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4 Exponential and Logarithmic Functions

In this chapter we study *exponential functions*. These are functions like $f(x) = 2^x$, where the independent variable is in the exponent. Exponential functions are used in modeling many different real-world phenomena, such as the growth of a population, the growth of an investment that earns compound interest, the spread of an infectious disease, or the decay of a radioactive substance. We also see how exponential functions are used in modeling the growth of a population in an environment with limited resources (food, water, living space), such as the buffalo herd pictured here. Once an exponential model has been obtained, we can use the model to predict—for any future time—the size of a population, calculate the amount of an investment, estimate the number of infected individuals, or determine the remaining amount of a radioactive substance.

The inverse functions of exponential functions are called *logarithmic functions*. With exponential models and logarithmic functions, we can answer questions such as these: When will world population reach a given level? When will my bank account have a million dollars? When will the number of new infections level off? When will radiation from a radioactive spill decay to a safe level?

In the *Focus on Modeling* at the end of the chapter we learn how to fit exponential and power curves to data.

4.1 Exponential Functions

■ Exponential Functions ■ Graphs of Exponential Functions ■ Compound Interest

In this chapter we study a class of functions called *exponential functions*. For example,

$$f(x) = 2^x$$

is an exponential function (with base 2). Notice how quickly the values of this function increase.

$$f(3) = 2^3 = 8$$
$$f(10) = 2^{10} = 1024$$
$$f(30) = 2^{30} = 1{,}073{,}741{,}824$$

Exponential functions get their name from the fact that the variable is in the exponent.

Compare this with the function $g(x) = x^2$, where $g(30) = 30^2 = 900$. The point is that when the variable is in the exponent, even a small change in the variable can cause a dramatic change in the value of the function.

■ Exponential Functions

To study exponential functions, we must first define what we mean by the exponential expression a^x when x is any real number. In Section 1.2 we defined a^x for $a > 0$ and x a rational number, but we have not yet defined irrational powers. So what is meant by $5^{\sqrt{3}}$ or 2^{π}? To define a^x when x is irrational, we approximate x by rational numbers.

For example, since

$$\sqrt{3} \approx 1.73205\ldots$$

is an irrational number, we successively approximate $a^{\sqrt{3}}$ by the following rational powers:

$$a^{1.7}, a^{1.73}, a^{1.732}, a^{1.7320}, a^{1.73205}, \ldots$$

Intuitively, we can see that these rational powers of a are getting closer and closer to $a^{\sqrt{3}}$. It can be shown by using advanced mathematics that there is exactly one number that these powers approach. We define $a^{\sqrt{3}}$ to be this number.

For example, using a calculator, we find

$$5^{\sqrt{3}} \approx 5^{1.732}$$
$$\approx 16.2411\ldots$$

The more decimal places of $\sqrt{3}$ we use in our calculation, the better our approximation of $5^{\sqrt{3}}$.

The Laws of Exponents are listed in Section 1.2.

It can be proved that the *Laws of Exponents are still true when the exponents are real numbers*.

Exponential Functions

The **exponential function with base a** is defined for all real numbers x by

$$f(x) = a^x$$

where $a > 0$ and $a \neq 1$.

We assume that $a \neq 1$ because the function $f(x) = 1^x = 1$ is just a constant function. Here are some examples of exponential functions:

$$\underbrace{f(x) = 2^x}_{\text{Base } 2} \qquad \underbrace{g(x) = 3^x}_{\text{Base } 3} \qquad \underbrace{h(x) = 10^x}_{\text{Base } 10}$$

Example 1 ■ Evaluating Exponential Functions

We use a calculator to find values of the exponential function $f(x) = 3^x$. Check to make sure you get the following answers on your own calculator.

	Calculator keystrokes	Output
(a) $f(5) = 3^5 = 243$	3 ^ 5 ENTER	243
(b) $f(-\frac{2}{3}) = 3^{-2/3} \approx 0.4807$	3 ^ ((−) 2 ÷ 3) ENTER	0.4807498
(c) $f(\pi) = 3^{\pi} \approx 31.544$	3 ^ π ENTER	31.5442807
(d) $f(\sqrt{2}) = 3^{\sqrt{2}} \approx 4.7288$	3 ^ √ 2 ENTER	4.7288043

Now Try Exercise 7 ■

■ Graphs of Exponential Functions

We first graph exponential functions by plotting points. We will see that the graphs of such functions have an easily recognizable shape.

Example 2 ■ Graphing Exponential Functions by Plotting Points

Draw the graph of each function.

(a) $f(x) = 3^x$ **(b)** $g(x) = \left(\dfrac{1}{3}\right)^x$

Solution We calculate values of $f(x)$ and $g(x)$ and plot points to sketch the graphs in Figure 1.

x	$f(x) = 3^x$	$g(x) = \left(\frac{1}{3}\right)^x$
-3	$\frac{1}{27}$	27
-2	$\frac{1}{9}$	9
-1	$\frac{1}{3}$	3
0	1	1
1	3	$\frac{1}{3}$
2	9	$\frac{1}{9}$
3	27	$\frac{1}{27}$

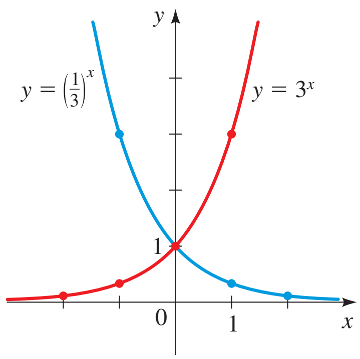


Figure 1

Now Try Exercise 17 ■

Reflecting graphs is explained in Section 2.6.

Note Figure 1 shows that the graph of g is the reflection of the graph of f about the y-axis. In general, the graph of $g(x) = \left(\frac{1}{a}\right)^x$ is the reflection of the graph of $f(x) = a^x$ about the y-axis because

$$g(x) = \left(\frac{1}{a}\right)^x = \frac{1}{a^x} = a^{-x} = f(-x)$$

So, in Example 2 we could have obtained the graph of $g(x) = \left(\frac{1}{3}\right)^x$ by reflecting the graph of $f(x) = 3^x$ about the y-axis.

The next figure shows graphs of the family of exponential functions $f(x) = a^x$ for various values of the base a. All these graphs have y-intercept 1 because $f(0) = a^0 = 1$

for $a \neq 0$. You can see from Figure 2 that there are two kinds of exponential functions: If $a > 1$, the exponential function increases rapidly, as in Figure 2(a). If $0 < a < 1$, the function decreases rapidly, as in Figure 2(b).

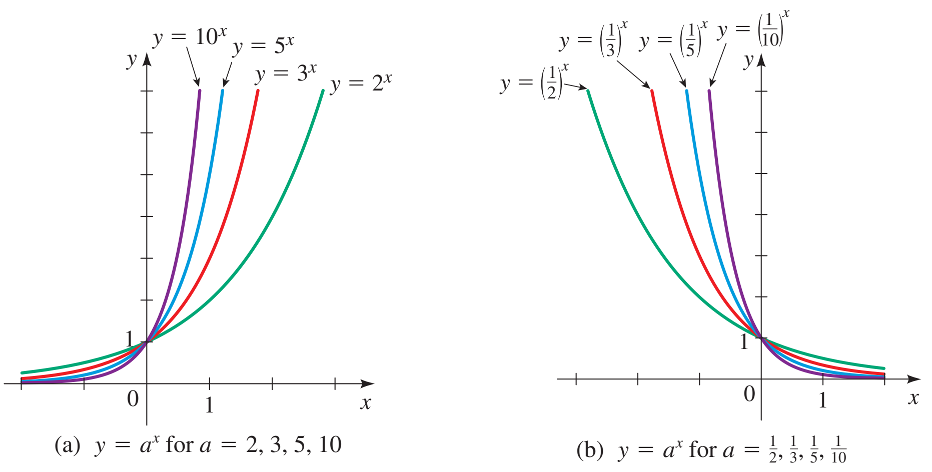


Figure 2 | Families of exponential functions

Arrow notation is explained in Section 3.6.

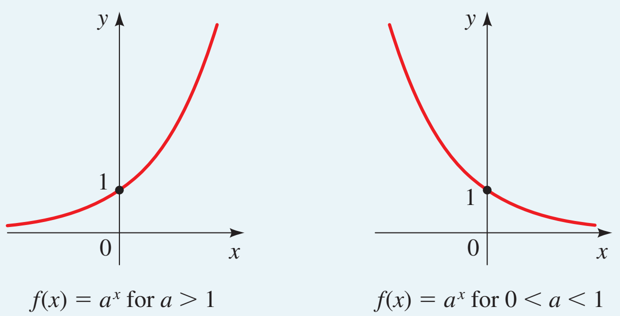
The x-axis is a horizontal asymptote for the exponential function $f(x) = a^x$. This is because when $a > 1$, we have $a^x \to 0$ as $x \to -\infty$, and when $0 < a < 1$, we have $a^x \to 0$ as $x \to \infty$. (See Figure 2.) Also, $a^x > 0$ for all $x \in \mathbb{R}$, so the function $f(x) = a^x$ has domain $\mathbb{R}$ and range $(0, \infty)$. These observations are summarized in the following box.

Graphs of Exponential Functions

The exponential function

$$f(x) = a^x \qquad (a > 0, a \neq 1)$$

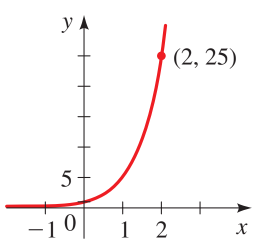
has domain $\mathbb{R}$, range $(0, \infty)$, and y-intercept 1. The line $y = 0$ (the x-axis) is a horizontal asymptote of f. The graph of f has one of the following shapes.



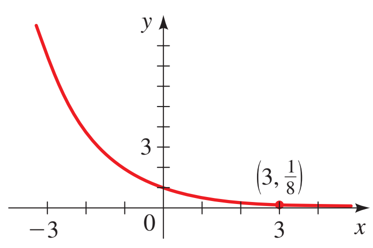
Example 3 ■ Identifying Graphs of Exponential Functions

Find the exponential function $f(x) = a^x$ whose graph is given.

(a)



(b)



Solution

(a) Since $f(2) = a^2 = 25$, we see that the base is $a = 5$. So $f(x) = 5^x$.

(b) Since $f(3) = a^3 = \frac{1}{8}$, we see that the base is $a = \frac{1}{2}$. So $f(x) = \left(\frac{1}{2}\right)^x$.

Now Try Exercise 21 ■

In the next example we see how to graph certain functions, not by plotting points, but by taking the basic graphs of the exponential functions in Figure 2 and applying the shifting and reflecting transformations that we studied in Section 2.6.

Example 4 ■ Transformations of Exponential Functions

Use the graph of $f(x) = 2^x$ to sketch the graph of each function. State the y-intercept, domain, range, and horizontal asymptote.

(a) $g(x) = 1 + 2^x$ **(b)** $h(x) = -2^x$ **(c)** $k(x) = 2^{x-1}$

Solution

(a) To obtain the graph of $g(x) = 1 + 2^x$, we start with the graph of $f(x) = 2^x$ and shift it upward 1 unit to get the graph shown in Figure 3(a). The y-intercept is $y = g(0) = 1 + 2^0 = 2$. From the graph we see that the domain of g is the set $\mathbb{R}$ of real numbers, the range is the interval $(1, \infty)$, and the line $y = 1$ is a horizontal asymptote.

(b) Again we start with the graph of $f(x) = 2^x$, but here we reflect about the x-axis to get the graph of $h(x) = -2^x$ shown in Figure 3(b). The y-intercept is $y = h(0) = -2^0 = -1$. From the graph we see that the domain of h is the set $\mathbb{R}$ of all real numbers, the range is the interval $(-\infty, 0)$, and the line $y = 0$ is a horizontal asymptote.

(c) This time we start with the graph of $f(x) = 2^x$ and shift it 1 unit to the right to get the graph of $k(x) = 2^{x-1}$ shown in Figure 3(c). The y-intercept is $y = k(0) = 2^{0-1} = \frac{1}{2}$. From the graph we see that the domain of k is the set $\mathbb{R}$ of all real numbers, the range is the interval $(0, \infty)$, and the line $y = 0$ is a horizontal asymptote.

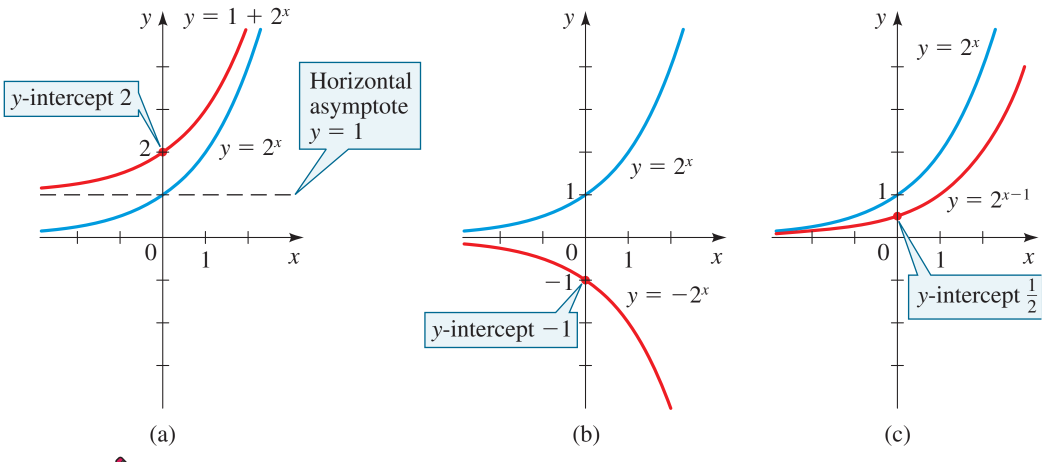


Figure 3

Now Try Exercises 27, 29, and 31 ■

Example 5 ■ Comparing Exponential and Power Functions

Compare the rates of growth of the exponential function $f(x) = 2^x$ and the power function $g(x) = x^2$ by drawing the graphs of both functions in the following viewing rectangles.

(a) $[0, 3]$ by $[0, 8]$ **(b)** $[0, 6]$ by $[0, 25]$ **(c)** $[0, 20]$ by $[0, 1000]$

Solution

(a) Figure 4(a) shows that the graph of $g(x) = x^2$ catches up with, and becomes higher than, the graph of $f(x) = 2^x$ at $x = 2$.

(b) The larger viewing rectangle in Figure 4(b) shows that the graph of $f(x) = 2^x$ overtakes that of $g(x) = x^2$ when $x = 4$.

(c) Figure 4(c) gives a more global view and shows that when x is large, $f(x) = 2^x$ is much larger than $g(x) = x^2$.

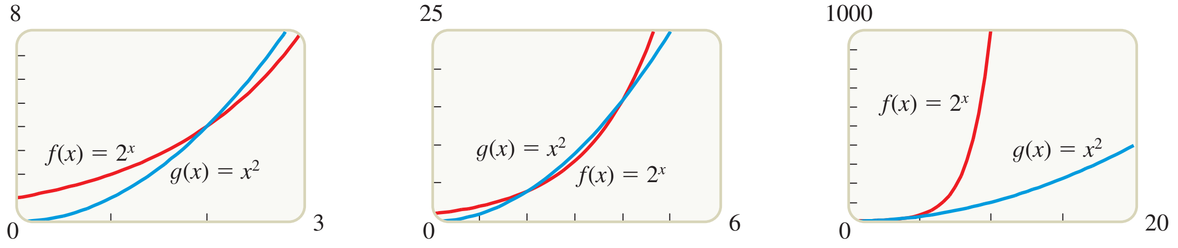


Figure 4 (a) (b) (c)

Now Try Exercise 45 ■

■ Compound Interest

Exponential functions occur in the calculation of compound interest. If an amount of money P, called the **principal**, is invested at an interest rate i per time period, then after one time period the interest is Pi, and the amount A of money is

$$A = P + Pi = P(1 + i)$$

If the interest is reinvested, then the new principal is $P(1 + i)$, and the amount after another time period is $A = P(1 + i)(1 + i) = P(1 + i)^2$. Similarly, after a third time period the amount is $A = P(1 + i)^3$. In general, after k periods the amount is

$$A = P(1 + i)^k$$

Notice that this is an exponential function with base $1 + i$.

If the annual interest rate is r and if interest is compounded n times per year, then in each time period the interest rate is $i = r/n$, and there are nt time periods in t years. This leads to the following formula for the amount after t years.

Compound Interest

Compound interest is calculated by the formula

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

where
$A(t)$ = amount after t years
P = principal
r = interest rate per year
n = number of times interest is compounded per year
t = number of years

r is often referred to as the nominal annual interest rate.

Example 6 ■ Calculating Compound Interest

A sum of \$1000 is invested at an interest rate of 12% per year. Find the amounts in the account after 3 years if interest is compounded annually, semiannually, quarterly, monthly, and daily.

Solution We use the compound interest formula with $P = \$1000$, $r = 0.12$, and $t = 3$.

Compounding	n	Amount After 3 Years
Annually	1	$1000\left(1 + \frac{0.12}{1}\right)^{1(3)} = \$1404.93$
Semiannually	2	$1000\left(1 + \frac{0.12}{2}\right)^{2(3)} = \$1418.52$
Quarterly	4	$1000\left(1 + \frac{0.12}{4}\right)^{4(3)} = \$1425.76$
Monthly	12	$1000\left(1 + \frac{0.12}{12}\right)^{12(3)} = \$1430.77$
Daily	365	$1000\left(1 + \frac{0.12}{365}\right)^{365(3)} = \$1433.24$

Now Try Exercise 57 ■

If an investment earns compound interest, then the **annual percentage yield** (APY) is the *simple* interest rate that yields the same amount at the end of one year.

Example 7 ■ Calculating the Annual Percentage Yield

Find the annual percentage yield for an investment that earns interest at a rate of 6% per year, compounded daily.

Solution After one year, a principal P will grow to the amount

$$A = P\left(1 + \frac{0.06}{365}\right)^{365} = P(1.06183)$$

Simple interest is studied in Section 1.7.

The formula for simple interest is

$$A = P(1 + r)$$

Comparing, we see that $1 + r = 1.06183$, so $r = 0.06183$. Thus the annual percentage yield is 6.183%.

Now Try Exercise 63 ■

Discovery Project ■ So You Want to Be a Millionaire?

In this project we explore how rapidly the values of an exponential function increase by examining some real-world situations. For example, if you save a penny today, two pennies tomorrow, four pennies the next day, and so on, how long do you have to continue saving in this way before you become a millionaire? You can find out the surprising answer to this and other questions by completing this discovery project. You can find the project at **www.stewartmath.com**.

4.1 Exercises

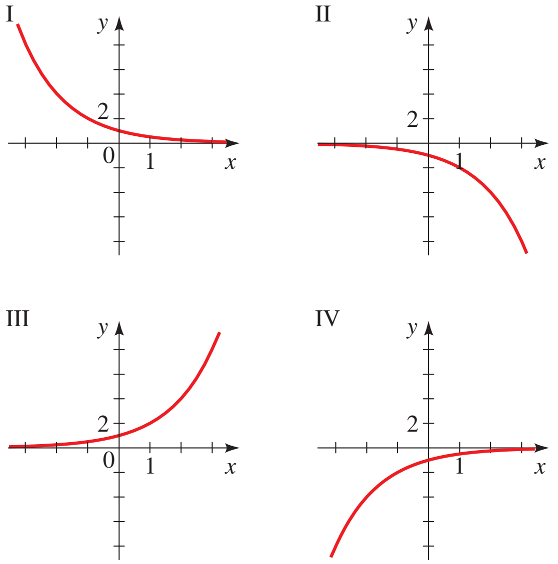
Concepts

1. The function $f(x) = 5^x$ is an exponential function with base ________; $f(-2) =$ ________, $f(0) =$ ________, $f(2) =$ ________, and $f(6) =$ ________.

2. Match the exponential function with one of the graphs labeled I, II, III, or IV, shown below.

(a) $f(x) = 2^x$ **(b)** $f(x) = 2^{-x}$

(c) $f(x) = -2^x$ **(d)** $f(x) = -2^{-x}$



3. (a) To obtain the graph of $g(x) = 2^x - 1$, we start with the graph of $f(x) = 2^x$ and shift it ________ (upward/downward) 1 unit.

(b) To obtain the graph of $h(x) = 2^{x-1}$, we start with the graph of $f(x) = 2^x$ and shift it 1 unit to the ________ (left/right).

4. In the formula $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$ for compound interest the letters P, r, n, and t stand for ________, ________, ________, and ________, respectively, and $A(t)$ stands for ________. So if \$100 is invested at an interest rate of 6% compounded quarterly, then the amount after 2 years is ________.

5. The exponential function $f(x) = \left(\frac{1}{2}\right)^x$ has the ________ asymptote $y =$ ________. This means that as $x \to \infty$, we have $\left(\frac{1}{2}\right)^x \to$ ________.

6. The exponential function $f(x) = \left(\frac{1}{2}\right)^x + 3$ has the ________ asymptote $y =$ ________. This means that as $x \to \infty$, we have $\left(\frac{1}{2}\right)^x + 3 \to$ ________.

Skills

7–10 ■ Evaluating Exponential Functions Use a calculator to evaluate the function at the indicated values. Round your answers to three decimal places.

7. $f(x) = 4^x$; $f\left(\frac{1}{2}\right), f\left(\sqrt{5}\right), f(-2), f(0.3)$

8. $f(x) = 3^{x-1}$; $f\left(\frac{1}{2}\right), f(2.5), f(-1), f\left(\frac{1}{4}\right)$

9. $g(x) = \left(\frac{1}{3}\right)^{x+1}$; $g\left(\frac{1}{2}\right), g\left(\sqrt{2}\right), g(-3.5), g(-1.4)$

10. $g(x) = \left(\frac{4}{3}\right)^{3x}$; $g\left(-\frac{1}{2}\right), g\left(\sqrt{6}\right), g(-3), g\left(\frac{4}{3}\right)$

11–16 ■ Graphing Exponential Functions Sketch the graph of the function by making a table of values. Use a calculator if necessary.

11. $f(x) = 6^x$ **12.** $f(x) = (0.1)^x$

13. $g(x) = \left(\frac{2}{3}\right)^x$ **14.** $g(x) = 4^x$

15. $h(x) = 5(2.2)^x$ **16.** $h(x) = 4\left(\frac{5}{8}\right)^x$

17–20 ■ Graphing Exponential Functions Graph both functions on one set of axes.

17. $f(x) = 4^x$ and $g(x) = 4^{-x}$

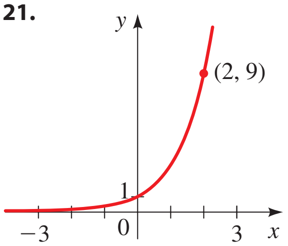
18. $f(x) = 8^{-x}$ and $g(x) = \left(\frac{1}{8}\right)^x$

19. $f(x) = 4^x$ and $g(x) = 7^x$

20. $f(x) = \left(\frac{3}{4}\right)^x$ and $g(x) = 1.5^x$

21–24 ■ Exponential Functions from a Graph Find the exponential function $f(x) = a^x$ whose graph is given.

21.



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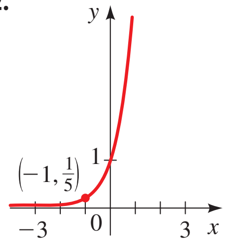

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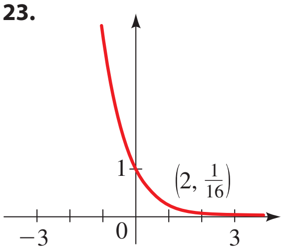

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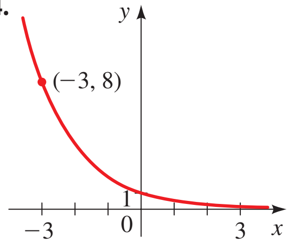
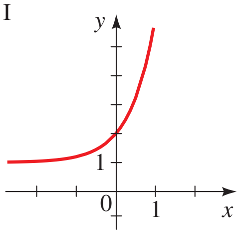


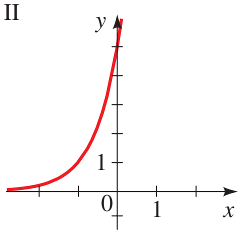
25–26 ■ Exponential Functions from a Graph Match the exponential function with one of the graphs labeled I or II.

25. $f(x) = 5^{x+1}$ **26.** $f(x) = 5^x + 1$

I



II



27–40 ■ Graphing Exponential Functions Graph the function, not by plotting points, but by starting from the graphs in Figure 2. State the y-intercept, domain, range, and horizontal asymptote.

27. $f(x) = 3^x + 1$ **28.** $f(x) = \left(\frac{1}{3}\right)^x - 2$

29. $g(x) = -5^x$ **30.** $g(x) = 5^{-x}$

31. $h(x) = 3^{x-2}$ **32.** $h(x) = 10^{x+1}$

33. $y = 2^{-x} + 3$ **34.** $y = -2^x + 3$

35. $y = -10^x - 1$ **36.** $y = \left(\frac{1}{2}\right)^x - 4$

37. $h(x) = 2^{x-4} + 1$ **38.** $y = 10^{x+1} - 5$

39. $g(x) = 1 - 3^{-x}$ **40.** $y = 3 - \left(\frac{1}{5}\right)^x$

41–42 ■ Comparing Exponential Functions In these exercises we compare the graphs of two exponential functions.

41. (a) Sketch the graphs of $f(x) = 2^x$ and $g(x) = 3(2^x)$.

(b) How are the graphs related?

42. (a) Sketch the graphs of $f(x) = 9^{x/2}$ and $g(x) = 3^x$.

(b) Use the Laws of Exponents to explain the relationship between these graphs.

43–44 ■ Comparing Exponential and Power Functions Compare the graphs of the power function f and exponential function g by evaluating both of them for $x = 0, 1, 2, 3, 4, 6, 8,$ and 10. Then draw the graphs of f and g on the same set of axes.

43. $f(x) = x^3$; $g(x) = 3^x$ **44.** $f(x) = x^4$; $g(x) = 4^x$

45–46 ■ Comparing Exponential and Power Functions In these exercises we use a graphing device to compare the rates of growth of the graphs of a power function and an exponential function.

45. (a) Compare the rates of growth of the functions $f(x) = 2^x$ and $g(x) = x^5$ by drawing the graphs of both functions in the following viewing rectangles.

(i) $[0, 5]$ by $[0, 20]$

(ii) $[0, 25]$ by $[0, 10^7]$

(iii) $[0, 50]$ by $[0, 10^8]$

(b) Find the solutions of the equation $2^x = x^5$, rounded to one decimal place.

46. (a) Compare the rates of growth of the functions $f(x) = 3^x$ and $g(x) = x^4$ by drawing the graphs of both functions in the following viewing rectangles:

(i) $[-4, 4]$ by $[0, 20]$

(ii) $[0, 10]$ by $[0, 5000]$

(iii) $[0, 20]$ by $[0, 10^5]$

(b) Find the solutions of the equation $3^x = x^4$, rounded to two decimal places.

Skills Plus

47–48 ■ Families of Functions Draw graphs of the given family of functions for $c = 0.25, 0.5, 1, 2, 4$. How are the graphs related?

47. $f(x) = c2^x$ **48.** $f(x) = 2^{cx}$

49–50 ■ Getting Information from a Graph Find, rounded to two decimal places, (a) the intervals on which the function is increasing or decreasing and (b) the range of the function.

49. $y = 10^{x-x^2}$ **50.** $y = x2^x$

51–52 ■ Difference Quotients These exercises involve a difference quotient for an exponential function.

51. If $f(x) = 10^x$, show that

$$\frac{f(x+h) - f(x)}{h} = 10^x\left(\frac{10^h - 1}{h}\right)$$

52. If $f(x) = 3^{x-1}$, show that

$$\frac{f(x+h) - f(x)}{h} = 3^{x-1}\left(\frac{3^h - 1}{h}\right)$$

Applications

53. Bacteria Growth A bacteria culture contains 1500 bacteria initially and doubles every hour.

(a) Find a function N that models the number of bacteria after t hours.

(b) Find the number of bacteria after 24 hours.

54. Mouse Population A certain breed of mouse was introduced onto a small island with an initial population of 320 mice, and scientists estimate that the mouse population is doubling every year.

(a) Find a function N that models the number of mice after t years.

(b) Estimate the mouse population after 8 years.

55–56 ■ Compound Interest An investment of \$5000 is deposited into an account in which interest is compounded monthly. Complete the table by filling in the amounts the investment grows to at the indicated times or interest rates.

55. $r = 4\%$

Time (years)	Amount
1	
2	
3	
4	
5	
6	

56. $t = 5$ years

Rate per Year	Amount
1%	
2%	
3%	
4%	
5%	
6%	

57. Compound Interest If \$8000 is invested at an interest rate of 6.25% per year, compounded semiannually, find the value of the investment after the given number of years.

(a) 5 years **(b)** 10 years **(c)** 15 years

58. Compound Interest If \$3500 is invested at an interest rate of 3.5% per year, compounded daily, find the value of the investment after the given number of years.

(a) 2 years **(b)** 3 years **(c)** 6 years

59. Compound Interest If \$1200 is invested at an interest rate of 2.75% per year, compounded quarterly, find the value of the investment after the given number of years.

(a) 1 year **(b)** 2 years **(c)** 10 years

60. Compound Interest If \$14,000 is borrowed at a rate of 5.25% interest per year, compounded quarterly, find the amount due at the end of the given number of years.

(a) 4 years **(b)** 6 years **(c)** 8 years

61–62 ■ Present Value The **present value** of a sum of money is the amount that must be invested now, at a given rate of interest, to produce the desired sum at a later date.

61. How much should be invested now (the present value) to have an amount of \$10,000, 3 years from now, if the amount is invested at an interest rate of 9% per year, compounded semiannually.

62. How much should be invested now (the present value) to have an amount of \$100,000, 5 years from now, if the amount is invested at an interest rate of 8% per year, compounded monthly.

63. Annual Percentage Yield Find the annual percentage yield for an investment that earns 8% per year, compounded monthly.

64. Annual Percentage Yield Find the annual percentage yield for an investment that earns $5\frac{1}{2}$% per year, compounded quarterly.

Discuss ■ Discover ■ Prove ■ Write

65. Discuss ■ Discover: Exponential Functions Increase To see just how quickly the exponential function $f(x) = 2^x$ increases, perform the following thought experiment: Imagine that you have a sheet of paper that is 0.001 inch thick, then visualize repeatedly folding the paper in half (so the thickness doubles with each fold). How thick is the folded paper after 50 folds? (Express your answer in miles.)

66. Discuss ■ Discover: The Height of the Graph of an Exponential Function Your mathematics instructor asks you to sketch a graph of the exponential function

$$f(x) = 2^x$$

for x between 0 and 40, using a scale of 10 units to one inch. What are the dimensions of the sheet of paper you will need to sketch this graph?

