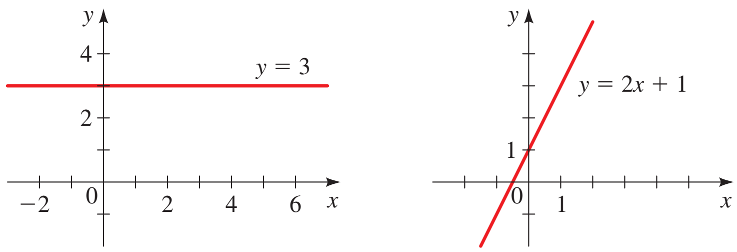


Figure 2 The constant function $f(x) = 3$ The linear function $f(x) = 2x + 1$

Functions of the form $f(x) = x^n$ are called **power functions**, and functions of the form $f(x) = x^{1/n}$ are called **root functions**. In the next example we graph two power functions and a root function.

Example 1 ■ Graphing Functions by Plotting Points

Sketch a graph of each of the following functions.

(a) $f(x) = x^2$ **(b)** $g(x) = x^3$ **(c)** $h(x) = \sqrt{x}$

Solution The graphs of these functions are the graphs of the equations $y = x^2$, $y = x^3$, and $y = \sqrt{x}$. To graph each of these equations, we make a table of values, plot the points corresponding to the ordered pairs in the table, and then join them by a smooth curve. The graphs are sketched in Figure 3.

x	$y = x^2$	(x, y)
-2	4	$(-2, 4)$
-1	1	$(-1, 1)$
$-\frac{1}{2}$	$\frac{1}{4}$	$(-\frac{1}{2}, \frac{1}{4})$
0	0	$(0, 0)$
$\frac{1}{2}$	$\frac{1}{4}$	$(\frac{1}{2}, \frac{1}{4})$
1	1	$(1, 1)$
2	4	$(2, 4)$

x	$y = x^3$	(x, y)
-2	-8	$(-2, -8)$
-1	-1	$(-1, -1)$
$-\frac{1}{2}$	$-\frac{1}{8}$	$(-\frac{1}{2}, -\frac{1}{8})$
0	0	$(0, 0)$
$\frac{1}{2}$	$\frac{1}{8}$	$(\frac{1}{2}, \frac{1}{8})$
1	1	$(1,1)$
2	8	$(2, 8)$

x	$y = \sqrt{x}$	(x, y)
0	0	$(0, 0)$
1	1	$(1, 1)$
2	$\sqrt{2}$	$(2, \sqrt{2})$
3	$\sqrt{3}$	$(3, \sqrt{3})$
4	2	$(4, 2)$
5	$\sqrt{5}$	$(5, \sqrt{5})$
6	$\sqrt{6}$	$(6, \sqrt{6})$

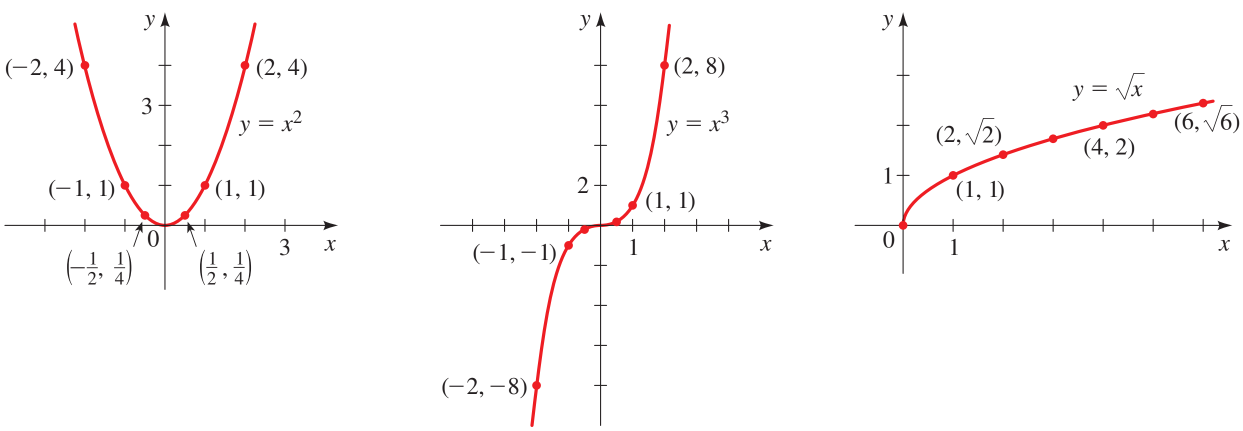


Figure 3 (a) $f(x) = x^2$ (b) $g(x) = x^3$ (c) $h(x) = \sqrt{x}$

Now Try Exercises 13, 19, and 23 ■

Graphing Functions with Graphing Devices

Graphing devices and viewing rectangles are introduced in Section 1.11. Graphing devices include graphing calculators as well as math apps for computers and smartphones. Familiarize yourself with the operation of the device you are using. If you are using a graphing calculator see Appendix C, *Graphing with a Graphing Calculator,* or Appendix D, *Using the TI-83/84 Graphing Calculator.* Go to **www.stewartmath.com**.

A convenient way to graph a function is to use a graphing device. To graph the function f, we use a device to graph the equation $y = f(x)$.

Example 2 ■ Graphing a Function with a Graphing Device

Use a graphing device to graph the function $f(x) = x^3 - 8x^2$ in an appropriate viewing rectangle.

Solution To graph the function $f(x) = x^3 - 8x^2$, we must graph the equation $y = x^3 - 8x^2$. A graphing device displays the graph in a default viewing rectangle, such as the one shown in Figure 4(a). But this graph appears to spill over the top and bottom of the screen. We need to expand the vertical axis to get a better representation of the graph. The viewing rectangle $[-4, 10]$ by $[-100, 100]$ gives a more complete picture of the graph, as shown in Figure 4(b).

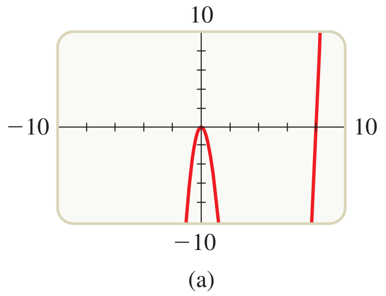


Figure 4 | Graphing the function $f(x) = x^3 - 8x^2$

Now Try Exercise 33 ■

Example 3 ■ A Family of Power Functions

(a) Graph the functions $f(x) = x^n$ for $n = 2, 4,$ and 6 in the viewing rectangle $[-2, 2]$ by $[-1, 3]$.

(b) Graph the functions $f(x) = x^n$ for $n = 1, 3,$ and 5 in the viewing rectangle $[-2, 2]$ by $[-2, 2]$.

(c) What conclusions can you make from these graphs?

Solution To graph the function $f(x) = x^n$, we graph the equation $y = x^n$. The graphs for parts (a) and (b) are shown in Figure 5.

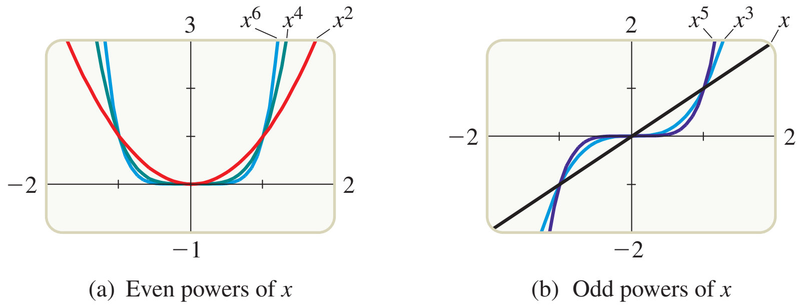


Figure 5 | A family of power functions: $f(x) = x^n$

(c) We see that the general shape of the graph of $f(x) = x^n$ depends on whether n is even or odd.

If n is even, the graph of $f(x) = x^n$ is similar to the parabola $y = x^2$.

If n is odd, the graph of $f(x) = x^n$ is similar to that of $y = x^3$.

Now Try Exercise 71 ■

Notice from Figure 5 that as n increases, the graph of $y = x^n$ becomes flatter near 0 and steeper when $x > 1$. When $0 < x < 1$, the lower powers of x are the "bigger" functions. But when $x > 1$, the higher powers of x are the dominant functions.

■ Graphing Piecewise-Defined Functions

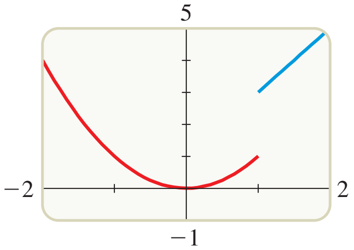
A piecewise-defined function is defined by different formulas on different parts of its domain. As you might expect, the graph of such a function consists of separate pieces.

Example 4 ■ Graph of a Piecewise-Defined Function

Sketch the graph of the function

$$f(x) = \begin{cases} x^2 & \text{if } x \le 1 \\ 2x + 1 & \text{if } x > 1 \end{cases}$$

Graphing devices can draw piecewise-defined functions like the function in Example 4. On many math apps the parts of the piecewise function are entered on the screen just as you see them displayed in the example. The device produces a graph like the one shown below.



Solution If $x \le 1$, then $f(x) = x^2$, so the part of the graph to the left of $x = 1$ coincides with the graph of $y = x^2$, which we sketched in Figure 3. If $x > 1$, then $f(x) = 2x + 1$, so the part of the graph to the right of $x = 1$ coincides with the line $y = 2x + 1$, which we graphed in Figure 2. This enables us to sketch the graph in Figure 6.

The solid dot at $(1, 1)$ indicates that this point is included in the graph; the open dot at $(1, 3)$ indicates that this point is excluded from the graph.

$$f(x) = \begin{cases} x^2 & \text{if } x \le 1 \\ 2x + 1 & \text{if } x > 1 \end{cases}$$

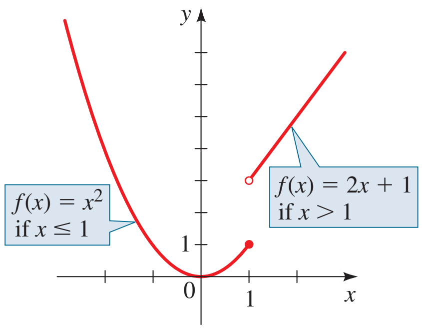


Figure 6

Now Try Exercise 37 ■

Example 5 ■ Graph of the Absolute-Value Function

Sketch a graph of the absolute-value function $f(x) = |x|$.

Solution Recall that

$$|x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}$$

Using the same method as in Example 4, we note that the graph of f coincides with the line $y = x$ to the right of the y-axis and coincides with the line $y = -x$ to the left of the y-axis (see Figure 7).

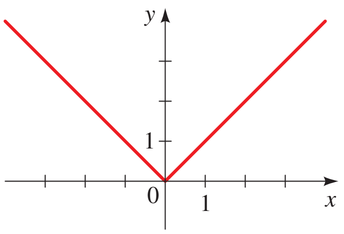


Figure 7 | Graph of $f(x) = |x|$

Now Try Exercise 27 ■

The **greatest integer function** is defined by

$$[\![x]\!] = \text{greatest integer less than or equal to } x$$

The greatest integer function $[\![x]\!]$ is also called the **floor function** and denoted by $\lfloor x \rfloor$.

For example, $[\![2]\!] = 2$, $[\![2.3]\!] = 2$, $[\![1.999]\!] = 1$, $[\![0.002]\!] = 0$, $[\![-3.5]\!] = -4$, and $[\![-0.5]\!] = -1$.

Example 6 ■ Graph of the Greatest Integer Function

Sketch a graph of $f(x) = [\![x]\!]$.

Solution The table shows the values of f for some values of x. Note that $f(x)$ is constant between consecutive integers, so the graph between integers is a horizontal line segment, as shown in Figure 8.

x	$[\![x]\!]$
$\vdots$	$\vdots$
$-2 \le x < -1$	-2
$-1 \le x < 0$	-1
$0 \le x < 1$	0
$1 \le x < 2$	1
$2 \le x < 3$	2
$\vdots$	$\vdots$

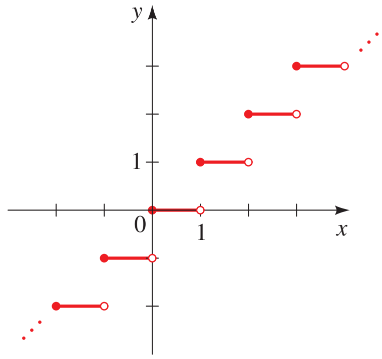


Figure 8 | The greatest integer function, $y = [\![x]\!]$

The greatest integer function is an example of a **step function**. The next example gives a real-world application of a step function.

Example 7 ■ The Cost Function for a Cell Phone Travel Plan

A cell phone company offers a travel plan for cell phone data usage in countries outside the United States. The travel plan has a monthly fee of \$100 for the first 5 gigabytes of data used, and \$20 for each additional gigabyte (or portion thereof). Draw a graph of the cost C (in dollars) as a function of the number of gigabytes x used per month.

Solution Let $C(x)$ be the cost of using x gigabytes of data in a month. Since $x \ge 0$ the domain of the function is $[0, \infty)$. From the given information we have

$$\begin{aligned} C(x) &= 100 && \text{if } 0 < x \le 5 \\ C(x) &= 100 + 20(1) = 120 && \text{if } 5 < x \le 6 \\ C(x) &= 100 + 20(2) = 140 && \text{if } 6 < x \le 7 \\ C(x) &= 100 + 20(3) = 160 && \text{if } 7 < x \le 8 \\ &\vdots && \vdots \end{aligned}$$

The graph is shown in Figure 9.

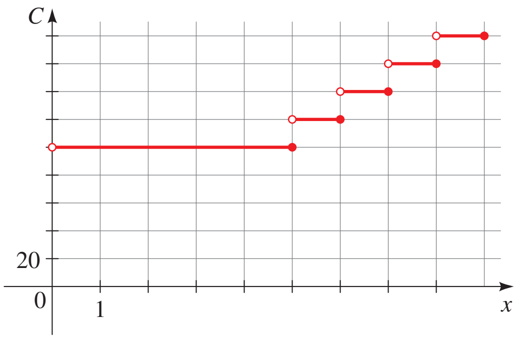


Figure 9 | Cost of data usage

Now Try Exercise 85

A function is called **continuous** if its graph has no "break" or "hole." The functions in Examples 1, 2, 3, and 5 are continuous; the functions in Examples 4, 6, and 7 are not continuous.

Which Graphs Represent Functions? The Vertical Line Test

The graph of a function is a curve in the xy-plane. But the question arises: Which curves in the xy-plane are graphs of functions? This is answered by the following test.

The Vertical Line Test

A curve in the coordinate plane is the graph of a function if and only if no vertical line intersects the curve more than once.

We can see from Figure 10 why the Vertical Line Test is true. If each vertical line $x = a$ intersects a curve only once at (a, b), then exactly one function value is defined by $f(a) = b$. But if a line $x = a$ intersects the curve twice, at (a, b) and at (a, c), then the curve cannot represent a function because a function cannot assign two different values to a.

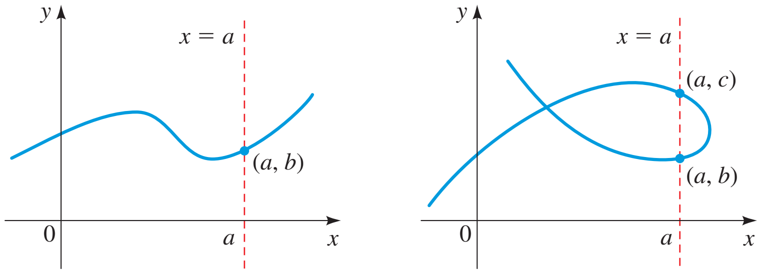


Figure 10 | Vertical Line Test

Example 8 ■ Using the Vertical Line Test

Using the Vertical Line Test, we see that the curves in parts (b) and (c) of Figure 11 represent functions, whereas those in parts (a) and (d) do not.

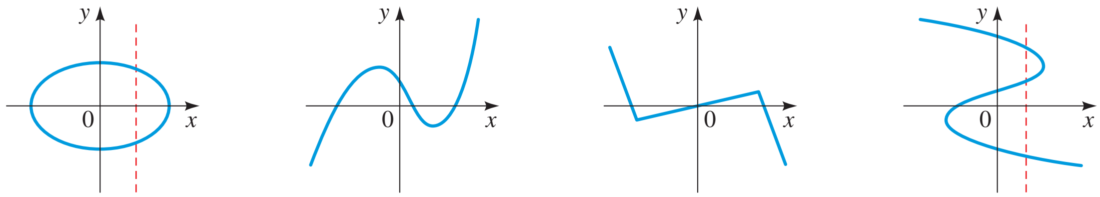
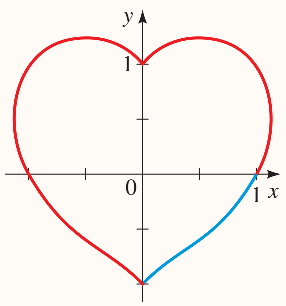


Figure 11

Now Try Exercise 51 ■



Discovery Project ■ **Implicit Functions**

Graphing the equation $(x^2 + y^2 - 1)^3 - x^2y^3 = 0$ with a graphing device gives the heart-shaped graph shown here. It is not the graph of a function, but parts of the graph (the part in the fourth quadrant, for instance) do represent functions. We say that the function whose graph is in the fourth quadrant is *implicitly* defined by the equation that produced the graph. In this project you will experiment with equations that produce interesting graphs and determine whether the equation can be solved to find *explicit* formulas for the functions that are implicit in the equation. You can find the project at **www.stewartmath.com**.

BBVA/NOTIMEX/Newscom

DONALD KNUTH was born in Milwaukee in 1938 and is Professor Emeritus of Computer Science at Stanford University. When Knuth was a high school student, he became fascinated with graphs of functions and laboriously drew many hundreds of them because he wanted to see the behavior of a great variety of functions. (Today, of course, it is far easier to use computers and graphing devices to do this.) While still a graduate student at Caltech, he started writing a monumental series of books entitled *The Art of Computer Programming*.

Knuth is famous for his invention of TEX, a system of computer-assisted typesetting. This system was used in the preparation of the manuscript for this textbook.

Knuth has received numerous honors, among them election as an associate of the French Academy of Sciences and as a Fellow of the Royal Society. Asteroid 21656 Knuth was named in his honor.

■ Which Equations Represent Functions?

Any equation in the variables x and y defines a relationship between these variables. For example, the equation

$$y - x^2 = 0$$

defines a relationship between y and x. Does this equation define y as a *function* of x? To find out, we solve for y and get

$$y = x^2 \qquad \text{Equation form}$$

We see that the equation defines a rule, or function, that gives one value of y for each value of x. We can express this rule in function notation as

$$f(x) = x^2 \qquad \text{Function form}$$

But not every equation defines y as a function of x, as the following example shows.

Example 9 ■ Equations That Define Functions

Does the equation define y as a function of x?

(a) $y - x^2 = 2$ **(b)** $x^2 + y^2 = 4$

Solution

(a) Solving for y in terms of x gives

$$y - x^2 = 2$$
$$y = x^2 + 2 \qquad \text{Add } x^2$$

The last equation is a rule that gives one value of y for each value of x, so it defines y as a function of x. We can write the function as $f(x) = x^2 + 2$.

(b) We try to solve for y in terms of x.

$$x^2 + y^2 = 4$$
$$y^2 = 4 - x^2 \qquad \text{Subtract } x^2$$
$$y = \pm\sqrt{4 - x^2} \qquad \text{Take square roots}$$

The last equation gives two values of y for a given value of x. Thus the equation does not define y as a function of x.

Now Try Exercises 53 and 59 ■

The graphs of the equations in Example 9 are shown in Figure 12. The Vertical Line Test shows graphically that the equation in Example 9(a) defines a function but the equation in Example 9(b) does not.

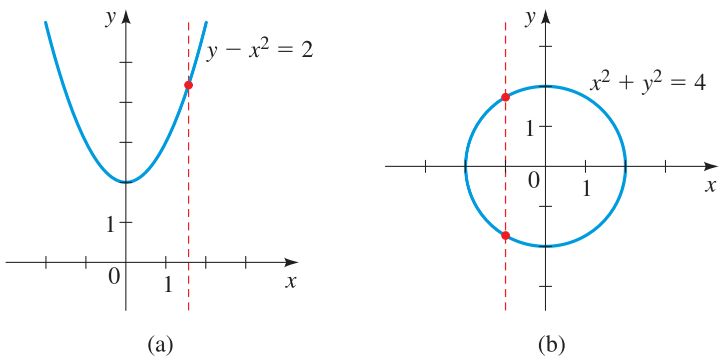


Figure 12

Which Relations Represent Functions?

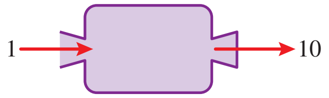
A function f can be represented as a two-column table (as shown in Example 2.1.7) or equivalently as a set of ordered pairs (x, y) where x is the input and $y = f(x)$ is the output. Conversely, given a two-column table (or a set of ordered pairs), how can we tell whether it represents a function? Let's begin by considering collections of ordered pairs.

> **Relations**
>
> Any collection of ordered pairs (x, y) is called a **relation**.

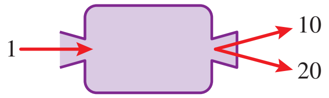
If we denote the ordered pairs in a relation by (x, y), then the set of x-values (or inputs) is the **domain** of the relation and the set of y-values (or outputs) is the **range** of the relation. Every function is a relation consisting of the ordered pairs (x, y), where $y = f(x)$. However, not every relation is a function, because a relation can have more than one output for a given input.

> A relation is a function if each input corresponds to exactly one output.

Here are two examples of relations:



Function



Not a function

Relation 1: $\{(1, 10), (2, 20), (3, 40), (4, 40)\}$

Relation 2: $\{(1, 10), (1, 20), (2, 30), (3,30), (4, 40)\}$

The diagrams in Figure 13 are visual representations of these relations. The first relation is a function because each input corresponds to exactly one output; the second relation is not a function because the input 1 corresponds to two different outputs, 10 and 20 (see the machine diagrams in the margin).

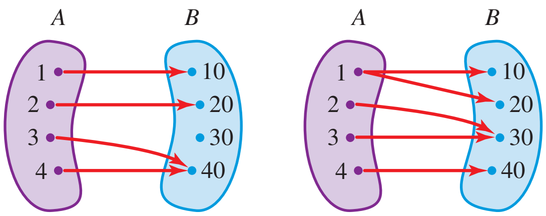


Figure 13 | Is the relation a function?

We can graph relations in the same way we graph functions. To graph a relation, plot the ordered pairs in the relation in a coordinate plane. Note that any graph in the coordinate plane defines a relation, namely the relation that consists of the set of coordinates (x, y) of the points on the graph. Also, any equation in the variables x and y defines a relation, namely the relation that consists of the ordered pairs (x, y) that satisfy the equation.

Example 10 ■ Is the Relation a Function?

Is the relation a function? Confirm your answer graphically. State the domain and range of the relation.

(a) $\{(-1, 2), (0, 3), (1, 1), (1, 2), (2, 3), (3, 1)\}$

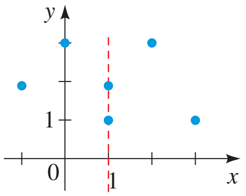
(b) $\{(-1, 3), (0, 1), (1, 2), (2, 1), (3, 1), (4, 3)\}$

Solution

(a) Since the ordered pairs $(1, 1)$ and $(1, 2)$ are in the relation, the input 1 corresponds to the two different outputs, 1 and 2. It follows that the relation is not a function. We can also verify this conclusion graphically by making a table of values and graphing the relation, as in Figure 14(a). From the graph we see that the vertical line $x = 1$ intersects the graph at the two points $(1, 1)$ and $(1, 2)$, so the Vertical Line Test confirms that the relation is not a function. The domain is $\{-1, 0, 1, 2, 3\}$ and the range is $\{1, 2, 3\}$.

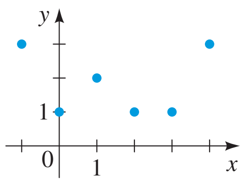
(b) The relation is a function because each input corresponds to exactly one output. That is, for each ordered pair the first component corresponds to exactly one second component. A graph of the relation is shown in Figure 14(b). Applying the Vertical Line Test to the graph shows that the relation is a function. The domain is $\{-1, 0, 1, 2, 3, 4\}$ and the range is $\{1, 2, 3\}$.

x	y
-1	2
0	3
1	1
1	2
2	3
3	1



(a)

x	y
-1	3
0	1
1	2
2	1
3	1
4	3



(b)

Figure 14

Now Try Exercise 67 ■

Example 11 ■ Is the Relation a Function?

A relation consists of the ordered pairs (x, y) that satisfy the equation $x^2 + y^2 = 4$. Show that the relation is not a function.

Solution The relation is not a function because there are ordered pairs that satisfy the equation and have different outputs for the same input; for instance, $(0, 2)$ and $(0, -2)$ or $(1, \sqrt{3})$ and $(1, -\sqrt{3})$.

Now Try Exercise 69 ■

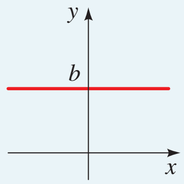
Note There are different ways to show that an equation does not represent a function. For instance, consider the equation $x^2 + y^2 = 4$. In Example 11 we showed that the relation defined by this equation is not a function. In Example 9(b) we solved the equation for y and found that the equation does not define y as a function of x. And, in Figure 12(b) we used the Vertical Line Test to show that the graph of the equation is not the graph of a function.

The following box shows the graphs of some functions that you will see frequently in this book.

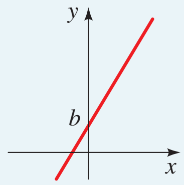
Some Functions and Their Graphs

Linear functions

$f(x) = mx + b$



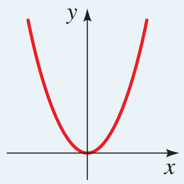
$f(x) = b$



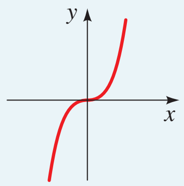
$f(x) = mx + b$

Power functions

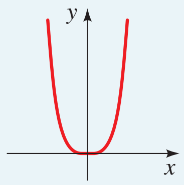
$f(x) = x^n$



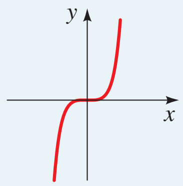
$f(x) = x^2$



$f(x) = x^3$



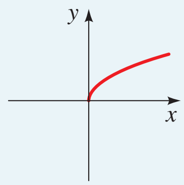
$f(x) = x^4$



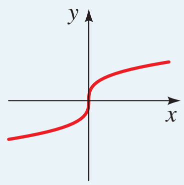
$f(x) = x^5$

Root functions

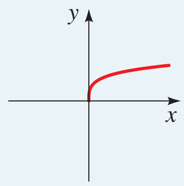
$f(x) = \sqrt[n]{x}$



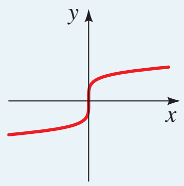
$f(x) = \sqrt{x}$



$f(x) = \sqrt[3]{x}$



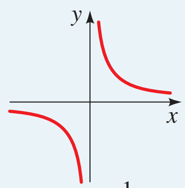
$f(x) = \sqrt[4]{x}$



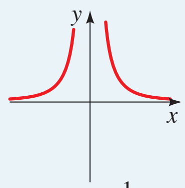
$f(x) = \sqrt[5]{x}$

Reciprocal functions

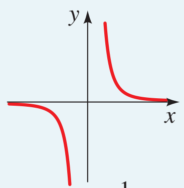
$f(x) = \dfrac{1}{x^n}$



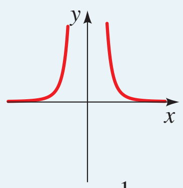
$f(x) = \dfrac{1}{x}$



$f(x) = \dfrac{1}{x^2}$



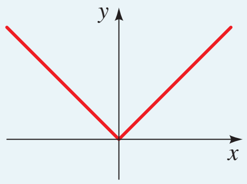
$f(x) = \dfrac{1}{x^3}$



$f(x) = \dfrac{1}{x^4}$

Absolute-value function

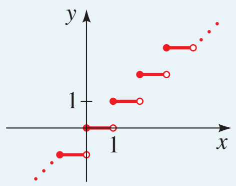
$f(x) = |x|$



$f(x) = |x|$

Greatest integer function

$f(x) = [\![x]\!]$



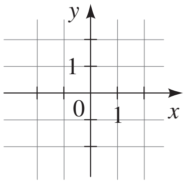
$f(x) = [\![x]\!]$

2.2 Exercises

Concepts

1. To graph the function f, we plot the points $(x, \_\_\_\_\_\_)$ in a coordinate plane. To graph $f(x) = x^2 - 2$, we plot the points $(x, \_\_\_\_\_\_)$. So the point $(3, \_\_\_\_\_\_)$ is on the graph of f. The height of the graph of f above the x-axis when $x = 3$ is ________. Complete the table, and sketch a graph of f.

x	$y = f(x)$	(x, y)
-2		
-1		
0		
1		
2		



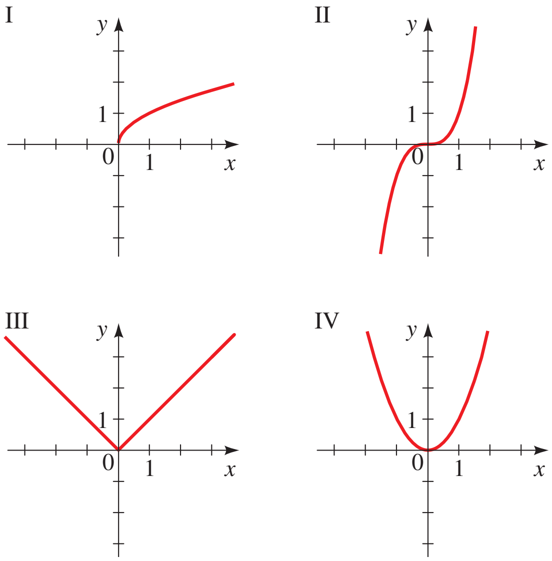
2. If $f(4) = 10$ then the point $(4, \_\_\_\_)$ is on the graph of f.

3. If the point $(3, 7)$ is on the graph of f, then $f(3) =$ ________.

4. Match the function with its graph.

(a) $f(x) = x^2$ **(b)** $f(x) = x^3$

(c) $f(x) = \sqrt{x}$ **(d)** $f(x) = |x|$



5–8 ■ Explain why the given graph, equation, set of ordered pairs (x, y), or table does *not* define y as a function of x.

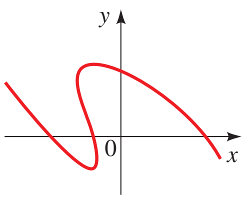
5. $\{(1, 1), (1, 2), (2, 3), (3, 4)\}$

6. $x = 4y^2$

7.

x	y
8	11
10	15
2	11
10	10

8.



Skills

9–32 ■ **Graphing Functions** Sketch a graph of the function by first making a table of values.

9. $f(x) = x + 2$ **10.** $f(x) = 4 - 2x$

11. $f(x) = -x + 3, \quad -3 \le x \le 3$

12. $f(x) = \dfrac{x - 3}{2}, \quad 0 \le x \le 5$

13. $f(x) = -x^2$ **14.** $f(x) = x^2 - 2$

15. $g(x) = x^2 - 6x + 9$ **16.** $g(x) = -(x + 3)^2$

17. $r(x) = 3x^4$ **18.** $r(x) = 20 - x^4$

19. $g(x) = x^3 + 8$ **20.** $g(x) = (x + 2)^3$

21. $k(x) = \sqrt[3]{-x}$ **22.** $k(x) = -\sqrt[3]{x}$

23. $f(x) = 2 - \sqrt{x}$ **24.** $f(x) = \sqrt{x + 4}$

25. $C(t) = \dfrac{1}{t^2}$ **26.** $C(t) = -\dfrac{1}{t + 1}$

27. $H(x) = |2x|$ **28.** $H(x) = |x - 2|$

29. $G(x) = |x| + x$ **30.** $G(x) = |x| - x$

31. $f(x) = |2x - 2|$ **32.** $f(x) = \dfrac{x}{|x|}$

33–36 ■ **Graphing Functions** Graph the function in each of the given viewing rectangles, and select the one that produces the most appropriate graph of the function.

33. $f(x) = 8x - x^2$

(a) $[-5, 5]$ by $[-5, 5]$

(b) $[-10, 10]$ by $[-10, 10]$

(c) $[-2, 10]$ by $[-5, 20]$

(d) $[-10, 10]$ by $[-100, 100]$

34. $f(x) = x^2 - 4x - 32$

(a) $[-3, 3] \times [-5, 5]$

(b) $[-10, 10] \times [-10, 10]$

(c) $[-7, 7] \times [-30, 5]$

(d) $[-6, 10] \times [-40, 5]$

35. $f(x) = 3x^3 - 9x - 20$

(a) $[-2, 2] \times [-10, 10]$

(b) $[-5, 5] \times [-10, 20]$

(c) $[-5, 5] \times [-20, 20]$

(d) $[-3, 3] \times [-40, 20]$

36. $f(x) = x^4 - 10x^2 + 5x$

(a) $[-1, 1] \times [-10, 10]$

(b) $[-5, 5] \times [-10, 10]$

(c) $[-5, 5] \times [-40, 20]$

(d) $[-10, 10] \times [-20, 20]$

37–46 ■ **Graphing Piecewise-Defined Functions** Sketch a graph of the piecewise-defined function.

37. $f(x) = \begin{cases} 2x & \text{if } x \le 0 \\ 2 & \text{if } x > 0 \end{cases}$ **38.** $f(x) = \begin{cases} 1 & \text{if } x \le 1 \\ x + 1 & \text{if } x > 1 \end{cases}$

39. $f(x) = \begin{cases} x & \text{if } x \le 0 \\ x + 1 & \text{if } x > 0 \end{cases}$

40. $f(x) = \begin{cases} 2x + 3 & \text{if } x < -1 \\ 3 - x & \text{if } x \ge -1 \end{cases}$

41. $f(x) = \begin{cases} 3 - x & \text{if } x \le -1 \\ 2x^2 & \text{if } x > -1 \end{cases}$

42. $f(x) = \begin{cases} 4 - x^2 & \text{if } x \le 1 \\ x - 5 & \text{if } x > 1 \end{cases}$

43. $f(x) = \begin{cases} 0 & \text{if } |x| \le 2 \\ 3 & \text{if } |x| > 2 \end{cases}$

44. $f(x) = \begin{cases} x^2 & \text{if } |x| \le 1 \\ 1 & \text{if } |x| > 1 \end{cases}$

45. $f(x) = \begin{cases} 4 & \text{if } x < -2 \\ x^2 & \text{if } -2 \le x \le 2 \\ -x + 6 & \text{if } x > 2 \end{cases}$

46. $f(x) = \begin{cases} -1 & \text{if } x < -1 \\ x & \text{if } -1 \le x \le 1 \\ 1 & \text{if } x > 1 \end{cases}$

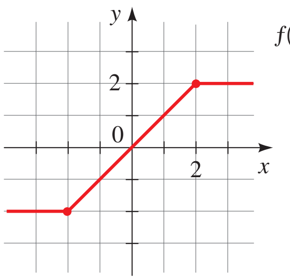
47–48 ■ Graphing Piecewise-Defined Functions Use a graphing device to draw a graph of the piecewise-defined function. (See the margin note near Example 4.)

47. $f(x) = \begin{cases} x^2 - 6x + 12 & \text{if } x \le 2 \\ 6 - x & \text{if } x > 2 \end{cases}$

48. $f(x) = \begin{cases} 2x - x^2 & \text{if } x > 1 \\ (x - 1)^3 & \text{if } x \le 1 \end{cases}$

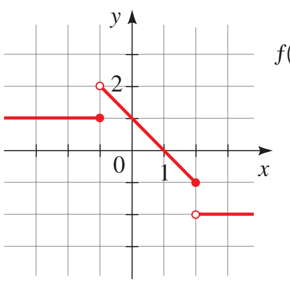
49–50 ■ Finding Piecewise-Defined Functions A graph of a piecewise-defined function is given. Find a formula for the function in the indicated form.

49.



$f(x) = \begin{cases} \underline{\qquad} & \text{if } x < -2 \\ \underline{\qquad} & \text{if } -2 \le x \le 2 \\ \underline{\qquad} & \text{if } x > 2 \end{cases}$

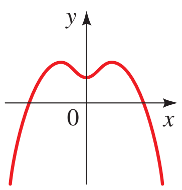
50.



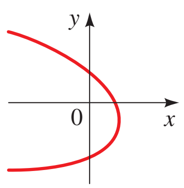
$f(x) = \begin{cases} \underline{\qquad} & \text{if } x \le -1 \\ \underline{\qquad} & \text{if } -1 < x \le 2 \\ \underline{\qquad} & \text{if } x > 2 \end{cases}$

51–52 ■ Graphs that Define Functions Use the Vertical Line Test to determine whether the curve is a graph of a function of x.

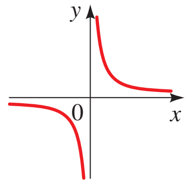
51. (a)



(b)



(c)



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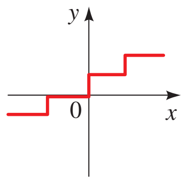
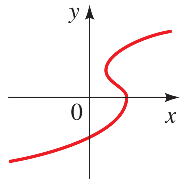


52. (a)



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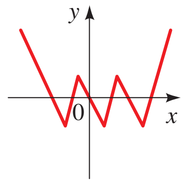

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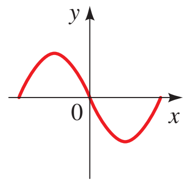

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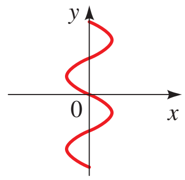


53–66 ■ Equations That Define Functions Determine whether the equation defines y as a function of x. (See Example 9.)

53. $x^2 - 3y = 7$

54. $10x - y = 5$

55. $y^3 - x = 5$

56. $x^2 - y^{1/3} = 1$

57. $x = y^2$

58. $x^2 + (y - 1)^2 = 4$

59. $2x - 4y^2 = 3$

60. $2x^2 - 4y^2 = 3$

61. $2xy - 5y^2 = 4$

62. $\sqrt{y} - x = 5$

63. $2|x| + y = 0$

64. $2x + |y| = 0$

65. $x = y^3$

66. $x = y^4$

67–68 ■ Relations That Define Functions A relation is given by a table or a set of ordered pairs. Graph the relation to determine whether it defines y as a function of x. State the domain and range of the relation.

67. $\{(0, 1), (1, 2), (1, 3), (4, 1), (5, 1), (6, 1)\}$

68.

x	1	2	3	4	5
y	10	5	15	10	20

69–70 ■ Relations That Define Functions Determine whether the relation defines y as a function of x. Give reasons for your answer.

69. The set of ordered pairs (x, y) that satisfy the equation $(y + 1)^3 = x$.

70. The set of ordered pairs of natural numbers (x, y) for which y/x is a natural number.

71–74 ■ Families of Functions A family of functions is given. In parts (a) and (b) graph all the given members of the family in the viewing rectangle indicated. In part (c) state the conclusions that you can make from your graphs.

71. $f(x) = x^2 + c$

(a) $c = 0, 2, 4, 6;\quad [-5, 5]$ by $[-10, 10]$

(b) $c = 0, -2, -4, -6$; $[-5, 5]$ by $[-10, 10]$

(c) How does the value of c affect the graph?

72. $f(x) = (x - c)^2$

(a) $c = 0, 1, 2, 3$; $[-5, 5]$ by $[-10, 10]$

(b) $c = 0, -1, -2, -3$; $[-5, 5]$ by $[-10, 10]$

(c) How does the value of c affect the graph?

73. $f(x) = cx^2$

(a) $c = 1, \frac{1}{2}, 2, 4$; $[-5, 5]$ by $[-10, 10]$

(b) $c = 1, -1, -\frac{1}{2}, -2$; $[-5, 5]$ by $[-10, 10]$

(c) How does the value of c affect the graph?

74. $f(x) = x^{1/n}$

(a) $n = 2, 4, 6$, $[-1, 4]$ by $[-1, 3]$

(b) $n = 3, 5, 7$, $[-3, 3]$ by $[-2, 2]$

(c) How does the value of n affect the graph?

Skills Plus

75–78 ■ Finding Functions for Certain Curves Find a function whose graph is the given curve.

