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12 Limits: A Preview of Calculus

In this chapter we study the central idea underlying calculus: the concept of a *limit*. Calculus is used in modeling real-life phenomena, particularly situations that involve change or motion. Limits are used in finding the instantaneous rate of change of a function as well as the area of a region with curved boundary. You will learn in calculus that these two apparently different problems are closely related. In this chapter we see how limits allow us to solve both problems.

In Chapter 2 we learned how to find the average rate of change of a function. For example, to find the average speed (like the speed of a moving soccer ball), we divide the total distance traveled by the total time elasped. But how can we find *instantaneous* speed—that is, the speed at a given instant? We can't divide the total distance by the total time because in an instant the total distance traveled is zero and the total time spent traveling is also zero! But we can find the average rate of change on smaller and smaller intervals, zooming in on the instant we want. In other words, the instantaneous speed is a *limit* of the average speeds.

In this chapter we also learn how to find areas of regions with curved sides by using the limit process.

12.1 Finding Limits Numerically and Graphically

■ Definition of Limit ■ Estimating Limits Numerically and Graphically ■ Limits That Fail to Exist ■ One-Sided Limits

In this section we use tables of values and graphs of functions to answer the question, What happens to the values $f(x)$ of a function f as the variable x approaches the number a?

■ Definition of Limit

We begin by investigating the behavior of the function f defined by

$$f(x) = x^2 - x + 2$$

for values of x near 2. The following tables give values of $f(x)$ for values of x close to 2 but not equal to 2.

x	$f(x)$
1.0	2.000000
1.5	2.750000
1.8	3.440000
1.9	3.710000
1.95	3.852500
1.99	3.970100
1.995	3.985025
1.999	3.997001

x	$f(x)$
3.0	8.000000
2.5	5.750000
2.2	4.640000
2.1	4.310000
2.05	4.152500
2.01	4.030100
2.005	4.015025
2.001	4.003001

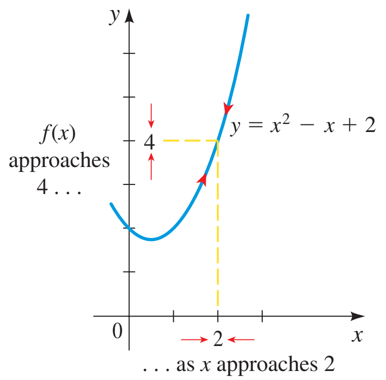


Figure 1

From the table and the graph of f (a parabola) shown in Figure 1 we see that when x is close to 2 (on either side of 2), $f(x)$ is close to 4. In fact, it appears that we can make the values of $f(x)$ as close as we like to 4 by taking x sufficiently close to 2. We express this by saying "the limit of the function $f(x) = x^2 - x + 2$ as x approaches 2 is equal to 4." The notation for this is

$$\lim_{x \to 2} (x^2 - x + 2) = 4$$

In general, we use the following notation.

Definition of the Limit of a Function

We write

$$\lim_{x \to a} f(x) = L$$

and say **"the limit of $f(x)$, as x approaches a**, equals L" if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by taking x to be sufficiently close to a, but not equal to a.

Roughly speaking, this says that the values of $f(x)$ get closer and closer to the number L as x gets closer and closer to the number a (from either side of a) but $x \neq a$.

An alternative notation for $\lim_{x \to a} f(x) = L$ is

$$f(x) \to L \qquad \text{as} \qquad x \to a$$

which is usually read "$f(x)$ approaches L as x approaches a." This is the notation we used in Section 3.6 when discussing asymptotes of rational functions.

Notice the phrase "but $x \neq a$" in the definition of limit. This means that in finding the limit of $f(x)$ as x approaches a, we never consider $x = a$. In fact, $f(x)$ need not even be defined when $x = a$. The only thing that matters is how f is defined *near* a.

Figure 2 shows the graphs of three functions. Note that in part (c), $f(a)$ is not defined, and in part (b), $f(a) \neq L$. But in each case, regardless of what happens at a, $\lim_{x \to a} f(x) = L$.

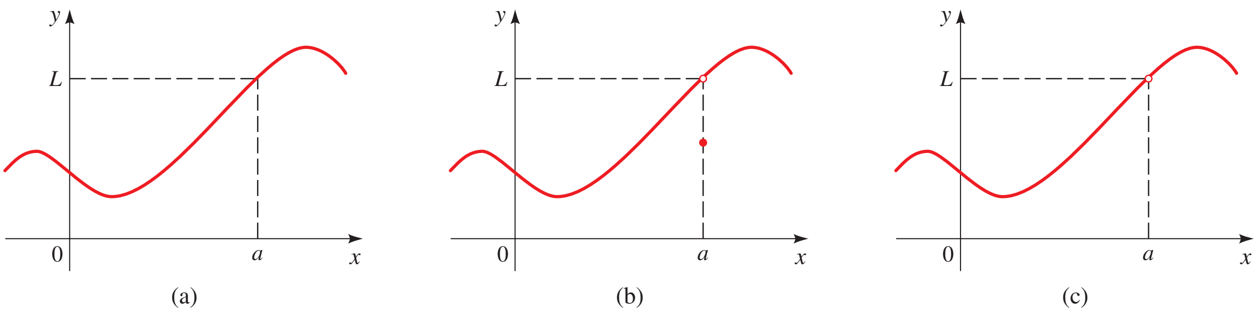


Figure 2 | $\lim_{x \to a} f(x) = L$ in all three cases

Estimating Limits Numerically and Graphically

In Section 12.2 we will develop techniques for finding exact values of limits. For now, we use tables and graphs to estimate limits of functions.

Example 1 ■ Estimating a Limit Numerically and Graphically

Estimate the value of the following limit by making a table of values. Check your work with a graph.

$$\lim_{x \to 1} \frac{x - 1}{x^2 - 1}$$

Solution Notice that the function $f(x) = (x - 1)/(x^2 - 1)$ is not defined when $x = 1$, but this doesn't matter because the definition of $\lim_{x \to a} f(x)$ says that we consider values of x that are close to a but not equal to a. The following tables give values of $f(x)$ (rounded to six decimal places) for values of x that approach 1 (but are not equal to 1).

$x < 1$	$f(x)$
0.5	0.666667
0.9	0.526316
0.99	0.502513
0.999	0.500250
0.9999	0.500025

$x > 1$	$f(x)$
1.5	0.400000
1.1	0.476190
1.01	0.497512
1.001	0.499750
1.0001	0.499975

On the basis of the values in the two tables we make the guess that

$$\lim_{x \to 1} \frac{x - 1}{x^2 - 1} = 0.5$$

As a graphical verification we use a graphing device to produce Figure 3. We see that when x is close to 1, y is close to 0.5. If we zoom in to get a closer look, as shown in Figure 4, we notice that as x gets closer to 1, y becomes closer to 0.5. This reinforces our conclusion.

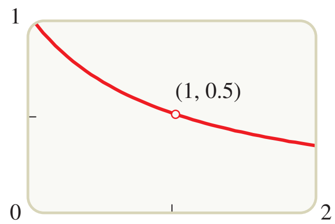


Figure 3

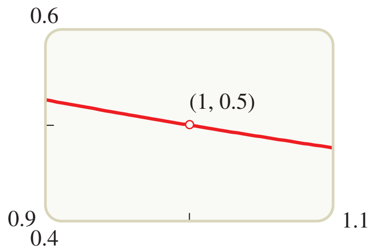


Figure 4

Now Try Exercise 3 ■

Example 2 ■ Finding a Limit from a Table

Find $\displaystyle\lim_{t \to 0} \frac{\sqrt{t^2 + 9} - 3}{t^2}$.

Solution The first table in the margin lists values of the function for several values of t near 0. As t approaches 0, the values of the function seem to approach 0.1666666 . . . , so we guess that

$$\lim_{t \to 0} \frac{\sqrt{t^2 + 9} - 3}{t^2} = \frac{1}{6}$$

Now Try Exercise 5 ■

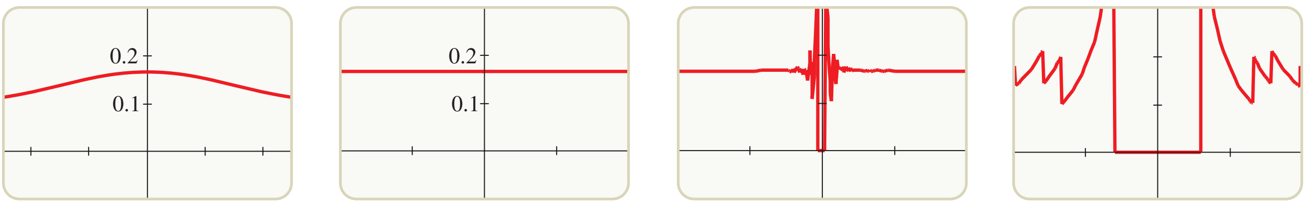
t	$\dfrac{\sqrt{t^2 + 9} - 3}{t^2}$
±1.0	0.16228
±0.5	0.16553
±0.1	0.16662
±0.05	0.16666
±0.01	0.16667

t	$\dfrac{\sqrt{t^2 + 9} - 3}{t^2}$
±0.0005	0.16800
±0.0001	0.20000
±0.00005	0.00000
±0.00001	0.00000

What would have happened in Example 2 if we had taken even smaller values of t? The second table in the margin shows the results from one calculator; you can see that something strange seems to be happening.

If you try these calculations on your own calculator, you might get different values, but eventually, you will get the value 0 if you make t sufficiently small. Does this mean that the answer is really 0 instead of $\frac{1}{6}$? No, the value of the limit is $\frac{1}{6}$, as we will show in the next section. The problem is that the calculator gave false values because $\sqrt{t^2 + 9}$ is very close to 3 when t is small. (In fact, when t is sufficiently small, a calculator's value for $\sqrt{t^2 + 9}$ is 3.000 . . . to as many digits as the calculator is capable of carrying.)

Something similar happens when we try to graph the function of Example 2 on a graphing device. Parts (a) and (b) of Figure 5 show quite accurate graphs of this function, and from these graphs we estimate that the limit is about $\frac{1}{6}$. But if we zoom in too far, as in parts (c) and (d), then we get inaccurate graphs, again because the calculator gave false values.



(a) $[-5, 5]$ by $[-0.1, 0.3]$ (b) $[-0.1, 0.1]$ by $[-0.1, 0.3]$ (c) $[-10^{-6}, 10^{-6}]$ by $[-0.1, 0.3]$ (d) $[-10^{-7}, 10^{-7}]$ by $[-0.1, 0.3]$

Figure 5

Limits That Fail to Exist

Functions do not necessarily approach a finite value at every point. In other words, it's possible for a limit not to exist. The next three examples illustrate ways in which this can happen.

Example 3 ■ A Limit That Fails to Exist (A Function with a Jump)

The Heaviside function H is defined by

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

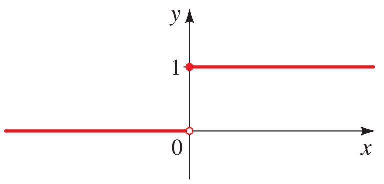


Figure 6

[This function, named after the electrical engineer Oliver Heaviside (1850–1925), can be used to describe an electric current that is switched on at time $t = 0$.] Its graph is shown in Figure 6. Notice the "jump" in the graph at $x = 0$.

As t approaches 0 from the left, $H(t)$ approaches 0. As t approaches 0 from the right, $H(t)$ approaches 1. There is no single number that $H(t)$ approaches as t approaches 0. Therefore $\lim_{t\to 0} H(t)$ does not exist.

Now Try Exercise 27 ■

Example 4 ■ A Limit That Fails to Exist (A Function That Oscillates)

Find $\lim_{x\to 0} \sin \dfrac{\pi}{x}$.

Solution The function $f(x) = \sin(\pi/x)$ is undefined at 0. Evaluating the function for some small values of x, we get

$$f(1) = \sin \pi = 0 \qquad f\left(\tfrac{1}{2}\right) = \sin 2\pi = 0$$
$$f\left(\tfrac{1}{3}\right) = \sin 3\pi = 0 \qquad f\left(\tfrac{1}{4}\right) = \sin 4\pi = 0$$
$$f(0.1) = \sin 10\pi = 0 \qquad f(0.01) = \sin 100\pi = 0$$

Similarly, $f(0.001) = f(0.0001) = 0$. On the basis of this information we might be tempted to guess that

$$\lim_{x\to 0} \sin \frac{\pi}{x} \stackrel{?}{=} 0$$

but this time our guess is wrong. Note that although $f(1/n) = \sin n\pi = 0$ for any integer n, it is also true that $f(x) = 1$ for infinitely many values of x that approach 0. (See the graph in Figure 7.)

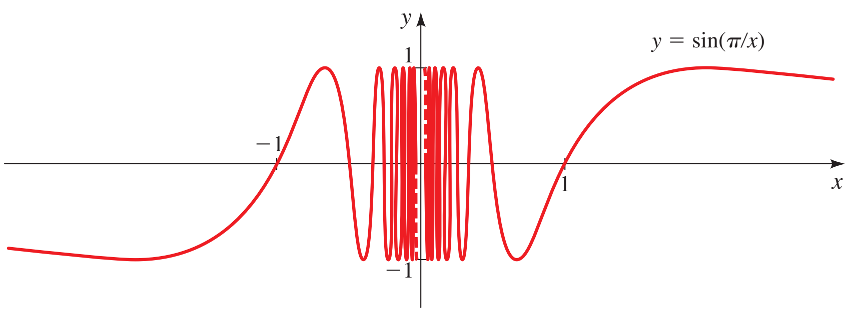


Figure 7

The dashed lines indicate that the values of $\sin(\pi/x)$ oscillate between 1 and -1 infinitely often as x approaches 0. Since the values of $f(x)$ do not approach a fixed number as x approaches 0,

$$\lim_{x\to 0} \sin \frac{\pi}{x} \quad \text{does not exist}$$

Now Try Exercise 25 ■

Example 4 illustrates some of the pitfalls in guessing the value of a limit. It is easy to guess the wrong value if we use inappropriate values of x, but it is difficult to know when to stop calculating values. And as the discussion after Example 2 shows, sometimes calculators and computers give incorrect values. In the next two sections, however, we will develop dependable methods for calculating limits.

Example 5 ■ A Limit That Fails to Exist (A Function with a Vertical Asymptote)

Find $\lim_{x \to 0} \dfrac{1}{x^2}$ if it exists.

x	$\dfrac{1}{x^2}$
± 1	1
± 0.5	4
± 0.2	25
± 0.1	100
± 0.05	400
± 0.01	10,000
± 0.001	1,000,000

Solution As x becomes close to 0, x^2 also becomes close to 0, and $1/x^2$ becomes very large. (See the table in the margin.) In fact, it appears from the graph of the function $f(x) = 1/x^2$ shown in Figure 8 that the values of $f(x)$ can be made arbitrarily large by taking x close enough to 0. Thus the values of $f(x)$ do not approach a number, so $\lim_{x \to 0}(1/x^2)$ does not exist.

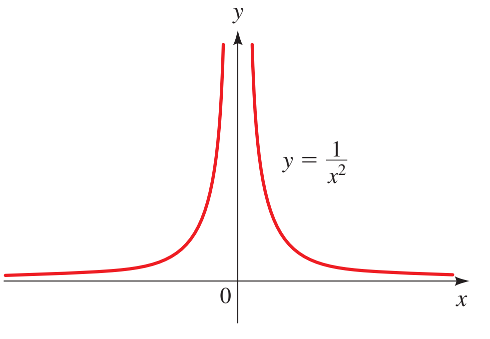


Figure 8

Now Try Exercise 23 ■

To indicate the kind of behavior exhibited in Example 5, we use the notation

$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

Vertical asymptotes are studied in Section 3.6.

This does not mean that we are regarding ∞ as a number. Nor does it mean that the limit exists. It simply expresses the particular way in which the limit does not exist: $1/x^2$ can be made as large as we like by taking x close enough to 0. Notice that the line $x = 0$ (the y-axis) is a vertical asymptote.

■ One-Sided Limits

We noticed in Example 3 that $H(t)$ approaches 0 as t approaches 0 from the left and $H(t)$ approaches 1 as t approaches 0 from the right. We indicate this situation symbolically by writing

$$\lim_{t \to 0^-} H(t) = 0 \qquad \text{and} \qquad \lim_{t \to 0^+} H(t) = 1$$

The symbol "$t \to 0^-$" indicates that we consider only values of t less than 0. Likewise, "$t \to 0^+$" indicates that we consider only values of t greater than 0.

We also say "the limit of $f(x)$, as x approaches a from the left, equals L."

Definition of a One-Sided Limit

We write

$$\lim_{x \to a^-} f(x) = L$$

and say "**the left-hand limit of $f(x)$ as x approaches a**, equals L" if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a and $x < a$.

Notice that this definition differs from the definition of a two-sided limit only in that we require x to be *less than a*. Similarly, if we require that x be *greater than a,* we write

$$\lim_{x \to a^+} f(x) = L$$

And say, "**the right-hand limit of $f(x)$, as x approaches a**, equals L".

Thus the symbol "$x \to a^+$" means that we consider only $x > a$. These definitions are illustrated in Figure 9.

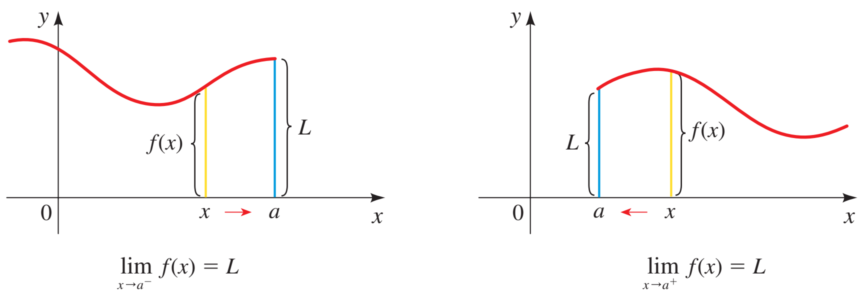


Figure 9

By comparing the definitions of two-sided and one-sided limits, we see that the following is true.

$$\lim_{x \to a} f(x) = L \quad \text{if and only if} \quad \lim_{x \to a^-} f(x) = L \quad \text{and} \quad \lim_{x \to a^+} f(x) = L$$

Thus if the left-hand and right-hand limits are different, the (two-sided) limit does not exist. We use this fact in the next two examples.

Example 6 ■ Limits from a Graph

The graph of a function g is shown in Figure 10. Use it to state the values (if they exist) of the following:

(a) $\lim_{x \to 2^-} g(x)$, $\lim_{x \to 2^+} g(x)$, $\lim_{x \to 2} g(x)$

(b) $\lim_{x \to 5^-} g(x)$, $\lim_{x \to 5^+} g(x)$, $\lim_{x \to 5} g(x)$

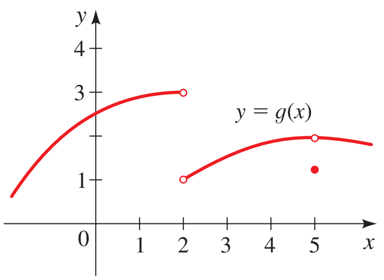


Figure 10

Solution

(a) From the graph we see that the values of $g(x)$ approach 3 as x approaches 2 from the left, but they approach 1 as x approaches 2 from the right. Therefore

$$\lim_{x \to 2^-} g(x) = 3 \qquad \text{and} \qquad \lim_{x \to 2^+} g(x) = 1$$

Since the left- and right-hand limits are different, we conclude that $\lim_{x \to 2} g(x)$ does not exist.

(b) The graph also shows that

$$\lim_{x\to 5^-} g(x) = 2 \qquad \text{and} \qquad \lim_{x\to 5^+} g(x) = 2$$

This time the left- and right-hand limits are the same, so we have

$$\lim_{x\to 5} g(x) = 2$$

Despite this fact, notice that $g(5) \neq 2$.

Now Try Exercise 19 ■

Example 7 ■ A Piecewise-Defined Function

Let f be the function defined by

$$f(x) = \begin{cases} 2x^2 & \text{if } x < 1 \\ 4 - x & \text{if } x \geq 1 \end{cases}$$

Graph f, and use the graph to find the following:

(a) $\lim_{x\to 1^-} f(x)$ **(b)** $\lim_{x\to 1^+} f(x)$ **(c)** $\lim_{x\to 1} f(x)$

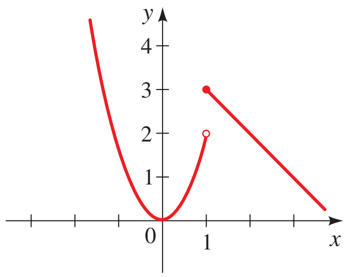


Figure 11

Solution The graph of f is shown in Figure 11. From the graph we see that the values of $f(x)$ approach 2 as x approaches 1 from the left, but they approach 3 as x approaches 1 from the right. Thus the left- and right-hand limits are not equal. So we have

(a) $\lim_{x\to 1^-} f(x) = 2$ **(b)** $\lim_{x\to 1^+} f(x) = 3$ **(c)** $\lim_{x\to 1} f(x)$ does not exist.

Now Try Exercise 29 ■

12.1 Exercises

Concepts

1. When we write $\lim_{x\to a} f(x) = L$ then, roughly speaking, the values of $f(x)$ get closer and closer to the number ______ as the values of x get closer and closer to ______. To determine $\lim_{x\to 5} \frac{x-5}{x-5}$, we try values for x closer and closer to ______ and find that the limit is ______.

2. We write $\lim_{x\to a^-} f(x) = L$ and say that the ______ of $f(x)$ as x approaches a from the ______ (left/right) is equal to ______. To find the left-hand limit, we try values for x that are ______ (less/greater) than a. A limit exists if and only if both the ______-hand and ______-hand limits exist and are ______.

Skills

3–4 ■ Estimating Limits Numerically and Graphically Estimate the value of the limit by making a table of values. Check your work with a graph.