4.2 The Natural Exponential Function

■ The Number e ■ The Natural Exponential Function ■ Continuously Compounded Interest

Any positive number can be used as a base for an exponential function. In this section we study the special base e, which is convenient for applications involving calculus.

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The **Gateway Arch** in St. Louis, Missouri, is shaped in the form of the graph of a combination of exponential functions (*not* a parabola, as it might first appear). Specifically, it is a **catenary**, which is the graph of an equation of the form

$$y = a(e^{bx} + e^{-bx})$$

(see Exercises 29 and 31). This shape was chosen because it is optimal for distributing the internal structural forces of the arch. Chains and cables suspended between two points (for example, the stretches of cable between pairs of telephone poles) hang in the shape of a catenary.

■ The Number e

The number e is defined as the value that $(1 + 1/n)^n$ approaches as n becomes large. (In calculus this idea is made more precise through the concept of a limit.) The table shows the values of the expression $(1 + 1/n)^n$ for increasingly large values of n.

n	$\left(1 + \frac{1}{n}\right)^n$
1	2.00000
5	2.48832
10	2.59374
100	2.70481
1000	2.71692
10,000	2.71815
100,000	2.71827
1,000,000	2.71828

It appears that, rounded to five decimal places, $e \approx 2.71828$; in fact, the approximate value to 20 decimal places is

$$\boxed{e \approx 2.71828182845904523536}$$

It can be shown that e is an irrational number, so we cannot write its exact value in decimal form.

The notation e was chosen by Leonhard Euler (see Section 1.6), probably because it is the first letter of the word *exponential*.

■ The Natural Exponential Function

The exponential function with base e is called the *natural exponential function*. Why use such a strange base for an exponential function? It might seem at first that a base such as 10 is easier to work with. We will see, however, that in certain applications the

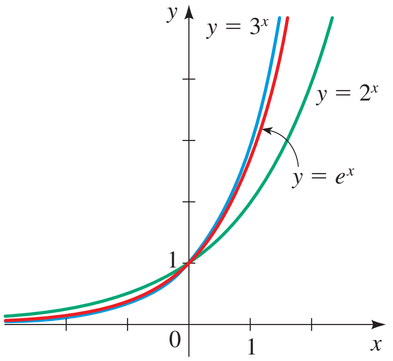


Figure 1 | Graph of the natural exponential function

number e is the best possible base. In this section we study how e occurs in the description of compound interest.

The Natural Exponential Function

The **natural exponential function** is the exponential function

$$f(x) = e^x$$

with base e. It is often referred to as *the* exponential function.

Since $2 < e < 3$, the graph of the natural exponential function lies between the graphs of $y = 2^x$ and $y = 3^x$, as shown in Figure 1.

Example 1 ■ Evaluating the Exponential Function

We use the $\boxed{e^x}$ key on a calculator to evaluate the natural exponential function. Check to make sure you get the following answers on your own calculator.

(a) $e^3 \approx 20.08554$ **(b)** $2e^{-0.53} \approx 1.17721$ **(c)** $e^{4.8} \approx 121.51042$

Now Try Exercise 3 ■

Example 2 ■ Graphing Exponential Functions

Sketch the graph of each function. State the y-intercept, domain, range, and horizontal asymptote.

(a) $f(x) = 3e^{0.5x}$ **(b)** $g(x) = e^{-x} - 2$

Solution

(a) We calculate several values, plot the resulting points, then connect the points with a smooth curve. The graph is shown in Figure 2. The y-intercept is $y = f(0) = 3e^0 = 3$. From the graph we see that the domain of f is the set $\mathbb{R}$ of all real numbers, the range is the interval $(0, \infty)$, and the line $y = 0$ is a horizontal asymptote.

x	$f(x) = 3e^{0.5x}$
-3	0.67
-2	1.10
-1	1.82
0	3.00
1	4.95
2	8.15
3	13.45

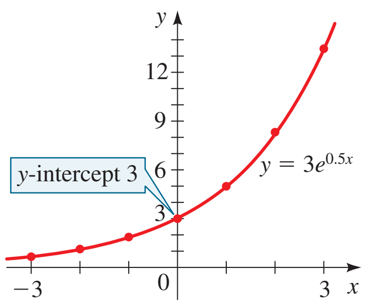


Figure 2

(b) We use transformations to sketch this graph. We start with the graph of $y = e^x$, reflect about the y-axis to obtain the graph of $y = e^{-x}$, and then shift downward 2 units to obtain the graph of $y = e^{-x} - 2$, as shown in Figure 3. The y-intercept is $y = g(0) = e^0 - 2 = -1$. From the graph we see that the domain of g is the set $\mathbb{R}$ of all real numbers, the range is the interval $(-2, \infty)$, and the line $y = -2$ is a horizontal asymptote.

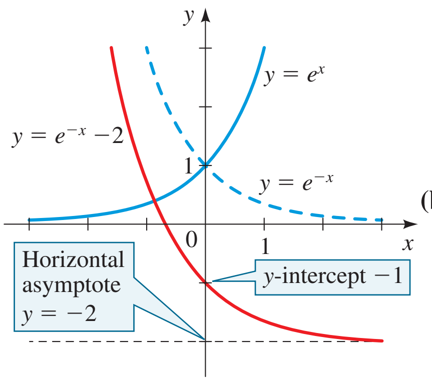


Figure 3

Now Try Exercises 5 and 7 ■

When we combine the exponential function with other functions we get new functions with a variety of different graphs, as shown in the next example.

Example 3 ■ Combinations Involving Exponential Functions

Use a graphing device to graph each function. State the y-intercept and use the graph to find the horizontal asymptote and any local extrema.

(a) $g(x) = e^{-x^2}$ **(b)** $l(x) = \dfrac{5}{1 + e^{-x}}$ **(c)** $s(x) = 10xe^{-0.1x}$

Solution Using a graphing device we get the graphs shown in Figure 4.

(a) The y-intercept is 1: when $x = 0$, $y = g(0) = e^0 = 1$. From the graph in Figure 4(a) we see that g has a local maximum value of 1 when $x = 0$, and the line $y = 0$ (the x-axis) is a horizontal asymptote.

To confirm that $y = 0$ is a horizontal asymptote, note that as x becomes large in absolute value, e^{x^2} also becomes large, so $1/e^{x^2}$ approaches 0. In symbols: $y = g(x) = e^{-x^2} = 1/e^{x^2} \to 0$ as $x \to \pm\infty$.

See Section 3.6 for a discussion of arrow notation.

(b) The y-intercept is $\frac{5}{2}$: when $x = 0$, $y = l(0) = 5/(1 + e^0) = \frac{5}{2}$. From the graph in Figure 4(b) we see that the function l has no local extrema and that the lines $y = 0$ and $y = 5$ are horizontal asymptotes.

To confirm that $y = 5$ is a horizontal asymptote, note that as $x \to \infty$,

$$y = l(x) = \frac{5}{1 + e^{-x}} \to \frac{5}{1 + 0} = 5$$

Also $y = 0$ is a horizontal asymptote because as $x \to -\infty$, $e^{-x} \to \infty$, so $5/(1 + e^{-x}) \to 0$.

(c) The y-intercept is 0: when $x = 0$, $y = s(0) = 10 \cdot 0 \cdot e^0 = 0$. From the graph in Figure 4(c) we see that the function s has a local maximum value of approximately 36.8 at $x \approx 10.0$. The x-axis is a horizontal asymptote.

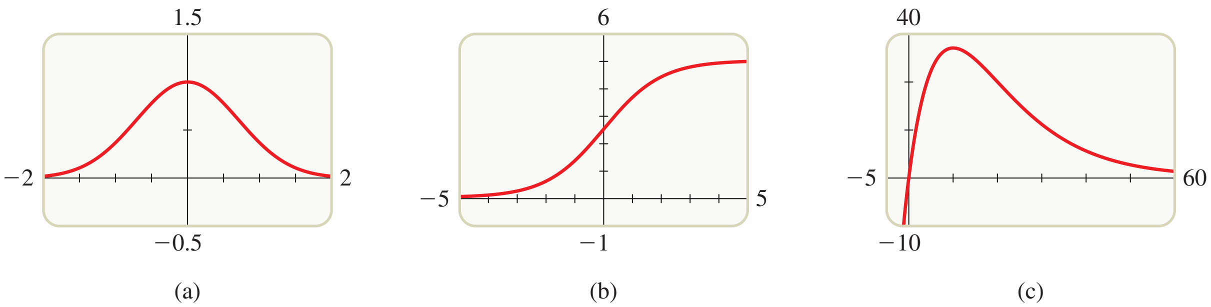


Figure 4

Now Try Exercises 19, 21, and 23 ■

The functions in Example 3 are used to model many real-world processes. The function g is a *Gaussian function* (see Exercise 38), the function l is a *logistic function* (see Exercise 41 and Section 4.6), and the function s is a *surge function* (see Exercise 37).

■ Continuously Compounded Interest

In Example 4.1.6 we saw that the interest paid on an investment increases as the number of compounding periods n increases. Let's see what happens as n increases indefinitely. If we let $m = n/r$, then

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt} = P\left[\left(1 + \frac{r}{n}\right)^{n/r}\right]^{rt} = P\left[\left(1 + \frac{1}{m}\right)^{m}\right]^{rt}$$

Recall that as m becomes large, the quantity $(1 + 1/m)^m$ approaches the number e. Thus the amount approaches $A = Pe^{rt}$, and this expression gives the amount when the interest is compounded at "every instant."

Continuously Compounded Interest

Continuously compounded interest is calculated by the formula

$$A(t) = Pe^{rt}$$

where
$A(t)$ = amount after t years
P = principal
r = interest rate per year
t = number of years

Example 4 ■ Calculating Continuously Compounded Interest

Find the amount after 3 years if $1000 is invested at an interest rate of 12% per year, compounded continuously.

Solution We use the formula for continuously compounded interest with $P = \$1000$, $r = 0.12$, and $t = 3$ to get

$$A(3) = 1000e^{(0.12)3} = 1000e^{0.36} = \$1433.33$$

Compare this amount with the amounts in Example 4.1.6.

Now Try Exercise 45 ■

4.2 Exercises

Concepts

1. The function $f(x) = e^x$ is called the ________ exponential function. The number e is approximately equal to ________.

2. In the formula $A(t) = Pe^{rt}$ for continuously compound interest, the letters P, r, and t stand for ________, ________, and ________, respectively, and $A(t)$ stands for ________. So if $100 is invested at an interest rate of 6% compounded continuously, then the amount after 2 years is ________.

Skills

3–4 ■ Evaluating Exponential Functions Use a calculator to evaluate the function at the indicated values. Round your answers to three decimal places.

3. $h(x) = e^x$; $h(1), h(\pi), h(-3), h(\sqrt{2})$

4. $h(x) = e^{-3x}$; $h(\frac{1}{3}), h(1.5), h(-1), h(-\pi)$

5–6 ■ Graphing Exponential Functions Complete the table of values, rounded to two decimal places, and sketch a graph of the function.

5.

x	$f(x) = 1.5e^x$
-2	
-1	
-0.5	
0	
0.5	
1	
2	

6.

x	$f(x) = 4e^{-x/3}$
-3	
-2	
-1	
0	
1	
2	
3	

7–18 ■ Graphing Exponential Functions Graph the function, not by plotting points, but by starting from the graph of $y = e^x$ in Figure 1. State the y-intercept, domain, range, and horizontal asymptote.

7. $f(x) = e^x + 3$

8. $f(x) = e^{-x} + 1$

9. $g(x) = e^{-x} - 3$

10. $h(x) = e^x - 4$

11. $f(x) = 3 - e^x$

12. $y = 2 - e^{-x}$

13. $y = 4 - 3e^{-x}$

14. $f(x) = 3e^x + 1$

15. $f(x) = e^{x-2}$

16. $y = e^{x-3} + 4$

17. $h(x) = e^{x+1} - 3$

18. $g(x) = -e^{x-1} - 2$

19–24 ■ Graphing Combinations of Exponential Functions Use a graphing device to graph the function. State the y-intercept, and use the graph to find the horizontal asymptote(s) and any local extrema.

19. $f(x) = 5e^{-x^2/2}$

20. $f(x) = 10e^{-(x-5)^2}$

21. $l(x) = \dfrac{20}{1 + 3e^{-1.5x}}$

22. $l(x) = \dfrac{100}{1 + 4e^{-0.5x}}$

23. $s(x) = 0.4xe^{-1.5x}$

24. $s(x) = 2.5xe^{-3.5x}$

25–28 ■ Expressing a Function as a Composition Find functions f and g such that $F = f \circ g$.

25. $F(x) = 2e^{(x-10)^2}$

26. $F(x) = e^x + e^{-x}$

27. $F(x) = \sqrt{1 + e^x}$

28. $F(x) = (3 + e^x)^3$

Skills Plus

29. Hyperbolic Cosine Function The *hyperbolic cosine function* is defined by

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

(a) Sketch the graphs of the functions $y = \frac{1}{2}e^x$ and $y = \frac{1}{2}e^{-x}$ on the same axes, and use graphical addition (see Section 2.7) to sketch the graph of $y = \cosh(x)$.

(b) Use the definition to show that $\cosh(-x) = \cosh(x)$.

30. Hyperbolic Sine Function The *hyperbolic sine function* is defined by

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

(a) Sketch the graph of this function using graphical addition as in Exercise 29.

(b) Use the definition to show that $\sinh(-x) = -\sinh(x)$.

31. Families of Functions

(a) Draw the graphs of the family of functions

$$f(x) = \frac{a}{2}(e^{x/a} + e^{-x/a})$$

for $a = 0.5$, 1, 1.5, and 2.

(b) How does a larger value of a affect the graph?

32. The Definition of *e* Illustrate the definition of the number e by graphing the curve $y = (1 + 1/x)^x$ and the line $y = e$ on the same screen, using the viewing rectangle $[0, 40]$ by $[0, 4]$.

33–34 ■ Local Extrema Find the local maximum and minimum values of the function and the value of x at which each occurs. State each answer rounded to two decimal places.

33. $g(x) = x^x \quad (x > 0)$

34. $g(x) = e^x + e^{-2x}$

Applications

35. Medical Drugs When a certain medical drug is administered to a patient, the number of milligrams remaining in the patient's bloodstream t hours after the drug is fully absorbed is modeled by

$$D(t) = 50e^{-0.2t}$$

How many milligrams of the drug remain in the patient's bloodstream after 3 hours?

36. Radioactive Decay A radioactive substance decays in such a way that the amount of mass remaining after t days is given by the function

$$m(t) = 13e^{-0.015t}$$

where $m(t)$ is measured in kilograms.

(a) Find the mass at time $t = 0$.

(b) How much of the mass remains after 45 days?

37. Drug Absorption When a certain drug is administered to a patient, the drug concentration in the bloodstream increases rapidly until it reaches a peak level, and then it decreases slowly. The concentration of the drug in the patient's bloodstream is modeled by the surge function

$$s(t) = 0.7te^{-1.2t}$$

where t is the time in hours since the drug was administered and $s(t)$ is the concentration of the drug in mg/mL.

(a) Draw a graph of the function s for $0 \le t \le 6$. Use the graph to describe how the drug's concentration in the bloodstream varies with time.

(b) After how many minutes does the amount of medication in the bloodstream reach its maximum level?

(c) Use the model to determine how long the patient must wait for the concentration of the drug to be only 0.01 mg/mL.

38. Distribution of Heights The distribution of the heights (in cm) of high-school players in a basketball league is modeled by the Gaussian function

$$g(x) = 0.005e^{-(x-185)^2/200}$$

(a) Draw a graph of the function g for $150 \le x \le 220$.

(b) Use the graph to estimate the average height of these players. (This is the value of x at which the graph g has a maximum.)

39. Skydiving A skydiver jumps from a reasonable height above the ground. The air resistance is proportional to the skydiver's velocity, and the constant of proportionality is 0.2. It can be shown that the downward velocity of the skydiver at time t is given by

$$v(t) = 180(1 - e^{-0.2t})$$

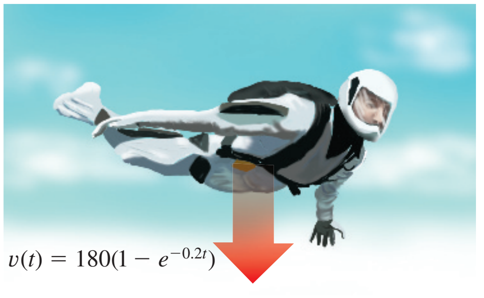
where t is measured in seconds (s) and $v(t)$ is measured in feet per second (ft/s).

(a) Find the initial velocity of the skydiver.

(b) Find the velocity after 5 s and after 10 s.

(c) Draw a graph of the velocity function $v(t)$.

(d) The maximum velocity of a falling object with wind resistance is called its *terminal velocity*. From the graph in part (c), find the terminal velocity of this skydiver.



40. Mixtures and Concentrations A 50-gallon barrel is filled completely with pure water. Salt water with a concentration of 0.3 lb/gal is then pumped into the barrel, and the resulting mixture overflows at the same rate. The amount of salt in the barrel at time t is given by

$$Q(t) = 15(1 - e^{-0.04t})$$

where t is measured in minutes and $Q(t)$ is measured in pounds.

(a) How much salt is in the barrel after 5 minutes?

(b) How much salt is in the barrel after 10 minutes?

(c) Draw a graph of the function $Q(t)$.

(d) Use the graph in part (c) to determine the value that the amount of salt in the barrel approaches as t becomes large. Is this what you would expect?

41. Axolotl Salamander The axolotl is a paedomorphic salamander that is invaluable for medical research into the healing process: it has the unique ability to regenerate limbs, gills, and parts of its eyes and brain. The species is critically endangered in the wild, so scientists are planning to introduce the species into a lake where it can thrive. The expected number $n(t)$ of axolotls in the lake t years after they are introduced is modeled by the logistic function

$$n(t) = \frac{5000}{1 + 39e^{-0.04t}}$$

(a) Find the initial number of axolotls introduced into the lake.

(b) Draw a graph of the function n.

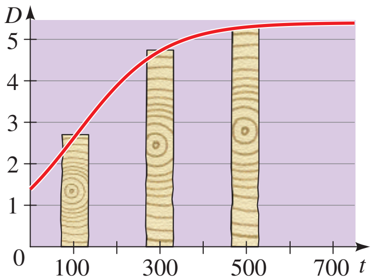
(c) What number does the axolotl population approach as time goes on?

axolotlowner/Shutterstock

42. Tree Diameter For a certain type of tree the diameter D (in feet) depends on the tree's age t (in years), and is modeled by the logistic function

$$D(t) = \frac{5.4}{1 + 2.9e^{-0.01t}}$$

Find the diameter of a 20-year-old tree.



43–44 ■ Compound Interest An investment of \$7000 is deposited into an account for which interest is compounded continuously. Complete the table by filling in the amounts to which the investment grows at the indicated times or interest rates.

43. $r = 3\%$

Time (years)	Amount
1	
2	
3	
4	
5	
6	

44. $t = 10$ years

Rate per Year	Amount
1%	
2%	
3%	
4%	
5%	
6%	

45. Compound Interest If \$2000 is invested at an interest rate of 3.5% per year, compounded continuously, find the value of the investment after the given number of years.

(a) 2 years **(b)** 4 years **(c)** 12 years

46. Compound Interest If \$3500 is invested at an interest rate of 6.25% per year, compounded continuously, find the value of the investment after the given number of years.

(a) 3 years **(b)** 6 years **(c)** 9 years

47. Compound Interest If \$600 is invested at an interest rate of 2.5% per year, find the amount of the investment at the end of 10 years for each compounding method.

(a) Annual **(b)** Semiannual
(c) Quarterly **(d)** Continuous

48. Compound Interest If \$8000 is invested in an account for which interest is compounded continuously, find the amount of the investment at the end of 12 years for the given interest rates.

(a) 2% **(b)** 3% **(c)** 4.5% **(d)** 7%

49. Compound Interest Which of the given interest rates and compounding periods would provide the best investment?

(a) $2\frac{1}{2}\%$ per year, compounded semiannually

(b) $2\frac{1}{4}\%$ per year, compounded monthly

(c) 2% per year, compounded continuously

50. Compound Interest Which of the given interest rates and compounding periods would provide the better investment?

(a) $5\frac{1}{8}\%$ per year, compounded semiannually

(b) 5% per year, compounded continuously

51. Investment A sum of \$5000 is invested at an interest rate of 9% per year, compounded continuously.

(a) Find the value $A(t)$ of the investment after t years.

(b) Draw a graph of $A(t)$.

(c) Use the graph of $A(t)$ to determine when this investment will amount to \$25,000.

Discuss ■ Discover ■ Prove ■ Write

52. Discuss ■ Discover: Minimum Value Show that the minimum value of $f(x) = e^x + e^{-x}$ is 2. (Do not use a graphing device.)

PS *Establish subgoals.* First find the minimum value for $x \geq 0$.

4.3 Logarithmic Functions

■ Logarithmic Functions ■ Graphs of Logarithmic Functions ■ Common Logarithms ■ Natural Logarithms

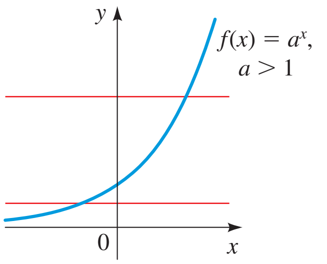


Figure 1 | $f(x) = a^x$ is one-to-one.

In this section we study the inverse functions of exponential functions.

Logarithmic Functions

Every exponential function $f(x) = a^x$, with $a > 0$ and $a \neq 1$, is a one-to-one function by the Horizontal Line Test (see Figure 1 for the case $a > 1$) and therefore has an inverse function. The inverse function f^{-1} is called the *logarithmic function with base a* and is denoted by $\log_a$. Recall from Section 2.8 that f^{-1} is defined by

$$f^{-1}(x) = y \quad \Leftrightarrow \quad f(y) = x$$

This leads to the following definition of logarithmic functions.

Definition of Logarithmic Functions

For $a > 0$ and $a \neq 1$, the **logarithmic function with base *a***, denoted by $\mathbf{log}_a$, is defined by

$$\log_a x = y \quad \Leftrightarrow \quad a^y = x$$

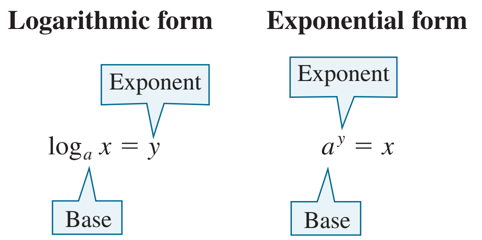
So $\log_a x$ is the *exponent* to which the base a must be raised to give x.

We read $\log_a x = y$ as "log base a of x is y."

By tradition the name of the logarithmic function is $\log_a$, not just a single letter. Also, we usually omit the parentheses in the function notation and write

$$\log_a(x) = \log_a x$$

When we use the definition of logarithms to switch back and forth between the **logarithmic form** $\log_a x = y$ and the **exponential form** $a^y = x$, it is helpful to notice that, in both forms, the base is the same.



Example 1 ■ Logarithmic and Exponential Forms

The logarithmic and exponential forms are equivalent equations: If one is true, then so is the other. So we can switch from one form to the other as in the following illustrations.

Logarithmic Form	Exponential Form
$\log_{10} 100{,}000 = 5$	$10^5 = 100{,}000$
$\log_2 8 = 3$	$2^3 = 8$
$\log_2\left(\frac{1}{8}\right) = -3$	$2^{-3} = \frac{1}{8}$
$\log_{1/3} 9 = -2$	$\left(\frac{1}{3}\right)^{-2} = 9$

Now Try Exercise 7 ■

In Example 1, the logarithmic form

$$\log_2 8 = 3$$

tells us that "2 raised to the power 3 is 8," so the exponential form is

$$2^3 = 8$$

x	$\log_{10} x$
10^4	4
10^3	3
10^2	2
10	1
1	0
10^{-1}	-1
10^{-2}	-2
10^{-3}	-3
10^{-4}	-4

Note It is important to understand that $\log_a x$ is an *exponent*. For example, the numbers in the right-hand column of the table in the margin are the logarithms (base 10) of the numbers in the left-hand column. This is the case for all bases, as the following example illustrates.

Example 2 ■ Evaluating Logarithms

(a) $\log_{10} 1000 = 3$ because $10^3 = 1000$

(b) $\log_2 32 = 5$ because $2^5 = 32$

(c) $\log_{10} 0.1 = -1$ because $10^{-1} = 0.1$

(d) $\log_{16} 4 = \frac{1}{2}$ because $16^{1/2} = 4$

(e) $\log_{1/2} 16 = -4$ because $\left(\frac{1}{2}\right)^{-4} = 16$

Now Try Exercises 9 and 11 ■

Inverse Function Property:

$$f^{-1}(f(x)) = x$$
$$f(f^{-1}(x)) = x$$

When we apply the cancellation equations of the Inverse Function Property described in Section 2.8 to $f(x) = a^x$ and $f^{-1}(x) = \log_a x$, we get

$$\log_a(a^x) = x \qquad (x \in \mathbb{R})$$
$$a^{\log_a x} = x \qquad (x > 0)$$

We list these and other properties of logarithms discussed in this section.

Properties of Logarithms

Property	Reason
1. $\log_a 1 = 0$	We must raise a to the power 0 to get 1.
2. $\log_a a = 1$	We must raise a to the power 1 to get a.
3. $\log_a a^x = x$	We must raise a to the power x to get a^x.
4. $a^{\log_a x} = x$	$\log_a x$ is the power to which a must be raised to get x.

Example 3 ■ Applying Properties of Logarithms

We illustrate the properties of logarithms when the base is 5.

$\log_5 1 = 0$ Property 1 $\qquad \log_5 5 = 1$ Property 2

$\log_5 5^8 = 8$ Property 3 $\qquad 5^{\log_5 12} = 12$ Property 4

Now Try Exercises 25 and 31 ■

■ Graphs of Logarithmic Functions

We first graph logarithmic functions by plotting points.

Example 4 ■ Graphing a Logarithmic Function by Plotting Points

Sketch the graph of $f(x) = \log_2 x$.

Solution To make a table of values, we choose the x-values to be powers of 2 so that we can easily find their logarithms. We plot these points and connect them with a smooth curve as in Figure 2 (on the next page).

x	$\log_2 x$
2^3	3
2^2	2
2	1
1	0
2^{-1}	-1
2^{-2}	-2
2^{-3}	-3
2^{-4}	-4

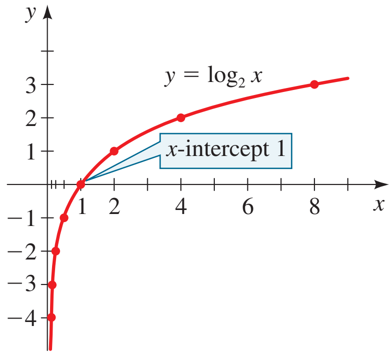
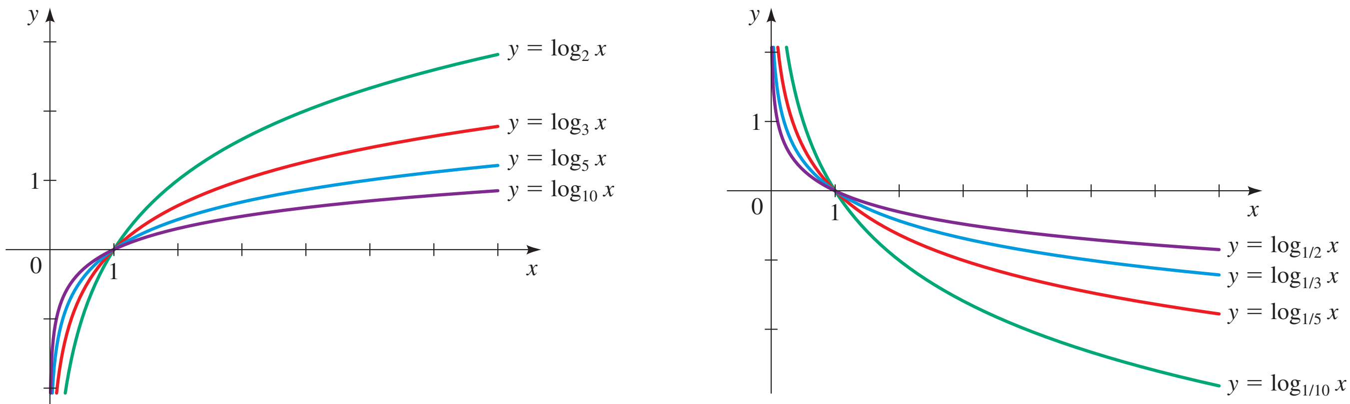


Figure 2

Now Try Exercise 49 ■

Figure 3 shows graphs of the family of logarithmic functions for various values of the base a.



(a) $y = \log_a x$ for $a = 2, 3, 5, 10$

(b) $y = \log_a x$ for $a = \frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{10}$

Figure 3 | Families of logarithmic functions

Note Each graph in Figure 3(b) is a reflection about the x-axis of the corresponding graph in Figure 3(a). This means that $\log_a x = -\log_{1/a} x$ (see Exercise 112). Thus, we mainly consider logarithmic functions with base $a > 1$.

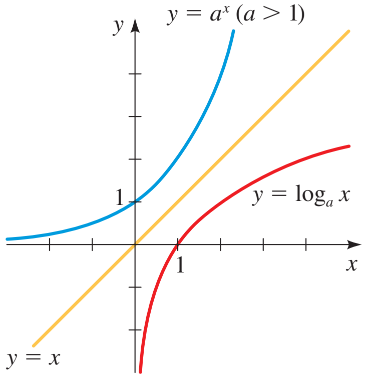


Figure 4 | Graph of the logarithmic function $f(x) = \log_a x$

Since $y = \log_a x$ is the inverse function of $y = a^x$, it follows that each logarithmic graph in Figure 3 is a reflection about the line $y = x$ of the corresponding exponential graph in Figure 4.1.2. This is illustrated in Figure 4 for the case $a > 1$. Also, since $f(x) = a^x$ has domain $\mathbb{R}$ and range $(0, \infty)$, it follows that its inverse function $y = \log_a x$ has domain $(0, \infty)$ and range $\mathbb{R}$.

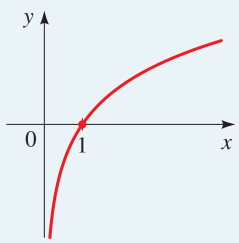
Since $\log_a 1 = 0$, the x-intercept of the graph of $y = \log_a x$ is 1. The y-axis is a vertical asymptote because $\log_a x \to -\infty$ as $x \to 0^+$. We summarize these observations in the following box.

Graphs of Logarithmic Functions $(a > 1)$

The logarithm function

$$f(x) = \log_a x \quad (a > 1)$$

has domain $(0, \infty)$, range $\mathbb{R}$, and x-intercept 1. The line $x = 0$ (the y-axis) is a vertical asymptote of f.