75. The line segment joining the points $(-2, 1)$ and $(4, -6)$

76. The line segment joining the points $(-3, -2)$ and $(6, 3)$

77. The top half of the circle $x^2 + y^2 = 9$

78. The bottom half of the circle $x^2 + y^2 = 9$

Applications

79. Weather Balloon As a weather balloon is inflated, the thickness T of its rubber skin is related to the radius of the balloon by

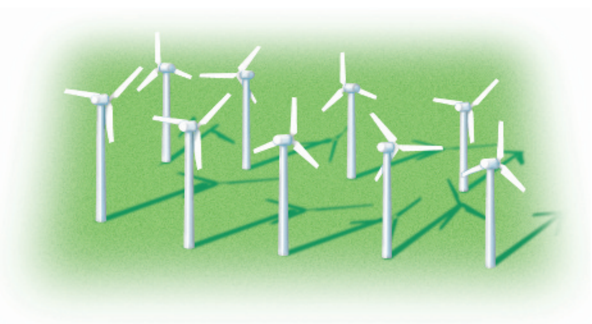
$$T(r) = \frac{0.5}{r^2}$$

where T and r are measured in centimeters. Graph the function T for values of r between 10 and 100.

80. Power from a Wind Turbine The power produced by a wind turbine depends on the speed of the wind. If a windmill has blades 3 meters long, then the power P produced by the turbine is modeled by

$$P(v) = 14.1v^3$$

where P is measured in watts (W) and v is measured in meters per second (m/s). Graph the function P for wind speeds between 1 m/s and 10 m/s.



81–84 ■ Graphing Applied Functions Graph the indicated function from Exercises 2.1 on the given interval.

81. Exercise 83; $0 \le t \le 20$ (Torricelli's law)

82. Exercise 86; $0 \le T \le 1000$ (Blackbody radiation)

83. Exercise 85; $0 < v \le 300{,}000$ (Relativity)

84. Exercise 88; $0 \le r \le 0.5$ (Blood flow)

85. Postage Rates The 2022 domestic postage rate for first-class letters weighing 3.5 oz or less is 60 cents for the first ounce (or less), plus 24 cents for each additional ounce (or part of an ounce). Express the postage P as a piecewise-defined function of the weight x of a letter, with $0 < x \le 3.5$, and sketch a graph of this function.

86. Utility Rates Westside Energy charges its electric customers a base rate of \$10.00 per month, plus 12¢ per kilowatt-hour (kWh) for the first 300 kWh used and 17¢ per kWh for all usage over 300 kWh. Suppose a customer uses x kWh of electricity in one month.

(a) Express the monthly cost E as a piecewise-defined function of x.

(b) Graph the function E for $0 \le x \le 600$.

Discuss ■ Discover ■ Prove ■ Write

87. Discover: When Does a Graph Represent a Function? For every integer n, the graph of the equation $y = x^n$ is the graph of a function, namely $f(x) = x^n$. Explain why the graph of $x = y^2$ is *not* the graph of a function of x. Is the graph of $x = y^3$ the graph of a function of x? If so, what function of x is it the graph of? Determine for what integers n the graph of $x = y^n$ is a graph of a function of x.

88. Discover: Graph of the Absolute Value of a Function

(a) Draw graphs of the functions

$$f(x) = x^2 + x - 6$$

and $$g(x) = |x^2 + x - 6|$$

How are the graphs of f and g related?

(b) Draw graphs of the functions $f(x) = x^4 - 6x^2$ and $g(x) = |x^4 - 6x^2|$. How are the graphs of f and g related?

(c) In general, if $g(x) = |f(x)|$, how are the graphs of f and g related? Draw graphs to illustrate your answer.

89. Discuss: Everyday Relations In everyday life we encounter many relations. For example, we match people with their telephone number(s), baseball players with their batting averages, or married people with their partners. Discuss whether the following everyday relations define y as a functions of x.

(a) x is the sibling of y.

(b) x is the birth mother of y.

(c) x is a student in your school and y is their ID number.

(d) x is the age of a student and y is their shoe size.

90. Discuss ■ Discover: Is *x* a Function of *y*? The equation $x = y^2$ does not define y as a function of x. In other words, if we view x as the independent variable (the input) and y as the dependent variable (the output), then the equation does not define a function. On the other hand, if we view y as the independent variable and x as the dependent variable, then the equation does define x as a function of y (because in this case each value of y determines exactly one value of x, namely y^2). Determine whether the relations in Exercises 5–8 define x as a function of y.

2.3 Getting Information from the Graph of a Function

■ Values of a Function; Domain and Range ■ Comparing Function Values: Solving Equations and Inequalities Graphically ■ Increasing and Decreasing Functions ■ Local Maximum and Minimum Values of a Function

Many properties of a function are more easily obtained from a graph than from the rule that describes the function. We see in this section how a graph tells us whether the values of a function are increasing or decreasing and also where the maximum and minimum values of a function are located.

■ Values of a Function; Domain and Range

A complete graph of a function contains all the information about a function because the graph tells us which input values correspond to which output values. To analyze the graph of a function, we must keep in mind that *the height of the graph is the value of the function*. So we can read off the values of a function from its graph.

Example 1 ■ Finding the Values of a Function from a Graph

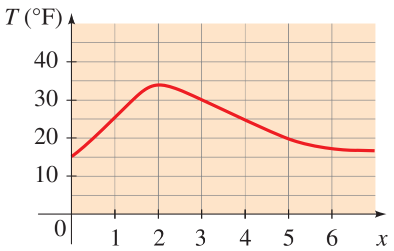


Figure 1 | Temperature function

The function T graphed in Figure 1 gives the temperature between noon and 6:00 P.M. at a certain weather station.

(a) Find $T(1)$, $T(3)$, and $T(5)$.

(b) Which is larger, $T(2)$ or $T(4)$?

(c) Find the value(s) of x for which $T(x) = 25$.

(d) Find the value(s) of x for which $T(x) \geq 25$.

(e) Find the net change in temperature from 1 P.M. to 3 P.M.

Solution

(a) $T(1)$ is the temperature at 1:00 P.M. It is represented by the height of the graph above the x-axis at $x = 1$. Thus $T(1) = 25$. Similarly, $T(3) = 30$ and $T(5) = 20$.

(b) Since the graph is higher at $x = 2$ than at $x = 4$, it follows that $T(2)$ is larger than $T(4)$.

(c) The height of the graph is 25 when x is 1 and when x is 4. In other words, the temperature is 25 at 1:00 P.M. and at 4:00 P.M.

(d) The graph is higher than 25 for x between 1 and 4. In other words, the temperature was 25 or greater between 1:00 P.M. and 4:00 P.M.

Net change is defined in Section 2.1.

(e) The net change in temperature is

$$T(3) - T(1) = 30 - 25 = 5$$

So there was a net increase of 5°F from 1 P.M. to 3 P.M.

Now Try Exercises 7 and 63 ■

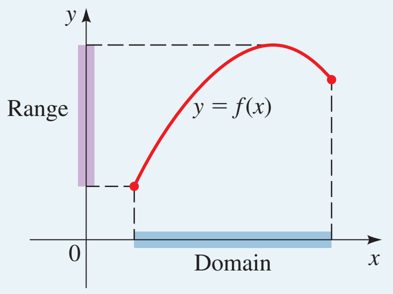
The graph of a function helps us to picture the domain and range of the function on the x-axis and y-axis, as shown in the box below.

Domain and Range from a Graph

The **domain** and **range** of a function

$$y = f(x)$$

can be obtained from a graph of f, as shown in the figure. The domain is the set of all x-values for which f is defined, and the range is all the corresponding y-values.



Example 2 ■ Finding the Domain and Range from a Graph

(a) Use a graphing device to draw the graph of $f(x) = \sqrt{4 - x^2}$.

(b) Find the domain and range of f.

Solution

(a) The graph is shown in Figure 2.

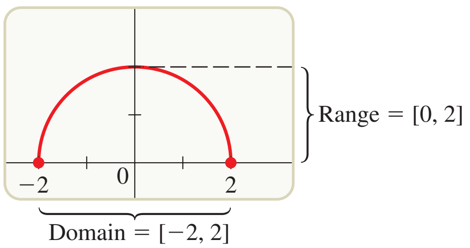


Figure 2 | Graph of $f(x) = \sqrt{4 - x^2}$

(b) From the graph in Figure 2 we see that the domain is $[-2, 2]$ and the range is $[0, 2]$.

Now Try Exercise 25 ■

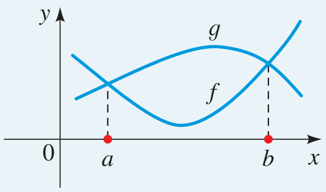
■ Comparing Function Values: Solving Equations and Inequalities Graphically

We can compare the values of two functions f and g visually by drawing their graphs. The points at which the graphs intersect are the points where the values of the two functions are equal. So the solutions of the equation $f(x) = g(x)$ are the values of x at which the two graphs intersect. The points at which the graph of g is higher than the graph of f are the points where the values of g are greater than the values of f. So the solutions of the inequality $f(x) < g(x)$ are the values of x at which the graph of g is *higher than* the graph of f.

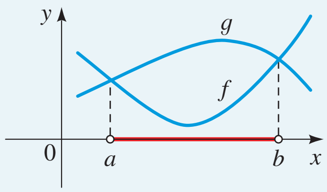
Solving Equations and Inequalities Graphically

The **solution(s) of the equation** $f(x) = g(x)$ are the values of x where the graphs of f and g intersect.

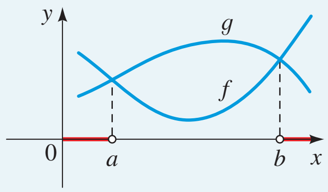
The **solution(s) of the inequality** $f(x) < g(x)$ are the values of x where the graph of g is higher than the graph of f. The solutions of the inequality $f(x) > g(x)$ are the values of x where the graph of f is higher than the graph of g.



The solutions of $f(x) = g(x)$ are the values a and b.



The solution of $f(x) < g(x)$ is the interval (a, b).



The solution of $f(x) > g(x)$ is a union of intervals $(-\infty, a) \cup (b, \infty)$.

We can use these observations to solve equations and inequalities graphically, as the next example illustrates.

Example 3 ■ Solving Graphically

Solve the given equation or inequality graphically.

(a) $2x^2 + 3 = 5x + 6$ **(b)** $2x^2 + 3 \le 5x + 6$ **(c)** $2x^2 + 3 > 5x + 6$

You can also solve the equations and inequalities algebraically. Check that your solutions match the solutions we obtained graphically.

Solution We first define functions f and g that correspond to the left-hand side and to the right-hand side of the equation or inequality. So we define

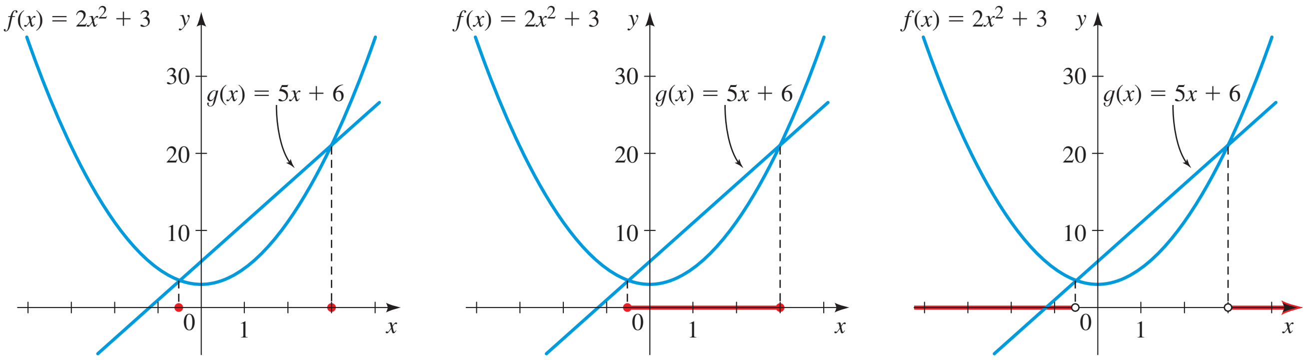
$$f(x) = 2x^2 + 3 \qquad \text{and} \qquad g(x) = 5x + 6$$

Next, we sketch graphs of f and g on the same set of axes.

(a) The given equation is equivalent to $f(x) = g(x)$. From the graph in Figure 3(a) we see that the solutions of the equation are $x = -0.5$ and $x = 3$.

(b) The given inequality is equivalent to $f(x) \le g(x)$. From the graph in Figure 3(b) we see that the solution is the interval $[-0.5, 3]$.

(c) The given inequality is equivalent to $f(x) > g(x)$. From the graph in Figure 3(c) we see that the solution is $(-\infty, -0.5) \cup (3, \infty)$.



(a) Solutions: $x = -0.5, 3$ (b) Solution: $[-0.5, 3]$ (c) Solution: $(-\infty, -0.5) \cup (3, \infty)$

Figure 3 | Graphs of $f(x) = 2x^2 + 3$ and $g(x) = 5x + 6$

Now Try Exercises 9 and 29 ■

Another way to solve an equation graphically is to first move all terms to one side of the equation and then graph the function that corresponds to the nonzero side of the equation. In this case the solutions of the equation are the x-intercepts of the graph. We can use this same method to solve inequalities graphically, as the following example shows.

Example 4 ■ Solving Graphically

Solve the given equation or inequality graphically.

(a) $x^3 + 6 = 2x^2 + 5x$ **(b)** $x^3 + 6 \geq 2x^2 + 5x$

Solution We first move all terms to one side to obtain an equivalent equation (or inequality). For the equation in part (a) we obtain

$$x^3 - 2x^2 - 5x + 6 = 0 \qquad \text{Move terms to LHS}$$

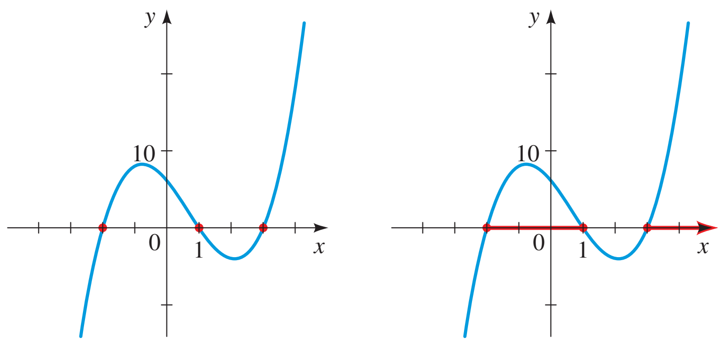
Then we define a function f by

$$f(x) = x^3 - 2x^2 - 5x + 6 \qquad \text{Define } f$$

Next, we use a graphing device to graph f, as shown in Figure 4.

(a) The given equation is the same as $f(x) = 0$, so the solutions are the x-intercepts of the graph. From Figure 4(a) we see that the solutions are $x = -2$, $x = 1$, and $x = 3$.

(b) The given inequality is the same as $f(x) \geq 0$, so the solutions are the x-values at which the graph of f is on or above the x-axis. From Figure 4(b) we see the solution is $[-2, 1] \cup [3, \infty)$.



(a) Solution: $x = -2, 1, 3$ (b) Solution: $[-2, 1] \cup [3, \infty)$

Figure 4 | Graphs of $f(x) = x^3 - 2x^2 - 5x + 6$

Now Try Exercise 33 ■

■ Increasing and Decreasing Functions

It is very useful to know where the graph of a function rises and where it falls. The graph shown in Figure 5 rises, falls, then rises again as we move from left to right: It rises from A to B, falls from B to C, and rises again from C to D. The function f is said to be *increasing* when its graph rises and *decreasing* when its graph falls.

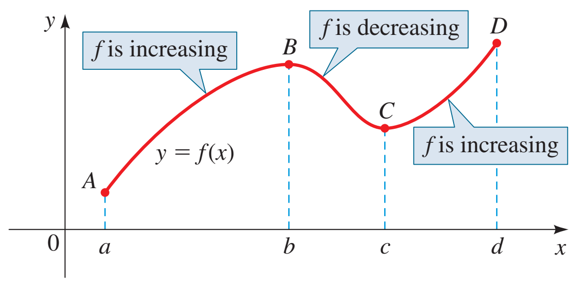


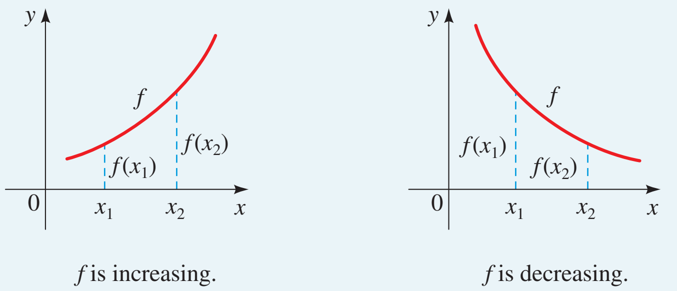
Figure 5 | f is increasing on (a, b) and (c, d); f is decreasing on (b, c)

We have the following definition.

From the definition we see that a function increases or decreases *on an interval*. It does not make sense to apply these definitions at a single point.

Definition of Increasing and Decreasing Functions

f is **increasing** on an interval I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in I.
f is **decreasing** on an interval I if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$ in I.



Example 5 ■ Intervals on Which a Function Increases or Decreases

The graph in Figure 6 gives the weight W of a person at age x. Determine the intervals on which the function W is increasing and on which it is decreasing.

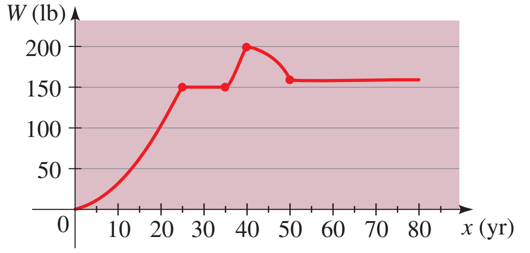


Figure 6 | Weight as a function of age

Solution The function W is increasing on the intervals $(0, 25)$ and $(35, 40)$. It is decreasing on $(40, 50)$. The function W is constant (neither increasing nor decreasing) on $(25, 35)$ and $(50, 80)$. This means that the person gained weight until age 25, then gained weight again between ages 35 and 40, and then lost weight between ages 40 and 50.

Now Try Exercise 65 ■

Note By convention we write the intervals on which a function is increasing or decreasing as open intervals. (It would also be true to say that the function is increasing or decreasing on the corresponding closed interval. So for instance, it is also correct to say that the function W in Example 5 is decreasing on $[40, 50]$.)

Example 6 ■ Finding Intervals on Which a Function Increases or Decreases

(a) Sketch a graph of the function $f(x) = 12x^2 + 4x^3 - 3x^4$.

(b) Find the domain and range of f.

(c) Find the intervals on which f is increasing and on which f is decreasing.

The Picture Art Collection/Alamy Stock Photo

DOROTHY VAUGHAN (1910–2008) is famous for her work in leading a team of "human computers" at NASA. Vaughan was recognized as a talented mathematician and teacher when she was hired in 1943 (at the height of World War II) to work with a group of mathematicians at NASA. Vaughn believed that the job was a temporary war job, but her exceptional expertise was soon recognized, and she was chosen to lead the team, becoming the first African American woman to do so. Vaughn used complex numerical methods to calculate flight trajectories of rockets and flight characteristics of aircraft. Electronic computers had not yet been invented, so computations were done using pencil and paper, hence the term "human computers." When NASA introduced digital computers, Vaughn taught her team how to compute with the new technology. Vaughan said that working as a "computer" during the Space Race felt like being on "the cutting edge of something very exciting." In 2019 Vaughn was posthumously awarded the Congressional Gold Medal for her work at NASA.

The 2016 hit film *Hidden Figures* features the work of Dorothy and her colleagues Kathrine Johnson and Mary Jackson as "human computers" at NASA.

Solution

(a) We use a graphing device to sketch the graph in Figure 7 in an appropriate viewing rectangle.

(b) The domain of f is $\mathbb{R}$ because f is defined for all real numbers. From the graph, we find that the highest value is $f(2) = 32$. So the range of f is $(-\infty, 32]$.

(c) From the graph we see that f is increasing on the intervals $(-\infty, -1)$ and $(0, 2)$ and is decreasing on $(-1, 0)$ and $(2, \infty)$.

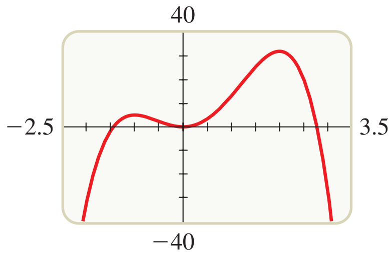


Figure 7 | Graph of $f(x) = 12x^2 + 4x^3 - 3x^4$

Now Try Exercise 41 ■

Example 7 ■ Finding Intervals Where a Function Increases and Decreases

(a) Sketch the graph of the function $f(x) = x^{2/3}$.

(b) Find the domain and range of the function.

(c) Find the intervals on which f is increasing and on which f is decreasing.

Solution

(a) We use a graphing device to sketch the graph in Figure 8.

(b) From the graph we observe that the domain of f is $\mathbb{R}$ and the range is $[0, \infty)$.

(c) From the graph we see that f is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$.

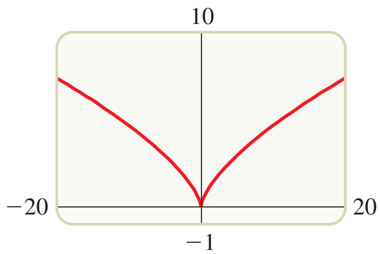


Figure 8 | Graph of $f(x) = x^{2/3}$

Now Try Exercise 47 ■

■ Local Maximum and Minimum Values of a Function

Finding the largest or smallest values of a function is important in many applications. For example, if a function represents revenue or profit, then we are interested in its maximum value. For a function that represents cost, we would want to find its minimum value. (See *Focus on Modeling: Modeling with Functions* at the end of this chapter for many such examples.) We can easily find these values from the graph of a function. We first define what we mean by a local maximum or minimum.

Local Maxima and Minima of a Function

1. The function value $f(a)$ is a **local maximum value** of f if

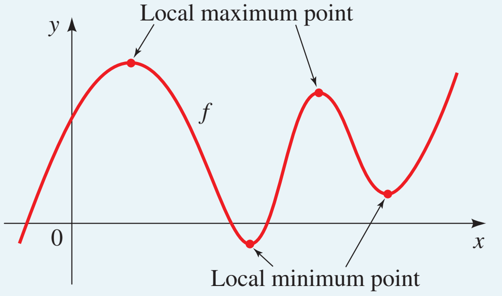
$$f(a) \geq f(x) \quad \text{when } x \text{ is near } a$$

(This means that $f(a) \geq f(x)$ for all x in some open interval containing a.) In this case we say that f has a **local maximum** at $x = a$.

2. The function value $f(a)$ is a **local minimum value** of f if

$$f(a) \leq f(x) \quad \text{when } x \text{ is near } a$$

(This means that $f(a) \leq f(x)$ for all x in some open interval containing a.) In this case we say that f has a **local minimum** at $x = a$.



We can find the local maximum and minimum values of a function using a graphing device. If there is a viewing rectangle such that the point $(a, f(a))$ is the highest point on the graph of f *within* the viewing rectangle (not on the edge), then the number $f(a)$ is a local maximum value of f. (See Figure 9.) Notice that $f(a) \geq f(x)$ for all numbers x that are close to a.

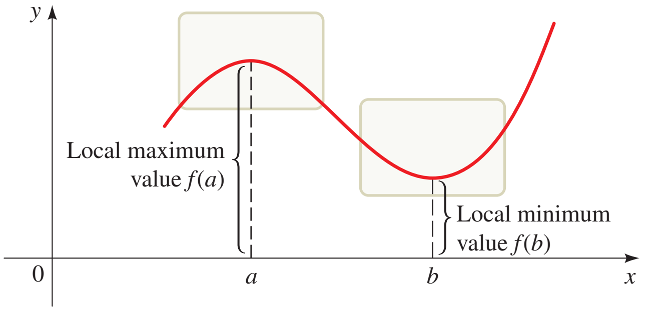


Figure 9

Similarly, if there is a viewing rectangle such that the point $(b, f(b))$ is the lowest point on the graph of f within the viewing rectangle, then the number $f(b)$ is a local minimum value of f. In this case $f(b) \leq f(x)$ for all numbers x that are close to b.

In the next two examples we use graphing devices to find local maxima and minima of functions from a graph.

On many math apps, you can find the coordinates of local maxima and minima by simply clicking on the appropriate point on the graph of the function.

Example 8 ■ Finding Local Maxima and Minima from a Graph

Find the local maximum and minimum values of the function $f(x) = x^3 - 8x + 1$, rounded to three decimal places.

Solution The graph of f is shown in Figure 10. There appears to be one local maximum between $x = -2$ and $x = -1$, and one local minimum between $x = 1$ and $x = 2$.

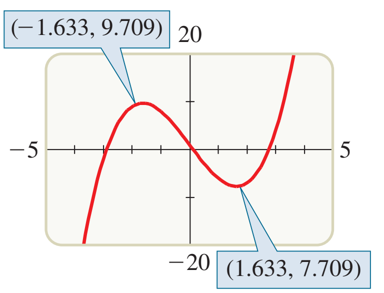


Figure 10 | Graph of $f(x) = x^3 - 8x + 1$

We see from the graph that the local maximum value of y is 9.709, and this value occurs when x is -1.633, rounded to three decimal places. The local minimum value is about -7.709, and this value occurs when $x \approx 1.633$.

Now Try Exercise 55

Example 9 ■ A Model for Managing Traffic

See the *Discovery Project* referenced in Section 3.6, for how this model is obtained.

A highway engineer develops a formula to estimate the number of cars that can safely travel a particular highway at a given speed. The engineer assumes that each car is 17 ft long, travels at a speed of x mi/h, and follows the car in front of it at the safe following distance for that speed. The number N of cars that can pass a given point per minute can be modeled by the function

$$N(x) = \frac{88x}{17 + 17\left(\dfrac{x}{20}\right)^2}$$

Graph the function in the viewing rectangle $[0, 100]$ by $[0, 60]$.

(a) Find the intervals on which the function N is increasing and on which it is decreasing.

(b) Find the maximum value of N. What is the maximum carrying capacity of the road, and at what speed is it achieved?

Solution The graph is shown in Figure 11.

(a) From the graph we see that the function N is increasing on $(0, 20)$ and decreasing on $(20, \infty)$.

(b) There appears to be a maximum between $x = 19$ and $x = 21$. From the graph we see that the maximum value of N is about 51.76, and it occurs when x is 20. So the maximum carrying capacity is about 52 cars per minute at a speed of 20 mi/h.

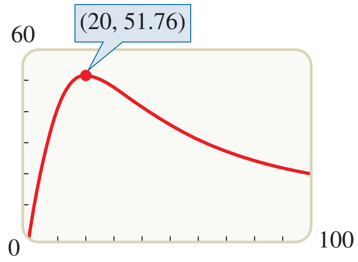
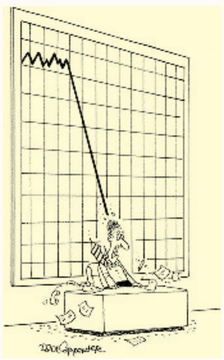


Figure 11 | Highway capacity at speed x

Now Try Exercise 73

Discovery Project ■ Every Graph Tells a Story



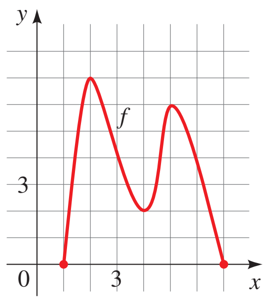
Dave Carpenter/CartoonStock.com

A graph can often describe a real-world "story" much more quickly and effectively than many words. For example, the stock market crash of 1929 is effectively described by a graph of the Dow Jones Industrial Average. No words are needed to convey the message in the cartoon shown here. In this project we describe, or tell the story, that corresponds to a given graph as well as make graphs that correspond to a real-world story. You can find the project at **www.stewartmath.com**.

2.3 Exercises

Concepts

1–5 ■ Refer to the graph of f shown below.



1. To find a function value $f(a)$ from the graph of f, we find the height of the graph above the x-axis at $x =$ ______. From the graph of f we see that $f(3) =$ ______ and $f(5) =$ ____. The net change in f between $x = 3$ and $x = 5$ is $f($____$) - f($____$) =$ ____.

2. The domain of the function f is all the ___-values of the points on the graph, and the range is all the corresponding ___-values. From the graph of f we see that the domain of f is the interval ______ and the range of f is the interval ______.

3. (a) If f is increasing on an interval, then the y-values of the points on the graph ______ as the x-values increase. From the graph of f we see that f is increasing on the intervals ______ and ______.

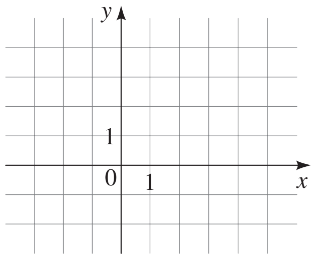
(b) If f is decreasing on an interval, then the y-values of the points on the graph ______ as the x-values increase. From the graph of f we see that f is decreasing on the intervals ______ and ______.

4. (a) A function value $f(a)$ is a local maximum value of f if $f(a)$ is the ______ value of f on some open interval containing a. From the graph of f we see that there are two local maximum values of f: One local maximum is ______, and it occurs when $x = 2$; the other local maximum is ______, and it occurs when $x =$ ______.

(b) The function value $f(a)$ is a local minimum value of f if $f(a)$ is the ______ value of f on some open interval containing a. From the graph of f we see that there is one local minimum value of f. The local minimum value is ______, and it occurs when $x =$ ______.

5. The solutions of the equation $f(x) = 0$ are the ____-intercepts of the graph of f. The solution of the inequality $f(x) \geq 0$ is the set of x-values at which the graph of f is on or above the ____-axis. From the graph of f we find that the solutions of the equation $f(x) = 0$ are $x =$ ____ and $x =$ ____, and the solution of the inequality $f(x) \geq 0$ is ______________.

6. (a) To solve the equation $2x + 1 = -x + 4$ graphically, we graph the functions $f(x) =$ ______________ and $g(x) =$ ______________ on the same set of axes and determine the values of x at which the graphs of f and g intersect. Graph f and g below, and use the graphs to solve the equation. The solution is $x =$ ________.



(b) To solve the inequality $2x + 1 < -x + 4$ graphically, we graph the functions $f(x) =$ ______________ and $g(x) =$ ______________ on the same set of axes and find the values of x at which the graph of g is ______________ (higher/lower) than the graph of f. From the graphs in part (a) we see that the solution of the inequality is the interval (____, ____).

Skills

7. Values of a Function The graph of a function h is given.

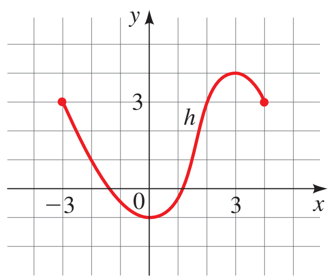
(a) Find $h(-2)$, $h(0)$, $h(2)$, and $h(3)$.

(b) Find the domain and range of h.

(c) Find the values of x for which $h(x) = 3$.

(d) Find the values of x for which $h(x) \leq 3$.

(e) Find the net change in h between $x = -3$ and $x = 3$.



8. Values of a Function The graph of a function g is given.

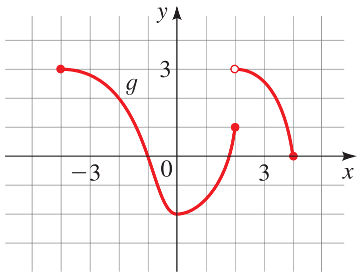
(a) Find $g(-4)$, $g(-2)$, $g(0)$, $g(2)$, and $g(4)$.

(b) Find the domain and range of g.

(c) Find the values of x for which $g(x) = 3$.

(d) Estimate the values of x for which $g(x) \leq 0$.

(e) Find the net change in g between $x = -1$ and $x = 2$.



9. Solving Equations and Inequalities Graphically Graphs of the functions f and g are given.

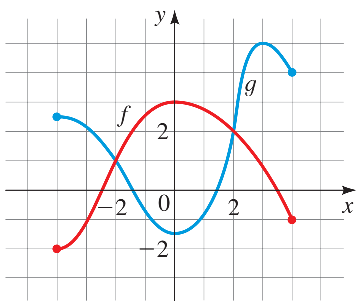
(a) Which is larger, $f(0)$ or $g(0)$?

(b) Which is larger, $f(-1)$ or $g(-1)$?

(c) For which values of x is $f(x) = g(x)$?

(d) Find the values of x for which $f(x) \leq g(x)$.

(e) Find the values of x for which $f(x) > g(x)$.



10. Solving Equations and Inequalities Graphically Graphs of the functions f and g are given.

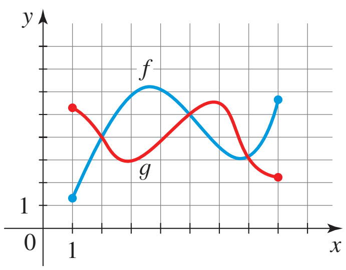
(a) Which is larger, $f(6)$ or $g(6)$?

(b) Which is larger, $f(3)$ or $g(3)$?

(c) Find the values of x for which $f(x) = g(x)$.

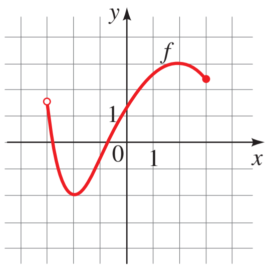
(d) Find the values of x for which $f(x) \leq g(x)$.

(e) Find the values of x for which $f(x) > g(x)$.



11–14 ■ Domain and Range from a Graph The graph of a function f is given. Use the graph to find the domain and range of f.

11.



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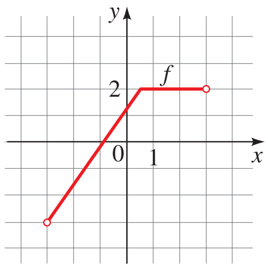

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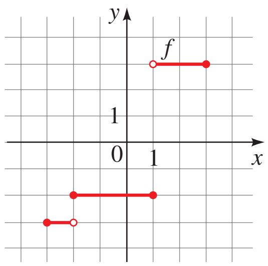

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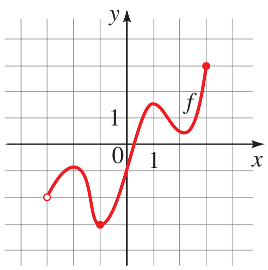


15–22 ■ Domain and Range from a Graph A function f is given. **(a)** Sketch a graph of f. **(b)** Use the graph to find the domain and range of f.

15. $f(x) = 2x + 3$ **16.** $f(x) = 3x - 2$

17. $f(x) = x^2 - 3$ **18.** $f(x) = x^2 + 2$

19. $f(x) = x - 2, \quad -2 \leq x \leq 5$

20. $f(x) = 4 - 2x, \quad 1 < x < 4$

21. $f(x) = x^2 - 1, \quad -3 \leq x \leq 3$

22. $f(x) = x^3 - 1; \quad -3 \leq x \leq 3$

23–28 ■ Finding Domain and Range Graphically A function f is given. **(a)** Use a graphing device to draw the graph of f. **(b)** Find the domain and range of f from the graph.

23. $f(x) = x^2 + 4x + 3$ **24.** $f(x) = \sqrt{x - 1}$

25. $f(x) = -\sqrt{36 - x^2}$ **26.** $f(x) = \dfrac{4}{x^2 + 2}$

27. $f(x) = \dfrac{2x}{x^2 + 1}$ **28.** $f(x) = x^4 - 6x^2$

29–32 ■ Solving Equations and Inequalities Graphically Solve the given equation or inequality graphically.

29. (a) $4x - 5 = 5 - x$ **(b)** $4x - 5 < 5 - x$

30. (a) $3 - 4x = 8x - 9$ **(b)** $3 - 4x \geq 8x - 9$

31. (a) $x^2 = 2 - x$ **(b)** $x^2 \leq 2 - x$

32. (a) $-x^2 = 3 - 4x$ **(b)** $-x^2 \geq 3 - 4x$

33–36 ■ Solving Equations and Inequalities Graphically Solve the given equation or inequality graphically. State your answers rounded to two decimals.

33. (a) $x^3 + 3x^2 = -x^2 + 3x + 7$

(b) $x^3 + 3x^2 \geq -x^2 + 3x + 7$

34. (a) $5x^2 - x^3 = -x^2 + 3x + 4$

(b) $5x^2 - x^3 \le -x^2 + 3x + 4$

35. (a) $16x^3 + 16x^2 = x + 1$

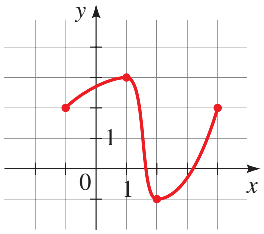
(b) $16x^3 + 16x^2 \ge x + 1$

36. (a) $1 + \sqrt{x} = \sqrt{x^2 + 1}$

(b) $1 + \sqrt{x} > \sqrt{x^2 + 1}$

37–40 ■ Increasing and Decreasing The graph of a function f is given. Use the graph to estimate the following. **(a)** The domain and range of f. **(b)** The intervals on which f is increasing and on which f is decreasing.

37.



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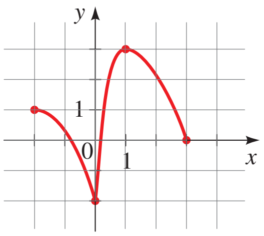

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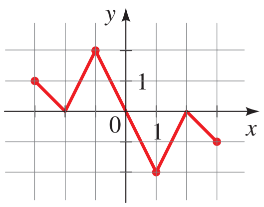

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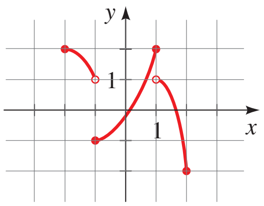


41–50 ■ Increasing and Decreasing A function f is given. **(a)** Use a graphing device to draw the graph of f. **(b)** Find the domain and range of f. **(c)** State approximately the intervals on which f is increasing and on which f is decreasing.

41. $f(x) = x^2 - 5x$

42. $f(x) = x^3 - 4x$

43. $f(x) = 2x^3 - 3x^2 - 12x$

44. $f(x) = x^4 - 16x^2$

45. $f(x) = x^3 + 2x^2 - x - 2$

46. $f(x) = x^4 - 4x^3 + 2x^2 + 4x - 3$

47. $f(x) = x^{2/5}$

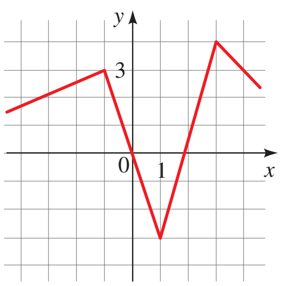
48. $f(x) = 4 - x^{2/3}$

49. $f(x) = 2 + \sqrt{x + 3}$

50. $f(x) = \sqrt{25 - x^2}$

51–54 ■ Local Maximum and Minimum Values The graph of a function f is given. Use the graph to estimate the following: **(a)** All the local maximum and minimum values of the function and the value of x at which each occurs. **(b)** The intervals on which the function is increasing and on which the function is decreasing.

51.



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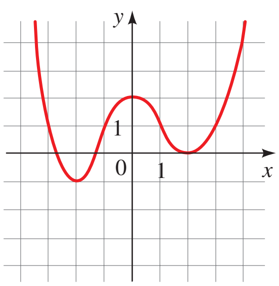

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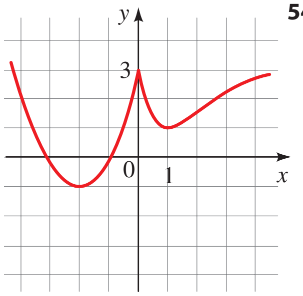

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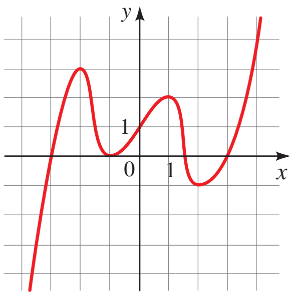


55–62 ■ Local Maximum and Minimum Values A function is given. **(a)** Find all the local maximum and minimum values of the function and the value of x at which each occurs. State each answer rounded to two decimal places. **(b)** Find the intervals on which the function is increasing and on which the function is decreasing. State each answer rounded to two decimal places.

55. $f(x) = x^3 - x$

56. $f(x) = 3 + x + x^2 - x^3$

57. $g(x) = x^4 - 2x^3 - 11x^2$

58. $g(x) = x^5 - 8x^3 + 20x$

59. $U(x) = x\sqrt{6 - x}$

60. $U(x) = x\sqrt{x - x^2}$

61. $V(x) = \dfrac{1 - x^2}{x^3}$

62. $V(x) = \dfrac{1}{x^2 + x + 1}$

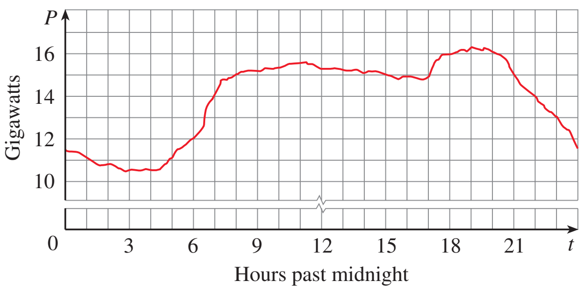
Applications

63. Power Consumption Shown below is a graph of the electric power consumption in the New England states (Connecticut, Maine, Massachusetts, New Hampshire, Rhode Island, and Vermont) for a day in October (P is measured in gigawatts and t is measured in hours starting at midnight).