3. $\lim_{x\to 5} \dfrac{x^2 - 25}{x - 5}$

4. $\lim_{x\to 3} \dfrac{x^2 - x - 6}{x - 3}$

5–10 ■ Estimating Limits Numerically Complete the table of values (to five decimal places), and use the table to estimate the value of the limit.

5. $\lim_{x\to 4} \dfrac{\sqrt{x} - 2}{x - 4}$

x	3.9	3.99	3.999	4.001	4.01	4.1
$f(x)$						

6. $\lim_{x\to 2} \dfrac{x - 2}{x^2 + x - 6}$

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$						

7. $\lim_{x\to 1} \dfrac{x - 1}{x^3 - 1}$

x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$						

8. $\displaystyle\lim_{x\to 0}\frac{e^x-1}{x}$

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$						

9. $\displaystyle\lim_{x\to 0}\frac{\sin x}{x}$

x	± 1	± 0.5	± 0.1	± 0.05	± 0.01
$f(x)$					

10. $\displaystyle\lim_{x\to 0^+} x\ln x$

x	0.1	0.01	0.001	0.0001	0.00001
$f(x)$					

11–16 ■ Estimating Limits Numerically and Graphically Use a table of values to estimate the value of the limit. Then use a graphing device to confirm your result graphically.

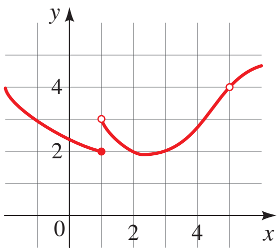
11. $\displaystyle\lim_{x\to -4}\frac{x+4}{x^2+7x+12}$ **12.** $\displaystyle\lim_{x\to 1}\frac{x^3-1}{x^2-1}$

13. $\displaystyle\lim_{x\to 0}\frac{5^x-3^x}{x}$ **14.** $\displaystyle\lim_{x\to 0}\frac{\sqrt{x+9}-3}{x}$

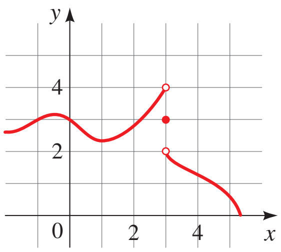
15. $\displaystyle\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)$ **16.** $\displaystyle\lim_{x\to 0}\frac{\tan 2x}{\tan 3x}$

17–20 ■ Limits from a Graph For the function f whose graph is given, state the value of the given quantity if it exists. If it does not exist, explain why.

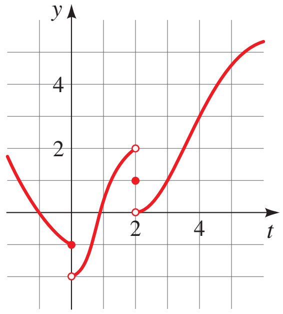
17. (a) $\displaystyle\lim_{x\to 1^-} f(x)$ **(b)** $\displaystyle\lim_{x\to 1^+} f(x)$ **(c)** $\displaystyle\lim_{x\to 1} f(x)$
(d) $\displaystyle\lim_{x\to 5} f(x)$ **(e)** $f(5)$



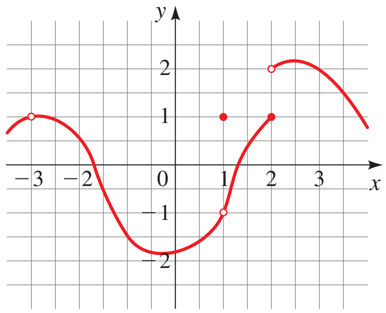
18. (a) $\displaystyle\lim_{x\to 0} f(x)$ **(b)** $\displaystyle\lim_{x\to 3^-} f(x)$ **(c)** $\displaystyle\lim_{x\to 3^+} f(x)$
(d) $\displaystyle\lim_{x\to 3} f(x)$ **(e)** $f(3)$



19. (a) $\displaystyle\lim_{t\to 0^-} f(t)$ **(b)** $\displaystyle\lim_{t\to 0^+} f(t)$ **(c)** $\displaystyle\lim_{t\to 0} f(t)$
(d) $\displaystyle\lim_{t\to 2^-} f(t)$ **(e)** $\displaystyle\lim_{t\to 2^+} f(t)$ **(f)** $\displaystyle\lim_{t\to 2} f(t)$
(g) $f(2)$ **(h)** $\displaystyle\lim_{t\to 4} f(t)$



20. (a) $\displaystyle\lim_{x\to 3} f(x)$ **(b)** $\displaystyle\lim_{x\to 1} f(x)$ **(c)** $\displaystyle\lim_{x\to -3} f(x)$
(d) $\displaystyle\lim_{x\to 2^-} f(x)$ **(e)** $\displaystyle\lim_{x\to 2^+} f(x)$ **(f)** $\displaystyle\lim_{x\to 2} f(x)$



21–28 ■ Estimating Limits Graphically Use a graphing device to determine whether the limit exists. If the limit exists, estimate its value to two decimal places.

21. $\displaystyle\lim_{x\to 1}\frac{x^3+x^2+3x-5}{2x^2-5x+3}$ **22.** $\displaystyle\lim_{x\to 0}\frac{x^2}{\cos 5x-\cos 4x}$

23. $\displaystyle\lim_{x\to 0}\ln(\sin^2 x)$ **24.** $\displaystyle\lim_{x\to 2}\frac{x^3+6x^2-5x+1}{x^3-x^2-8x+12}$

25. $\displaystyle\lim_{x\to 0}\cos\frac{1}{x}$ **26.** $\displaystyle\lim_{x\to 0}\sin\frac{2}{x}$

27. $\displaystyle\lim_{x\to 3}\frac{|x-3|}{x-3}$ **28.** $\displaystyle\lim_{x\to 0}\frac{1}{1+e^{1/x}}$

29–32 ■ One-Sided Limits Graph the piecewise-defined function and use your graph to find the values of the limits, if they exist.

29. $f(x)=\begin{cases} x^2 & \text{if } x\le 2 \\ 6-x & \text{if } x>2\end{cases}$

(a) $\displaystyle\lim_{x\to 2^-} f(x)$ **(b)** $\displaystyle\lim_{x\to 2^+} f(x)$ **(c)** $\displaystyle\lim_{x\to 2} f(x)$

30. $f(x)=\begin{cases} 2 & \text{if } x<0 \\ x+1 & \text{if } x\ge 0\end{cases}$

(a) $\displaystyle\lim_{x\to 0^-} f(x)$ **(b)** $\displaystyle\lim_{x\to 0^+} f(x)$ **(c)** $\displaystyle\lim_{x\to 0} f(x)$

31. $f(x) = \begin{cases} -x + 3 & \text{if } x < -1 \\ 3 & \text{if } x \geq -1 \end{cases}$

(a) $\lim_{x \to -1^-} f(x)$ **(b)** $\lim_{x \to -1^+} f(x)$ **(c)** $\lim_{x \to -1} f(x)$

32. $f(x) = \begin{cases} 2x + 10 & \text{if } x \leq -2 \\ -x + 4 & \text{if } x > -2 \end{cases}$

(a) $\lim_{x \to -2^-} f(x)$ **(b)** $\lim_{x \to -2^+} f(x)$ **(c)** $\lim_{x \to -2} f(x)$

■ Discuss ■ Discover ■ Prove ■ Write

33. Discuss: A Function with Specified Limits Sketch the graph of an example of a function f that satisfies all of the following conditions.

$$\lim_{x \to 0^-} f(x) = 2 \qquad \lim_{x \to 0^+} f(x) = 0$$

$$\lim_{x \to 2} f(x) = 1 \qquad f(0) = 2 \qquad f(2) = 3$$

How many such functions are there?

34. Discuss: Graphing Device Pitfalls

(a) Evaluate

$$h(x) = \frac{\tan x - x}{x^3}$$

for $x = 1, 0.5, 0.1, 0.05, 0.01$, and 0.005.

(b) Guess the value of $\lim_{x \to 0} \dfrac{\tan x - x}{x^3}$.

(c) Evaluate $h(x)$ for successively smaller values of x until you finally get a value of 0 for $h(x)$. Are you still confident that your guess in part (b) is correct? Explain why you eventually got a value of 0 for $h(x)$.

(d) Graph the function h in the viewing rectangle $[-1, 1]$ by $[0, 1]$. Then zoom in toward the point where the graph crosses the y-axis to estimate the limit of $h(x)$ as x approaches 0. Continue to zoom in until you observe distortions in the graph of h. Compare with your results in part (c).

12.2 Finding Limits Algebraically

■ Limit Laws ■ Applying the Limit Laws ■ Finding Limits Using Algebra and the Limit Laws ■ Using Left- and Right-Hand Limits

In Section 12.1 we used calculators and graphs to guess the values of limits, but we saw that such methods don't always lead to the correct answer. In this section we use algebraic methods to find limits exactly.

■ Limit Laws

We use the following properties of limits, called the *Limit Laws*, to calculate limits.

Limit Laws

Suppose that c is a constant and that the following limits exist:

$$\lim_{x \to a} f(x) \quad \text{and} \quad \lim_{x \to a} g(x)$$

Then

1. $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$ — Limit of a Sum

2. $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$ — Limit of a Difference

3. $\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$ — Limit of a Constant Multiple

4. $\lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$ — Limit of a Product

5. $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ if $\lim_{x \to a} g(x) \neq 0$ — Limit of a Quotient

These five laws can be stated verbally as follows:

Limit of a Sum
1. The limit of a sum is the sum of the limits.

Limit of a Difference
2. The limit of a difference is the difference of the limits.

Limit of a Constant Multiple
3. The limit of a constant times a function is the constant times the limit of the function.

Limit of a Product
4. The limit of a product is the product of the limits.

Limit of a Quotient
5. The limit of a quotient is the quotient of the limits (provided that the limit of the denominator is not 0).

It's easy to believe that these properties are true. For instance, if $f(x)$ is close to L and $g(x)$ is close to M, it is reasonable to conclude that $f(x) + g(x)$ is close to $L + M$. This gives us an intuitive basis for believing that Law 1 is true.

If we use Law 4 (Limit of a Product) repeatedly with $g(x) = f(x)$, we obtain the following Law 6 for the limit of a power. A similar law holds for roots.

Limit Laws

6. $\lim_{x \to a} [f(x)]^n = [\lim_{x \to a} f(x)]^n$ where n is a positive integer — Limit of a Power

7. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$ where n is a positive integer — Limit of a Root

[If n is even, we assume that $\lim_{x \to a} f(x) > 0$.]

In words, these laws say the following:

Limit of a Power
6. The limit of a power is the power of the limit.

Limit of a Root
7. The limit of a root is the root of the limit.

Example 1 ■ Using the Limit Laws

Use the Limit Laws and the graphs of f and g in Figure 1 to evaluate the following limits if they exist.

(a) $\lim_{x \to -2} [f(x) + 5g(x)]$ **(b)** $\lim_{x \to 1} [f(x)g(x)]$

(c) $\lim_{x \to 2} \dfrac{f(x)}{g(x)}$ **(d)** $\lim_{x \to 1} [f(x)]^3$

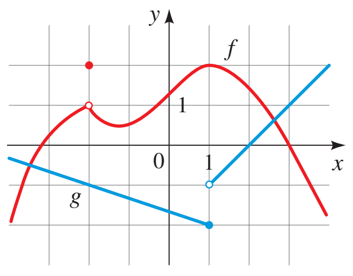


Figure 1

Solution

(a) From the graphs of f and g we see that

$$\lim_{x \to -2} f(x) = 1 \qquad \text{and} \qquad \lim_{x \to -2} g(x) = -1$$

Therefore we have

$$\begin{aligned}\lim_{x\to -2}[f(x)+5g(x)] &= \lim_{x\to -2} f(x) + \lim_{x\to -2}[5g(x)] && \text{Limit of a Sum}\\ &= \lim_{x\to -2} f(x) + 5\lim_{x\to -2} g(x) && \text{Limit of a Constant Multiple}\\ &= 1 + 5(-1) = -4\end{aligned}$$

(b) We see that $\lim_{x\to 1} f(x) = 2$. But $\lim_{x\to 1} g(x)$ does not exist because the left- and right-hand limits are different:

$$\lim_{x\to 1^-} g(x) = -2 \qquad \lim_{x\to 1^+} g(x) = -1$$

So we can't use Law 4 (Limit of a Product). The given limit does not exist because the left-hand limit is not equal to the right-hand limit.

(c) The graphs show that

$$\lim_{x\to 2} f(x) \approx 1.4 \qquad \text{and} \qquad \lim_{x\to 2} g(x) = 0$$

Because the limit of the denominator is 0, we can't use Law 5 (Limit of a Quotient). The given limit does not exist because the denominator approaches 0 whereas the numerator approaches a nonzero number.

(d) Since $\lim_{x\to 1} f(x) = 2$, we use Law 6 to get

$$\begin{aligned}\lim_{x\to 1}[f(x)]^3 &= [\lim_{x\to 1} f(x)]^3 && \text{Limit of a Power}\\ &= 2^3 = 8\end{aligned}$$

Now Try Exercise 3 ■

■ Applying the Limit Laws

In applying the Limit Laws, we need to use four special limits.

Some Special Limits

1. $\lim_{x\to a} c = c$
2. $\lim_{x\to a} x = a$
3. $\lim_{x\to a} x^n = a^n$ where n is a positive integer
4. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer and $a > 0$

Special Limits 1 and 2 are intuitively clear—looking at the graphs of $y = c$ and $y = x$ will convince you of their validity. Special Limits 3 and 4 are particular cases of Limit Laws 6 and 7 (Limits of a Power and of a Root).

Example 2 ■ Using the Limit Laws

Evaluate the following limits, and justify each step.

(a) $\displaystyle\lim_{x\to 5}(2x^2 - 3x + 4)$ **(b)** $\displaystyle\lim_{x\to -2}\frac{x^3 + 2x^2 - 1}{5 - 3x}$

Solution

(a)

$$\lim_{x\to 5}(2x^2 - 3x + 4) = \lim_{x\to 5}(2x^2) - \lim_{x\to 5}(3x) + \lim_{x\to 5} 4$$

Limits of a Difference and Sum

$$= 2\lim_{x\to 5} x^2 - 3\lim_{x\to 5} x + \lim_{x\to 5} 4$$

Limit of a Constant Multiple

$$= 2(5^2) - 3(5) + 4$$

Special Limits 3, 2, and 1

$$= 39$$

(b) We start by using Law 5, but its use is fully justified only at the final stage when we see that the limits of the numerator and denominator exist and the limit of the denominator is not 0.

$$\lim_{x\to -2}\frac{x^3 + 2x^2 - 1}{5 - 3x} = \frac{\lim_{x\to -2}(x^3 + 2x^2 - 1)}{\lim_{x\to -2}(5 - 3x)}$$

Limit of a Quotient

$$= \frac{\lim_{x\to -2} x^3 + 2\lim_{x\to -2} x^2 - \lim_{x\to -2} 1}{\lim_{x\to -2} 5 - 3\lim_{x\to -2} x}$$

Limits of Sums, Differences, and Constant Multiples

$$= \frac{(-2)^3 + 2(-2)^2 - 1}{5 - 3(-2)}$$

Special Limits 3, 2, and 1

$$= -\frac{1}{11}$$

Now Try Exercises 9 and 11 ■

If we let $f(x) = 2x^2 - 3x + 4$, then $f(5) = 39$. In Example 2(a) we found that $\lim_{x\to 5} f(x) = 39$. In other words, we would have gotten the correct answer by substituting 5 for x. Similarly, direct substitution provides the correct answer in part (b). The functions in Example 2 are a polynomial and a rational function, respectively, and similar use of the Limit Laws proves that direct substitution always works for such functions. We state this fact as follows.

Limits by Direct Substitution

If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x\to a} f(x) = f(a)$$

Functions with this direct substitution property are called **continuous at** a. You will learn more about continuous functions when you study calculus.

Example 3 ■ Finding Limits by Direct Substitution

Evaluate the following limits.

(a) $\displaystyle\lim_{x\to 3}(2x^3 - 10x - 8)$ **(b)** $\displaystyle\lim_{x\to -1}\frac{x^2 + 5x}{x^4 + 2}$

Solution

(a) The function $f(x) = 2x^3 - 10x - 8$ is a polynomial, so we can find the limit by direct substitution.

$$\lim_{x\to 3}(2x^3 - 10x - 8) = 2(3)^3 - 10(3) - 8 = 16$$

(b) The function $f(x) = (x^2 + 5x)/(x^4 + 2)$ is a rational function, and $x = -1$ is in its domain (because the denominator is not zero for $x = -1$). Thus we can find the limit by direct substitution.

$$\lim_{x\to -1}\frac{x^2 + 5x}{x^4 + 2} = \frac{(-1)^2 + 5(-1)}{(-1)^4 + 2} = -\frac{4}{3}$$

Now Try Exercise 13 ■

■ Finding Limits Using Algebra and the Limit Laws

As we saw in Example 3, evaluating limits by direct substitution is straightforward. But not all limits can be evaluated this way. In fact, most of the situations in which limits are useful require us to work harder to evaluate the limit. The next three examples illustrate how we can use algebra to find limits.

Example 4 ■ Finding a Limit by Canceling a Common Factor

Find $\displaystyle\lim_{x\to 1}\frac{x - 1}{x^2 - 1}$.

Solution Let $f(x) = (x - 1)/(x^2 - 1)$. We can't find the limit by substituting $x = 1$ because $f(1)$ isn't defined. Nor can we apply Law 5 (Limit of a Quotient) because the limit of the denominator is 0. Instead, we need to do some preliminary algebra. We factor the denominator as a difference of squares:

$$\frac{x - 1}{x^2 - 1} = \frac{x - 1}{(x - 1)(x + 1)}$$

The numerator and denominator have a common factor of $x - 1$. When we take the limit as x approaches 1, we have $x \neq 1$, and so $x - 1 \neq 0$. Therefore we can cancel the common factor and compute the limit as follows.

$$\begin{aligned}
\lim_{x\to 1}\frac{x - 1}{x^2 - 1} &= \lim_{x\to 1}\frac{x - 1}{(x - 1)(x + 1)} && \text{Factor}\\
&= \lim_{x\to 1}\frac{1}{x + 1} && \text{Cancel}\\
&= \frac{1}{1 + 1} = \frac{1}{2} && \text{Let } x \to 1
\end{aligned}$$

This calculation confirms algebraically the answer we got numerically and graphically in Example 12.1.1.

Now Try Exercise 19 ■

Example 5 ■ Finding a Limit by Simplifying

Evaluate $\displaystyle\lim_{h\to 0}\frac{(3 + h)^2 - 9}{h}$.

Solution We can't use direct substitution to evaluate this limit because the limit of the denominator is 0. So we first simplify the limit algebraically.

$$\begin{aligned}
\lim_{h\to 0}\frac{(3+h)^2-9}{h} &= \lim_{h\to 0}\frac{(9+6h+h^2)-9}{h} && \text{Expand}\\
&= \lim_{h\to 0}\frac{6h+h^2}{h} && \text{Simplify}\\
&= \lim_{h\to 0}(6+h) && \text{Cancel } h\\
&= 6 && \text{Let } h\to 0
\end{aligned}$$

Now Try Exercise 25 ■

Example 6 ■ Finding a Limit by Rationalizing

Find $\displaystyle\lim_{t\to 0}\frac{\sqrt{t^2+9}-3}{t^2}$.

Solution We can't apply Law 5 (Limit of a Quotient) immediately, since the limit of the denominator is 0. Here the preliminary algebra consists of rationalizing the numerator.

$$\begin{aligned}
\lim_{t\to 0}\frac{\sqrt{t^2+9}-3}{t^2} &= \lim_{t\to 0}\frac{\sqrt{t^2+9}-3}{t^2}\cdot\frac{\sqrt{t^2+9}+3}{\sqrt{t^2+9}+3} && \text{Rationalize numerator}\\
&= \lim_{t\to 0}\frac{(t^2+9)-9}{t^2\left(\sqrt{t^2+9}+3\right)} = \lim_{t\to 0}\frac{t^2}{t^2\left(\sqrt{t^2+9}+3\right)}\\
&= \lim_{t\to 0}\frac{1}{\sqrt{t^2+9}+3} = \frac{1}{\sqrt{\lim_{t\to 0}(t^2+9)}+3} = \frac{1}{3+3} = \frac{1}{6}
\end{aligned}$$

This calculation confirms the guess that we made in Example 12.1.2.

Now Try Exercise 27 ■

■ Using Left- and Right-Hand Limits

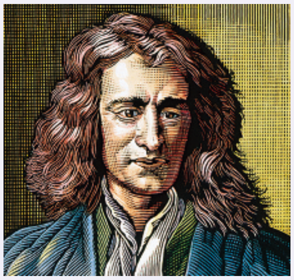
Some limits are best calculated by first finding the left- and right-hand limits. The following theorem is a reminder of what we discovered in Section 12.1. It says that *a two-sided limit exists if and only if both of the one-sided limits exist and are equal.*

$$\lim_{x\to a} f(x) = L \quad \text{if and only if} \quad \lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$$

When computing one-sided limits, we use the fact that the Limit Laws also hold for one-sided limits.

Example 7 ■ Comparing Right and Left Limits

Show that $\displaystyle\lim_{x\to 0}|x| = 0$.


B. Sanerson/Science Source

SIR ISAAC NEWTON (1642–1727) is universally regarded as one of the giants of physics and mathematics. He is well known for discovering the laws of motion and gravity and for inventing calculus, but he also proved the Binomial Theorem and the laws of optics, and he developed methods for solving polynomial equations to any desired accuracy. He was born a few months after the death of his father. After an unhappy childhood, he entered Cambridge University, where he learned mathematics by studying the writings of Euclid and Descartes.

During the plague years of 1665 and 1666, when the university was closed, Newton thought and wrote about ideas that, once published, instantly revolutionized the sciences. Imbued with a pathological fear of criticism, he published these writings only after many years of encouragement from Edmund Halley (who discovered the famous comet) and other colleagues.