$f(x) = \log_a x$ for $a > 1$

For $a > 1$, the fact that $y = a^x$ is a very rapidly increasing function for $x > 0$ implies that $y = \log_a x$ is a very slowly increasing function for $x > 1$ (see Exercise 108).

In the next two examples we graph logarithmic functions starting with the basic graph of $y = \log_a x$ for $a > 1$ and use the transformations of Section 2.6.

Example 5 ■ Reflecting Graphs of Logarithmic Functions

Sketch the graph of each function. State the x-intercept, domain, range, and vertical asymptote.

(a) $g(x) = -\log_2 x$ **(b)** $h(x) = \log_2(-x)$

Solution

(a) We start with the graph of $f(x) = \log_2 x$ and reflect about the x-axis to get the graph of $g(x) = -\log_2 x$ in Figure 5(a). In general, an x-intercept is unchanged when we reflect a graph about the x-axis, so in this case the x-intercept is 1. From the graph we see that the domain of g is $(0, \infty)$, the range is the set $\mathbb{R}$ of all real numbers, and the line $x = 0$ is a vertical asymptote.

(b) We start with the graph of $f(x) = \log_2 x$ and reflect about the y-axis to get the graph of $h(x) = \log_2(-x)$ in Figure 5(b). We observe from the graph that the x-intercept is -1 because the point $(-1, 0)$ is the reflection about the y-axis of the point $(1, 0)$. From the graph we see that the domain of h is $(-\infty, 0)$, the range is the set $\mathbb{R}$ of all real numbers, and the line $x = 0$ is a vertical asymptote.

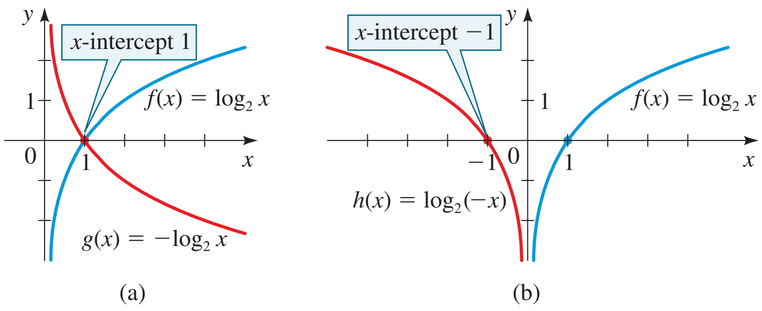


Figure 5

Now Try Exercise 63 ■

Example 6 ■ Shifting Graphs of Logarithmic Functions

Sketch the graph of each function. State the domain, range, and vertical asymptote.

(a) $g(x) = 2 + \log_5 x$ **(b)** $h(x) = \log_{10}(x - 3)$

Solution

(a) The graph of g is obtained from the graph of $f(x) = \log_5 x$ [Figure 3(a)] by shifting upward 2 units, as shown in Figure 6. Note, in particular, that the point $(1, 0)$ on the graph of f shifts to the point $(1, 2)$ on the graph of g. From the graph we see that the domain of g is $(0, \infty)$, the range is the set $\mathbb{R}$ of all real numbers, and the line $x = 0$ is a vertical asymptote.

(b) The graph of h is obtained from the graph of $f(x) = \log_{10} x$ [Figure 3(a)] by shifting to the right 3 units, as shown in Figure 7. Note, in particular, that the point $(1, 0)$ on the graph of f shifts to point $(4, 0)$ on the graph of h. From the

Mathematics in the Modern World

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Law Enforcement

Mathematics aids law enforcement in numerous and surprising ways, from the reconstruction of bullet trajectories to determining the time of death to calculating the probability that a DNA sample is from a particular person. One interesting use is in the search for missing persons. Someone who has been missing for several years might look quite different from their most recent available photograph. This is particularly true if the missing person is a child. Have you ever wondered what you will look like 5, 10, or 15 years from now?

Researchers have found that different parts of the body grow at different rates. For example, you have no doubt noticed that a baby's head is much larger relative to its body than an adult's. As another example, the ratio of arm length to height is $\frac{1}{3}$ in a child but about $\frac{2}{5}$ in an adult. By collecting data and analyzing the graphs, researchers are able to determine functions that model growth. As in all growth phenomena, exponential and logarithmic functions play a crucial role. For instance, the formula that relates arm length l to height h is $l = ae^{kh}$ where a and k are constants. By studying various physical characteristics of a person, mathematical biologists model each characteristic by a function that describes how it changes over time. Models of facial characteristics can be programmed into a computer to give a picture of how a person's appearance changes over time. These pictures aid law enforcement agencies in locating missing persons.

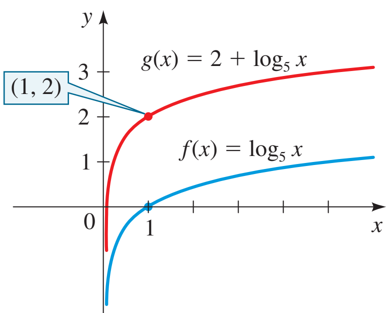

Figure 6

graph we see that the domain of h is $(3, \infty)$, the range is the set $\mathbb{R}$ of all real numbers, and the line $x = 3$ is a vertical asymptote.

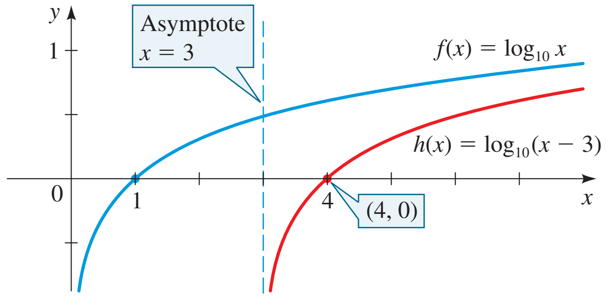

Figure 7

Now Try Exercises 65 and 69 ■

■ Common Logarithms

We now study logarithms with base 10.

Common Logarithm

The logarithm with base 10 is called the **common logarithm** and is denoted by omitting the base:

$$\log x = \log_{10} x$$

From the definition of logarithms we can find that

$$\log 10 = 1 \qquad \text{and} \qquad \log 100 = 2$$

But how do we find log 50? We need to find the exponent y such that $10^y = 50$. Clearly, 1 is too small and 2 is too large. So

$$1 < \log 50 < 2$$

To get a better approximation, we can experiment to find a power of 10 closer to 50. Fortunately, scientific calculators are equipped with a $\boxed{\text{LOG}}$ key that directly gives values of common logarithms.

Example 7 ■ Evaluating Common Logarithms

Use a calculator to find appropriate values of $f(x) = \log x$, and use these values to sketch the graph.

Solution We make a table of values, using a calculator to evaluate the function at those values of x that are not powers of 10. We plot those points and connect them by a smooth curve as shown in Figure 8.

x	$\log x$
0.01	−2
0.1	−1
0.5	−0.301
1	0
4	0.602
5	0.699
10	1

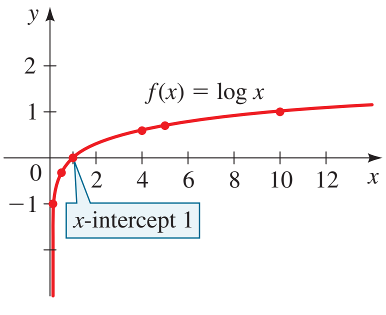

Figure 8

Now Try Exercise 51 ■

Human response to sound and light intensity is logarithmic.

Scientists model human response to stimuli (such as sound, light, or pressure) using logarithmic functions. For example, the intensity of a sound must be increased manyfold before we "feel" that the loudness has simply doubled. The psychologist Gustav Fechner formulated the law as

$$S = k \log \frac{I}{I_0}$$

where S is the subjective intensity of the stimulus, I is the physical intensity of the stimulus, I_0 stands for the threshold physical intensity, and k is a constant that is different for each sensory stimulus.

We study the decibel scale in more detail in Section 4.7.

Example 8 ■ Common Logarithms and Sound

The perception of the loudness B (in decibels, dB) of a sound with physical intensity I (in W/m^2) is given by

$$B = 10 \log \frac{I}{I_0}$$

where I_0 is the physical intensity of a barely audible sound. Find the decibel level (loudness) of a sound whose physical intensity I is 100 times that of I_0.

Solution We find the decibel level B by using the fact that $I = 100I_0$.

$$\begin{aligned} B &= 10 \log \frac{I}{I_0} && \text{Definition of } B \\ &= 10 \log \frac{100I_0}{I_0} && I = 100I_0 \\ &= 10 \log 100 && \text{Cancel } I_0 \\ &= 10 \cdot 2 = 20 && \text{Definition of log} \end{aligned}$$

The loudness of the sound is 20 dB.

Now Try Exercise 103 ■

■ Natural Logarithms

Of all possible bases a for logarithms, it turns out that the most convenient choice for the purposes of calculus is the number e, which we defined in Section 4.2.

The notation ln is an abbreviation for the Latin name *logarithmus naturalis*.

Natural Logarithm

The logarithm with base e is called the **natural logarithm** and is denoted by **ln**:

$$\ln x = \log_e x$$

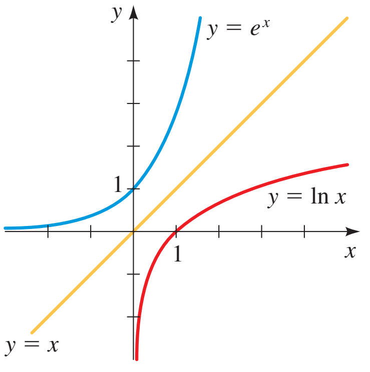

Figure 9 | Graph of the natural logarithmic function

The natural logarithmic function $y = \ln x$ is the inverse function of the natural exponential function $y = e^x$. Both functions are graphed in Figure 9. By the definition of inverse functions we have

$$\boxed{\ln x = y \quad \Leftrightarrow \quad e^y = x}$$

If we substitute $a = e$ and write "ln" for "$\log_e$" in the properties of logarithms mentioned earlier, we obtain the following properties of natural logarithms.

Properties of Natural Logarithms

Property	Reason
1. $\ln 1 = 0$	We must raise e to the power 0 to get 1.
2. $\ln e = 1$	We must raise e to the power 1 to get e.
3. $\ln e^x = x$	We must raise e to the power x to get e^x.
4. $e^{\ln x} = x$	$\ln x$ is the power to which e must be raised to get x.

Calculators are equipped with an LN key that directly gives the values of natural logarithms.

Example 9 ■ Evaluating the Natural Logarithm Function

(a) $\ln e^8 = 8$ Definition of natural logarithm

(b) $\ln \dfrac{1}{e^2} = \ln e^{-2} = -2$ Definition of natural logarithm

(c) $\ln 5 \approx 1.609$ Calculator

Now Try Exercise 47 ■

Example 10 ■ Finding the Domain of a Logarithmic Function

Find the domain of the function $f(x) = \ln(4 - x^2)$.

Solution As with any logarithmic function, $\ln x$ is defined when $x > 0$. Thus, the domain of f is

$$\{x \mid 4 - x^2 > 0\} = \{x \mid x^2 < 4\} = \{x \mid |x| < 2\}$$
$$= \{x \mid -2 < x < 2\} = (-2, 2)$$

Now Try Exercise 75 ■

Example 11 ■ Drawing the Graph of a Logarithmic Function

Draw the graph of the function $y = x \ln(4 - x^2)$, and use it to find the asymptotes and local maximum and minimum values.

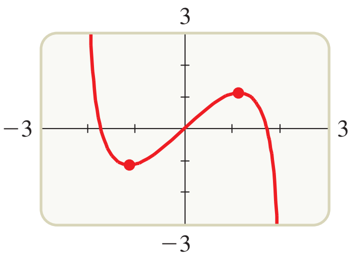

Figure 10 | $y = x \ln(4 - x^2)$

Solution As in Example 10 the domain of this function is the interval $(-2, 2)$, so we choose the viewing rectangle $[-3, 3]$ by $[-3, 3]$. The graph is shown in Figure 10, and from it we see that the lines $x = -2$ and $x = 2$ are vertical asymptotes.

The function has a local maximum point to the right of $x = 1$ and a local minimum point to the left of $x = -1$. From the graph we find that the local maximum value is

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Discovery Project ■ **Orders of Magnitude**

In this project we explore how to compare the sizes of real-world objects using logarithms. For example, how much bigger is an elephant than a flea? How much smaller is a man than a giant redwood? It is difficult to compare objects of such enormously varying sizes. In this project we learn how logarithms can be used to define the concept of "order of magnitude," which provides a simple and meaningful way of comparison. You can find the project at **www.stewartmath.com**.

approximately 1.13 and this occurs when $x \approx 1.15$. Similarly (or by noticing that the function is odd), we find that the local minimum value is about -1.13, and it occurs when $x \approx -1.15$.

Now Try Exercise 81

4.3 Exercises

Concepts

1. $\log x$ is the exponent to which the base 10 must be raised to get ________. So we can complete the following table for $\log x$.

x	10^3	10^2	10^1	10^0	10^{-1}	10^{-2}	10^{-3}	$10^{1/2}$
$\log x$								

2. The function $f(x) = \log_9 x$ is the logarithm function with base ________. So $f(9) =$ ________, $f(1) =$ ________, $f(\frac{1}{9}) =$ ________, $f(81) =$ ________, and $f(3) =$ ________.

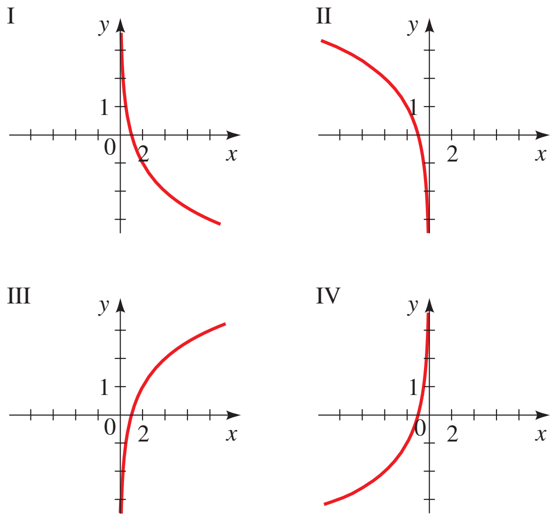
3. (a) $5^3 = 125$, so $\log_{\square} \square = \square$

(b) $\log_5 25 = 2$, so $\square^{\square} = \square$

4. Match the logarithmic function with its graph.

(a) $f(x) = \log_2 x$ **(b)** $f(x) = \log_2(-x)$

(c) $f(x) = -\log_2 x$ **(d)** $f(x) = -\log_2(-x)$



5. The natural logarithmic function $f(x) = \ln x$ has the ________ asymptote $x =$ ____.

6. The logarithmic function $f(x) = \ln(x - 1)$ has the ________ asymptote $x =$ ____.

Skills

7–8 ■ Logarithmic and Exponential Forms Complete the table by finding the appropriate logarithmic or exponential form of the equation, as in Example 1.

7.

Logarithmic Form	Exponential Form
$\log_8 8 = 1$	
$\log_8 64 = 2$	
	$8^{2/3} = 4$
	$8^3 = 512$
$\log_8(\frac{1}{8}) = -1$	
	$8^{-2} = \frac{1}{64}$

8.

Logarithmic Form	Exponential Form
	$4^3 = 64$
$\log_4 2 = \frac{1}{2}$	
	$4^{3/2} = 8$
$\log_4(\frac{1}{16}) = -2$	
$\log_4(\frac{1}{2}) = -\frac{1}{2}$	
	$4^{-5/2} = \frac{1}{32}$

9–16 ■ Exponential Form Express the equation in exponential form.

9. (a) $\log_3 81 = 4$ **(b)** $\log_{1/3} 1 = 0$

10. (a) $\log_5(\frac{1}{5}) = -1$ **(b)** $\log_{1/5} 5 = -1$

11. (a) $\log_8 2 = \frac{1}{3}$ **(b)** $\log_{10} 0.01 = -2$

12. (a) $\log_5(\frac{1}{125}) = -3$ **(b)** $\log_8 4 = \frac{2}{3}$

13. (a) $\log_3 5 = x$ **(b)** $\log_{1/6}(2y) = 3$

14. (a) $\log_{1/10} z = 2$ **(b)** $\log_{10} 3 = 2t$

15. (a) $\ln 10 = 2y$ **(b)** $\ln(3x + 1) = -2$

16. (a) $\ln(x - 2) = 3$ **(b)** $\ln(2x - 3) = 1$

17–24 ■ Logarithmic Form Express the equation in logarithmic form.

17. (a) $10^4 = 10{,}000$ **(b)** $5^{-2} = \frac{1}{25}$

18. (a) $6^2 = 36$ **(b)** $10^{-1} = \frac{1}{10}$

19. **(a)** $8^{-1} = \frac{1}{8}$ **(b)** $2^{-3} = \frac{1}{8}$

20. **(a)** $4^{-3/2} = 0.125$ **(b)** $\left(\frac{1}{2}\right)^{-3} = 8$

21. **(a)** $4^x = 70$ **(b)** $\left(\frac{1}{2}\right)^3 = w$

22. **(a)** $3^{2x} = 10$ **(b)** $10^{-4x} = 0.1$

23. **(a)** $e^x = 2$ **(b)** $e^3 = y$

24. **(a)** $e^{x+1} = 0.5$ **(b)** $e^{0.5x} = t$

25–34 ■ Evaluating Logarithms Evaluate the expression.

25. **(a)** $\log_2 2$ **(b)** $\log_5 1$ **(c)** $\log_{1/2} 2$

26. **(a)** $\log_3 3^7$ **(b)** $\log_4 64$ **(c)** $\log_{1/2} 0.25$

27. **(a)** $\log_6 36$ **(b)** $\log_9 81$ **(c)** $\log_7 7^{10}$

28. **(a)** $\log_2 32$ **(b)** $\log_5 5^{13}$ **(c)** $\log_6 1$

29. **(a)** $\log_3\left(\frac{1}{27}\right)$ **(b)** $\log_{1/3} 27$ **(c)** $\log_7 \sqrt{7}$

30. **(a)** $\log_5 125$ **(b)** $\log_{49} 7$ **(c)** $\log_9 \sqrt{3}$

31. **(a)** $3^{\log_3 5}$ **(b)** $5^{\log_5 27}$ **(c)** $e^{\ln 10}$

32. **(a)** $e^{\ln\sqrt{3}}$ **(b)** $e^{\ln(1/\pi)}$ **(c)** $10^{\log 13}$

33. **(a)** $\log_8 0.25$ **(b)** $\ln e^4$ **(c)** $\ln(1/e)$

34. **(a)** $\log_4 \sqrt{2}$ **(b)** $\log_4\left(\frac{1}{2}\right)$ **(c)** $\log_4 8$

35–44 ■ Logarithmic Equations Use the definition of the logarithmic function to find x.

35. **(a)** $\log_6 x = 2$ **(b)** $\log_{10} 0.001 = x$

36. **(a)** $\log_{1/3} x = 0$ **(b)** $\log_4 1 = x$

37. **(a)** $\ln x = 3$ **(b)** $\ln e^2 = x$

38. **(a)** $\ln x = -1$ **(b)** $\ln(1/e) = x$

39. **(a)** $\log_4\left(\frac{1}{64}\right) = x$ **(b)** $\log_{1/2} x = 3$

40. **(a)** $\log_9\left(\frac{1}{3}\right) = x$ **(b)** $\log_9 x = 0.5$

41. **(a)** $\log_2\left(\frac{1}{2}\right) = x$ **(b)** $\log_{10} x = -3$

42. **(a)** $\log_x 1000 = 3$ **(b)** $\log_x 25 = 2$

43. **(a)** $\log_x 16 = 4$ **(b)** $\log_x 8 = \frac{3}{2}$

44. **(a)** $\log_x 6 = \frac{1}{2}$ **(b)** $\log_x 3 = \frac{1}{3}$

45–48 ■ Evaluating Logarithms Use a calculator to evaluate the expression, correct to four decimal places.

45. **(a)** $\log 2$ **(b)** $\log 35.2$ **(c)** $\log\left(\frac{2}{3}\right)$

46. **(a)** $\log 50$ **(b)** $\log \sqrt{2}$ **(c)** $\log(3\sqrt{2})$

47. **(a)** $\ln 5$ **(b)** $\ln 25.3$ **(c)** $\ln(1 + \sqrt{3})$

48. **(a)** $\ln 27$ **(b)** $\ln 7.39$ **(c)** $\ln 54.6$

49–54 ■ Graphing Logarithmic Functions Sketch the graph of the function by plotting points.

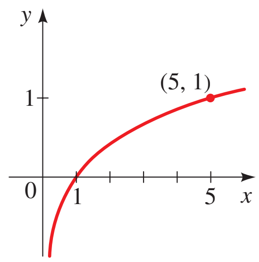
49. $f(x) = \log_3 x$ **50.** $g(x) = \log_4 x$

51. $f(x) = 2 \log x$ **52.** $f(x) = \log_{1/2} x$

53. $g(x) = 1 + \log x$ **54.** $g(x) = -2 + \log_2 x$

55–58 ■ Finding Logarithmic Functions Find the function of the form $y = \log_a x$ whose graph is given.

55.



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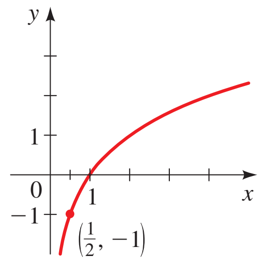

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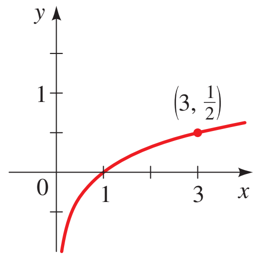

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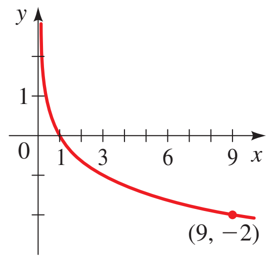
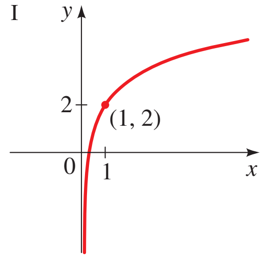
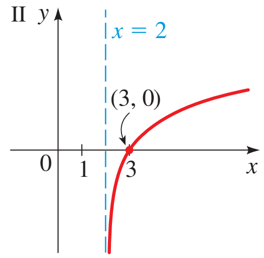


59–60 ■ Graphing Logarithmic Functions Match the logarithmic function with one of the graphs labeled I or II.

59. $f(x) = 2 + \ln x$ **60.** $f(x) = \ln(x - 2)$





61. Graphing Draw the graph of $y = 4^x$, then use it to draw the graph of $y = \log_4 x$.

62. Graphing Draw the graph of $y = 3^x$, then use it to draw the graph of $y = \log_3 x$.

63–74 ■ Graphing Logarithmic Functions Graph the function, not by plotting points, but by starting from the graphs in Figures 3 and 9. State the domain, range, and vertical asymptote.

63. $g(x) = \log_5(-x)$ **64.** $f(x) = -\log_{10} x$

65. $f(x) = \log_2(x - 4)$ **66.** $g(x) = \ln(x + 2)$

67. $h(x) = \ln(x + 5)$ **68.** $f(x) = 2 - \log_{1/3} x$

69. $y = 2 + \log_3 x$ **70.** $y = 1 - \log_{10} x$

71. $y = \log_3(x - 1) - 2$ **72.** $y = 1 + \ln(-x)$

73. $y = |\ln x|$ **74.** $y = \ln|x|$

75–80 ■ Domain Find the domain of the function.

75. $f(x) = \log(x + 3)$ **76.** $f(x) = \log_5(8 - 2x)$

77. $g(x) = \log_3(x^2 - 1)$ **78.** $g(x) = \ln(x - x^2)$

79. $h(x) = \ln x + \ln(2 - x)$

80. $h(x) = \sqrt{x - 2} - \log_5(10 - x)$

81–86 ■ Graphing Logarithmic Functions Draw the graph of the function in a suitable viewing rectangle, and use it to find the domain, the asymptotes, and the local maximum and minimum values.

81. $y = \log(1 - x^2)$ **82.** $y = \ln(x^2 - x)$

83. $y = x + \ln x$ **84.** $y = x(\ln x)^2$

85. $y = \dfrac{\ln x}{x}$ **86.** $y = x\log(x + 10)$

Skills Plus

87–90 ■ Expressing a Function as a Composition Find functions f and g such that $F = f \circ g$.

87. $F(x) = \ln(x^2 + 1)$ **88.** $F(x) = (\ln x)^3$

89. $F(x) = \sqrt{1 + |\ln x|}$ **90.** $F(x) = 5 - \log\sqrt{x}$

91–94 ■ Domain of a Composition Find the functions $f \circ g$ and $g \circ f$ and their domains.

91. $f(x) = 2^x, \quad g(x) = x + 1$

92. $f(x) = 3^x, \quad g(x) = x^2 + 1$

93. $f(x) = \log_2 x, \quad g(x) = x - 2$

94. $f(x) = \log x, \quad g(x) = x^2$

95. Rates of Growth Compare the rates of growth of the functions $f(x) = \ln x$ and $g(x) = \sqrt{x}$ by drawing their graphs on a common screen using the viewing rectangle $[-1, 30]$ by $[-1, 6]$.

96. Rates of Growth

(a) By drawing the graphs of the functions

$$f(x) = 1 + \ln(1 + x) \qquad \text{and} \qquad g(x) = \sqrt{x}$$

in a suitable viewing rectangle, show that even when a logarithmic function starts out higher than a root function, it is ultimately overtaken by the root function.

(b) Find, rounded to two decimal places, the solutions of the equation $\sqrt{x} = 1 + \ln(1 + x)$.

97–98 ■ Family of Functions A family of functions is given. **(a)** Draw graphs of the family for $c = 1, 2, 3$, and 4. **(b)** How are the graphs in part (a) related?

97. $f(x) = \log(cx)$ **98.** $f(x) = c\log x$

99–100 ■ Inverse Functions A function $f(x)$ is given. **(a)** Find the domain of the function f. **(b)** Find the inverse function of f.

99. $f(x) = \log_2(\log_{10} x)$ **100.** $f(x) = \ln(\ln(\ln x))$

101. Inverse Functions

(a) Find the inverse of the function $f(x) = \dfrac{2^x}{1 + 2^x}$.

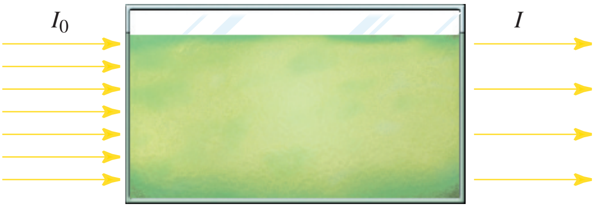
(b) What is the domain of the inverse function?

Applications

102. Absorption of Light A spectrophotometer measures the concentration of a sample dissolved in water by shining a light through it and recording the amount of light that emerges. In other words, if we know the amount of light that is absorbed, we can calculate the concentration of the sample. For a certain substance the concentration (in moles per liter, mol/L) is found by using the formula

$$C = -2500\ln\frac{I}{I_0}$$

where I_0 is the intensity of the incident light and I is the intensity of light that emerges. Find the concentration of the substance if the intensity I is 70% of I_0.



103. Carbon Dating The age of an ancient artifact can be determined by the amount of radioactive carbon-14 remaining in it. If D_0 is the original amount of carbon-14 and D is the amount remaining, then the artifact's age A (in years) is given by

$$A = -8267\ln\frac{D}{D_0}$$

Find the age of an object if the amount D of carbon-14 that remains in the object is 73% of the original amount D_0.

104. Bacteria Colony A certain strain of bacteria divides every 3 hours. If a colony is started with 50 bacteria, then the time t (in hours) required for the colony to grow to N bacteria is

$$t = 3\,\frac{\log(N/50)}{\log 2}$$

Find the time required for the colony to grow to a million bacteria.

105. Investment The time required to double the amount of an investment at an interest rate r, compounded continuously, is

$$t = \frac{\ln 2}{r}$$

Find the time required to double an investment at 6%, 7%, and 8%.

106. Charging a Battery The rate at which a battery charges is slower the closer the battery is to its maximum charge C_0. The time (in hours) required to charge a fully discharged battery to a charge C is given by

$$t = -k\ln\left(1 - \frac{C}{C_0}\right)$$

where k is a positive constant that depends on the battery. For a certain battery, $k = 0.25$. If this battery is fully discharged, how long will it take to charge to 90% of its maximum charge C_0?

107. Difficulty of a Task The difficulty in "acquiring a target" (such as using a mouse to click on an icon on a computer screen) depends on the distance to the target and the size of the target. According to Fitts's Law, the index of difficulty (ID) is given by

$$\text{ID} = \frac{\log(2A/W)}{\log 2}$$

where W is the width of the target and A is the distance to the center of the target. Find the ID of clicking on an icon that is 5 mm wide and the ID for an icon that is 10 mm wide. In each case assume that the mouse pointer is 100 mm from the icon. Which task is more difficult?

Discuss ■ Discover ■ Prove ■ Write

108. Discuss: The Height of the Graph of a Logarithmic Function Suppose that the graph of $y = 2^x$ is drawn on a coordinate plane where the unit of measurement is an inch.

(a) Show that at a distance 2 ft to the right of the origin the height of the graph is about 265 mi.

(b) If the graph of $y = \log_2 x$ is drawn on the same set of axes, how far to the right of the origin do we have to go before the height of the curve reaches 2 ft?

109. Discuss: The Googolplex A **googol** is 10^{100}, and a **googolplex** is 10^{googol}. Find

$$\log(\log(\text{googol})) \qquad \text{and} \qquad \log(\log(\log(\text{googolplex})))$$

110. Discuss: Comparing Logarithms Without using a calculator, determine which is larger, $\log_5 24$ or $\log_4 17$.

PS *Try to recognize something familiar.* Consider integers close to 24 or 17 for which you can easily calculate their respective logarithms.

111. Discuss ■ Discover: The Number of Digits in an Integer Compare log 1000 to the number of digits in 1000. Do the same for 10,000. How many digits does any number between 1000 and 10,000 have? Between what two values must the common logarithm of such a number lie? Use your observations to explain why the number of digits in any positive integer x is $[\![\log x]\!] + 1$. (The symbol $[\![n]\!]$ is the greatest integer function defined in Section 2.2.) How many digits does the number 2^{100} have?

112. Discover ■ Prove: A Logarithmic Identity For $a > 0$, prove that for all $x > 0$,

$$\log_{1/a} x = -\log_a x$$

How are the graphs of $f(x) = \log_a x$ and $g(x) = \log_{1/a} x$ related? Graph f and g for $a = 4$ to confirm your answer.