(a) What was the power consumption at 5:00 A.M.? At 10:00 P.M.?

(b) At what times was the power consumption the lowest? The highest?

(c) Find the net change in the power consumption from 5:00 A.M. to 10:00 P.M.



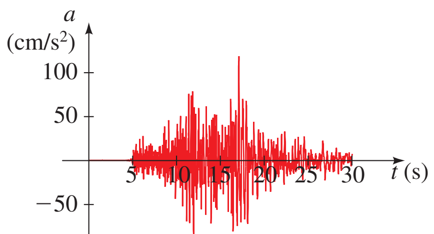
Source: US Energy Information Administration

64. Earthquake The graph shows the vertical acceleration of the ground from the 1994 Northridge earthquake in Los Angeles, as measured by a seismograph. (Here t represents the time in seconds.)

(a) At what time t did the earthquake first make noticeable movements of the earth?

(b) At what time t did the earthquake seem to end?

(c) At what time t was the maximum intensity of the earthquake reached?



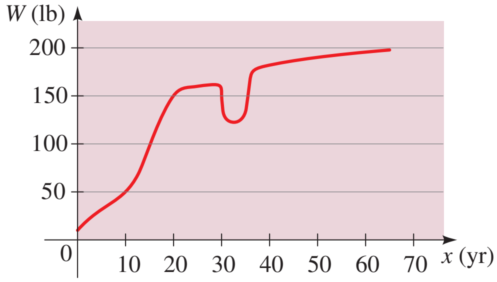
Source: California Department of Mines and Geology

65. Weight Function The graph gives the weight W of a person at age x.

(a) Determine the intervals on which the function W is increasing and those on which it is decreasing.

(b) What do you think happened when this person was 30 years old?

(c) Find the net change in the person's weight W from age 10 to age 20.

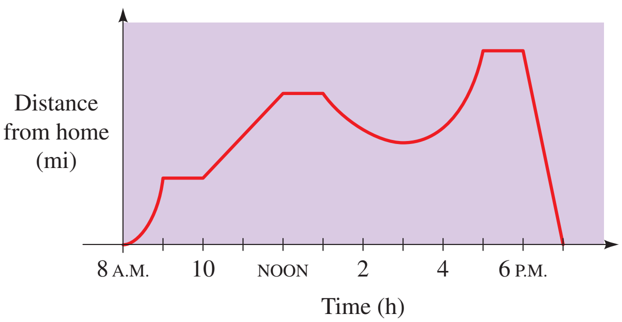


66. Distance Function The graph gives a sales representative's distance from home as a function of time on a certain day.

(a) Determine the time intervals on which the distance from home was increasing and those on which it was decreasing.

(b) Describe in words what the graph indicates about the representative's travels on this day.

(c) Find the net change in distance from home between noon and 1:00 P.M.

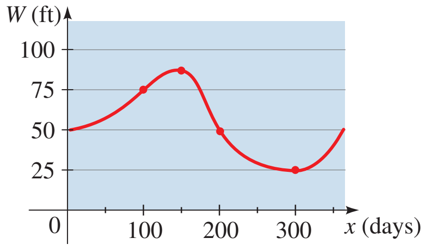


67. Changing Water Levels The graph shows the depth of water W in a reservoir over a one-year period as a function of the number of days x since the beginning of the year.

(a) Determine the intervals on which the function W is increasing and on which it is decreasing.

(b) At what value of x does W achieve a local maximum? A local minimum?

(c) Find the net change in the depth W from 100 days to 300 days.

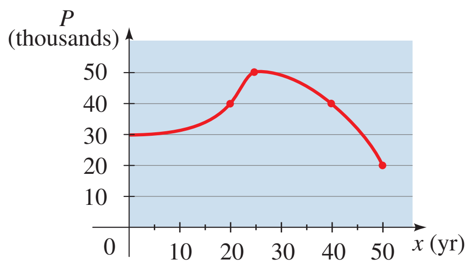


68. Population Growth and Decline The graph shows the population P in a small industrial city from 1970 to 2020. The variable x represents the number of years since 1970.

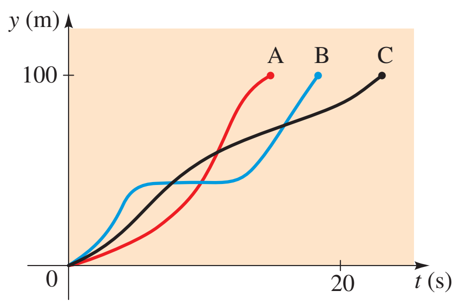
(a) Determine the intervals on which the function P is increasing and on which it is decreasing.

(b) What was the maximum population, and in what year was it attained?

(c) Find the net change in the population P from 1990 to 2010.



69. Hurdle Race Three runners compete in a 100-meter hurdle race. The graph depicts the distance run as a function of time for each runner. Describe in words what the graph tells you about this race. Who won the race? Did each runner finish the race? What do you think happened to Runner B?



70. Gravity Near the Moon We can use Newton's Law to calculate the acceleration a due to gravity of a spacecraft at a distance x km from the center of the moon:

$$a(x) = \frac{5 \times 10^6}{x^2}$$

where a is measured in m/s^2.

(a) The radius r of the moon is 1737 km. Graph the function a for values of x between r and $2r$.

(b) Use the graph to describe the behavior of the gravitational acceleration a as the distance x increases.

71. Radii of Stars Astronomers infer the radii of stars using the Stefan Boltzmann Law:

$$E(T) = (5.67 \times 10^{-8})T^4$$

where E is the energy radiated per unit of surface area measured in Watts per square meter (W/m^2) and T is the absolute temperature measured in kelvins (K).

(a) Graph the function E for temperatures T between 100 K and 300 K.

(b) Use the graph to describe the change in energy E as the temperature T increases.

72. Volume of Water Between 0°C and 30°C, the volume V (in cm^3) of 1 kg of water at a temperature T is given by the formula

$$V = 999.87 - 0.06426T + 0.0085043T^2 - 0.0000679T^3$$

Find the temperature at which the volume of 1 kg of water is a minimum.

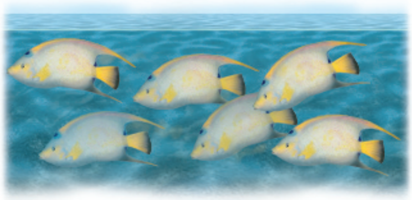
[*Source: Physics* by D. Halliday and R. Resnick]

73. Migrating Fish A fish swims at a speed v relative to the water, against a current of 5 mi/h. Using a mathematical model of energy expenditure, it can be shown that the total energy E required to swim a distance of 10 miles is given by

$$E(v) = 2.73v^3 \frac{10}{v - 5}$$

Biologists believe that migrating fish try to minimize the total energy required to swim a fixed distance. Find the value of v that minimizes the energy required.

[*Note:* This result has been verified; migrating fish swim against a current at a speed 50% greater than the speed of the current.]



74. Coughing When a foreign object that is lodged in the trachea (windpipe) forces a person to cough, the diaphragm thrusts upward, causing an increase in pressure in the lungs. At the same time, the trachea contracts, causing the expelled air to move faster and increasing the pressure on the foreign object. According to a mathematical model of coughing, the velocity v (in cm/s) of the airstream through an average-sized person's trachea is related to the radius r of the trachea (in cm) by the function

$$v(r) = 3.2(1 - r)r^2 \qquad \tfrac{1}{2} \le r \le 1$$

Determine the value of r for which v is a maximum.

Discuss ■ Discover ■ Prove ■ Write

75. Discuss: Functions That Are Always Increasing or Decreasing Sketch rough graphs of functions that are defined for all real numbers and that exhibit the indicated behavior (or explain why the behavior is impossible).

(a) f is always increasing, and $f(x) > 0$ for all x.

(b) f is always decreasing, and $f(x) > 0$ for all x.

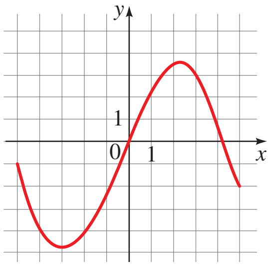
(c) f is always increasing, and $f(x) < 0$ for all x.

(d) f is always decreasing, and $f(x) < 0$ for all x.

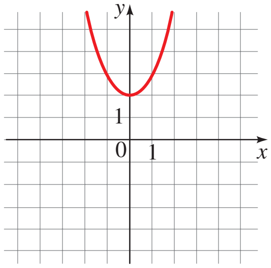
76. Discuss ■ Discover: Fixed Points A *fixed point* of a function f is a number x for which $f(x) = x$. How would you go about finding the fixed points of a function algebraically? Graphically? Find the fixed points of the function f.

(a) $f(x) = 5x - x^2$ **(b)** $f(x) = x^3 + x + 1$

(c)



(d)



PS *Introduce something extra.* To find the fixed points algebraically, solve the equation $f(x) = x$. To find fixed points graphically, introduce the graph of $y = x$.

2.4 Average Rate of Change of a Function

■ Average Rate of Change ■ Linear Functions Have Constant Rate of Change

Functions are often used to model changing quantities. In this section we learn how to find the rate at which the values of a function change as the input variable changes.

■ Average Rate of Change

We are all familiar with the concept of speed: If you drive a distance of 120 miles in 2 hours, then your average speed, or rate of travel, is $\frac{120\text{ mi}}{2\text{ h}} = 60$ mi/h. Now suppose you take a car trip and record the distance that you travel every few minutes. The distance s you have traveled is a function of the time t:

$$s(t) = \text{total distance traveled at time } t$$

We graph the function s as shown in Figure 1. The graph shows that you have traveled a total of 50 miles after 1 hour, 75 miles after 2 hours, 140 miles after 3 hours, and so on. To find your *average* speed between any two points on the trip, we divide the distance traveled by the time elapsed.

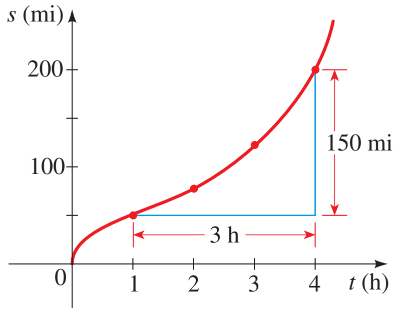


Figure 1 | Average speed

Let's calculate your average speed between 1:00 P.M. and 4:00 P.M. The time elapsed is $4 - 1 = 3$ hours. To find the distance you traveled, we subtract the distance at 1:00 P.M. from the distance at 4:00 P.M., that is, $200 - 50 = 150$ mi. Thus your average speed is

$$\text{average speed} = \frac{\text{distance traveled}}{\text{time elapsed}} = \frac{150\text{ mi}}{3\text{ h}} = 50\text{ mi/h}$$

The average speed we have just calculated can be expressed by using function notation:

$$\text{average speed} = \frac{s(4) - s(1)}{4 - 1} = \frac{200 - 50}{3} = 50\text{ mi/h}$$

Observe that the average speed is different over different time intervals. For example, between 2:00 P.M. and 3:00 P.M. we find that

$$\text{average speed} = \frac{s(3) - s(2)}{3 - 2} = \frac{140 - 75}{1} = 65\text{ mi/h}$$

Finding average rates of change is important in many contexts. For instance, we might be interested in knowing how quickly the air temperature is dropping as a storm approaches or how fast revenues are increasing from the sale of a new product. So we need to know how to determine the average rate of change of the functions that model these quantities. In fact, the concept of average rate of change can be defined for any function.

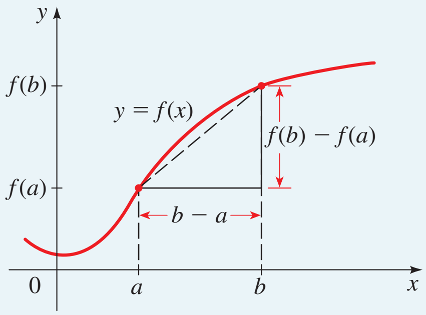
The expression $\dfrac{f(b) - f(a)}{b - a}$ is called a *difference quotient*.

Average Rate of Change

The **average rate of change** of the function $y = f(x)$ between $x = a$ and $x = b$ is

$$\text{average rate of change} = \frac{\text{change in } y}{\text{change in } x} = \frac{f(b) - f(a)}{b - a}$$

The average rate of change is the slope of the **secant line** between $x = a$ and $x = b$ on the graph of f, that is, the line that passes through $(a, f(a))$ and $(b, f(b))$.



In the expression for average rate of change, the numerator $f(b) - f(a)$ is the net change in the value of f between $x = a$ and $x = b$ (see Section 2.1).

Example 1 ■ Calculating the Average Rate of Change

For the function $f(x) = (x - 3)^2$, whose graph is shown in Figure 2, find the net change and the average rate of change between the following values of x.

(a) $x = 1$ and $x = 3$

(b) $x = 4$ and $x = 7$

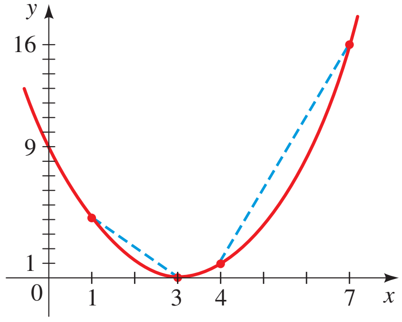


Figure 2 | $f(x) = (x - 3)^2$

Solution

(a)
$$\begin{aligned} \text{Net change} &= f(3) - f(1) && \text{Definition} \\ &= (3 - 3)^2 - (1 - 3)^2 && \text{Use } f(x) = (x-3)^2 \\ &= -4 && \text{Calculate} \end{aligned}$$

$$\text{Average rate of change} = \frac{f(3) - f(1)}{3 - 1} \qquad \text{Definition}$$

$$= \frac{-4}{2} = -2 \qquad \text{Calculate}$$

(b) Net change $= f(7) - f(4)$ Definition

$$= (7 - 3)^2 - (4 - 3)^2 \qquad f(x) = (x - 3)^2$$

$$= 15 \qquad \text{Calculate}$$

$$\text{Average rate of change} = \frac{f(7) - f(4)}{7 - 4} \qquad \text{Definition}$$

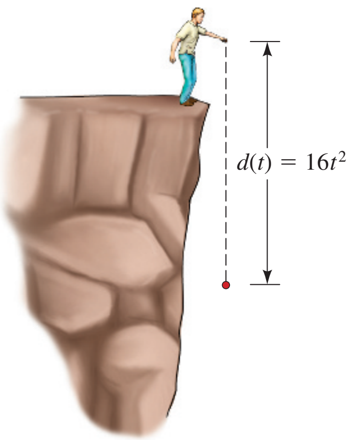
$$= \frac{15}{3} = 5 \qquad \text{Calculate}$$

Now Try Exercise 13 ■

Example 2 ■ Average Speed of a Falling Object

If an object is dropped from a high cliff or a tall building, then the distance (in feet) it has fallen after t seconds is given by the function $d(t) = 16t^2$. Find its average speed (average rate of change) over the following intervals.

(a) Between 1 s and 5 s **(b)** Between $t = a$ and $t = a + h$



Function: In t seconds the stone falls $16t^2$ ft.

Solution

(a)

$$\text{Average rate of change} = \frac{d(5) - d(1)}{5 - 1} \qquad \text{Definition}$$

$$= \frac{16(5)^2 - 16(1)^2}{5 - 1} \qquad d(t) = 16t^2$$

$$= \frac{400 - 16}{4} \qquad \text{Calculate}$$

$$= 96 \text{ ft/s} \qquad \text{Calculate}$$

(b)

$$\text{Average rate of change} = \frac{d(a + h) - d(a)}{(a + h) - a} \qquad \text{Definition}$$

$$= \frac{16(a + h)^2 - 16(a)^2}{(a + h) - a} \qquad d(t) = 16t^2$$

$$= \frac{16(a^2 + 2ah + h^2 - a^2)}{h} \qquad \text{Expand and factor 16}$$

$$= \frac{16(2ah + h^2)}{h} \qquad \text{Simplify numerator}$$

$$= \frac{16h(2a + h)}{h} \qquad \text{Factor } h$$

$$= 16(2a + h) \qquad \text{Simplify}$$

Now Try Exercises 15 and 19 ■

Note The expression for the average rate of change (or the difference quotient) as in Example 2(b) is used in calculus to calculate an *instantaneous rate of change* (see Exercise 39 and Section 12.3).

The graphs in Figure 3 show that if a function is increasing on an interval, then the average rate of change between any two points in that interval is positive, whereas if a function is decreasing on an interval, then the average rate of change between any two points in that interval is negative.

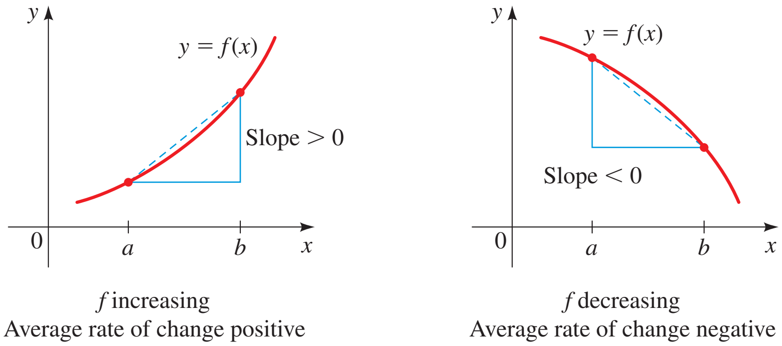


Figure 3

Example 3 ■ Average Rate of Temperature Change

Time	Temperature (°F)
8:00 A.M.	38
9:00 A.M.	40
10:00 A.M.	44
11:00 A.M.	50
12:00 NOON	56
1:00 P.M.	62
2:00 P.M.	66
3:00 P.M.	67
4:00 P.M.	64
5:00 P.M.	58
6:00 P.M.	55
7:00 P.M.	51

The table in the margin gives the outdoor temperatures observed by a science student on a spring day. Draw a graph of the data, and find the average rate of change of temperature between the following times.

(a) 8:00 A.M. and 9:00 A.M.

(b) 1:00 P.M. and 3:00 P.M.

(c) 4:00 P.M. and 7:00 P.M.

Solution A graph of the temperature data is shown in Figure 4. Let t represent time, measured in hours since midnight (so, for example, 2:00 P.M. corresponds to $t = 14$). Define the function F by $F(t) =$ temperature at time t.

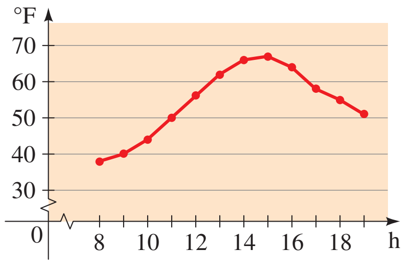


Figure 4

Temperature at 9:00 A.M.   Temperature at 8:00 A.M.

(a) Average rate of change $= \dfrac{F(9) - F(8)}{9 - 8} = \dfrac{40 - 38}{9 - 8} = 2$

The average rate of change was 2°F per hour.

(b) Average rate of change $= \dfrac{F(15) - F(13)}{15 - 13} = \dfrac{67 - 62}{2} = 2.5$

The average rate of change was 2.5°F per hour.

(c) Average rate of change $= \dfrac{F(19) - F(16)}{19 - 16} = \dfrac{51 - 64}{3} \approx -4.3$

The average rate of change was about -4.3°F per hour during this time interval. The negative sign indicates that the temperature was dropping.

Now Try Exercise 31 ■

Linear Functions Have Constant Rate of Change

Recall that a function of the form $f(x) = mx + b$ is a linear function (see Section 2.2). Its graph is a line with slope m. On the other hand, if a function f has constant rate of change, then it must be a linear function. (You are asked to prove these facts in Exercises 2.5.53 and 2.5.54) In general, the average rate of change of a linear function between any two x-values is the constant m. In the next example we find the average rate of change for a particular linear function.

Example 4 ■ Linear Functions Have Constant Rate of Change

Let $f(x) = 3x - 5$. Find the average rate of change of f between the following values of x.

(a) $x = 0$ and $x = 1$

(b) $x = 3$ and $x = 7$

(c) $x = a$ and $x = b$

What conclusion can you draw from your answers?

Solution

(a) Average rate of change $= \dfrac{f(1) - f(0)}{1 - 0} = \dfrac{(3 \cdot 1 - 5) - (3 \cdot 0 - 5)}{1}$

$$= \frac{(-2) - (-5)}{1} = 3$$

(b) Average rate of change $= \dfrac{f(7) - f(3)}{7 - 3} = \dfrac{(3 \cdot 7 - 5) - (3 \cdot 3 - 5)}{4}$

$$= \frac{16 - 4}{4} = 3$$

(c) Average rate of change $= \dfrac{f(b) - f(a)}{b - a} = \dfrac{(3b - 5) - (3a - 5)}{b - a}$

$$= \frac{3b - 3a - 5 + 5}{b - a} = \frac{3(b - a)}{b - a} = 3$$

It appears that the average rate of change is always 3 for this function. In fact, part (c) proves that the rate of change between any two arbitrary points $x = a$ and $x = b$ is 3.

Now Try Exercise 25 ■

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Discovery Project ■ Concavity: When Rates of Change Change

In the real world, rates of change often themselves change. A statement like “inflation is rising, but at a slower rate” involves a change of a rate of change. Also, the speed (rate of change of distance) of a car increases when a driver accelerates and decreases when a driver decelerates. From Example 4 we see that functions whose graph is a line (linear functions) have constant rate of change. In this project we explore how the shape of a graph corresponds to a changing rate of change. You can find the project at **www.stewartmath.com**.

2.4 Exercises

Concepts

1. If you travel 100 miles in two hours, then your average speed for the trip is

$$\text{average speed} = \frac{\boxed{}}{\boxed{}} = \underline{\qquad\qquad}$$

2. The average rate of change of a function f between $x = a$ and $x = b$ is

$$\text{average rate of change} = \frac{\boxed{}}{\boxed{}}$$

3. The average rate of change of the function $f(x) = x^2$ between $x = 1$ and $x = 5$ is

$$\text{average rate of change} = \frac{\boxed{}}{\boxed{}} = \underline{\qquad\qquad}$$

4. (a) The average rate of change of a function f between $x = a$ and $x = b$ is the slope of the ______________ line between $(a, f(a))$ and $(b, f(b))$.

(b) The average rate of change of the linear function $f(x) = 3x + 5$ between any two x-values is ________.

5–6 ■ *Yes or No*? If *No*, give a reason.

5. (a) Is the average rate of change of a function between $x = a$ and $x = b$ the slope of the secant line through $(a, f(a))$ and $(b, f(b))$?

(b) Is the average rate of change of a linear function the same for all intervals?

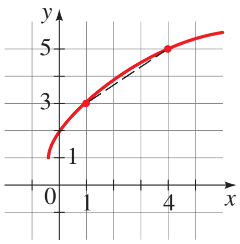
6. (a) Can the average rate of change of an increasing function ever be negative?

(b) If the average rate of change of a function between $x = a$ and $x = b$ is negative, then is the function necessarily decreasing on the interval (a, b)?

Skills

7–10 ■ **Net Change and Average Rate of Change** The graph of a function is given. Determine **(a)** the net change and **(b)** the average rate of change between the indicated points on the graph.

7.



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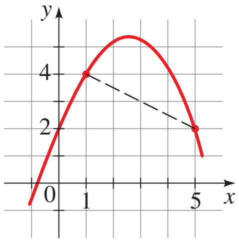

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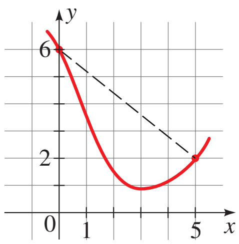

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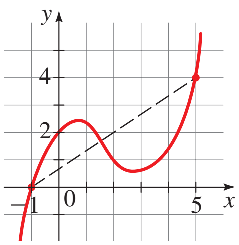


11–18 ■ **Net Change and Average Rate of Change** A function is given. Determine **(a)** the net change and **(b)** the average rate of change between the given values of the variable.

11. $f(t) = 5t + 3;\quad t = 4, t = 7$

12. $s(t) = 4 - 2t;\quad t = 1, t = 5$

13. $g(x) = 2 - \frac{1}{2}x;\quad x = -6, x = 10$

14. $h(x) = \dfrac{3x - 4}{5};\quad x = 2, x = 5$

15. $f(t) = 3t^2 + t;\quad t = 1, t = 3$

16. $f(z) = 3z - z^2;\quad z = -3, z = 0$

17. $f(x) = x^3 - 4x^2;\quad x = 0, x = 10$

18. $g(t) = t^4 - t^3 + t^2;\quad t = -2, t = 2$

19–24 ■ **Difference Quotient** A function is given. Find an expression for the difference quotient (or average rate of change) between $x = a$ and $x = a + h$. Simplify your answer.

19. $f(x) = 4x^2$

20. $f(x) = 3 - 10x^2$

21. $f(x) = \dfrac{1}{x}$

22. $f(x) = \dfrac{2}{x + 1}$

23. $f(x) = \sqrt{x}$

24. $f(x) = \dfrac{2}{x^2}$

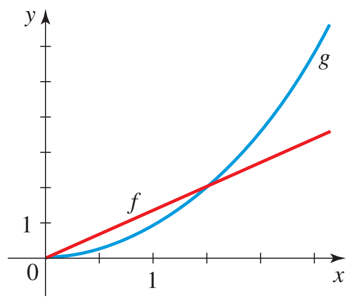
25–26 ■ **Average Rate of Change of a Linear Function** A linear function is given. **(a)** Find the average rate of change of the function between $x = a$ and $x = b$. **(b)** Show that the average rate of change is the same as the slope of the line.

25. $f(x) = \frac{1}{2}x + 3$

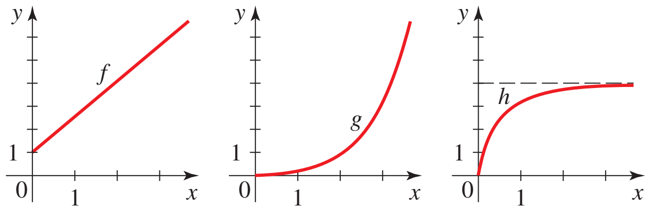
26. $g(x) = -4x + 2$

Skills Plus

27. Average Rate of Change The graphs of the functions f and g are shown. The function ____ (f or g) has a greater average rate of change between $x = 0$ and $x = 1$. The function ____ (f or g) has a greater average rate of change between $x = 1$ and $x = 2$. The functions f and g have the same average rate of change between $x =$ ____ and $x =$ ____.



28. Average Rate of Change Graphs of the functions f, g, and h are shown below. What can you say about the average rate of change of each function on the successive intervals $[0, 1]$, $[1, 2]$, $[2, 3]$, . . .?

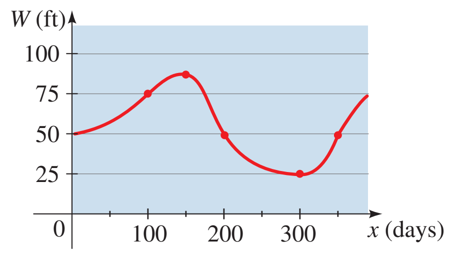


Applications

29. Changing Water Levels The graph shows the depth of water W in a reservoir over a one-year period as a function of the number of days x since the beginning of the year.

(a) What was the average rate of change of W between $x = 100$ and $x = 200$? What does the sign of your answer indicate?

(b) Identify an interval where the average rate of change is 0.

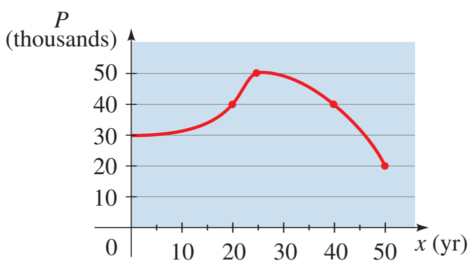


30. Population Growth and Decline The graph shows the population P in a small industrial city from 1970 to 2020. The variable x represents the number of years since 1970.

(a) What was the average rate of change of P between $x = 20$ and $x = 25$?

(b) Interpret the value of the average rate of change that you found in part (a).

(c) Identify a time period where the average rate of change is 0.



31. Population Growth and Decline The table gives the population in a small coastal community for the period 2002–2020. Figures shown are for January 1 in each year.

(a) What was the average rate of change of population between 2008 and 2012?

(b) What was the average rate of change of population between 2014 and 2018?

(c) For what period of time was the population increasing?

(d) For what period of time was the population decreasing?

Year	Population
2002	3220
2004	3645
2006	4357
2008	4869
2010	5871
2012	6375
2014	6288
2016	5318
2018	4921
2020	4636

32. Running Speed A runner is sprinting on a circular track that is 200 m in circumference. An observer uses a stopwatch to record the runner's time at the end of each lap, obtaining the data in the following table.

(a) What was the runner's average speed (rate) between 68 s and 152 s?

(b) What was the runner's average speed between 263 s and 412 s?

(c) Calculate the runner's speed for each lap. Is the runner slowing down, speeding up, or neither?

Time (s)	Distance (m)
32	200
68	400
108	600
152	800
203	1000
263	1200
335	1400
412	1600

33. Snack Cake Sales The table shows the number of creme-filled snack cakes sold in a small convenience store for the period 2010–2020.

(a) What was the average rate of change of sales between 2010 and 2020?

(b) What was the average rate of change of sales between 2011 and 2012?

(c) What was the average rate of change of sales between 2012 and 2013?

(d) Between which two successive years did snack cake sales *increase* most quickly? *Decrease* most quickly?

Year	Snack Cakes Sold
2010	1146
2011	1042
2012	638
2013	1145
2014	1738
2015	1804
2016	1121
2017	1987
2018	2533
2019	2983
2020	3629

34. Book Collection Between 2000 and 2020 a rare book collector purchased books at the rate of 60 books per year. Use this information to complete the following table. (Note that not every year is given in the table.)

Year	Number of Books	Year	Number of Books
2000	530	2015	
2001	590	2017	
2002		2018	
2006		2019	
2010		2020	1730
2012			

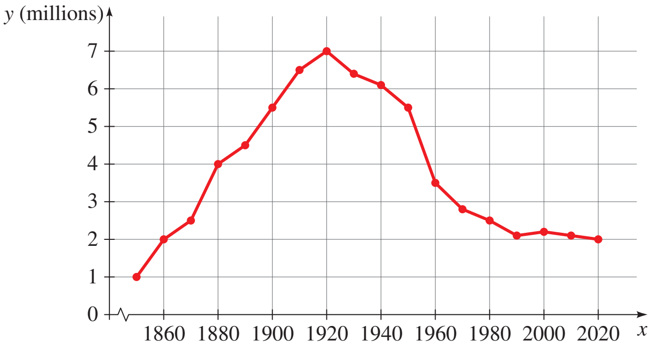
35. Cooling Soup When a bowl of hot soup is left in a room, the soup eventually cools to room temperature. The temperature T of the soup is a function of time t. The table below gives the temperature (in °F) of a bowl of soup t minutes after it was set on the table. Find the average rate of change of the temperature of the soup over the first 20 minutes and over the next 20 minutes. During which interval did the soup cool more quickly?

t (min)	T (°F)	t (min)	T (°F)
0	200	35	94
5	172	40	89
10	150	50	81
15	133	60	77
20	119	90	72
25	108	120	70
30	100	150	70

36. Farms in the United States The graph gives the number of farms in the United States from 1850 to 2020.

(a) Estimate the average rate of change in the number of farms between the years (i) 1860 and 1890 and (ii) 1950 and 1980.

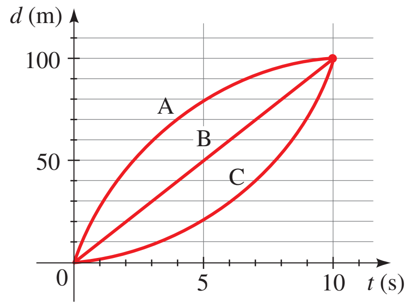
(b) In which decade did the number of farms experience the greatest average rate of decline?



37. Three-Way Tie A downhill skiing race ends in a three-way tie for first place. The graph shows distance as a function of time for each of the three winners, A, B, and C.

(a) Find the average speed for each skier.

(b) Describe the differences between the ways in which the three participants skied the race.

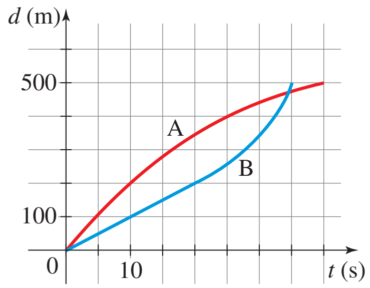


38. Speed Skating Two speed skaters, A and B, are racing in a 500-meter event. The graph shows the distance they have traveled as a function of the time from the start of the race.

(a) Who won the race?

(b) Find the average speed during the first 10 s for each skater.

(c) Find the average speed during the last 15 s for each skater.



■ Discuss ■ Discover ■ Prove ■ Write

39. Discuss ■ Discover: Limiting Behavior of Average Speed An object is dropped from a high cliff, and the distance (in feet) it has fallen after t seconds is given by the function $d(t) = 16t^2$. Complete the table to find the average speed during the given time intervals. Use the table to determine what value the average speed approaches as the time intervals get smaller and smaller. Is it reasonable to say that this value is the speed of the object at the instant $t = 3$? Explain.

$t = a$	$t = b$	Average Speed $= \dfrac{d(b) - d(a)}{b - a}$
3	3.5	
3	3.1	
3	3.01	
3	3.001	
3	3.0001	

2.5 Linear Functions and Models

■ Linear Functions ■ Slope and Rate of Change ■ Making and Using Linear Models

In this section we study the simplest functions that can be expressed by an algebraic expression: linear functions.

■ Linear Functions

Recall that a *linear function* is a function of the form $f(x) = ax + b$. So in the expression defining a linear function the variable occurs to the first power only. We can also express a linear function in equation form as $y = ax + b$. From Section 1.10 we know that the graph of this equation is a line with slope a and y-intercept b.

Linear Functions

A **linear function** is a function of the form $f(x) = ax + b$.

The graph of a linear function is a line with slope a and y-intercept b.

Example 1 ■ Identifying Linear Functions

Determine whether the given function is linear. If the function is linear, express the function in the form $f(x) = ax + b$.

(a) $f(x) = 2 + 3x$ **(b)** $g(x) = 3(1 - 2x)$

(c) $h(x) = x(4 + 3x)$ **(d)** $k(x) = \dfrac{1 - 5x}{4}$

Solution

(a) We have $f(x) = 2 + 3x = 3x + 2$. So f is a linear function in which a is 3 and b is 2.

(b) We have $g(x) = 3(1 - 2x) = -6x + 3$. So g is a linear function in which a is -6 and b is 3.

(c) We have $h(x) = x(4 + 3x) = 4x + 3x^2$, which is not a linear function because the variable x is squared in the second term of the expression for h.

(d) We have $k(x) = \dfrac{1 - 5x}{4} = -\dfrac{5}{4}x + \dfrac{1}{4}$. So k is a linear function in which a is $-\frac{5}{4}$ and b is $\frac{1}{4}$.

Now Try Exercise 7 ■

Example 2 ■ Graphing a Linear Function

Let f be the linear function defined by $f(x) = 3x + 2$.

(a) Make a table of values, and sketch a graph.

(b) What is the slope of the graph of f?

Solution

x	$f(x)$
-2	-4
-1	-1
0	2
1	5
2	8
3	11
4	14
5	17

(a) A table of values is shown in the margin. Since f is a linear function, its graph is a line. So to obtain the graph of f, we plot any two points from the table and draw the straight line that contains the points. We use the points $(1, 5)$ and $(4, 14)$. The graph is the line shown in Figure 1. You can check that the other points in the table of values also lie on the line.

From the box on the previous page, you can see that the slope of the graph of $f(x) = 3x + 2$ is 3.

(b) Using the points given in Figure 1, we see that the slope is

$$\text{slope} = \frac{14 - 5}{4 - 1} = 3$$

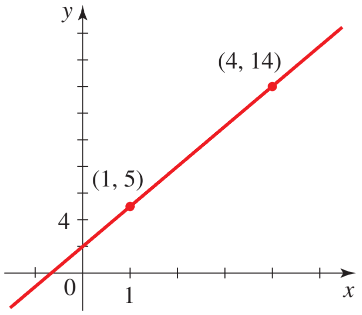


Figure 1 | Graph of the linear function $f(x) = 3x + 2$

Now Try Exercise 15 ■

■ Slope and Rate of Change

In Exercise 54 we prove that every function with constant rate of change is linear.

Let $f(x) = ax + b$ be a linear function. If x_1 and x_2 are two different values for x and if $y_1 = f(x_1)$ and $y_2 = f(x_2)$, then the points (x_1, y_1) and (x_2, y_2) lie on the graph of f. From the definitions of slope and average rate of change we have

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \text{average rate of change}$$

From Section 1.10 we know that the *slope* of a linear function is the same between any two points. From the equation above we conclude that the *average rate of change* of a linear function is the same between any two points. Moreover, the average rate of change is equal to the slope (see Exercise 53). Since the average rate of change of a linear function is the same between any two points, it is called simply the **rate of change**.

Slope and Rate of Change

For the linear function $f(x) = ax + b$, the slope of the graph of f and the rate of change of f are both equal to a, the coefficient of x.

$$a = \text{slope of graph of } f = \text{rate of change of } f$$

Note The difference between "slope" and "rate of change" is simply a difference in point of view. For example, to describe how a reservoir fills up over time, it is natural to talk about the rate at which the water level is rising, but we can also think of the slope of the graph of the water level (see Example 3). To describe the steepness of a staircase, it is natural to talk about the slope of the trim board of the staircase, but we can also think of the rate at which the stairs rise (see Example 5).

Example 3 ■ Slope and Rate of Change

A dam is built on a river to create a reservoir. The water level $f(t)$ in the reservoir at time t is given by

$$f(t) = 4.5t + 28$$

where t is the number of years since the dam was constructed and $f(t)$ is measured in feet.

(a) Sketch a graph of f.

(b) What is the slope of the graph?

(c) At what rate is the water level in the reservoir changing?

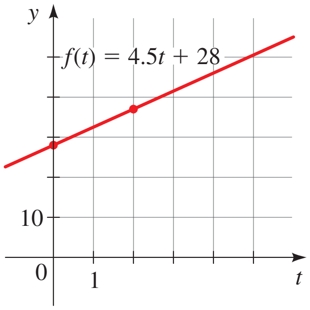


Figure 2 | Water level as a function of time

Solution

(a) A graph of f is shown in Figure 2.

(b) The graph is a line with slope 4.5, the coefficient of t.

(c) The rate of change of f is 4.5, the coefficient of t. Since time t is measured in years and the water level $f(t)$ is measured in feet, the water level in the reservoir is changing at the rate of 4.5 ft per year. Since this rate of change is positive, the water level is rising.

Now Try Exercises 19 and 39 ■

■ Making and Using Linear Models

When a linear function is used to model the relationship between two quantities, the slope of the graph of the function is the rate of change of the one quantity with respect to the other. For example, the graph in Figure 3(a) gives the amount of gas in a tank that is being filled. The slope between the indicated points is

$$a = \frac{6 \text{ gal}}{3 \text{ min}} = 2 \text{ gal/min}$$

The slope is the rate at which the tank is being filled, 2 gallons per minute.

In Figure 3(b) the tank is being drained at the rate of 0.03 gallons per minute, and the slope is -0.03.

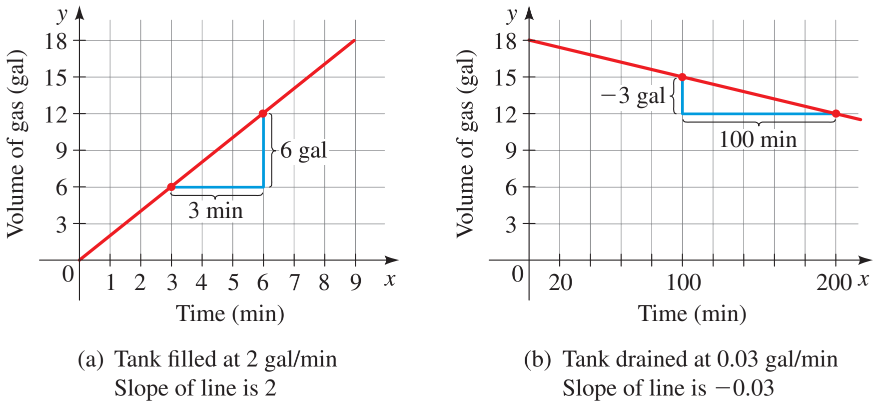


Figure 3 | Amount of gas as a function of time

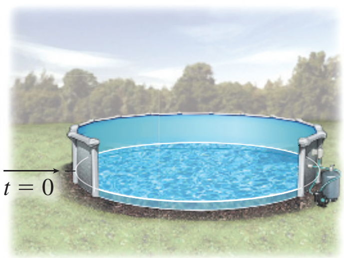
In the following examples we model real-world situations using linear functions. In each of these examples the model involves a constant rate of change (or a constant slope).