Newton's works brought him enormous fame and prestige. Even poets were moved to praise; Alexander Pope wrote:

Nature and Nature's Laws
lay hid in Night.
God said, "Let Newton be"
and all was Light.

Newton was far more modest about his accomplishments. He said, "I do not know what I may appear to the world, but to myself I seem to have been only like a boy playing on the seashore . . . while the great ocean of truth lay all undiscovered before me." Newton was knighted by Queen Anne in 1705 and was buried with great honor in Westminster Abbey.

Solution Recall that

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

The result of Example 7 looks plausible from Figure 2.

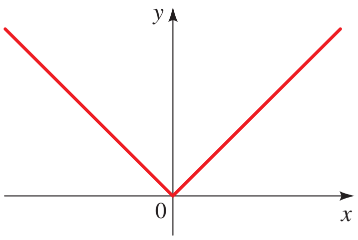


Figure 2 | $y = |x|$

Since $|x| = x$ for $x > 0$, we have

$$\lim_{x \to 0^+} |x| = \lim_{x \to 0^+} x = 0$$

For $x < 0$ we have $|x| = -x$, so

$$\lim_{x \to 0^-} |x| = \lim_{x \to 0^-} (-x) = 0$$

Because the left- and right-hand limits exist and are equal, we have

$$\lim_{x \to 0} |x| = 0$$

Now Try Exercise 37 ■

Example 8 ■ Comparing Right and Left Limits

Prove that $\lim_{x \to 0} \dfrac{|x|}{x}$ does not exist.

Solution Since $|x| = x$ for $x > 0$ and $|x| = -x$ for $x < 0$, we have

$$\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = \lim_{x \to 0^+} 1 = 1$$

$$\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x} = \lim_{x \to 0^-} (-1) = -1$$

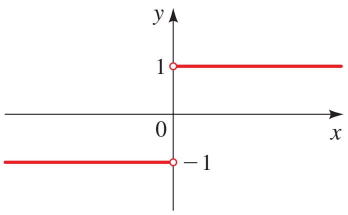


Figure 3 | $y = \dfrac{|x|}{x}$

Since the left-hand and right-hand limits exist and are different, it follows that $\lim_{x \to 0} |x|/x$ does not exist. The graph of the function $f(x) = |x|/x$ shown in Figure 3 confirms the limits that we found.

Now Try Exercise 39 ■

Example 9 ■ The Limit of a Piecewise-Defined Function

Let

$$f(x) = \begin{cases} \sqrt{x - 4} & \text{if } x > 4 \\ 8 - 2x & \text{if } x < 4 \end{cases}$$

Determine whether $\lim_{x \to 4} f(x)$ exists.

Solution Since $f(x) = \sqrt{x - 4}$ for $x > 4$, we have

$$\lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} \sqrt{x - 4} = \sqrt{4 - 4} = 0$$

Since $f(x) = 8 - 2x$ for $x < 4$, we have

$$\lim_{x \to 4^-} f(x) = \lim_{x \to 4^-} (8 - 2x) = 8 - 2 \cdot 4 = 0$$

The left- and right-hand limits are equal. Thus the limit exists, and

$$\lim_{x \to 4} f(x) = 0$$

The graph of f is shown in Figure 4.

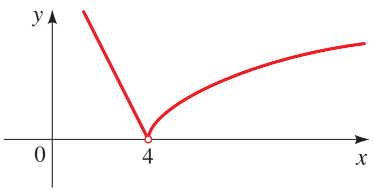


Figure 4

Now Try Exercise 43 ■

12.2 Exercises

Concepts

1. Suppose the following limits exist:

$$\lim_{x\to a} f(x) \quad \text{and} \quad \lim_{x\to a} g(x)$$

Then $\lim_{x\to a}[f(x)+g(x)] = $ ________ + ________, and

$\lim_{x\to a}[f(x)g(x)] = $ ________ · ________.

These formulas can be stated verbally as follows: The limit of a sum is the ________ of the limits, and the limit of a product is the ________ of the limits.

2. If f is a polynomial or a rational function and a is in the domain of f, then $\lim_{x\to a} f(x) = $ ________.

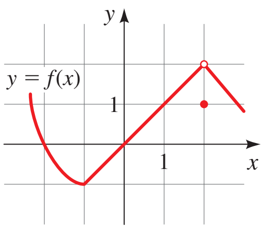
Skills

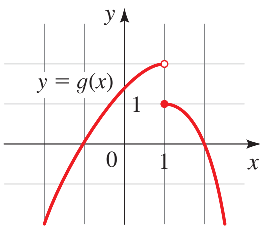
3. Limits from a Graph The graphs of f and g are given. Use them to evaluate each limit if it exists. If the limit does not exist, explain why.

(a) $\lim_{x\to 2}[f(x)+g(x)]$ **(b)** $\lim_{x\to 1}[f(x)+g(x)]$

(c) $\lim_{x\to 0}[f(x)g(x)]$ **(d)** $\lim_{x\to -1}\dfrac{f(x)}{g(x)}$

(e) $\lim_{x\to 2} x^3 f(x)$ **(f)** $\lim_{x\to 1}\sqrt{3+f(x)}$





4. Using Limit Laws Suppose that

$$\lim_{x\to a} f(x) = -3 \qquad \lim_{x\to a} g(x) = 0 \qquad \lim_{x\to a} h(x) = 8$$

Find the value of the given limit. If the limit does not exist, explain why.

(a) $\lim_{x\to a}[f(x)+h(x)]$ **(b)** $\lim_{x\to a}[f(x)]^2$

(c) $\lim_{x\to a}\sqrt[3]{h(x)}$ **(d)** $\lim_{x\to a}\dfrac{1}{f(x)}$

(e) $\lim_{x\to a}\dfrac{f(x)}{h(x)}$ **(f)** $\lim_{x\to a}\dfrac{g(x)}{f(x)}$

(g) $\lim_{x\to a}\dfrac{f(x)}{g(x)}$ **(h)** $\lim_{x\to a}\dfrac{2f(x)}{h(x)-f(x)}$

5–18 ■ Using Limit Laws Evaluate the limit and justify each step by indicating the appropriate Limit Law(s).

5. $\lim_{x\to 5} x$ **6.** $\lim_{x\to 0} 3$

7. $\lim_{t\to 3} 4t$ **8.** $\lim_{t\to 2}(1-3t)$

9. $\lim_{x\to 4}(5x^2-2x+3)$ **10.** $\lim_{x\to 0}(3x^3-2x^2+5)$

11. $\lim_{x\to -1}\dfrac{x-2}{x^2+4x-3}$ **12.** $\lim_{x\to 2}\dfrac{2-x}{x^2+1}$

13. $\lim_{x\to 3}(x^3+2)(x^2-5x)$ **14.** $\lim_{t\to -2}(t+1)^9(t^2-1)$

15. $\lim_{x\to 1}\left(\dfrac{x^4+x^2-6}{x^4+2x+3}\right)^2$

16. $\lim_{x\to 0}\left(\dfrac{-5x^{20}-2x^2+3000}{x^2-1}\right)^{1/3}$

17. $\lim_{x\to 12}\left(\sqrt{x^2+25}-\sqrt{3x}\right)$ **18.** $\lim_{u\to -2}\sqrt{u^4+3u+6}$

19–32 ■ Finding Limits Evaluate the limit, if it exists.

19. $\lim_{x\to 2}\dfrac{x^2+x-6}{x-2}$ **20.** $\lim_{x\to -4}\dfrac{x^2+5x+4}{x^2+3x-4}$

21. $\lim_{x\to -2}\dfrac{x^2-x+6}{x+2}$ **22.** $\lim_{x\to 1}\dfrac{x^3-1}{x^2-1}$

23. $\lim_{t\to -3}\dfrac{t^2-9}{2t^2+7t+3}$ **24.** $\lim_{x\to 2}\dfrac{x^4-16}{x-2}$

25. $\lim_{h\to 0}\dfrac{(2+h)^2-4}{h}$ **26.** $\lim_{h\to 0}\dfrac{(2+h)^3-8}{h}$

27. $\lim_{x\to 7}\dfrac{\sqrt{x+2}-3}{x-7}$ **28.** $\lim_{h\to 0}\dfrac{\sqrt{1+h}-1}{h}$

29. $\lim_{x\to -4}\dfrac{\frac{1}{4}+\frac{1}{x}}{4+x}$ **30.** $\lim_{t\to 0}\left(\dfrac{1}{t}-\dfrac{1}{t^2+t}\right)$

31. $\lim_{h\to 0}\dfrac{(3+h)^{-1}-3^{-1}}{h}$ **32.** $\lim_{t\to 4}\dfrac{\frac{1}{\sqrt{t}}-\frac{1}{2}}{t-4}$

33–36 ■ Finding Limits Find the limit, and use a graphing device to confirm your result graphically.

33. $\lim_{x\to 1}\dfrac{x^2-1}{\sqrt{x}-1}$ **34.** $\lim_{x\to 0}\dfrac{(4+x)^3-64}{x}$

35. $\lim_{x\to -1}\dfrac{x^2-x-2}{x^3-x}$ **36.** $\lim_{x\to 1}\dfrac{x^8-1}{x^5-1}$

37–42 ■ Does the Limit Exist? Find the limit, if it exists. If the limit does not exist, explain why.

37. $\lim_{x\to -4}|x+4|$ **38.** $\lim_{x\to -4^-}\dfrac{|x+4|}{x+4}$

39. $\lim_{x\to 2}\dfrac{|x-2|}{x-2}$ **40.** $\lim_{x\to 1.5}\dfrac{2x^2-3x}{|2x-3|}$

41. $\lim_{x\to 0^-}\left(\dfrac{1}{x}-\dfrac{1}{|x|}\right)$ **42.** $\lim_{x\to 0^+}\left(\dfrac{1}{x}-\dfrac{1}{|x|}\right)$

43. Does the Limit Exist? Let

$$f(x) = \begin{cases} x-1 & \text{if } x<2 \\ x^2-4x+6 & \text{if } x\ge 2 \end{cases}$$

(a) Find $\lim_{x\to 2^-} f(x)$ and $\lim_{x\to 2^+} f(x)$.

(b) Does $\lim_{x\to 2} f(x)$ exist?

(c) Sketch the graph of f.

44. Does the Limit Exist? Let

$$h(x) = \begin{cases} x & \text{if } x < 0 \\ x^2 & \text{if } 0 < x \le 2 \\ 8 - x & \text{if } x > 2 \end{cases}$$

(a) Evaluate each limit if it exists.

(i) $\lim_{x \to 0^+} h(x)$ (ii) $\lim_{x \to 0} h(x)$

(iii) $\lim_{x \to 1} h(x)$ (iv) $\lim_{x \to 2^-} h(x)$

(v) $\lim_{x \to 2^+} h(x)$ (vi) $\lim_{x \to 2} h(x)$

(b) Sketch the graph of h.

■ Skills Plus

45. Finding Limits Numerically and Graphically

(a) Estimate the value of

$$\lim_{x \to 0} \frac{x}{\sqrt{1 + 3x} - 1}$$

by graphing the function $f(x) = x/(\sqrt{1 + 3x} - 1)$.

(b) Make a table of values of $f(x)$ for x close to 0, and guess the value of the limit.

(c) Use the Limit Laws to prove that your guess is correct.

46. Finding Limits Numerically and Graphically

(a) Use a graph of

$$f(x) = \frac{\sqrt{3 + x} - \sqrt{3}}{x}$$

to estimate the value of $\lim_{x \to 0} f(x)$ to two decimal places.

(b) Use a table of values of $f(x)$ to estimate the limit to four decimal places.

(c) Use the Limit Laws to find the exact value of the limit.

■ Discuss ■ Discover ■ Prove ■ Write

47. Discuss: Cancellation and Limits

(a) What is wrong with the following equation?

$$\frac{x^2 + x - 6}{x - 2} = x + 3$$

(b) In view of part (a), explain why the equation

$$\lim_{x \to 2} \frac{x^2 + x - 6}{x - 2} = \lim_{x \to 2} (x + 3)$$

is correct.

48. Discuss: The Lorentz Contraction In the theory of relativity the Lorentz contraction formula

$$L = L_0 \sqrt{1 - v^2/c^2}$$

expresses the length L of an object as a function of its velocity v with respect to an observer, where L_0 is the length of the object at rest and c is the speed of light. Find $\lim_{v \to c^-} L$, and interpret the result. Why is a left-hand limit necessary?

49. Discuss ■ Prove: Limits of Sums and Products

(a) Show by means of an example that $\lim_{x \to a} [f(x) + g(x)]$ may exist even though neither $\lim_{x \to a} f(x)$ nor $\lim_{x \to a} g(x)$ exists.

(b) Show by means of an example that $\lim_{x \to a} [f(x)g(x)]$ may exist even though neither $\lim_{x \to a} f(x)$ nor $\lim_{x \to a} g(x)$ exists.

12.3 Tangent Lines and Derivatives

■ Tangent Lines ■ Derivatives ■ Instantaneous Rates of Change

In this section we see how limits arise when we attempt to find the tangent line to a curve or the instantaneous rate of change of a function.

■ Tangent Lines

A *tangent line* is a line that *just* touches a curve. For instance, Figure 1 shows the parabola $y = x^2$ and the tangent line t that touches the parabola at the point $P(1, 1)$. We will be able to find an equation of the tangent line t as soon as we know its slope m. The difficulty is that we know only one point, P, on t, whereas we need two points to compute the slope. But observe that we can compute an approximation to m by choosing a nearby point $Q(x, x^2)$ on the parabola (as in Figure 2) and computing the slope m_{PQ} of the secant line PQ.

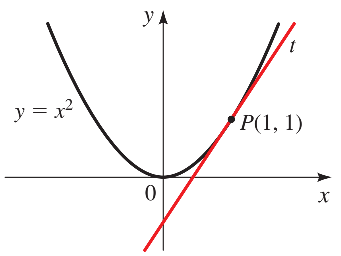


Figure 1

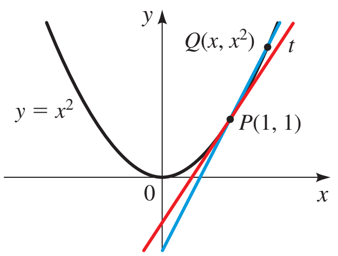


Figure 2

We choose $x \neq 1$ so that $Q \neq P$. Then

$$m_{PQ} = \frac{x^2 - 1}{x - 1}$$

Now we let x approach 1, so Q approaches P along the parabola. Figure 3 shows how the corresponding secant lines rotate about P and approach the tangent line t.

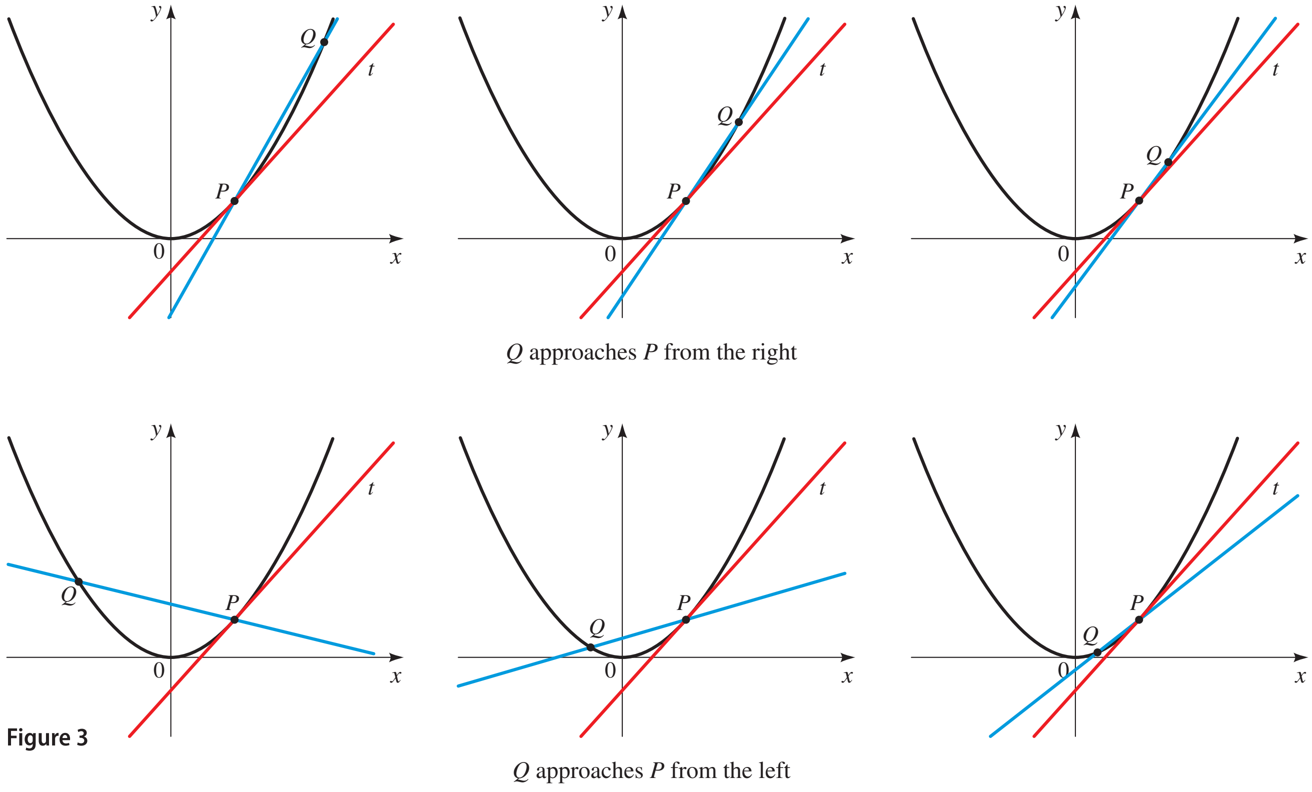


Figure 3

The slope of the tangent line is the limit of the slopes of the secant lines:

$$m = \lim_{Q \to P} m_{PQ}$$

So using the method of Section 12.2, we have

$$m = \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x - 1)(x + 1)}{x - 1}$$

$$= \lim_{x \to 1} (x + 1) = 1 + 1 = 2$$

The point-slope form for the equation of a line through the point (x_1, y_1) with slope m is

$$y - y_1 = m(x - x_1)$$

(See Section 1.10.)

Now that we know the slope of the tangent line is $m = 2$, we can use the point-slope form of the equation of a line to find its equation.

$$y - 1 = 2(x - 1) \qquad \text{or} \qquad y = 2x - 1$$

We sometimes refer to the slope of the tangent line to a curve at a point as the **slope of the curve** at the point. The idea is that if we zoom in far enough toward the point, the curve looks almost like a straight line. Figure 4 illustrates this procedure for the curve $y = x^2$. The more we zoom in, the more the parabola looks like a line. In other words, the curve becomes almost indistinguishable from its tangent line.

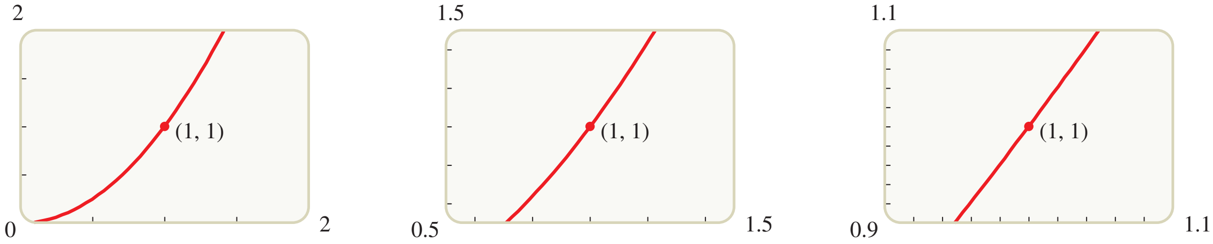


Figure 4 | Zooming in toward the point $(1, 1)$ on the parabola $y = x^2$

If we have a general curve C with equation $y = f(x)$ and we want to find the tangent line to C at the point $P(a, f(a))$, then we consider a nearby point $Q(x, f(x))$, where $x \neq a$, and compute the slope of the secant line PQ.

$$m_{PQ} = \frac{f(x) - f(a)}{x - a}$$

Then we let Q approach P along the curve C by letting x approach a. If m_{PQ} approaches a number m, then we define the *tangent* t to be the line through P with slope m. (This amounts to saying that the tangent line is the limiting position of the secant line PQ as Q approaches P. See Figure 5.)

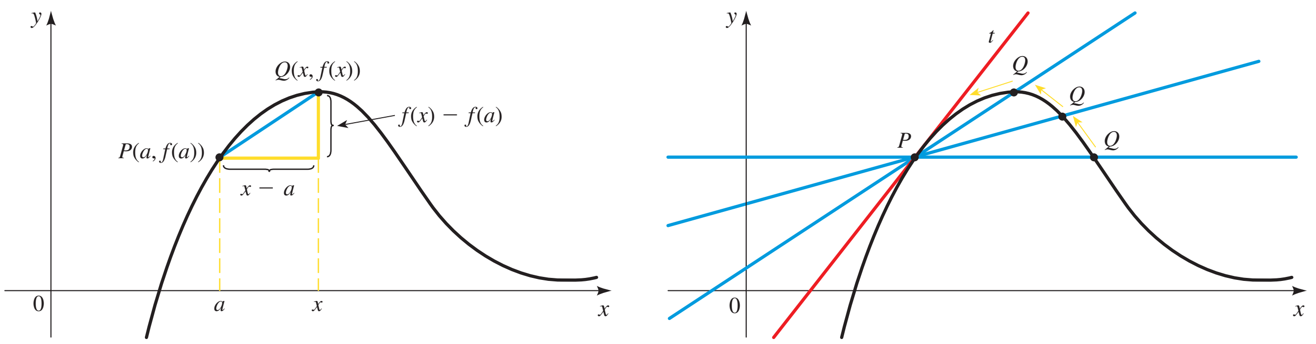


Figure 5

Definition of a Tangent Line

The **tangent line** to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.