PS *Try to recognize something familiar.* Relate the familiar identity $a^{-1} = 1/a$ to the definition of the logarithm.

4.4 Laws of Logarithms

■ Laws of Logarithms ■ Expanding and Combining Logarithmic Expressions
■ Change of Base Formula

In this section we study properties of logarithms. These properties give logarithmic functions a wide range of applications, as we will see in Sections 4.6 and 4.7.

■ Laws of Logarithms

Since logarithms are exponents, the Laws of Exponents give rise to the following Laws of Logarithms.

Laws of Logarithms

Let a be a positive number, with $a \neq 1$. Let A, B, and C be any real numbers with $A > 0$ and $B > 0$.

Law	Description
1. $\log_a(AB) = \log_a A + \log_a B$	The logarithm of a product of numbers is the sum of the logarithms of the numbers.
2. $\log_a\left(\dfrac{A}{B}\right) = \log_a A - \log_a B$	The logarithm of a quotient of numbers is the difference of the logarithms of the numbers.
3. $\log_a(A^C) = C \log_a A$	The logarithm of a power of a number is the exponent times the logarithm of the number.

Proof We make use of the property $\log_a a^x = x$ from Section 4.3.

Law 1 Let $\log_a A = u$ and $\log_a B = v$. When written in exponential form, these equations become

$$a^u = A \qquad \text{and} \qquad a^v = B$$

Thus
$$\log_a(AB) = \log_a(a^u a^v) = \log_a(a^{u+v})$$
$$= u + v = \log_a A + \log_a B$$

Law 2 Using Law 1, we have

$$\log_a A = \log_a\left[\left(\frac{A}{B}\right)B\right] = \log_a\left(\frac{A}{B}\right) + \log_a B$$

so
$$\log_a\left(\frac{A}{B}\right) = \log_a A - \log_a B$$

Law 3 Let $\log_a A = u$. Then $a^u = A$, so

$$\log_a(A^C) = \log_a(a^u)^C = \log_a(a^{uC}) = uC = C\log_a A$$

■

Example 1 ■ Using the Laws of Logarithms to Evaluate Expressions

Evaluate each expression (without using a calculator).

(a) $\log_4 2 + \log_4 32$ **(b)** $\log_2 80 - \log_2 5$ **(c)** $-\frac{1}{3}\log 1000$

Solution

(a) $\log_4 2 + \log_4 32 = \log_4(2 \cdot 32)$ Law 1

$= \log_4 64 = 3$ Because $64 = 4^3$

(b) $\log_2 80 - \log_2 5 = \log_2(\frac{80}{5})$ Law 2

$= \log_2 16 = 4$ Because $16 = 2^4$

(c) $-\frac{1}{3}\log 1000 = \log 1000^{-1/3}$ Law 3

$= \log\frac{1}{10}$ Property of negative exponents

$= -1$ Because $\frac{1}{10} = 10^{-1}$

Now Try Exercises 9, 11, and 13 ■

■ Expanding and Combining Logarithmic Expressions

The Laws of Logarithms allow us to write the logarithm of a product or a quotient as the sum or difference of logarithms. This process, called *expanding* a logarithmic expression, is illustrated in the next example.

Example 2 ■ Expanding Logarithmic Expressions

Use the Laws of Logarithms to expand each expression.

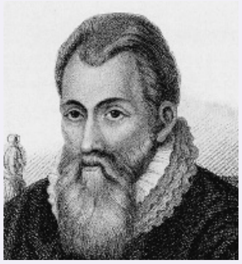
(a) $\log_2(6x)$ **(b)** $\log_5(x^3y^6)$ **(c)** $\ln\left(\frac{ab}{\sqrt[3]{c}}\right)$

Solution

(a) $\log_2(6x) = \log_2 6 + \log_2 x$ Law 1

(b) $\log_5(x^3y^6) = \log_5 x^3 + \log_5 y^6$ Law 1

$= 3\log_5 x + 6\log_5 y$ Law 3



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JOHN NAPIER (1550–1617) was a Scottish landowner for whom mathematics was a hobby. We know him today because of his key invention: logarithms. He published his ideas in 1614 under the title *A Description of the Marvelous Rule of Logarithms*. In Napier's time, logarithms were used exclusively for simplifying complicated calculations. For example, to multiply two large numbers, we would write them as powers of 10. The exponents are simply the logarithms of the numbers. For instance,

$$4532 \times 57783$$
$$\approx 10^{3.65629} \times 10^{4.76180}$$
$$= 10^{8.41809}$$
$$\approx 261{,}872{,}564$$

The idea is that multiplying powers of 10 is easy (we simply add their exponents). Napier produced extensive tables giving the logarithms (or exponents) of numbers. Since the advent of calculators and computers, logarithms are no longer used for this purpose. The logarithmic functions, however, have found many applications, some of which are described in this chapter.

Napier wrote on many topics. One of his more colorful works is a book entitled *A Plaine Discovery of the Whole Revelation of Saint John*, in which he predicted that the world would end in the year 1700.

(c) $$\ln\left(\frac{ab}{\sqrt[3]{c}}\right) = \ln(ab) - \ln\sqrt[3]{c} \qquad \text{Law 2}$$
$$= \ln a + \ln b - \ln c^{1/3} \qquad \text{Law 1}$$
$$= \ln a + \ln b - \tfrac{1}{3}\ln c \qquad \text{Law 3}$$

Now Try Exercises 23, 31, and 37 ■

The Laws of Logarithms also allow us to reverse the process of expanding that was illustrated in Example 2. That is, we can write sums and differences of logarithms as a single logarithm. This process, called *combining* logarithmic expressions, is illustrated in the next example.

Example 3 ■ Combining Logarithmic Expressions

Use the Laws of Logarithms to combine each expression into a single logarithm.

(a) $3\log x + \frac{1}{2}\log(x+1)$ **(b)** $3\ln s + \frac{1}{2}\ln t - 4\ln(t^2+1)$

Solution

(a) $$3\log x + \tfrac{1}{2}\log(x+1) = \log x^3 + \log(x+1)^{1/2} \qquad \text{Law 3}$$
$$= \log(x^3(x+1)^{1/2}) \qquad \text{Law 1}$$

(b) $$3\ln s + \tfrac{1}{2}\ln t - 4\ln(t^2+1) = \ln s^3 + \ln t^{1/2} - \ln(t^2+1)^4 \qquad \text{Law 3}$$
$$= \ln(s^3 t^{1/2}) - \ln(t^2+1)^4 \qquad \text{Law 1}$$
$$= \ln\frac{s^3\sqrt{t}}{(t^2+1)^4} \qquad \text{Law 2}$$

Now Try Exercises 51 and 53 ■

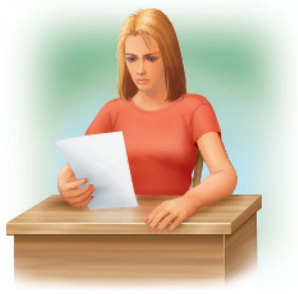
Warning Although the Laws of Logarithms tell us how to compute the logarithm of a product or a quotient, *there is no corresponding rule for the logarithm of a sum or a difference*. For instance,

$$\log_a(x+y) \neq \log_a x + \log_a y$$

In fact, we know that the right side is equal to $\log_a(xy)$. Also, don't improperly simplify quotients or powers of logarithms. For instance,

$$\frac{\log 6}{\log 2} \neq \log\frac{6}{2} \qquad \text{and} \qquad (\log_2 x)^3 \neq 3\log_2 x$$

Logarithmic functions are used to model a variety of situations involving human behavior. One such behavior is how quickly we forget things we have learned. For example, if you learn algebra at a certain performance level (say, 90% on a test) and then don't use algebra for a while, how much will you retain after a week, a month, or a year? Hermann Ebbinghaus (1850–1909) studied this phenomenon and formulated the law described in the next example.



Forgetting what we've learned depends on how long ago we learned it.

Example 4 ■ The Law of Forgetting

If a task is learned at a performance level P_0, then after a time interval t the performance level P satisfies

$$\log P = \log P_0 - c\log(t+1)$$

where c is a constant that depends on the type of task and t is measured in months.

(a) Solve for P.

(b) If your score on a history test is 90, what score would you expect to get on a similar test after two months? After a year? (Assume that $c = 0.2$.)

Solution

(a) We first combine the right-hand side.

$$\begin{aligned} \log P &= \log P_0 - c\log(t+1) && \text{Given equation} \\ \log P &= \log P_0 - \log(t+1)^c && \text{Law 3} \\ \log P &= \log \frac{P_0}{(t+1)^c} && \text{Law 2} \\ P &= \frac{P_0}{(t+1)^c} && \text{Because log is one-to-one} \end{aligned}$$

(b) Here $P_0 = 90$, $c = 0.2$, and t is measured in months.

$$\text{In 2 months:} \qquad t = 2 \qquad \text{and} \qquad P = \frac{90}{(2+1)^{0.2}} \approx 72$$

$$\text{In 1 year:} \qquad t = 12 \qquad \text{and} \qquad P = \frac{90}{(12+1)^{0.2}} \approx 54$$

Your expected scores after 2 months and after 1 year are 72 and 54, respectively.

Now Try Exercise 73 ■

■ Change of Base Formula

For some purposes we find it useful to change from logarithms in one base to logarithms in another base.

We may write the Change of Base Formula as

$$\log_b x = \frac{1}{\log_a b} \cdot \log_a x$$

So $\log_b x$ is just a constant multiple of $\log_a x$; the constant is $\dfrac{1}{\log_a b}$.

Change of Base Formula

$$\log_b x = \frac{\log_a x}{\log_a b} \qquad (a > 0,\ b > 0)$$

Proof Let $y = \log_b x$ and consider the following.

$$\begin{aligned} y &= \log_b x && \text{Given} \\ b^y &= x && \text{Exponential form} \\ \log_a b^y &= \log_a x && \text{Take } \log_a \text{ of each side} \\ y\log_a b &= \log_a x && \text{Law 3} \\ y &= \frac{\log_a x}{\log_a b} && \text{Divide by } \log_a b \end{aligned}$$

So, if $y = \log_b x$, then $y = (\log_a x)/(\log_a b)$, and this proves the formula. ■

In particular, if we put $x = a$, then $\log_a a = 1$, and this formula becomes

$$\log_b a = \frac{1}{\log_a b}$$

We can now evaluate a logarithm to *any* base by using the Change of Base Formula to express the logarithm in terms of common logarithms or natural logarithms.

Example 5 ■ Evaluating Logarithms with the Change of Base Formula

Use the Change of Base Formula to express each logarithm in terms of common or natural logarithms, and then evaluate, rounded to five decimal places.

(a) $\log_8 5$ **(b)** $\log_9 20$

Solution

Check that you get the same answers in Example 5 if you use either log or ln with the Change of Base Formula.

(a) We use the Change of Base Formula with $b = 8$ and $a = 10$:

$$\log_8 5 = \frac{\log_{10} 5}{\log_{10} 8} \approx 0.77398$$

(b) We use the Change of Base Formula with $b = 9$ and $a = e$:

$$\log_9 20 = \frac{\ln 20}{\ln 9} \approx 1.36342$$

Now Try Exercises 59 and 61 ■

Example 6 ■ Using the Change of Base Formula

Let $f(x) = \log_6 x$ and $g(x) = (\ln x)/(\ln 6)$. Explain why $f(x) = g(x)$, and confirm this graphically by graphing both functions on the same screen.

Solution By the Change of Base formula we have

$$f(x) = \log_6 x = \frac{\ln x}{\ln 6} = g(x)$$

The graphs of f and g in Figure 1 confirm that $f(x) = g(x)$ for every x in their domain.

Now Try Exercise 67 ■

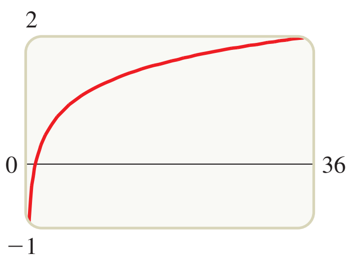


Figure 1 | $f(x) = \log_6 x = \dfrac{\ln x}{\ln 6} = g(x)$

4.4 Exercises

■ Concepts

1. The logarithm of a product of two numbers is the same as the ________ of the logarithms of these numbers. So $\log_5(25 \cdot 125) =$ ________ $+$ ________.

2. The logarithm of a quotient of two numbers is the same as the ________ of the logarithms of these numbers. So $\log_5\left(\frac{25}{125}\right) =$ ________ $-$ ________.

3. The logarithm of a number raised to a power is the same as the ________ times the logarithm of the number. So $\log_5 25^{10} =$ ________ $\cdot$ ________.

4. We can expand $\log \dfrac{x^2 y}{z}$ to get ________________.

5. We can combine $2 \log x + \log y - \log z$ to get ________.

6. (a) To express $\log_7 12$ in terms of common logarithms, we use the Change of Base Formula to write

$$\log_7 12 = \frac{\log \square}{\log \square} \approx \text{________}$$

(b) Do we get the same answer if we perform the calculation in part (a) using ln in place of log?

7–8 ■ *True or False*?

7. (a) $\log(A + B)$ is the same as $\log A + \log B$.

(b) $\log AB$ is the same as $\log A + \log B$.

8. (a) $\log \dfrac{A}{B}$ is the same as $\log A - \log B$.

(b) $\dfrac{\log A}{\log B}$ is the same as $\log A - \log B$.

Skills

9–22 ■ Evaluating Logarithms Use the Laws of Logarithms to evaluate the expression.

9. $\log 50 + \log 200$
10. $\log_6 9 + \log_6 24$
11. $\log_2 60 - \log_2 15$
12. $\log_3 135 - \log_3 45$
13. $\frac{1}{4}\log_3 81$
14. $-\frac{1}{3}\log_3 27$
15. $\log_5 \sqrt{5}$
16. $\log_5 \dfrac{1}{\sqrt{125}}$
17. $\log_2 6 - \log_2 15 + \log_2 20$
18. $\log_3 100 - \log_3 18 - \log_3 50$
19. $\log_4 16^{100}$
20. $\log_2 8^{33}$
21. $\log(\log 10^{10,000})$
22. $\ln(\ln e^{e^{200}})$

23–48 ■ Expanding Logarithmic Expressions Use the Laws of Logarithms to expand the expression.

23. $\log_3(8x)$
24. $\log_6(7r)$
25. $\log_3(2xy)$
26. $\log_5(4st)$
27. $\ln a^3$
28. $\log\sqrt{t^5}$
29. $\log_3\sqrt{xyz}$
30. $\log_5(xy)^6$
31. $\ln(a^3b^2)$
32. $\ln(y^2\sqrt{x})$
33. $\log_2\left(\dfrac{4a}{b}\right)$
34. $\log_6\left(\dfrac{y}{6z}\right)$
35. $\log_8\left(\dfrac{a^3b^2}{c}\right)$
36. $\log_7\left(\dfrac{3x^4y^2}{2z^3}\right)$
37. $\log_3\left(\dfrac{\sqrt{3x^5}}{y}\right)$
38. $\log \dfrac{y^3}{\sqrt{2x}}$
39. $\log \dfrac{x^3y^4}{z^6}$
40. $\log_a\left(\dfrac{x^2}{yz^3}\right)$
41. $\ln\sqrt{x^4+2}$
42. $\log\sqrt[3]{x^2+4}$
43. $\log\sqrt{\dfrac{x+z}{y}}$
44. $\ln\dfrac{4x^2}{x^2+3}$
45. $\ln\sqrt[3]{\dfrac{x^2+y^2}{x+y}}$
46. $\ln\dfrac{x^2}{\sqrt{x+1}}$
47. $\log\sqrt{\dfrac{x^2+4}{(x^2+1)(x^3-7)^2}}$
48. $\log\sqrt{x\sqrt{y\sqrt{z}}}$

49–58 ■ Combining Logarithmic Expressions Use the Laws of Logarithms to combine the expression.

49. $\log_4 6 + 2\log_4 7$
50. $\frac{1}{2}\log_2 5 - 2\log_2 7$
51. $2\log x - 3\log(x+1)$
52. $3\ln 2 + 2\ln x - \frac{1}{2}\ln(x+4)$
53. $\log(x+1) + \log(x-1) - 3\log x$
54. $\ln(x+3) - \ln(x^2-9)$
55. $\frac{1}{2}[\log_5(x+2) - \log_5(x^2+4) - \log_5 x]$
56. $4(\log_3 a - 3\log_3 b + 2\log_3 c)$
57. $\frac{1}{3}\log(x+2)^3 + \frac{1}{2}[\log x^4 - \log(x^2-x-6)^2]$
58. $\log_a b + c\log_a d - r\log_a s$

59–66 ■ Change of Base Formula Use the Change of Base Formula to express the given logarithm in terms of common or natural logarithms, and then evaluate. State your answer rounded to six decimal places.

59. $\log_5 10$
60. $\log_{14} 7$
61. $\log_9 4$
62. $\log_5 30$
63. $\log_7 2.61$
64. $\log_6 532$
65. $\log_4 125$
66. $\log_{12} 2.5$

67. Change of Base Formula Use the Change of Base Formula to show that

$$\log_3 x = \frac{\ln x}{\ln 3}$$

Then use this fact to draw the graph of the function $f(x) = \log_3 x$.

Skills Plus

68. Families of Functions Draw graphs of the family of functions $y = \log_a x$ for $a = 2, e, 5$, and 10 on the same screen, using the viewing rectangle $[0, 5]$ by $[-3, 3]$. How are these graphs related?

69. Change of Base Formula Use the Change of Base Formula to show that

$$\log e = \frac{1}{\ln 10}$$

70–72 ■ Logarithmic Identities Let $a, b, c, d > 0$ and n a positive integer. Prove the identity.

70. $(\log_a b)(\log_b c)(\log_c d) = \dfrac{\log d}{\log a}$

71. $\dfrac{1}{\log_a x} + \dfrac{1}{\log_b x} + \dfrac{1}{\log_c x} = \dfrac{1}{\log_{abc} x}$

72. $\log_{a^n} x = \dfrac{1}{n}\log_a x$

Applications

73. Wealth Distribution Vilfredo Pareto (1848–1923) observed that most of the wealth of a country is owned by a few members of the population. **Pareto's principle** is

$$\log P = \log c - k\log W$$

where W is the wealth level (how much money a person has) and P is the number of people in the population having that much money.

(a) Solve the equation for P.

(b) Assume that $k = 2.1$ and $c = 8000$ and that W is measured in millions of dollars. Use part (a) to find the number of people who have \$2 million or more. How many people have \$10 million or more?

74. Forgetting Use the Law of Forgetting (Example 4) to estimate a student's score on a biology test two years after the student got a score of 80 on a test covering the same material. Assume that $c = 0.3$ and t is measured in months.

75. Magnitude of Stars The magnitude M of a star is a measure of how bright a star appears to the human eye. It is defined by

$$M = -2.5 \log \frac{B}{B_0}$$

where B is the actual brightness of the star and B_0 is a constant.

(a) Expand the right-hand side of the equation.

(b) Use part (a) to show that the brighter a star, the less its magnitude.

(c) Betelgeuse is about 100 times brighter than Albiero. Use part (a) to show that Betelgeuse is 5 magnitudes less bright than Albiero.

76. Biodiversity Some biologists model the number of species S in a fixed area A (such as an island) by the species-area relationship

$$\log S = \log c + k \log A$$

where c and k are positive constants that depend on the type of species and habitat.

(a) Solve the equation for S.

(b) Use part (a) to show that if $k = 3$, then doubling the area increases the number of species eightfold.

■ Discuss ■ Discover ■ Prove ■ Write

77. Discuss: True or False? Discuss each equation, and determine whether it is true for all possible values of the variables. (Ignore values of the variables for which any term is undefined.)

(a) $\log \dfrac{x}{y} = \dfrac{\log x}{\log y}$

(b) $\log_2(x - y) = \log_2 x - \log_2 y$

(c) $\log_5\left(\dfrac{a}{b^2}\right) = \log_5 a - 2\log_5 b$

(d) $\log 2^z = z \log 2$

(e) $(\log P)(\log Q) = \log P + \log Q$

(f) $\dfrac{\log a}{\log b} = \log a - \log b$

(g) $(\log_2 7)^x = x \log_2 7$

(h) $\log_a a^a = a$

(i) $\log(x - y) = \dfrac{\log x}{\log y}$

(j) $-\ln \dfrac{1}{A} = \ln A$

78. Discuss: Find the Error What is wrong with the following argument?

$$\begin{aligned} \log 0.1 &< 2 \log 0.1 \\ &= \log(0.1)^2 \\ &= \log 0.01 \\ \log 0.1 &< \log 0.01 \\ 0.1 &< 0.01 \end{aligned}$$

79. Prove: Shifting, Shrinking, and Stretching Graphs of Functions Let $f(x) = x^2$. Show that

$$f(2x) = 4f(x)$$

and explain how this shows that shrinking the graph of f horizontally has the same effect as stretching it vertically. Then use the identities $e^{2+x} = e^2e^x$ and $\ln(2x) = \ln 2 + \ln x$ to show that for $g(x) = e^x$ a horizontal shifting is the same as a vertical stretching and for $h(x) = \ln x$ a horizontal shrinking is the same as a vertical shifting.

80. Prove: A Logarithmic Identity Show that

$$-\ln\left(x - \sqrt{x^2 - 1}\right) = \ln\left(x + \sqrt{x^2 - 1}\right)$$

PS *Work backward.* Assume the equation holds; then use the properties of logarithms and the rules of algebra to arrive at an equivalent true equation.

81–82 ■ Discuss ■ Prove: Linearizing Exponential and Power Curves These exercises are about "straightening" an exponential or power curve by applying a logarithmic function to the appropriate variable(s). Prove the statement.

81. If the points (x, y) are on the exponential curve $y = Ce^{kx}$, then the points $(x, \ln y)$ are on the line

$$Y = kX + \ln C$$

82. If the points (x, y) are on the power curve $y = ax^n$, then the points $(\ln x, \ln y)$ are on the line

$$Y = nX + \ln a.$$

4.5 Exponential and Logarithmic Equations

■ Exponential Equations ■ Logarithmic Equations ■ Compound Interest

In this section we solve equations that involve exponential or logarithmic functions. In the next two sections we use the techniques that we develop here.

■ Exponential Equations

An *exponential equation* is an equation in which the variable occurs in the exponent. Some exponential equations can be solved by using the fact that exponential functions are one-to-one. This means that

$$a^x = a^y \quad \Rightarrow \quad x = y$$

We use this property in the first example.

Example 1 ■ Exponential Equations

Solve each exponential equation.

(a) $5^x = 125$ **(b)** $5^{2x} = 5^{x+1}$

Solution

(a) We first express 125 as a power of 5 and then use the fact that the exponential function $f(x) = 5^x$ is one-to-one.

$$\begin{aligned} 5^x &= 125 && \text{Given equation} \\ 5^x &= 5^3 && \text{Because } 125 = 5^3 \\ x &= 3 && \text{One-to-one property} \end{aligned}$$

The solution is $x = 3$.

(b) We first use the fact that the function $f(x) = 5^x$ is one-to-one.

$$\begin{aligned} 5^{2x} &= 5^{x+1} && \text{Given equation} \\ 2x &= x + 1 && \text{One-to-one property} \\ x &= 1 && \text{Solve for } x \end{aligned}$$

The solution is $x = 1$.

Now Try Exercises 3 and 7 ■

Law 3: $\log_a A^C = C \log_a A$

The equations in Example 1 were solved by comparing exponents. This method is not suitable for solving an equation like $5^x = 160$ because 160 is not easily expressed as a power of the base 5. To solve such equations, we take the logarithm of each side and use Law 3 of logarithms to "bring down the exponent."

Guidelines for Solving Exponential Equations

1. Isolate the exponential expression on one side of the equation.
2. Take the logarithm of each side, then use the Laws of Logarithms to "bring down the exponent."
3. Solve for the variable.

Example 2 ■ Solving an Exponential Equation

Consider the exponential equation $3^{x+2} = 7$.

(a) Find the exact solution of the equation expressed in terms of logarithms.

(b) Use a calculator to find an approximation to the solution rounded to six decimal places.

Solution

(a) We take the common logarithm of each side and use Law 3.

$$
\begin{aligned}
3^{x+2} &= 7 && \text{Given equation}\\
\log(3^{x+2}) &= \log 7 && \text{Take log of each side}\\
(x+2)\log 3 &= \log 7 && \text{Law 3 (bring down exponent)}\\
x+2 &= \frac{\log 7}{\log 3} && \text{Divide by log 3}\\
x &= \frac{\log 7}{\log 3} - 2 && \text{Subtract 2}
\end{aligned}
$$

The exact solution is $x = \dfrac{\log 7}{\log 3} - 2$.

(b) Using a calculator, we find the decimal approximation $x \approx -0.228756$.

Now Try Exercise 15 ■

We could have used natural logarithms instead of common logarithms. In fact, using the same steps, we get

$$x = \frac{\ln 7}{\ln 3} - 2 \approx -0.228756$$

Check Your Answer

Substituting $x = -0.228756$ into the original equation and using a calculator, we get

$$3^{(-0.228756)+2} \approx 7 \checkmark$$

Example 3 ■ Solving an Exponential Equation

Solve the equation $8e^{2x} = 20$.

Solution We first divide by 8 to isolate the exponential term on one side of the equation.

$$
\begin{aligned}
8e^{2x} &= 20 && \text{Given equation}\\
e^{2x} &= \tfrac{20}{8} && \text{Divide by 8}\\
\ln e^{2x} &= \ln 2.5 && \text{Take ln of each side}\\
2x &= \ln 2.5 && \text{Property of ln}\\
x &= \frac{\ln 2.5}{2} && \text{Divide by 2 (exact solution)}\\
&\approx 0.458 && \text{Calculator (approximate solution)}
\end{aligned}
$$

Now Try Exercise 17 ■

Check Your Answer

Substituting $x = 0.458$ into the original equation and using a calculator, we get

$$8e^{2(0.458)} \approx 20 \checkmark$$

Example 4 ■ Solving an Exponential Equation Algebraically and Graphically

Solve the equation $e^{3-2x} = 4$ algebraically and graphically.

Solution 1: Algebraic

Since the base of the exponential term is e, we use natural logarithms to solve this equation.

$$
\begin{aligned}
e^{3-2x} &= 4 && \text{Given equation}\\
\ln(e^{3-2x}) &= \ln 4 && \text{Take ln of each side}\\
3-2x &= \ln 4 && \text{Property of ln}\\
-2x &= -3 + \ln 4 && \text{Subtract 3}\\
x &= \tfrac{1}{2}(3 - \ln 4) \approx 0.807 && \text{Multiply by } -\tfrac{1}{2}
\end{aligned}
$$

You should check that this answer satisfies the original equation.

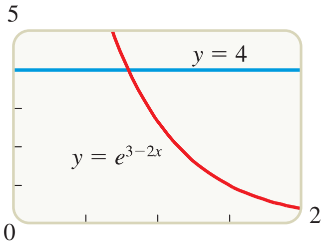


Figure 1

Solution 2: Graphical

We graph the equations $y = e^{3-2x}$ and $y = 4$ in the same viewing rectangle as in Figure 1. The solutions occur where the graphs intersect. In this case the only solution is $x \approx 0.81$.

Now Try Exercise 21 ■

Example 5 ■ An Exponential Equation of Quadratic Type

Solve the equation $e^{2x} - e^x - 6 = 0$.

Solution To isolate the exponential term, we factor.

$$e^{2x} - e^x - 6 = 0 \quad \text{Given equation}$$

$$(e^x)^2 - e^x - 6 = 0 \quad \text{Law of Exponents}$$

$$(e^x - 3)(e^x + 2) = 0 \quad \text{Factor (a quadratic in } e^x\text{)}$$

$$e^x - 3 = 0 \quad \text{or} \quad e^x + 2 = 0 \quad \text{Zero-Product Property}$$

$$e^x = 3 \qquad e^x = -2$$

If we let $w = e^x$, we get the quadratic equation

$$w^2 - w - 6 = 0$$

which factors as

$$(w - 3)(w + 2) = 0$$

The equation $e^x = 3$ leads to $x = \ln 3$. But the equation $e^x = -2$ has no solution because $e^x > 0$ for all x. Thus $x = \ln 3 \approx 1.0986$ is the only solution. You should check that this answer satisfies the original equation.

Now Try Exercise 37 ■

Example 6 ■ An Equation Involving Exponential Functions

Solve the equation $3xe^x + x^2e^x = 0$.

Solution First we factor the left side of the equation.

$$3xe^x + x^2e^x = 0 \quad \text{Given equation}$$

$$x(3 + x)e^x = 0 \quad \text{Factor out common factors}$$

$$x(3 + x) = 0 \quad \text{Divide by } e^x \text{ (because } e^x \neq 0\text{)}$$

$$x = 0 \quad \text{or} \quad 3 + x = 0 \quad \text{Zero-Product Property}$$

Thus the solutions are $x = 0$ and $x = -3$.

Now Try Exercise 43 ■

Check Your Answer

$x = 0$:

$3(0)e^0 + 0^2e^0 = 0$ ✓

$x = -3$:

$3(-3)e^{-3} + (-3)^2e^{-3}$

$= -9e^{-3} + 9e^{-3} = 0$ ✓

■ Logarithmic Equations

A *logarithmic equation* is an equation in which a logarithm of the variable occurs. Some logarithmic equations can be solved by using the fact that logarithmic functions are one-to-one. This means that

$$\log_a x = \log_a y \quad \Rightarrow \quad x = y$$

We use this property in the next example.

Example 7 ■ Solving a Logarithmic Equation

Solve the equation $\log_5(x^2 + 1) = \log_5(x - 2) + \log_5(x + 3)$.

Solution First we combine the logarithms on the right-hand side, and then we use the one-to-one property of logarithms.

$$\begin{aligned}
\log_5(x^2 + 1) &= \log_5(x - 2) + \log_5(x + 3) && \text{Given equation} \\
\log_5(x^2 + 1) &= \log_5[(x - 2)(x + 3)] && \text{Law 1: } \log_a AB = \log_a A + \log_a B \\
\log_5(x^2 + 1) &= \log_5(x^2 + x - 6) && \text{Expand} \\
x^2 + 1 &= x^2 + x - 6 && \text{log is one-to-one (or raise 5 to each side)} \\
x &= 7 && \text{Solve for } x
\end{aligned}$$

The solution is $x = 7$. (You can check that $x = 7$ satisfies the original equation.)

Now Try Exercise 47 ■

The method of Example 7 is not suitable for solving an equation like $\log_5 x = 13$ because the right-hand side is not expressed as a logarithm (base 5). To solve such equations, we use the following guidelines.

Guidelines for Solving Logarithmic Equations

1. Isolate the logarithmic term on one side of the equation; you might first need to combine the logarithmic terms.
2. Write the equation in exponential form (or raise the base to each side of the equation).
3. Solve for the variable.