Example 4 ■ Making a Linear Model from a Rate of Change

Water is being pumped into a swimming pool at the rate of 5 gal per min. Initially, the pool contains 200 gal of water.

(a) Find a linear function V that models the volume of water in the pool at any time t.

(b) If the pool has a capacity of 600 gal, how long does it take to fill the pool?



There are 200 gallons of water in the pool at time $t = 0$.

Solution

(a) We need to find a linear function

$$V(t) = at + b$$

that models the volume $V(t)$ of water in the pool after t minutes. The rate of change of volume is 5 gal/min, so $a = 5$. Since the pool contains 200 gal to begin with, we have $V(0) = a \cdot 0 + b = 200$, so $b = 200$. Now that we know a and b, we get the model

$$V(t) = 5t + 200$$

(b) We want to find the time t at which $V(t) = 600$. So we need to solve the equation

$$600 = 5t + 200$$

Solving for t, we get $t = 80$. So it takes 80 min to fill the pool.

Now Try Exercise 41 ■

Example 5 ■ Making a Linear Model from a Slope

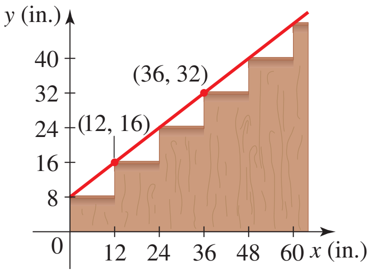


Figure 4 | Slope of a staircase

In Figure 4 we have placed a staircase in a coordinate plane, with the origin at the bottom left corner. The red line in the figure is the edge of the trim board of the staircase.

(a) Find a linear function H that models the height of the trim board above the floor.

(b) If the space available to build a staircase is 11 ft wide, how high does the staircase reach?

Solution

(a) We need to find a function

$$H(x) = ax + b$$

that models the red line in the figure. First we find the value of a, the slope of the line. From Figure 4 we see that two points on the line are $(12, 16)$ and $(36, 32)$, so the slope is

$$a = \frac{32 - 16}{36 - 12} = \frac{2}{3}$$

Another way to find the slope is to observe that each of the steps is 8 in. high (the rise) and 12 in. deep (the run), so the slope of the line is $\frac{8}{12} = \frac{2}{3}$. From Figure 4 we see that the y-intercept is 8, so $b = 8$. So the model we want is

$$H(x) = \tfrac{2}{3}x + 8$$

(b) Since 11 ft is 132 in., we need to evaluate the function H when x is 132. We have

$$H(132) = \tfrac{2}{3}(132) + 8 = 96$$

So the staircase reaches a height of 96 in., or 8 ft.

Now Try Exercise 43 ■

Example 6 ■ Making Linear Models Involving Speed

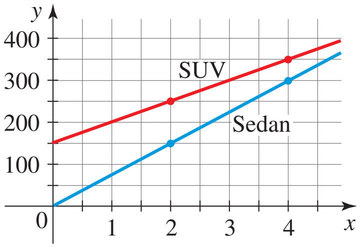


Figure 5

An SUV and a sedan are traveling westward along I-76 at constant speeds. The graphs in Figure 5 show the distance y (in miles) that they have traveled from Philadelphia at time x (in hours), where $x = 0$ corresponds to noon. (Note that at noon the SUV has a 150-mile head start.)

(a) At what speed is each vehicle traveling? Which vehicle is traveling faster, and how does this show up in the graph?

(b) For each vehicle, find a function that models the distance traveled as a function of x.

(c) How far has each vehicle traveled at 5:00 P.M.?

(d) For what time period is the sedan behind the SUV? Will the sedan overtake the SUV? If so, at what time?

Solution

(a) From the graph we see that the SUV has traveled 250 mi at 2:00 P.M. and 350 mi at 4:00 P.M. The speed is the rate of change of distance with respect to time. So the speed is the slope of the graph. Therefore the speed of the SUV is

$$\frac{350 \text{ mi} - 250 \text{ mi}}{4 \text{ h} - 2 \text{ h}} = 50 \text{ mi/h} \qquad \text{Speed of SUV}$$

The sedan has traveled 150 mi at 2:00 P.M. and 300 mi at 4:00 P.M., so we calculate the speed of the sedan to be

$$\frac{300 \text{ mi} - 150 \text{ mi}}{4 \text{ h} - 2 \text{ h}} = 75 \text{ mi/h} \qquad \text{Speed of sedan}$$

The sedan is traveling faster than the SUV. We can see this from the graph because the line for the sedan is steeper (has greater slope) than the one for the SUV.

(b) Let $f(x)$ be the distance the SUV has traveled at time x. Since the speed (average rate of change) is constant, it follows that f is a linear function. Thus we can write f in the form $f(x) = ax + b$. From part (a) we know that the slope a is 50, and from the graph we see that the y-intercept b is 150. Thus the distance that the SUV has traveled at time x is modeled by the linear function

$$f(x) = 50x + 150 \qquad \text{Model for SUV distance}$$

Similarly, the sedan is traveling at 75 mi/h, and the y-intercept is 0. Thus the distance the sedan has traveled at time x is modeled by the linear function

$$g(x) = 75x \qquad \text{Model for sedan distance}$$

(c) Replacing x by 5 in the models that we obtained in part (b), we find that at 5:00 P.M. the SUV has traveled $f(5) = 50(5) + 150 = 400$ mi and the sedan has traveled $g(5) = 75(5) = 375$ mi.

(d) The sedan overtakes the SUV at the time when each has traveled the same distance, that is, at the time x when $f(x) = g(x)$. So we must solve the equation

$$50x + 150 = 75x \qquad \text{SUV distance} = \text{Sedan distance}$$

Solving this equation, we get $x = 6$. So the sedan overtakes the SUV after 6 h, that is, at 6:00 P.M. We can confirm our solution graphically by drawing the graphs of f and g on a larger domain as shown in Figure 6. The graphs intersect when $x = 6$. From the graph we see that the sedan is behind the SUV (has traveled a shorter distance) from $x = 0$ to $x = 6$, that is, from noon until 6:00 P.M.

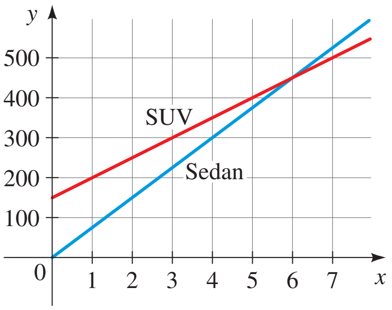
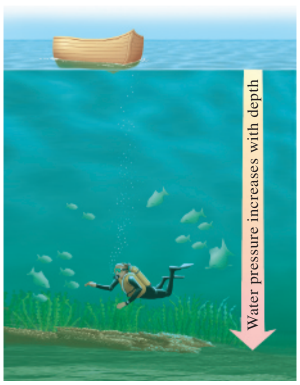


Figure 6

Now Try Exercise 45 ■

Deep sea divers know that the deeper they dive, the higher the water pressure becomes. But how deep can one dive before water pressure becomes dangerously high? How much pressure must a submarine be able to withstand if it is to dive to the deepest parts of the ocean? These questions can be answered using the model in the next example.



Example 7 ■ Modeling Pressure at Depth

The pressure at the surface of the ocean is 14.7 pounds per square inch (psi) and the pressure increases by 4.34 psi for every 10-foot descent below the surface.

(a) Find a linear function P that models the water pressure at depth x feet below the surface of the ocean.

(b) What is the pressure one mile below the surface?

Solution

(a) We want to find a function $P(x) = ax + b$ that models the pressure $P(x)$ at depth x. The rate of change of pressure is $(4.34 \text{ psi})/(10 \text{ ft}) = 0.434 \text{ psi/ft}$, so $a = 0.434$. Because the pressure is 14.7 psi when $x = 0$, it follows that $b = 14.7$. So the model is

$$P(x) = 0.434x + 14.7$$

(b) We can use the model to estimate water pressure at any ocean depth. Because 1 mi = 5280 ft, the pressure one mile below the surface is

$$P(5280) = 0.434(5280) + 14.7 \approx 2306 \text{ psi}$$

Now Try Exercise 49 ■

2.5 Exercises

Concepts

1. Let f be a function with constant rate of change. Then

(a) f is a ________ function and f is of the form $f(x) =$ ____ $x +$ ____.

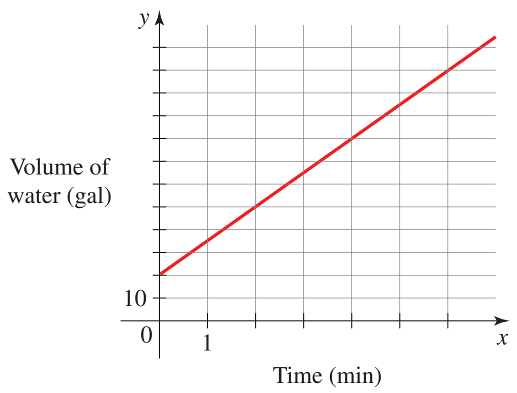
(b) The graph of f is a ________.

2. Let f be the linear function $f(x) = -5x + 7$.

(a) The rate of change of f is ________.

(b) The graph of f is a ________ with slope ________ and y-intercept ________.

3–4 ■ A swimming pool is being filled. The graph shows the number of gallons y in the pool after x minutes.



3. What is the slope of the graph?

4. At what rate is the pool being filled?

5. If a linear function has positive rate of change, does its graph slope upward or downward?

6. Is $f(x) = 3$ a linear function? If so, what are the slope and the rate of change?

Skills

7–14 ■ **Identifying Linear Functions** Determine whether the given function is linear. If the function is linear, express the function in the form $f(x) = ax + b$.

7. $f(x) = \sqrt{5} + 2x$ **8.** $f(x) = \frac{1}{2}(x + 4)$

9. $f(x) = \dfrac{20 - x}{5}$ **10.** $f(x) = \dfrac{2x - 4}{x}$

11. $f(x) = x(2 - 3x)$ **12.** $f(x) = -3(6 - 5x)$

13. $f(x) = \sqrt{x} + 1$ **14.** $f(x) = (2x - 5)^2$

15–18 ■ **Graphing Linear Functions** For the given linear function, make a table of values and use it to sketch the graph. What is the slope of the graph?

15. $f(x) = -2x + 3$ **16.** $g(x) = 3x - 1$

17. $r(t) = -\frac{2}{3}t + 2$ **18.** $h(t) = \frac{1}{2} - \frac{3}{4}t$

19–26 ■ **Slope and Rate of Change** A linear function is given. **(a)** Sketch the graph. **(b)** Find the slope of the graph. **(c)** Find the rate of change of the function.

19. $f(x) = 2x - 6$ **20.** $g(z) = -3z - 9$

21. $f(x) = 2 - 3x$ **22.** $g(z) = -(z - 3)$

23. $h(t) = \dfrac{5 - 2t}{10}$ **24.** $s(w) = 0.5w + 2$

25. $f(t) = -\frac{3}{2}t + 2$ **26.** $g(x) = \frac{5}{4}x - 10$

27–30 ■ **Linear Functions Given Verbally** A verbal description of a linear function f is given. Express the function f in the form $f(x) = ax + b$.

27. The linear function f has rate of change 5 and initial value 10.

28. The linear function f has rate of change -3 and initial value -1.

29. The graph of the linear function f has slope $\frac{1}{2}$ and y-intercept 3.

30. The graph of the linear function f has slope $-\frac{4}{5}$ and y-intercept -2.

31–32 ■ **Linear Functions Given Numerically** A table of values for a linear function f is given. **(a)** Find the rate of change of f. **(b)** Express f in the form $f(x) = ax + b$.

31.

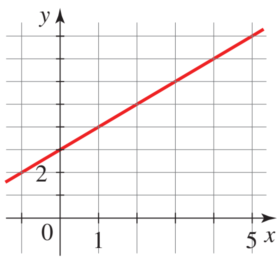
x	$f(x)$
0	7
2	10
4	13
6	16
8	19

32.

x	$f(x)$
-3	11
0	2
2	-4
5	-13
7	-19

33–36 ■ Linear Functions Given Graphically The graph of a linear function f is given. **(a)** Find the rate of change of f. **(b)** Express f in the form $f(x) = ax + b$.

33.



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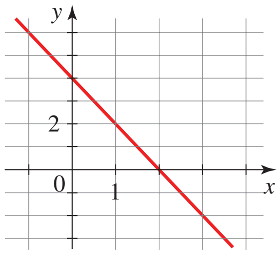

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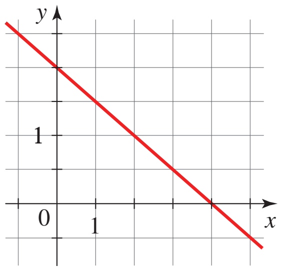

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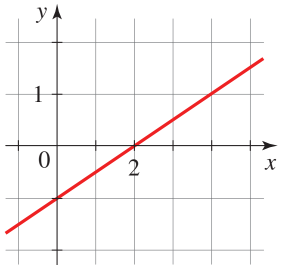


Skills Plus

37. Families of Linear Functions Graph $f(x) = ax$ for $a = \frac{1}{2}$, $a = 1$, and $a = 2$, all on the same set of axes. How does increasing the value of a affect the graph of f? What about the rate of change of f?

38. Families of Linear Functions Graph $f(x) = x + b$ for $b = \frac{1}{2}$, $b = 1$, and $b = 2$, all on the same set of axes. How does increasing the value of b affect the graph of f? What about the rate of change of f?

Applications

39. Landfill The amount of trash in a county landfill is modeled by the function

$$T(x) = 150x + 32$$

where x is the number of years since 2010 and $T(x)$ is measured in thousands of tons.

(a) Sketch a graph of T.

(b) What is the slope of the graph?

(c) At what rate is the amount of trash in the landfill increasing per year?

40. Copper Mining The amount of copper ore produced from a copper mine in Arizona is modeled by the function

$$f(x) = 200 + 32x$$

where x is the number of years since 2015 and $f(x)$ is measured in thousands of tons.

(a) Sketch a graph of f.

(b) What is the slope of the graph?

(c) At what rate is the amount of ore produced changing?

41. Weather Balloon Weather balloons are filled with hydrogen and released at various sites to measure and transmit data about conditions such as air pressure and temperature. A weather balloon is filled with hydrogen at the rate of $0.5\ \text{ft}^3/\text{s}$. Initially, the balloon contains $2\ \text{ft}^3$ of hydrogen.

(a) Find a linear function V that models the volume of hydrogen in the balloon at any time t.

(b) If the balloon has a capacity of $15\ \text{ft}^3$, how long does it take to completely fill the balloon?

42. Filling a Pond A large koi pond is filled from a garden hose at the rate of 10 gal/min. Initially, the pond contains 300 gal of water.

(a) Find a linear function V that models the volume of water in the pond at any time t.

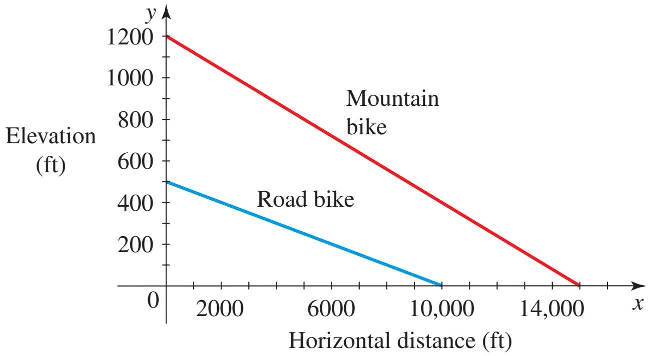
(b) If the pond has a capacity of 1300 gal, how long does it take to completely fill the pond?

43. Wheelchair Ramp A local diner must build a wheelchair ramp to provide accessibility to the restaurant. Federal building codes require that a wheelchair ramp must have a maximum rise of 1 in. for every horizontal distance of 12 in.

(a) What is the maximum allowable slope for a wheelchair ramp? Assuming that the ramp has maximum rise, find a linear function H that models the height of the ramp above the ground as a function of the horizontal distance x.

(b) If the space available to build a ramp is 150 in. wide, how high does the ramp reach?

44. Mountain Biking Two cyclists are traveling down straight roads with steep grades, one on a mountain bike and the other on a road bike. The graphs give a representation of the elevation of the road on which each of them cycles. Find the grade of each road. (The grade of a road is the slope expressed as a percentage.)

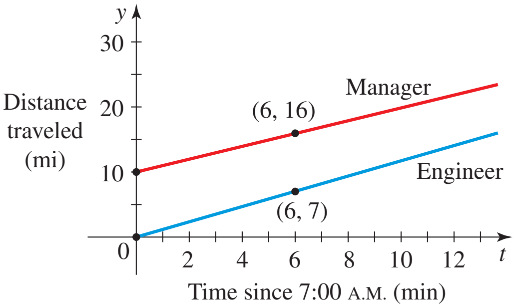


45. Commute to Work Two employees of a software company commute from the same housing complex each morning. One works as a software engineer and the other is a product manager. One morning the product manager left for work at 6:50 A.M., and the software engineer left 10 minutes later. Both drove at a constant speed. The following graphs show the distance (in miles) each of them had traveled on I-10 at time t (in minutes), where $t = 0$ is 7:00 A.M.

(a) Use the graph to decide which of them was traveling faster.

(b) Find the speed (in mi/h) at which each of them was driving.

(c) Find linear functions f and g that model the distance each employee traveled as a function of time t (in minutes).



46. Distance, Speed, and Time A bus leaves Detroit at 2:00 P.M. and drives at a constant speed, traveling west on I-90. It passes Ann Arbor, 40 mi from Detroit, at 2:50 P.M.

(a) Find a linear function d that models the distance (in mi) the bus has traveled after t min.

(b) Draw a graph of d. What is the slope of this line?

(c) At what speed (in mi/h) is the bus traveling?

47. Grade of Road West of Albuquerque, New Mexico, Route 40 eastbound is straight and makes a steep descent toward the city. The highway has a 6% grade, which means that its slope is $-\frac{6}{100}$. Driving on this road, you notice from elevation signs that you have descended a distance of 1000 ft. What is the change in your horizontal distance in miles?

48. Sedimentation Geologists study sedimentation by drilling tubular core samples. Studies show that the mean sedimentation rate at the bottom of Devil's Lake, North Dakota, is about 0.24 cm per year. In 1980, the total thickness of the sedimentary layers at a certain location was 20 cm.

(a) Find a linear function D that models the total thickness of the sedimentary layers x years after 1980.

(b) Sketch a graph of D.

(c) What is the slope of the graph?

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49–50 ■ Pressure, Boiling Point, and Elevation Mountain climbers know that the atmospheric pressure and the boiling point of water decrease as elevation increases. Although these properties depend on many factors, in these exercises we find approximate linear models that relate them. We measure the pressure in kilopascals (kPa), the boiling point in degrees Celsius (°C), and the elevation in kilometers (km).

49. If atmospheric pressure is 100 kPa at sea level and decreases by about 12 kPa for each kilometer increase in elevation, find a linear function f that models atmospheric pressure at elevation x kilometers above sea level. Estimate the atmospheric pressure at the peak of Mt. Rainier, 4.4 km above sea level.

50. At a pressure of 100 kPa the boiling point of water is100°C and drops by about 3.75°C for each 10-kPa drop in atmospheric pressure. Find a linear function g that models the boiling point of water at an atmospheric pressure of x kilopascals. Estimate the boiling point of water if the atmospheric pressure is 88 kPa.

51. Cost of Driving The monthly cost of driving a car depends on the number of miles driven. In one month the cost was \$380 for driving 480 mi and in the next month the cost was \$460 for driving 800 mi. Assume that there is a linear relationship between the monthly cost C of driving a car and the distance x driven.

(a) Find a linear function C that models the cost of driving x miles per month.

(b) Draw a graph of C. What is the slope of this line?

(c) At what rate does the cost increase for every additional mile driven?

52. Manufacturing Cost The manager of a furniture factory finds that it costs \$2200 to produce 100 chairs in one day and \$4800 to produce 300 chairs in one day.

(a) Assuming that the relationship between cost and the number of chairs produced is linear, find a linear function C that models the cost of producing x chairs in one day.

(b) Draw a graph of C. What is the slope of this line?

(c) At what rate does the factory's cost increase for every additional chair produced?

■ Discuss ■ Discover ■ Prove ■ Write

53. Prove: Linear Functions Have Constant Rate of Change Suppose that $f(x) = ax + b$ is a linear function.

(a) Use the definition of the average rate of change of a function to calculate the average rate of change of f between any two real numbers x_1 and x_2.

(b) Use your calculation in part (a) to show that the average rate of change of f is the same as the slope a.

54. Prove: Functions with Constant Rate of Change Are Linear Suppose that the function f has the same average rate of change c between any two values of the variable.

(a) Find the average rate of change of f between the values a and x to show that

$$c = \frac{f(x) - f(a)}{x - a}$$

(b) Rearrange the equation in part (a) to show that

$$f(x) = cx + [f(a) - ca]$$

How does this show that f is a linear function? What is the slope, and what is the y-intercept?

2.6 Transformations of Functions

■ Vertical Shifting ■ Horizontal Shifting ■ Reflecting Graphs ■ Vertical Stretching and Shrinking ■ Horizontal Stretching and Shrinking ■ Even and Odd Functions

In this section we study how certain transformations of a function affect its graph. This will give us a better understanding of how to graph functions. The transformations that we study are shifting, reflecting, and stretching.

■ Vertical Shifting

Adding a constant to a function shifts its graph vertically: upward if the constant is positive, and downward if it is negative.

In general, suppose we know the graph of $y = f(x)$. How do we obtain from it the graphs of

Recall that the graph of the function f is the same as the graph of the equation $y = f(x)$.

$$y = f(x) + c \qquad \text{and} \qquad y = f(x) - c \qquad (c > 0)$$

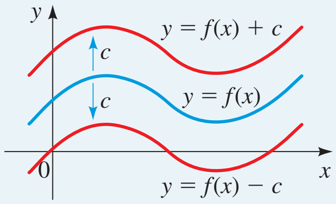
The y-coordinate of each point on the graph of $y = f(x) + c$ is c units above the y-coordinate of the corresponding point on the graph of $y = f(x)$. So we obtain the graph of $y = f(x) + c$ by shifting the graph of $y = f(x)$ upward c units. Similarly, we obtain the graph of $y = f(x) - c$ by shifting the graph of $y = f(x)$ downward c units.

Vertical Shifts of Graphs

Suppose $c > 0$.

To graph $y = f(x) + c$, shift the graph of $y = f(x)$ upward c units.

To graph $y = f(x) - c$, shift the graph of $y = f(x)$ downward c units.



Example 1 ■ Vertical Shifts of Graphs

Use the graph of $f(x) = x^2$ to sketch the graph of each function.

(a) $g(x) = x^2 + 3$ **(b)** $h(x) = x^2 - 2$

Solution The function $f(x) = x^2$ was graphed in Example 2.2.1(a). It is sketched again in Figure 1.

(a) Observe that

$$g(x) = x^2 + 3 = f(x) + 3$$

So the y-coordinate of each point on the graph of g is 3 units above the corresponding point on the graph of f. This means that to graph g, we shift the graph of f upward 3 units, as in Figure 1.

(b) Similarly, to graph h we shift the graph of f downward 2 units, as shown in Figure 1.

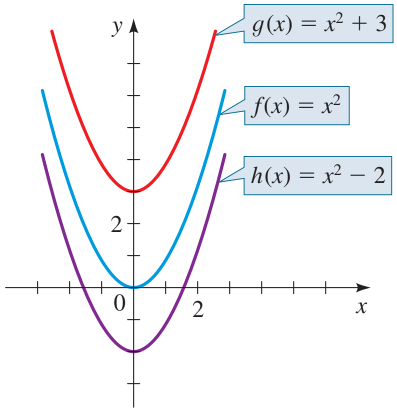


Figure 1

Now Try Exercises 27 and 29 ■

■ Horizontal Shifting

Suppose that we know the graph of $y = f(x)$. How do we use it to obtain the graphs of

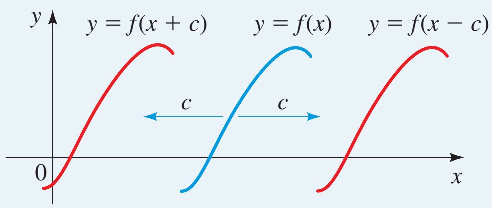
$$y = f(x + c) \qquad \text{and} \qquad y = f(x - c) \qquad (c > 0)$$

The value of $f(x - c)$ at x is the same as the value of $f(x)$ at $x - c$. Since $x - c$ is c units to the left of x, it follows that the graph of $y = f(x - c)$ is just the graph of $y = f(x)$ shifted to the right c units. Similar reasoning shows that the graph of $y = f(x + c)$ is the graph of $y = f(x)$ shifted to the left c units.

Horizontal Shifts of Graphs

Suppose $c > 0$.

To graph $y = f(x - c)$, shift the graph of $y = f(x)$ to the right c units.
To graph $y = f(x + c)$, shift the graph of $y = f(x)$ to the left c units.



Example 2 ■ Horizontal Shifts of Graphs

Use the graph of $f(x) = x^2$ to sketch the graph of each function.

(a) $g(x) = (x + 4)^2$ **(b)** $h(x) = (x - 2)^2$

Solution

(a) To graph g, we shift the graph of f to the left 4 units.

(b) To graph h, we shift the graph of f to the right 2 units.

The graphs of g and h are sketched in Figure 2.

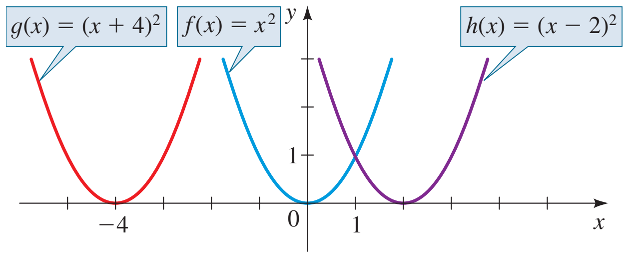


Figure 2

Now Try Exercises 31 and 33

Example 3 ■ Combining Horizontal and Vertical Shifts

Sketch the graph of $f(x) = |x + 1| - 3$

Solution The equation involves two transformations of the graph of $y = |x|$ (Example 2.2.5). In Figure 3, we graph the equation using the following steps.

① Start with the graph of $y = |x|$.

② Shift the graph obtained in ① to the left 1 unit to obtain the graph of $y = |x + 1|$.

③ Shift the graph obtained in ② downward 3 units to obtain the graph of $y = |x + 1| - 3$.

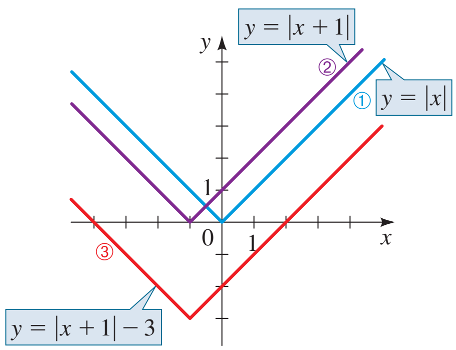


Figure 3

Now Try Exercise 43

■ Reflecting Graphs

Suppose we know the graph of $y = f(x)$. How do we use it to obtain the graphs of $y = -f(x)$ and $y = f(-x)$? The y-coordinate of each point on the graph of $y = -f(x)$ is the negative of the y-coordinate of the corresponding point on the graph of $y = f(x)$. So the desired graph is the reflection of the graph of $y = f(x)$ about the x-axis. On the other

Discovery Project ■ **Transformation Stories**

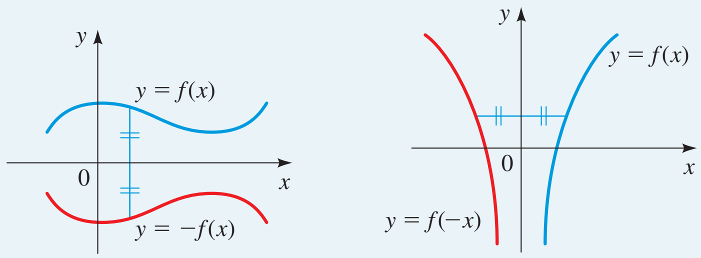
If a real-world situation, or "story," is modeled by a function, how does transforming the function change the story? For example, if the distance traveled on a road trip is modeled by a function, then how does shifting or stretching the function change the story of the trip? How does changing the story of the trip transform the function that models the trip? In this project we explore some real-world stories and transformations of these stories. You can find the project at **www.stewartmath.com**.

hand, the value of $y = f(-x)$ at x is the same as the value of $y = f(x)$ at $-x$, so the desired graph here is the reflection of the graph of $y = f(x)$ about the y-axis.

Reflecting Graphs

To graph $y = -f(x)$, reflect the graph of $y = f(x)$ about the x-axis.

To graph $y = f(-x)$, reflect the graph of $y = f(x)$ about the y-axis.



Example 4 ■ Reflecting Graphs

Sketch the graph of each function.

(a) $f(x) = -x^2$ **(b)** $g(x) = \sqrt{-x}$

Solution

(a) We start with the graph of $y = x^2$. The graph of $f(x) = -x^2$ is the graph of $y = x^2$ reflected about the x-axis (see Figure 4).

(b) We start with the graph of $y = \sqrt{x}$ [Example 2.2.1(c)]. The graph of $g(x) = \sqrt{-x}$ is the graph of $y = \sqrt{x}$ reflected about the y-axis (see Figure 5). Note that the domain of the function $g(x) = \sqrt{-x}$ is $\{x \mid x \le 0\}$.

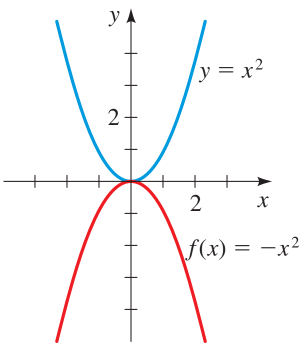


Figure 4

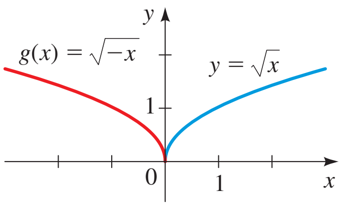


Figure 5

Now Try Exercises 35 and 37 ■

Library of Congress Prints and Photographs Division

RENÉ DESCARTES (1596–1650) was born in the town of La Haye in France. From an early age Descartes liked mathematics because of "the certainty of its results and the clarity of its reasoning." He believed that to arrive at truth, one must begin by doubting everything, including one's own existence; this led him to formulate perhaps the best-known sentence in all of philosophy: "I think, therefore I am." In his book *Discourse on Method* he described what is now called the Cartesian plane. This idea of combining algebra and geometry enabled mathematicians for the first time to graph functions and thus "see" the equations they were studying. The philosopher John Stuart Mill called this invention "the greatest single step ever made in the progress of the exact sciences." Descartes liked to get up late and spend the morning in bed thinking and writing. He invented the coordinate plane while lying in bed watching a fly crawl on the ceiling, reasoning that he could describe the exact location of the fly by knowing its distance from two perpendicular walls. In 1649 Descartes became the tutor of Queen Christina of Sweden. She liked her lessons at 5 o'clock in the morning, when, she said, her mind was sharpest. However, the change from his usual habits and the ice-cold library where they studied proved too much for Descartes. In February 1650, after just two months of this regimen, he caught pneumonia and died. The 1974 television film *Cartesius* chronicles the life of Rene Descartes.

Vertical Stretching and Shrinking

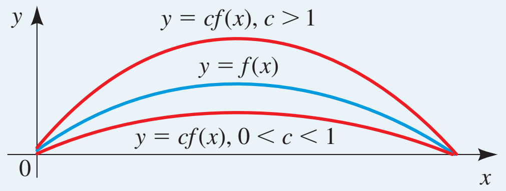
Suppose we know the graph of $y = f(x)$. How do we use it to obtain the graph of $y = cf(x)$? The y-coordinate of $y = cf(x)$ at x is the same as the corresponding y-coordinate of $y = f(x)$ multiplied by c. Multiplying the y-coordinates by c has the effect of vertically stretching or shrinking the graph by a factor of c (if $c > 0$).

Vertical Stretching and Shrinking of Graphs

To graph $y = cf(x)$:

If $c > 1$, stretch the graph of $y = f(x)$ vertically by a factor of c.

If $0 < c < 1$, shrink the graph of $y = f(x)$ vertically by a factor of c.



Example 5 ■ Vertical Stretching and Shrinking of Graphs

Use the graph of $f(x) = x^2$ to sketch the graph of each function.

(a) $g(x) = 3x^2$ **(b)** $h(x) = \frac{1}{3}x^2$

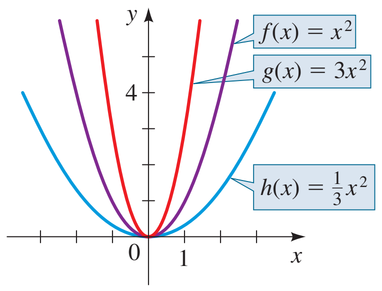


Figure 6

Solution

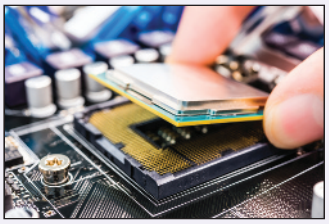
(a) The graph of g is obtained by multiplying the y-coordinate of each point on the graph of f by 3. That is, to obtain the graph of g, we stretch the graph of f vertically by a factor of 3. The result is the narrowest parabola in Figure 6.

(b) The graph of h is obtained by multiplying the y-coordinate of each point on the graph of f by $\frac{1}{3}$. That is, to obtain the graph of h, we shrink the graph of f vertically by a factor of $\frac{1}{3}$. The result is the widest parabola in Figure 6.

Now Try Exercises 39 and 41 ■

We illustrate the effect of combining shifts, reflections, and stretching in the following example.

Mathematics in the Modern World



Georgii Shipin/Shutterstock.com

Computers

For centuries machines have been designed to perform specific tasks. For example, a washing machine washes clothes, a weaving machine weaves cloth, an adding machine adds numbers, and so on. The computer has changed all that.

The computer is a machine that does nothing—until it is given instructions on what to do. So your computer can play games, draw pictures, or calculate π to a million decimal places; it all depends on what program (or instructions) you give the computer. The computer can do all this because it is able to accept instructions and logically change those instructions based on incoming data. This versatility makes computers useful in nearly every aspect of human endeavor.

The idea of a computer was described theoretically in the 1940s by the mathematician Alan Turing (see the biography in this section) in what he called a *universal machine*. In 1945 the mathematician John Von Neumann, extending Turing's ideas, built one of the first electronic computers.

Mathematicians continue to develop new theoretical bases for the design of computers. The heart of the computer is the "chip," (or CPU), which is capable of processing logical instructions. To get an idea of the chip's complexity, consider that modern computer chips contain more than ten billion transistors making up their logic circuits.

Example 6 ■ Combining Shifting, Stretching, and Reflecting

Sketch the graph of the function $f(x) = 1 - 2(x - 3)^2$.

Solution The equation involves several transformations of the graph of $y = x^2$. In Figure 7, we graph the equation using the following steps; in each step we transform the graph that we obtained in the preceding step.

① Start with the graph of $y = x^2$.

② Shift the graph in ① to the right 3 units to get the graph of $y = (x - 3)^2$.

③ Reflect the graph in ② about the x-axis and stretch vertically by a factor of 2 to obtain the graph of $y = -2(x - 3)^2$.

④ Finally, shift the graph in ③ upward 1 unit to get the graph of $y = 1 - 2(x - 3)^2$.

Note that the shifts and stretches follow the normal order of operations. In particular, the upward shift must be performed *last*.

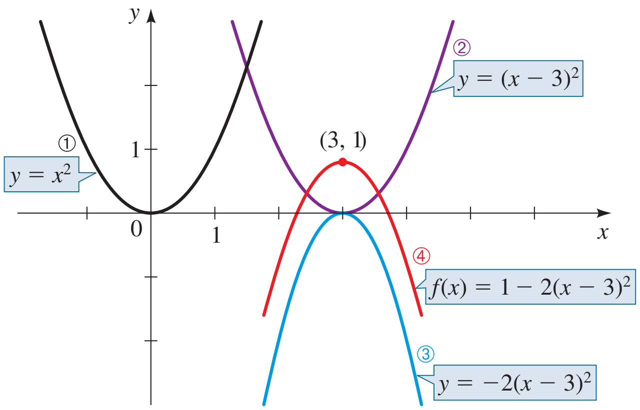


Figure 7

Now Try Exercise 45

■ Horizontal Stretching and Shrinking

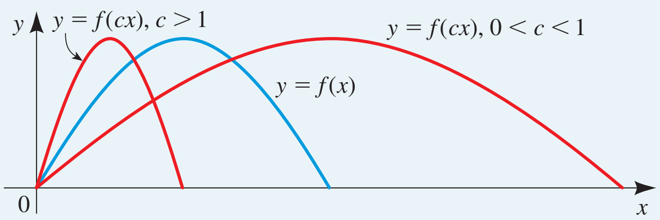
Now we consider horizontal shrinking and stretching of graphs. If we know the graph of $y = f(x)$, then how is the graph of $y = f(cx)$ related to it? The y-coordinate of $y = f(cx)$ at x is the same as the y-coordinate of $y = f(x)$ at cx. Thus the x-coordinates in the graph of $y = f(x)$ correspond to the x-coordinates in the graph of $y = f(cx)$ multiplied by c. Looking at this the other way around, we see that the x-coordinates in the graph of $y = f(cx)$ are the x-coordinates in the graph of $y = f(x)$ multiplied by $1/c$. In other words, to change the graph of $y = f(x)$ to the graph of $y = f(cx)$, we must shrink (or stretch) the graph horizontally by a factor of $1/c$ (if $c > 0$).

Horizontal Shrinking and Stretching of Graphs

To graph $y = f(cx)$:

If $c > 1$, shrink the graph of $y = f(x)$ horizontally by a factor of $1/c$.

If $0 < c < 1$, stretch the graph of $y = f(x)$ horizontally by a factor of $1/c$.



Science History Images/Alamy Stock Photo

ALAN TURING (1912–1954) was at the center of two pivotal events of the 20th century: World War II and the invention of computers. At the age of 23 Turing made his mark on mathematics by solving an important problem in the foundations of mathematics. In this research he invented a theoretical machine—now called a Turing machine,—which was the inspiration for modern digital computers. During World War II, Turing was in charge of the British effort to decipher secret German messages enciphered by the Enigma machine. His complete success in this endeavor played a decisive role in the Allied victory. To carry out the numerous logical steps that are required to break a coded message, Turing developed decision procedures similar to modern computer programs. After the war he helped to develop the first electronic computers in Britain. He also did pioneering work on artificial intelligence and computer models of biological processes. The 2014 film *The Imitation Game* is based on Turing's work in cracking the Enigma code.

Example 7 ■ Horizontal Stretching and Shrinking of Graphs

The graph of $y = f(x)$ is shown in Figure 8. Sketch the graph of each function.

(a) $y = f(2x)$ **(b)** $y = f(\frac{1}{2}x)$

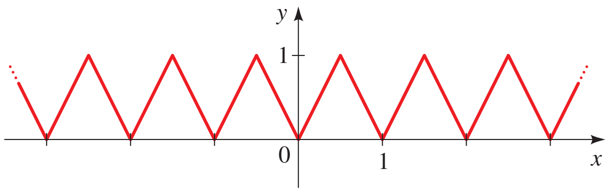


Figure 8 | $y = f(x)$

Solution

(a) We *shrink* the graph horizontally by the factor $\frac{1}{2}$ to obtain the graph in Figure 9.

(b) We *stretch* the graph horizontally by the factor 2 to obtain the graph in Figure 10.

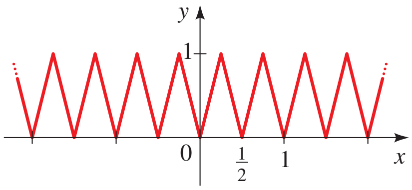


Figure 9 | $y = f(2x)$

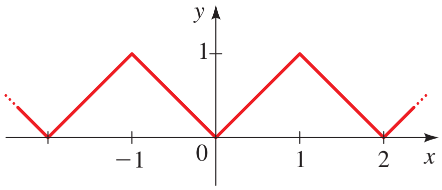


Figure 10 | $y = f(\frac{1}{2}x)$

Now Try Exercise 69 ■

Example 8 ■ Finding an Equation for a Combination of Transformations

Apply the following transformations (in order) to the graph of $f(x) = \sqrt{x}$: shift 1 unit to the left, shrink vertically by a factor of $\frac{1}{2}$, then reflect about the y-axis. Write an equation for the final transformed graph.

Solution We apply the given transformations in order, as shown in Figure 11. In each step we apply the required transformation to the graph obtained in the preceding step.

① Start with the graph of $y = \sqrt{x}$.

② Shift the graph 1 unit to the left, to obtain the graph of $y = \sqrt{x + 1}$.

③ Shrink the graph vertically by a factor of $\frac{1}{2}$, to obtain the graph of $y = \frac{1}{2}\sqrt{x + 1}$.