Example 1 ■ Finding a Tangent Line to a Hyperbola

Find an equation of the tangent line to the hyperbola $y = 3/x$ at the point $(3, 1)$.

Solution Let $f(x) = 3/x$. Then the slope of the tangent line at $(3, 1)$ is

$$
\begin{aligned}
m &= \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3} && \text{Definition of } m \\
&= \lim_{x \to 3} \frac{\dfrac{3}{x} - 1}{x - 3} && f(x) = \frac{3}{x} \\
&= \lim_{x \to 3} \frac{3 - x}{x(x - 3)} && \text{Multiply numerator and denominator by } x \\
&= \lim_{x \to 3} \left(-\frac{1}{x}\right) && \text{Cancel } x - 3 \\
&= -\frac{1}{3} && \text{Let } x \to 3
\end{aligned}
$$

Therefore an equation of the tangent line at the point $(3, 1)$ is

$$y - 1 = -\tfrac{1}{3}(x - 3)$$

which simplifies to

$$x + 3y - 6 = 0$$

The hyperbola and its tangent at the point $(3, 1)$ are shown in Figure 6.

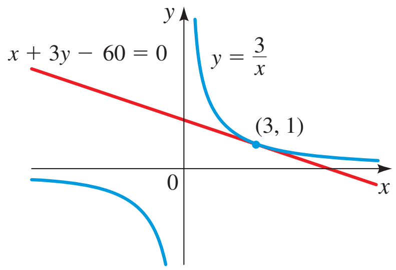


Figure 6

Now Try Exercises 3 and 11

There is another expression for the slope of a tangent line that is sometimes easier to use. Let $h = x - a$. Then $x = a + h$, so the slope of the secant line PQ is

$$m_{PQ} = \frac{f(a + h) - f(a)}{h}$$

See Figure 7, in which the case $h > 0$ is illustrated and Q is to the right of P. If it happened that $h < 0$, however, Q would be to the left of P.

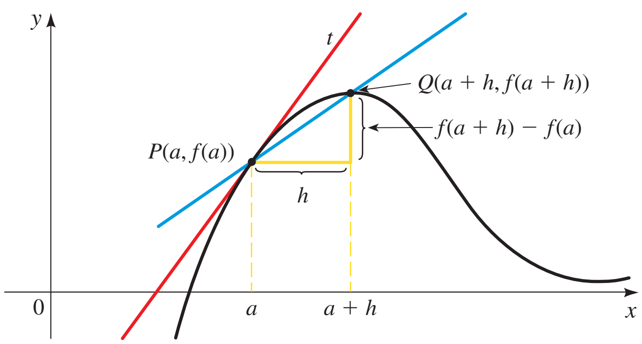


Figure 7

Notice that as x approaches a, h approaches 0 (because $h = x - a$), so the expression for the slope of the tangent line is the limit of difference quotients:

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Example 2 ■ Finding a Tangent Line

Find an equation of the tangent line to the curve $y = x^3 - 2x + 3$ at the point $(1, 2)$.

Solution If $f(x) = x^3 - 2x + 3$, then the slope of the tangent line where $a = 1$ is

$$\begin{aligned} m &= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} && \text{Definition of } m \\ &= \lim_{h \to 0} \frac{[(1+h)^3 - 2(1+h) + 3] - [1^3 - 2(1) + 3]}{h} && f(x) = x^3 - 2x + 3 \\ &= \lim_{h \to 0} \frac{1 + 3h + 3h^2 + h^3 - 2 - 2h + 3 - 2}{h} && \text{Expand numerator} \\ &= \lim_{h \to 0} \frac{h + 3h^2 + h^3}{h} && \text{Simplify} \\ &= \lim_{h \to 0} (1 + 3h + h^2) && \text{Cancel } h \\ &= 1 && \text{Let } h \to 0 \end{aligned}$$

So an equation of the tangent line at $(1, 2)$ is

$$y - 2 = 1(x - 1) \qquad \text{or} \qquad y = x + 1$$

Now Try Exercise 13 ■

■ Derivatives

Recall from Section 2.4 that the expression

$$\frac{f(a+h) - f(a)}{h}$$

is called a difference quotient and represents the average rate of change of f between $x = a$ and $x = a + h$.

We have seen that the slope of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$ can be written as

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

It turns out that this expression arises in many other contexts as well, such as finding velocities and other rates of change. Because this type of limit occurs so widely, it is given a special name and notation.

Definition of a Derivative

The **derivative of a function f at a number a**, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists.

Example 3 ■ Finding a Derivative at a Point

Find the derivative of the function $f(x) = 5x^2 + 3x - 1$ at the number 2.

Solution According to the definition of a derivative, with $a = 2$, we have

$$\begin{aligned}
f'(2) &= \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} && \text{Definition of } f'(2)\\
&= \lim_{h\to 0} \frac{[5(2+h)^2 + 3(2+h) - 1] - [5(2)^2 + 3(2) - 1]}{h} && f(x) = 5x^2 + 3x - 1\\
&= \lim_{h\to 0} \frac{20 + 20h + 5h^2 + 6 + 3h - 1 - 25}{h} && \text{Expand}\\
&= \lim_{h\to 0} \frac{23h + 5h^2}{h} && \text{Simplify}\\
&= \lim_{h\to 0} (23 + 5h) && \text{Cancel } h\\
&= 23 && \text{Let } h \to 0
\end{aligned}$$

Now Try Exercise 19 ■

Note We see from the definition of a derivative that the number $f'(a)$ is the slope of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$. So the result of Example 3 shows that the slope of the tangent line to the parabola $y = 5x^2 + 3x - 1$ at the point $(2, 25)$ is $f'(2) = 23$.

Example 4 ■ Finding a Derivative

Let $f(x) = \sqrt{x}$.

(a) Find $f'(a)$. **(b)** Find $f'(1)$, $f'(4)$, and $f'(9)$.

Solution

(a) We use the definition of the derivative at a.

$$\begin{aligned}
f'(a) &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} && \text{Definition of derivative}\\
&= \lim_{h\to 0} \frac{\sqrt{a+h} - \sqrt{a}}{h} && f(x) = \sqrt{x}\\
&= \lim_{h\to 0} \frac{\sqrt{a+h} - \sqrt{a}}{h} \cdot \frac{\sqrt{a+h} + \sqrt{a}}{\sqrt{a+h} + \sqrt{a}} && \text{Rationalize numerator}\\
&= \lim_{h\to 0} \frac{(a+h) - a}{h\left(\sqrt{a+h} + \sqrt{a}\right)} && \text{Difference of squares}\\
&= \lim_{h\to 0} \frac{h}{h\left(\sqrt{a+h} + \sqrt{a}\right)} && \text{Simplify numerator}\\
&= \lim_{h\to 0} \frac{1}{\sqrt{a+h} + \sqrt{a}} && \text{Cancel } h\\
&= \frac{1}{\sqrt{a} + \sqrt{a}} = \frac{1}{2\sqrt{a}} && \text{Let } h \to 0
\end{aligned}$$

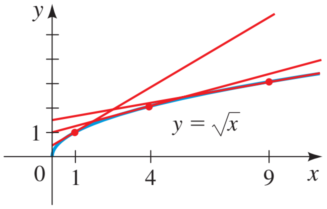


Figure 8

(b) Substituting $a = 1$, $a = 4$, and $a = 9$ into the result of part (a), we get

$$f'(1) = \frac{1}{2\sqrt{1}} = \frac{1}{2} \qquad f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4} \qquad f'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{6}$$

These values of the derivative are the slopes of the tangent lines shown in Figure 8.

Now Try Exercises 25 and 27 ■

Why are we interested in *instantaneous* speed? Because, for example, if we drop a ball from a high cliff, the ball falls faster and faster, its speed increasing at each *instant*. In order to model the motion of the ball we need to know its speed at each *instant*. Knowing this information allows us to completely describe all quantities related to the motion of the ball, including its speed, acceleration, and the distance it has traveled at any given time. You will learn more about these concepts in your Calculus course.

■ Instantaneous Rates of Change

In Section 2.4 we defined the average rate of change of a function f between the numbers a and x as

$$\text{average rate of change} = \frac{\text{change in } y}{\text{change in } x} = \frac{f(x) - f(a)}{x - a}$$

Suppose we consider the average rate of change over smaller and smaller intervals by letting x approach a. The limit of these average rates of change is called the instantaneous rate of change.

Instantaneous Rate of Change

If $y = f(x)$, the **instantaneous rate of change of y with respect to x** at $x = a$ is the limit of the average rates of change as x approaches a:

$$\text{instantaneous rate of change} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)$$

Notice that we now have two ways of interpreting the derivative:

- $f'(a)$ is the slope of the tangent line to $y = f(x)$ at $x = a$
- $f'(a)$ is the instantaneous rate of change of y with respect to x at $x = a$

In the special case in which $x = t =$ time and $s = f(t) =$ displacement (directed distance) at time t of an object traveling in a straight line, the instantaneous rate of change is called the **instantaneous velocity**.

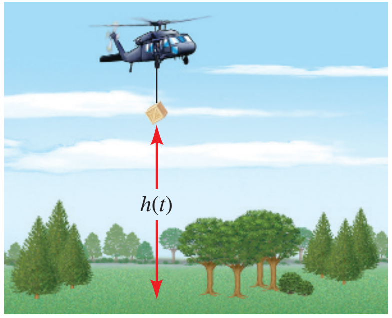
Example 5 ■ Instantaneous Velocity of a Falling Object

If an object is dropped from a height of 3000 ft, its distance above the ground (in feet) after t seconds is given by $h(t) = 3000 - 16t^2$. Find the object's instantaneous velocity after 4 seconds.

Lianys/Shutterstock.com

Discovery Project ■ Designing a Roller Coaster

To ensure an exhilarating ride, a roller coaster ought to consist of steep rises and drops joined by thrilling curves. For a safe ride, these curves must fit together "smoothly." In designing a roller coaster, you can choose where to locate the ascents and drops. We'll explore how the derivative can help us join these ascents and drops smoothly. You can find the project at **www.stewartmath.com**.



Solution After 4 seconds have elapsed, the height is $h(4) = 2744$ feet. The instantaneous velocity is

$$\begin{aligned} h'(4) &= \lim_{t \to 4} \frac{h(t) - h(4)}{t - 4} && \text{Definition of } h'(4) \\ &= \lim_{t \to 4} \frac{3000 - 16t^2 - 2744}{t - 4} && h(t) = 3000 - 16t^2 \\ &= \lim_{t \to 4} \frac{256 - 16t^2}{t - 4} && \text{Simplify} \\ &= \lim_{t \to 4} \frac{16(4 - t)(4 + t)}{t - 4} && \text{Factor numerator} \\ &= \lim_{t \to 4} -16(4 + t) && \text{Cancel } t - 4 \\ &= -16(4 + 4) = -128 \text{ ft/s} && \text{Let } t \to 4 \end{aligned}$$

The negative sign indicates that the height is *decreasing* at a rate of 128 ft/s.

Now Try Exercise 37 ■

US Population

t	$P(t)$ (millions)
2012	313.9
2014	318.4
2016	323.1
2018	326.8
2020	329.5

Source: US Census Bureau

t	$\dfrac{P(t) - P(2016)}{t - 2016}$
2012	2.30
2014	2.35
2018	1.85
2020	1.60

Here, we have estimated the derivative by averaging the slopes of two secant lines. Another method is to plot the population function and estimate the slope of the tangent line when $t = 2016$.

Example 6 ■ Estimating an Instantaneous Rate of Change

Let $P(t)$ be the population of the United States at time t. The first table in the margin gives approximate values of this function by providing midyear population estimates from 2012 to 2020. Interpret and estimate the value of $P'(2016)$.

Solution The derivative $P'(2016)$ means the rate of change of P with respect to t when $t = 2016$, that is, the rate of increase of the population in 2016.

According to the definition of a derivative, we have

$$P'(2016) = \lim_{t \to 2016} \frac{P(t) - P(2016)}{t - 2016}$$

So we compute and tabulate values of the difference quotient (the average rates of change) as shown in the second table in the margin. We see that $P'(2016)$ lies somewhere between 2.35 and 1.85 million. (Here we are making the reasonable assumption that the population didn't fluctuate wildly between 2012 and 2020.) We estimate that the rate of increase of the US population in 2016 was the average of these two numbers, namely,

$$P'(2016) \approx 2.10 \text{ million people/year}$$

Now Try Exercise 43 ■

12.3 Exercises

Concepts

1. The derivative of a function f at a number a is

$$f'(a) = \lim_{h \to 0} \frac{\square - \square}{\square}$$

if the limit exists. The derivative $f'(a)$ is the ________ of the tangent line to the curve $y = f(x)$ at the point $(\square, \square)$.

2. If $y = f(x)$, the average rate of change of f between the numbers x and a is $\dfrac{\square - \square}{\square - \square}$. The limit of the average rates of change as x approaches a is the ________ rate of change of y with respect to x at $x = a$; this is also the derivative $f'(\square)$.

Skills

3–10 ■ Slope of a Tangent Line Find the slope of the tangent line to the graph of f at the given point.

3. $f(x) = 3x + 4$, at $(1, 7)$

4. $f(x) = 5 - 2x$, at $(-3, 11)$

5. $f(x) = 4x^2 - 3x$, at $(-1, 7)$

6. $f(x) = 1 + 2x - 3x^2$, at $(1, 0)$

7. $f(x) = 2x^3$, at $(2, 16)$

8. $f(x) = x^3 + 1$, at $(2, 9)$

9. $f(x) = \dfrac{5}{x+2}$, at $(3, 1)$

10. $f(x) = \dfrac{6}{x+1}$, at $(2, 2)$

11–18 ■ Equation of a Tangent Line Find an equation of the tangent line to the curve at the given point. Graph the curve and the tangent line.

11. $f(x) = -2x^2 + 1$, at $(2, -7)$

12. $f(x) = 4x^2 - 3$, at $(-1, 1)$

13. $y = x + x^2$, at $(-1, 0)$

14. $y = 2x - x^3$, at $(1, 1)$

15. $y = \dfrac{x}{x-1}$, at $(2, 2)$

16. $y = \dfrac{1}{x^2}$, at $(-1, 1)$

17. $y = \sqrt{x+3}$, at $(1, 2)$

18. $y = \sqrt{1+2x}$, at $(4, 3)$

19–26 ■ The Derivative at a Number Find the derivative of the function at the given number.

19. $f(x) = 1 - 3x^2$, at 2

20. $f(x) = 2 - 3x + x^2$, at -1

21. $f(x) = x - 3x^2$, at -1

22. $f(x) = x + x^3$, at 1

23. $f(x) = \dfrac{1}{x+1}$, at 2

24. $f(x) = \dfrac{x}{2-x}$, at -3

25. $F(x) = \dfrac{1}{\sqrt{x}}$, at 4

26. $G(x) = 1 + 2\sqrt{x}$, at 4

27–30 ■ Evaluating Derivatives Find the following for the given function f: **(a)** $f'(a)$, where a is in the domain of f, and **(b)** $f'(3)$ and $f'(4)$.

27. $f(x) = x^2 + 2x$

28. $f(x) = -\dfrac{1}{x^2}$

29. $f(x) = \dfrac{x}{x+1}$

30. $f(x) = \sqrt{x-2}$

Skills Plus

31. Tangent Lines

(a) If $f(x) = x^3 - 2x + 4$, find $f'(a)$.

(b) Find equations of the tangent lines to the graph of f at the points whose x-coordinates are 0, 1, and 2.

(c) Graph f and the three tangent lines.

32. Tangent Lines

(a) If $g(x) = 1/(2x-1)$, find $g'(a)$.

(b) Find equations of the tangent lines to the graph of g at the points whose x-coordinates are -1, 0, and 1.

(c) Graph g and the three tangent lines.

33–36 ■ Which Derivative Does the Limit Represent? The given limit represents the derivative of a function f at a number a. Find f and a.

33. $\displaystyle\lim_{h\to 0} \frac{(1+h)^{10} - 1}{h}$

34. $\displaystyle\lim_{x\to 5} \frac{2^x - 32}{x-5}$

35. $\displaystyle\lim_{t\to 1} \frac{\sqrt{t+1} - \sqrt{2}}{t-1}$

36. $\displaystyle\lim_{h\to 0} \frac{\cos(\pi + h) + 1}{h}$

Applications

37. Velocity of a Ball If a ball is thrown straight up with a velocity of 40 ft/s, its height (in ft) after t seconds is given by $y = 40t - 16t^2$. Find the instantaneous velocity when $t = 2$.

38. Velocity on the Moon If an arrow is shot upward on the moon with a velocity of 58 m/s, its height (in meters) after t seconds is given by $H = 58t - 0.83t^2$.

(a) Find the instantaneous velocity of the arrow after 1 second.

(b) Find the instantaneous velocity of the arrow when $t = a$.

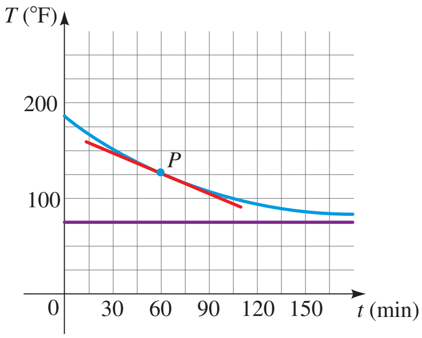
(c) At what time t will the arrow hit the moon?

(d) With what velocity will the arrow hit the moon?

39. Velocity of a Particle The displacement s (in meters) of a particle moving in a straight line is given by the equation of motion $s = 4t^3 + 6t + 2$, where t is measured in seconds. Find the instantaneous velocity of the particle s at times $t = a$, $t = 1$, $t = 2$, $t = 3$.

40. Inflating a Balloon A spherical balloon is being inflated. Find the rate of change of the surface area $(S = 4\pi r^2)$ with respect to the radius r when $r = 2$ ft.

41. Temperature Change A roast turkey is taken from an oven when its temperature has reached 185°F and is placed on a table in a room where the temperature is 75°F. The graph shows how the temperature of the turkey decreases and eventually approaches room temperature. By measuring the slope of the tangent, estimate the rate of change of the temperature after 60 minutes.



42. Heart Rate A cardiac monitor is used to measure the heart rate of a patient after surgery. It compiles the number of heartbeats after t min. When the data in the table are graphed, the slope of the tangent line represents the heart rate in beats per minute.

t (min)	Heartbeats
36	2530
38	2661
40	2806
42	2948
44	3080

(a) Find the average heart rates (slopes of the secant lines) over the time intervals $[40, 42]$ and $[42, 44]$.

(b) Estimate the patient's heart rate after 42 min by averaging the slopes of these two secant lines.

43. Water Flow A tank holds 1000 gallons of water, which drains from the bottom of the tank in half an hour. The values in the table show the volume V of water remaining in the tank (in gal) after t minutes.

t (min)	V (gal)
5	694
10	444
15	250
20	111
25	28
30	0

(a) Find the average rates at which water flows from the tank (slopes of secant lines) for the time intervals $[10, 15]$ and $[15, 20]$.

(b) The slope of the tangent line at the point $(15, 250)$ represents the rate at which water is flowing from the tank after 15 min. Estimate this rate by averaging the slopes of the secant lines in part (a).

44. World Population Growth The table gives approximate values for the world population by providing midyear population estimates for the years 1900–2020. Estimate the rate of population growth in 1920 and in 2010 by averaging the slopes of two secant lines.

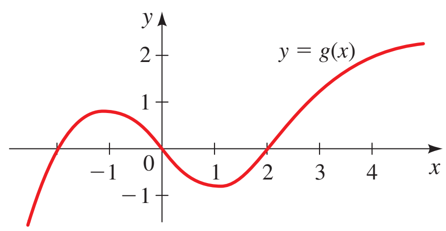
Year	Population (millions)	Year	Population (millions)
1900	1650	1970	3710
1910	1750	1980	4450
1920	1860	1990	5290
1930	2070	2000	6090
1940	2300	2010	6870
1950	2560	2020	7757
1960	3040		

Source: US Census Bureau

■ Discuss ■ Discover ■ Prove ■ Write

45. Discuss: Estimating Derivatives from a Graph For the function g whose graph is given, arrange the following numbers in increasing order, and explain your reasoning.

$$0 \qquad g'(-2) \qquad g'(0) \qquad g'(2) \qquad g'(4)$$



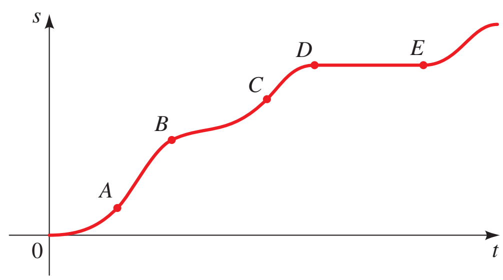
46. Discuss: Estimating Velocities from a Graph The graph shows the position function of a car. Use the shape of the graph to explain your answers to the following questions.

(a) What was the initial velocity of the car?

(b) Was the car going faster at B or at C?

(c) Was the car slowing down or speeding up at A, B, and C?

(d) What happened between D and E?



12.4 Limits at Infinity; Limits of Sequences

■ Limits at Infinity ■ Limits of Sequences

In this section we study a special kind of limit called a *limit at infinity*. We examine the limit of a function $f(x)$ as x becomes large. We also examine the limit of a sequence a_n as n becomes large. Limits of sequences will be used in Section 12.5 to help us find the area under the graph of a function.

■ Limits at Infinity

Let's investigate the behavior of the function f defined by

$$f(x) = \frac{x^2 - 1}{x^2 + 1}$$

x	$f(x)$
0	-1.000000
± 1	0.000000
± 2	0.600000
± 3	0.800000
± 4	0.882353
± 5	0.923077
± 10	0.980198
± 50	0.999200
± 100	0.999800
± 1000	0.999998

as x becomes large. The table in the margin gives values of this function rounded to six decimal places, and the graph of f has been drawn by a computer in Figure 1.