Example 8 ■ Solving Logarithmic Equations

Solve each equation for x.

(a) $\ln x = 8$ **(b)** $\log_2(25 - x) = 3$

Solution

(a)

$$\begin{aligned}
\ln x &= 8 && \text{Given equation} \\
x &= e^8 && \text{Exponential form}
\end{aligned}$$

Therefore $x = e^8 \approx 2981$.

We can also solve this problem another way.

$$\begin{aligned}
\ln x &= 8 && \text{Given equation} \\
e^{\ln x} &= e^8 && \text{Raise } e \text{ to each side} \\
x &= e^8 && \text{Property of ln}
\end{aligned}$$

(b) The first step is to rewrite the equation in exponential form.

$$\begin{aligned}
\log_2(25 - x) &= 3 && \text{Given equation} \\
25 - x &= 2^3 && \text{Exponential form (or raise 2 to each side)} \\
25 - x &= 8 \\
x &= 25 - 8 = 17
\end{aligned}$$

Check Your Answer

If $x = 17$, we get

$\log_2(25 - 17) = \log_2 8 = 3$ ✓

Now Try Exercises 53 and 57 ■

Example 9 ■ Solving a Logarithmic Equation

Solve the equation $4 + 3\log(2x) = 16$.

Solution We first isolate the logarithmic term. This allows us to write the equation in exponential form.

$$\begin{aligned} 4 + 3\log(2x) &= 16 && \text{Given equation} \\ 3\log(2x) &= 12 && \text{Subtract 4} \\ \log(2x) &= 4 && \text{Divide by 3} \\ 2x &= 10^4 && \text{Exponential form (or raise 10 to each side)} \\ x &= 5000 && \text{Divide by 2} \end{aligned}$$

Check Your Answer

If $x = 5000$, we get

$$\begin{aligned} 4 + 3\log(2 \cdot 5000) &= 4 + 3\log 10{,}000 \\ &= 4 + 3(4) \\ &= 16 \end{aligned}$$

Now Try Exercise 59

Example 10 ■ Solving a Logarithmic Equation Algebraically and Graphically

Solve the equation $\log(x + 2) + \log(x - 1) = 1$ algebraically and graphically.

Solution 1: Algebraic

We first combine the logarithmic terms, using the Laws of Logarithms.

$$\begin{aligned} \log[(x + 2)(x - 1)] &= 1 && \text{Law 1} \\ (x + 2)(x - 1) &= 10 && \text{Exponential form (or raise 10 to each side)} \\ x^2 + x - 2 &= 10 && \text{Expand left side} \\ x^2 + x - 12 &= 0 && \text{Subtract 10} \\ (x + 4)(x - 3) &= 0 && \text{Factor} \end{aligned}$$

$$x = -4 \quad \text{or} \quad x = 3$$

We check these potential solutions in the original equation and find that $x = -4$ is not a solution (because logarithms of negative numbers are undefined), but $x = 3$ is a solution. (See *Check Your Answers* in the margin.)

Check Your Answer

$x = -4$:

$$\begin{aligned} &\log(-4 + 2) + \log(-4 - 1) \\ &\quad = \log(-2) + \log(-5) \\ &\qquad \text{undefined} \end{aligned}$$

$x = 3$:

$$\begin{aligned} &\log(3 + 2) + \log(3 - 1) \\ &\quad = \log 5 + \log 2 = \log(5 \cdot 2) \\ &\quad = \log 10 = 1 \end{aligned}$$

Solution 2: Graphical

We first move all terms to one side of the equation:

$$\log(x + 2) + \log(x - 1) - 1 = 0$$

Then we graph

$$y = \log(x + 2) + \log(x - 1) - 1$$

as shown in Figure 2. The solutions are the x-intercepts of the graph. Thus the only solution is $x \approx 3$.

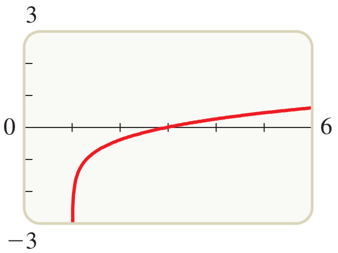


Figure 2

Now Try Exercise 61

Example 11 ■ Solving a Logarithmic Equation Graphically

In Example 11 it's not possible to isolate x algebraically, so we must solve the equation graphically.

Solve the equation $x^2 = 2\ln(x + 2)$.

Solution We first move all terms to one side of the equation.

$$x^2 - 2\ln(x + 2) = 0$$

Then we graph

$$y = x^2 - 2\ln(x + 2)$$

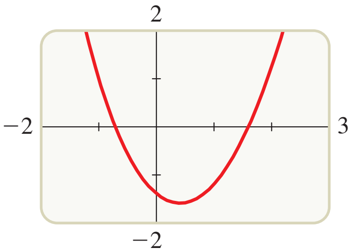


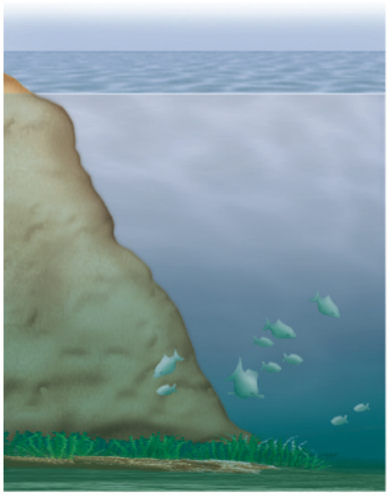
Figure 3

as in Figure 3. The solutions are the x-intercepts of the graph:

$$x \approx -0.71 \qquad \text{and} \qquad x \approx 1.60$$

Now Try Exercise 67 ■

Logarithmic equations are used in determining the amount of light that reaches various depths in a lake. (This information helps biologists to determine the types of life a lake can support.) As light passes through water (or other transparent materials such as glass or plastic), some of the light is absorbed. It's easy to see that the murkier the water, the more light is absorbed. The exact relationship between light absorption and the distance light travels in a material is described in the next example.



The intensity of light in a lake diminishes with depth. Environmental scientists are interested in the "transparency" of the water in a lake because certain levels of transparency are required to support biodiversity of the submerged macrophyte population.

Example 12 ■ Transparency of a Lake

If I_0 and I denote the intensity of light (in lumens, lm) before and after going through a material and x is the distance (in feet) the light travels in the material, then according to the **Beer-Lambert Law**,

$$-\frac{1}{k}\ln\frac{I}{I_0} = x$$

where k is a constant depending on the type of material.

(a) Solve the equation for I.

(b) For a certain lake $k = 0.025$, and the light intensity is $I_0 = 14$ lm. Find the light intensity at a depth of 20 ft.

Solution

(a) We first isolate the logarithmic term.

$$\begin{aligned} -\frac{1}{k}\ln\frac{I}{I_0} &= x && \text{Given equation} \\ \ln\frac{I}{I_0} &= -kx && \text{Multiply by } -k \\ \frac{I}{I_0} &= e^{-kx} && \text{Exponential form} \\ I &= I_0 e^{-kx} && \text{Multiply by } I_0 \end{aligned}$$

(b) We find I using the formula from part (a).

$$\begin{aligned} I &= I_0 e^{-kx} && \text{From part (a)} \\ &= 14e^{(-0.025)(20)} && I_0 = 14,\ k = 0.025,\ x = 20 \\ &\approx 8.49 && \text{Calculator} \end{aligned}$$

The light intensity at a depth of 20 ft is about 8.5 lm.

Now Try Exercise 97 ■

■ Compound Interest

Recall the formulas for interest that we found in Section 4.1. If a principal P is invested at an interest rate r for a period of t years, then the amount A of the investment is given by

$$\begin{aligned} A &= P(1 + r) && \text{Simple interest (for one year)} \\ A(t) &= P\left(1 + \frac{r}{n}\right)^{nt} && \text{Interest compounded } n \text{ times per year} \\ A(t) &= Pe^{rt} && \text{Interest compounded continuously} \end{aligned}$$

Radiocarbon dating is a method that archaeologists use to determine the age of ancient objects. The carbon dioxide in the atmosphere always contains a fixed fraction of radioactive carbon, carbon-14 (^{14}C), with a half-life of about 5730 years. Plants absorb carbon dioxide from the atmosphere, which then makes its way to animals through the food chain. Thus, all living creatures contain the same fixed proportion of ^{14}C to nonradioactive ^{12}C as in the atmosphere.

After an organism dies, it stops assimilating ^{14}C, and the amount of ^{14}C in it begins to decay exponentially. We can determine the time that has elapsed since the death of the organism by measuring the amount of ^{14}C left in it.

For example, if a donkey bone contains 73% as much ^{14}C as a living donkey and it died t years ago, then by the formula for radioactive decay (Section 4.6),

$$0.73 = (1.00)e^{-(t \ln 2)/5730}$$

We solve this exponential equation to find $t \approx 2600$, so the bone is about 2600 years old.

We can use logarithms to determine the time it takes for the principal to increase to a given amount.

Example 13 ■ Finding the Term for an Investment to Double

A sum of \$5000 is invested at an interest rate of 5% per year. Find the time required for the money to double if the interest is compounded according to the following methods.

(a) Semiannual **(b)** Continuous

Solution

(a) We use the formula for compound interest with $P = \$5000$, $A(t) = \$10{,}000$, $r = 0.05$, and $n = 2$, and solve the resulting exponential equation for t.

$$\begin{aligned}
5000\left(1 + \frac{0.05}{2}\right)^{2t} &= 10{,}000 && P\left(1+\frac{r}{n}\right)^{nt} = A \\
(1.025)^{2t} &= 2 && \text{Divide by 5000} \\
\log(1.025)^{2t} &= \log 2 && \text{Take log of each side} \\
2t \log 1.025 &= \log 2 && \text{Law 3 (bring down the exponent)} \\
t &= \frac{\log 2}{2 \log 1.025} && \text{Divide by 2 log 1.025} \\
t &\approx 14.04 && \text{Calculator}
\end{aligned}$$

The money will double in about 14 years.

(b) We use the formula for continuously compounded interest with $P = \$5000$, $A(t) = \$10{,}000$, and $r = 0.05$, and solve the resulting exponential equation for t.

$$\begin{aligned}
5000e^{0.05t} &= 10{,}000 && Pe^{rt} = A \\
e^{0.05t} &= 2 && \text{Divide by 5000} \\
\ln e^{0.05t} &= \ln 2 && \text{Take ln of each side} \\
0.05t &= \ln 2 && \text{Property of ln} \\
t &= \frac{\ln 2}{0.05} && \text{Divide by 0.05} \\
t &\approx 13.86 && \text{Calculator}
\end{aligned}$$

The money will double in about 13 years 10 months.

Now Try Exercise 87 ■

Example 14 ■ Time Required to Grow an Investment

A sum of \$1000 is invested at an interest rate of 4% per year. Find the time required for the amount to grow to \$5000 if interest is compounded continuously.

Solution We use the formula for continuously compounded interest with $P = \$1000$, $A(t) = \$5000$, and $r = 0.04$, and solve the resulting exponential equation for t.

$$\begin{aligned}
1000e^{0.04t} &= 5000 && Pe^{rt} = A \\
e^{0.04t} &= 5 && \text{Divide by 1000} \\
0.04t &= \ln 5 && \text{Take ln of each side} \\
t &= \frac{\ln 5}{0.04} && \text{Divide by 0.04} \\
t &\approx 40.24 && \text{Calculator}
\end{aligned}$$

The amount will be \$5000 in about 40 years 3 months.

Now Try Exercise 89 ■

4.5 Exercises

Concepts

1. Let's solve the exponential equation $2e^x = 50$.

(a) First, we isolate e^x to get the equivalent equation ________.

(b) Next, we take ln of each side to get the equivalent equation ________.

(c) Now we use a calculator to find $x \approx$ ________.

2. Let's solve the logarithmic equation

$$\log 3 + \log(x - 2) = \log x$$

(a) First, we combine the logarithms on the LHS to get the equivalent equation ________.

(b) Next, we use the fact that log is one-to-one to get the equivalent equation ________.

(c) Now we find $x =$ ________.

Skills

3–10 ■ Exponential Equations Find the solution of the exponential equation, as in Example 1.

3. $5^{x-1} = 625$ **4.** $e^{x^2} = e^9$

5. $5^{2x-3} = 1$ **6.** $10^{2x-3} = \frac{1}{10}$

7. $7^{2x-3} = 7^{6+5x}$ **8.** $e^{1-2x} = e^{3x-5}$

9. $6^{x^2-1} = 6^{1-x^2}$ **10.** $10^{2x^2-3} = 10^{9-x^2}$

11–36 ■ Exponential Equations **(a)** Find the exact solution of the exponential equation in terms of logarithms. **(b)** Use a calculator to find an approximation to the solution, rounded to six decimal places.

11. $e^x = 16$ **12.** $e^{-2x} = 5$

13. $10^{-x} = 6$ **14.** $10^{5x} = 24$

15. $3^{x+5} = 4$ **16.** $e^{2-3x} = 11$

17. $3 \cdot 6^{1-x} = 15$ **18.** $5 \cdot 4^{3-2x} = 8$

19. $200(1.02)^{4t} = 1500$ **20.** $25(1.015)^{12t} = 60$

21. $3e^{5-t} = 12$ **22.** $5\left(\frac{1}{2}\right)^{2t-3} = 24$

23. $2^{x/10} = 0.3$ **24.** $2^{-x/50} = 0.6$

25. $4(1 + 10^{5x}) = 9$ **26.** $2(5 + 3^{x+1}) = 100$

27. $8 + e^{1-4x} = 20$ **28.** $1 + e^{4x+1} = 20$

29. $4^x + 2^{1+2x} = 50$ **30.** $125^x + 5^{3x+1} = 200$

31. $5^t = 10^{3t+2}$ **32.** $e^t = 3^{1-t}$

33. $5^{x/3} = 3^{x+1}$ **34.** $3^{2x-1} = 2^{4x+1}$

35. $\dfrac{50}{1 + e^{-x}} = 4$ **36.** $\dfrac{10}{1 + e^{-x}} = 2$

37–42 ■ Exponential Equations of Quadratic Type Solve the equation.

37. $e^{2x} + 5e^x - 6 = 0$ **38.** $e^{2x} + 3e^x - 10 = 0$

39. $e^{4x} + 4e^{2x} - 21 = 0$ **40.** $3^{4x} - 3^{2x} - 6 = 0$

41. $2^x - 10(2^{-x}) + 3 = 0$ **42.** $e^x + 15e^{-x} - 8 = 0$

43–46 ■ Equations Involving Exponential Functions Solve the equation.

43. $x^2 2^x - 2^x = 0$ **44.** $x^2 10^x - x10^x = 2(10^x)$

45. $4x^3 e^{-3x} - 3x^4 e^{-3x} = 0$ **46.** $x^2 e^x + xe^x - e^x = 0$

47–52 ■ Logarithmic Equations Solve the logarithmic equation for x, as in Example 7.

47. $\log(x + 2) + \log(x - 3) = \log(4x)$

48. $\log_5(x + 2) + \log_5(x - 5) = \log_5(6x)$

49. $2 \log x = \log 2 + \log(3x - 4)$

50. $\ln\left(x - \frac{1}{2}\right) + \ln 2 = 2 \ln x$

51. $\log_2 3 + \log_2 x = \log_2 5 + \log_2(x - 2)$

52. $\log_4(x + 2) + \log_4 3 = \log_4 5 + \log_4(2x - 3)$

53–66 ■ Logarithmic Equations Solve the logarithmic equation for x.

53. $\log x = 9$

54. $\log(x + 3) = 4$

55. $\ln(4 - x) = 1$

56. $\ln(3x + 1) = 0$

57. $\log_3(5 - x) = -1$

58. $\log_{1/2}(3x + 2) = -1$

59. $4 - \log(3 - x) = 3$

60. $\log_2(x^2 - x - 2) = 2$

61. $\log_2 x + \log_2(x - 3) = 2$

62. $\log x + \log(x - 3) = 1$

63. $\log_9(x - 5) + \log_9(x + 3) = 1$

64. $\ln(x - 1) + \ln(x + 1) = 0$

65. $\log_{1/5}(x - 1) - \log_{1/5}(x + 1) = 2$

66. $\log_3(x + 15) - \log_3(x - 1) = 2$

67–74 ■ Solving Equations Graphically Use a graphing device to find all solutions of the equation, rounded to two decimal places.

67. $\ln x = 3 - x$ **68.** $\log x = x^2 - 2$

69. $x^3 - x = \log(x + 1)$ **70.** $x = \ln(4 - x^2)$

71. $e^x = -x$ **72.** $2^{-x} = x - 1$

73. $4^{-x} = \sqrt{x}$ **74.** $e^{x^2} - 2 = x^3 - x$

Skills Plus

75–78 ■ Solving Inequalities Solve the inequality.

75. $\log(x - 2) + \log(9 - x) < 1$

76. $3 \le \log_2 x \le 4$

77. $2 < 10^x < 5$

78. $x^2 e^x - 2e^x < 0$

79–82 ■ Inverse Functions Find the inverse function of f.

79. $f(x) = 2^{2x}$

80. $f(x) = 3^{x+1}$

81. $f(x) = \log_2(x - 1)$

82. $f(x) = \log 3x$

83–86 ■ Special Exponential and Logarithmic Equations Find the value(s) of x for which the equation is true.

83. $2^{2/\log_5 x} = \frac{1}{16}$

84. $\log_2(\log_3 x) = 4$

85. $(\log x)^4 + (\log x)^3 = 0$

86. $(\log x)^3 = 3 \log x$

Applications

87. Compound Interest Suppose you invest \$5000 in an account that pays 2.25% interest per year, compounded quarterly.

(a) Find the amount after 5 years.

(b) How long will it take for the investment to double?

88. Compound Interest Suppose you invest \$6500 in an account that pays 4.5% interest per year, compounded continuously.

(a) What is the amount after 4 years?

(b) How long will it take for the amount to be \$8000?

89. Compound Interest Find the time required for an investment of \$5000 to grow to \$8000 at an interest rate of 3.5% per year, compounded quarterly.

90. Compound Interest Find the time required for an investment of \$3000 to grow to \$5000 at an interest rate of 6.5%, compounded semiannually.

91. Doubling an Investment How long will it take for an investment of \$1000 to double in value if the interest rate is 8.5% per year, compounded continuously?

92. Interest Rate A sum of \$1000 was invested for 4 years, and the interest was compounded semiannually. If this sum amounted to \$1435.77 in the given time, what was the interest rate?

93. Radioactive Decay A 15-gram sample of radioactive iodine decays in such a way that the mass remaining after t days is given by

$$m(t) = 15e^{-0.087t}$$

where $m(t)$ is measured in grams. After how many days are there only 5 g remaining?

94. Skydiving The velocity of a skydiver t seconds after jumping is given by

$$v(t) = 80(1 - e^{-0.2t})$$

After how many seconds is the velocity 70 ft/s?

95. Fish Population A small lake is stocked with a certain species of fish. The fish population is modeled by the function

$$P = \frac{10}{1 + 4e^{-0.8t}}$$

where P is the number of fish (in thousands) and t is measured in years since the lake was stocked.

(a) Find the fish population after 3 years.

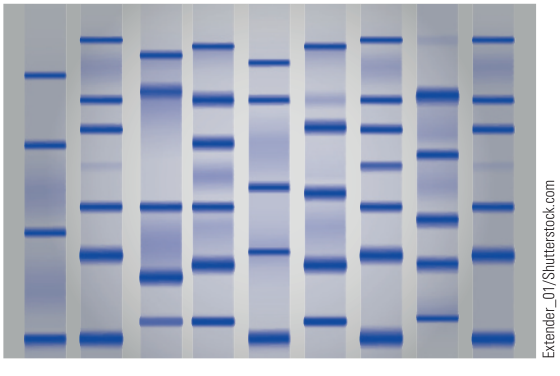
(b) After how many years will the fish population reach 5000 fish?

96. DNA Gel Electrophoresis Gel electrophoresis is a DNA analysis technique in which an electric field is used to propel charged DNA fragments through a porous gel. Shorter strands of DNA move further through the gel than the longer strands. Each DNA sample starts in its own well on one side of the electric field (at the top in the image). The length y of a DNA fragment (the number of base-pairs in the fragment) is inversely proportional to 2^x where x is the distance the fragment travels in the gel. Fragments of different lengths can be distinguished by how far they travel through the gel. (The dark lines in the image are locations where large numbers of identical fragments end up together.) Suppose that fragments of length 200 base-pairs travel 8 cm in the gel.

(a) Find an equation that expresses the inverse relationship of the length y of a DNA fragment and the distance x that the fragment travels in the gel.

(b) What is the length of a fragment that travels 5 cm?

(c) What is the distance traveled by a fragment of length 2000 base-pairs?



97. Atmospheric Pressure Atmospheric pressure P (in kilopascals, kPa) at altitude h (in kilometers, km) is governed by the formula

$$\ln\frac{P}{P_0} = -\frac{h}{k}$$

where $k = 7$ and $P_0 = 100$ kPa are constants.

(a) Solve the equation for P.

(b) Use part (a) to find the pressure P at an altitude of 4 km.

98. Cooling an Engine Suppose a car is driven on a cold winter day (20°F outside) and the engine overheats (at about 220°F). When the car is parked, the engine begins to cool down. The temperature T of the engine t minutes after the car is parked satisfies the equation

$$\ln\frac{T - 20}{200} = -0.11t$$

(a) Solve the equation for T.

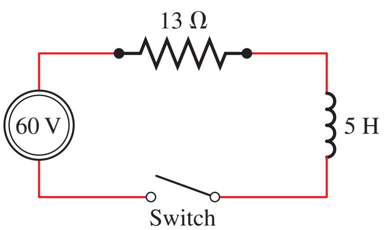
(b) Use part (a) to find the temperature of the engine after 20 min ($t = 20$).

99. Electric Circuits An electric circuit contains a battery that produces a voltage of 60 volts (V), a resistor with a resistance of 13 ohms (Ω), and an inductor with an inductance of 5 henrys (H), as shown in the figure. Using calculus, it can be shown that the current $I = I(t)$ (in amperes, A) t seconds after the switch is closed is

$$I = \tfrac{60}{13}(1 - e^{-13t/5})$$

(a) Use this equation to express the time t as a function of the current I.

(b) After how many seconds is the current 2 A?



100. Learning Curve A *learning curve* is a graph of a function $P(t)$ that measures the performance of someone learning a skill as a function of the training time t. At first, the rate of learning is rapid. Then, as performance increases and approaches a maximal value M, the rate of learning decreases. It has been found that the function

$$P(t) = M - Ce^{-kt}$$

where k and C are positive constants and $C < M$ is a reasonable model for learning.

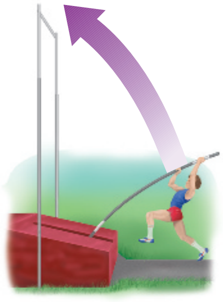
(a) Express the learning time t as a function of the performance level P.

(b) For a particular pole-vaulter in training, the learning curve is given by

$$P(t) = 20 - 14e^{-0.024t}$$

where $P(t)$ is the height the pole-vaulter is able to vault after t months. After how many months of training is the pole-vaulter able to vault 12 ft?

(c) Draw a graph of the learning curve in part (b).



■ Discuss ■ Discover ■ Prove ■ Write

101. Discuss: Estimating a Solution Without actually solving the equation, find two whole numbers between which the solution of $9^x = 20$ must lie. Do the same for $9^x = 100$. Explain how you reached your conclusions.

102. Discuss ■ Discover: A Surprising Equation Take logarithms to show that the equation

$$x^{1/\log x} = 5$$

has no solution. For what values of k does the equation

$$x^{1/\log x} = k$$

have a solution? What does this tell us about the graph of the function $f(x) = x^{1/\log x}$? Confirm your answer using a graphing device.

103. Discuss: Disguised Equations Each of these equations can be transformed into an equation of linear or quadratic type by applying the hint. Solve each equation.

(a) $(x - 1)^{\log(x-1)} = 100(x - 1)$
[*Hint:* Take log of each side.]

(b) $\log_2 x + \log_4 x + \log_8 x = 11$
[*Hint:* Change all logs to base 2.]

(c) $4^x - 2^{x+1} = 3$
[*Hint:* Write as a quadratic in 2^x.]

4.6 Modeling with Exponential Functions

■ Exponential Growth (Doubling Time) ■ Exponential Growth (Relative Growth Rate) ■ Logistic Growth ■ Radioactive Decay ■ Newton's Law of Cooling

Many processes that occur in nature—such as population growth, radioactive decay, heat diffusion, and numerous others—can be modeled by using exponential functions. In this section we study exponential models.

■ Exponential Growth (Doubling Time)

Suppose we start with a single bacterium, which divides every hour. After one hour we have 2 bacteria, after two hours we have 2^2 or 4 bacteria, after three hours we have 2^3 or 8 bacteria, and so on (see Figure 1). We see that we can model the bacteria population after t hours by $f(t) = 2^t$.

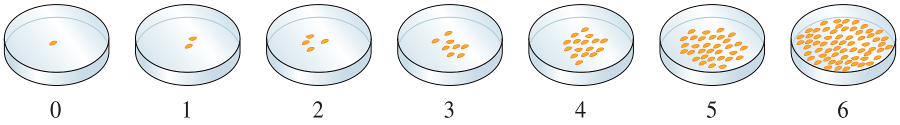


Figure 1 | Bacteria population

If we start with 10 of these bacteria, then the population is modeled by $f(t) = 10 \cdot 2^t$. A slower-growing strain of bacteria doubles every 3 hours; in this case the population is modeled by $f(t) = 10 \cdot 2^{t/3}$. In general, we have the following.

Exponential Growth (Doubling Time)

If the initial size of a population is n_0 and the doubling time is a, then the size of the population at time t is

$$n(t) = n_0 2^{t/a}$$

where a and t are measured in the same time units (minutes, hours, days, years, and so on).

Example 1 ■ Bacteria Population

Christoph Burgstedt/Shutterstock.com

Under ideal conditions a certain bacteria population doubles every four hours. Initially, there are 1000 bacteria in a colony.

(a) Find a model for the bacteria population after t hours.

(b) How many bacteria are in the colony after 24 hours?

(c) After how many hours will the bacteria count reach one million?

Solution

(a) The population at time t is modeled by

$$n(t) = 1000 \cdot 2^{t/4} = 1000 \cdot 2^{0.25t}$$

where t is measured in hours.

(b) After 24 hours the number of bacteria is

$$n(24) = 1000 \cdot 2^{0.25(24)} = 64{,}000$$

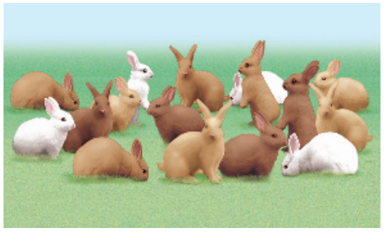
(c) We set $n(t) = 1{,}000{,}000$ in the model that we found in part (a) and solve the resulting exponential equation for t.

$$
\begin{aligned}
1{,}000{,}000 &= 1000 \cdot 2^{0.25t} && n(t) = 1000 \cdot 2^{0.25t} \\
1000 &= 2^{0.25t} && \text{Divide by 1000} \\
\log 1000 &= \log 2^{0.25t} && \text{Take log of each side} \\
3 &= (0.25t)\log 2 && \text{Properties of log} \\
t &= \frac{3}{0.25\log 2} \approx 39.86 && \text{Solve for } t
\end{aligned}
$$

The bacteria level reaches one million in about 40 hours.

Now Try Exercise 1 ■

Example 2 ■ Rabbit Population



A certain breed of rabbit was introduced into a grassland habitat. After 8 months the rabbit population is estimated to be 4100 and doubling every 3 months.

(a) What was the initial size of the rabbit population?

(b) Estimate the population 1 year after the rabbits were introduced to the grassland habitat.

(c) Sketch a graph of the rabbit population.

Solution

(a) The doubling time is $a = 3$, so the population after t months is

$$n(t) = n_0 2^{t/3} \qquad \text{Model}$$

where n_0 is the initial population. Since the population is 4100 after 8 months, we have

$$
\begin{aligned}
n(8) &= n_0 2^{8/3} && \text{From model} \\
4100 &= n_0 2^{8/3} && \text{Because } n(8) = 4100 \\
n_0 &= \frac{4100}{2^{8/3}} && \text{Divide by } 2^{8/3} \text{ and switch sides} \\
n_0 &\approx 645 && \text{Calculator}
\end{aligned}
$$

Thus we estimate that 645 rabbits were introduced to the grassland habitat.

(b) From part (a) we know that the initial population is $n_0 = 645$, so we can model the population after t months by

$$n(t) = 645 \cdot 2^{t/3} \qquad \text{Model}$$

After 1 year $t = 12$, so

$$n(12) = 645 \cdot 2^{12/3} = 10{,}320$$

Thus after 1 year there would be about 10,000 rabbits.

(c) We first note that the domain is $t \geq 0$. The graph is shown in Figure 2.

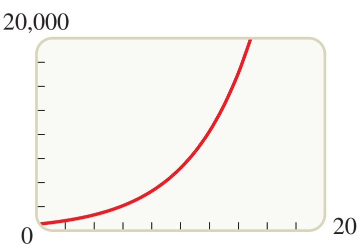


Figure 2 | $n(t) = 645 \cdot 2^{t/3}$

Now Try Exercise 3 ■

■ Exponential Growth (Relative Growth Rate)

We have used an exponential function with base 2 to model population growth (in terms of the doubling time). We could also model the same population with an exponential function with base 3 (in terms of the tripling time). In fact, we can find an exponential

The growth of a population with relative growth rate r is analogous to the growth of an investment with continuously compounded interest rate r.

model with any base. If we use the base e, we get a population model in terms of the **relative growth rate** r: the rate of population growth expressed as a proportion of the population at any time. In this case r is the "instantaneous" growth rate. (In calculus the concept of instantaneous rate is given a precise meaning.) For instance, if $r = 0.02$, then at any time t the growth rate is 2% of the population at time t.

Exponential Growth (Relative Growth Rate)

A population that experiences **exponential growth** increases according to the model

$$n(t) = n_0 e^{rt}$$

where
$n(t)$ = population at time t
n_0 = initial size of the population
r = relative rate of growth (expressed as a proportion of the population)

Note The formula for population growth is the same as that for continuously compounded interest. In fact, the same principle is at work in both cases: The growth of a population (or an investment) per time period is proportional to the size of the population (or the amount of the investment). A population of 1,000,000 will increase more in one year than a population of 1000; in exactly the same way, an investment of \$1,000,000 will increase more in one year than an investment of \$1000.

In the following examples we assume that the populations grow exponentially.

Example 3 ■ Predicting the Size of a Population

The initial bacterium count in a culture is 500. A biologist later makes a sample count of bacteria in the culture and finds that the relative rate of growth is 40% per hour.