④ Finally, reflect the graph about the y-axis to obtain the graph of $y = \frac{1}{2}\sqrt{-x + 1}$, or $y = \frac{1}{2}\sqrt{1 - x}$.

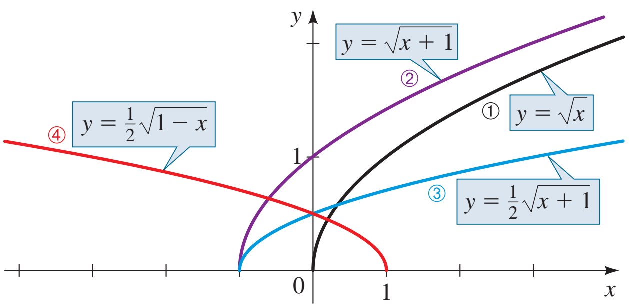


Figure 11

Now Try Exercises 51 and 59 ■


INTERFOTO/Alamy Stock Photo

SOFYA KOVALEVSKAYA (1850–1891) is considered to be one of the most important mathematicians of the 19th century. She was born in Moscow to an aristocratic family. As a child, she was exposed to the principles of calculus in a very unusual way: Her bedroom was temporarily wallpapered with the pages of a calculus book. She later wrote that she "spent many hours in front of that wall, trying to understand it." Since Russian law forbade women from studying in universities, she entered a marriage of convenience, which allowed her to travel to Germany and obtain a doctorate in mathematics from the University of Göttingen. Eventually she was awarded a full professorship at the University of Stockholm, where she taught for eight years before dying in an influenza epidemic at the age of 41. Her research was instrumental in helping to put the ideas and applications of functions and calculus on a sound and logical foundation. She received many accolades and prizes for her research work. The 1983 film *A Hill on the Dark Side of the Moon* is about the life of Sofya Kovalevskaya.

Note When we apply more than one transformation to a graph, order can matter. For example, starting with the function $y = \sqrt{x}$, reflecting about the x-axis, then shifting upward 1 unit results in the equation $y = -\sqrt{x} + 1$, whereas if we first shift upward 1 unit, then reflect about the x-axis, we obtain the equation $y = -(\sqrt{x} + 1) = -\sqrt{x} - 1$.

Even and Odd Functions

If a function f satisfies $f(-x) = f(x)$ for every number x in its domain, then f is called an **even function**. For instance, the function $f(x) = x^2$ is even because

$$f(-x) = (-x)^2 = (-1)^2x^2 = x^2 = f(x)$$

The graph of an even function is symmetric with respect to the y-axis (see Figure 12). This means that if we have plotted the graph of f for $x \geq 0$, then we can obtain the entire graph simply by reflecting this portion about the y-axis.

If f satisfies $f(-x) = -f(x)$ for every number x in its domain, then f is called an **odd function**. For example, the function $f(x) = x^3$ is odd because

$$f(-x) = (-x)^3 = (-1)^3x^3 = -x^3 = -f(x)$$

The graph of an odd function is symmetric with respect to the origin (see Figure 13). If we have plotted the graph of f for $x \geq 0$, then we can obtain the entire graph by rotating this portion through 180° about the origin. (This is equivalent to reflecting first about the x-axis and then about the y-axis.)

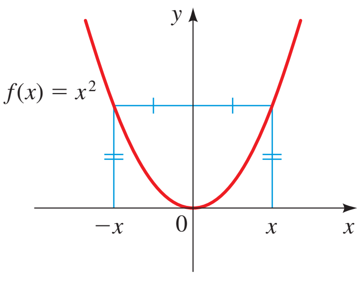


Figure 12 | $f(x) = x^2$ is an even function.

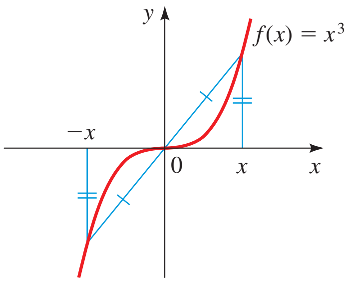


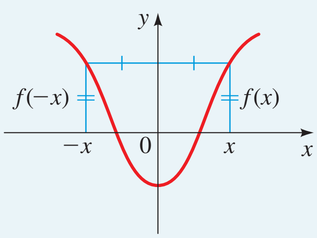
Figure 13 | $f(x) = x^3$ is an odd function.

Even and Odd Functions

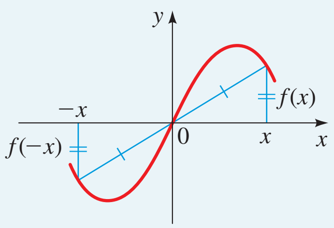
Let f be a function.

f is **even** if $f(-x) = f(x)$ for all x in the domain of f.

f is **odd** if $f(-x) = -f(x)$ for all x in the domain of f.



The graph of an even function is symmetric with respect to the y-axis.



The graph of an odd function is symmetric with respect to the origin.

Example 9 ■ Even and Odd Functions

Determine whether the function is even, odd, or neither even nor odd.

(a) $f(x) = x^5 + x$ **(b)** $g(x) = 1 - x^4$ **(c)** $h(x) = 2x - x^2$

Solution

(a)
$$\begin{aligned} f(-x) &= (-x)^5 + (-x) \\ &= -x^5 - x = -(x^5 + x) \\ &= -f(x) \end{aligned}$$

Therefore f is an odd function.

(b) $g(-x) = 1 - (-x)^4 = 1 - x^4 = g(x)$

So g is even.

(c) $h(-x) = 2(-x) - (-x)^2 = -2x - x^2$

Since $h(-x) \neq h(x)$ and $h(-x) \neq -h(x)$, we conclude that h is neither even nor odd.

Now Try Exercises 81, 83, and 85 ■

The graphs of the functions in Example 9 are shown in Figure 14. The graph of f is symmetric with respect to the origin, and the graph of g is symmetric with respect to the y-axis. The graph of h is not symmetric with respect to either the y-axis or the origin.

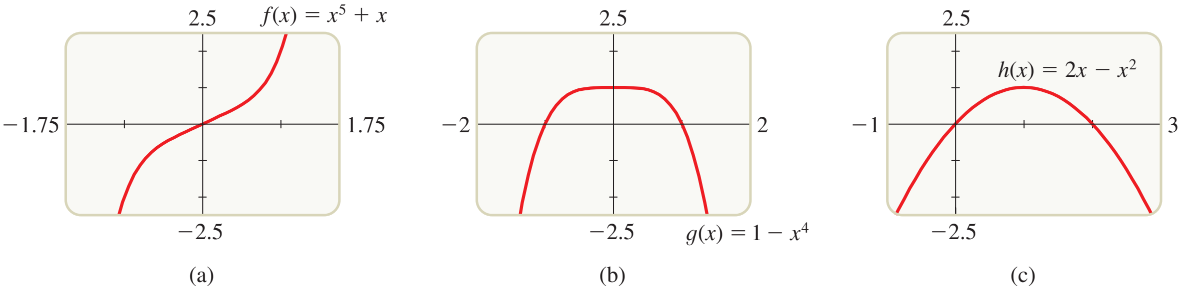


Figure 14

2.6 Exercises

Concepts

1–2 ■ Fill in the blank with the appropriate direction (left, right, upward, or downward).

1. (a) The graph of $y = f(x) + 3$ is obtained from the graph of $y = f(x)$ by shifting ________ 3 units.

(b) The graph of $y = f(x + 3)$ is obtained from the graph of $y = f(x)$ by shifting ________ 3 units.

2. (a) The graph of $y = f(x) - 3$ is obtained from the graph of $y = f(x)$ by shifting ________ 3 units.

(b) The graph of $y = f(x - 3)$ is obtained from the graph of $y = f(x)$ by shifting ________ 3 units.

3. Fill in the blank with the appropriate axis (x-axis or y-axis).

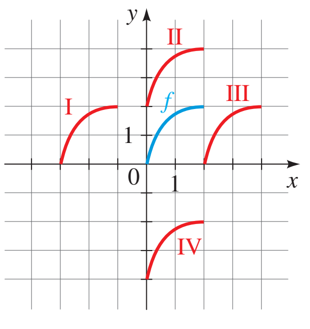
(a) The graph of $y = -f(x)$ is obtained from the graph of $y = f(x)$ by reflecting about the ________.

(b) The graph of $y = f(-x)$ is obtained from the graph of $y = f(x)$ by reflecting about the ________.

4. A graph of a function f is given. Match each equation with one of the graphs labeled I–IV.

(a) $f(x) + 2$ **(b)** $f(x + 3)$

(c) $f(x - 2)$ **(d)** $f(x) - 4$



5. If a function f is an even function, then what type of symmetry does the graph of f have?

6. If a function f is an odd function, then what type of symmetry does the graph of f have?

Skills

7–16 ■ Describing Transformations Suppose the graph of f is given. Describe how the graph of each function can be obtained from the graph of f.

7. (a) $y = f(x) + 11$ **(b)** $y = f(x + 8)$

8. (a) $y = f(x - 7)$ **(b)** $y = f(x) - 10$

9. (a) $y = \frac{1}{4}f(-x)$ **(b)** $y = -5f(x)$

10. (a) $y = -6f(x)$ **(b)** $y = \frac{2}{3}f(-x)$

11. (a) $y = f(x - 1) - 5$ **(b)** $y = f(x + 2) - 4$

12. (a) $y = f(x - 4) + 6$ **(b)** $y = f(x + 2) + 9$

13. (a) $y = 5 + f(-x)$ **(b)** $y = 3 - \frac{1}{2}f(x + 2)$

14. (a) $y = 10 - f(x + 1)$ **(b)** $y = 4f(-x + 5) - 8$

15. (a) $y = 2 - f(5x)$ **(b)** $y = 1 + f\left(\frac{1}{2}(x + 1)\right)$

16. (a) $y = f\left(\frac{1}{3}x\right) - 2$ **(b)** $y = f(2(x - 3)) - 1$

17–20 ■ Describing Transformations Explain how the graph of g is obtained from the graph of f.

17. (a) $f(x) = x^2, \quad g(x) = (x + 2)^2$
(b) $f(x) = x^2, \quad g(x) = x^2 + 2$

18. (a) $f(x) = x^3, \quad g(x) = (x - 4)^3$
(b) $f(x) = x^3, \quad g(x) = x^3 - 4$

19. (a) $f(x) = |x|, \quad g(x) = |x + 2| - 2$
(b) $f(x) = |x|, \quad g(x) = |x - 2| + 2$

20. (a) $f(x) = \sqrt{x}, \quad g(x) = -\sqrt{x} + 1$
(b) $f(x) = \sqrt{x}, \quad g(x) = \sqrt{-x} + 1$

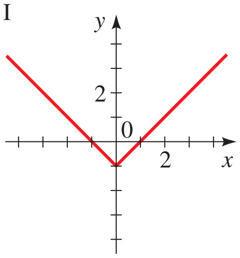
21. Graphing Transformations Use the graph of $y = x^2$ in Figure 4 to graph the following equations.
(a) $g(x) = x^2 - 4$
(b) $g(x) = 2(x + 3)^2$
(c) $g(x) = 1 - x^2$
(d) $g(x) = (x + 1)^2 - 3$

22. Graphing Transformations Use the graph of $y = \sqrt{x}$ in Figure 5 to graph the given function.
(a) $g(x) = \sqrt{x - 2}$
(b) $g(x) = \sqrt{x} + 1$
(c) $g(x) = \sqrt{x + 2} + 2$
(d) $g(x) = -\sqrt{x} + 1$

23–26 ■ Identifying Transformations Match the function with its graph in I–IV, and state the range of the function. (See the graph of $y = |x|$ in Section 1.9.)

23. $y = |x + 1|$ **24.** $y = |x - 1|$

25. $y = |x| - 1$ **26.** $y = -|x|$

I


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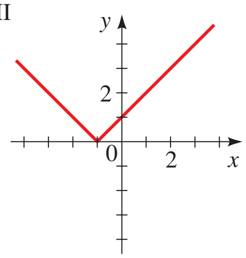

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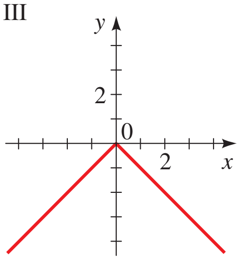

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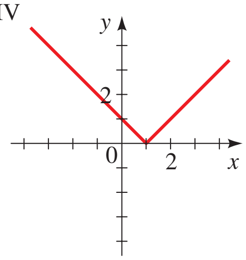


27–50 ■ Graphing Transformations Sketch the graph of the function, not by plotting points, but by starting with the graph of a standard function and applying transformations.

27. $f(x) = x^2 - 5$ **28.** $f(x) = x^2 + 2$

29. $f(x) = \sqrt{x} + 3$ **30.** $f(x) = |x| - 5$

31. $f(x) = (x - 5)^2$ **32.** $f(x) = (x + 1)^2$

33. $f(x) = |x + 2|$ **34.** $f(x) = \sqrt{x - 4}$

35. $f(x) = -x^3$ **36.** $f(x) = -|x|$

37. $f(x) = \sqrt[4]{-x}$ **38.** $f(x) = \sqrt[3]{-x}$

39. $f(x) = 5x^2$ **40.** $f(x) = \frac{1}{3}|x|$

41. $f(x) = -\frac{1}{5}\sqrt{x}$ **42.** $f(x) = 2\sqrt{-x}$

43. $f(x) = |x - 4| + 2$ **44.** $f(x) = (x + 1)^2 - 3$

45. $f(x) = -2\sqrt{x + 4} + 3$ **46.** $f(x) = 1 - \frac{1}{2}|x - 2|$

47. $f(x) = \frac{1}{2}(x + 2)^2 - 3$ **48.** $f(x) = 2\sqrt{x - 1} + 3$

49. $f(x) = \frac{1}{2}\sqrt{x + 4} - 3$ **50.** $f(x) = 3 - 2(x - 1)^2$

51–60 ■ Finding Equations for Transformations A function f is given, and the indicated transformations are applied to its graph (in the given order). Write an equation for the final transformed graph.

51. $f(x) = x^2$; shift upward 10 units

52. $f(x) = \sqrt{x}$; shift downward 4 units

53. $f(x) = x^4$; shift 3 units to the right

54. $f(x) = x^3$; shift 8 units to the left

55. $f(x) = |x|$; shift 2 units to the left and shift downward 5 units

56. $f(x) = |x|$; reflect about the x-axis, shift 4 units to the right, and shift upward 3 units

57. $f(x) = \sqrt[4]{x}$; reflect about the y-axis and shift upward 1 unit

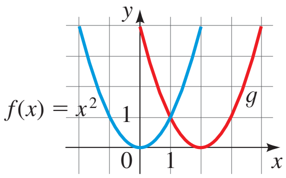
58. $f(x) = x^2$; shift 2 units to the left and reflect about the x-axis

59. $f(x) = x^2$; stretch vertically by a factor of 2, shift downward 2 units, and shift 3 units to the right

60. $f(x) = |x|$; shrink vertically by a factor of $\frac{1}{2}$, shift 1 unit to the left, and shift upward 3 units

61–66 ■ Finding Formulas for Transformations The graphs of f and g are given. Find a formula for the function g.

61.



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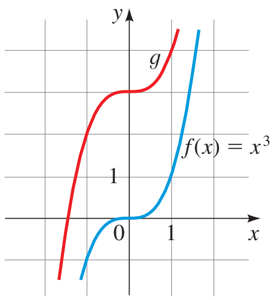

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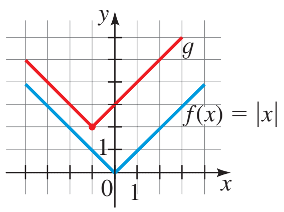

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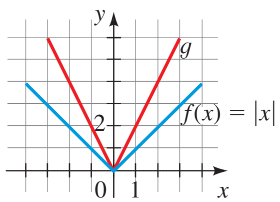

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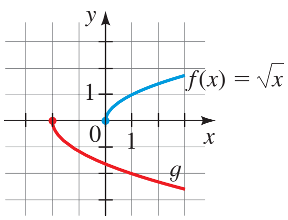

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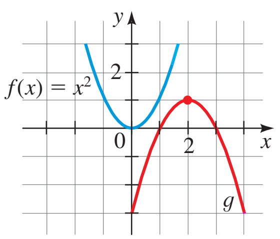
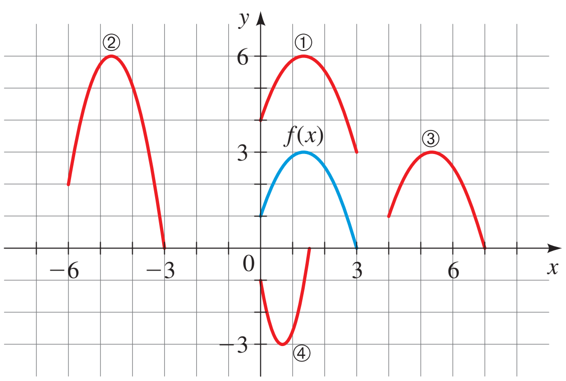


67–68 ■ Identifying Transformations The graph of $y = f(x)$ is given. Match each equation with its graph.

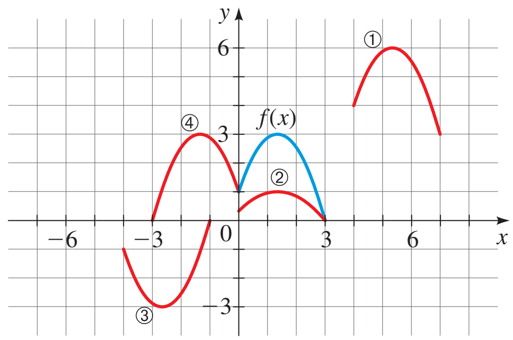
67. (a) $y = f(x - 4)$ **(b)** $y = f(x) + 3$

(c) $y = 2f(x + 6)$ **(d)** $y = -f(2x)$



68. (a) $y = \frac{1}{3}f(x)$ **(b)** $y = -f(x + 4)$

(c) $y = f(x - 4) + 3$ **(d)** $y = f(-x)$

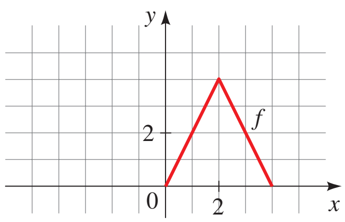


69–72 ■ Graphing Transformations The graph of a function f is given. Sketch the graphs of the following transformations of f.

69. (a) $y = f(x - 2)$ **(b)** $y = f(x) - 2$

(c) $y = 2f(x)$ **(d)** $y = -f(x) + 3$

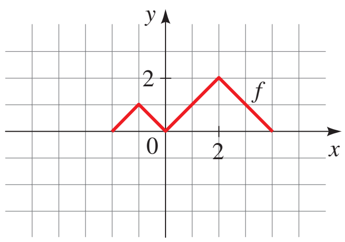
(e) $y = f(-x)$ **(f)** $y = \frac{1}{2}f(x - 1)$



70. (a) $y = f(x + 1)$ **(b)** $y = f(-x)$

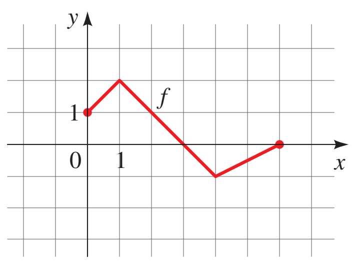
(c) $y = f(x - 2)$ **(d)** $y = f(x) - 2$

(e) $y = -f(x)$ **(f)** $y = 2f(x)$



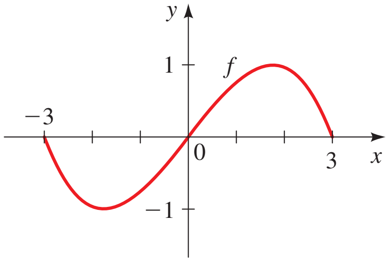
71. (a) $y = f(2x)$ **(b)** $y = f(\frac{1}{2}x)$

(c) $y = 2f(2x)$ **(d)** $y = -2f(\frac{1}{2}x)$



72. **(a)** $y = f(3x)$ **(b)** $y = f(\frac{1}{3}x)$

(c) $y = f(3(x + 1))$ **(d)** $y = 1 - f(\frac{1}{3}x)$



73–74 ■ Graphing Transformations Use the graph of $f(x) = [\![x]\!]$ described in Section 2.2 to graph the indicated function.

73. $y = [\![2x]\!]$ **74.** $y = [\![\frac{1}{4}x]\!]$

75–78 ■ Graphing Transformations Graph the functions on the same screen using the given viewing rectangle. How is each graph related to the graph in part (a)?

75. Viewing rectangle $[-8, 8]$ by $[-2, 8]$

(a) $y = \sqrt[4]{x}$ **(b)** $y = \sqrt[4]{x + 5}$

(c) $y = 2\sqrt[4]{x + 5}$ **(d)** $y = 4 + 2\sqrt[4]{x + 5}$

76. Viewing rectangle $[-8, 8]$ by $[-6, 6]$

(a) $y = |x|$ **(b)** $y = -|x|$

(c) $y = -3|x|$ **(d)** $y = -3|x - 5|$

77. Viewing rectangle $[-4, 6]$ by $[-4, 4]$

(a) $y = x^6$ **(b)** $y = \frac{1}{3}x^6$

(c) $y = -\frac{1}{3}x^6$ **(d)** $y = -\frac{1}{3}(x - 4)^6$

78. Viewing rectangle $[-6, 6]$ by $[-4, 4]$

(a) $y = \dfrac{1}{\sqrt{x}}$ **(b)** $y = \dfrac{1}{\sqrt{x + 3}}$

(c) $y = \dfrac{1}{2\sqrt{x + 3}}$ **(d)** $y = \dfrac{1}{2\sqrt{x + 3}} - 3$

79–80 ■ Graphing Transformations If $f(x) = \sqrt{2x - x^2}$, graph the following functions in the viewing rectangle $[-5, 5]$ by $[-4, 4]$. How is each graph related to the graph in part (a)?

79. **(a)** $y = f(x)$ **(b)** $y = f(2x)$ **(c)** $y = f(\frac{1}{2}x)$

80. **(a)** $y = f(x)$ **(b)** $y = f(-x)$

(c) $y = -f(-x)$ **(d)** $y = f(-2x)$

(e) $y = f(-\frac{1}{2}x)$

81–88 ■ Even and Odd Functions Determine whether the function f is even, odd, or neither. If f is even or odd, use symmetry to sketch its graph.

81. $f(x) = x^4$ **82.** $f(x) = x^3$

83. $f(x) = x^2 + x$ **84.** $f(x) = x^4 - 4x^2$

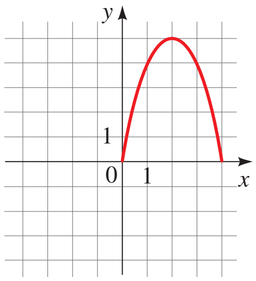
85. $f(x) = x^3 - x$ **86.** $f(x) = 3x^3 + 2x^2 + 1$

87. $f(x) = 1 - \sqrt[3]{x}$ **88.** $f(x) = x + \dfrac{1}{x}$

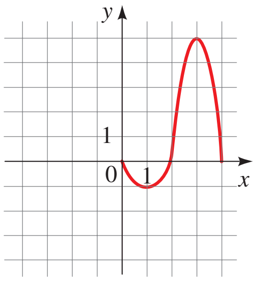
Skills Plus

89–90 ■ Graphing Even and Odd Functions The graph of a function defined for $x \geq 0$ is given. Complete the graph for $x < 0$ to make **(a)** an even function and **(b)** an odd function.

89.



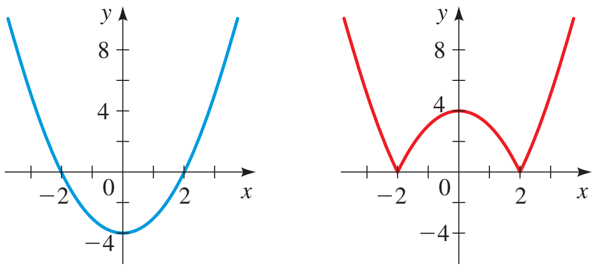
90.



91. Graphing the Absolute Value of a Function This exercise shows how the graph of $y = |f(x)|$ is obtained from the graph of $y = f(x)$.

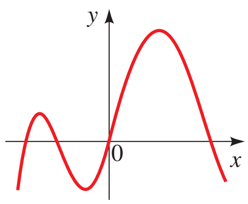
(a) The graphs of $f(x) = x^2 - 4$ and $g(x) = |x^2 - 4|$ are shown. Explain how the graph of g is obtained from the graph of f.

(b) Sketch the graph of the functions $g(x) = |4x - x^2|$ and $h(x) = |x^3|$.



$f(x) = x^2 - 4$ $g(x) = |x^2 - 4|$

92. Graphs with Absolute Value The graph of a function $y = f(x)$ is given. Draw graphs of **(a)** $y = |f(x)|$ and **(b)** $y = f(|x|)$.



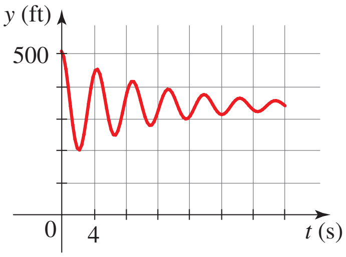
Applications

93. Bungee Jumping A bungee jumper jumps off a 500-ft-high bridge. The graph shows the bungee jumper's height $h(t)$ (in ft) after t seconds.

(a) Describe in words what the graph indicates about the bungee jump.

(b) Suppose a bungee jumper jumps off a 400-ft-high bridge. Sketch a new graph that shows the bungee jumper's height $H(t)$ after t seconds.

(c) What transformation must be performed on the function h to obtain the function H? Express the function H in terms of h.

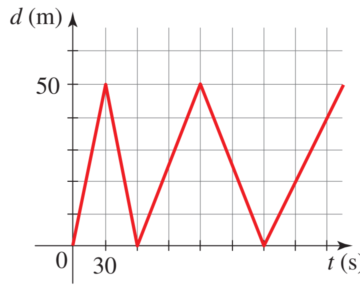


94. Swimming Laps The function $y = f(t)$ graphed below gives the distance (in meters) of a swimmer from the starting edge of a pool t seconds after the swimmer starts swimming laps.

(a) Describe in words the swimmer's lap practice. What is the average speed for the first 30 s?

(b) Graph the function $y = 1.2f(t)$. How is the graph of the new function related to the graph of the original function?

(c) What is the swimmer's new average speed for the first 30 s?

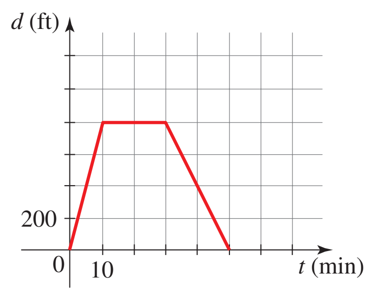


95. Field Trip A class of fourth graders walks to a park on a field trip. The function $y = f(t)$ graphed below gives their distance from school (in ft) t minutes after they left the building.

(a) What is the average speed walking to the park? How long was the class at the park? How far away from the school is the park?

(b) Graph the function $y = 0.5f(t)$. How is the graph of the new function related to the graph of the original function? What is the average speed going to the new park? How far away from the school is the new park?

(c) Graph the function $y = f(t - 10)$. How is the graph of the new function related to the graph of the original function? How does the field trip descibed by this function differ from the original trip?

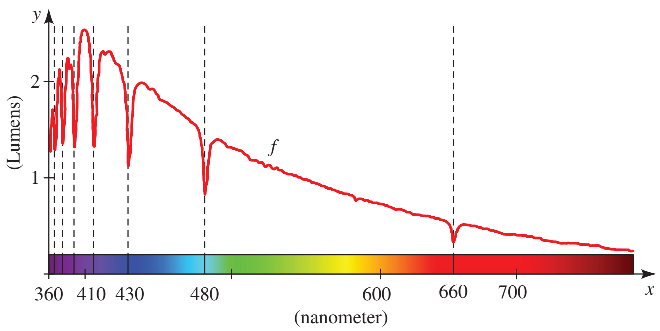


96. Redshift The colors in a light spectrum correspond to different wavelengths of light. If a galaxy is rapidly moving away from the earth, we see the spectrum of light from the galaxy "shifted" toward the red (long) end of the wavelength spectrum. A *redshift* $z > 0$ of a distant galaxy means that light emitted from the galaxy (the source) at wavelength x is measured from the earth to have wavelength $(1 + z)x$. (See also Exercise 1.R.159.) The spectrum shown below is of light from a distant galaxy. The function $y = f(x)$ graphed below gives the intensity (or brightness) of the light at wavelength x nanometers (nm) as measured from the earth. Let $y = g(x)$ be the intensity of the light at wavelength x as emitted from the source. Then $g(x) = f((1 + z)x)$. The largest measured redshift of any known galaxy is about $z = 11$. Assume $z = 11$ for the spectrum shown below.

(a) Describe how the graph of g is obtained from the graph of f.

(b) There is a dip in the graph at wavelength $x = 480$ nm. At what wavelength would this dip be at the source?

(c) Estimate the wavelengths at the source at several of the dips in the graph. Use your results to sketch a rough graph of the intensity function g.



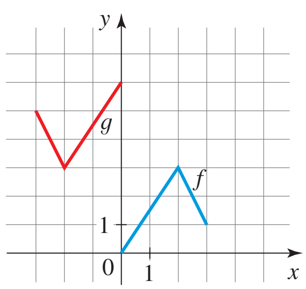
■ Discuss ■ Discover ■ Prove ■ Write

97–98 ■ Discuss: Obtaining Transformations Can the function g be obtained from f by transformations? If so, describe the transformations needed.

97. The functions f and g are described algebraically as follows:

$$f(x) = (x + 2)^2 \qquad g(x) = (x - 2)^2 + 5$$

98. The functions f and g are described graphically in the figure.



99. Discover: Stretched Step Functions Sketch graphs of the functions $f(x) = [\![x]\!]$, $g(x) = [\![2x]\!]$, and $h(x) = [\![3x]\!]$ on separate sets of axes. How are the graphs related? If n is a positive integer, what does a graph of $k(x) = [\![nx]\!]$ look like? What about the graphs of $k(x) = \left[\!\!\left[\frac{1}{n}x\right]\!\!\right]$?

100. Discuss: Sums of Even and Odd Functions If f and g are both even functions, is $f + g$ necessarily even? If both are odd, is their sum necessarily odd? What can you say about the sum if one is odd and one is even? In each case, prove your answer.

101. Discuss: Products of Even and Odd Functions Answer the same questions as in Exercise 100, except this time consider the product of f and g instead of the sum.

102. Discuss: Even and Odd Power Functions What must be true about the integer n if the function

$$f(x) = x^n$$

is an even function? If it is an odd function? Why do you think the names "even" and "odd" were chosen for these function properties?

103. Discuss ■ Discover: A Constant Function? Let f be a function for which $f(x) = f(x + 1)$ for every real number x. A constant function satisfies this property. Find a nonconstant function f with this property.

PS *Draw a diagram.* Think about defining a function graphically. That is, try to draw the graph of a function that satisfies the given condition.

2.7 Combining Functions

■ Sums, Differences, Products, and Quotients ■ Composition of Functions
■ Applications of Composition

In this section we study different ways of combining functions to make new functions.

■ Sums, Differences, Products, and Quotients

Two functions f and g can be combined to form new functions $f + g$, $f - g$, fg, and f/g in a manner similar to the way we add, subtract, multiply, and divide real numbers. For example, we define the function $f + g$ by

$$(f + g)(x) = f(x) + g(x)$$

The new function $f + g$ is called the **sum** of the functions f and g; its value at x is $f(x) + g(x)$. Of course, the sum on the right-hand side makes sense only if both $f(x)$ and $g(x)$ are defined, that is, if x belongs to the domain of f and also to the domain of g. So if the domain of f is A and the domain of g is B, then the domain of $f + g$ is the intersection of these domains, that is, $A \cap B$. Similarly, we can define the **difference** $f - g$, the **product** fg, and the **quotient** f/g of the functions f and g. Their domains are $A \cap B$, but in the case of the quotient we must remember not to divide by 0.

The sum of f and g is defined by

$$(f + g)(x) = f(x) + g(x)$$

The name of the new function is "$f + g$." So this $+$ sign stands for the operation of addition of *functions*. The $+$ sign on the right side, however, stands for addition of the *numbers* $f(x)$ and $g(x)$.

Algebra of Functions

Let f and g be functions with domains A and B. Then the functions $f + g$, $f - g$, fg, and f/g are defined as follows.

$(f + g)(x) = f(x) + g(x)$	Domain $A \cap B$
$(f - g)(x) = f(x) - g(x)$	Domain $A \cap B$
$(fg)(x) = f(x)g(x)$	Domain $A \cap B$
$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$	Domain $\{x \in A \cap B \mid g(x) \neq 0\}$

Example 1 ■ Combinations of Functions and Their Domains

Let $f(x) = \dfrac{1}{x-2}$ and $g(x) = \sqrt{x}$.

(a) Find the functions $f + g$, $f - g$, fg, and f/g and their domains.
(b) Find $(f + g)(4)$, $(f - g)(4)$, $(fg)(4)$, and $(f/g)(4)$.

Solution

(a) The domain of f is $\{x \mid x \neq 2\}$, and the domain of g is $\{x \mid x \geq 0\}$. The intersection of the domains of f and g is

$$\{x \mid x \geq 0 \text{ and } x \neq 2\} = [0, 2) \cup (2, \infty)$$

Thus we have

$$(f+g)(x) = f(x) + g(x) = \frac{1}{x-2} + \sqrt{x} \qquad \text{Domain } \{x \mid x \geq 0 \text{ and } x \neq 2\}$$

$$(f-g)(x) = f(x) - g(x) = \frac{1}{x-2} - \sqrt{x} \qquad \text{Domain } \{x \mid x \geq 0 \text{ and } x \neq 2\}$$

$$(fg)(x) = f(x)g(x) = \frac{\sqrt{x}}{x-2} \qquad \text{Domain } \{x \mid x \geq 0 \text{ and } x \neq 2\}$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{1}{(x-2)\sqrt{x}} \qquad \text{Domain } \{x \mid x > 0 \text{ and } x \neq 2\}$$

To divide fractions, invert the denominator and multiply:

$$\frac{1/(x-2)}{\sqrt{x}} = \frac{1/(x-2)}{\sqrt{x}/1} = \frac{1}{x-2} \cdot \frac{1}{\sqrt{x}} = \frac{1}{(x-2)\sqrt{x}}$$

Note that in the domain of f/g we exclude 0 because $g(0) = 0$.

(b) Each of these values exist because $x = 4$ is in the domain of each function:

$$(f+g)(4) = f(4) + g(4) = \frac{1}{4-2} + \sqrt{4} = \frac{5}{2}$$

$$(f-g)(4) = f(4) - g(4) = \frac{1}{4-2} - \sqrt{4} = -\frac{3}{2}$$

$$(fg)(4) = f(4)g(4) = \left(\frac{1}{4-2}\right)\sqrt{4} = 1$$

$$\left(\frac{f}{g}\right)(4) = \frac{f(4)}{g(4)} = \frac{1}{(4-2)\sqrt{4}} = \frac{1}{4}$$

Now Try Exercise 9 ■

The graph of the function $f + g$ can be obtained from the graphs of f and g by **graphical addition**. This means that we add corresponding y-coordinates, as illustrated in the next example.

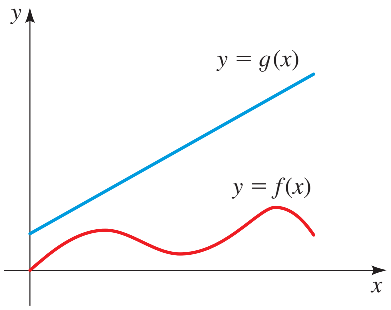


Figure 1

Example 2 ■ Using Graphical Addition

The graphs of f and g are shown in Figure 1. Use graphical addition to graph the function $f + g$.

Solution We obtain the graph of $f + g$ by "graphically adding" the value of $f(x)$ to $g(x)$ as shown in Figure 2. This is implemented by copying the line segment PQ on top of PR to obtain the point S on the graph of $f + g$.

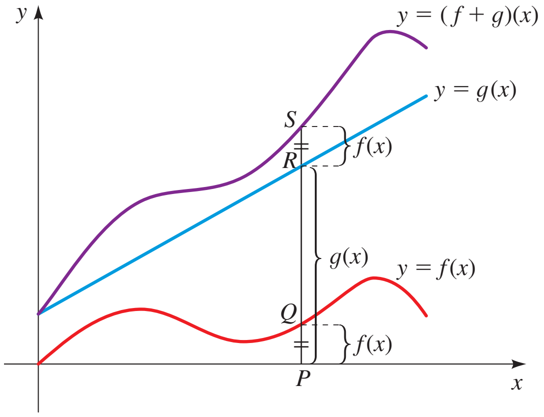


Figure 2 | Graphical addition

Now Try Exercise 21 ■

Composition of Functions

Now let's consider an important way of combining two functions to get a new function. Suppose $f(x) = \sqrt{x}$ and $g(x) = x^2 + 1$. We may define a new function h as

$$h(x) = f(g(x)) = f(x^2 + 1) = \sqrt{x^2 + 1}$$

The function h is made up of the functions f and g in an interesting way: Given a number x, we first apply the function g to it, then apply f to the result. In this case, f is the rule "take the square root," g is the rule "square, then add 1," and h is the rule "square, then add 1, then take the square root." In other words, we get the rule h by applying the rule g and then the rule f. Figure 3 shows a machine diagram for h.

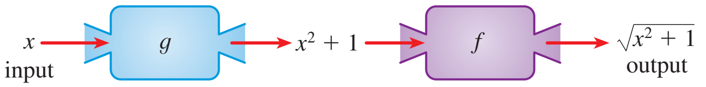


Figure 3 | The h machine is composed of the g machine (first) and then the f machine.

In general, given any two functions f and g, we start with a number x in the domain of g and find its image $g(x)$. If this number $g(x)$ is in the domain of f, we can then calculate the value of $f(g(x))$. The result is a new function $h(x) = f(g(x))$, which is obtained by substituting g into f. It is called the *composition* (or *composite*) of f and g and is denoted by $f \circ g$ ("f composed with g").

Discovery Project ■ **Iteration and Chaos**

The *iterates* of a function f at a point x are the numbers

$$f(x),\ f(f(x)),\ f(f(f(x))), \ldots$$

We examine iterates of the *logistic function,* which models the population of a species with limited potential for growth (such as lizards on an island or fish in a pond). Iterates of the model can help us to predict whether the population will eventually stabilize or whether it will fluctuate chaotically. You can find the project at **www.stewartmath.com**.

Composition of Functions

Given two functions f and g, the **composite function** $f \circ g$ (also called the **composition** of f and g) is defined by

$$(f \circ g)(x) = f(g(x))$$

The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f. In other words, $(f \circ g)(x)$ is defined whenever both $g(x)$ and $f(g(x))$ are defined. We can picture $f \circ g$ using an arrow diagram (Figure 4).

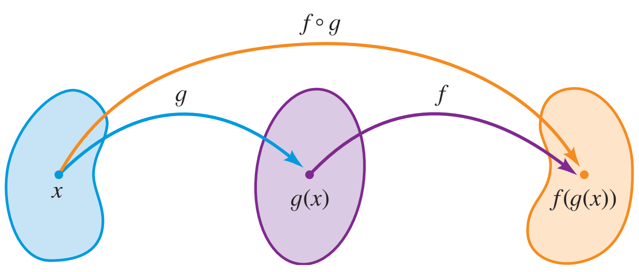


Figure 4 | Arrow diagram for $f \circ g$

Example 3 ■ Finding the Composition of Functions

Let $f(x) = x^2$ and $g(x) = x - 3$.

(a) Find the functions $f \circ g$ and $g \circ f$ and their domains.

(b) Find $(f \circ g)(5)$ and $(g \circ f)(7)$.

In Example 3, f is the rule "square," and g is the rule "subtract 3." The function $f \circ g$ *first* subtracts 3 and *then* squares; the function $g \circ f$ *first* squares and *then* subtracts 3.

Solution

(a) We have

$$\begin{aligned} (f \circ g)(x) &= f(g(x)) && \text{Definition of } f \circ g \\ &= f(x - 3) && \text{Definition of } g \\ &= (x - 3)^2 && \text{Definition of } f \end{aligned}$$

and

$$\begin{aligned} (g \circ f)(x) &= g(f(x)) && \text{Definition of } g \circ f \\ &= g(x^2) && \text{Definition of } f \\ &= x^2 - 3 && \text{Definition of } g \end{aligned}$$

The domains of both $f \circ g$ and $g \circ f$ are $\mathbb{R}$.