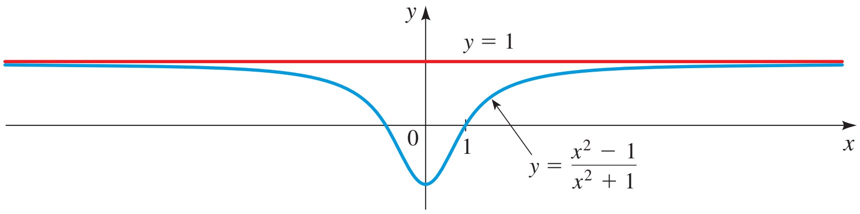


Figure 1

As x grows larger and larger, you can see that the values of $f(x)$ get closer and closer to 1. In fact, it seems that we can make the values of $f(x)$ as close as we like to 1 by taking x sufficiently large. This situation is expressed symbolically by writing

$$\lim_{x \to \infty} \frac{x^2 - 1}{x^2 + 1} = 1$$

In general, we use the notation $\lim_{x \to \infty} f(x) = L$ to indicate that the values of $f(x)$ become closer and closer to L as x becomes larger and larger.

Definition of a Limit at Infinity

We write

$$\lim_{x \to \infty} f(x) = L$$

and say that **"the limit of $f(x)$, as x approaches infinity**, equals L" if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently large.

We also say "the limit of $f(x)$, as x increases without bound, equals L."

Another notation for $\lim_{x \to \infty} f(x) = L$ is

$$f(x) \to L \qquad \text{as} \qquad x \to \infty$$

Limits at infinity are also discussed in Section 3.6.

Graphical illustrations of limits at infinity are shown in Figure 2. Notice that there are many ways for the graph of f to approach the line $y = L$ as we look to the far right of each graph.

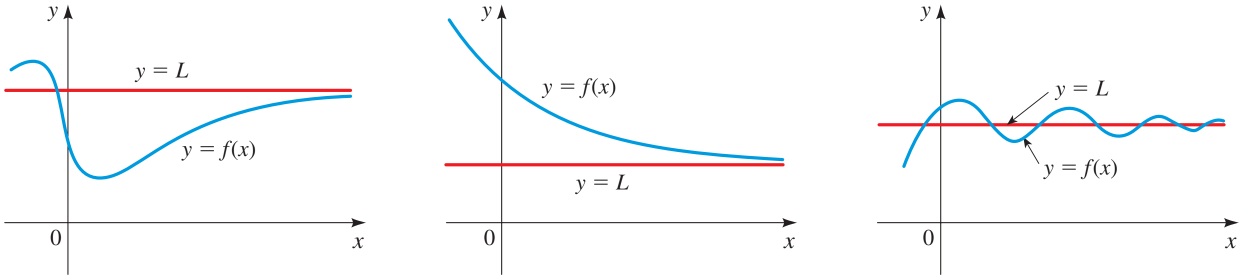


Figure 2 | Examples illustrating $\lim_{x \to \infty} f(x) = L$

Referring back to Figure 1, we see that for numerically large negative values of x, the values of $f(x)$ are close to 1. This is expressed by writing

$$\lim_{x \to -\infty} \frac{x^2 - 1}{x^2 + 1} = 1$$

Definition of a Limit at Negative Infinity

We write

$$\lim_{x \to -\infty} f(x) = L$$

and say that **"the limit of $f(x)$, as x approaches negative infinity**, equals L" if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently large negative.

We also say "the limit of $f(x)$, as x decreases (through negative values) without bound, equals L."

This definition is illustrated in Figure 3. Notice how each graph approaches the line $y = L$ as we look to the far left.

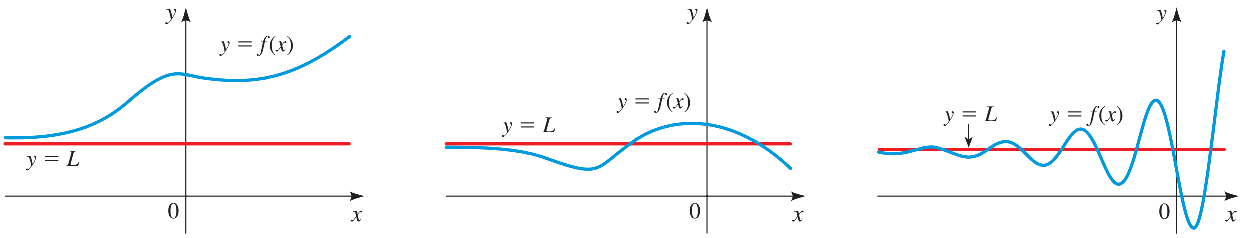


Figure 3 | Examples illustrating $\lim_{x \to -\infty} f(x) = L$

Note The symbols ∞ and $-\infty$ do not represent numbers. When we write "$x \to \infty$" we mean that x increases without bound. Graphically, this means that x is allowed to move to the right on the real line indefinitely. Similarly, when we write "$x \to -\infty$" we mean that x decreases (through negative numbers) without bound. Graphically, this means that x can move to the left indefinitely.

In Section 3.6 we studied horizontal asymptotes of rational function. We now define the concept of a horizontal asymptote for any function by using limits.

Horizontal Asymptote

The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = L \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = L$$

For instance, the curve in Figure 1 has the line $y = 1$ as a horizontal asymptote because

$$\lim_{x\to\infty} \frac{x^2 - 1}{x^2 + 1} = 1$$

As we discovered in Section 5.5, an example of a curve with two horizontal asymptotes is $y = \tan^{-1}x$. (See Figure 4.) In fact,

$$\lim_{x\to-\infty} \tan^{-1}x = -\frac{\pi}{2} \qquad \text{and} \qquad \lim_{x\to\infty} \tan^{-1}x = \frac{\pi}{2}$$

so both of the lines $y = -\pi/2$ and $y = \pi/2$ are horizontal asymptotes.

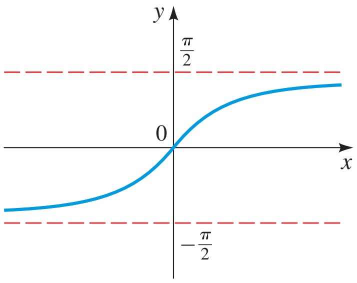


Figure 4 | $y = \tan^{-1}x$

Example 1 ■ Limits at Infinity

Find $\lim_{x\to\infty} \frac{1}{x}$ and $\lim_{x\to-\infty} \frac{1}{x}$.

We first investigated horizontal asymptotes and limits at infinity for rational functions in Section 3.6.

Solution Observe that when x is large, $1/x$ is small. For instance,

$$\frac{1}{100} = 0.01 \qquad \frac{1}{10{,}000} = 0.0001 \qquad \frac{1}{1{,}000{,}000} = 0.000001$$

In fact, by taking x large enough, we can make $1/x$ as close to 0 as we please. Therefore

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

Similar reasoning shows that when x is large negative, $1/x$ is small negative, so we also have

$$\lim_{x\to-\infty} \frac{1}{x} = 0$$

It follows that the line $y = 0$ (the x-axis) is a horizontal asymptote of the curve $y = 1/x$. (See Figure 5.)

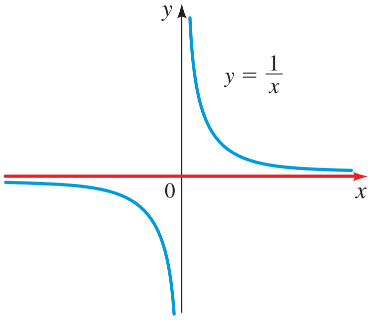


Figure 5 | $\lim_{x\to\infty} \frac{1}{x} = 0, \ \lim_{x\to-\infty} \frac{1}{x} = 0$

Now Try Exercise 5 ■

The Limit Laws that we studied in Section 12.2 also hold for limits at infinity. In particular, if we combine Law 6 (Limit of a Power) with the results of Example 1, we obtain the following important rule for calculating limits.

> If k is any positive integer, then
>
> $$\lim_{x\to\infty} \frac{1}{x^k} = 0 \qquad \text{and} \qquad \lim_{x\to-\infty} \frac{1}{x^k} = 0$$

Example 2 ■ Finding a Limit at Infinity

Evaluate $\lim_{x\to\infty} \dfrac{3x^2 - x - 2}{5x^2 + 4x + 1}$.

Solution To evaluate the limit at infinity of a rational function, we first divide both the numerator and denominator by the highest power of x that occurs in the denominator. (We may assume that $x \neq 0$, since we are interested only in large values of x.) In this case the highest power of x in the denominator is x^2, so we have

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \lim_{x\to\infty} \frac{3 - \dfrac{1}{x} - \dfrac{2}{x^2}}{5 + \dfrac{4}{x} + \dfrac{1}{x^2}} \qquad \text{Divide numerator and denominator by } x^2$$

$$= \frac{\lim\limits_{x\to\infty}\left(3 - \dfrac{1}{x} - \dfrac{2}{x^2}\right)}{\lim\limits_{x\to\infty}\left(5 + \dfrac{4}{x} + \dfrac{1}{x^2}\right)} \qquad \text{Limit of a Quotient}$$

$$= \frac{\lim\limits_{x\to\infty} 3 - \lim\limits_{x\to\infty} \dfrac{1}{x} - 2\lim\limits_{x\to\infty} \dfrac{1}{x^2}}{\lim\limits_{x\to\infty} 5 + 4\lim\limits_{x\to\infty} \dfrac{1}{x} + \lim\limits_{x\to\infty} \dfrac{1}{x^2}} \qquad \text{Limits of Sums, Differences, and Constant Multiples}$$

$$= \frac{3 - 0 - 0}{5 + 0 + 0} = \frac{3}{5} \qquad \text{Let } x \to \infty$$

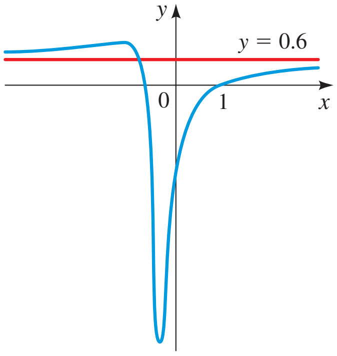


Figure 6

Similarly, the limit as $x \to -\infty$ is also $\frac{3}{5}$. Figure 6 confirms that the graph of the given rational function approaches the horizontal asymptote $y = \frac{3}{5}$.

Now Try Exercise 9 ■

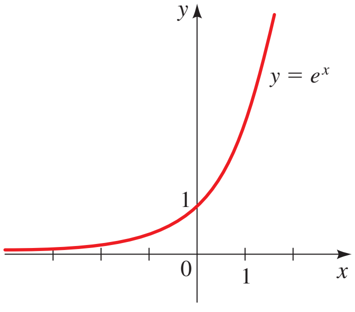
Example 3 ■ A Limit at Negative Infinity

Use numerical and graphical methods to find $\lim\limits_{x\to-\infty} e^x$.

Solution From the graph of the natural exponential function $y = e^x$ in Figure 7 and the corresponding table of values we see that

$$\lim_{x\to-\infty} e^x = 0$$

It follows that the line $y = 0$ (the x-axis) is a horizontal asymptote.



x	e^x
0	1.00000
−1	0.36788
−2	0.13534
−3	0.04979
−5	0.00674
−8	0.00034
−10	0.00005

Figure 7

Now Try Exercise 19 ■

Example 4 ■ A Function with No Limit at Infinity

Evaluate $\lim\limits_{x\to\infty} \sin x$.

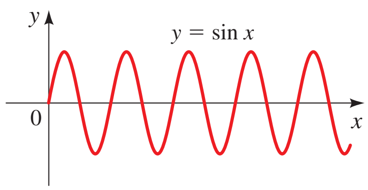


Figure 8

Solution From the graph in Figure 8 and the periodic nature of the sine function we see that as x increases, the values of $\sin x$ oscillate between 1 and -1 infinitely often, so they don't approach any definite number. Therefore $\lim_{x\to\infty} \sin x$ does not exist.

Now Try Exercise 17 ■

■ Limits of Sequences

In Section 11.1 we introduced the idea of a sequence of numbers $a_1, a_2, a_3, \ldots$. Here we are interested in their behavior as n becomes large. For instance, the sequence defined by

$$a_n = \frac{n}{n+1}$$

is pictured in Figure 9(a) by plotting its terms on a number line and in Figure 9(b) by plotting its graph. From Figure 9 it appears that the terms of the sequence $a_n = n/(n+1)$ are approaching 1 as n becomes large. We indicate this by writing

$$\lim_{n\to\infty} \frac{n}{n+1} = 1$$

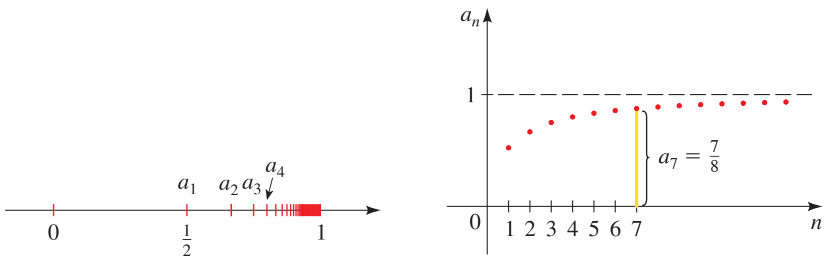


Figure 9 (a) (b)

Definition of the Limit of a Sequence

A sequence $a_1, a_2, a_3, \ldots$ has the **limit** L and we write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \quad \text{as} \quad n \to \infty$$

if the nth term a_n of the sequence can be made arbitrarily close to L by taking n sufficiently large. If $\lim_{n\to\infty} a_n$ exists, we say the sequence **converges** (or is **convergent**). Otherwise, we say the sequence **diverges** (or is **divergent**).

This definition is illustrated by Figure 10.

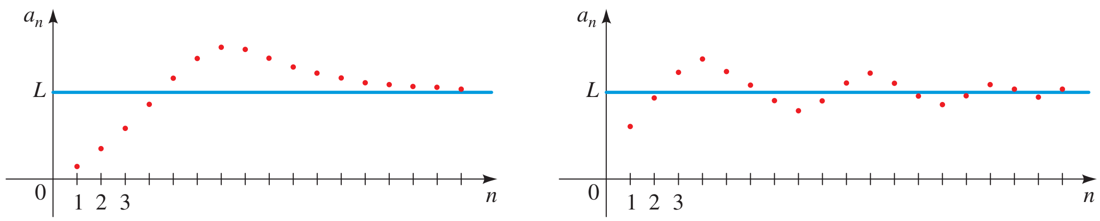


Figure 10 | Graphs of two sequences with $\lim_{n\to\infty} a_n = L$

If we compare the definitions of $\lim_{n\to\infty} a_n = L$ and $\lim_{x\to\infty} f(x) = L$, we see that the only difference is that n is required to be an integer. Thus we get the following result which is illustrated in Figure 11.

> If $\lim\limits_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim\limits_{n\to\infty} a_n = L$.

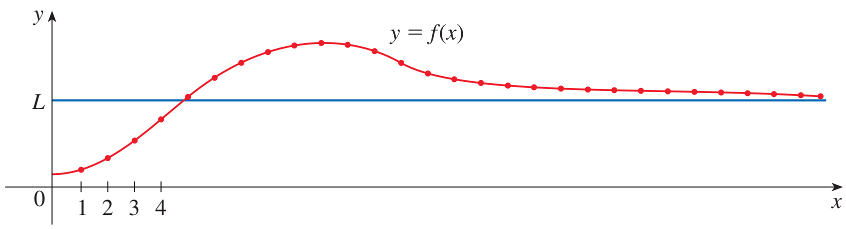


Figure 11 | $\lim\limits_{x\to\infty} f(x) = L$ and $\lim\limits_{x\to\infty} f(n) = L$

In particular, since we know that $\lim_{x\to\infty}(1/x^k) = 0$ when k is a positive integer, we have

$$\lim_{n\to\infty} \frac{1}{n^k} = 0 \qquad \text{if } k \text{ is a positive integer}$$

Note that the Limit Laws given in Section 12.2 also hold for limits of sequences.

Example 5 ■ Finding the Limit of a Sequence

Find $\lim\limits_{n\to\infty} \dfrac{n}{n+1}$.

Solution The method is similar to the one we used in Example 2: Divide the numerator and denominator by the highest power of n, and then use the Limit Laws.

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\dfrac{1}{n}} \qquad \text{Divide numerator and denominator by } n$$

$$= \frac{\lim\limits_{n\to\infty} 1}{\lim\limits_{n\to\infty} 1 + \lim\limits_{n\to\infty} \dfrac{1}{n}} \qquad \text{Limits of a Quotient and a Sum}$$

$$= \frac{1}{1+0} = 1 \qquad \text{Let } n \to \infty$$

This result shows that the guesses we made earlier from Figure 9(a) and Figure 9(b) were correct.

Therefore the sequence $a_n = n/(n+1)$ is convergent.

Now Try Exercise 23 ■

Example 6 ■ A Sequence That Diverges

Determine whether the sequence $a_n = (-1)^n$ is convergent or divergent.

Solution If we write out the terms of the sequence, we obtain

$$-1, 1, -1, 1, -1, 1, -1, \ldots$$

The graph of this sequence is shown in Figure 12. Since the terms oscillate between 1 and -1 infinitely often, a_n does not approach any number. Thus $\lim_{n\to\infty}(-1)^n$ does not exist; that is, the sequence $a_n = (-1)^n$ is divergent.

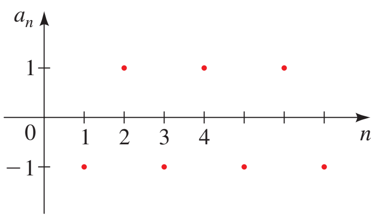


Figure 12

Now Try Exercise 29 ■

Example 7 ■ Finding the Limit of a Sequence

Find the limit of the sequence given by

$$a_n = \frac{15}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$$

Solution Before calculating the limit, let's first simplify the expression for a_n. Because $n^3 = n \cdot n \cdot n$, we place a factor of n beneath each factor in the numerator that contains an n:

$$a_n = \frac{15}{6}\cdot\frac{n}{n}\cdot\frac{n+1}{n}\cdot\frac{2n+1}{n} = \frac{5}{2}\cdot 1\cdot\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$

Now we can compute the limit.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{5}{2}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) \qquad \text{Definition of } a_n$$

$$= \frac{5}{2}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)\lim_{n\to\infty}\left(2+\frac{1}{n}\right) \qquad \text{Limit of a Product}$$

$$= \frac{5}{2}(1)(2) = 5 \qquad \text{Let } n \to \infty$$

Now Try Exercise 31

12.4 Exercises

Concepts

1. Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x\to\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to ________ by taking ________ sufficiently large. In this case the line $y = L$ is called a ________ ________ of the function $y = f(x)$. For example, $\lim_{x\to\infty}\frac{1}{x} =$ ________, and the line $y =$ ________ is a horizontal asymptote.

2. A sequence $a_1, a_2, a_3, \ldots$ has the limit L if the nth term a_n of the sequence can be made arbitrarily close to ________ by taking n to be sufficiently ________. If the limit exists, we say that the sequence ________________; otherwise, the sequence ________________.

Skills

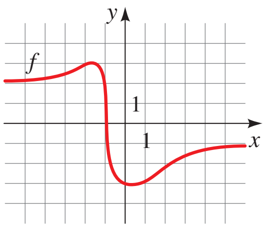
3–4 ■ Limits from a Graph

(a) Use the graph of f to find the following limits.

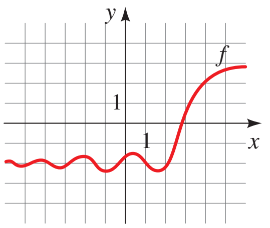
(i) $\lim_{x\to\infty} f(x)$ (ii) $\lim_{x\to-\infty} f(x)$

(b) State the equations of the horizontal asymptotes.

3.



4.



5–18 ■ Limits at Infinity Find the limit.

5. $\lim_{x\to\infty}\frac{6}{x}$

6. $\lim_{x\to\infty}\frac{3}{x^4}$

7. $\lim_{x\to\infty}\frac{2x+1}{5x-1}$

8. $\lim_{x\to\infty}\frac{2-3x}{4x+5}$

9. $\lim_{x\to-\infty}\frac{4x^2+1}{2+3x^2}$

10. $\lim_{x\to-\infty}\frac{x^2+2}{x^3+x+1}$

11. $\lim_{t\to\infty}\frac{8t^3+t}{(2t-1)(2t^2+1)}$

12. $\lim_{r\to\infty}\frac{4r^3-r^2}{(r+1)^3}$

13. $\lim_{x\to\infty}\frac{x^4}{1-x^2+x^3}$

14. $\lim_{t\to\infty}\left(\frac{1}{t}-\frac{2t}{t-1}\right)$

15. $\lim_{x\to-\infty}\left(\frac{x-1}{x+1}+6\right)$

16. $\lim_{x\to-\infty}\left(\frac{3-x}{3+x}-2\right)$

17. $\lim_{x\to\infty}\cos x$

18. $\lim_{x\to\infty}\sin^2 x$

19–22 ■ Estimating Limits Numerically and Graphically Use a table of values to estimate the limit. Then use a graphing device to confirm your result graphically.

19. $\lim_{x \to -\infty} \dfrac{\sqrt{x^2 + 4x}}{4x + 1}$

20. $\lim_{x \to \infty} \left(\sqrt{9x^2 + x} - 3x\right)$

21. $\lim_{x \to \infty} \dfrac{x^5}{e^x}$

22. $\lim_{x \to \infty} \left(1 + \dfrac{2}{x}\right)^{3x}$

23–34 ■ Limits of Sequences If the sequence with the given nth term is convergent, find its limit. If it is divergent, explain why.