(a) Find a function that models the number of bacteria after t hours.
(b) What is the estimated count after 10 hours?
(c) After how many hours will the bacteria count reach 80,000?
(d) Sketch a graph of the function $n(t)$.

Solution

(a) We use the exponential growth model with $n_0 = 500$ and $r = 0.4$ to get

$$n(t) = 500e^{0.4t}$$

where t is measured in hours.

(b) Using the function in part (a), we find that the bacterium count after 10 hours is

$$n(10) = 500e^{0.4(10)} = 500e^4 \approx 27{,}300$$

(c) We set $n(t) = 80{,}000$ and solve the resulting exponential equation for t.

$$80{,}000 = 500 \cdot e^{0.4t} \qquad n(t) = 500 \cdot e^{0.4t}$$
$$160 = e^{0.4t} \qquad \text{Divide by 500}$$
$$\ln 160 = 0.4t \qquad \text{Take ln of each side}$$
$$t = \frac{\ln 160}{0.4} \approx 12.68 \qquad \text{Solve for } t$$

The bacteria level reaches 80,000 in about 12.7 hours.

(d) The graph is shown in Figure 3.

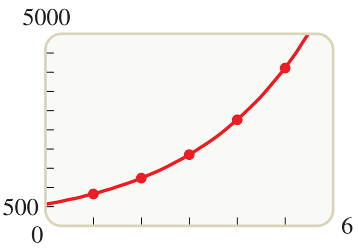


Figure 3 | $y = 500e^{0.4t}$

Now Try Exercise 5

The relative growth rate of world population has been declining over the past few decades—from 1.32% in 2000 to 1.04% in 2020.

Standing Room Only
The population of the world was about 7.8 billion in 2020 and was increasing at 1.04% per year. Assuming that each person occupies an average of 4 ft^2 of the surface of the earth, the exponential model for population growth projects that by the year 3074 there will be standing room only! (The total land surface area of the world is about 1.8×10^{15} ft^2.)

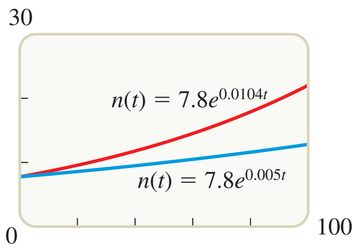


Figure 4

Example 4 ■ Comparing Different Rates of Population Growth

In 2020 the population of the world was 7.8 billion, and the relative rate of growth was 1.04% per year. It is claimed that a rate of 0.50% per year would make a significant difference in the total population in just a few decades. Test this claim by estimating the population of the world in the year 2050 using a relative rate of growth of (a) 1.04% per year and (b) 0.50% per year.

Graph the population functions for the next 100 years for the two relative growth rates in the same viewing rectangle.

Solution

(a) By the exponential growth model we have

$$n(t) = 7.8e^{0.0104t}$$

where $n(t)$ is measured in billions and t is measured in years since 2020. Because the year 2050 is 30 years after 2020, we find

$$n(30) = 7.8e^{0.0104(30)} = 7.8e^{0.312} \approx 10.7$$

The estimated population in the year 2050 is about 10.7 billion.

(b) We use the function

$$n(t) = 7.8e^{0.005t}$$

and find

$$n(30) = 7.8e^{0.005(30)} = 7.8e^{0.15} \approx 9.1$$

The estimated population in the year 2050 is about 9.1 billion.

The graphs in Figure 4 show that a small change in the relative rate of growth will, over time, make a large difference in population size.

Now Try Exercise 7 ■

Example 5 ■ Expressing a Model in Terms of *e*

A population of a certain species of fish starts with 3000 fish and the number doubles every 3.75 years. Find a function of the given form that models the number of fish after t years, and graph the function.

(a) $n(t) = n_0 2^{t/a}$ **(b)** $n(t) = n_0 e^{rt}$

Solution

(a) The initial population is $n_0 = 3000$ and the doubling time is $a = 3.75$ years. The model is

$$n(t) = 3000 \cdot 2^{t/3.75}$$

The model is graphed in Figure 5.

(b) Since the initial population is $n_0 = 3000$, the model we want has the form $n(t) = 3000e^{rt}$. To find the relative growth rate r we equate the two models and solve for r:

$$3000e^{rt} = 3000 \cdot 2^{t/3.75} \quad \text{Equate models}$$

$$e^{rt} = 2^{t/3.75} \quad \text{Divide by 3000}$$

$$rt = \ln 2^{t/3.75} \quad \text{Take ln of each side}$$

$$rt = \frac{t \ln 2}{3.75} \quad \text{Property of ln}$$

$$r = \frac{\ln 2}{3.75} \quad \text{Divide by } t \text{ (for } t \neq 0)$$

$$r \approx 0.1848 \quad \text{Calculator}$$

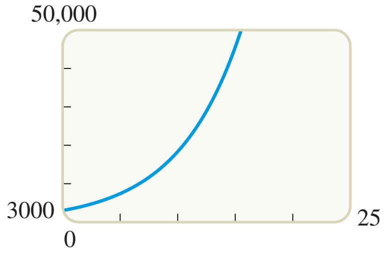


Figure 5 | Graph of $y = 3000 \cdot 2^{t/3.75}$

Now that we know the relative growth rate r we can find the model:

$$n(t) = 3000 \cdot e^{0.1848t}$$

The model is graphed in Figure 6.

The graphs of the functions in Figures 5 and 6 are the same because the two functions are just two different ways of expressing the same model.

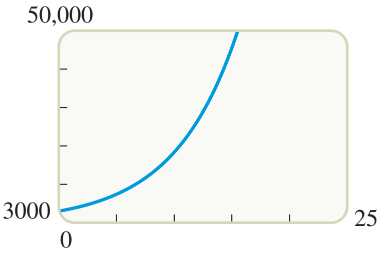


Figure 6 | Graph of $y = 3000 \cdot e^{0.1848t}$

Now Try Exercise 9 ■

Note In general, a model of the form $y = C \cdot b^t$ can be expressed in the form

$$y = C \cdot e^{kt}$$

To find the appropriate value of k, we note that for the models to be the same, we must have $C \cdot b^t = C \cdot e^{kt}$. So

$$b^t = e^{kt} \Leftrightarrow b^t = (e^k)^t \Leftrightarrow b = e^k \Leftrightarrow k = \ln b$$

It follows that the model $y = C \cdot b^t$ is the same as the model

$$y = C \cdot e^{(\ln b)t}$$

■ Logistic Growth

The population models that we've studied so far are appropriate only under ideal conditions that allow for unlimited growth. In the real world, a given environment has limited resources (food, water, living space, and so on) and can support a maximum population M, called the *carrying capacity*. Populations tend to first increase exponentially and then level off as they approach M. Under such conditions the rate of growth of the population $P(t)$ at any time t is jointly proportional to $P(t)$ and to $M - P(t)$. It can be shown using calculus (see Stewart *Calculus* 9e, Sections 9.1 and 9.4) that under such conditions, the population is modeled by a *logistic function*, as follows.

Proportionality is studied in Section 1.12.

Logistic Growth

A population that experiences **logistic growth** increases according to the model

$$n(t) = \frac{M}{1 + Ae^{-rt}}$$

where $n(t)$ = population at time t

M = carrying capacity

$A = (M - n_0)/n_0$, where n_0 is the initial population

r = initial relative growth rate

Example 6 ■ Logistic Population Growth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Suppose that the fish population in Example 5 exists in a small lake that can support a maximum of 30,000 fish. An initial population of 3000 fish is introduced into the lake.

(a) Model the population with a logistic growth model.

(b) Draw a graph of the logistic growth model and compare with the exponential growth model in Example 5.

Solution

(a) The carrying capacity is $M = 30{,}000$ and the initial population is $n_0 = 3000$. From Example 5 we know that for these fish the initial relative growth rate is $r = 0.1848$. We calculate $A = (30{,}000 - 3000)/3000 = 9$, so

$$n(t) = \frac{30{,}000}{1 + 9e^{-0.1848t}}$$

(b) Figure 7 shows a graph of the logistic growth model. Figure 8 shows a graph of both the logistic and exponential growth models for comparison. We see that with the logistic growth model the fish population initially grows exponentially (when plenty of resources are available) and then the rate of growth slows down as the population approaches the carrying capacity of 30,000. With the exponential growth model, the fish population continues to grow exponentially without bound.

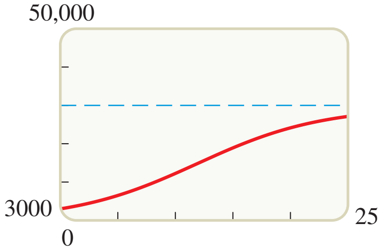


Figure 7 | $n(t) = \dfrac{30{,}000}{1 + 9e^{-0.1848t}}$

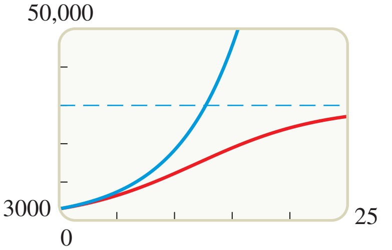


Figure 8 | Comparison of logistic and exponential models

Now Try Exercise 17 ■

■ Radioactive Decay

Radioactive substances decay by spontaneously emitting radiation. The rate of decay is proportional to the mass of the substance. This is analogous to population growth except that the mass *decreases*. Physicists express the rate of decay in terms of **half-life**, the time it takes for a sample of the substance to decay to half its original mass. For example, the half-life of radium-226 is 1600 years, so a 100-gram sample decays to 50 g $\left(\text{or } \frac{1}{2} \times 100 \text{ g}\right)$ in 1600 years, then to 25 g $\left(\text{or } \frac{1}{2} \times \frac{1}{2} \times 100 \text{ g}\right)$ in 3200 years, and so on. In general, for a radioactive substance with mass m_0 and half-life h, the amount remaining at time t is modeled by

$$\boxed{m(t) = m_0 2^{-t/h}}$$

where h and t are measured in the same time units (minutes, hours, days, years, and so on).

Discovery Project ■ **Modeling Radiation with Coins and Dice**

Radioactive elements decay when their atoms spontaneously emit radiation and change into smaller, stable atoms. But if atoms decay randomly, how is it possible to find a function that models their behavior? We'll try to answer this question by experiments with randomly tossed coins and randomly rolled dice. The experiments allow us to experience how a very large number of random events can result in predictable exponential results. You can find the project at **www.stewartmath.com**.

The half-lives of **radioactive elements** vary from very long to very short. Here are some examples.

Element	Half-life
thorium-232	14 billion years
uranium-238	4.5 billion years
uranium-235	704 million years
thorium-230	75,380 years
plutonium-239	24,100 years
carbon-14	5,730 years
radium-226	1,600 years
americium-241	432 years
cesium-137	30 years
strontium-90	28 years
polonium-210	138 days
thorium-234	24 days
iodine-131	8 days
radon-222	3.8 days
lead-211	36 minutes
krypton-91	10 seconds

It is customary to express the exponential model for radioactive decay in the form $m(t) = m_0 e^{-rt}$. To do this, we need to find the relative decay rate r. Since h is the half-life, we have

$$m(t) = m_0 e^{-rt} \qquad \text{Model}$$

$$\frac{m_0}{2} = m_0 e^{-rh} \qquad h \text{ is the half-life}$$

$$\frac{1}{2} = e^{-rh} \qquad \text{Divide by } m_0$$

$$\ln \frac{1}{2} = -rh \qquad \text{Take ln of each side}$$

$$r = \frac{\ln 2}{h} \qquad \text{Solve for } r \text{ } (\ln(\tfrac{1}{2}) = -\ln 2)$$

This last equation allows us to find the relative decay rate r from the half-life h.

Radioactive Decay Model

If m_0 is the initial mass of a radioactive substance with half-life h, then the mass remaining at time t is modeled by the function

$$m(t) = m_0 e^{-rt}$$

where $r = \dfrac{\ln 2}{h}$ is the **relative decay rate**.

Example 7 ■ Radioactive Decay

Polonium-210 (^{210}Po) has a half-life of 138 days. Suppose a sample of this substance has a mass of 300 mg.

(a) Find a function $m(t) = m_0 2^{-t/h}$ that models the mass remaining after t days.

(b) Find a function $m(t) = m_0 e^{-rt}$ that models the mass remaining after t days.

(c) Find the mass remaining after one year.

(d) How long will it take for the sample to decay to a mass of 200 mg?

(e) Draw a graph of the sample mass as a function of time.

Solution

(a) We have $m_0 = 300$ and $h = 138$, so the amount remaining after t days is

$$m(t) = 300 \cdot 2^{-t/138}$$

(b) We have $m_0 = 300$ and $r = \ln 2/138 \approx 0.00502$, so the amount remaining after t days is

$$m(t) = 300 \cdot e^{-0.00502t}$$

In parts (c) and (d) we can also use the model found in part (a). Check that the result is the same using either model.

(c) We use the function we found in part (a) with $t = 365$ (1 year):

$$m(365) = 300e^{-0.00502(365)} \approx 48.014$$

Thus approximately 48 mg of ^{210}Po remains after 1 year.

(d) We use the function that we found in part (b) with $m(t) = 200$ and solve the resulting exponential equation for t:

$$200 = 300e^{-0.00502t} \qquad m(t) = m_0 e^{-rt}$$

$$\tfrac{2}{3} = e^{-0.00502t} \qquad \text{Divide by 300}$$

$$\ln \tfrac{2}{3} = \ln e^{-0.00502t} \qquad \text{Take ln of each side}$$

$$\ln \tfrac{2}{3} = -0.00502t \qquad \text{Property of ln}$$

$$t = \frac{-\ln \frac{2}{3}}{0.00502} \qquad \text{Solve for } t$$

$$t \approx 80.8 \qquad \text{Calculator}$$

The time required for the sample to decay to 200 mg is about 81 days.

(e) We can graph the model in part (a) or the one in part (b). The graphs are identical. See Figure 9.

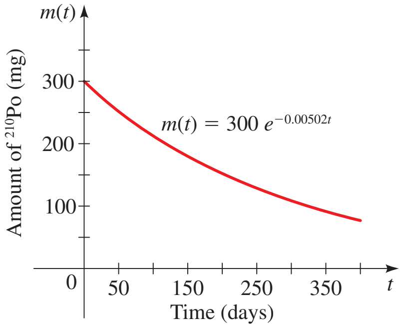


Figure 9

Now Try Exercise 21

Newton's Law of Cooling

Newton's Law of Cooling states that the rate at which an object cools is proportional to the temperature difference between the object and its surroundings, provided that the temperature difference is not too large. By using calculus, the following model can be deduced from this law.

Newton's Law of Cooling

If D_0 is the initial temperature difference between an object and its surroundings, and if its surroundings have temperature T_s, then the temperature of the object at time t is modeled by the function

$$T(t) = T_s + D_0 e^{-kt}$$

where k is a positive constant that depends on the type of object.

Example 8 ■ Newton's Law of Cooling

A cup of coffee has a temperature of 200°F and is placed in a room that has a temperature of 70°F. After 10 minutes the temperature of the coffee is 150°F.

(a) Find a function that models the temperature of the coffee after t minutes.

(b) Find the temperature of the coffee after 15 minutes.

(c) After how long will the coffee have cooled to 100°F?

(d) Illustrate by drawing a graph of the temperature function.

Solution

(a) The temperature of the room is $T_s = 70$°F, and the initial temperature difference is

$$D_0 = 200 - 70 = 130°\text{F}$$

So by Newton's Law of Cooling, the temperature after t minutes is modeled by the function

$$T(t) = 70 + 130e^{-kt}$$

We need to find the constant k associated with this cup of coffee. To do this, we use the fact that when $t = 10$, the temperature is $T(10) = 150$. So

$$\begin{aligned} 70 + 130e^{-10k} &= 150 && T_s + D_0 e^{-kt} = T(t) \\ 130e^{-10k} &= 80 && \text{Subtract 70} \\ e^{-10k} &= \tfrac{8}{13} && \text{Divide by 130} \\ -10k &= \ln \tfrac{8}{13} && \text{Take ln of each side} \\ k &= -\tfrac{1}{10} \ln \tfrac{8}{13} && \text{Solve for } k \\ k &\approx 0.04855 && \text{Calculator} \end{aligned}$$

Substituting this value of k into the expression for $T(t)$, we get

$$T(t) = 70 + 130e^{-0.04855t}$$

(b) We use the function that we found in part (a) with $t = 15$:

$$T(15) = 70 + 130e^{-0.04855(15)} \approx 133°\text{F}$$

(c) We use the function that we found in part (a) with $T(t) = 100$ and solve the resulting exponential equation for t.

$$\begin{aligned} 70 + 130e^{-0.04855t} &= 100 && T_s + D_0 e^{-kt} = T(t) \\ 130e^{-0.04855t} &= 30 && \text{Subtract 70} \\ e^{-0.04855t} &= \tfrac{3}{13} && \text{Divide by 130} \\ -0.04855t &= \ln \tfrac{3}{13} && \text{Take ln of each side} \\ t &= \frac{\ln \frac{3}{13}}{-0.04855} && \text{Solve for } t \\ t &\approx 30.2 && \text{Calculator} \end{aligned}$$

The coffee will have cooled to 100°F after about half an hour.

(d) The graph of the temperature function is sketched in Figure 10. Notice that the line $t = 70$ is a horizontal asymptote, so as we would expect, the temperature of the coffee decreases to the temperature of the surroundings.

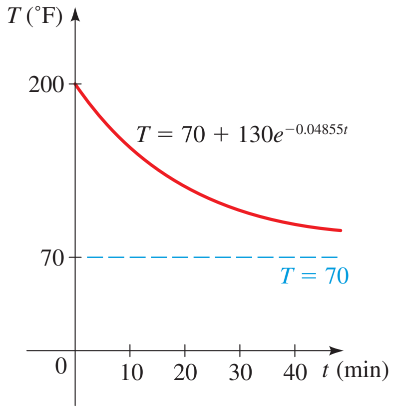


Figure 10 | Temperature of coffee after t minutes

Now Try Exercise 33 ■

4.6 Exercises

Applications

1–16 ■ Exponential Growth These exercises use the exponential growth model.

1. Bacteria Culture A certain culture of the bacterium *Streptococcus A* initially has 10 bacteria and is observed to double every 1.5 hours.

(a) Find an exponential model $n(t) = n_0 2^{t/a}$ for the number of bacteria in the culture after t hours.

(b) Estimate the number of bacteria after 35 hours.

(c) After how many hours will the bacteria count reach 10,000?

SciePro/Shutterstock.com

Streptococcus A
(12,000 × magnification)

2. Bacteria Culture A certain culture of the bacterium *Rhodobacter sphaeroides* initially has 25 bacteria and is observed to double every 5 hours.

(a) Find an exponential model

$$n(t) = n_0 2^{t/a}$$

for the number of bacteria in the culture after t hours.

(b) Estimate the number of bacteria after 18 hours.

(c) After how many hours will the bacteria count reach 1 million?

3. Squirrel Population A grey squirrel population was introduced in a certain county of Great Britain 30 years ago. Biologists observe that the population doubles every 6 years, and now the population is 100,000.

(a) What was the initial size of the squirrel population?

(b) Estimate the squirrel population 10 years from now.

(c) Sketch a graph of the squirrel population.

4. Bird Population A species of bird was introduced in a certain county 25 years ago. Biologists observe that the population doubles every 10 years, and now the population is 13,000.

(a) What was the initial size of the bird population?

(b) Estimate the bird population 5 years from now.

(c) Sketch a graph of the bird population.

5. Beaver Population Beavers are sometimes seen as pests, but lately scientists have discovered the importance of this dam-building species to maintaining the viability of freshwater ecosystems. For instance, beaver dams create ponds and wetlands, help store water for farms and ranches, and help filter out water pollution. It is estimated that for a certain northeastern ecosystem a beaver population has a relative growth rate of 12% per year and the population in 2005 was 12,800.

(a) Find a function

$$n(t) = n_0 e^{rt}$$

that models the population t years after 2005.

(b) Use the model from part (a) to estimate the beaver population in 2010.

(c) After how many years will the population reach 50,000?

(d) Sketch a graph of the beaver population function for the years 2005 to 2020.

P Harstela/Shutterstock.com

6. Prairie Dog Population Decline Ecologists have identified prairie dogs as a keystone species of the grasslands ecosystem of the West. Researchers have observed that the population of a certain species of prairie dog is declining at a relative rate of 6% per year in a certain county of South Dakota. (This means the relative growth rate is -0.06.) It is estimated that the population in 2010 was 450,000.

(a) Find a function $n(t) = n_0 e^{rt}$ that models the population t years after 2010.

(b) Use the model from part (a) to estimate the prairie dog population in 2025.

(c) After how many years will the population decrease to 300,000?

(d) Sketch a graph of the prairie dog population function for the years 2010 to 2025.

7. Population of a Country The population of a country has a relative growth rate of 3% per year. The government is trying to reduce the growth rate to 2%. The population in 2011 was approximately 110 million. Find the projected population for the year 2036 for the following conditions.

(a) The relative growth rate remains at 3% per year.

(b) The relative growth rate is reduced to 2% per year.

8. Bacteria Culture It is observed that a certain bacteria culture has a relative growth rate of 12% per hour, but in the presence of an antibiotic the relative growth rate is reduced to 5% per hour. The initial number of bacteria in the culture is 22. Find the projected population after 24 hours for the following conditions.

(a) No antibiotic is present, so the relative growth rate is 12%.

(b) An antibiotic is present in the culture, so the relative growth rate is reduced to 5%.

9. Population of a City The population of a certain city was 112,000 in 2014, and the observed doubling time for the population is 18 years.

(a) Find an exponential model $n(t) = n_0 2^{t/a}$ for the population t years after 2014.

(b) Find an exponential model $n(t) = n_0 e^{rt}$ for the population t years after 2014.

(c) Sketch a graph of the population at time t.

(d) Estimate how long it takes the population to reach 500,000.

10. Bat Population The bat population in a certain Midwestern county was 350,000 in 2012, and the observed doubling time for the population is 25 years.

(a) Find an exponential model $n(t) = n_0 2^{t/a}$ for the population t years after 2012.

(b) Find an exponential model $n(t) = n_0 e^{rt}$ for the population t years after 2012.

(c) Sketch a graph of the population at time t.

(d) Estimate how long it takes the population to reach 2 million.

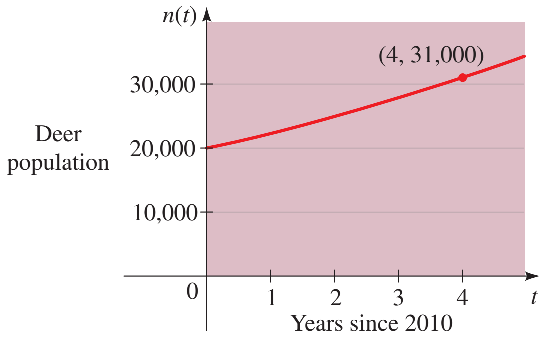
11. Deer Population The graph shows the deer population in a Pennsylvania county between 2010 and 2014. Assume that the population grows exponentially.

(a) What was the deer population in 2010?

(b) Find a function that models the deer population t years after 2010.

(c) What is the projected deer population in 2018?

(d) Estimate how long it takes the population to reach 100,000.



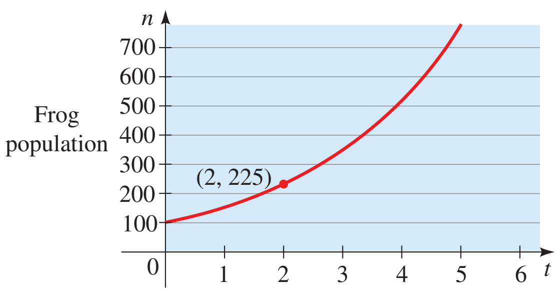
12. **Frog Population** Some bullfrogs were introduced into a small pond. The graph shows the bullfrog population for the next few years. Assume that the population grows exponentially.

(a) What was the initial bullfrog population?

(b) Find a function that models the bullfrog population t years since the bullfrogs were put into the pond.

(c) What is the projected bullfrog population after 15 years?

(d) Estimate how long it takes the population to reach 75,000.



13. **Bacteria Culture** A culture starts with 8600 bacteria. After 1 hour the count is 10,000.

(a) Find a function that models the number of bacteria $n(t)$ after t hours.

(b) Find the number of bacteria after 2 hours.

(c) After how many hours will the number of bacteria double?

14. **Bacteria Culture** The count in a culture of bacteria was 400 after 2 hours and 25,600 after 6 hours.

(a) What is the relative rate of growth of the bacteria population? Express your answer as a percentage.

(b) What was the initial size of the culture?

(c) Find a function that models the number of bacteria $n(t)$ after t hours.

(d) Find the number of bacteria after 4.5 hours.

(e) After how many hours will the number of bacteria reach 50,000?

15. **Population Decline** The population of a certain country was 49 million in 2000 and 44 million in 2019. Assume that the population continues to decline at this rate.

(a) Find a function that models the population (in millions) t years after 2000. Use the model to estimate in what year the population will decline to 35 million.

(b) In how many years will the population be cut in half?

16. **World Population** The population of the world was 7.8 billion in 2020, and the observed relative growth rate was 1.04% per year. Assume that the world population continues to grow at this rate. Use an exponential model to estimate each of the following.

(a) The year in which year the population will reach 10 billion.

(b) The time required for the population to double.

17–20 ■ Logistic Growth These exercises use the logistic growth model.

17. **World Population** The relative growth rate of world population has been decreasing steadily in recent years. Based on this, some population models predict that world population will eventually stabilize around 11 billion. In 1950 the world population was 2.5 billion and the observed relative growth rate was 1.89% per year.

(a) Assuming the carrying capacity of the earth is 11 billion, model the population (in billions) with a logistic growth model, where t is the number of years since 1950. Use the model to estimate the year in which the population will reach 10 billion.

(b) Draw a graph of the logistic growth model.

18. **Beaver Population** Suppose that the beaver population in Exercise 5 exists on an island that can support a maximum of 80,000 beavers. An initial population of 12,800 beavers is introduced onto the island in 2005.

(a) Model the population $n(t)$ with a logistic growth model, where t is the number of years since 2005.

(b) Draw a graph of the logistic growth model and compare with the exponential growth model in Exercise 5 for the years 2005 to 2045.

19. **Spread of a Disease** An infectious disease begins to spread in a small city of population 10,000. Initially, 8 people in the city are infected with the disease. Researchers have observed that the initial relative growth rate of the disease is 0.57 per day, and the number of people that get infected levels off when 60% of the population have been infected. (So the carrying capacity of the disease is 6000.)

(a) Model the number of infections $n(t)$ with a logistic growth model, where t is the number of days since the initial 8 people were infected.

(b) Draw a graph of the logistic growth model. After how many days does the number of infections reach 5900?

20. **Overstocked Pond** A pond in a fish farm has a carrying capacity of 10,000 fish of a certain species of fish that has relative growth rate $r = 0.2$ per year. The pond is inadvertently stocked with 18,000 fish.

(a) Find a logistic model $n(t)$ for the fish population t years since the pond was stocked.

(b) Draw a graph of the model you found and use it to describe how the population changes over many years. What does $n(t)$ approach as $t \to \infty$?

21–31 ■ Radioactive Decay These exercises use the radioactive decay model.

21. Radioactive Radium The half-life of radium-226 is 1600 years. Suppose we have a 22-milligram sample.

(a) Find a function $m(t) = m_0 2^{-t/h}$ that models the mass remaining after t years.

(b) Find a function $m(t) = m_0 e^{-rt}$ that models the mass remaining after t years.

(c) How much of the sample will remain after 4000 years?

(d) After how many years will only 18 mg of the sample remain?

22. Radioactive Cesium The half-life of cesium-137 is 30 years. Suppose we have a 10-gram sample.

(a) Find a function $m(t) = m_0 2^{-t/h}$ that models the mass remaining after t years.

(b) Find a function $m(t) = m_0 e^{-rt}$ that models the mass remaining after t years.

(c) How much of the sample will remain after 80 years?

(d) After how many years will only 2 g of the sample remain?

23. Radioactive Strontium The half-life of strontium-90 is 29 years. How long will it take a 50-milligram sample to decay to a mass of 32 mg?

24. Radioactive Radium Radium-221 has a half-life of 30 s. How long will it take for 95% of a sample to decay?

25. Finding Half-Life If 250 mg of a radioactive element decays to 200 mg in 48 hours, find the half-life of the element.

26. Radioactive Radon After 3 days a sample of radon-222 has decayed to 58% of its original amount.

(a) What is the half-life of radon-222?

(b) How long will it take the sample to decay to 20% of its original amount?

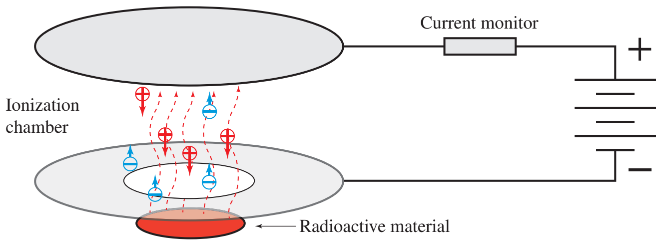
27. Carbon-14 Dating A wooden artifact from an ancient tomb contains 65% of the carbon-14 that is present in living trees. How long ago was the artifact made? (The half-life of carbon-14 is 5730 years.)

28. Carbon-14 Dating The burial cloth of an Egyptian mummy is estimated to contain 59% of the carbon-14 it contained originally. How long ago was the mummy buried? (The half-life of carbon-14 is 5730 years.)

29. Uranium-Lead Dating Radioactive uranium-238 atoms decay to lead-206 atoms with a half-life of 4.5 billion years. A sample of zircon crystal contains 0.04 moles of uranium-238 and 0.01 moles of lead-206. Assuming that the lead-206 atoms were the product of the decay of uranium-238 atoms, it follows that the initial number of uranium-238 atoms in the sample was $m_0 = 0.04 + 0.01 = 0.05$ mole. Determine the age of the sample using the uranium-238 to lead-206 decay pathway.

30. Concordant Dating Scientists use different radioactive decay pathways to arrive at a *concordant age* for a sample, that is, an age for which the different pathways reasonably agree. It is known that uranium-235 decays to lead-207 with a half-life of 700 million years. Suppose the sample in Exercise 29 is found to also contain 0.0005 mole of uranium-235 and 0.0015 mole of lead-207. Determine the age of the sample using the uranium-235 to lead-207 decay pathway. Does your answer reasonably agree with the answer to Exercise 29? Would you say that the calculated ages are concordant?

31. Smoke Detectors Smoke alarms use a small amount of radioactive americium-241, which ionizes the air between two electrically charged plates, causing current to flow between the plates. When smoke enters the chamber, it disrupts the current, thus activating the alarm. Suppose that a smoke detector contains 0.29 microgram of americium-241. How long does it take for the radioactive material to decrease to 80% of its initial mass? (The half-life of americium-241 is 432 years.)