(b) We have

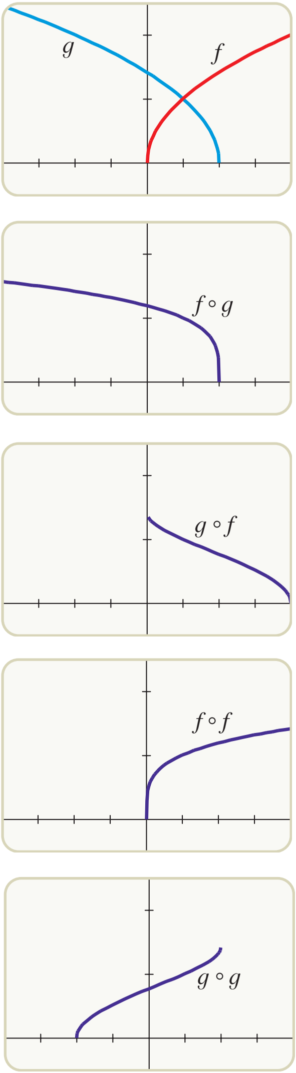
$$(f \circ g)(5) = f(g(5)) = f(2) = 2^2 = 4$$

$$(g \circ f)(7) = g(f(7)) = g(49) = 49 - 3 = 46$$

Now Try Exercises 27 and 49 ■

You can see from Example 3 that, in general, $f \circ g \neq g \circ f$. Remember that the notation $f \circ g$ means that the function g is applied first and then f is applied.

The graphs of f and g of Example 4, as well as those of $f \circ g$, $g \circ f$, $f \circ f$, and $g \circ g$, are shown below. These graphs indicate that the operation of composition can produce functions that are quite different from the original functions.



Example 4 ■ Finding the Composition of Functions

If $f(x) = \sqrt{x}$ and $g(x) = \sqrt{2 - x}$, find the following functions and their domains.

(a) $f \circ g$ **(b)** $g \circ f$ **(c)** $f \circ f$ **(d)** $g \circ g$

Solution

(a)
$$\begin{aligned}(f \circ g)(x) &= f(g(x)) && \text{Definition of } f \circ g\\ &= f(\sqrt{2-x}) && \text{Definition of } g\\ &= \sqrt{\sqrt{2-x}} && \text{Definition of } f\\ &= \sqrt[4]{2-x} && \text{Property of } n\text{th roots}\end{aligned}$$

The domain of $f \circ g$ is $\{x \mid 2 - x \geq 0\} = \{x \mid x \leq 2\} = (-\infty, 2]$.

(b)
$$\begin{aligned}(g \circ f)(x) &= g(f(x)) && \text{Definition of } g \circ f\\ &= g(\sqrt{x}) && \text{Definition of } f\\ &= \sqrt{2-\sqrt{x}} && \text{Definition of } g\end{aligned}$$

For $\sqrt{x}$ to be defined, we must have $x \geq 0$. For $\sqrt{2 - \sqrt{x}}$ to be defined, we must have $2 - \sqrt{x} \geq 0$, that is, $\sqrt{x} \leq 2$, or $x \leq 4$. Thus we have $0 \leq x \leq 4$, so the domain of $g \circ f$ is the closed interval $[0, 4]$.

(c)
$$\begin{aligned}(f \circ f)(x) &= f(f(x)) && \text{Definition of } f \circ f\\ &= f(\sqrt{x}) && \text{Definition of } f\\ &= \sqrt{\sqrt{x}} && \text{Definition of } f\\ &= \sqrt[4]{x} && \text{Property of } n\text{th roots}\end{aligned}$$

The domain of $f \circ f$ is $[0, \infty)$.

(d)
$$\begin{aligned}(g \circ g)(x) &= g(g(x)) && \text{Definition of } g \circ g\\ &= g(\sqrt{2-x}) && \text{Definition of } g\\ &= \sqrt{2-\sqrt{2-x}} && \text{Definition of } g\end{aligned}$$

This expression is defined when both $2 - x \geq 0$ and $2 - \sqrt{2 - x} \geq 0$. The first inequality means $x \leq 2$, and the second is equivalent to $\sqrt{2 - x} \leq 2$, or $2 - x \leq 4$, or $x \geq -2$. Thus $-2 \leq x \leq 2$, so the domain of $g \circ g$ is $[-2, 2]$.

Now Try Exercise 53 ■

It is possible to take the composition of three or more functions. For instance, the composite function $f \circ g \circ h$ is found by first applying h, then g, and then f as follows:

$$(f \circ g \circ h)(x) = f(g(h(x)))$$

Example 5 ■ A Composition of Three Functions

Find $f \circ g \circ h$ if $f(x) = x/(x + 1)$, $g(x) = x^{10}$, and $h(x) = x + 3$.

Solution

$$\begin{aligned}(f \circ g \circ h)(x) &= f(g(h(x))) && \text{Definition of } f \circ g \circ h\\ &= f(g(x+3)) && \text{Definition of } h\\ &= f((x+3)^{10}) && \text{Definition of } g\\ &= \frac{(x+3)^{10}}{(x+3)^{10}+1} && \text{Definition of } f\end{aligned}$$

Now Try Exercise 61 ■

So far, we have used composition to build complicated functions from simpler ones. In calculus it is also useful to be able to "decompose" a complicated function into simpler ones, as shown in the following example.

Example 6 ■ Recognizing a Composition of Functions

Given $F(x) = \sqrt[4]{x + 9}$, find functions f and g such that $F = f \circ g$.

Solution Since the formula for F says to first add 9 and then take the fourth root, we can let

$$g(x) = x + 9 \qquad \text{and} \qquad f(x) = \sqrt[4]{x}$$

Then

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) && \text{Definition of } f \circ g\\ &= f(x + 9) && \text{Definition of } g\\ &= \sqrt[4]{x + 9} && \text{Definition of } f\\ &= F(x)\end{aligned}$$

Now Try Exercise 65

Applications of Composition

When working with functions that model real-world situations, we can name the variables using letters that suggest the quantity being modeled. We may use t for time, d for distance, V for volume, and so on. For example, if air is being pumped into a balloon, then the radius R of the balloon is a function of the volume V of air pumped into the balloon, say, $R = f(V)$. Also the volume V is a function of the time t that the pump has been working, say, $V = g(t)$. It follows that the radius R is a function of the time t given by $R = f(g(t))$.

Example 7 ■ An Application of Composition of Functions

A ship is traveling at 20 mi/h parallel to a straight shoreline. The ship is 5 mi from shore. It passes a lighthouse at noon.

(a) Express the distance s between the lighthouse and the ship as a function of d, the distance the ship has traveled since noon; that is, find f so that $s = f(d)$.

(b) Express d as a function of t, the time elapsed since noon; that is, find g so that $d = g(t)$.

(c) Find $f \circ g$. What does this function represent?

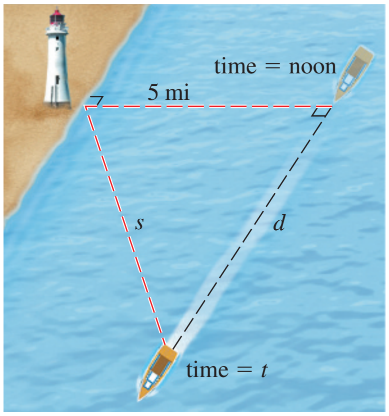


Figure 5

Solution We first draw a diagram as in Figure 5.

(a) We can relate the distances s and d by the Pythagorean Theorem. Thus s can be expressed as a function of d by

$$s = f(d) = \sqrt{25 + d^2}$$

(b) Since the ship is traveling at 20 mi/h, the distance d that it has traveled is a function of t as follows:

distance = rate × time

$$d = g(t) = 20t$$

(c) We have

$$\begin{aligned}(f \circ g)(t) &= f(g(t)) && \text{Definition of } f \circ g\\ &= f(20t) && \text{Definition of } g\\ &= \sqrt{25 + (20t)^2} && \text{Definition of } f\end{aligned}$$

The function $f \circ g$ gives the distance of the ship from the lighthouse as a function of time.

Now Try Exercise 83

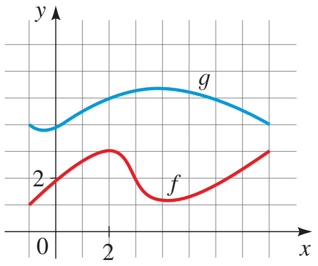
2.7 Exercises

Concepts

1. From the graphs of f and g in the figure, we find

$(f + g)(2) =$ ______ $(f - g)(2) =$ ______

$(fg)(2) =$ ______ $\left(\dfrac{f}{g}\right)(2) =$ ______



2. By definition, $(f \circ g)(x) =$ ______. So if $g(2) = 5$ and $f(5) = 12$, then $(f \circ g)(2) =$ ______.

3. If the rule of the function f is "add one" and the rule of the function g is "multiply by 2," then the rule of $f \circ g$ is

"______________________________."

and the rule of $g \circ f$ is

"______________________________."

4. We can express the functions in Exercise 3 algebraically as

$f(x) =$ ______ $g(x) =$ ______

$(f \circ g)(x) =$ ______ $(g \circ f)(x) =$ ______

5–6 ■ Let f and g be functions.

5. **(a)** The function $(f + g)(x)$ is defined for all values of x that are in the domains of both ______ and ______.

(b) The function $(fg)(x)$ is defined for all values of x that are in the domains of both ______ and ______.

(c) The function $(f/g)(x)$ is defined for all values of x that are in the domains of both ______ and ______, and $g(x)$ is not equal to ______.

6. The composition $(f \circ g)(x)$ is defined for all values of x for which x is in the domain of ______ and $g(x)$ is in the domain of ______.

Skills

7–16 ■ **Combining Functions** Find $f + g$, $f - g$, fg, and f/g and their domains.

7. $f(x) = 3x, \quad g(x) = 1 - x$

8. $f(x) = 3 - 2x, \quad g(x) = 2x + 1$

9. $f(x) = x^3 + x^2, \quad g(x) = x^2$

10. $f(x) = x^2 + 1, \quad g(x) = x^2 + 2$

11. $f(x) = 5 - x, \quad g(x) = x^2 - 3x$

12. $f(x) = x^2 + 2x, \quad g(x) = 3x^2 - 1$

13. $f(x) = \sqrt{25 - x^2}, \quad g(x) = \sqrt{x + 3}$

14. $f(x) = \sqrt{16 - x^2}, \quad g(x) = \sqrt{x^2 - 1}$

15. $f(x) = \dfrac{1}{x + 1}, \quad g(x) = \dfrac{3}{x - 2}$

16. $f(x) = \dfrac{3}{x - 3}, \quad g(x) = \dfrac{x}{x + 3}$

17–20 ■ **Domain** Find the domain of the function.

17. $f(x) = \sqrt{x} + \sqrt{3 - x}$

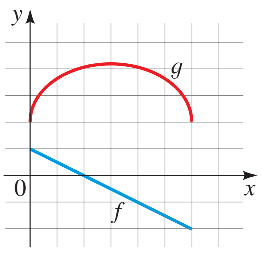
18. $f(x) = \sqrt{x + 4} - \dfrac{\sqrt{1 - x}}{x}$

19. $h(x) = (x - 3)^{-1/4}$

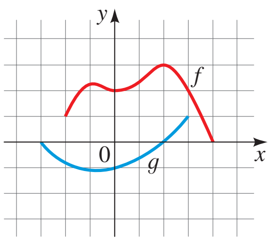
20. $k(x) = \dfrac{\sqrt{x + 3}}{x - 1}$

21–22 ■ **Graphical Addition** Use graphical addition to sketch the graph of $f + g$.

21.



22.



23–26 ■ **Graphical Addition** Draw the graphs of f, g, and $f + g$ on a common screen to illustrate graphical addition.

23. $f(x) = \sqrt{1 + x}, \quad g(x) = \sqrt{1 - x}$

24. $f(x) = x^2, \quad g(x) = \sqrt{x}$

25. $f(x) = x^2, \quad g(x) = \frac{1}{3}x^3$

26. $f(x) = \sqrt[4]{1 - x}, \quad g(x) = \sqrt{1 - \dfrac{x^2}{9}}$

27–32 ■ **Evaluating Composition of Functions** Use $f(x) = 4x + 5$ and $g(x) = x^2 + 2$ to evaluate the expression.

27. **(a)** $f(g(1))$ **(b)** $g(f(1))$

28. **(a)** $f(f(0))$ **(b)** $g(g(-1))$

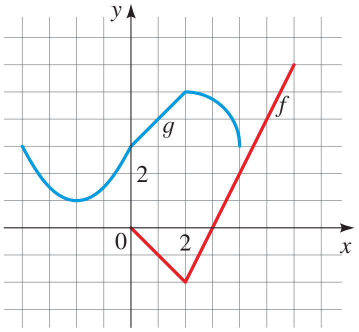
29. **(a)** $(f \circ g)(-2)$ **(b)** $(g \circ f)(-1)$

30. **(a)** $(f \circ f)(1)$ **(b)** $(g \circ g)(0)$

31. **(a)** $(f \circ g)(x)$ **(b)** $(g \circ f)(x)$

32. **(a)** $(f \circ f)(x)$ **(b)** $(g \circ g)(x)$

33–38 ■ Composition Using a Graph Use the given graphs of f and g to evaluate the expression.



33. $f(g(2))$

34. $g(f(0))$

35. $(g \circ f)(4)$

36. $(f \circ g)(0)$

37. $(g \circ g)(-2)$

38. $(f \circ f)(4)$

39–46 ■ Composition Using a Table Use the table to evaluate the expression.

x	1	2	3	4	5	6
$f(x)$	2	3	5	1	6	3
$g(x)$	3	5	6	2	1	4

39. $f(g(2))$

40. $g(f(2))$

41. $f(f(1))$

42. $g(g(2))$

43. $(f \circ g)(6)$

44. $(g \circ f)(2)$

45. $(f \circ f)(5)$

46. $(g \circ g)(3)$

47–60 ■ Composition of Functions Find the functions $f \circ g$, $g \circ f$, $f \circ f$, and $g \circ g$ and their domains.

47. $f(x) = 2x + 3, \quad g(x) = 4x - 1$

48. $f(x) = 6x - 5, \quad g(x) = \dfrac{x}{2}$

49. $f(x) = x^2, \quad g(x) = x + 1$

50. $f(x) = \sqrt[3]{x}, \quad g(x) = \dfrac{1}{x^3}$

51. $f(x) = x^2 + 1, \quad g(x) = \dfrac{1}{\sqrt{x}}$

52. $f(x) = x - 4, \quad g(x) = |x + 4|$

53. $f(x) = \dfrac{x}{x+1}, \quad g(x) = 2x - 1$

54. $f(x) = \dfrac{x}{x+1}, \quad g(x) = \dfrac{1}{x}$

55. $f(x) = \dfrac{2}{x}, \quad g(x) = \dfrac{x}{x+2}$

56. $f(x) = \dfrac{1}{x-1}, \quad g(x) = \dfrac{1}{x^2+1}$

57. $f(x) = \dfrac{1}{\sqrt{x}}, \quad g(x) = x^2 - 4x$

58. $f(x) = x^2, \quad g(x) = \sqrt{x - 3}$

59. $f(x) = 1 - \sqrt{x}, \quad g(x) = \sqrt[3]{x}$

60. $f(x) = \sqrt{x^2 - 1}, \quad g(x) = \sqrt{1 - x}$

61–64 ■ Composition of Three Functions Find $f \circ g \circ h$.

61. $f(x) = x - 1, \quad g(x) = \sqrt{x}, \quad h(x) = x - 1$

62. $f(x) = \dfrac{1}{x}, \quad g(x) = x^3, \quad h(x) = x^2 + 2$

63. $f(x) = x^4 + 1, \quad g(x) = x - 5, \quad h(x) = \sqrt{x}$

64. $f(x) = \sqrt{x}, \quad g(x) = \dfrac{x}{x-1}, \quad h(x) = \sqrt[3]{x}$

65–72 ■ Expressing a Function as a Composition Find functions f and g such that $F = f \circ g$.

65. $F(x) = (x - 9)^5$

66. $F(x) = \sqrt{x} + 1$

67. $F(x) = \dfrac{x^2}{x^2 + 4}$

68. $F(x) = \dfrac{1}{x + 3}$

69. $F(x) = |1 - x^3|$

70. $F(x) = \sqrt{1 + \sqrt{x}}$

71. $F(x) = 1 - \sqrt{x^3 + 1}$

72. $F(x) = \dfrac{1}{\sqrt[3]{x^2 + x + 1}}$

73–78 ■ Expressing a Function as a Composition Find functions f, g, and h such that $F = f \circ g \circ h$.

73. $F(x) = \dfrac{1}{x^2 + 1}$

74. $F(x) = \sqrt[3]{\sqrt{x} - 1}$

75. $F(x) = (4 + \sqrt[3]{x})^9$

76. $F(x) = \dfrac{2}{(3 + \sqrt{x})^2}$

77. $F(x) = \left(\dfrac{\sqrt{x}}{\sqrt{x} - 1}\right)^3$

78. $F(x) = \dfrac{1}{1 + \dfrac{1}{\sqrt{x^2 + 1}}}$

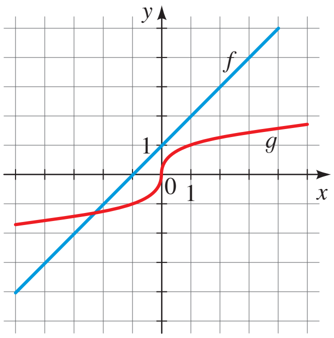
Skills Plus

79. Composing Linear Functions The graphs of the functions

$$f(x) = m_1 x + b_1$$
$$g(x) = m_2 x + b_2$$

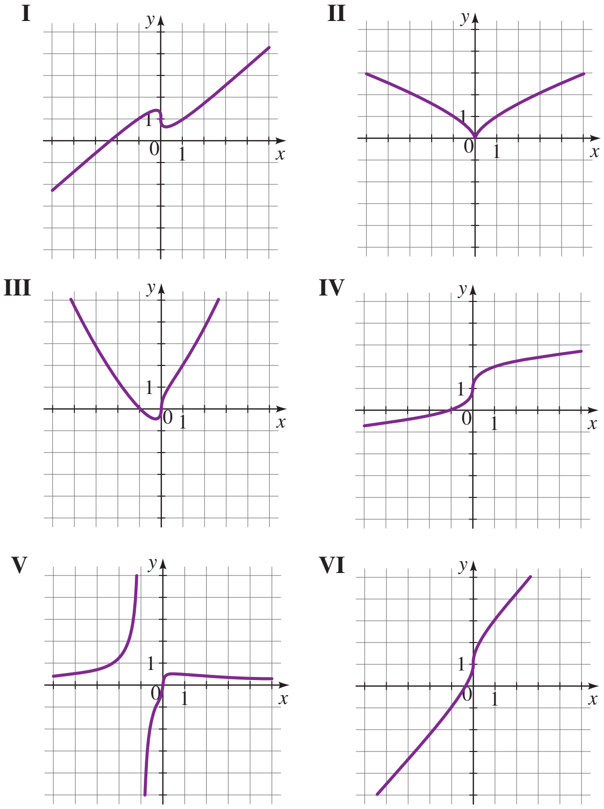
are lines with slopes m_1 and m_2, respectively. Is the graph of $f \circ g$ a line? If so, what is its slope?

80. Combining Functions Graphically Graphs of the functions f and g are given. Match each combination of these functions given in parts (a)–(f) with its graph in I–VI. Give reasons for your answers.



(a) $f(x) + g(x)$ **(b)** $f(x) - g(x)$ **(c)** $f(x)g(x)$

(d) $g^2(x)$ **(e)** $g(x)/f(x)$ **(f)** $(f \circ g)(x)$



Applications

81–82 ■ Revenue, Cost, and Profit A print shop makes bumper stickers for election campaigns. If x stickers are ordered (where $x < 10{,}000$), then the price per bumper sticker is $0.15 - 0.000002x$ dollars, and the total cost of producing the order is $0.095x - 0.0000005x^2$ dollars.

81. Use the fact that

$$\boxed{\text{revenue}} = \boxed{\text{price per item}} \times \boxed{\text{number of items sold}}$$

to express $R(x)$, the revenue from an order of x stickers, as a product of two functions of x.

82. Use the fact that

$$\boxed{\text{profit}} = \boxed{\text{revenue}} - \boxed{\text{cost}}$$

to express $P(x)$, the profit on an order of x stickers, as a difference of two functions of x.

83. Area of a Ripple A stone is dropped into a lake, creating a circular ripple that travels outward at a speed of 60 cm/s.

(a) Find a function g that models the radius as a function of time.

(b) Find a function f that models the area of the circle as a function of the radius.

(c) Find $f \circ g$. What does this function represent?

84. Inflating a Balloon A spherical balloon is being inflated. The radius of the balloon is increasing at the rate of 2 cm/s.

(a) Find a function f that models the radius as a function of time.

(b) Find a function g that models the volume as a function of the radius. (Formulas for volume are given on the inside front cover of this book.)

(c) Find $g \circ f$. What does this function represent?

85. Surface Area of a Balloon A spherical weather balloon is being inflated. The radius of the balloon is increasing at the rate of 2 cm/s. Express the surface area of the balloon as a function of time t (in seconds).

86. Elevation, Pressure, and Boiling Point At higher elevations, atmospheric pressure decreases, causing water to boil at lower temperatures than at sea level. Nineteenth century explorers used the boiling point of water to estimate elevation. In Exercises 2.5.49–50 you found a function f that models the atmospheric pressure at a given elevation and a function g that models the boiling point of water at a given atmospheric pressure.

(a) Find the function $h = g \circ f$. Describe the inputs and outputs of h. What does h model?

(b) Estimate the boiling point of water at the peak of Mt. Rainier, 4.4 km above sea level.

(c) If water boils at 91°C, estimate the elevation.

Thye-Wee Gn/Shutterstock.com

87. Multiple Discounts You have a \$50 coupon from the manufacturer that is good for the purchase of a cell phone. The store where you are purchasing your cell phone is offering a 20% discount on all phones. Let x represent the regular price of the phone.

(a) Suppose only the 20% discount applies. Find a function f that models the purchase price of the cell phone as a function of the regular price x.

(b) Suppose only the \$50 coupon applies. Find a function g that models the purchase price of the cell phone as a function of the full price x.

(c) If you can use both the coupon and the discount, then the purchase price is either $(f \circ g)(x)$ or $(g \circ f)(x)$, depending on the order in which they are applied to the price. Find both $(f \circ g)(x)$ and $(g \circ f)(x)$. Which composition gives the lower price?

88. Multiple Discounts An appliance dealer advertises a 10% discount on all washing machines. In addition, the manufacturer offers a \$100 rebate on the purchase of a washing machine. Let x represent the full price of the washing machine.

(a) Suppose only the 10% discount applies. Find a function f that models the purchase price of the washer as a function of the full price x.

(b) Suppose only the \$100 rebate applies. Find a function g that models the purchase price of the washer as a function of the full price x.

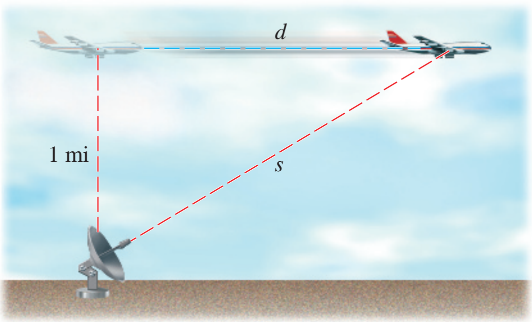
(c) Find $f \circ g$ and $g \circ f$. What do these functions represent? Which is the better deal?

89. Airplane Trajectory An airplane is flying at a speed of 350 mi/h at an altitude of one mile. The plane passes directly above a radar station at time $t = 0$.

(a) Express the distance s (in miles) between the plane and the radar station as a function of the horizontal distance d (in miles) that the plane has flown.

(b) Express d as a function of the time t (in hours) that the plane has flown.

(c) Use composition to express s as a function of t.



■ Discuss ■ Discover ■ Prove ■ Write

90. Discover: Compound Interest A savings account earns 5% interest compounded annually. If you invest x dollars in such an account, then the amount $A(x)$ of the investment after one year is the initial investment plus 5%; that is,

$$A(x) = x + 0.05x = 1.05x$$

Find

$$A \circ A$$
$$A \circ A \circ A$$
$$A \circ A \circ A \circ A$$

What do these compositions represent? Find a formula for what you get when you compose n copies of A.

91. Discover: Solving an Equation for an Unknown Function Suppose that

$$g(x) = 2x + 1$$
$$h(x) = 4x^2 + 4x + 7$$

Find a function f such that $f \circ g = h$. (Think about what operations you would have to perform on the formula for g to end up with the formula for h.) Now suppose that

$$f(x) = 3x + 5$$
$$h(x) = 3x^2 + 3x + 2$$

Use the same sort of reasoning to find a function g such that $f \circ g = h$.

92. Discuss: Compositions of Odd and Even Functions Suppose that

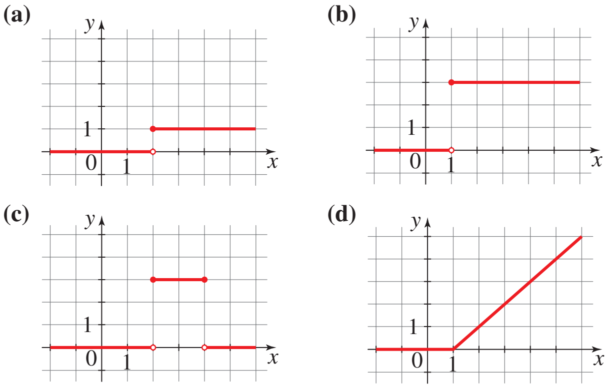
$$h = f \circ g$$

If g is an even function, is h necessarily even? If g is odd, is h odd? What if g is odd and f is odd? What if g is odd and f is even?

93. Discover: The Unit Step Function The *unit step function* u is defined by

$$u(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

Draw graphs of the following functions: $u(x)$, $2u(x)$, $u(x - 3)$, $u(x - 1) - u(x - 2)$, and $xu(x)$. Now find combinations and transformations involving the unit step function that yield the following graphs.



PS *Try to recognize something familiar.* We are familiar with what happens when we multiply a number by zero or one. Think about how subtracting two unit step functions or multiplying a function by a unit step function changes its graph.

94. Discover: Let $f_0(x) = 1/(1 - x)$ and $f_{n+1}(x) = (f_0 \circ f_n)(x)$ for $n = 0, 1, 2, \ldots$. Find $f_{1000}(x)$.

PS *Try to recognize a pattern.* First find $f_1(x)$, $f_2(x)$, $f_3(x)$, $f_4(x)$, . . . , simplify, and find a pattern.

2.8 One-to-One Functions and Their Inverses

■ One-to-One Functions : The Horizontal Line Test ■ The Inverse of a Function ■ Finding the Inverse of a Function ■ Graphing the Inverse of a Function ■ Applications of Inverse Functions

The *inverse* of a function is a rule that acts on the output of the function and produces the corresponding input. So the inverse "undoes" or reverses what the function has done. Not all functions have inverses; those that do are called *one-to-one*.

■ One-to-One Functions: The Horizontal Line Test

Let's compare the functions f and g whose arrow diagrams are shown in Figure 1. Note that f never takes on the same value twice (any two numbers in A have different images), whereas g does take on the same value twice (both 2 and 3 have the same image, 4). In symbols, $g(2) = g(3)$ but $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$. Functions that have this latter property are called *one-to-one*.

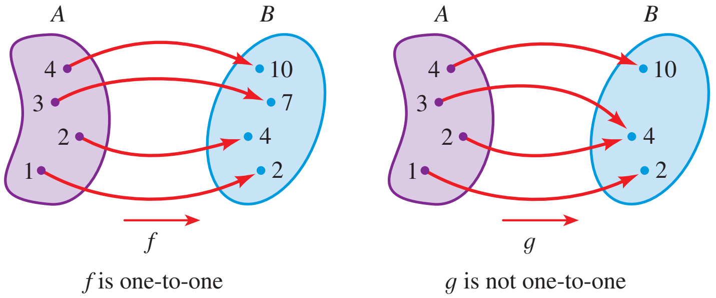


Figure 1

Definition of a One-to-one Function

A function is called a **one-to-one function** if no two elements in its domain have the same image, that is,

$$f(x_1) \neq f(x_2) \quad \text{whenever } x_1 \neq x_2$$

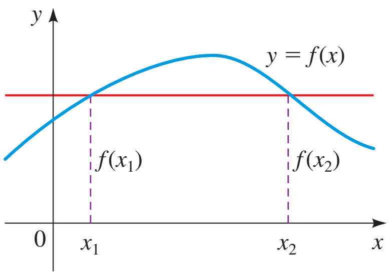


Figure 2 | This function is not one-to-one because $f(x_1) = f(x_2)$.

An equivalent way of writing the condition for a one-to-one function is:

$$\text{if } f(x_1) = f(x_2), \text{ then } x_1 = x_2.$$

If a horizontal line intersects the graph of f at more than one point, then we see from Figure 2 that there are numbers $x_1 \neq x_2$ such that $f(x_1) = f(x_2)$. This means that f is not one-to-one. Therefore we have the following geometric method for determining whether a function is one-to-one.

Horizontal Line Test

A function is one-to-one if and only if no horizontal line intersects its graph more than once.

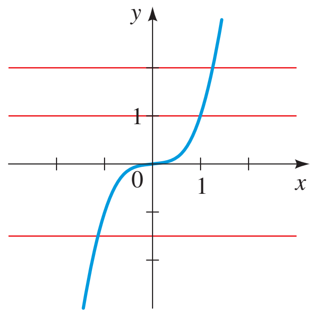


Figure 3 | $f(x) = x^3$ is one-to-one.

Example 1 ■ Deciding Whether a Function Is One-to-One

Is the function $f(x) = x^3$ one-to-one?

Solution 1 If $x_1 \neq x_2$, then $x_1^3 \neq x_2^3$ (two different numbers cannot have the same cube). Therefore $f(x) = x^3$ is one-to-one.

Solution 2 From Figure 3 we see that no horizontal line intersects the graph of $f(x) = x^3$ more than once. Therefore, by the Horizontal Line Test, f is one-to-one.

Now Try Exercise 15 ■

Notice that the function f of Example 1 is increasing and is also one-to-one. In fact, it can be proved that *every increasing function and every decreasing function is one-to-one*.

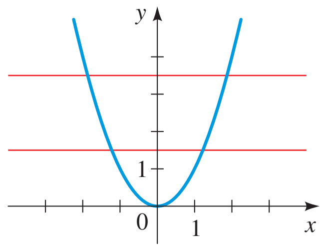


Figure 4 | $g(x) = x^2$ is not one-to-one.

Example 2 ■ Deciding Whether a Function Is One-to-One

Is the function $g(x) = x^2$ one-to-one?

Solution 1 This function is not one-to-one because, for instance,

$$g(1) = 1 \qquad \text{and} \qquad g(-1) = 1$$

so 1 and -1 have the same image.

Solution 2 From Figure 4 we see that there are horizontal lines that intersect the graph of g more than once. Therefore, by the Horizontal Line Test, g is not one-to-one.

Now Try Exercise 17 ■

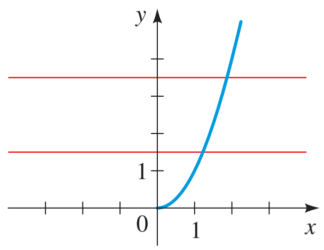


Figure 5 | $h(x) = x^2$ $(x \geq 0)$ is one-to-one.

Note Although the function g in Example 2 is not one-to-one, it is possible to restrict its domain so that the resulting function *is* one-to-one. In fact, if we define

$$h(x) = x^2 \qquad (x \geq 0)$$

then h is one-to-one, as you can see from Figure 5 and the Horizontal Line Test.

Example 3 ■ Showing That a Function Is One-to-One

Show that the function $f(x) = 3x + 4$ is one-to-one.

Solution Suppose there are numbers x_1 and x_2 such that $f(x_1) = f(x_2)$. Then

$$\begin{aligned} 3x_1 + 4 &= 3x_2 + 4 && \text{Suppose } f(x_1) = f(x_2) \\ 3x_1 &= 3x_2 && \text{Subtract 4} \\ x_1 &= x_2 && \text{Divide by 3} \end{aligned}$$

Therefore f is one-to-one.

Now Try Exercise 13 ■

■ The Inverse of a Function

One-to-one functions are important because they are precisely the functions that possess inverse functions according to the following definition.

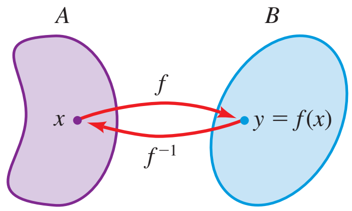


Figure 6

Definition of the Inverse of a Function

Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y$$

for any y in B.

This definition says that if f takes x to y, then f^{-1} takes y back to x. (If f were not one-to-one, then f^{-1} would not be defined uniquely.) The arrow diagram in Figure 6 indicates that f^{-1} reverses the effect of f. From the definition we have

$$\text{domain of } f^{-1} = \text{range of } f$$

$$\text{range of } f^{-1} = \text{domain of } f$$

Example 4 ■ Finding f^{-1} for Specific Values

Don't mistake the -1 in f^{-1} for an exponent.

$$f^{-1}(x) \quad \textit{does not mean} \quad \frac{1}{f(x)}$$

The reciprocal $1/f(x)$ is written as $(f(x))^{-1}$.

If $f(1) = 5$, $f(3) = 7$, and $f(8) = -10$, find $f^{-1}(5)$, $f^{-1}(7)$, and $f^{-1}(-10)$.

Solution From the definition of f^{-1} we have

$$f^{-1}(5) = 1 \quad \text{because} \quad f(1) = 5$$

$$f^{-1}(7) = 3 \quad \text{because} \quad f(3) = 7$$

$$f^{-1}(-10) = 8 \quad \text{because} \quad f(8) = -10$$

Figure 7 shows how f^{-1} reverses the effect of f in this case.

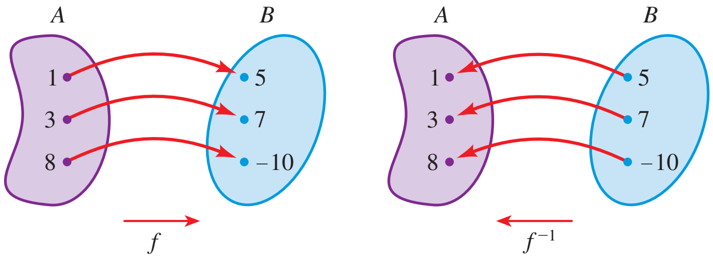


Figure 7

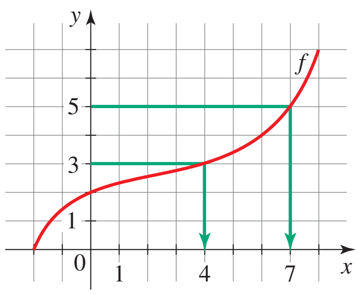
Now Try Exercise 25 ■

Example 5 ■ Finding Values of an Inverse Function

We can find specific values of an inverse function from a table or graph of the function itself. A graph of a function f is shown in Figure 8, as well as a table of values for f. From the table, we can see that $f^{-1}(3) = 4$ and $f^{-1}(5) = 7$. We can also "read" these values of f^{-1} from the graph.

x	$f(x)$
0	2
4 ←	3
6	4
7 ←	5
8	7

Finding values of f^{-1} from a table of f

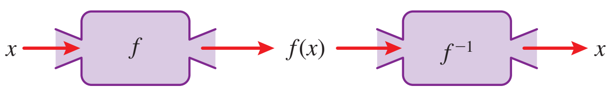


Finding values of f^{-1} from a graph of f

Figure 8

Now Try Exercises 29 and 31 ■

By definition the inverse function f^{-1} undoes what f does: If we start with x, apply f, and then apply f^{-1}, we arrive back at x, where we started (see the following machine diagram).



Similarly, f undoes what f^{-1} does. In general, any function that reverses the effect of f in this way must be the inverse of f. These observations are expressed precisely as follows.

Inverse Function Property

Let f be a one-to-one function with domain A and range B. The inverse function f^{-1} satisfies the following cancellation equations:

$$f^{-1}(f(x)) = x \quad \text{for every } x \text{ in } A$$

$$f(f^{-1}(x)) = x \quad \text{for every } x \text{ in } B$$

Conversely, any function f^{-1} satisfying these equations is the inverse of f.

These properties indicate that f is the inverse function of f^{-1}, so we say that f and f^{-1} are *inverses of each other*.

Example 6 ■ Verifying That Two Functions Are Inverses

Show that $f(x) = x^3$ and $g(x) = x^{1/3}$ are inverses of each other.

Solution Note that the domain and range of both f and g are $\mathbb{R}$. We show that f and g satisfy the cancellation equations of the Inverse Function Property. We have

$$g(f(x)) = g(x^3) = (x^3)^{1/3} = x$$

$$f(g(x)) = f(x^{1/3}) = (x^{1/3})^3 = x$$

So by the Inverse Function Property, f and g are inverses of each other. These equations simply say that the cube function and the cube root function, when composed, cancel each other.

Now Try Exercise 39 ■

■ Finding the Inverse of a Function

Now let's examine how we compute inverse functions. We first observe from the definition of f^{-1} that

$$y = f(x) \quad \Leftrightarrow \quad f^{-1}(y) = x$$

So if $y = f(x)$ and if we are able to solve this equation for x in terms of y, then we must have $x = f^{-1}(y)$. If we then interchange x and y, we have $y = f^{-1}(x)$, which is the desired equation.

How to Find the Inverse of a One-to-One Function

1. Write $y = f(x)$.
2. Solve this equation for x in terms of y (if possible).
3. Interchange x and y. The resulting equation is $y = f^{-1}(x)$.

Note that Steps 2 and 3 can be reversed. In other words, we can interchange x and y first and then solve for y in terms of x.

Example 7 ■ Finding the Inverse of a Function

Find the inverse of the function $f(x) = 3x - 2$.

Solution First we write $y = f(x)$.

$$y = 3x - 2$$

Then we solve this equation for x:

$$3x = y + 2 \qquad \text{Add 2}$$

$$x = \frac{y+2}{3} \qquad \text{Divide by 3}$$

Finally, we interchange x and y:

$$y = \frac{x+2}{3}$$

Therefore the inverse function is $f^{-1}(x) = \dfrac{x+2}{3}$.

In Example 7 note how f^{-1} reverses the effect of f. The function f is the rule "Multiply by 3, then subtract 2," whereas f^{-1} is the rule "Add 2, then divide by 3."

Check Your Answer

We use the Inverse Function Property:

$$f^{-1}(f(x)) = f^{-1}(3x-2) = \frac{(3x-2)+2}{3} = \frac{3x}{3} = x$$

$$f(f^{-1}(x)) = f\left(\frac{x+2}{3}\right) = 3\left(\frac{x+2}{3}\right) - 2 = x + 2 - 2 = x$$

Both cancellation equations are satisfied. ✓

Now Try Exercise 49 ■

Example 8 ■ Finding the Inverse of a Function

Find the inverse of the function $f(x) = \dfrac{x^5 - 3}{2}$.

Solution We first write $y = (x^5 - 3)/2$ and solve for x.

$$y = \frac{x^5 - 3}{2} \qquad \text{Equation defining function}$$

$$2y = x^5 - 3 \qquad \text{Multiply by 2}$$

$$x^5 = 2y + 3 \qquad \text{Add 3 (and switch sides)}$$

$$x = (2y+3)^{1/5} \qquad \text{Take fifth root of each side}$$

Then we interchange x and y to get $y = (2x + 3)^{1/5}$. Therefore the inverse function is $f^{-1}(x) = (2x + 3)^{1/5}$.

In Example 8 note how f^{-1} reverses the effect of f. The function f is the rule "Take the fifth power, subtract 3, then divide by 2," whereas f^{-1} is the rule "Multiply by 2, add 3, then take the fifth root."

Check Your Answer

We use the Inverse Function Property:

$$f^{-1}(f(x)) = f^{-1}\left(\frac{x^5-3}{2}\right) = \left[2\left(\frac{x^5-3}{2}\right) + 3\right]^{1/5} = (x^5 - 3 + 3)^{1/5} = (x^5)^{1/5} = x$$

$$f(f^{-1}(x)) = f((2x+3)^{1/5}) = \frac{[(2x+3)^{1/5}]^5 - 3}{2} = \frac{2x + 3 - 3}{2} = \frac{2x}{2} = x$$

Both cancellation equations are satisfied. ✓

Now Try Exercise 63 ■

A **rational function** is a function defined by a rational expression. (See Section 3.6.) In the next example we find the inverse of a rational function.

Example 9 ■ Finding the Inverse of a Rational Function

Find the inverse of the function $f(x) = \dfrac{2x + 3}{x - 1}$.

Solution We first write $y = (2x + 3)/(x - 1)$ and solve for x.

$$
\begin{aligned}
y &= \frac{2x+3}{x-1} && \text{Equation defining function}\\
y(x-1) &= 2x + 3 && \text{Multiply by } x - 1\\
yx - y &= 2x + 3 && \text{Expand}\\
yx - 2x &= y + 3 && \text{Bring } x\text{-terms to LHS}\\
x(y-2) &= y + 3 && \text{Factor } x\\
x &= \frac{y+3}{y-2} && \text{Divide by } y - 2
\end{aligned}
$$

Therefore the inverse function is $f^{-1}(x) = \dfrac{x + 3}{x - 2}$.