23. $a_n = \dfrac{1 + n}{n + n^2}$

24. $a_n = \dfrac{5n}{n + 5}$

25. $a_n = \dfrac{n^2}{n + 1}$

26. $a_n = \dfrac{n - 1}{n^3 + 1}$

27. $a_n = \dfrac{1}{3^n}$

28. $a_n = \dfrac{(-1)^n}{n}$

29. $a_n = \sin(n\pi/2)$

30. $a_n = \cos n\pi$

31. $a_n = \dfrac{3}{n^2}\left[\dfrac{n(n + 1)}{2}\right]$

32. $a_n = \dfrac{12}{n^4}\left[\dfrac{n(n + 1)}{2}\right]^2$

33. $a_n = \dfrac{24}{n^3}\left[\dfrac{n(n + 1)(2n + 1)}{6}\right]$

34. $a_n = \dfrac{5}{n}\left(n + \dfrac{4}{n}\left[\dfrac{n(n + 1)}{2}\right]\right)$

Skills Plus

35–36 ■ A Function from a Description Find a formula for a function f that satisfies the following conditions.

35. Vertical asymptotes $x = 1$ and $x = 3$ and horizontal asymptote $y = 1$

36. $\lim_{x \to \infty} f(x) = 0, \quad \lim_{x \to 0} f(x) = -\infty, \quad f(2) = 0,$

$\lim_{x \to 3^-} f(x) = \infty, \quad \lim_{x \to 3^+} f(x) = -\infty$

37. Asymptote Behavior How close to -3 do we have to take x so that

$$\frac{1}{(x + 3)^2} > 10{,}000$$

38. Equivalent Limits Show that

$$\lim_{x \to \infty} f(x) = \lim_{t \to 0^+} f\left(\frac{1}{t}\right)$$

and
$$\lim_{x \to -\infty} f(x) = \lim_{t \to 0^-} f\left(\frac{1}{t}\right)$$

if these limits exist.

Applications

39. Salt Concentration

(a) A tank contains 5000 liters of pure water. Brine that contains 30 grams of salt per liter of water is pumped into the tank at a rate of 25 L/min. Show that the concentration of salt after t minutes (in g/L) is

$$C(t) = \frac{30t}{200 + t}$$

(b) What happens to the concentration as $t \to \infty$?

40. Velocity of a Raindrop The downward velocity (in m/s) of a falling raindrop at time t is modeled by the function

$$v(t) = 9.1(1 - e^{-1.2t})$$

(a) Find the terminal velocity of the raindrop by evaluating $\lim_{t \to \infty} v(t)$.

(b) Graph $v(t)$, and use the graph to estimate how long it takes for the velocity of the raindrop to reach 99% of its terminal velocity.

Discuss ■ Discover ■ Prove ■ Write

41. Discuss ■ Discover: The Tail of a Sequence Let a_n be a sequence. Explain why

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} a_{n+1}$$

That is, removing the first term of a sequence does not affect the limit. Does removing the first 100 terms of a sequence affect its limit? Conclude that convergence and divergence are properties of the "tail" of a sequence.

42. Discuss ■ Discover: Limit of a Recursive Sequence A sequence is defined recursively by $a_1 = 0$ and $a_{n+1} = \sqrt{2 + a_n}$. Calculate several terms of the sequence; do you think the sequence converges? Assume that the sequence does converge and that $\lim \lim_{n \to \infty} a_n = L$. Use Exercise 41 to show that L satisfies the equation $L = \sqrt{2 + L}$. Solve this equation to find the limit L.

43. Discover ■ Prove: The Fibonacci Sequence and the Golden Ratio Let F_n denote the nth term of the Fibonacci sequence (Section 11.1). Assume that the sequence F_{n+1}/F_n converges. Show that limit of this sequence is the Golden Ratio:

$$\lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \frac{1 + \sqrt{5}}{2}$$

PS *Introduce something extra.* Assume that the limit is L. Use the fact that $F_{n+1} = F_{n-1} + F_n$ and the property in Exercise 41 to find an equation that L must satisfy.

12.5 Areas

■ The Area Problem ■ Definition of Area

We have seen that limits are needed to compute the slope of a tangent line or an instantaneous rate of change. Here we will see that they are also needed to find the area of a region with a curved boundary. The problem of finding such areas has consequences far beyond simply finding area. (See the *Focus on Modeling* at the end of the chapter.)

■ The Area Problem

One of the central problems in calculus is the *area problem*: Find the area of the region S that lies under the curve $y = f(x)$ from a to b. This means that S, illustrated in Figure 1, is bounded by the graph of a function f (where $f(x) \geq 0$), the vertical lines $x = a$ and $x = b$, and the x-axis.

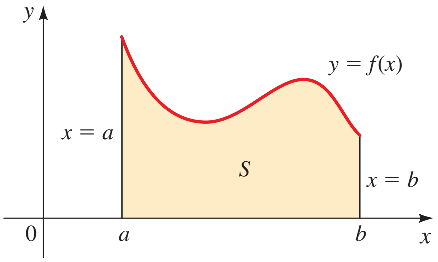


Figure 1

In trying to solve the area problem, we have to ask ourselves: What is the meaning of the word *area*? Let's start by answering this question for regions with straight sides. For a rectangle, the area is defined as the product of the length and the width. The area of a triangle is half the base times the height. The area of a polygon is found by dividing it into triangles (as in Figure 2) and adding the areas of the triangles.

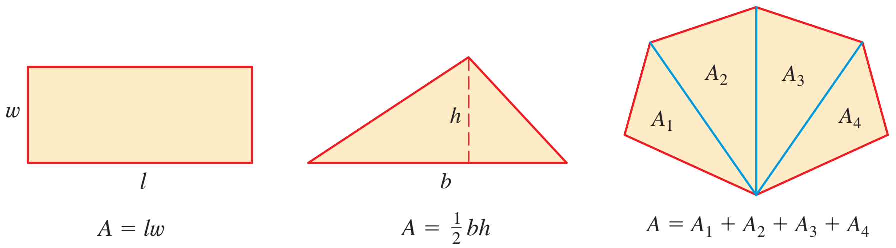


Figure 2

However, it is not so easy to find the area of a region with curved sides. We all have an intuitive idea of what the area of a region is. But part of the area problem is to make this intuitive idea precise by giving an exact definition of area.

Recall that in defining a tangent, we first approximated the slope of the tangent line by slopes of secant lines, and then we took the limit of these approximations. We pursue a similar idea for areas. We first approximate the region S by rectangles, and then we take the limit of the areas of these rectangles as we increase the number of rectangles. The following example illustrates the procedure.

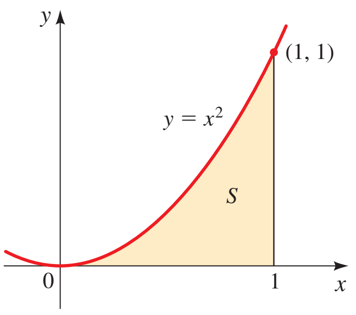


Figure 3

Example 1 ■ Estimating an Area Using Rectangles

Use rectangles to estimate the area under the parabola $y = x^2$ from 0 to 1 (the parabolic region S illustrated in Figure 3).

Solution We first notice that the area of S must be somewhere between 0 and 1 because S is contained in a square with side length 1, but we can certainly do better than that. Suppose we divide S into four strips $S1$, $S2$, $S3$, and $S4$ by drawing the vertical lines $x = \frac{1}{4}$, $x = \frac{1}{2}$, and $x = \frac{3}{4}$, as in Figure 4(a). We can approximate each strip by a rectangle whose base is the same as the strip and whose height is the same as the right edge of the strip [see Figure 4(b)]. In other words, the heights of these rectangles are the values of the function $f(x) = x^2$ at the right endpoints of the following subintervals: $\left[0, \frac{1}{4}\right]$, $\left[\frac{1}{4}, \frac{1}{2}\right]$, $\left[\frac{1}{2}, \frac{3}{4}\right]$, and $\left[\frac{3}{4}, 1\right]$.

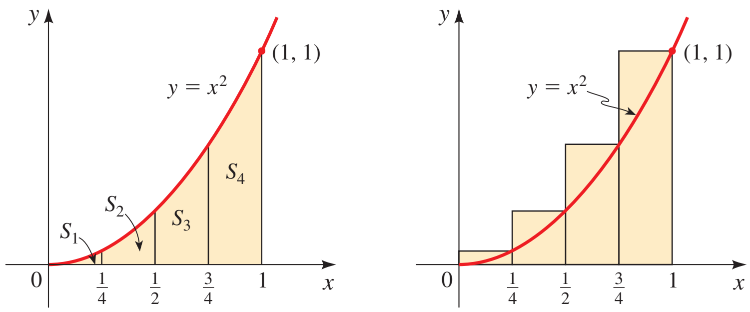


Figure 4 (a) (b)

Each rectangle has width $\frac{1}{4}$, and the heights are $\left(\frac{1}{4}\right)^2$, $\left(\frac{1}{2}\right)^2$, $\left(\frac{3}{4}\right)^2$, and 1^2. If we let R_4 be the sum of the areas of these approximating rectangles, we get

$$R_4 = \tfrac{1}{4}\cdot\left(\tfrac{1}{4}\right)^2 + \tfrac{1}{4}\cdot\left(\tfrac{1}{2}\right)^2 + \tfrac{1}{4}\cdot\left(\tfrac{3}{4}\right)^2 + \tfrac{1}{4}\cdot 1^2 = \tfrac{15}{32} = 0.46875$$

From Figure 4(b) we see that the area A of S is less than R_4, so

$$A < 0.46875$$

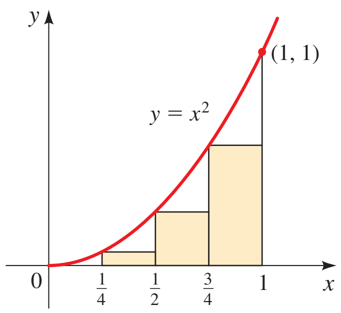


Figure 5

Instead of using the rectangles in Figure 4(b), we could use the smaller rectangles in Figure 5 whose heights are the values of f at the left endpoints of the subintervals. (The leftmost rectangle has collapsed because its height is 0.) The sum of the areas of these approximating rectangles is

$$L_4 = \tfrac{1}{4}\cdot 0^2 + \tfrac{1}{4}\cdot\left(\tfrac{1}{4}\right)^2 + \tfrac{1}{4}\cdot\left(\tfrac{1}{2}\right)^2 + \tfrac{1}{4}\cdot\left(\tfrac{3}{4}\right)^2 = \tfrac{7}{32} = 0.21875$$

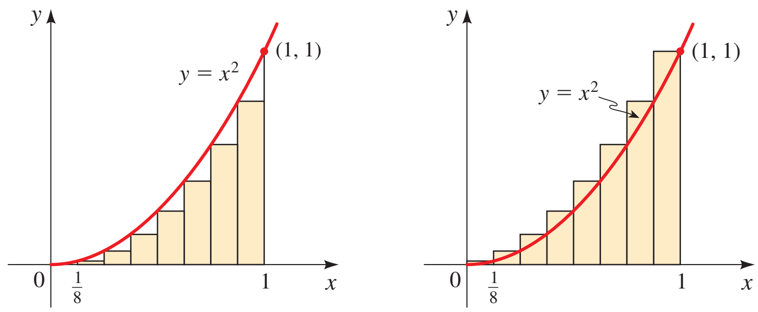
We see that the area of S is larger than L_4, so we have lower and upper estimates for A:

$$0.21875 < A < 0.46875$$

We can repeat this procedure with a larger number of strips. Figure 6 shows what happens when we divide the region S into eight strips of equal width. By computing the sum of the areas of the smaller rectangles (L_8) and the sum of the areas of the larger rectangles (R_8), we obtain better lower and upper estimates for A:

$$0.2734375 < A < 0.3984375$$

So one possible answer to the question is to say that the true area of S lies somewhere between 0.2734375 and 0.3984375.



(a) Using left endpoints (b) Using right endpoints

Figure 6 | Approximating S with eight rectangles

n	L_n	R_n
10	0.2850000	0.3850000
20	0.3087500	0.3587500
30	0.3168519	0.3501852
50	0.3234000	0.3434000
100	0.3283500	0.3383500
1000	0.3328335	0.3338335

We could obtain better estimates by increasing the number of strips. The table in the margin shows the results of similar calculations (with a computer) using n rectangles whose heights are found with left endpoints (L_n) or right endpoints (R_n). In particular, we see by using 50 strips that the area lies between 0.3234 and 0.3434. With 1000 strips we narrow it down even more: A lies between 0.3328335 and 0.3338335. A good estimate is obtained by averaging these numbers: $A \approx 0.3333335$.

Now Try Exercise 3 ■

From the values in the table it looks as if R_n is approaching $\frac{1}{3}$ as n increases. We confirm this in the next example.

Example 2 ■ The Limit of Approximating Sums

For the region S in Example 1, show that the sum of the areas of the upper approximating rectangles approaches $\frac{1}{3}$, that is,

$$\lim_{n\to\infty} R_n = \frac{1}{3}$$

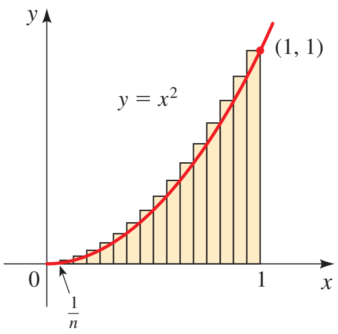


Figure 7

Solution Let R_n be the sum of the areas of the n rectangles shown in Figure 7. Each rectangle has width $1/n$, and the heights are the values of the function $f(x) = x^2$ at the points $1/n, 2/n, 3/n, \ldots, n/n$. That is, the heights are $(1/n)^2, (2/n)^2, (3/n)^2, \ldots, (n/n)^2$. Thus

$$\begin{aligned} R_n &= \frac{1}{n}\left(\frac{1}{n}\right)^2 + \frac{1}{n}\left(\frac{2}{n}\right)^2 + \frac{1}{n}\left(\frac{3}{n}\right)^2 + \cdots + \frac{1}{n}\left(\frac{n}{n}\right)^2 \\ &= \frac{1}{n}\cdot\frac{1}{n^2}(1^2 + 2^2 + 3^2 + \cdots + n^2) \\ &= \frac{1}{n^3}(1^2 + 2^2 + 3^2 + \cdots + n^2) \end{aligned}$$

Here we need the formula for the sum of the squares of the first n positive integers:

This formula was discussed in Section 11.4.

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Putting the preceding formula into our expression for R_n, we get

$$R_n = \frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6n^2}$$

Thus we have

$$\begin{aligned} \lim_{n\to\infty} R_n &= \lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n^2} \\ &= \lim_{n\to\infty} \frac{1}{6}\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right) \\ &= \lim_{n\to\infty} \frac{1}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) \\ &= \tfrac{1}{6}\cdot 1 \cdot 2 = \tfrac{1}{3} \end{aligned}$$

Now Try Exercise 13 ■

It can be shown that the lower approximating sums also approach $\frac{1}{3}$, that is,

$$\lim_{n\to\infty} L_n = \frac{1}{3}$$

From Figures 8 and 9 it appears that as n increases, both R_n and L_n become better and better approximations to the area of S. Therefore we *define* the area A to be the limit of the sums of the areas of the approximating rectangles, that is,

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n = \tfrac{1}{3}$$

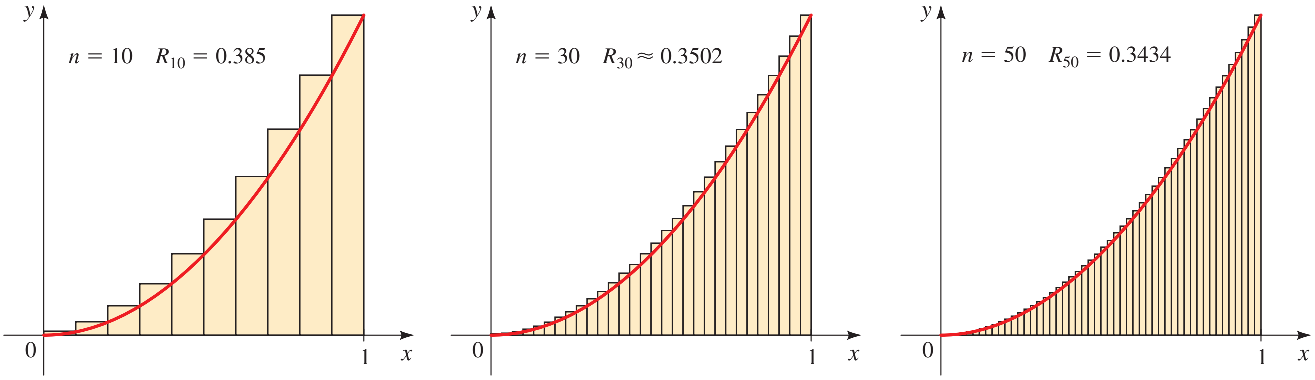


Figure 8

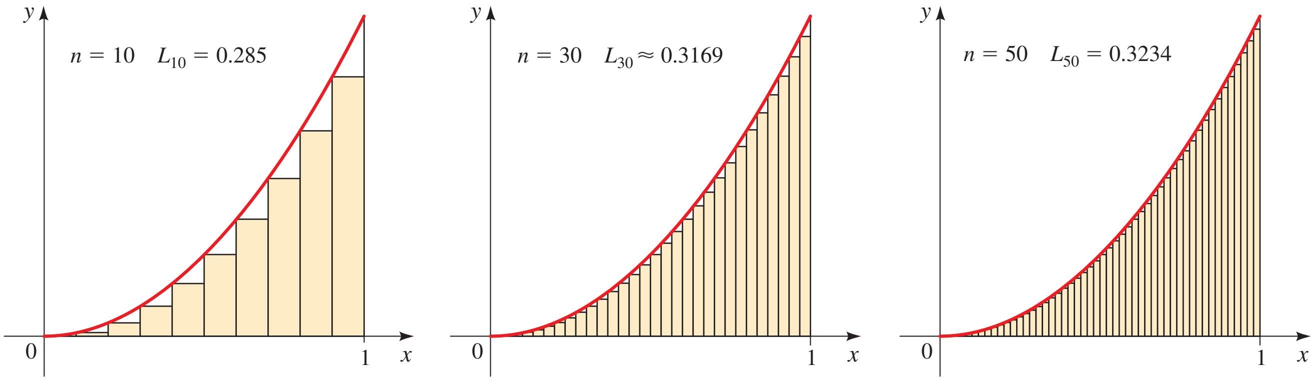


Figure 9

■ Definition of Area

Let's apply the idea of Examples 1 and 2 to the more general region S of Figure 1. We start by subdividing S into n strips $S_1, S_2, \ldots, S_n$ of equal width, as shown in Figure 10.

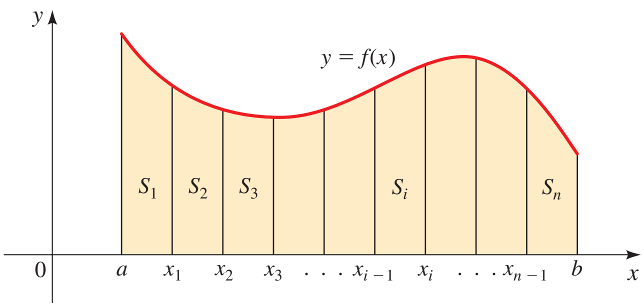


Figure 10

The width of the interval $[a, b]$ is $b - a$, so the width of each of the n strips is

$$\Delta x = \frac{b - a}{n}$$

These strips divide the interval $[a, b]$ into n subintervals

$$[x_0, x_1], \quad [x_1, x_2], \quad [x_2, x_3], \quad \ldots, \quad [x_{n-1}, x_n]$$

where $x_0 = a$ and $x_n = b$. The right endpoints of the subintervals are

$$x_1 = a + \Delta x, \quad x_2 = a + 2\,\Delta x, \quad x_3 = a + 3\,\Delta x, \quad \ldots, \quad x_k = a + k\,\Delta x, \quad \ldots$$

Let's approximate the kth strip S_k by a rectangle with width Δx and height $f(x_k)$, which is the value of f at the right endpoint (see Figure 11). Then the area of the kth rectangle is $f(x_k)\,\Delta x$. What we think of intuitively as the area of S is approximated by the sum of the areas of these rectangles, which is

$$R_n = f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x$$

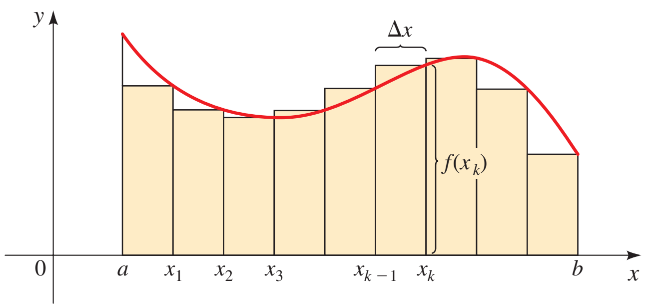


Figure 11

Figure 12 shows this approximation for $n = 2$, 4, 8, and 12.

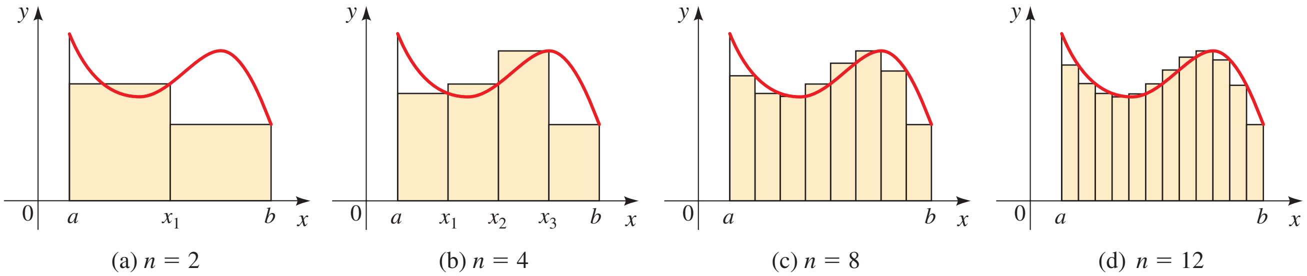


Figure 12

Notice that this approximation appears to become better and better as the number of strips increases, that is, as $n \to \infty$. Therefore we define the area A of the region S in the following way.