Note: The National Fire Protection Association (NFPA) recommends that smoke detectors be replaced after 10 years and that batteries be replaced every six months. The americium-241 will outlast every other component of the detector.

32–35 ■ Law of Cooling These exercises use Newton's Law of Cooling.

32. Time of Death Newton's Law of Cooling is used in homicide investigations to determine the time of death. The normal body temperature is 98.6°F. Immediately following death, the body begins to cool. It has been determined experimentally that the constant in Newton's Law of Cooling is approximately $k = 0.1947$, assuming that time is measured in hours. Suppose that the temperature of the surroundings is 60°F.

(a) Find a function $T(t)$ that models the temperature t hours after death.

(b) If the temperature of the body is now 72°F, how long ago was the time of death?

33. Cooling Soup A hot bowl of soup is served at a dinner party. It starts to cool according to Newton's Law of Cooling, so its temperature at time t is given by

$$T(t) = 65 + 145e^{-0.05t}$$

where t is measured in minutes and T is measured in °F.

(a) What is the initial temperature of the soup?

(b) What is the temperature after 10 min?

(c) After how long will the temperature be 100°F?

34. Cooling Turkey A roasted turkey is taken from an oven when its temperature has reached 185°F and is placed on a table in a room where the temperature is 75°F.

(a) If the temperature of the turkey is 150°F after half an hour, what is its temperature after 45 min?

(b) After how many hours has the turkey cooled to 100°F?

35. Boiling Water A kettle full of water is brought to a boil in a room with temperature 20°C. After 15 minutes the temperature of the water has decreased from 100°C to 75°C. Find the temperature after another 10 minutes. Illustrate by graphing the temperature function.

4.7 Logarithmic Scales

■ The pH Scale ■ The Richter Scale ■ The Decibel Scale

Animal	W (kg)	$\log W$
Ant	0.000003	-5.5
Elephant	4000	3.6
Whale	170,000	5.2

When a physical quantity varies over a very large range, it is often convenient to take its logarithm in order to work with more manageable numbers. On a **logarithmic scale**, numbers are represented by their logarithms. For example, the table in the margin gives the weights W of some animals (in kilograms) and their logarithms ($\log W$).

The weights (W) vary enormously, but on a logarithmic scale, the weights are represented by more manageable numbers ($\log W$), as illustrated in Figure 1.

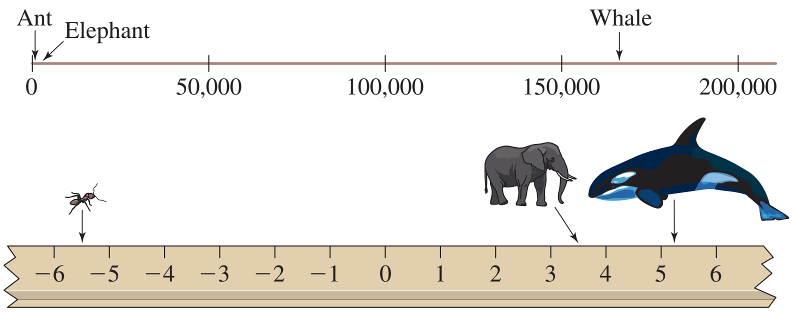


Figure 1 | Weight graphed on the real line (top) and on a logarithmic scale (bottom)

We discuss three commonly used logarithmic scales: the pH scale, which measures acidity; the Richter scale, which measures the intensity of earthquakes; and the decibel scale, which measures the loudness of sounds. Other quantities that are measured on logarithmic scales are light intensity, information capacity, and radiation.

■ The pH Scale

Chemists measured the acidity of a solution by giving its hydrogen ion concentration until Søren Peter Lauritz Sørensen, in 1909, proposed a more convenient measure. He defined

$$\boxed{\text{pH} = -\log[\text{H}^+]}$$

where $[\text{H}^+]$ is the concentration of hydrogen ions measured in moles per liter (M). He did this to avoid very small numbers and negative exponents. For instance,

$$\text{if} \qquad [\text{H}^+] = 10^{-4}\ \text{M}, \qquad \text{then} \qquad \text{pH} = -\log_{10}(10^{-4}) = -(-4) = 4$$

Solutions with a pH of 7 are defined as *neutral*, those with pH < 7 are *acidic*, and those with pH > 7 are *basic*. Notice that when the pH increases by one unit, $[\text{H}^+]$ decreases by a factor of 10.

pH for Some Common Substances

Substance	pH
Milk of magnesia	10.5
Seawater	8.0–8.4
Human blood	7.3–7.5
Crackers	7.0–8.5
Hominy	6.9–7.9
Cow's milk	6.4–6.8
Spinach	5.1–5.7
Tomatoes	4.1–4.4
Oranges	3.0–4.0
Apples	2.9–3.3
Limes	1.3–2.0
Battery acid	1.0

Example 1 ■ pH Scale and Hydrogen Ion Concentration

(a) The hydrogen ion concentration of a sample of human blood was measured to be $[H^+] = 3.16 \times 10^{-8}$ M. Find the pH, and classify the blood as acidic or basic.

(b) The most acidic rainfall ever measured occurred in Scotland in 1974; its pH was 2.4. Find the hydrogen ion concentration.

Solution

(a) A calculator gives

$$\text{pH} = -\log[H^+] = -\log(3.16 \times 10^{-8}) \approx 7.5$$

Since this is greater than 7, the blood is basic.

(b) To find the hydrogen ion concentration, we need to solve for $[H^+]$ in the logarithmic equation

$$\log[H^+] = -\text{pH}$$

so we write it in exponential form:

$$[H^+] = 10^{-\text{pH}}$$

In this case pH = 2.4, so

$$[H^+] = 10^{-2.4} \approx 4.0 \times 10^{-3}\text{ M}$$

Now Try Exercises 1 and 3 ■

■ The Richter Scale

In 1935 the American geologist Charles Richter (1900–1985) defined the magnitude M of an earthquake to be

$$\boxed{M = \log \frac{I}{S}}$$

where I is the intensity of the earthquake (measured by the amplitude of a seismograph reading taken 100 km from the epicenter of the earthquake) and S is the intensity of a "standard" earthquake (whose amplitude is 1 micron $= 10^{-4}$ cm). (In practice, seismograph stations may not be exactly 100 km from the epicenter, so appropriate adjustments are made in calculating the magnitude of an earthquake.) The magnitude of a standard earthquake is

$$M = \log \frac{S}{S} = \log 1 = 0$$

Robert Vos/AFP/Getty Images

Discovery Project ■ The Even-Tempered Clavier

Poets, writers, philosophers, and even politicians have extolled the virtues of music—its beauty and its power to communicate emotion. But at the heart of music is a logarithmic scale. The tones that we are familiar with from our everyday listening can all be reproduced by the keys of a piano. The keys of a piano, in turn, are "evenly tempered" using a logarithmic scale. In this project we explore how exponential and logarithmic functions are used in properly tuning a piano. You can find the project at **www.stewartmath.com**.

Largest Earthquakes

Location	Date	Magnitude
Chile	1960	9.5
Alaska	1964	9.2
Japan	2011	9.1
Sumatra	2004	9.1
Kamchatka	1952	9.0
Chile	2010	8.8
Ecuador	1906	8.8
Alaska	1965	8.7
Alaska	1957	8.6
Sumatra	2005	8.6
Sumatra	2012	8.6
Tibet	1950	8.6
Indonesia	1938	8.5
Kamchatka	1923	8.5

Source: US Geological Society

Richter studied many earthquakes that occurred between 1900 and 1950. The largest had magnitude 8.9 on the Richter scale, and the smallest had magnitude 0. This corresponds to a ratio of intensities of 800,000,000, so the Richter scale provides more manageable numbers to work with. For instance, an earthquake of magnitude 6 is ten times stronger than an earthquake of magnitude 5.

Example 2 ■ Magnitude and Intensity

(a) Find the magnitude of an earthquake that has an intensity of 3.75 (that is, the amplitude of the seismograph reading is 3.75 cm).

(b) An earthquake was measured to have a magnitude of 5.1 on the Richter scale. Find the intensity of the earthquake.

Solution

(a) From the definition of magnitude we see that

$$M = \log \frac{I}{S} = \log \frac{3.75}{10^{-4}} = \log 37500 \approx 4.6$$

Thus the magnitude is 4.6 on the Richter scale.

(b) To find the intensity, we need to solve for I in the logarithmic equation

$$M = \log \frac{I}{S}$$

so we write it in exponential form:

$$10^M = \frac{I}{S}$$

In this case $S = 10^{-4}$ and $M = 5.1$, so

$$\begin{aligned} 10^{5.1} &= \frac{I}{10^{-4}} && M = 5.1,\ S = 10^{-4} \\ (10^{-4})(10^{5.1}) &= I && \text{Multiply by } 10^{-4} \\ I &= 10^{1.1} \approx 12.6 && \text{Add exponents} \end{aligned}$$

Thus the intensity of the earthquake is about 12.6, which means that the amplitude of the seismograph reading is about 12.6 cm.

Now Try Exercise 9 ■

Example 3 ■ Magnitude of Earthquakes

There are several other logarithmic scales used to calculate the magnitude of earthquakes. For instance, the US Geological Survey uses the *moment magnitude scale*.

The 1906 earthquake in San Francisco had an estimated magnitude of 8.3 on the Richter scale. In the same year a powerful earthquake occurred on the Colombia-Ecuador border that was four times as intense. What was the magnitude of the Colombia-Ecuador earthquake on the Richter scale?

Solution If I is the intensity of the San Francisco earthquake, then from the definition of magnitude we have

$$M = \log \frac{I}{S} = 8.3$$

The intensity of the Colombia-Ecuador earthquake was $4I$, so its magnitude was

$$M = \log \frac{4I}{S} = \log 4 + \log \frac{I}{S} = \log 4 + 8.3 \approx 8.9$$

Now Try Exercise 11 ■

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Example 4 ■ Intensity of Earthquakes

The 1989 Loma Prieta earthquake that shook San Francisco had a magnitude of 7.1 on the Richter scale. How many times more intense was the 1906 earthquake (see Example 3) than the 1989 event?

Solution If I_1 and I_2 are the intensities of the 1906 and 1989 earthquakes, then we are required to find I_1/I_2. To relate this to the definition of magnitude, we divide the numerator and denominator by S.

$$\begin{aligned}
\log \frac{I_1}{I_2} &= \log \frac{I_1/S}{I_2/S} && \text{Divide numerator and denominator by } S \\
&= \log \frac{I_1}{S} - \log \frac{I_2}{S} && \text{Law 2 of logarithms} \\
&= 8.3 - 7.1 = 1.2 && \text{Definition of earthquake magnitude}
\end{aligned}$$

Therefore

$$\frac{I_1}{I_2} = 10^{\log(I_1/I_2)} = 10^{1.2} \approx 16$$

The 1906 earthquake was about 16 times as intense as the 1989 earthquake.

Now Try Exercise 13 ■

■ The Decibel Scale

The ear is sensitive to an extremely wide range of sound intensities I (measured in W/m^2). We take as a reference intensity $I_0 = 10^{-12}\ W/m^2$ (watts per square meter) at a frequency of 1000 hertz, which measures a sound that is just barely audible (the threshold of hearing). The psychological sensation of loudness varies with the logarithm of the intensity (the Weber-Fechner Law), so the **decibel level** B, measured in decibels (dB), is defined as

$$\boxed{B = 10 \log \frac{I}{I_0}}$$

The decibel level of the barely audible reference sound is

$$B = 10 \log \frac{I_0}{I_0} = 10 \log 1 = 0 \text{ dB}$$

Example 5 ■ Decibel Level and Intensity

(a) Find the decibel level of a jet engine at takeoff if the intensity was measured at $100\ W/m^2$.

(b) Find the intensity level of a motorcycle engine at full throttle if the decibel level was measured at 90 dB.

Solution

(a) From the definition of decibel level we see that

$$B = 10 \log \frac{I}{I_0} = 10 \log \frac{10^2}{10^{-12}} = 10 \log 10^{14} = 140$$

Thus the decibel level is 140 dB.

The **decibel levels of sounds** that we can hear vary from very loud to very soft. Here are some examples of the decibel levels of commonly heard sounds.

Source of Sound	B (dB)
Jet takeoff	140
Jackhammer	130
Rock concert	120
Subway	100
Heavy traffic	80
Ordinary traffic	70
Normal conversation	50
Whisper	30
Rustling leaves	10–20
Threshold of hearing	0

(b) To find the intensity, we need to solve for I in the logarithmic equation

$$B = 10 \log \frac{I}{I_0} \qquad \text{Definition of decibel level}$$

$$\frac{B}{10} = \log I - \log 10^{-12} \qquad \text{Divide by 10, } I_0 = 10^{-12}$$

$$\frac{B}{10} = \log I + 12 \qquad \text{Definition of logarithm}$$

$$\frac{B}{10} - 12 = \log I \qquad \text{Subtract 12}$$

$$\log I = \frac{90}{10} - 12 = -3 \qquad B = 90$$

$$I = 10^{-3} \qquad \text{Exponential form}$$

Thus the intensity is 10^{-3} W/m^2.

Now Try Exercises 15 and 17 ■

The table in the margin lists decibel levels for some common sounds ranging from the threshold of human hearing to the jet takeoff of Example 5. The threshold of pain is about 120 dB.

4.7 Exercises

Applications

1. Finding pH The hydrogen ion concentration of a sample of each substance is given. Calculate the pH of the substance.

(a) Lemon juice: $[H^+] = 5.0 \times 10^{-3}$ M

(b) Tomato juice: $[H^+] = 3.2 \times 10^{-4}$ M

(c) Seawater: $[H^+] = 5.0 \times 10^{-9}$ M

2. Finding pH An unknown substance has a hydrogen ion concentration of $[H^+] = 3.1 \times 10^{-8}$ M. Find the pH and classify the substance as acidic or basic.

3. Ion Concentration The pH reading of a sample of each substance is given. Calculate the hydrogen ion concentration of the substance.

(a) Vinegar: pH = 3.0 **(b)** Milk: pH = 6.5

4. Ion Concentration The pH reading of a glass of liquid is given. Find the hydrogen ion concentration of the liquid.

(a) Beer: pH = 4.6 **(b)** Water: pH = 7.3

5. Finding pH The hydrogen ion concentrations in cheeses range from 4.0×10^{-7} M to 1.6×10^{-5} M. Find the corresponding range of pH readings.

6. Finding pH of Wine The hydrogen ion concentrations for wines vary from 1.58×10^{-4} M to 1.58×10^{-3} M. Find the corresponding range of pH readings.

7. pH of Wine If the pH of a wine is too high, say, 4.0 or above, the wine becomes unstable and has a flat taste.

(a) A certain California red wine has a pH of 3.2, and a certain Italian white wine has a pH of 2.9. Find the corresponding hydrogen ion concentrations of the two wines.

(b) Which wine has the lower hydrogen ion concentration?

8. pH of Saliva The pH of saliva is normally in the range of 6.4 to 7.0. However, when a person is ill, the person's saliva becomes more acidic.

(a) Suppose the pH of the saliva of a patient is 5.5. What is the hydrogen ion concentration of the patient's saliva?

(b) After the patient recovers, the pH of their saliva is 6.5. Was the saliva more acidic or less acidic when the patient was sick?

(c) Will the hydrogen ion concentration of the saliva increase or decrease as the patient recovers?

9. Earthquake Magnitude and Intensity

(a) Find the magnitude of an earthquake that has an intensity of 31.25 (that is, the amplitude of the seismograph reading is 31.25 cm).

(b) An earthquake was measured to have a magnitude of 4.8 on the Richter scale. Find the intensity of the earthquake.

10. Earthquake Magnitude and Intensity

(a) Find the magnitude of an earthquake that has an intensity of 72.1 (that is, the amplitude of the seismograph reading is 72.1 cm).

(b) An earthquake was measured to have a magnitude of 5.8 on the Richter scale. Find the intensity of the earthquake.

11. Earthquake Magnitudes If one earthquake is 20 times as intense as another, how much larger is its magnitude on the Richter scale?

12. Earthquake Magnitudes The 1906 earthquake in San Francisco had a magnitude of 8.3 on the Richter scale. At the same time in Japan an earthquake with magnitude 4.9 caused only minor damage. How many times more intense was the San Francisco earthquake than the Japan earthquake?

13. Earthquake Magnitudes The Japan earthquake of 2011 had a magnitude of 9.1 on the Richter scale. How many times more intense was this than the 1906 San Francisco earthquake? (See Exercise 12.)

14. Earthquake Magnitudes The Northridge, California, earthquake of 1994 had a magnitude of 6.8 on the Richter scale. A year later, a 7.2-magnitude earthquake struck Kobe, Japan. How many times more intense was the Kobe earthquake than the Northridge earthquake?

15. Traffic Noise The intensity of the sound of traffic at a busy intersection was measured at 2.0×10^{-5} W/m^2. Find the decibel level.

16. Leaf Blower The intensity of the sound from a certain leaf blower is measured at 3.2×10^{-2} W/m^2. Find the decibel level.

17. Hair Dryer The decibel level of the sound from a certain hair dryer is measured at 70 dB. Find the intensity of the sound.

18. Subway Noise The decibel level of the sound of a subway train was measured at 98 dB. Find the intensity of the sound.

19. Hearing Loss from Headphones Recent research has shown that the use of earbuds can cause permanent hearing loss.

(a) The intensity of the sound from the speakers of a certain audio system is measured at 3.1×10^{-5} W/m^2. Find the decibel level.

(b) If earbuds are used with the audio system in part (a), the decibel level is 90 dB. Find the intensity.

(c) Find the ratio of the intensity of the sound from the audio system with earbuds to that of the sound without earbuds.

20. Comparing Decibel Levels The noise from a power mower was measured at 106 dB. The noise level at a rock concert was measured at 120 dB. Find the ratio of the intensity of the rock music to that of the power mower.

Discuss ■ Discover ■ Prove ■ Write

21. Prove: Inverse Square Law for Sound A law of physics states that the intensity of sound is inversely proportional to the square of the distance d from the source: $I = k/d^2$.

(a) Use this model and the equation

$$B = 10 \log \frac{I}{I_0}$$

(described in this section) to show that the decibel levels B_1 and B_2 at distances d_1 and d_2 from a sound source are related by the equation

$$B_2 = B_1 + 20 \log \frac{d_1}{d_2}$$

(b) The intensity level at a rock concert is 120 dB at a distance 2 m from the speakers. Find the intensity level at a distance of 10 m.

Chapter 4 Review

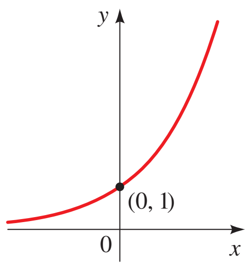
Properties and Formulas

Exponential Functions | Section 4.1

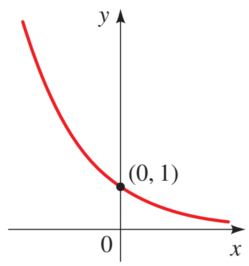
The **exponential function** f with base a (where $a > 0$, $a \neq 1$) is defined for all real numbers x by

$$f(x) = a^x$$

The domain of f is $\mathbb{R}$, and the range of f is $(0, \infty)$ The graph of f has one of the following shapes, depending on the value of a:



$f(x) = a^x$ for $a > 1$



$f(x) = a^x$ for $0 < a < 1$

The Natural Exponential Function | Section 4.2

The **natural exponential function** is the exponential function with base e:

$$f(x) = e^x$$

The number e is defined to be the number that the expression $(1 + 1/n)^n$ approaches as $n \to \infty$. An approximate value for the irrational number e is

$$e \approx 2.7182818284590\ldots$$

Compound Interest | Sections 4.1 and 4.2

If a principal P is invested in an account paying an annual interest rate r, compounded n times a year, then after t years the **amount** $A(t)$ in the account is

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

If the interest is compounded **continuously**, then the amount is

$$A(t) = Pe^{rt}$$

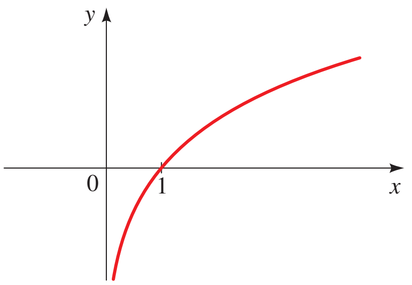
Logarithmic Functions | Section 4.3

The **logarithmic function** $\log_a$ with base a (where $a > 0, a \neq 1$) is defined for $x > 0$ by

$$\log_a x = y \quad \Leftrightarrow \quad a^y = x$$

So $\log_a x$ is the exponent to which the base a must be raised to give y.

The domain of $\log_a$ is $(0, \infty)$, and the range is $\mathbb{R}$. For $a > 1$, the graph of the function $\log_a$ has the following shape:



$y = \log_a x$ for $a > 1$

Common and Natural Logarithms | Section 4.3

The logarithm function with base 10 is called the **common logarithm** and is denoted **log**. So

$$\log x = \log_{10} x$$

The logarithm function with base e is called the **natural logarithm** and is denoted **ln**. So

$$\ln x = \log_e x$$

Properties of Logarithms | Section 4.3

1. $\log_a 1 = 0$

2. $\log_a a = 1$

3. $\log_a a^x = x$

4. $a^{\log_a x} = x$

Laws of Logarithms | Section 4.4

Let a be a logarithm base $(a > 0, a \neq 1)$, and let A, B, and C be any real numbers or algebraic expressions that represent real numbers, with $A > 0$ and $B > 0$. Then:

1. $\log_a(AB) = \log_a A + \log_a B$

2. $\log_a(A/B) = \log_a A - \log_a B$

3. $\log_a(A^C) = C \log_a A$

Change of Base Formula | Section 4.4

$$\log_b x = \frac{\log_a x}{\log_a b}$$

Guidelines for Solving Exponential Equations | Section 4.5

1. Isolate the exponential term on one side of the equation.

2. Take the logarithm of each side, and use the Laws of Logarithms to "bring down the exponent."

3. Solve for the variable.

Guidelines for Solving Logarithmic Equations | Section 4.5

1. Isolate the logarithmic term(s) on one side of the equation, and use the Laws of Logarithms to combine logarithmic terms if necessary.

2. Rewrite the equation in exponential form.

3. Solve for the variable.

Exponential Growth Model | Section 4.6

A population experiences **exponential growth** if it can be modeled by the exponential function

$$n(t) = n_0 e^{rt}$$

where $n(t)$ is the population at time t, n_0 is the initial population (at time $t = 0$), and r is the relative growth rate (expressed as a proportion of the population).

Logistic Growth Model | Section 4.6

A population experiences **logistic growth** if it can be modeled by a function of the form

$$n(t) = \frac{M}{1 + Ae^{-rt}}$$

where $n(t)$ is the population at time t, r is the initial relative growth rate, M is the **carrying capacity** of the environment, and $A = (M - n_0)/n_0$, where n_0 is the initial population.

Radioactive Decay Model | Section 4.6

If a **radioactive substance** with half-life h has initial mass m_0, then at time t the mass $m(t)$ of the substance that remains is modeled by the exponential function

$$m(t) = m_0 e^{-rt}$$

where $r = \dfrac{\ln 2}{h}$.

Newton's Law of Cooling | Section 4.6

If an object has an initial temperature that is D_0 degrees warmer than the surrounding temperature T_s, then at time t the temperature $T(t)$ of the object is modeled by the function

$$T(t) = T_s + D_0 e^{-kt}$$

where the constant $k > 0$ depends on the size and type of the object.

Logarithmic Scales | Section 4.7

The **pH scale** measures the acidity of a solution:

$$\text{pH} = -\log[\text{H}^+]$$

The **Richter scale** measures the intensity of earthquakes:

$$M = \log \frac{I}{S}$$

The **decibel scale** measures the intensity of sound:

$$B = 10 \log \frac{I}{I_0}$$

Concept Check

1. Let f be the exponential function with base a.

(a) Write an equation that defines f.

(b) Write an equation for the exponential function f with base 3.

2. Let f be the exponential function $f(x) = a^x$, where $a > 0$.

(a) What is the domain of f?

(b) What is the range of f?

(c) Sketch graphs of f for the following cases.

(i) $a > 1$ **(ii)** $0 < a < 1$

3. If x is large, which function grows faster, $f(x) = 2^x$ or $g(x) = x^2$?

4. (a) How is the number e defined?

(b) Give an approximate value of e, correct to four decimal places.

(c) What is the natural exponential function?

5. (a) How is $\log_a x$ defined?

(b) Find $\log_3 9$.

(c) What is the natural logarithm?

(d) What is the common logarithm?

(e) Write the exponential form of the equation $\log_7 49 = 2$.

6. Let f be the logarithmic function $f(x) = \log_a x$.

(a) What is the domain of f?

(b) What is the range of f?

(c) Sketch a graph of the logarithmic function for the case $a > 1$.

7. State the three Laws of Logarithms.

8. (a) State the Change of Base Formula.

(b) Find $\log_7 30$.

9. (a) What is an exponential equation?

(b) How do you solve an exponential equation?

(c) Solve for x: $2^x = 19$

10. (a) What is a logarithmic equation?

(b) How do you solve a logarithmic equation?

(c) Solve for x: $4 \log_3 x = 7$

11. Suppose that an amount P is invested at an interest rate r and $A(t)$ is the amount of the investment after t years. Write a formula for $A(t)$ in the following cases.

(a) Interest is compounded n times per year.

(b) Interest is compounded continuously.

12. Suppose that the initial size of a population is n_0 and the population grows exponentially. Let $n(t)$ be the size of the population at time t.

(a) Write a formula for $n(t)$ in terms of the doubling time a.

(b) Write a formula for $n(t)$ in terms of the relative growth rate r.

13. Suppose that the initial size of a population is n_0, the initial growth rate is r, and the population is in an environment with carrying capacity M. Find a logistic model for $n(t)$, the size of the population at time t.

14. Suppose that the initial mass of a radioactive substance is m_0 and the half-life of the substance is h. Let $m(t)$ be the mass remaining at time t.

(a) What is meant by the half-life h?

(b) Write a formula for $m(t)$ in terms of the half-life h.

(c) Write a formula for the relative decay rate r in terms of the half-life h.

(d) Write a formula for $m(t)$ in terms of the relative decay rate r.

15. Suppose that the initial temperature difference between an object and its surroundings is D_0 and the surroundings have temperature T_s. Let $T(t)$ be the temperature at time t. State Newton's Law of Cooling for $T(t)$.

16. What is a logarithmic scale? If we use a logarithmic scale with base 10, what do the following numbers correspond to on the logarithmic scale?

(i) 100 **(ii)** 100,000 **(iii)** 0.0001

17. (a) What does the pH scale measure?

(b) Define the pH of a substance with hydrogen ion concentration $[H^+]$.

18. (a) What does the Richter scale measure?

(b) Define the magnitude M of an earthquake in terms of the intensity I of the earthquake and the intensity S of a standard earthquake.

19. (a) What does the decibel scale measure?

(b) Define the decibel level B of a sound in terms of the intensity I of the sound and the intensity I_0 of a barely audible sound.

Answers to the Concept Check can be found at the book companion website stewartmath.com.

Exercises

1–4 ■ Evaluating Exponential Functions Use a calculator to find the indicated values of the exponential function, rounded to three decimal places.

1. $f(x) = 5^x$; $f(-1.5), f(\sqrt{2}), f(2.5)$

2. $f(x) = 3 \cdot 2^x$; $f(-2.2), f(\sqrt{7}), f(5.5)$

3. $g(x) = 4e^{x-2}$; $g(-0.7), g(1), g(\pi)$

4. $g(x) = \frac{7}{4}e^{x+1}$; $g(-2), g(\sqrt{3}), g(3.6)$

5–18 ■ Graphing Exponential and Logarithmic Functions Sketch the graph of the function. State the domain, range, and asymptote.

5. $f(x) = 3^x + 1$

6. $f(x) = \left(\frac{1}{2}\right)^x - 5$

7. $g(x) = 4^{-(x+1)}$

8. $g(x) = -2^{x-1}$

9. $h(x) = -e^{x+2} - 1$

10. $h(x) = 3e^{-x} + 1$

11. $f(x) = \log_3(x - 2)$

12. $f(x) = -\log_4(x + 2)$

13. $f(x) = \log_{1/3} x + 1$

14. $f(x) = 1 - \log_{1/2}(x + 2)$

15. $g(x) = \log_2(-x) - 2$

16. $g(x) = -\log_3(x + 3) + 2$

17. $g(x) = 2 \ln x$

18. $g(x) = \ln(x^2)$

19–22 ■ Domain Find the domain of the function.

19. $f(x) = 10^{x^2} + \log(1 - 2x)$

20. $g(x) = \log(2 + x - x^2)$

21. $h(x) = \ln(x^2 - 4)$

22. $k(x) = \ln|x|$

23–26 ■ Exponential Form Write the equation in exponential form.

23. $\log_2 1024 = 10$

24. $\log_6 37 = x$

25. $\log x = y$

26. $\ln c = 17$

27–30 ■ Logarithmic Form Write the equation in logarithmic form.

27. $2^6 = 64$

28. $49^{-1/2} = \frac{1}{7}$

29. $10^x = 74$

30. $e^k = m$

31–46 ■ Evaluating Logarithmic Expressions Evaluate the expression without using a calculator.

31. $\log_2 128$

32. $\log_8 1$

33. $10^{\log 45}$

34. $\log 0.000001$

35. $\ln e^6$

36. $\log_4 8$

37. $\log_3\left(\frac{1}{27}\right)$

38. $2^{\log_2 13}$

39. $\log_5 \sqrt{5}$

40. $e^{2 \ln 7}$

41. $\log 25 + \log 4$

42. $\log_3 \sqrt{243}$

43. $\log_2 16^{23}$

44. $\log_5 250 - \log_5 2$

45. $\log_8 6 - \log_8 3 + \log_8 2$

46. $\log(\log 10^{100})$

47–52 ■ Expanding Logarithmic Expressions Expand the logarithmic expression.

47. $\log(AB^2C^3)$

48. $\log_2\left(x\sqrt{x^2 + 1}\right)$

49. $\ln\sqrt{\dfrac{x^2 - 1}{x^2 + 1}}$

50. $\log\left(\dfrac{4x^3}{y^2(x - 1)^5}\right)$

51. $\log_5\left(\dfrac{x^2(1 - 5x)^{3/2}}{\sqrt{x^3 - x}}\right)$

52. $\ln\left(\dfrac{\sqrt[3]{x^4 + 12}}{(x + 16)\sqrt{x - 3}}\right)$

53–58 ■ Combining Logarithmic Expressions Combine into a single logarithm.