Now Try Exercise 57 ■

■ Graphing the Inverse of a Function

The principle of interchanging x and y to find the inverse function also gives us a method for obtaining the graph of f^{-1} from the graph of f. If $f(a) = b$, then $f^{-1}(b) = a$. Thus the point (a, b) is on the graph of f if and only if the point (b, a) is on the graph of f^{-1}. But we get the point (b, a) from the point (a, b) by reflecting about the line $y = x$ (see Figure 9). Therefore, as Figure 10 illustrates, the following is true.

> The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.

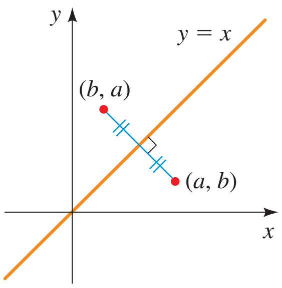


Figure 9

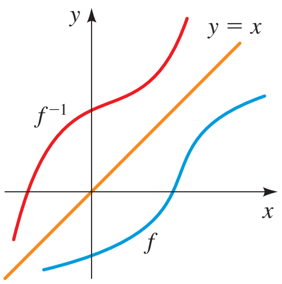


Figure 10

Example 10 ■ Graphing the Inverse of a Function

In Example 10 note how f^{-1} reverses the effect of f. The function f is the rule "Subtract 2, then take the square root," whereas f^{-1} is the rule "Square, then add 2."

(a) Sketch the graph of $f(x) = \sqrt{x - 2}$.

(b) Use the graph of f to sketch the graph of f^{-1}.

(c) Find an equation for f^{-1}.

Solution

(a) Using the transformations from Section 2.6, we sketch the graph of $y = \sqrt{x - 2}$ by plotting the graph of the function $y = \sqrt{x}$ [Example 2.2.1(c)] and shifting it 2 units to the right as shown in Figure 11.

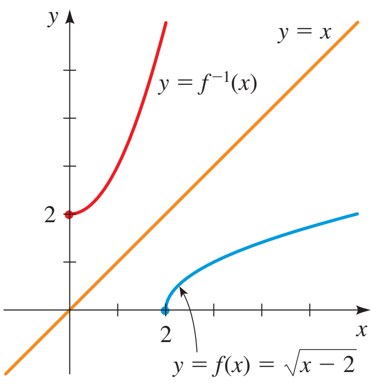


Figure 11

(b) The graph of f^{-1} is obtained from the graph of f in part (a) by reflecting it about the line $y = x$, as shown in Figure 11.

(c) Solve $y = \sqrt{x-2}$ for x, noting that $y \geq 0$.

$$\sqrt{x-2} = y$$

$$x - 2 = y^2 \qquad \text{Square each side}$$

$$x = y^2 + 2 \qquad (y \geq 0) \qquad \text{Add 2}$$

Interchange x and y, as follows:

$$y = x^2 + 2 \qquad (x \geq 0)$$

Thus $$f^{-1}(x) = x^2 + 2 \qquad (x \geq 0)$$

This expression shows that the graph of f^{-1} is the right half of the parabola $y = x^2 + 2$, and from the graph shown in Figure 11 this seems reasonable.

Now Try Exercise 71 ■

■ Applications of Inverse Functions

When working with functions that model real-world situations, we name the variables using letters that suggest the quantity being modeled. For instance we may use t for time, d for distance, V for volume, and so on. When using inverse functions, we follow this convention. For example, suppose that the variable R is a function of the variable N, say, $R = f(N)$. Then $f^{-1}(R) = N$. So the function f^{-1} defines N as a function of R.

Example 11 ■ An Inverse Function

At a local pizza restaurant the daily special is \$12 for a plain cheese pizza plus \$2 for each additional topping.

(a) Find a function f that models the price of a pizza with n toppings.

(b) Find the inverse of the function f. What does f^{-1} represent?

(c) If a pizza costs \$22, how many toppings does it have?

Solution Note that the price p of a pizza is a function of the number n of toppings.

(a) The price of a pizza with n toppings is given by the function

$$f(n) = 12 + 2n$$

(b) To find the inverse function, we first write $p = f(n)$, where we use the letter p instead of our usual y because $f(n)$ is the price of the pizza. We have

$$p = 12 + 2n$$

Next we solve for n:

$$p = 12 + 2n$$

$$p - 12 = 2n$$

$$n = \frac{p - 12}{2}$$

So $n = f^{-1}(p) = \dfrac{p-12}{2}$. The function f^{-1} gives the number n of toppings for a pizza with price p.

(c) We have $n = f^{-1}(22) = (22 - 12)/2 = 5$. So the pizza has five toppings.

Now Try Exercise 99 ■

2.8 Exercises

Concepts

1. A function f is one-to-one if different inputs produce ________ outputs. You can tell from the graph that a function is one-to-one by using the ________________ Test.

2. (a) For a function to have an inverse, it must be ________. Which one of the following functions has an inverse?

$$f(x) = x^2 \qquad g(x) = x^3$$

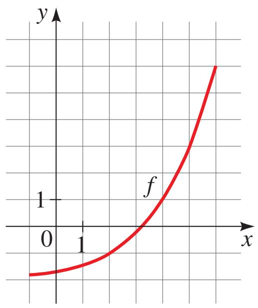
(b) What is the inverse of the function that you chose in part (a)?

3. A function f has the following verbal description: "Multiply by 3, add 5, and then take the third power of the result."

(a) Write a verbal description for f^{-1}.

(b) Find algebraic formulas that express f and f^{-1} in terms of the input x.

4. A graph of a function f is given. Does f have an inverse? If so, find $f^{-1}(1) =$ ________ and $f^{-1}(3) =$ ________.



5. If the point $(3, 4)$ is on the graph of the function f, then the point (____, ____) is on the graph of f^{-1}.

6. *True or false?*

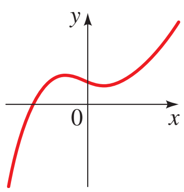
(a) If f has an inverse, then $f^{-1}(x)$ is always the same as $\dfrac{1}{f(x)}$.

(b) If f has an inverse, then $f^{-1}(f(x)) = x$.

Skills

7–12 ■ One-to-One Function? A graph of a function f is given. Determine whether f is one-to-one.

7.



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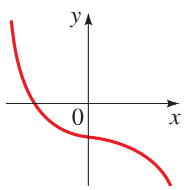

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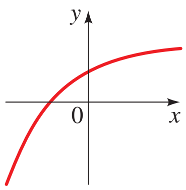

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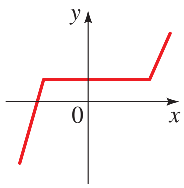

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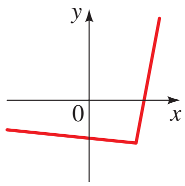
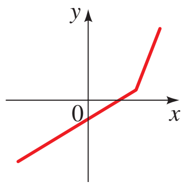


12.



13–24 ■ One-to-One Function Determine whether the function is one-to-one.

13. $f(x) = -2x + 4$ **14.** $f(x) = 3x - 2$

15. $g(x) = \sqrt{x}$ **16.** $g(x) = |x|$

17. $h(x) = x^2 - 2x$ **18.** $h(x) = x^3 + 8$

19. $f(x) = x^4 + 5$

20. $f(x) = x^4 + 5, \quad 0 \le x \le 2$

21. $r(t) = t^6 - 3, \quad 0 \le t \le 5$

22. $r(t) = t^4 - 1$

23. $f(x) = \dfrac{1}{x^2}$ **24.** $f(x) = \dfrac{1}{x}$

25–28 ■ Finding Values of an Inverse Function Assume that f is a one-to-one function.

25. (a) If $f(5) = -9$, find $f^{-1}(-9)$.

(b) If $f^{-1}(10) = 0$, find $f(0)$.

26. (a) If $f(-3) = 6$, find $f^{-1}(6)$.

(b) If $f^{-1}(12) = 8$, find $f(8)$.

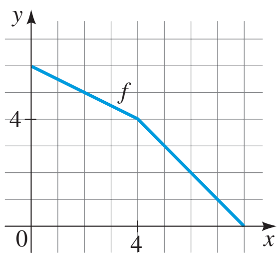
27. If $f(x) = 5 - 2x$, find $f^{-1}(3)$.

28. If $g(x) = x^2 + 4x$ with $x \ge -2$, find $g^{-1}(5)$.

29–30 ■ Finding Values of an Inverse from a Graph A graph of a function f is given. Use the graph to find the indicated values.

(a) $f^{-1}(2)$ **(b)** $f^{-1}(5)$ **(c)** $f^{-1}(6)$

29.



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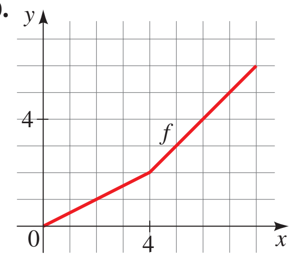


31–36 ■ Finding Values of an Inverse Using a Table A table of values for a one-to-one function is given. Find the indicated values.

31. $f^{-1}(5)$ **32.** $f^{-1}(0)$

33. $f^{-1}(f(1))$ **34.** $f(f^{-1}(6))$

35. $f^{-1}(f^{-1}(1))$ **36.** $f^{-1}(f^{-1}(0))$

x	1	2	3	4	5	6
$f(x)$	4	6	2	5	0	1

37–48 ■ Inverse Function Property Use the Inverse Function Property to show that f and g are inverses of each other.

37. $f(x) = \frac{1}{4}x + 5; \quad g(x) = 4(x - 5)$

38. $f(x) = \frac{x}{3} + 1; \quad g(x) = 3x - 3$

39. $f(x) = \frac{2}{3}x + 6; \quad g(x) = \frac{3}{2}x - 9$

40. $f(x) = 4x - 7; \quad g(x) = \frac{x + 7}{4}$

41. $f(x) = \frac{1}{x}; \quad g(x) = \frac{1}{x}$

42. $f(x) = x^5; \quad g(x) = \sqrt[5]{x}$

43. $f(x) = x^2 - 9, \quad x \geq 0; \quad g(x) = \sqrt{x + 9}, \quad x \geq -9$

44. $f(x) = x^3 + 1; \quad g(x) = (x - 1)^{1/3}$

45. $f(x) = \frac{1}{x - 1}; \quad g(x) = \frac{1}{x} + 1$

46. $f(x) = \sqrt{4 - x^2}, \quad 0 \leq x \leq 2;$
$g(x) = \sqrt{4 - x^2}, \quad 0 \leq x \leq 2$

47. $f(x) = \frac{x + 2}{x - 2}; \quad g(x) = \frac{2x + 2}{x - 1}$

48. $f(x) = \frac{x - 5}{3x + 4}; \quad g(x) = \frac{5 + 4x}{1 - 3x}$

49–68 ■ Finding Inverse Functions Find the inverse function of f. Check your answer by using the Inverse Function Property.

49. $f(x) = 3x + 15$

50. $f(x) = 8 - 3x$

51. $f(x) = \frac{3}{4}x - 12$

52. $f(x) = \frac{3 - x}{10}$

53. $f(x) = 5 - 4x^3$

54. $f(x) = 3x^3 + 8$

55. $f(x) = \frac{1}{x + 2}$

56. $f(x) = \frac{x - 2}{x + 2}$

57. $f(x) = \frac{x}{2 - x}$

58. $f(x) = \frac{4x}{x + 5}$

59. $f(x) = \frac{2x + 5}{x - 7}$

60. $f(x) = \frac{4x - 2}{3x + 1}$

61. $f(x) = \frac{2x + 3}{1 - 5x}$

62. $f(x) = \frac{3 - 4x}{8x - 1}$

63. $f(x) = \frac{x^3 + 1}{3}$

64. $f(x) = (x^5 - 6)^7$

65. $f(x) = 2 + \sqrt[3]{x}$

66. $f(x) = \sqrt[3]{6x - 5}$

67. $f(x) = x^{3/2} + 1$

68. $f(x) = (x - 2)^{3/5}$

69–74 ■ Graph of an Inverse Function A function f is given. **(a)** Sketch the graph of f. **(b)** Use the graph of f to sketch the graph of f^{-1}. **(c)** Find f^{-1}.

69. $f(x) = 3x - 6$

70. $f(x) = 16 - x^2, \quad x \geq 0$

71. $f(x) = x^3 - 1$

72. $f(x) = \sqrt{x + 1}$

73. $f(x) = 3 + \sqrt{x - 1}$

74. $f(x) = 2 + \sqrt{x + 1}$

75–80 ■ One-to-One Functions from a Graph Draw the graph of f, and use it to determine whether the function is one-to-one.

75. $f(x) = x^3 - x$

76. $f(x) = x^3 + x$

77. $f(x) = \frac{x + 12}{x - 6}$

78. $f(x) = \sqrt{x^3 - 4x + 1}$

79. $f(x) = |x| - |x - 6|$

80. $f(x) = x \cdot |x|$

81–84 ■ Finding Inverse Functions A one-to-one function is given. **(a)** Find the inverse of the function. **(b)** Graph both the function and its inverse on the same screen to verify that the graphs are reflections of each other about the line $y = x$.

81. $f(x) = 2 + x$

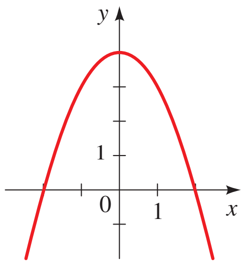
82. $f(x) = 2 - \frac{1}{2}x$

83. $g(x) = \sqrt{x + 3}$

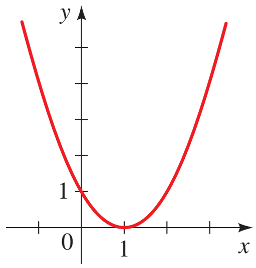
84. $g(x) = x^2 + 1, \quad x \geq 0$

85–88 ■ Restricting the Domain The given function is not one-to-one. Restrict its domain so that the resulting function *is* one-to-one and has the same range as the given function. Find the inverse of the function with the restricted domain.

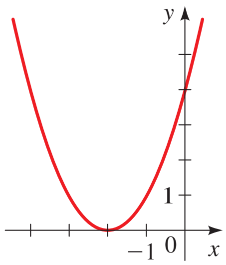
85. $f(x) = 4 - x^2$



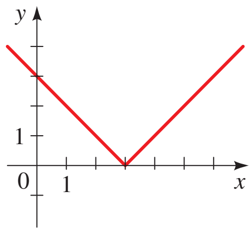
86. $g(x) = (x - 1)^2$



87. $h(x) = (x + 2)^2$

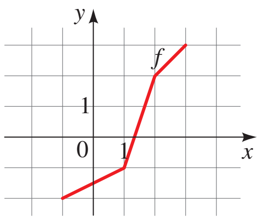


88. $k(x) = |x - 3|$



89–90 ■ Graph of an Inverse Function Use the graph of f to sketch the graph of f^{-1}.

89.



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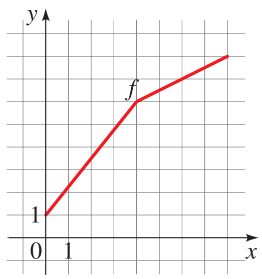


Skills Plus

91–96 ■ Inverse Functions Find the inverse function f^{-1} and state its domain. Check that the range of f is the same as the domain of the inverse function you found.

91. $f(x) = x^2 - 9;\quad x \geq 0$

92. $f(x) = x^2 - 2x + 1;\quad x \geq 1$

93. $f(x) = \dfrac{1}{x^4},\quad x > 0$

94. $f(x) = \dfrac{1}{x^2 + 1};\quad x \geq 0$

95. $f(x) = \sqrt{x},\quad 0 \leq x \leq 9$

96. $f(x) = x^2 - 6x,\quad x \geq 3$

97–98 ■ Functions That Are Their Own Inverse If a function f is its own inverse, then the graph of f is symmetric with respect to the line $y = x$. **(a)** Graph the given function. **(b)** Does the graph indicate that f and f^{-1} are the same function? **(c)** Find the function f^{-1}. Use your result to verify your answer to part (b).

97. $f(x) = \dfrac{1}{x}$

98. $f(x) = \dfrac{x + 3}{x - 1}$

Applications

99. Pizza Cost A popular pizza place charges a base price of \$16 for a large cheese pizza plus \$1.50 for each topping.

(a) Find a function f that models the price of a pizza with n toppings.

(b) Find the inverse of the function f. What does f^{-1} represent?

(c) If a pizza costs \$25, how many toppings does it have?

100. Fee for Service A private investigator requires a \$500 retainer fee plus \$80 per hour. Let x represent the number of hours the investigator spends working on a case.

(a) Find a function f that models the investigator's fee as a function of x.

(b) Find f^{-1}. What does f^{-1} represent?

(c) Find $f^{-1}(1220)$. What does your answer represent?

101. Torricelli's Law A tank holds 100 gallons of water, which drains from a leak at the bottom, causing the tank to empty in 40 minutes. According to Torricelli's Law, the volume V of water remaining in the tank after t min is given by the function

$$V = f(t) = 100\left(1 - \frac{t}{40}\right)^2$$

(a) Find f^{-1}. What does f^{-1} represent?

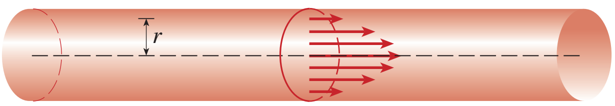
(b) Find $f^{-1}(15)$. What does your answer represent?

102. Blood Flow As blood moves through a vein or artery, its velocity v is greatest along the central axis and decreases as the distance r from the central axis increases (see the figure below). For an artery with radius 0.5 cm, v (in cm/s) is given as a function of r (in cm) by

$$v = g(r) = 18{,}500(0.25 - r^2)$$

(a) Find g^{-1}. What does g^{-1} represent?

(b) Find $g^{-1}(30)$. What does your answer represent?



103. Demand Function The amount of a commodity that is sold is called the *demand* for the commodity. The demand D for a certain commodity is a function of the price given by

$$D = f(p) = -3p + 150$$

(a) Find f^{-1}. What does f^{-1} represent?

(b) Find $f^{-1}(30)$. What does your answer represent?

104. Temperature Scales The relationship between the Fahrenheit (F) and Celsius (C) scales is given by

$$F = g(C) = \tfrac{9}{5}C + 32$$

(a) Find g^{-1}. What does g^{-1} represent?

(b) Find $g^{-1}(86)$. What does your answer represent?

105. Exchange Rates The relative value of currencies fluctuates every day. When this problem was written, one Canadian dollar was worth 0.79 US dollars.

(a) Find a function f that gives the US dollar value $f(x)$ of x Canadian dollars.

(b) Find f^{-1}. What does f^{-1} represent?

(c) How much Canadian money would \$12,250 in US currency be worth?

106. Income Tax In a certain EU country the tax on incomes less than or equal to €20,000 is 10%. For incomes that are more than €20,000 the tax is €2000 plus 20% of the amount over €20,000. (Currency units are euros.)

(a) Find a function f that gives the income tax on an income x. Express f as a piecewise-defined function.

(b) Find f^{-1}. What does f^{-1} represent?

(c) How much income would require paying a tax of €10,000?

107. Multiple Discounts A car dealership advertises a 15% discount on all its new cars. In addition, the manufacturer offers a \$1000 rebate on the purchase of a new car. Let x represent the sticker price of the car.

(a) Suppose that only the 15% discount applies. Find a function f that models the purchase price of the car as a function of the sticker price x.

(b) Suppose that only the \$1000 rebate applies. Find a function g that models the purchase price of the car as a function of the sticker price x.

(c) Find a formula for $H = f \circ g$.

(d) Find H^{-1}. What does H^{-1} represent?

(e) Find $H^{-1}(13{,}000)$. What does your answer represent?

Discuss ■ Discover ■ Prove ■ Write

108. Discuss: Determining When a Linear Function Has an Inverse For the linear function $f(x) = mx + b$ to be one-to-one, what must be true about its slope? If it is one-to-one, find its inverse. Is the inverse linear? If so, what is its slope?

109. Discuss: Finding an Inverse "in Your Head" In the margin notes in this section we pointed out that the inverse of a function can be found by simply reversing the operations that make up the function. For instance, in Example 7 we saw that the inverse of

$$f(x) = 3x - 2 \qquad \text{is} \qquad f^{-1}(x) = \frac{x+2}{3}$$

because the "reverse" of "Multiply by 3 and subtract 2" is "Add 2 and divide by 3." Use the same procedure to find the inverse of the following functions.

(a) $f(x) = \dfrac{2x+1}{5}$ **(b)** $f(x) = 3 - \dfrac{1}{x}$

(c) $f(x) = \sqrt{x^3 + 2}$ **(d)** $f(x) = (2x - 5)^3$

Now consider another function:

$$f(x) = x^3 + 2x + 6$$

Is it possible to use the same sort of simple reversal of operations to find the inverse of this function? If so, do it. If not, explain what is different about this function that makes this task difficult.

110. Prove: The Identity Function The function $I(x) = x$ is called the *identity function*. Show that for any function f we have $f \circ I = f$, $I \circ f = f$, and $f \circ f^{-1} = f^{-1} \circ f = I$. (This means that the identity function I behaves for functions and composition just the way the number 1 behaves for real numbers and multiplication.)

111. Discuss: Solving an Equation for an Unknown Function In Exercise 2.7.91 you were asked to solve equations in which the unknowns are functions. Now that we know about inverses and the identity function (see Exercise 110), we can use algebra to solve such equations. For instance, to solve $f \circ g = h$ for the unknown function f, we perform the following steps:

$f \circ g = h$	Problem: Solve for f
$f \circ g \circ g^{-1} = h \circ g^{-1}$	Compose with g^{-1} on the right
$f \circ I = h \circ g^{-1}$	Because $g \circ g^{-1} = I$
$f = h \circ g^{-1}$	Because $f \circ I = f$

So the solution is $f = h \circ g^{-1}$. Use this technique to solve the equation $f \circ g = h$ for the indicated unknown function.

(a) Solve for f, where

$$g(x) = 2x + 1$$
$$h(x) = 4x^2 + 4x + 7$$

(b) Solve for g, where

$$f(x) = 3x + 5$$
$$h(x) = 3x^2 + 3x + 2$$

112. Prove: The Inverse of a Composition of Functions Show that the inverse function of $f \circ g$ is the function $g^{-1} \circ f^{-1}$.

PS *Try to recognize something familiar.* Show that these functions satisfy the cancellation properties of inverse functions.

Chapter 2 Review

Properties and Formulas

Function Notation | Section 2.1

If a function is given by the formula $y = f(x)$, then x is the independent variable and denotes the **input**; y is the dependent variable and denotes the **output**; the **domain** is the set of all possible inputs x; the **range** is the set of all possible outputs y.

Net Change | Section 2.1

The **net change** in the value of the function f between $x = a$ and $x = b$ is

$$\text{net change} = f(b) - f(a)$$

The Graph of a Function | Section 2.2

The graph of a function f is the graph of the equation $y = f(x)$ that defines f.

The Vertical Line Test | Section 2.2

A curve in the coordinate plane is the graph of a function if and only if no vertical line intersects the graph more than once.

Relations and Functions | Section 2.2

A **relation** is any collection of ordered pairs (x, y). The x-values are inputs and the corresponding y-values are outputs. A relation is a function if every input corresponds to exactly one output.

Increasing and Decreasing Functions | Section 2.3

A function f is **increasing** on an interval if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in the interval.

A function f is **decreasing** on an interval if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$ in the interval.

Local Maximum and Minimum Values | Section 2.3

The function value $f(a)$ is a **local maximum value** of the function f if $f(a) \geq f(x)$ for all x near a. In this case we also say that f has a **local maximum** at $x = a$.

The function value $f(b)$ is a **local minimum value** of the function f if $f(b) \leq f(x)$ for all x near b. In this case we also say that f has a **local minimum** at $x = b$.

Average Rate of Change | Section 2.4

The **average rate of change** of the function f between $x = a$ and $x = b$ is the slope of the **secant** line between $(a, f(a))$ and $(b, f(b))$:

$$\text{average rate of change} = \frac{f(b) - f(a)}{b - a}$$

Linear Functions | Section 2.5

A **linear function** is a function of the form $f(x) = ax + b$. The graph of f is a line with slope a and y-intercept b. The average rate of change of f has the constant value a between any two points.

$$a = \text{slope of graph of } f = \text{rate of change of } f$$

Vertical and Horizontal Shifts of Graphs | Section 2.6

Let c be a positive constant.

To graph $y = f(x) + c$, shift the graph of $y = f(x)$ **upward** by c units.

To graph $y = f(x) - c$, shift the graph of $y = f(x)$ **downward** by c units.

To graph $y = f(x - c)$, shift the graph of $y = f(x)$ **to the right** by c units.

To graph $y = f(x + c)$, shift the graph of $y = f(x)$ **to the left** by c units.

Reflecting Graphs | Section 2.6

To graph $y = -f(x)$, **reflect** the graph of $y = f(x)$ about the ***x*-axis**.

To graph $y = f(-x)$, **reflect** the graph of $y = f(x)$ about the ***y*-axis**.

Vertical and Horizontal Stretching and Shrinking of Graphs | Section 2.6

If $c > 1$, then to graph $y = cf(x)$, **stretch** the graph of $y = f(x)$ **vertically** by a factor of c.

If $0 < c < 1$, then to graph $y = cf(x)$, **shrink** the graph of $y = f(x)$ **vertically** by a factor of c.

If $c > 1$, then to graph $y = f(cx)$, **shrink** the graph of $y = f(x)$ **horizontally** by a factor of $1/c$.

If $0 < c < 1$, then to graph $y = f(cx)$, **stretch** the graph of $y = f(x)$ **horizontally** by a factor of $1/c$.

Even and Odd Functions | Section 2.6

A function f is

even if $f(-x) = f(x)$

odd if $f(-x) = -f(x)$

for every x in the domain of f.

Composition of Functions | Section 2.7

Given two functions f and g, the **composition** of f and g is the function $f \circ g$ defined by

$$(f \circ g)(x) = f(g(x))$$

The **domain** of $f \circ g$ is the set of all x for which both $g(x)$ and $f(g(x))$ are defined.

One-to-One Functions | Section 2.8

A function f is **one-to-one** if $f(x_1) \neq f(x_2)$ whenever x_1 and x_2 are *different* elements of the domain of f.

Horizontal Line Test | Section 2.8

A function is one-to-one if and only if no horizontal line intersects its graph more than once.

Inverse of a Function | Section 2.8

Let f be a one-to-one function with domain A and range B.

The **inverse** of f is the function f^{-1} defined by

$$f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y$$

The inverse function f^{-1} has domain B and range A.

The functions f and f^{-1} satisfy the following **cancellation equations**:

$$f^{-1}(f(x)) = x \quad \text{for every } x \text{ in } A$$
$$f(f^{-1}(x)) = x \quad \text{for every } x \text{ in } B$$

Concept Check

1. Define each concept.
   (a) Function
   (b) Domain and range of a function
   (c) Graph of a function
   (d) Independent and dependent variables

2. Describe the four ways of representing a function.

3. Sketch graphs of the following functions by hand.
   (a) $f(x) = x^2$ **(b)** $g(x) = x^3$
   (c) $h(x) = |x|$ **(d)** $k(x) = \sqrt{x}$

4. What is a piecewise-defined function? Give an example.

5. What is a relation? How do you determine whether a relation is a function? Give an example of a relation that is not a function.

6. **(a)** What is the Vertical Line Test, and what is it used for?
   (b) What is the Horizontal Line Test, and what is it used for?

7. Define each concept, and give an example of each.
   (a) Increasing function
   (b) Decreasing function
   (c) Constant function

8. Suppose we know that the point $(3, 5)$ is a point on the graph of a function f. Explain how to find $f(3)$ and $f^{-1}(5)$.

9. What does it mean to say that $f(4)$ is a local maximum value of f?

10. Explain how to find the average rate of change of a function f between $x = a$ and $x = b$.

11. **(a)** What is the slope of a linear function? How do you find it? What is the rate of change of a linear function?
   (b) Is the rate of change of a linear function constant? Explain.
   (c) Give an example of a linear function, and sketch its graph.

12. Suppose the graph of a function f is given. Write an equation for each of the graphs that are obtained from the graph of f as follows.
   (a) Shift upward 3 units
   (b) Shift downward 3 units
   (c) Shift 3 units to the right
   (d) Shift 3 units to the left
   (e) Reflect about the x-axis
   (f) Reflect about the y-axis

(g) Stretch vertically by a factor of 3

(h) Shrink vertically by a factor of $\frac{1}{3}$

(i) Shrink horizontally by a factor of $\frac{1}{3}$

(j) Stretch horizontally by a factor of 3

13. (a) What is an even function? How can you tell that a function is even by looking at its graph? Give an example of an even function.

(b) What is an odd function? How can you tell that a function is odd by looking at its graph? Give an example of an odd function.

14. Suppose that f has domain A and g has domain B. What are the domains of the following functions?

(a) Domain of $f + g$

(b) Domain of fg

(c) Domain of f/g

15. (a) How is the composition function $f \circ g$ defined? What is its domain?

(b) If $g(a) = b$ and $f(b) = c$, then explain how to find $(f \circ g)(a)$.

16. (a) What is a one-to-one function?

(b) How can you tell from the graph of a function whether it is one-to-one?

(c) Suppose that f is a one-to-one function with domain A and range B. How is the inverse function f^{-1} defined? What are the domain and range of f^{-1}?

(d) If you are given a formula for f, how do you find a formula for f^{-1}? Find the inverse of the function $f(x) = 2x$.

(e) If you are given a graph of f, how do you find a graph of the inverse function f^{-1}?

Answers to the Concept Check can be found at the book companion website stewartmath.com.

Exercises

1–2 ■ Function Notation A verbal description of a function f is given. Find a formula that expresses f in function notation.

1. "Square, then subtract 5."

2. "Divide by 2, then add 9."

3–4 ■ Function in Words A formula for a function f is given. Give a verbal description of the function.

3. $f(x) = 3(x + 10)$ **4.** $f(x) = \sqrt{6x - 10}$

5–6 ■ Table of Values Complete the table of values for the given function.

5. $g(x) = x^2 - 4x$

x	$g(x)$
-1	
0	
1	
2	
3	

6. $h(x) = 3x^2 + 2x - 5$

x	$h(x)$
-2	
-1	
0	
1	
2	

7. Printing Cost A publisher estimates that the cost $C(x)$ of printing a run of x copies of a certain mathematics textbook is given by the function $C(x) = 5000 + 30x - 0.001x^2$.

(a) Find $C(1000)$ and $C(10{,}000)$.

(b) What do your answers in part (a) represent?

(c) Find $C(0)$. What does this number represent?

(d) Find the net change and the average rate of change of the cost C between $x = 1000$ and $x = 10{,}000$.

8. Earnings An electronics store pays each of their sales staff a weekly base salary plus a commission based on the retail price of the goods they have sold. If a salesperson sells x dollars of goods in a week, their earnings for that week are given by the function $E(x) = 400 + 0.03x$.

(a) Find $E(2000)$ and $E(15{,}000)$.

(b) What do your answers in part (a) represent?

(c) Find $E(0)$. What does this number represent?

(d) Find the net change and the average rate of change of the salesperson's earnings E between $x = 2000$ and $x = 15{,}000$.

(e) From the formula for E, determine what percentage the salesperson earns on the goods sold.

9–10 ■ Evaluating Functions Evaluate the function at the indicated values.

9. $f(x) = x^2 - 4x + 6$; $f(0), f(2), f(-2), f(a), f(-a), f(x + 1), f(2x)$

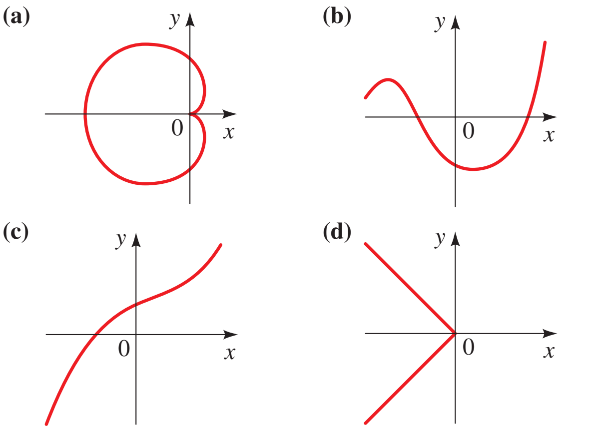
10. $f(x) = 4 - \sqrt{3x - 6}$; $f(5), f(9), f(a + 2), f(-x), f(x^2)$

11–12 ■ Difference Quotient Find $f(a)$, $f(a + h)$, and the difference quotient $\dfrac{f(a + h) - f(a)}{h}$.

11. $f(x) = x^2 + 8$

12. $f(x) = \dfrac{1}{x - 2}$

13. Functions Given by a Graph Which of the following figures are graphs of functions? Which of the functions are one-to-one?



14. Getting Information from a Graph A graph of a function f is given.

(a) Find $f(-2)$ and $f(2)$.

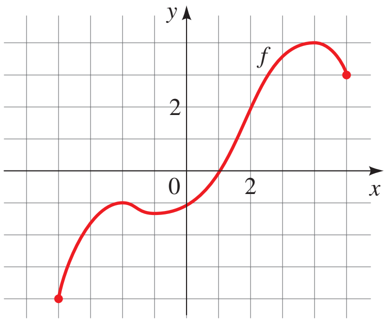
(b) Find the net change and the average rate of change of f between $x = -2$ and $x = 2$.

(c) Find the domain and range of f.

(d) On what intervals is f increasing? On what intervals is f decreasing?

(e) What are the local maximum values of f?

(f) Is f one-to-one?



15–16 ■ Domain and Range Find the domain and range of the function.

15. $f(x) = \sqrt{x - 5}$ **16.** $f(x) = \dfrac{1}{x - 2}$

17–24 ■ Domain Find the domain of the function.

17. $f(x) = 7x + 15$ **18.** $f(x) = \dfrac{2x + 1}{2x - 1}$

19. $f(x) = \sqrt{x^2 + 4}$ **20.** $f(x) = 3x - \dfrac{2}{\sqrt{x + 1}}$

21. $f(x) = \dfrac{1}{x} + \dfrac{1}{x + 1} + \dfrac{1}{x + 2}$

22. $g(x) = \dfrac{2x^2 + 5x + 3}{2x^2 - 5x - 3}$

23. $h(x) = \sqrt{4 - x} + \sqrt{x^2 - 1}$

24. $f(x) = \dfrac{\sqrt[3]{2x + 1}}{\sqrt[3]{2x} + 2}$

25–42 ■ Graphing Functions Sketch a graph of the function. Use transformations of functions whenever possible.

25. $f(x) = 2 + \frac{3}{4}x$

26. $f(x) = 3(1 - 2x), -2 \le x \le 2$

27. $f(x) = -3x^2 + 4$ **28.** $f(x) = \frac{1}{2}x^2 - 8$

29. $f(x) = \sqrt{x - 5}$ **30.** $f(x) = \sqrt{3(x + 1)}$

31. $f(x) = \frac{1}{3}(x + 3)^2 + 2$ **32.** $f(x) = -2\sqrt{x + 4} - 3$

33. $f(x) = 4\sqrt{-x} - 2$ **34.** $f(x) = -\frac{1}{2}(x - 1)^2 + 2$

35. $f(x) = \frac{1}{2}x^3$ **36.** $f(x) = \sqrt[3]{-x}$

37. $f(x) = 5 - |x|$ **38.** $f(x) = 3 - |x + 2|$

39. $f(x) = -\dfrac{1}{x^2}$ **40.** $f(x) = \dfrac{1}{(x - 1)^3}$

41. $f(x) = \begin{cases} 1 - x & \text{if } x < 0 \\ 1 & \text{if } x \ge 0 \end{cases}$

42. $f(x) = \begin{cases} -x & \text{if } x < 0 \\ x^2 & \text{if } 0 \le x < 2 \\ 1 & \text{if } x \ge 2 \end{cases}$

43–46 ■ Equations That Represent Functions Determine whether the equation defines y as a function of x.

43. $x + y^2 = 14$ **44.** $3x - \sqrt{y} = 8$

45. $x^3 - y^3 = 27$ **46.** $2x = y^4 - 16$

47. Relations That Define Functions A relation is given by a table. List the ordered pairs in the relation, graph the relation, and determine whether the relation defines y as a function of x. State the domain and range of the relation.

(a)

x	y
-3	-3
-2	0
0	-1
2	3
3	-3

(b)

x	y
-3	3
2	1
0	-2
2	5
3	3

48. Relations That Define Functions The ordered pairs (x, y) in a relation are described. Determine whether the relation defines y as a function of x. Give reasons for your answer.

(a) $\{(-10, 20)(0, 70), (10, 50), (5, 30), (-5, 0)\}$

(b)

x Height (in)	64	68	72	71	68
y Student ID Number	23745	12933	10834	12772	91836

(c) The set of ordered pairs of real numbers (x, y) for which xy is an integer.

(d) The set of ordered pairs (x, y) that satisfy the equation $x = y^4$.

49–52 ■ Domain and Range from a Graph Draw a graph of the function f and use the graph to find the following.

(a) The domain and range of f

(b) The value(s) of x for which $f(x) = 0$

(c) The intervals on which $f(x) > 1$

49. $f(x) = \sqrt{9 - x^2}$

50. $f(x) = \sqrt{x^2 - 3}$

51. $f(x) = \sqrt{x^3 - 4x + 1}$

52. $f(x) = x^4 - x^3 + x^2 + 3x - 6$

53–58 ■ Getting Information From a Graph Draw a graph of the function f and use the graph to find the following.

(a) The local maximum and minimum values of f and the values of x at which they occur

(b) The intervals on which f is increasing and on which f is decreasing

53. $f(x) = 2x^2 - 4x + 5$

54. $f(x) = 1 - x - x^2$

55. $f(x) = 3.3 + 1.6x - 2.5x^3$

56. $f(x) = x^3 - 4x^2$

57. $f(x) = x^{2/3}(6 - x)^{1/3}$

58. $f(x) = |x^4 - 16|$

59–64 ■ Net Change and Average Rate of Change A function is given (either numerically, graphically, or algebraically). Find the net change and the average rate of change of the function between the indicated values.

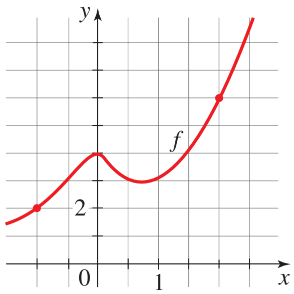
59. Between $x = 4$ and $x = 8$

x	$f(x)$
2	14
4	12
6	12
8	8
10	6

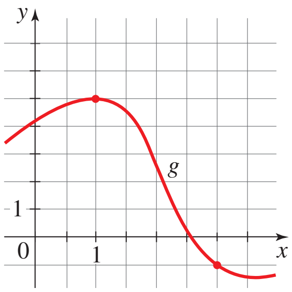
60. Between $x = 10$ and $x = 30$

x	$g(x)$
0	25
10	-5
20	-2
30	30
40	0

61. Between $x = -1$ and $x = 2$



62. Between $x = 1$ and $x = 3$



63. $f(x) = x^2 - 2x$; between $x = 1$ and $x = 4$

64. $g(x) = (x + 1)^2$; between $x = a$ and $x = a + h$

65–66 ■ Linear Functions? Determine whether the function is linear.

65. $f(x) = (2 + 3x)^2$

66. $f(x) = \dfrac{2x - 10}{\sqrt{5}}$

67–68 ■ Linear Functions A linear function is given. **(a)** Sketch a graph of the function. **(b)** What is the slope of the graph? **(c)** What is the rate of change of the function?

67. $f(x) = 3x + 2$

68. $g(x) = 3 - \frac{1}{2}x$

69–74 ■ Linear Functions A linear function is described either verbally, numerically, or graphically. Express f in the form $f(x) = ax + b$.

69. The function has rate of change -2 and initial value 3.

70. The graph of the function has slope $\frac{1}{2}$ and y-intercept -1.

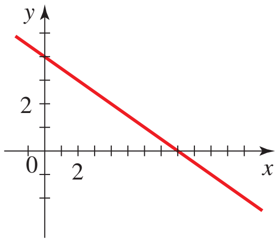
71.

x	$f(x)$
0	3
1	5
2	7
3	9
4	11

72.

x	$f(x)$
0	6
2	5.5
4	5
6	4.5
8	4

73.



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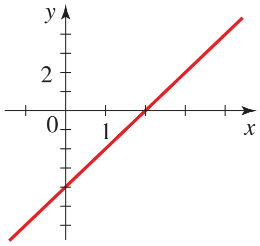


75–78 ■ Average Rate of Change A function f is given. **(a)** Find the average rate of change of f between $x = 0$ and $x = 2$, and the average rate of change of f between $x = 15$ and $x = 50$. **(b)** Were the two average rates of change that you found in part (a) the same? **(c)** Is the function linear? If so, what is its rate of change?