Definition of Area

The **area** A of the region S that lies under the graph of the continuous function f on $[a, b]$ is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x]$$

Using sigma notation, we write this as follows:

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\,\Delta x$$

In using this formula for area, remember that Δx is the width of an approximating rectangle, x_k is the right endpoint of the kth rectangle, and $f(x_k)$ is its height. So

$$\text{Width:} \qquad \Delta x = \frac{b-a}{n}$$

$$\text{Right endpoint:} \qquad x_k = a + k\,\Delta x$$

$$\text{Height:} \qquad f(x_k) = f(a + k\,\Delta x)$$

When working with sums, we will need the following properties from Section 11.1.

$$\sum_{k=1}^{n}(a_k \pm b_k) = \sum_{k=1}^{n} a_k \pm \sum_{k=1}^{n} b_k \qquad \sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$$

We will also need the following formulas for the sums of the powers of the first n natural numbers from Section 11.4.

$$\sum_{k=1}^{n} c = nc \qquad \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

Example 3 ■ Finding the Area Under a Curve

Find the area of the region that lies under the parabola $y = x^2$, where $0 \le x \le 5$.

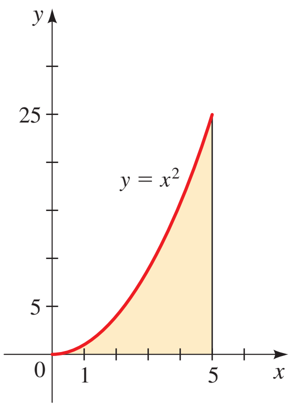


Figure 13

Solution The region is graphed in Figure 13. To find the area, we first find the dimensions of the approximating rectangles at the nth stage.

$$\text{Width:} \qquad \Delta x = \frac{b-a}{n} = \frac{5-0}{n} = \frac{5}{n}$$

$$\text{Right endpoint:} \qquad x_k = a + k\,\Delta x = 0 + k\left(\frac{5}{n}\right) = \frac{5k}{n}$$

$$\text{Height:} \qquad f(x_k) = f\left(\frac{5k}{n}\right) = \left(\frac{5k}{n}\right)^2 = \frac{25k^2}{n^2}$$

Now we substitute these values into the definition of area.

$$\begin{aligned}
A &= \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\,\Delta x && \text{Definition of area}\\
&= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{25k^2}{n^2}\cdot\frac{5}{n} && f(x_k) = \frac{25k^2}{n^2},\ \Delta x = \frac{5}{n}\\
&= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{125k^2}{n^3} && \text{Simplify}\\
&= \lim_{n\to\infty}\frac{125}{n^3}\sum_{k=1}^{n}k^2 && \text{Factor } \frac{125}{n^3}\\
&= \lim_{n\to\infty}\frac{125}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} && \text{Sum of Squares Formula}\\
&= \lim_{n\to\infty}\frac{125(2n^2+3n+1)}{6n^2} && \text{Cancel } n\text{, and expand the numerator}\\
&= \lim_{n\to\infty}\frac{125}{6}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) && \text{Divide the numerator and denominator by } n^2\\
&= \frac{125}{6}(2+0+0) = \frac{125}{3} && \text{Let } n \to \infty
\end{aligned}$$

We can also calculate the limit by writing

$$\frac{125}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} = \frac{125}{6}\left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)$$

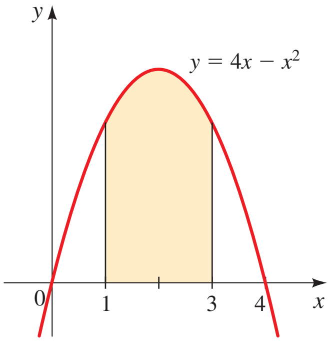
as in Example 2.

Thus the area of the region is $\frac{125}{3} \approx 41.7$. ■

Now Try Exercise 15

Example 4 ■ Finding the Area Under a Curve

The figure below shows the region whose area is computed in Example 4.



Find the area of the region that lies under the parabola $y = 4x - x^2$, where $1 \le x \le 3$.

Solution We start by finding the dimensions of the approximating rectangles at the nth stage.

$$\text{Width:} \qquad \Delta x = \frac{b-a}{n} = \frac{3-1}{n} = \frac{2}{n}$$

$$\text{Right endpoint:} \qquad x_k = a + k\,\Delta x = 1 + k\left(\frac{2}{n}\right) = 1 + \frac{2k}{n}$$

$$\begin{aligned}\text{Height:} \qquad f(x_k) &= f\left(1 + \frac{2k}{n}\right) = 4\left(1 + \frac{2k}{n}\right) - \left(1 + \frac{2k}{n}\right)^2 \\ &= 4 + \frac{8k}{n} - 1 - \frac{4k}{n} - \frac{4k^2}{n^2} \\ &= 3 + \frac{4k}{n} - \frac{4k^2}{n^2}\end{aligned}$$

Thus according to the definition of area, we get

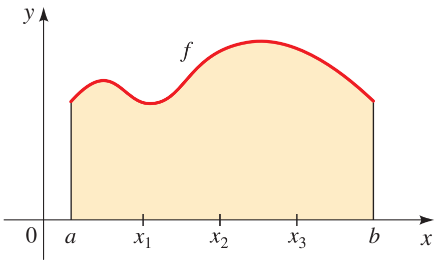
$$\begin{aligned}A &= \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\,\Delta x = \lim_{n\to\infty} \sum_{k=1}^{n}\left(3 + \frac{4k}{n} - \frac{4k^2}{n^2}\right)\left(\frac{2}{n}\right) \\ &= \lim_{n\to\infty}\left(\sum_{k=1}^{n} 3 + \frac{4}{n}\sum_{k=1}^{n} k - \frac{4}{n^2}\sum_{k=1}^{n} k^2\right)\left(\frac{2}{n}\right) \\ &= \lim_{n\to\infty}\left(\frac{2}{n}\sum_{k=1}^{n} 3 + \frac{8}{n^2}\sum_{k=1}^{n} k - \frac{8}{n^3}\sum_{k=1}^{n} k^2\right) \\ &= \lim_{n\to\infty}\left(\frac{2}{n}(3n) + \frac{8}{n^2}\left[\frac{n(n+1)}{2}\right] - \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]\right) \\ &= \lim_{n\to\infty}\left(6 + 4\cdot\frac{n}{n}\cdot\frac{n+1}{n} - \frac{4}{3}\cdot\frac{n}{n}\cdot\frac{n+1}{n}\cdot\frac{2n+1}{n}\right) \\ &= \lim_{n\to\infty}\left[6 + 4\left(1 + \frac{1}{n}\right) - \frac{4}{3}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)\right] \\ &= 6 + 4\cdot 1 - \frac{4}{3}\cdot 1\cdot 2 = \frac{22}{3}\end{aligned}$$

Now Try Exercise 17 ■

12.5 Exercises

Concepts

1–2 ■ The graph of a function f is shown below.



1. To find the area under the graph of f, we first approximate the area by ________. Approximate the area by drawing four rectangles. The area R_4 of this approximation is

$$R_4 = \boxed{} + \boxed{} + \boxed{} + \boxed{}$$

2. Let R_n be the approximation obtained by using n rectangles of equal width. The exact area under the graph of f is

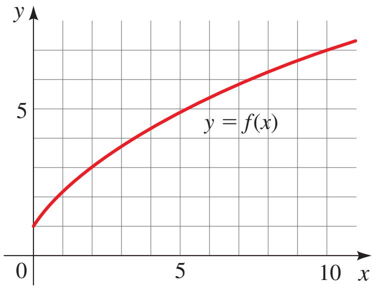
$$A = \lim_{n\to\infty} \boxed{}$$

Skills

3. Estimating an Area Using Rectangles

(a) By reading values from the given graph of f, use five rectangles to find a lower estimate and an upper estimate for the area under the given graph of f from $x = 0$ to $x = 10$. In each case, sketch the rectangles that you use.

(b) Find new estimates using 10 rectangles in each case.

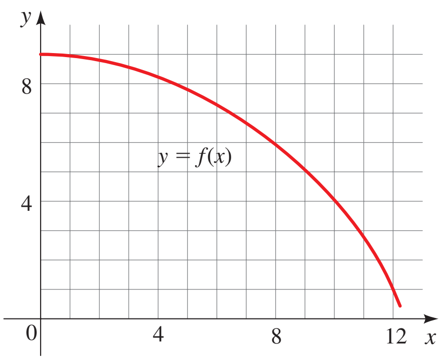


4. Estimating an Area Using Rectangles

(a) Use six rectangles to find estimates of each type for the area under the given graph of f from $x = 0$ to $x = 12$.
(i) L_6 (using left endpoints)
(ii) R_6 (using right endpoints)

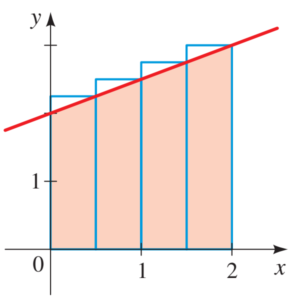
(b) Is L_6 an underestimate or an overestimate of the true area?

(c) Is R_6 an underestimate or an overestimate of the true area?

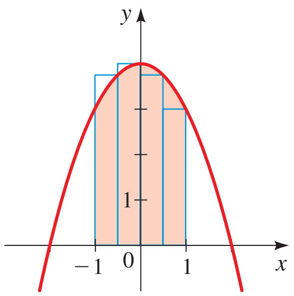


5–8 ■ Estimating Areas Using Rectangles Approximate the area of the shaded region under the graph of the given function by using the indicated rectangles. (The rectangles have equal width.)

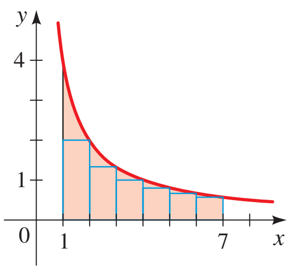
5. $f(x) = \frac{1}{2}x + 2$



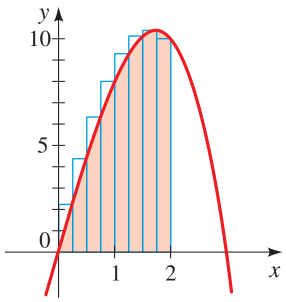
6. $f(x) = 4 - x^2$



7. $f(x) = \dfrac{4}{x}$



8. $f(x) = 9x - x^3$



9–12 ■ Estimating Areas Using Rectangles In these exercises we estimate the area under the graph of a function by using rectangles.

9. (a) Estimate the area under the graph of $f(x) = 1/x$ from $x = 1$ to $x = 5$ using four approximating rectangles and right endpoints. Sketch the graph and the rectangles. Is your estimate an underestimate or an overestimate?

(b) Repeat part (a), using left endpoints.

10. (a) Estimate the area under the graph of $f(x) = 25 - x^2$ from $x = 0$ to $x = 5$ using five approximating rectangles and right endpoints. Sketch the graph and the rectangles. Is your estimate an underestimate or an overestimate?

(b) Repeat part (a) using left endpoints.

11. (a) Estimate the area under the graph of $f(x) = 1 + x^2$ from $x = -1$ to $x = 2$ using three rectangles and right endpoints. Then improve your estimate by using six rectangles. Sketch the curve and the approximating rectangles.

(b) Repeat part (a) using left endpoints.

12. (a) Estimate the area under the graph of $f(x) = e^{-x}$, $0 \le x \le 4$, using four approximating rectangles and
(i) right endpoints
(ii) left endpoints
In each case, sketch the curve and the rectangles.

(b) Improve your estimates in part (a) by using eight rectangles.

13–14 ■ Finding the Area Under A Curve Use the definition of area as a limit to find the area of the region that lies under the curve. Check your answer by sketching the region and using geometry.

13. $y = 3x, \quad 0 \le x \le 5$

14. $y = 2x + 1, \quad 1 \le x \le 3$

15–20 ■ Finding the Area Under a Curve Find the area of the region that lies under the graph of f over the given interval.

15. $f(x) = 3x^2, \quad 0 \le x \le 2$

16. $f(x) = x + x^2, \quad 0 \le x \le 1$

17. $f(x) = x^3 + 2, \quad 0 \le x \le 5$

18. $f(x) = 4x^3, \quad 2 \le x \le 5$

19. $f(x) = x + 6x^2, \quad 1 \le x \le 4$

20. $f(x) = 20 - 2x^2, \quad 2 \le x \le 3$

Discuss ■ Discover ■ Prove ■ Write

21. Discuss: Approximating Area with a Calculator The following TI-84 program finds the approximate area under the graph of f on the interval $[a, b]$ using n rectangles. To use the program, first store the function f in `Y₁`. The program prompts you to enter `N`, the number of rectangles, and `A` and `B`, the endpoints of the interval. Use the program to approximate the area under the given function using 10, 20, and 100 rectangles.

(a) $f(x) = x^5 + 2x + 3, \quad \text{on } [1, 3]$

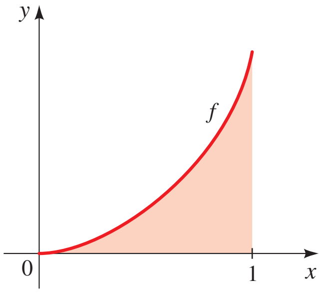
(b) $f(x) = \sin x, \quad \text{on } [0, \pi]$

(c) $f(x) = e^{-x^2}, \quad \text{on } [-1, 1]$

```
PROGRAM:AREA
:Prompt N
:Prompt A
:Prompt B
:(B-A)/N→D
:0→S
:A→X
:For (K,1,N)
:X+D→X
:S+D*Y₁→S
:End
:Disp "AREA IS"
:Disp S
```

22. Discuss ■ Prove: Area Under the Graph of a Function Let f be a continuous one-to-one function defined on $[0, 1]$ such that $f(0) = 0$ and $f(1) = 1$, as shown in the figure. Show that

$$(\text{Area under } f) + (\text{Area under } f^{-1}) = 1$$



PS *Draw a diagram.* Draw a graph of f^{-1}. Argue from the graphs of f and f^{-1} that the given equation is true.

Chapter 12 Review

Properties and Formulas

Limits | Section 12.1

We say that the **limit of a function** f, as x approaches a, equals L, and we write

$$\lim_{x \to a} f(x) = L$$

provided that the values of $f(x)$ can be made arbitrarily close to L by taking x to be sufficiently close to a.

The **left-hand** and **right-hand** limits of f, as x approaches a, are defined similarly:

$$\lim_{x \to a^-} f(x) = L \qquad \lim_{x \to a^+} f(x) = L$$

The limit of f, as x approaches a, exists if and only if both left- and right-hand limits exist: $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$.

Algebraic Properties of Limits | Section 12.2

The following **Limit Laws** hold.

1. $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

2. $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

3. $\lim_{x \to a} cf(x) = c \lim_{x \to a} f(x)$

4. $\lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

5. $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}, \quad \text{if } \lim_{x \to a} g(x) \neq 0$

6. $\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n$ **7.** $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$

The following **special limits** hold.

1. $\lim_{x \to a} c = c$ **2.** $\lim_{x \to a} x = a$

3. $\lim_{x \to a} x^n = a^n$ **4.** $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$

If f is a polynomial or a rational function and a is in the domain of f, then $\lim_{x \to a} f(x) = f(a)$.

Derivatives | Section 12.3

Let $y = f(x)$ be a function. The **derivative of f at a**, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

Equivalently, the derivative $f'(a)$ is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

The derivative of f at a is the **slope of the tangent line** to the curve $y = f(x)$ at the point $P(a, f(a))$.

The derivative of f at a is the **instantaneous rate of change of y with respect to x** at $x = a$.

Limits at Infinity | Section 12.4

We say that **the limit of a function** f, as x **approaches infinity**, is L, and write

$$\lim_{x \to \infty} f(x) = L$$

provided that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large.

We say that **the limit of a function** f, as x **approaches negative infinity**, is L, and we write

$$\lim_{x \to -\infty} f(x) = L$$

provided that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large negative.

The line $y = L$ is a horizontal asymptote of the curve $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = L \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = L$$

The following special limits hold, where $k > 0$:

$$\lim_{x \to \infty} \frac{1}{x^k} = 0 \qquad \text{and} \qquad \lim_{x \to -\infty} \frac{1}{x^k} = 0$$

Limits of Sequences | Section 12.4

We say that a sequence $a_1, a_2, a_3, \ldots$ has the limit L, and we write

$$\lim_{n \to \infty} a_n = L$$

provided that the nth term a_n of the sequence can be made arbitrarily close to L by taking n sufficiently large.

If $\lim_{x \to \infty} f(x) = L$ and if $f(n) = a_n$ when n is an integer, then $\lim_{n \to \infty} a_n = L$.

Area | Section 12.5

Let f be a continuous function defined on the interval $[a, b]$. The area A of the region that lies under the graph of f is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n \to \infty} [f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x]$$

$$= \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k)\,\Delta x$$

where

$$\Delta x = \frac{b - a}{n} \qquad \text{and} \qquad x_k = a + k\,\Delta x$$

Summation Formulas | Section 12.5

The following summation formulas are useful for calculating areas.

$$\sum_{k=1}^{n} c = nc \qquad \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

Concept Check

1. **(a)** Explain what is meant by $\lim_{x \to a} f(x) = L$.

(b) If $\lim_{x \to 2} f(x) = 5$, is it possible that $f(2) = 3$?

(c) Find $\lim_{x \to 2} x^2$.

2. To evaluate the limit of a function, we often need to first rewrite the function using the rules of algebra. What is the logical first step in evaluating each of the following limits?

(a) $\displaystyle \lim_{x \to 2} \frac{x^2 - 4}{x - 2}$ **(b)** $\displaystyle \lim_{h \to 0} \frac{(5 + h)^2 - 25}{h}$

(c) $\displaystyle \lim_{x \to 3} \frac{\sqrt{x+1} - 2}{x - 3}$ **(d)** $\displaystyle \lim_{x \to 7} \frac{\left(\frac{1}{7} - \frac{1}{x}\right)}{x - 7}$

3. **(a)** Explain what it means to say:

$$\lim_{x \to 3^-} f(x) = 5 \qquad \lim_{x \to 3^+} f(x) = 10$$

(b) If the two equations in part (a) are true, is it possible that $\lim_{x \to 3} f(x) = 5$?

(c) Find $\lim_{x \to 2^-} f(x)$ and $\lim_{x \to 2^+} f(x)$, where f is defined as follows.

$$f(x) = \begin{cases} 1 & \text{if } x \le 2 \\ x & \text{if } x > 2 \end{cases}$$

(d) For f as given in part (c), does $\lim_{x \to 2} f(x)$ exist?

4. **(a)** Define the derivative $f'(a)$ of a function f at $x = a$.

(b) State an equivalent formulation for $f'(a)$.

(c) Find the derivative of $f(x) = x^2$ at $x = 3$.

5. **(a)** Give two different interpretations of the derivative of the function $y = f(x)$ at $x = a$.

(b) For the function $f(x) = x^2$, find the slope of the tangent line to the graph of f at the point $(3, 9)$ on the graph.

(c) For the function $y = x^2$, find the instantaneous rate of change of y with respect to x when $x = 3$.

(d) Write expressions for the average rate of change of y with respect to x between a and x and for the instantaneous rate of change of y with respect to x at $x = a$.

6. **(a)** Explain what is meant by $\lim_{x \to \infty} f(x) = L$. Draw sketches to illustrate different ways in which this can happen.

(b) Find $\displaystyle \lim_{x \to \infty} \frac{3x^2 + x}{x^2 + 1}$.

(c) Explain why $\lim_{x \to \infty} \sin x$ does not exist.

7. **(a)** If $a_1, a_2, a_3, \ldots$ is a sequence, what is meant by $\lim_{n \to \infty} a_n = L$? What is a convergent sequence?

(b) Find $\lim_{n \to \infty} (-1)^n/n$.

8. **(a)** Suppose S is the region under the graph of the function $y = f(x)$ and above the x-axis, where $a \le x \le b$. Explain how this area is approximated by rectangles, and write an expression for the area of S as a limit of sums.

(b) Find the area under the graph of $f(x) = x^2$ and above the x-axis, between $x = 0$ and $x = 3$.

Answers to the Concept Check can be found at the book companion website stewartmath.com.

Exercises

1–6 ■ Estimating Limits Numerically and Graphically Use a table of values to estimate the value of the limit. Then use a graphing device to confirm your result graphically.

1. $\displaystyle\lim_{x\to 2} \frac{x-2}{x^2-3x+2}$

2. $\displaystyle\lim_{t\to -1} \frac{t+1}{t^3-t}$

3. $\displaystyle\lim_{x\to 0} \frac{2^x-1}{x}$

4. $\displaystyle\lim_{x\to 0} \frac{\sin 2x}{x}$

5. $\displaystyle\lim_{x\to 1^+} \ln\sqrt{x-1}$

6. $\displaystyle\lim_{x\to 0^-} \frac{\tan x}{|x|}$

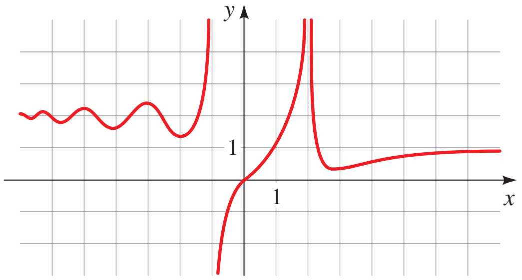
7. Limits from a Graph The graph of f is shown in the figure. Find each limit, or explain why it does not exist.