53. $\log 6 + 4 \log 2$

54. $\log x + \log(x^2 y) + 3 \log y$

55. $\frac{3}{2}\log_2(x - y) - 2\log_2(x^2 + y^2)$

56. $\log_5 2 + \log_5(x + 1) - \frac{1}{3}\log_5(3x + 7)$

57. $\log(x - 2) + \log(x + 2) - \frac{1}{2}\log(x^2 + 4)$

58. $\frac{1}{2}[\ln(x - 4) + 5\ln(x^2 + 4x)]$

59–72 ■ Exponential and Logarithmic Equations Solve the equation. Find the exact solution if possible; otherwise, use a calculator to approximate to two decimal places.

59. $2^{6x-3} = 8$

60. $\left(\frac{1}{3}\right)^{1-x} = \frac{1}{9}$

61. $5^{3x+2} = 2$

62. $10^{4-3x} = 5$

63. $2^{5x+1} = 3^{4-x}$

64. $10^{2x/5} = e^{3x+1}$

65. $x^3 \cdot 3^{4x} - x^2 \cdot 3^{4x} = 6x \cdot 3^{4x}$

66. $e^{2x} - 6e^x + 9 = 0$

67. $\log x + \log(x + 1) = \log 12$

68. $\ln(x - 2) + \ln 3 = \ln(5x - 7)$

69. $\log_2(1 - x) = 4$

70. $\ln(2x - 3) + 1 = 0$

71. $\log_3(x - 8) + \log_3 x = 2$

72. $\log_{1/2}(x - 5) - \log_{1/2}(x + 2) = 1$

73–76 ■ Exponential Equations Use a calculator to find the solution of the equation, rounded to six decimal places.

73. $5^{-2x/3} = 0.63$

74. $2^{3x-5} = 7$

75. $5^{2x+1} = 3^{4x-1}$

76. $e^{-15k} = 10{,}000$

77–80 ■ Local Extrema and Asymptotes Draw a graph of the function and use it to determine the asymptotes and the local maximum and minimum values.

77. $y = e^{x/(x+2)}$

78. $y = 10^x - 5^x$

79. $y = \log(x^3 - x)$

80. $y = 2x^2 - \ln x$

81–82 ■ Solving Equations Find the solution(s) of the equation, rounded to two decimal places.

81. $3 \log x = 6 - 2x$

82. $4 - x^2 = e^{-2x}$

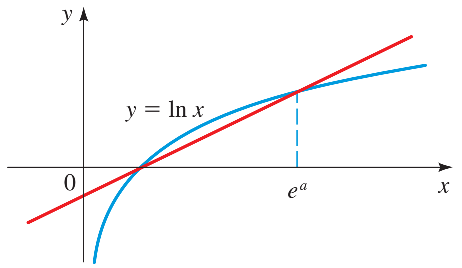
83–84 ■ Solving Inequalities Solve the inequality graphically.

83. $\ln x > x - 2$

84. $e^x < 4x^2$

85. Increasing and Decreasing Use a graph of $f(x) = e^x - 3e^{-x} - 4x$ to find, approximately, the intervals on which f is increasing and on which f is decreasing.

86. Equation of a Line Find an equation of the line shown in the figure.



87–90 ■ Change of Base Use the Change of Base Formula to express the given logarithm in terms of common or natural logarithms, and then evaluate, rounded to six decimal places.

87. $\log_4 15$

88. $\log_7\left(\frac{3}{4}\right)$

89. $\log_9 0.28$

90. $\log_{100} 250$

91. Comparing Logarithms Which is larger, $\log_4 258$ or $\log_5 620$?

92. Inverse Function Find the inverse of the function $f(x) = 2^{3^x}$, and state its domain and range.

93. Compound Interest If \$12,000 is invested at an interest rate of 10% per year, find the amount of the investment at the end of 3 years for each compounding method.

(a) Semiannual **(b)** Monthly

(c) Daily **(d)** Continuous

94. Compound Interest A sum of \$5000 is invested at an interest rate of $8\frac{1}{2}\%$ per year, compounded semiannually.

(a) Find the amount of the investment after $1\frac{1}{2}$ years.

(b) After what period of time will the investment amount to \$7000?

(c) If interest were compounded continously instead of semiannually, how long would it take for the amount to grow to \$7000?

95. Compound Interest A money market account pays 5.2% annual interest, compounded daily. If \$100,000 is invested in this account, how long will it take for the account to accumulate \$10,000 in interest?

96. Compound Interest A retirement savings plan pays 4.5% interest, compounded continuously. How long will it take for an investment in this plan to double?

97–98 ■ APY Determine the annual percentage yield (APY) for the given nominal annual interest rate and compounding frequency.

97. 4.25%; daily

98. 3.2%; monthly

99. Cat Population The stray-cat population in a small town grows exponentially. In 2022 the town had 30 stray cats, and the relative growth rate was 15% per year.

(a) Find a function that models the stray-cat population $n(t)$ after t years.

(b) Find the projected population after 4 years.

(c) Find the number of years required for the stray-cat population to reach 500.

100. Bacterial Growth A culture contains 10,000 bacteria initially. After 1 hour the bacteria count is 25,000.

(a) Find the doubling period.

(b) Find the number of bacteria after 3 hours.

101. Radioactive Decay Thorium-230 has a half-life of 75,380 years.

(a) If a sample has a mass of 150 mg, find a function that models the mass that remains after t years.

(b) Find the mass that will remain after 1000 years.

(c) After how many years will only 50 mg remain?

102. Radioactive Decay A sample of bismuth-210 decayed to 33% of its original mass after 8 days.

(a) Find the half-life of this element.

(b) Find the mass remaining after 12 days.

103. Radioactive Decay The half-life of palladium-100 is 4 days. After 20 days a sample has been reduced to a mass of 0.375 g.

(a) What was the initial mass of the sample?

(b) Find a function that models the mass remaining after t days.

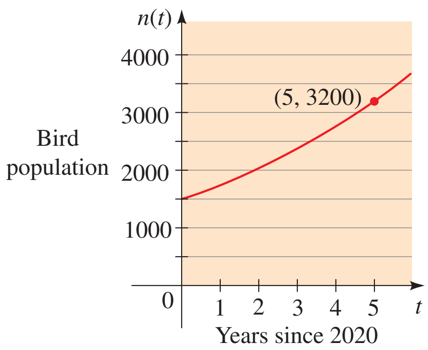
(c) What is the mass after 3 days?

(d) After how many days will only 0.15 g remain?

104. Bird Population The graph shows the population $n(t)$ of a rare species of bird in a nature reserve, where t represents years since 2020.

(a) Find a function that models the bird population at time t.

(b) What is the bird population expected to be in the year 2031?

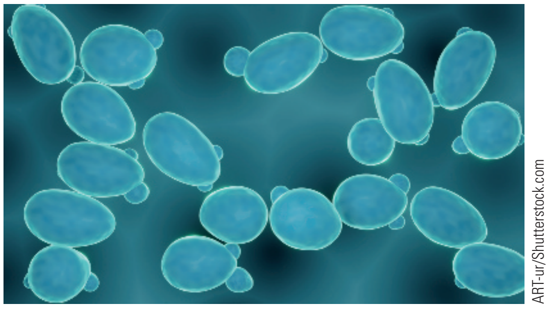


105. Logistic Growth of Yeast The yeast fungus *Saccharomyces cerevisiae* is used in leavening dough for making bread. In a recent research article it was reported that the doubling time of *S. cerevisiae* colonies is about 1.5 hours. About 100 colonies were placed in a Petri dish with a carrying capacity of 1400 colonies.

(a) Find the initial relative growth rate r.

(b) Find a logistic model for the number of yeast colonies $n(t)$ at time t.

(c) Graph the model that you found in part (b) and determine from the graph how long it takes for the yeast colonies to reach half the carrying capacity.



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106. Law of Cooling A car engine runs at a temperature of 190°F. When the engine is turned off, it cools according to Newton's Law of Cooling with constant $k = 0.0341$, where the time is measured in minutes. Find the time needed for the engine to cool to 90°F if the surrounding temperature is 60°F.

107. pH Scale The hydrogen ion concentration of fresh egg whites was measured as

$$[\mathrm{H}^+] = 1.3 \times 10^{-8}\ \mathrm{M}$$

Find the pH, and classify the substance as acidic or basic.

108. pH Scale The pH of lime juice is 1.9. Find the hydrogen ion concentration.

109. Richter Scale If one earthquake has magnitude 6.5 on the Richter scale, what is the magnitude of another quake that is 35 times as intense?

110. Decibel Scale The drilling of a jackhammer was measured at 132 dB. The sound of whispering was measured at 28 dB. Find the ratio of the intensity of the drilling to that of the whispering.

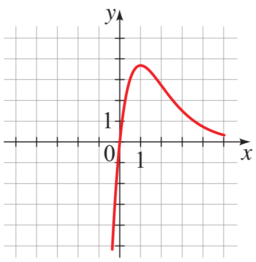
Matching

111. Equations and Their Graphs Match each equation with its graph. Give reasons for your answers. (Don't use a graphing device.)

(a) $y = 2^x$ **(b)** $y = -\ln x$ **(c)** $2x + 3y = 6$ **(d)** $y = 1 - \dfrac{1}{x^3}$

(e) $y = \log_2 x$ **(f)** $y = 4e^{-x^2/4}$ **(g)** $y = 2 - 2x - x^2$ **(h)** $y = 10xe^{-x}$

I



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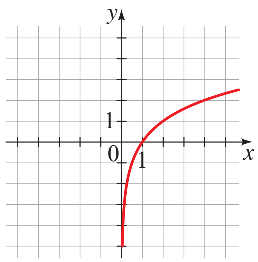

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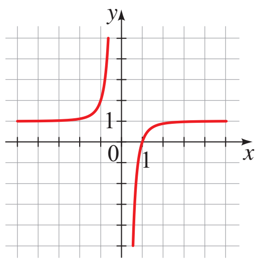

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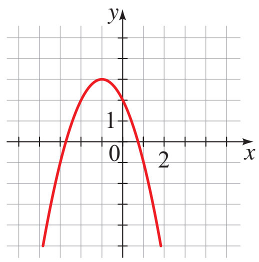

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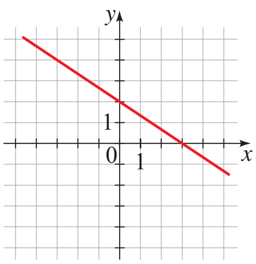

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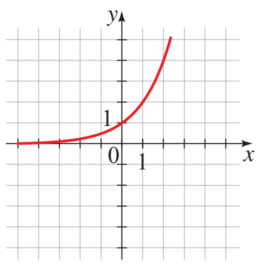
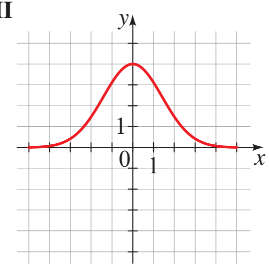


VII



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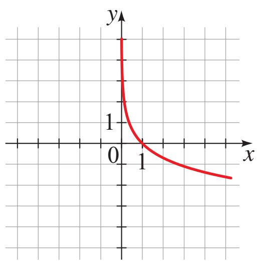

Chapter 4 | Test

1. Sketch the graph of each function, and state its domain, range, and asymptote. Show the x- and y-intercepts on the graph.

(a) $f(x) = 3 - 3^x$ **(b)** $g(x) = \log_3(x + 3)$

2. Find the domain of each function.

(a) $f(t) = \ln(2t - 3)$ **(b)** $g(x) = \log(x^2 - 1)$

3. (a) Write the equation $6^{2x} = 25$ in logarithmic form.

(b) Write the equation $\ln A = 3$ in exponential form.

4. Find the exact value of each expression.

(a) $10^{\log 36}$ **(b)** $\ln e^3$ **(c)** $\log_3 \sqrt{27}$

(d) $\log_2 80 - \log_2 10$ **(e)** $\log_8 4$ **(f)** $\log_6 4 + \log_6 9$

5. Use the Laws of Logarithms to expand each expression.

(a) $\log\left(\dfrac{xy^3}{z^2}\right)$ **(b)** $\ln\sqrt{\dfrac{x}{y}}$ **(c)** $\log\sqrt{\dfrac{x^2+1}{x^3(x-1)}}$

6. Use the Laws of Logarithms to combine each expression into a single logarithm.

(a) $\log a + 2 \log b$ **(b)** $\ln(x^2 - 25) - \ln(x + 5)$ **(c)** $\log_3 x - 2 \log_3 (x + 1) + 3 \log_3 y$

7. Find the solution of each exponential equation, rounded to two decimal places.

(a) $3^{4x} = 3^{100}$ **(b)** $e^{3x-2} = e^{x^2}$ **(c)** $5\left(\frac{2}{3}\right)^{3x+1} = 6$ **(d)** $10^{x+3} = 6^{2x}$

8. Solve each logarithmic equation for x.

(a) $\log(2x) = 3$ **(b)** $\log(x + 1) + \log 2 = \log(5x)$

(c) $5 \ln(3 - x) = 4$ **(d)** $\log_4(x + 3) - \log_4(x - 1) = 2$

9. Use the Change of Base Formula to express the logarithm $\log_{12} 27$ in terms of common or natural logarithms, and then evaluate your result.

10. The initial size of a culture of bacteria is 1000. After 1 hour the bacteria count is 8000.

(a) Find a function $n(t) = n_0 e^{rt}$ that models the population after t hours.

(b) Find the population after 1.5 hours.

(c) After how many hours will the number of bacteria reach 15,000?

(d) Sketch the graph of the population function.

11. Suppose that \$12,000 is invested in a savings account paying 5.6% interest per year.

(a) Write the formula for the amount in the account after t years if interest is compounded monthly.

(b) Find the amount in the account after 3 years if interest is compounded daily.

(c) How long will it take for the amount in the account to grow to \$20,000 if interest is compounded continuously?

12. The half-life of krypton-91 (^{91}Kr) is 10 s. At time $t = 0$ a heavy canister contains 3 g of this radioactive gas.

(a) Find a function $m(t) = m_0 2^{-t/h}$ that models the amount of ^{91}Kr remaining in the canister after t seconds.

(b) Find a function $m(t) = m_0 e^{-rt}$ that models the amount of ^{91}Kr remaining in the canister after t seconds.

(c) How much ^{91}Kr remains after 1 minute?

(d) After how long will the amount of ^{91}Kr remaining be reduced to 1 μg (1 microgram, or 10^{-6} g)?

13. An earthquake measuring 6.4 on the Richter scale struck Japan in July 2007, causing extensive damage. Earlier that year, a minor earthquake measuring 3.1 on the Richter scale was felt in parts of Pennsylvania. How many times more intense was the Japanese earthquake than the Pennsylvania earthquake?

A Cumulative Review Test for Chapters 2, 3, and 4 can be found at the book companion website www.stewartmath.com.

When we model real-world data, the shape of a scatter plot is one factor that helps us choose an appropriate model. In Figure 1, the first scatter plot suggests a linear model. But what type of function fits the second plot? A quadratic or an exponential? It's not easy to tell by just looking at the scatter plot. In this section we learn how to fit exponential and power curves to data.

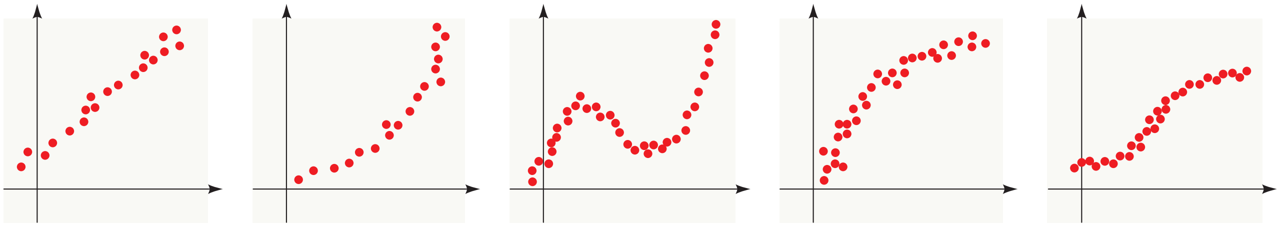

Figure 1

There are other factors to be considered in choosing a type of function to model real-world data. You can explore some of these in the Discovery Project *The Art of Modeling* at **www.stewartmath.com**.

Modeling with Exponential Functions

If a scatter plot shows that the data increase rapidly, we might want to model the data using an exponential model, that is, a function of the form

$$\boxed{f(x) = a \cdot b^x}$$

where a and b are constants.

Example 1 ■ An Exponential Model for World Population

Table 1 gives the population of the world since 1900.

(a) Draw a scatter plot of the data. Is a linear model appropriate?

(b) Find an exponential function that models the data and draw a graph of the function that you found, together with the scatter plot.

(c) What does the model predict for world population in the year 2030?

Table 1
World Population

Years Since 1900 (t)	Population in Millions (P)
0	1650
10	1750
20	1860
30	2070
40	2300
50	2520
60	3020
70	3700
80	4450
90	5300
100	6060
110	6920
120	7790

Solution

(a) The scatter plot in Figure 2(a) does not appear to lie along a line, so a linear model is not appropriate.

(b) Using the `Exponential Regression` command on a graphing device [see Figure 2(b)] we get an exponential model for the population in millions.

$$P(t) = (1427.6) \cdot (1.0141)^t$$

A graph of this model together with the scatter plot is shown in Figure 2(c).

(c) The model predicts that the world population in 2030 (130 years after 1900) will be

$$P(130) = (1427.6) \cdot (1.0141)^{130} \approx 8812.7 \text{ million}$$

Thus the predicted population is about 8.8 billion.

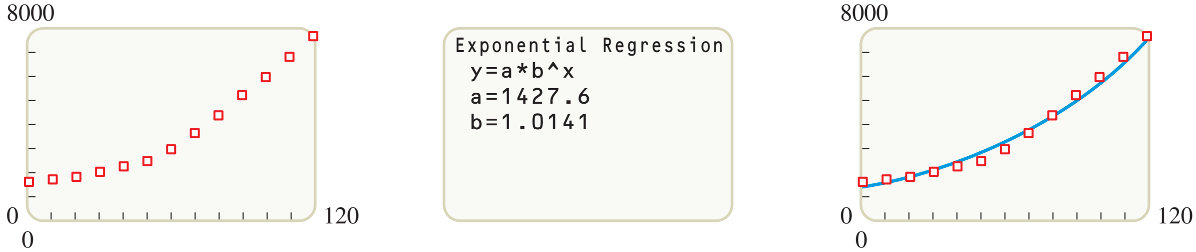

Figure 2 (a) Scatter plot (b) Exponential model (c) Scatter plot and model

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Note The model in Example 1 has the form $y = a \cdot b^t$. If we want to find the relative growth rate, we need to express the model in the form $y = Ce^{kt}$. By the note following Example 4.6.5 we have $k = \ln b$. In this case $k = \ln 1.0141 \approx 0.014$ and the model is

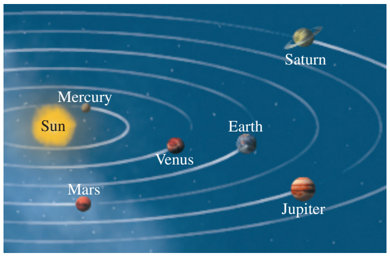
$$P(t) = (1427.6) \cdot e^{0.014t}$$

From this form of the model we see that the relative growth rate is $r = 0.014$, or 1.4%.

Modeling with Power Functions

If the scatter plot of the data we are studying resembles the graph of $y = ax^2$, $y = ax^{1.32}$, or some other power function, then we use a *power model*, that is, a function of the form

$$\boxed{f(x) = a \cdot x^b}$$



where a is a positive constant and b is any real number.

In the next example we find a power model that relates the period of a planet to its distance from the sun. Distance in the solar system is measured in astronomical units; an *astronomical unit* (AU) is the mean distance between the earth and the sun. The *period* of a planet is the time it takes the planet to make a complete revolution around the sun (measured in earth years).

Example 2 ■ A Power Model for Planetary Periods

Table 2 (on the next page) gives the mean distance d of each planet from the sun in astronomical units and its period T in years.

(a) Draw a scatter plot of the data. Is a linear model appropriate?

(b) Find a power function that models the data and draw a graph of the function that you found, together with the scatter plot. How well does the model fit the data?

(c) Use the model to find the period of an asteroid whose mean distance from the sun is 5 AU.

Table 2
Distances and Periods of the Planets

Planet	d	T
Mercury	0.387	0.241
Venus	0.723	0.615
Earth	1.000	1.000
Mars	1.523	1.881
Jupiter	5.203	11.861
Saturn	9.541	29.457
Uranus	19.190	84.008
Neptune	30.086	164.784
Pluto*	39.507	248.350

*Pluto is a "dwarf planet."

Solution

(a) The scatter plot in Figure 3(a) shows that the data do not appear to lie along a line, so a linear model is not appropriate.

(b) Using the `Power Regression` command on a graphing device [see Figure 3(b)] we obtain the power model $T = 1.000396d^{1.49966}$. Correct to two decimal places the model is

$$T = d^{1.5}$$

The graph of the model and scatter plot are shown in Figure 3(c). The model appears to fit the data well.

(c) With $d = 5$ AU the model gives

$$T = 5^{1.5} \approx 11.18$$

The period of the asteroid is about 11.2 years.

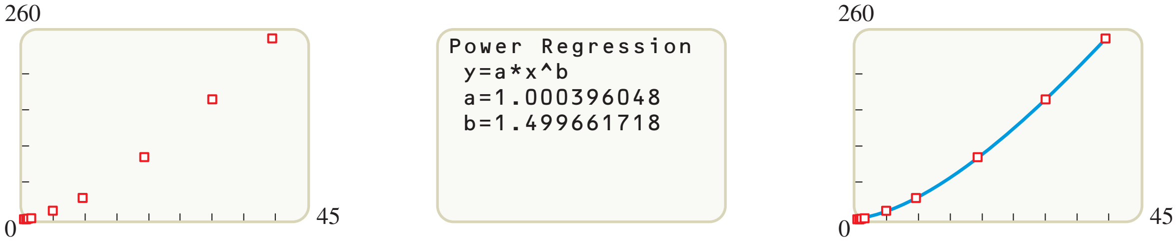


Figure 3 (a) Scatter plot (b) Power model (c) Scatter plot and model

Johannes Kepler (see Section 10.4) first discovered the formula in Example 2 by analyzing planetary data. Later Isaac Newton (see Section 12.2) derived the formula from his inverse square law of gravity.

An Exponential or Power Model?

It is often difficult to visually determine from a scatter plot whether a power or exponential function best fits the data. To help us decide, we can *linearize the data* by applying a function that "straightens" the scatter plot.

For the data points (x, y), a scatter plot of the points $(x, \ln y)$ is called a **semi-log plot** and a scatter plot of the points $(\ln x, \ln y)$ is called a **log-log plot**. If the data points (x, y) lie on an exponential curve $y = Ce^{kx}$, then a semi-log plot of the data will lie on a line (see Exercise 4.4.81). If the data points (x, y) lie on a power curve $y = a \cdot x^b$, then a log-log plot of the data will lie on a line (see Exercise 4.4.82). In the next example we use semi-log and log-log plots to help us decide whether an exponential or power model fits the data better.

Example 3 ■ An Exponential or Power Model?

Table 3 contains several data points (x, y). The table also has the values of $\ln x$ and $\ln y$ for each data point.

(a) Draw a scatter plot, a semi-log plot, and a log-log-plot of the data.

(b) Is an exponential function or a power function appropriate for modeling these data?

(c) Find and graph an appropriate model for the data.

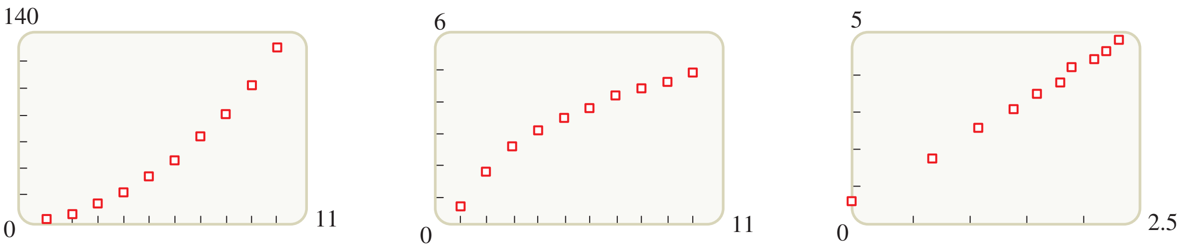
Table 3

x	y	$\ln x$	$\ln y$
1	2	0	0.7
2	6	0.7	1.8
3	14	1.1	2.6
4	22	1.4	3.1
5	34	1.6	3.5
6	46	1.8	3.8
7	64	1.9	4.2
8	80	2.1	4.4
9	102	2.2	4.6
10	130	2.3	4.9

Solution

(a) We use the values from Table 3 to graph the plots in Figures 4(a–c).

(b) The log-log plot in Figure 4(c) is very nearly linear so a power model is appropriate.



(a) Scatter plot (b) Semi-log Plot (c) Log-log Plot

Figure 4

(c) Using the `Power Regression` command on a graphing device, we find the power function that best fits the data: $y = 1.85x^{1.82}$. The graph of this function and the scatter plot of the data are shown in Figure 5.

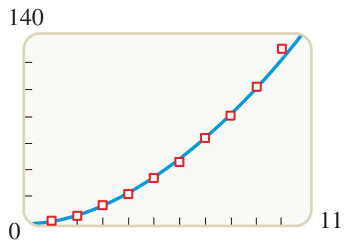


Figure 5

■

You can explore more properties of semi-log and log-log plots in the Discovery Project *Semi-log and Log-Log Plots* at the textbook website: **www.stewartmath.com**.

Problems

1. **Half-Life of Radioactive Iodine** A student is trying to determine the half-life of radioactive iodine-131. The student measures the amount of iodine-131 in a sample solution every 8 hours. The data are shown in the following table.

   (a) Make a scatter plot of the data.

(b) Use a graphing device to find an exponential model.

(c) Use your model to find the half-life of iodine-131.

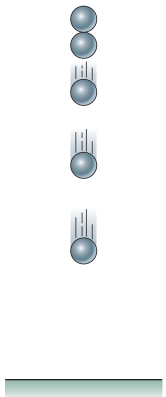
Time (h)	Amount of ^{131}I (g)
0	4.80
8	4.66
16	4.51
24	4.39
32	4.29
40	4.14
48	4.04

2. A Falling Ball In a physics experiment a lead ball is dropped from a height of 5 m. The students record the distance the ball has fallen every one-tenth of a second. (This can be done by using a camera and a strobe light.) Their data are shown in the table below.

(a) Make a scatter plot of the data.

(b) Use a graphing device to find a power model.

(c) Use your model to predict how far a dropped ball would fall in 3 s.



Time (s)	Distance (m)
0.1	0.048
0.2	0.197
0.3	0.441
0.4	0.882
0.5	1.227
0.6	1.765
0.7	2.401
0.8	3.136
0.9	3.969
1.0	4.902

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The number of different bat species in a cave is related to the size of the cave by a power function.

3. Modeling the Species-Area Relation The table gives the areas of several caves in central Mexico and the number of bat species that live in each cave.*

(a) Use a graphing device to find a power function that models the data. Draw a graph of the function and a scatter plot of the data on the same screen. Does the model fit the data well?

(b) The cave called El Sapo near Puebla, Mexico, has a surface area of $A = 205$ m^2. Use the model to estimate the number of bat species you would expect to find in that cave.

Cave	Area (m^2)	Number of Species
La Escondida	18	1
El Escorpion	19	1
El Tigre	58	1
Misión Imposible	60	2
San Martin	128	5
El Arenal	187	4
La Ciudad	344	6
Virgen	511	7

*A. K. Brunet and R. A. Medallin, "The Species-Area Relationship in Bat Assemblages of Tropical Caves." *Journal of Mammalogy,* 82(4):1114–1122, 2001.

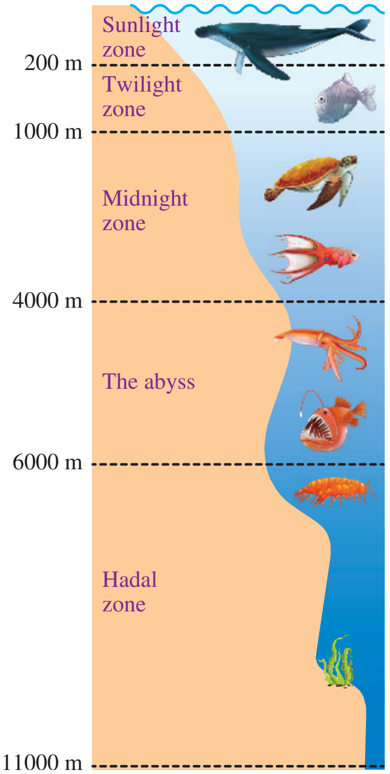
Ostrich and Sandhill crane

4. Bird Flight: Exponential or Power Model? Ornithologists have catalogued the weights and wingspans of many species of birds that fly. The table gives some of the data.

(a) Make semi-log and log-log plots of the data to determine whether an exponential or power function best fits these data, and then find an appropriate model. Graph the model and a scatter plot of the data on the same screen.

(b) Ostriches weigh about 300 lb and have a wingspan of about 72 inches. How does the model you found explain why ostriches can't fly?

Bird	Weight (lb)	Wingspan (in.)
Turkey vulture	4.40	69
Bald eagle	6.82	84
Great horned owl	3.08	44
Cooper's hawk	1.03	28
Sandhill crane	9.02	79
Atlantic puffin	0.95	24
California condor	17.82	109
Common loon	7.04	48
Yellow warbler	0.022	8
Common grackle	0.20	16
Wood stork	5.06	63
Mallard	2.42	35



5. Sunlight in the Twilight Zone As sunlight passes through ocean water its intensity diminishes. The intensity I at depth x is modeled by the Beer-Lambert Law: $I = I_0 e^{-kx}$, where I_0 is the intensity at the surface and k is a constant that depends on the "murkiness" of the water. The data in the table were obtained by a marine biologist.

(a) Find an exponential function of the form given by the Beer-Lambert Law to model these data. What is the "murkiness" constant k in the model you found?

(b) Some species of deep-sea fish need light intensity of at least $3 \times 10^{-12}\ \mathrm{W/m^2}$ to be able to see.* Can these species thrive in the twilight zone (see the graphic in the margin)?

Depth (m)	0	100	200	300	400	600	1000
Intensity ($\mathrm{W/m^2}$)	300	15.1	0.75	0.04	0.002	4.5×10^{-6}	2.9×10^{-11}

6. Logistic Population Growth The table gives the population of black flies in a closed laboratory container over an 18-day period.

(a) Use the `Logistic` command on a graphing device to find a logistic model for these data. Graph the model and a scatter plot of the data on the same screen. Does the model appear to fit the data well?

(b) Use the model to estimate the carrying capacity of the container.

Time (days)	0	2	4	6	8	10	12	16	18
Number of Flies	10	25	66	144	262	374	446	492	498

*_Seawater: Its Composition, Properties and Behaviour_, The Open University, 1995.