75. $f(x) = \frac{1}{2}x - 6$

76. $f(x) = 8 - 3x$

77. $f(x) = (x - 1)^2$

78. $f(x) = \dfrac{1}{x + 3}$

79. Transformations Suppose the graph of f is given. **(i)** Describe in words how the graph of each of the following functions can be obtained from the graph of f. **(ii)** Find a formula for the function you described in part (i) for the case $f(x) = x^3$.

(a) $y = f(x) + 8$

(b) $y = f(x + 8)$

(c) $y = 1 + 2f(x)$

(d) $y = f(x - 2) - 2$

(e) $y = f(-x)$

(f) $y = -f(-x)$

(g) $y = -f(x)$

(h) $y = f^{-1}(x)$

80. Transformations The graph of f is given. Draw the graph of each of the following functions.

(a) $y = f(x - 2)$

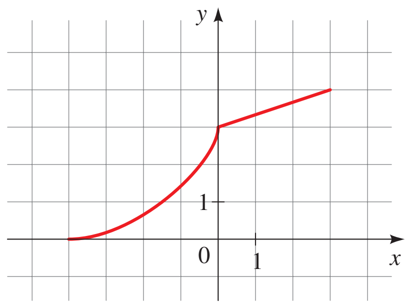
(b) $y = -f(x)$

(c) $y = 3 - f(x)$

(d) $y = \frac{1}{2}f(x) - 1$

(e) $y = f^{-1}(x)$

(f) $y = f(-x)$



81. Even and Odd Functions Determine whether f is even, odd, or neither.

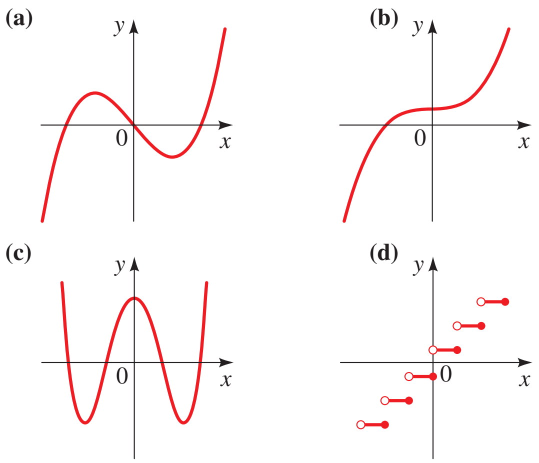
(a) $f(x) = 2x^5 - 3x^2 + 2$

(b) $f(x) = x^3 - x^7$

(c) $f(x) = \dfrac{1 - x^2}{1 + x^2}$

(d) $f(x) = \dfrac{1}{x + 2}$

82. Even and Odd Functions Determine whether the function in the figure is even, odd, or neither.



83. Getting Information From a Graph Match each description with the appropriate graph(s). Explain your choices. (A graph may satisfy more than one description.)

(a) Average rate of change is the same between any two points

(b) Increasing on $(-\infty, 2)$ and decreasing on $(2, \infty)$

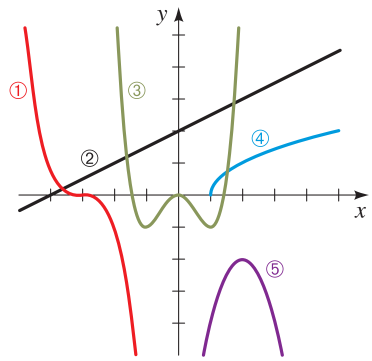
(c) Domain $[1, \infty)$

(d) The function is even

(e) Decreasing on $(-\infty, \infty)$

(f) Has two local minima

(g) Has an inverse function



84. Maximum Height of Projectile A stone is thrown upward from the top of a building. Its height (in feet) above the ground after t seconds is given by

$$h(t) = -16t^2 + 48t + 32$$

What maximum height does it reach?

85–86 ■ Volume of Weather Balloon A pump is used to fill an approximately spherical weather balloon with helium at the rate of 0.8 ft^3/s. If the balloon is empty at time $t = 0$, then the function $V(t) = 0.8t$ models the volume of the balloon at time $t \geq 0$.

85. Find a transformation of V that models the given situation.

(a) The pump fills the balloon with helium at double the rate.

(b) The pump malfunctions for the first 4 s and then starts pumping.

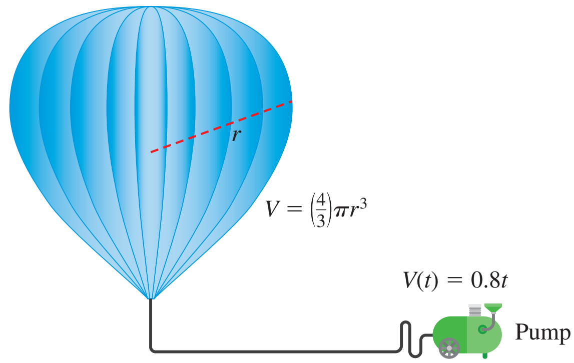
(c) The balloon already contains 5.0 ft^3 of helium at $t = 0$.

86. As the balloon is being filled with helium, its radius increases.

(a) Find a function $r(V)$ that models the radius r of a sphere as a function of its volume V. [*Hint:* The volume of a sphere is $V = \frac{4}{3}\pi r^3$; solve for r.]

(b) Find $(r \circ V)(t)$. What does this composite function model?

(c) Find the radius of the balloon at time $t = 50$ s.



87. Weight of an Astronaut If an astronaut weighs 144 lb on the earth, then the astronaut's weight h miles above the surface of earth is given by the function

$$w(h) = 144\left(\frac{3960}{3960 + h}\right)^2$$

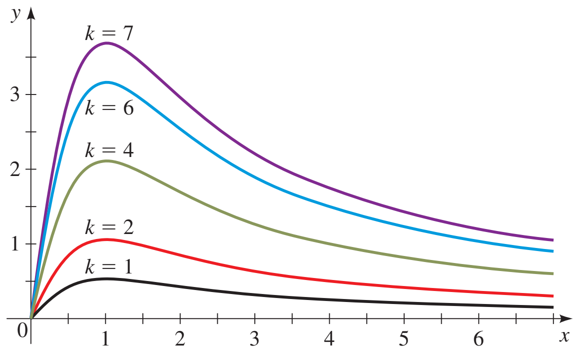
(a) Find w^{-1}. What does w^{-1} represent?

(b) Find $w^{-1}(64)$. Interpret your answer.

88. Crop Yield A model used for the yield Y (tons per acre) of an agricultural crop as a function of the nitrogen level x in the soil (measured in parts per million, ppm) is

$$Y(x) = \frac{kx}{1 + x^2} \qquad (x \geq 0)$$

where k is a constant that depends on the type of crop. A graph of this family of functions is shown for $k = 1, 2, 4, 6$, and 7. Does the value of k affect the maximum yield? Does it affect the nitrogen level at which the maximum yield occurs? Find the maximum crop yield for a crop with $k = 5$.



89–90 ■ Graphical Addition Functions, f and g are given. Draw graphs of f, g, and $f + g$ on the same coordinate axes to illustrate the concept of graphical addition.

89. $f(x) = x + 2, \quad g(x) = x^2$

90. $f(x) = x^2 + 1, \quad g(x) = 3 - x^2$

91. Combining Functions If $f(x) = x^2 - 3x + 2$ and $g(x) = 4 - 3x$, find the following functions.

(a) $f + g$ **(b)** $f - g$ **(c)** fg

(d) f/g **(e)** $f \circ g$ **(f)** $g \circ f$

92. If $f(x) = 1 + x^2$ and $g(x) = \sqrt{x - 1}$, find the following.

(a) $f \circ g$ **(b)** $g \circ f$ **(c)** $(f \circ g)(2)$

(d) $(f \circ f)(2)$ **(e)** $f \circ g \circ f$ **(f)** $g \circ f \circ g$

93–94 ■ Composition of Functions Find the functions $f \circ g$, $g \circ f$, $f \circ f$, and $g \circ g$ and their domains.

93. $f(x) = \sqrt{x} + 1, \quad g(x) = x - x^2$

94. $f(x) = \sqrt{x}, \quad g(x) = \dfrac{2}{x - 4}$

95. Finding a Composition Find $f \circ g \circ h$, where $f(x) = \sqrt{1 - x}$, $g(x) = 1 - x^2$, and $h(x) = 1 + \sqrt{x}$.

96. Finding a Composition If $T(x) = \dfrac{1}{\sqrt{1 + \sqrt{x}}}$, find functions f, g, and h such that $f \circ g \circ h = T$.

97–102 ■ One-to-One Functions Determine whether the function is one-to-one.

97. $f(x) = 3 + x^3$ **98.** $g(x) = 2 - 2x + x^2$

99. $h(x) = \dfrac{1}{x^4}$ **100.** $r(x) = 2 + \sqrt{x + 3}$

101. $p(x) = 3.3 + 1.6x - 2.5x^3$

102. $q(x) = 3.3 + 1.6x + 2.5x^3$

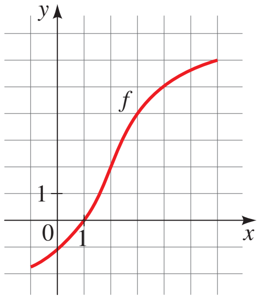
103–106 ■ Finding Inverse Functions Find the inverse of the function.

103. $f(x) = 3x - 2$ **104.** $f(x) = \dfrac{2x + 1}{3}$

105. $f(x) = (x + 1)^3$ **106.** $f(x) = 1 + \sqrt[5]{x - 2}$

107–108 ■ Inverse Functions From a Graph A graph of a function f is given. Does f have an inverse? If so, find $f^{-1}(0)$ and $f^{-1}(4)$.

107.



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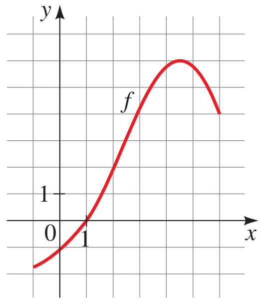


109. Graphing Inverse Functions

(a) Sketch a graph of the function

$$f(x) = x^2 - 4 \qquad (x \geq 0)$$

(b) Use part (a) to sketch the graph of f^{-1}.

(c) Find an equation for f^{-1}.

110. Graphing Inverse Functions

(a) Show that the function $f(x) = 1 + \sqrt[4]{x}$ is one-to-one.

(b) Sketch the graph of f.

(c) Use part (b) to sketch the graph of f^{-1}.

(d) Find an equation for f^{-1}.

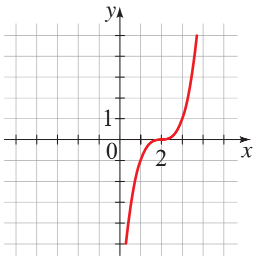
Matching

111. Equations and Their Graphs Match each equation with its graph in I–VIII, and state whether the equation defines y as a function of x. (Don't use a graphing device.)

(a) $y = |x| - 2$ **(b)** $y = (x + 1)^2 - 2$ **(c)** $(x - 2)^2 + (y + 1)^2 = 4$ **(d)** $y = (x - 2)^3$

(e) $x = y^2 - 3$ **(f)** $y = \dfrac{1}{x}$ **(g)** $y = -3(x + 3)^2 + 3$ **(h)** $y = \sqrt{2 - x}$

I



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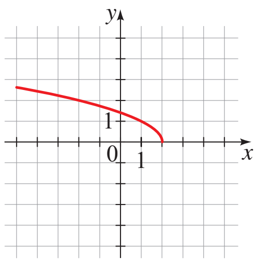

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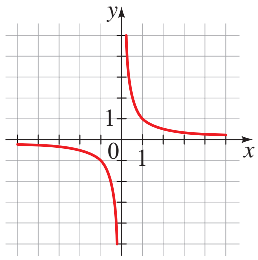

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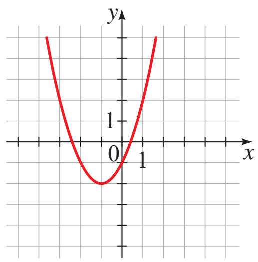

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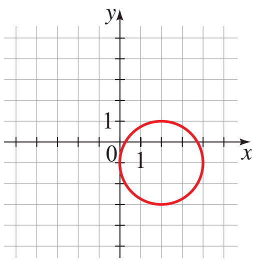

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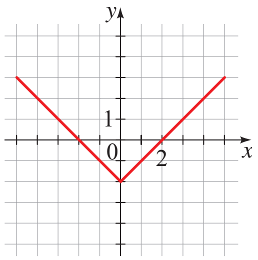
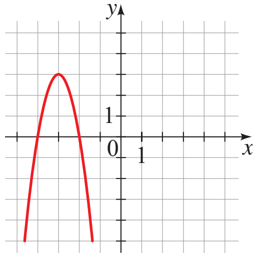
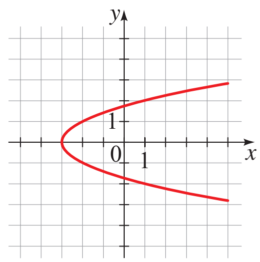


VII



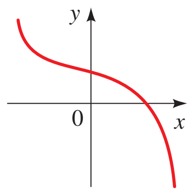
VIII



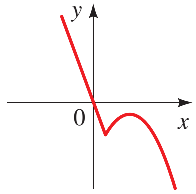
Chapter 2 | Test

1. Which of the following are graphs of functions? If the graph is that of a function, is it one-to-one?

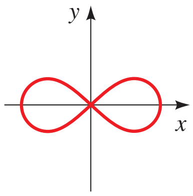
(a)



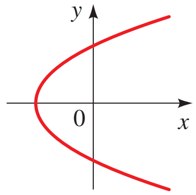
(b)



(c)



(d)



2. Let $f(x) = \dfrac{\sqrt{x}}{x+1}$.

   (a) Evaluate $f(0)$, $f(2)$, and $f(a+2)$.

   (b) Find the domain of f.

   (c) What is the average rate of change of f between $x = 2$ and $x = 10$?

3. A function f has the following verbal description: "Subtract 2, then cube."

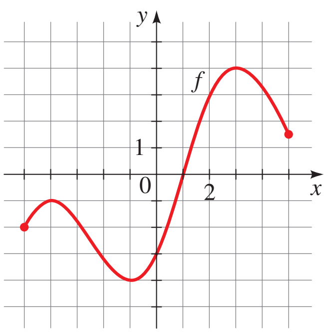
   (a) Find a formula that expresses f algebraically.

   (b) Make a table of values of f, for the inputs -1, 0, 1, 2, 3, and 4.

   (c) Sketch a graph of f, using the table of values from part (b) to help.

   (d) How do you know that f has an inverse? Give a verbal description for f^{-1}.

   (e) Find a formula that expresses f^{-1} algebraically.



4. A graph of a function f is given in the margin.

   (a) Find $f(-3)$ and $f(2)$.

   (b) Find the net change and the average rate of change of f between $x = -3$ and $x = 2$.

   (c) Find the domain and range of f.

   (d) On what intervals is f increasing? On what intervals is f decreasing?

   (e) What are the local maximum and local minimum values of f?

   (f) Is f one-to-one? Give reasons for your answer.

5. A fund-raising group sells chocolate bars to finance a swimming pool for their school. The group finds that when they set the price at x dollars per bar (where $0 < x \le 5$), the total sales revenue (in dollars) is given by the function $R(x) = -500x^2 + 3000x$.

   (a) Evaluate $R(2)$ and $R(4)$. What do these values represent?

   (b) Use a graphing device to draw a graph of R. What does the graph tell you about what happens to revenue as the price increases from 0 to 5 dollars?

   (c) What is the maximum revenue, and at what price is it achieved?

6. Determine the net change and the average rate of change for the function $f(t) = t^2 - 2t$ between $t = 2$ and $t = 2 + h$.

7. Let $f(x) = (x+5)^2$ and $g(x) = 1 - 5x$.

   (a) Only one of the two functions f and g is linear. Which one is linear, and why is the other one not linear?

(b) Sketch a graph of each function.

(c) What is the rate of change of the linear function?

8. (a) Sketch the graph of the function $f(x) = x^2$.

(b) Use part (a) to graph the function $g(x) = f(x) = (x + 4)^2 - 1$.

9–10 ■ Suppose the graph of f is given.

(a) Describe in words how the graph of the given function can be obtained from the graph of f.

(b) Find a formula for the function you described in part (a) for the case $f(x) = \sqrt{x}$.

9. $y = f(x - 3) + 2$ **10.** $y = f(-x)$

11. Let $f(x) = \begin{cases} 1 - x & \text{if } x \le 1 \\ 2x + 1 & \text{if } x > 1 \end{cases}$

(a) Evaluate $f(-2)$ and $f(1)$.

(b) Sketch the graph of f.

12. If $f(x) = x^2 + x + 1$ and $g(x) = x - 3$, find the following.

(a) $f + g$ (b) $f - g$ (c) $f \circ g$ (d) $g \circ f$

(e) $f(g(2))$ (f) $g(f(2))$ (g) $g \circ g \circ g$

13. Determine whether the function is one-to-one.

(a) $f(x) = x^3 + 1$ (b) $g(x) = |x + 1|$

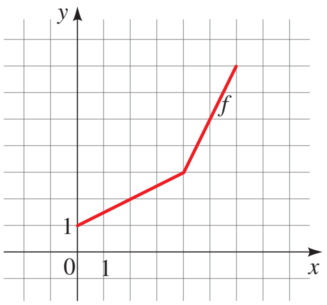
14. Use the Inverse Function Property to show that $f(x) = \dfrac{1}{x - 2}$ is the inverse of $g(x) = \dfrac{1}{x} + 2$.

15. Find the inverse function of $f(x) = \dfrac{x - 3}{2x + 5}$.

16. (a) If $f(x) = \sqrt{3 - x}$, find the inverse function f^{-1}.

(b) Sketch the graphs of f and f^{-1} on the same coordinate axes.

17–22 ■ A graph of a function f is given below.



17. Find the domain and range of f.

18. Find $f(0)$ and $f(4)$.

19. Graph $f(x - 2)$ and $f(x) + 2$ on the same set of coordinate axes as f.

20. Find the net change and the average rate of change of f between $x = 2$ and $x = 6$.

21. Find $f^{-1}(1)$ and $f^{-1}(3)$.

22. Sketch the graph of f^{-1}.

23. Let $f(x) = 3x^4 - 14x^2 + 5x - 3$.

(a) Draw the graph of f in an appropriate viewing rectangle.

(b) Is f one-to-one?

(c) Find the local maximum and minimum values of f and the values of x at which they occur. State each answer correct to two decimal places.

(d) Use the graph to determine the range of f.

(e) Find the intervals on which f is increasing and on which f is decreasing.

Many objects or processes that are studied in the physical and social sciences involve understanding how one quantity varies with respect to another. Finding a function that describes the dependence of one quantity on another is called *modeling*. For example, a biologist who observes that the number of bacteria in a certain culture increases with time tries to model this phenomenon by finding the precise function (or rule) that relates the bacteria population to the elapsed time.

In this *Focus on Modeling* we learn how to find models that can be constructed using geometric or algebraic properties of the object or process under study. Once the model is found, we use it to analyze and predict properties of that object or process.

Modeling with Functions

We begin by giving some general guidelines for making a function model.

Guidelines for Modeling with Functions

1. **Express the Model in Words.** Identify the quantity you want to model, and express it, in words, as a function of the other quantities in the problem.
2. **Choose the Variable.** Identify all the variables that are used to express the function in Step 1. Assign a symbol, such as x, to one variable, and express the other variables in terms of this symbol.
3. **Set up the Model.** Express the function in the language of algebra by writing it as a function of the single variable chosen in Step 2.
4. **Use the Model.** Use the function to answer the questions posed in the problem. (To find a maximum or a minimum, use the methods described in Section 2.3.)

Example 1 ■ Fencing a Garden

A gardener has 140 feet of fencing to fence in a rectangular vegetable garden.

(a) Find a function that models the area of the garden that can be fenced.

(b) For what range of widths is the area greater than 825 ft^2?

(c) Can the garden have a fenced area of 1250 ft^2?

(d) Find the dimensions of the largest area that can be fenced.

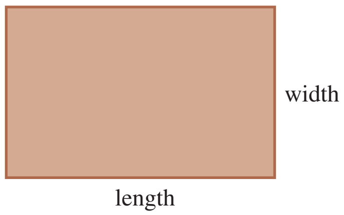
Thinking About the Problem

If the gardener fences a plot with width 10 ft, then the length must be 60 ft, because $10 + 10 + 60 + 60 = 140$. So the area is

$$A = \text{width} \times \text{length} = 10 \cdot 60 = 600 \text{ ft}^2$$

The table shows various choices for fencing the garden. We see that as the width increases, the fenced area increases, then decreases.

Width	Length	Area
10	60	600
20	50	1000
30	40	1200
40	30	1200
50	20	1000
60	10	600



Solution

(a) The model that we want is a function that gives the area that can be fenced.

Express the model in words. We know that the area of a rectangular garden is

$$\boxed{\text{area}} = \boxed{\text{width}} \times \boxed{\text{length}}$$

Choose the variable. There are two varying quantities: width and length. Because the function we want depends on only one variable, we let

$$x = \text{width of the garden}$$

Then we must express the length in terms of x. The perimeter is fixed at 140 ft, so the length is determined once we choose the width. If we let the length be l, as in Figure 1, then $2x + 2l = 140$, so $l = 70 - x$. We summarize these facts:

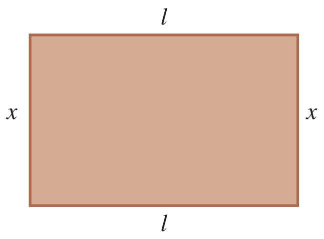


Figure 1

In Words	In Algebra
Width	x
Length	$70 - x$

Set up the model. The model is the function A that gives the area of the garden for any width x.

$$\boxed{\text{area}} = \boxed{\text{width}} \times \boxed{\text{length}}$$

$$A(x) = x(70 - x)$$

$$A(x) = 70x - x^2$$

The area that can be fenced is modeled by the function $A(x) = 70x - x^2$.

Use the model. We use the model to answer the questions in parts (b)–(d).

(b) We need to solve the inequality $A(x) \geq 825$. To solve graphically, we graph $y = 70x - x^2$ and $y = 825$ in the same viewing rectangle (see Figure 2). We see that $15 \leq x \leq 55$.

(c) From Figure 3 we see that the graph of $A(x)$ always lies below the line $y = 1250$, so an area of 1250 ft^2 is never attained.

Maximum values of functions are discussed in Section 2.3.

(d) We need to find where the maximum value of the function $A(x) = 70x - x^2$ occurs. Using a graphing device, we find that the function achieves its maximum value at $x = 35$ (see Figure 4). So the area that can be fenced is maximized when the garden's width is 35 ft and its length is $70 - 35 = 35$ ft. The maximum area then is $35 \times 35 = 1225\ \text{ft}^2$.

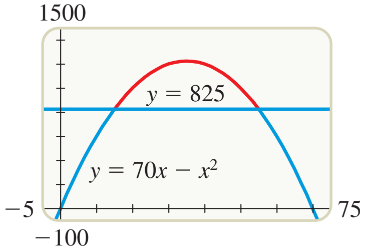


Figure 2

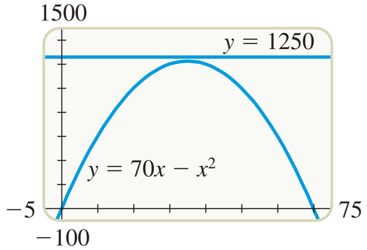


Figure 3

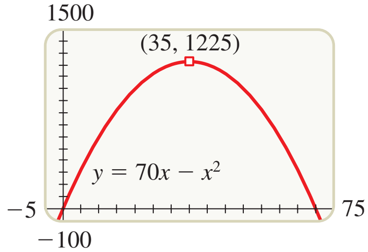


Figure 4

Example 2 ■ Minimizing the Metal in a Can

A manufacturer makes a cylindrical metal can that holds 1 L (liter) of oil. What radius minimizes the amount of metal in the can?

Thinking About the Problem

To use the least amount of metal, we must minimize the surface area of the can, that is, the area of the top, bottom, and the side. The combined area of the top and bottom is $2\pi r^2$ and the area of the side is $2\pi rh$ (see Figure 5), so the surface area of the can is

$$S = 2\pi r^2 + 2\pi rh$$

The radius and height of the can must be chosen so that the volume is exactly 1 L, or 1000 cm^3. If we want a small radius, say, $r = 3$, then the height must be just tall enough to make the total volume 1000 cm^3. In other words, we must have

$$\pi(3)^2 h = 1000 \qquad \text{Volume of the can is } \pi r^2 h$$

$$h = \frac{1000}{9\pi} \approx 35.37 \text{ cm} \qquad \text{Solve for } h$$

Now that we know the radius and height, we can find the surface area of the can:

$$\text{surface area} = 2\pi(3)^2 + 2\pi(3)(35.4) \approx 723.2 \text{ cm}^3$$

If we want a different radius, we can find the corresponding height and surface area in a similar fashion.

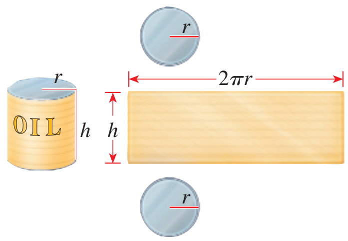


Figure 5

Solution The model that we want is a function that gives the surface area of the can.

Express the model in words. We know that for a cylindrical can

$$\boxed{\text{surface area}} = \boxed{\text{area of top and bottom}} + \boxed{\text{area of side}}$$

Choose the variable. There are two varying quantities: radius and height. Because the function we want depends on the radius, we let

$$r = \text{radius of can}$$

Next, we must express the height in terms of the radius r. Because the volume of a cylindrical can is $V = \pi r^2 h$ and the volume must be 1000 cm^3, we have

$$\pi r^2 h = 1000 \qquad \text{Volume of can is 1000 cm}^3$$

$$h = \frac{1000}{\pi r^2} \qquad \text{Solve for } h$$

We can now express the areas of the top, bottom, and side in terms of r only:

In Words	In Algebra
Radius of can	r
Height of can	$\dfrac{1000}{\pi r^2}$
Area of top and bottom	$2\pi r^2$
Area of side ($2\pi rh$)	$2\pi r\left(\dfrac{1000}{\pi r^2}\right)$

Set up the model. The model is the function S that gives the surface area of the can as a function of the radius r.

$$\boxed{\text{surface area}} = \boxed{\text{area of top and bottom}} + \boxed{\text{area of side}}$$

$$S(r) = 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right)$$

$$S(r) = 2\pi r^2 + \frac{2000}{r}$$

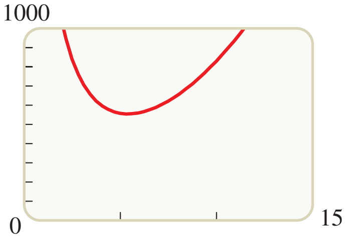


Figure 6 | $S(r) = 2\pi r^2 + \dfrac{2000}{r}$

Use the model. We use the model to find the minimum surface area of the can. We graph S in Figure 6 and find that the minimum value of S is about 554 cm^2 and this value occurs when the radius is about 5.4 cm.

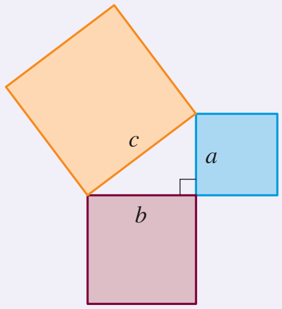
Problems

1–18 ■ Finding Models In these problems you are asked to find a function that models a real-life situation. Use the principles of modeling described in this *Focus* to help you.

1. **Area** A rectangular building lot is three times as long as it is wide. Find a function that models its area A in terms of its width w.
2. **Area** A poster is 10 in. longer than it is wide. Find a function that models its area A in terms of its width w.
3. **Volume** A rectangular box has a square base. Its height is half the width of the base. Find a function that models its volume V in terms of its width w.
4. **Volume** The height of a cylinder is four times its radius. Find a function that models the volume V of the cylinder in terms of its radius r.
5. **Area** A rectangle has a perimeter of 20 ft. Find a function that models its area A in terms of the length x of one of its sides.
6. **Perimeter** A rectangle has an area of 16 m^2. Find a function that models its perimeter P in terms of the length x of one of its sides.
7. **Area** Find a function that models the area A of an equilateral triangle in terms of the length x of one of its sides.
8. **Area** Find a function that models the surface area S of a cube in terms of its volume V.
9. **Radius** Find a function that models the radius r of a circle in terms of its area A.
10. **Area** Find a function that models the area A of a circle in terms of its circumference C.

PYTHAGORAS (circa 580–500 B.C.) founded a school in Croton in southern Italy, devoted to the study of arithmetic, geometry, music, and astronomy. The Pythagoreans, as they were called, were a secret society with peculiar rules and initiation rites. They wrote down nothing and were instructed not to reveal to anyone what they had learned from the Master. Although women were barred by law from attending public meetings, Pythagoras allowed women in his school, and his most famous student was Theano (whom he later married).

According to Aristotle, the Pythagoreans were convinced that "the principles of mathematics are the principles of all things." Their motto was "Everything is Number," by which they meant *whole* numbers. The outstanding contribution of Pythagoras is the theorem that bears his name: In a right triangle the area of the square on the hypotenuse is equal to the sum of the areas of the square on the other two sides.

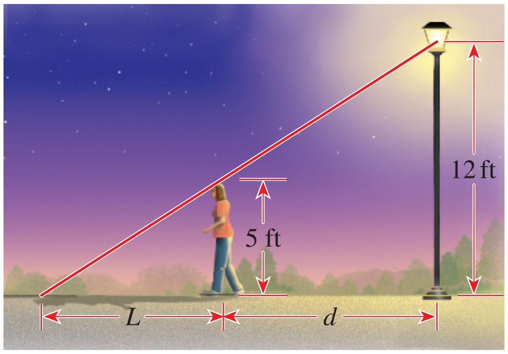


$c^2 = a^2 + b^2$

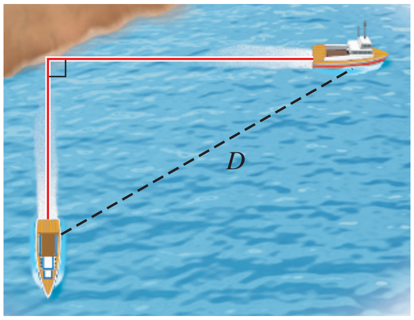
The converse of Pythagoras's Theorem is also true; that is, a triangle whose sides a, b, and c satisfy $a^2 + b^2 = c^2$ is a right triangle.

11. Area A rectangular box with a volume of 60 ft^3 has a square base. Find a function that models its surface area S in terms of the length x of one side of its base.

12. Length A five-foot-tall person is standing near a street lamp that is 12 ft tall, as shown in the figure. Find a function that models the length L of the person's shadow in terms of the distance d from the person to the base of the lamp.



13. Distance Two ships leave port at the same time. One sails south at 15 mi/h, and the other sails east at 20 mi/h. Find a function that models the distance D between the ships in terms of the time t (in hours) elapsed since their departure from the port.

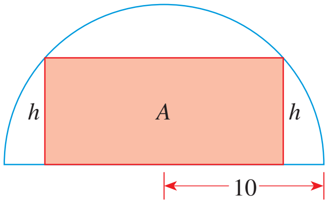


14. Product The sum of two positive numbers is 60. Find a function that models their product P in terms of x, one of the numbers.

15. Area An isosceles triangle has a perimeter of 8 cm. Find a function that models its area A in terms of the length of its base b.

16. Perimeter A right triangle has one leg twice as long as the other. Find a function that models its perimeter P in terms of the length x of the shorter leg.

17. Area A rectangle is inscribed in a semicircle of radius 10, as shown in the figure. Find a function that models the area A of the rectangle in terms of its height h.



18. Height The volume of a cone is 100 in^3. Find a function that models the height h of the cone in terms of its radius r.

19–33 ■ Using Models to Find Maxima and Minima In these problems you are asked to find a function that models a real-life situation and then use the model to answer questions about the situation. Use the *Guidelines for Modeling with Functions* to help you answer these questions.

19. Maximizing a Product Consider the following problem: Find two numbers whose sum is 19 and whose product is as large as possible.

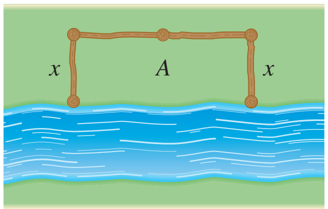
(a) Experiment with the problem by making a table like the one following, showing the product of different pairs of numbers that add up to 19. On the basis of the evidence in your table, estimate the answer to the problem.

First Number	Second Number	Product
1	18	18
2	17	34
3	16	48
⋮	⋮	⋮

(b) Find a function that models the product in terms of one of the two numbers.

(c) Use your model to solve the problem, and compare with your answer to part (a).

20. Minimizing a Sum Find two positive numbers whose sum is 100 and the sum of whose squares is a minimum.

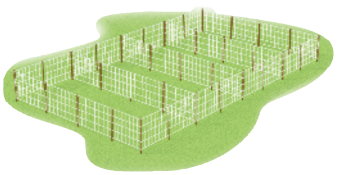


21. Fencing a Field Consider the following problem: A farmer has 2400 feet of fencing and wants to fence off a rectangular field that borders a straight river. The farmer does not need a fence along the river (see the figure). What are the dimensions of the field of largest area that can be fenced?

(a) Experiment with the problem by drawing several diagrams illustrating the situation. Calculate the area of each configuration, and use your results to estimate the dimensions of the largest possible field.

(b) Find a function that models the area of the field in terms of one of its sides.

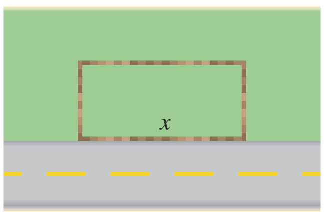
(c) Use your model to solve the problem, and compare with your answer to part (a).



22. Dividing a Pen A rancher with 750 ft of fencing wants to enclose a rectangular area and then divide it into four pens with fencing parallel to one side of the rectangle (see the figure).

(a) Find a function that models the total area of the four pens.

(b) Find the largest possible total area of the four pens.



23. Fencing a Garden Plot A property owner wants to fence a garden plot adjacent to a road, as shown in the figure. The fencing next to the road must be sturdier and costs \$5 per foot, but the other fencing costs just \$3 per foot. The garden is to have an area of 1200 ft^2.

(a) Find a function that models the cost of fencing the garden.

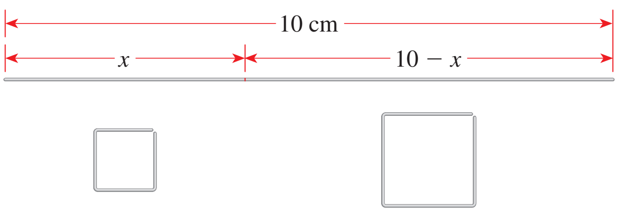
(b) Find the garden dimensions that minimize the cost of fencing.

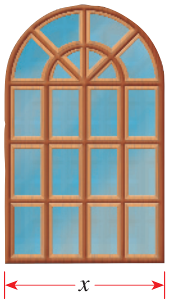
(c) If the owner has at most \$600 to spend on fencing, find the range of lengths that can be fenced along the road.

24. Minimizing Area A wire 10 cm long is cut into two pieces, one of length x and the other of length $10 - x$, as shown in the figure. Each piece is bent into the shape of a square.

(a) Find a function that models the total area enclosed by the two squares.

(b) Find the value of x that minimizes the total area of the two squares.





25. Light from a Window A Norman window has the shape of a rectangle surmounted by a semicircle, as shown in the figure to the left. A Norman window with perimeter 30 ft is to be constructed.

(a) Find a function that models the area of the window.

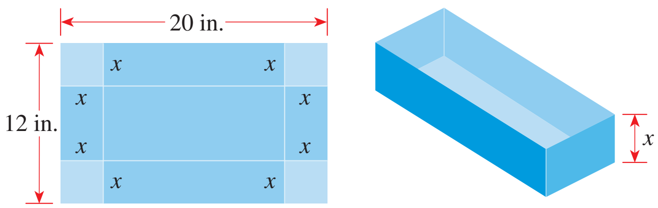
(b) Find the dimensions of the window that admits the greatest amount of light.

26. Volume of a Box A box with an open top is to be constructed from a rectangular piece of cardboard with dimensions 12 in. by 20 in. by cutting out equal squares of side x at each corner and then folding up the sides (see the figure).

(a) Find a function that models the volume of the box.

(b) Find the values of x for which the volume is greater than 200 in^3.

(c) Find the largest volume that such a box can have.

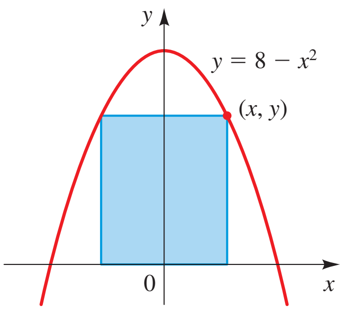


27. Area of a Box An open box with a square base of length x is to have a volume of 12 ft^3.

(a) Find a function that models the surface area of the box.

(b) Find the box dimensions that minimize the amount of material used.

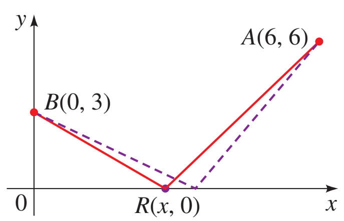
28. Inscribed Rectangle Find the dimensions that give the largest area for the rectangle shown in the figure. Its base is on the x-axis, and its other two vertices are above the x-axis, lying on the parabola $y = 8 - x^2$.



29. Minimizing Costs A rancher wants to build a rectangular pen with width x and an area of 100 m^2.

(a) Find a function that models the length of fencing required.

(b) Find the pen dimensions that require the minimum amount of fencing.



30. Minimizing a Distance Suppose that two villages are located at points $A(6, 6)$ and $B(0, 3)$, and the x-axis is a river. You are at point A and you need to walk to a point R on the river to get water and then walk to point B.

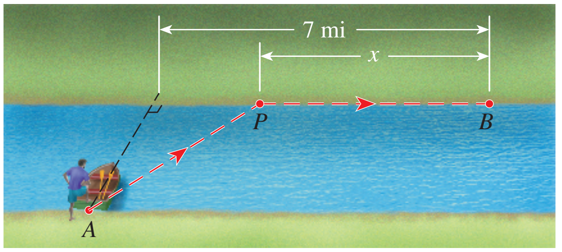
(a) Find a function of x that models the total distance from A to B via R.

(b) Find the value for x that minimizes the total distance you have to walk.

31. Minimizing Time You are standing at a point A on the bank of a straight river, that is 2 miles wide. To reach point B, 7 miles downstream on the opposite bank, you first row your boat to point P on the opposite bank and then walk the remaining distance x to B, as shown in the figure. You can row at a speed of 2 mi/h and walk at a speed of 5 mi/h.

(a) Find a function that models the time needed for your trip.

(b) Where should you land so that you reach B as soon as possible?



32. Bird Flight A bird is released from point A on an island, 5 miles from the nearest point B on a straight shoreline. The bird flies to a point C on the shoreline and then flies along the shoreline to its nesting area D (see the figure). Suppose the bird requires 10 kcal/mi of energy to fly over land and 14 kcal/mi to fly over water.

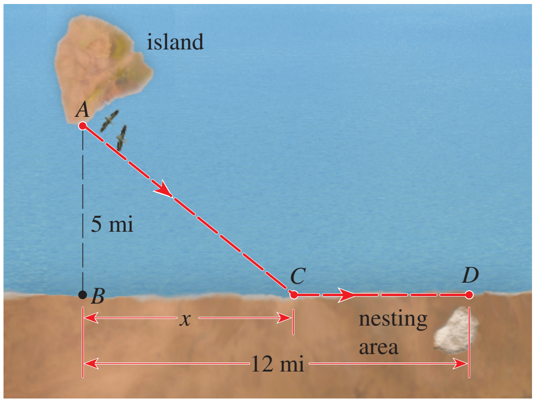
(a) Use the fact that

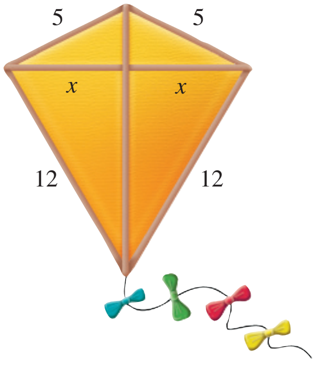
$$\text{energy used} = \text{energy per mile} \times \text{miles flown}$$

to show that the total energy used by the bird is modeled by the function

$$E(x) = 14\sqrt{x^2 + 25} + 10(12 - x)$$

(b) If the bird instinctively chooses a path that minimizes its energy expenditure, what point does it fly to?





33. Area of a Kite A kite frame is to be made from six pieces of wood. The four pieces that form its border have been cut to the lengths indicated in the figure. Let x be as shown in the figure.

(a) Show that the area of the kite is given by the function

$$A(x) = x\left(\sqrt{25 - x^2} + \sqrt{144 - x^2}\right)$$

(b) How long should each of the two crosspieces be to maximize the area of the kite?