(a) $\displaystyle\lim_{x\to 2^+} f(x)$ **(b)** $\displaystyle\lim_{x\to -3^+} f(x)$

(c) $\displaystyle\lim_{x\to -3^-} f(x)$ **(d)** $\displaystyle\lim_{x\to -3} f(x)$

(e) $\displaystyle\lim_{x\to 4} f(x)$ **(f)** $\displaystyle\lim_{x\to \infty} f(x)$

(g) $\displaystyle\lim_{x\to -\infty} f(x)$ **(h)** $\displaystyle\lim_{x\to 0} f(x)$



8. One-Sided Limits Let

$$f(x) = \begin{cases} 2 & \text{if } x < -1 \\ x^2 & \text{if } -1 \le x \le 2 \\ x+2 & \text{if } x > 2 \end{cases}$$

Find each limit, or explain why it does not exist.

(a) $\displaystyle\lim_{x\to -1^-} f(x)$ **(b)** $\displaystyle\lim_{x\to -1^+} f(x)$

(c) $\displaystyle\lim_{x\to -1} f(x)$ **(d)** $\displaystyle\lim_{x\to 2^-} f(x)$

(e) $\displaystyle\lim_{x\to 2^+} f(x)$ **(f)** $\displaystyle\lim_{x\to 2} f(x)$

(g) $\displaystyle\lim_{x\to 0} f(x)$ **(h)** $\displaystyle\lim_{x\to 3} (f(x))^2$

9–20 ■ Finding Limits Evaluate the limit, if it exists. Use the Limit Laws when possible.

9. $\displaystyle\lim_{x\to 2} \frac{x+1}{x-3}$

10. $\displaystyle\lim_{t\to 1} (t^3-3t+6)$

11. $\displaystyle\lim_{x\to 3} \frac{x^2+x-12}{x-3}$

12. $\displaystyle\lim_{x\to -2} \frac{x^2-4}{x^2+x-2}$

13. $\displaystyle\lim_{u\to 0} \frac{(u+1)^2-1}{u}$

14. $\displaystyle\lim_{z\to 9} \frac{\sqrt{z}-3}{z-9}$

15. $\displaystyle\lim_{x\to 3^-} \frac{x-3}{|x-3|}$

16. $\displaystyle\lim_{x\to 0} \left(\frac{1}{x}+\frac{2}{x^2-2x}\right)$

17. $\displaystyle\lim_{x\to \infty} \frac{2x}{x-4}$

18. $\displaystyle\lim_{x\to \infty} \frac{x^2+1}{x^4-3x+6}$

19. $\displaystyle\lim_{x\to \infty} \cos^2 x$

20. $\displaystyle\lim_{t\to -\infty} \frac{t^4}{t^3-1}$

21–24 ■ Derivative of a Function Find the derivative of the function at the given number.

21. $f(x) = 3x - 5$, at 4

22. $g(x) = 2x^2 - 1$, at -1

23. $f(x) = \sqrt{x}$, at 16

24. $f(x) = \dfrac{x}{x+1}$, at 1

25–28 ■ Evaluating Derivatives **(a)** Find $f'(a)$. **(b)** Find $f'(2)$ and $f'(-2)$.

25. $f(x) = 6 - 2x$

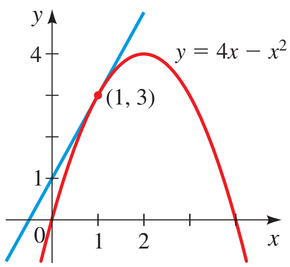
26. $f(x) = x^2 - 3x$

27. $f(x) = \sqrt{x+6}$

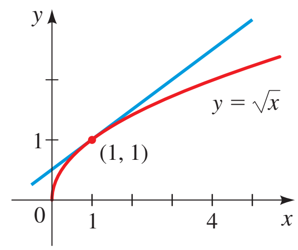
28. $f(x) = \dfrac{4}{x}$

29–30 ■ Equation of a Tangent Line Find an equation of the tangent line shown in the figure.

29.



30.



31–34 ■ Equation of a Tangent Line Find an equation of the line tangent to the graph of f at the given point.

31. $f(x) = 2x$, at $(3, 6)$

32. $f(x) = x^2 - 3$, at $(2, 1)$

33. $f(x) = \dfrac{1}{x}$, at $\left(2, \dfrac{1}{2}\right)$

34. $f(x) = \sqrt{x+1}$, at $(3, 2)$

35. Velocity of a Dropped Stone A stone is dropped from the roof of a building 640 feet above the ground. The height of the stone (in ft) after t seconds is given by $h(t) = 640 - 16t^2$.

(a) Find the velocity of the stone when $t = 2$.

(b) Find the velocity of the stone when $t = a$.

(c) At what time t will the stone hit the ground?

(d) With what velocity will the stone hit the ground?

36. Instantaneous Rate of Change If a gas is confined in a fixed volume, then according to Boyle's Law the product of the pressure P and the temperature T is a constant. For a certain gas, $PT = 100$, where P is measured in lb/in^2 and T is measured in kelvins (K).

(a) Express P as a function of T.

(b) Find the instantaneous rate of change of P with respect to T when $T = 300$ K.

37–42 ■ Limit of a Sequence If the sequence is convergent, find its limit. If it is divergent, explain why.

37. $a_n = \dfrac{n}{5n + 1}$

38. $a_n = \dfrac{n^3}{n^3 + 1}$

39. $a_n = \dfrac{n(n + 1)}{2n^2}$

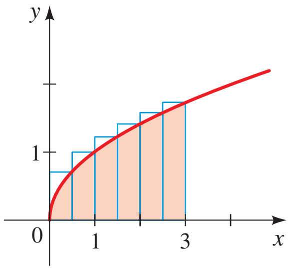
40. $a_n = \dfrac{n^3}{2n + 6}$

41. $a_n = \cos\left(\dfrac{n\pi}{2}\right)$

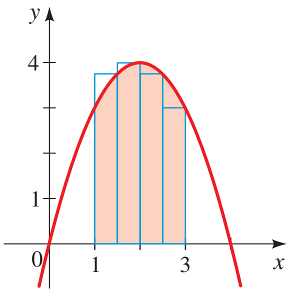
42. $a_n = \dfrac{10}{3^n}$

43–44 ■ Estimating Areas Using Rectangles Approximate the area of the shaded region under the graph of the given function by using the indicated rectangles. (The rectangles have equal width.)

43. $f(x) = \sqrt{x}$



44. $f(x) = 4x - x^2$



45–48 ■ Area Under a Curve Use the limit definition of area to find the area of the region that lies under the graph of f over the given interval.

45. $f(x) = 2x + 3, \quad 0 \le x \le 2$

46. $f(x) = x^2 + 1, \quad 0 \le x \le 3$

47. $f(x) = x^2 - x, \quad 1 \le x \le 2$

48. $f(x) = x^3, \quad 1 \le x \le 2$

Chapter 12 | Test

1. (a) Use a table of values to estimate the limit

$$\lim_{x \to 0} \frac{x}{\sin 2x}$$

(b) Use a graphing device to confirm your answer graphically.

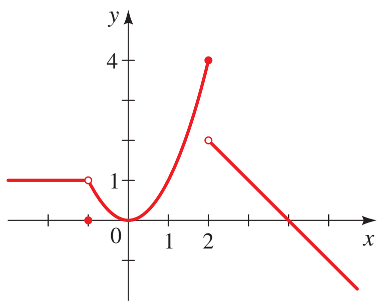
2. For the piecewise-defined function f whose graph is shown, find:

(a) $\lim_{x \to -1^-} f(x)$ **(b)** $\lim_{x \to -1^+} f(x)$ **(c)** $\lim_{x \to -1} f(x)$

(d) $\lim_{x \to 0^-} f(x)$ **(e)** $\lim_{x \to 0^+} f(x)$ **(f)** $\lim_{x \to 0} f(x)$

(g) $\lim_{x \to 2^-} f(x)$ **(h)** $\lim_{x \to 2^+} f(x)$ **(i)** $\lim_{x \to 2} f(x)$

$$f(x) = \begin{cases} 1 & \text{if } x < -1 \\ 0 & \text{if } x = -1 \\ x^2 & \text{if } -1 < x \le 2 \\ 4 - x & \text{if } 2 < x \end{cases}$$



3. Evaluate the limit, if it exists.

(a) $\lim_{x \to 2} \frac{x^2 + 2x - 8}{x - 2}$ **(b)** $\lim_{x \to 2} \frac{x^2 - 2x - 8}{x + 2}$ **(c)** $\lim_{x \to 2} \frac{1}{x - 2}$

(d) $\lim_{x \to 2} \frac{x - 2}{|x - 2|}$ **(e)** $\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4}$ **(f)** $\lim_{x \to \infty} \frac{2x^2 - 4}{x^2 + x}$

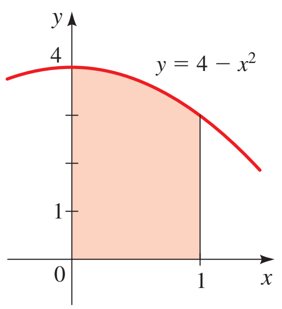
4. Let $f(x) = x^2 - 2x$. Find:

(a) $f'(a)$ **(b)** $f'(-1), f'(1), f'(2)$

5. Find the equation of the line tangent to the graph of $f(x) = \sqrt{x}$ at the point where $x = 9$.

6. Find the limit of the sequence.

(a) $a_n = \frac{n}{n^2 + 4}$ **(b)** $a_n = \sec n\pi$



7. The region sketched in the figure in the margin lies under the graph of $f(x) = 4 - x^2$, above the interval $0 \le x \le 1$.

(a) Approximate the area of the region with five rectangles, equally spaced along the x-axis, using right endpoints to determine the heights of the rectangles.

(b) Use the limit definition of area to find the exact value of the area of the region.

A Cumulative Review Test for Chapters 11 and 12 can be found at the book companion website **stewartmath.com.**

The area under the graph of a function is used to model many quantities in physics, economics, engineering, and other fields. That is why the area problem is so important. Here, we will show how the concept of work (see Section 8.6) is modeled by area. Several other applications are explored in the problems.

Recall that the work W done in moving an object is the product of the force F applied to the object and the distance d that the object moves:

$$W = Fd \qquad \text{work} = \text{force} \times \text{distance}$$

This formula is used if the force is *constant*. For example, suppose you are pushing a crate across a floor, moving along the positive x-axis from $x = a$ to $x = b$, and you apply a constant force $F = k$. The graph of F as a function of the distance x is shown in Figure 1(a). Notice that the work done is $W = Fd = k(b - a)$, which is the area under the graph of F. [See Figure 1(b).]

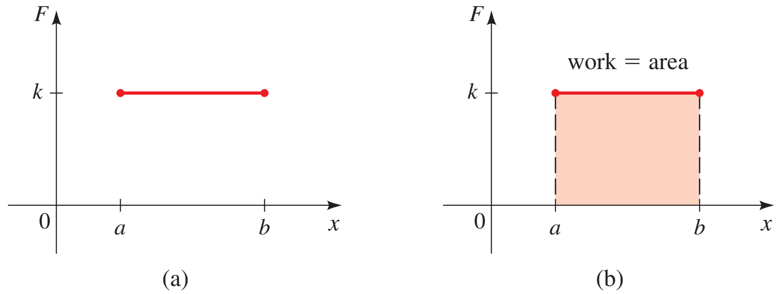


Figure 1 | A constant force F

But what if the force is *not* constant? For example, suppose the force you apply to the crate varies with distance (you push harder at certain places than you do at others). More precisely, suppose that you push the crate along the x-axis in the positive direction, from $x = a$ to $x = b$, and at each point x between a and b you apply a force $f(x)$ to the crate. Figure 2 shows a graph of the force f as a function of the distance x.

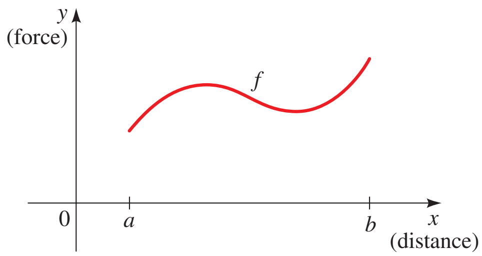


Figure 2 | A variable force

How much work was done? We can't apply the formula for work directly because the force is not constant. So let's divide the interval $[a, b]$ into n subintervals with endpoints $x_0, x_1, \ldots, x_n$ and equal width Δx, as shown in Figure 3(a) on the next page. The force at the right endpoint of the interval $[x_{k-1}, x_k]$ is $f(x_k)$. If n is large, then Δx is small, so the values of f don't change very much over the interval $[x_{k-1}, x_k]$. In other words f is almost constant on the interval, so the work W_k that is done in moving the crate from x_{k-1} to x_k is approximately

$$W_k \approx f(x_k)\,\Delta x$$

Thus we can approximate the work done in moving the crate from $x = a$ to $x = b$ by

$$W \approx \sum_{k=1}^{n} f(x_k)\,\Delta x$$

It seems that this approximation becomes better as we make n larger (and so make the interval $[x_{k-1}, x_k]$ smaller). Therefore we define the work done in moving an object from a to b as the limit of this quantity as $n \to \infty$:

$$W = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k)\, \Delta x$$

Notice that this is precisely the area under the graph of f between $x = a$ and $x = b$ as defined in Section 12.5. [See Figure 3(b).]

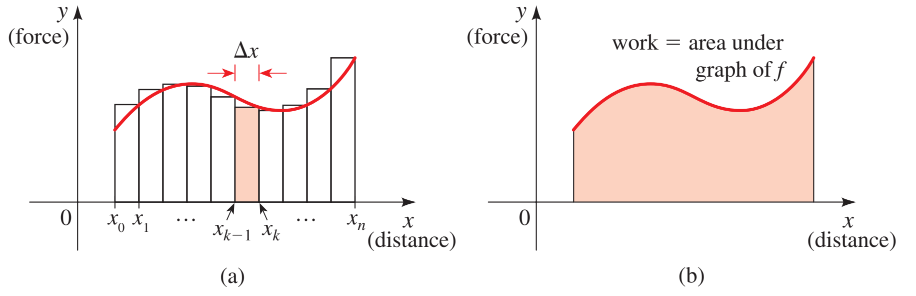


Figure 3 | Approximating work

Example ■ The Work Done by a Variable Force

A worker pushes a crate along a straight path a distance of 18 feet. At a distance x from the starting point, the force applied is given by $f(x) = 340 - x^2$. Find the work done by the worker.

Solution The graph of f between $x = 0$ and $x = 18$ is shown in Figure 4. Notice how the force varies: The worker starts by pushing with a force of 340 lb but steadily applies less force.

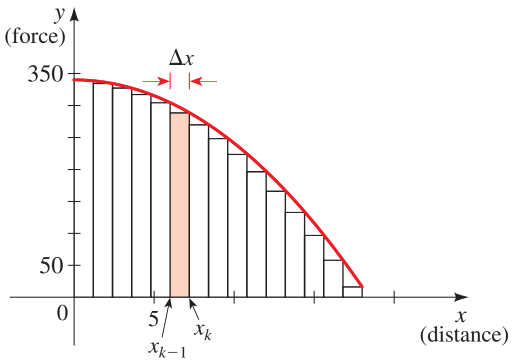


Figure 4

The work done is the area under the graph of f on the interval $[0, 18]$. To find this area, we start by finding the dimensions of the approximating rectangles at the nth stage.

Width: $\qquad \Delta x = \dfrac{b - a}{n} = \dfrac{18 - 0}{n} = \dfrac{18}{n}$

Right endpoint: $\qquad x_k = a + k\,\Delta x = 0 + k\left(\dfrac{18}{n}\right) = \dfrac{18k}{n}$

Height: $\qquad f(x_k) = f\left(\dfrac{18k}{n}\right) = 340 - \left(\dfrac{18k}{n}\right)^2$

$$= 340 - \frac{324k^2}{n^2}$$

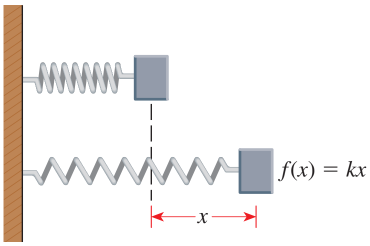
Thus according to the definition of work, we get

$$
\begin{aligned}
W &= \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\,\Delta x = \lim_{n\to\infty}\sum_{k=1}^{n}\left(340 - \frac{324k^2}{n^2}\right)\left(\frac{18}{n}\right)\\
&= \lim_{n\to\infty}\left(\frac{18}{n}\sum_{k=1}^{n}340 - \frac{(18)(324)}{n^3}\sum_{k=1}^{n}k^2\right)\\
&= \lim_{n\to\infty}\left(\frac{18}{n}340n - \frac{5832}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]\right)\\
&= \lim_{n\to\infty}\left(6120 - 972\cdot\frac{n}{n}\cdot\frac{n+1}{n}\cdot\frac{2n+1}{n}\right)\\
&= 6120 - 972\cdot 1\cdot 1\cdot 2 = 4176
\end{aligned}
$$

So the work done by the worker in moving the crate is 4176 ft-lb. ■

Problems

1. Work Done by a Winch A motorized winch is being used to pull a felled tree to a logging truck. The motor exerts a force of $f(x) = 1500 + 10x - \frac{1}{2}x^2$ lb on the tree at the instant when the tree has moved x feet. The tree must be moved a distance of 40 feet, from $x = 0$ to $x = 40$. How much work is done by the winch in moving the tree?

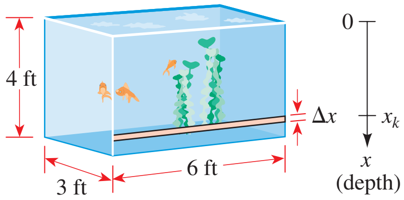


2. Work Done by a Spring Hooke's law states that when a spring is stretched, it pulls back with a force proportional to the amount of the stretch. The constant of proportionality is a characteristic of the spring known as the **spring constant**. Thus a spring with spring constant k exerts a force $f(x) = kx$ when it is stretched a distance x.

A certain spring has spring constant $k = 20$ lb/ft. Find the work done when the spring is pulled so that the amount by which it is stretched increases from $x = 0$ to $x = 2$ ft.

3. Force of Water As any diver knows, an object submerged in water experiences pressure, and as depth increases, so does the water pressure. At a depth of x feet, the water pressure is $p(x) = 62.5x$ lb/ft^2. To find the force exerted by the water on a surface, we multiply the pressure by the area of the surface:

$$\text{force} = \text{pressure} \times \text{area}$$



Suppose an aquarium that is 3 ft wide, 6 ft long, and 4 ft high is full of water. The bottom of the aquarium has area $3 \times 6 = 18$ ft^2, and it experiences water pressure of $p(4) = 62.5 \times 4 = 250$ lb/ft^2. Thus the total force exerted by the water on the bottom is $250 \times 18 = 4500$ lb.

The water also exerts a force on the sides of the aquarium, but this is not as easy to calculate because the pressure increases from top to bottom. To calculate the force on one of the 4 ft by 6 ft sides, we divide its area into n thin horizontal strips of width Δx, as shown in the figure. The area of each strip is

$$\text{length} \times \text{width} = 6\,\Delta x$$

If the bottom of the kth strip is at the depth x_k, then it experiences water pressure of approximately $p(x_k) = 62.5x_k$ lb/ft^2—the thinner the strip, the more accurate the approximation. Thus on each strip, the water exerts a force of

$$\text{pressure} \times \text{area} = 62.5x_k \times 6\,\Delta x = 375x_k\,\Delta x \text{ lb}$$

(a) Explain why the total force exerted by the water on the 4 ft by 6 ft sides of the aquarium is

$$\lim_{n\to\infty}\sum_{k=1}^{n} 375x_k\,\Delta x$$

where $\Delta x = 4/n$ and $x_k = 4k/n$.

(b) What area does the limit in part (a) represent?

(c) Evaluate the limit in part (a) to find the force exerted by the water on one of the 4 ft by 6 ft sides of the aquarium.

(d) Use the same technique to find the force exerted by the water on one of the 4 ft by 3 ft sides of the aquarium.

[*Note:* Engineers use the technique outlined in this problem to find the total force exerted on a dam by the water in the reservoir behind the dam.]

4. Distance Traveled by a Car Since distance = speed × time, a car moving, say, at a constant speed of 70 mi/h for 5 hours will travel a distance of 350 miles. But what if the speed varies, as it usually does in practice?

(a) Suppose the speed of a moving object at time t is $v(t)$. Explain why the distance traveled by the object between times $t = a$ and $t = b$ is the area under the graph of v between $t = a$ and $t = b$.

(b) The speed of a car t seconds after it starts moving is given by the function

$$v(t) = 6t + 0.1t^3 \text{ ft/s}$$

Find the distance traveled by the car from $t = 0$ to $t = 5$ seconds.

5. Heating Capacity If the outdoor temperature reaches a maximum of 90°F one day and only 80°F the next, then we would probably say that the first day was hotter than the second. Suppose, however, that on the first day the temperature was below 60°F for most of the day, reaching the high only briefly, whereas on the second day the temperature stayed above 75°F all the time. Now which day is the hotter one? To better measure how hot a particular day is, scientists use the concept of **heating degree-hour**. If the temperature is a constant D degrees for t hours, then the "heating capacity" generated over this period is Dt heating degree-hours.

$$\text{heating degree-hours} = \text{temperature} \times \text{time}$$

If the temperature is not constant, then the number of heating degree-hours equals the area under the graph of the temperature function over the time period in question.

(a) On a particular day the temperature (in °F) was modeled by the function $D(t) = 61 + \frac{6}{5}t - \frac{1}{25}t^2$, where t was measured in hours since midnight. How many heating degree-hours were experienced on this day, from $t = 0$ to $t = 24$?

(b) What was the maximum temperature on the day described in part (a)?

(c) On another day the temperature (in °F) was modeled by the function $E(t) = 50 + 5t - \frac{1}{4}t^2$. How many heating degree-hours were experienced on this day?

(d) What was the maximum temperature on the day described in part (c)?

(e) Which day was "hotter"?