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3 Polynomial and Rational Functions

Functions defined by polynomial expressions are called *polynomial functions.* The graphs of polynomial functions can have many peaks and valleys. This property makes them suitable models for many real-world situations. For example, if a factory increases the number of workers, productivity increases, but if there are too many workers, productivity begins to decrease. This situation is modeled by a polynomial function of degree 2 (a quadratic function). The growth of many animal species follows a predictable pattern, beginning with a period of rapid growth, followed by a period of slow growth and then a final growth spurt. This variability in growth is modeled by a polynomial of degree 3.

In the *Focus on Modeling* at the end of this chapter we explore different ways in which polynomials are used to model real-world situations.

3.1 Quadratic Functions and Models

■ Graphing Quadratic Functions Using the Vertex Form ■ Maximum and Minimum Values of Quadratic Functions ■ Modeling with Quadratic Functions

A polynomial function is a function that is defined by a polynomial expression. So a **polynomial function of degree *n*** is a function of the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \qquad (a_n \neq 0)$$

Polynomial expressions are defined in Section 1.3.

We have already studied polynomial functions of degree 0 and 1. These are functions of the form $P(x) = a_0$ and $P(x) = a_1 x + a_0$, respectively, whose graphs are lines. In this section we study polynomial functions of degree 2. These are called quadratic functions.

Quadratic Functions

A **quadratic function** is a polynomial function of degree 2. So a quadratic function is a function of the form

$$f(x) = ax^2 + bx + c \qquad (a \neq 0)$$

We see in this section how quadratic functions model many real-world phenomena. We begin by analyzing the graphs of quadratic functions.

■ Graphing Quadratic Functions Using the Vertex Form

For a geometric definition of parabolas, see Section 10.1.

If we take $a = 1$ and $b = c = 0$ in the quadratic function $f(x) = ax^2 + bx + c$, we get the quadratic function $f(x) = x^2$, whose graph is the parabola graphed in Example 2.2.1. In fact, the graph of any quadratic function is a **parabola**; it can be obtained from the graph of $f(x) = x^2$ by the transformations given in Section 2.6.

Vertex Form of a Quadratic Function

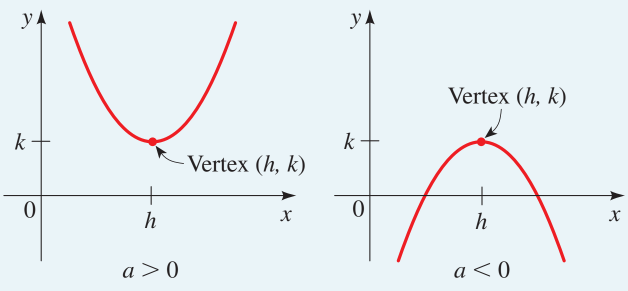
A quadratic function $f(x) = ax^2 + bx + c$ can be expressed in the **vertex form**

$$f(x) = a(x - h)^2 + k$$

by completing the square. The graph of f is a parabola with vertex (h, k).

If $a > 0$ the parabola **opens upward**

If $a < 0$ the parabola **opens downward**



Example 1 ■ Vertex Form of a Quadratic Function

Let $f(x) = 2x^2 - 12x + 13$.

(a) Express f in vertex form.

(b) Find the vertex and x- and y-intercepts of f.

(c) Sketch a graph of f.

(d) Find the domain and range of f.

Solution

(a) Since the coefficient of x^2 is not 1, we must factor this coefficient from the terms involving x before we complete the square.

$$\begin{aligned} f(x) &= 2x^2 - 12x + 13 & & \\ &= 2(x^2 - 6x) + 13 & & \text{Factor 2 from the } x\text{-terms} \\ &= 2(x^2 - 6x + 9) + 13 - 2\cdot 9 & & \text{Complete the square: Add 9 inside parentheses, subtract } 2\cdot 9 \text{ outside} \\ &= 2(x-3)^2 - 5 & & \text{Factor and simplify} \end{aligned}$$

Completing the square is discussed in Section 1.5.

The vertex form is $f(x) = 2(x-3)^2 - 5$.

(b) From the vertex form of f we can see that the vertex of f is $(3, -5)$. The y-intercept is $f(0) = 13$. To find the x-intercepts, we set $f(x) = 0$ and solve the resulting equation. We can solve a quadratic equation by any of the methods we studied in Section 1.5. In this case we solve the equation by using the Quadratic Formula.

$$\begin{aligned} 0 &= 2x^2 - 12x + 13 & & \text{Set } f(x) = 0 \\ x &= \frac{12 \pm \sqrt{144 - 4\cdot 2\cdot 13}}{4} & & \text{Solve for } x \text{ using the Quadratic Formula} \\ x &= \frac{6 \pm \sqrt{10}}{2} & & \text{Simplify} \end{aligned}$$

Thus the x-intercepts are $x = (6 \pm \sqrt{10})/2$, or approximately 1.42 and 4.58.

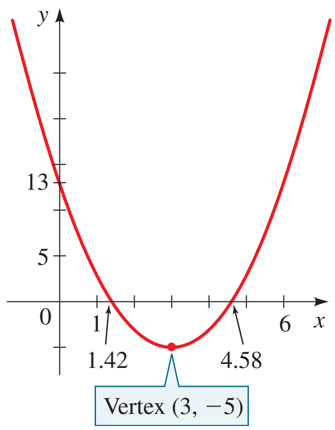


Figure 1 | $f(x) = 2x^2 - 12x + 13$

(c) The vertex form tells us that we get the graph of f by taking the parabola $y = x^2$, shifting it 3 units to the right, stretching it vertically by a factor of 2, and moving it downward 5 units. We sketch a graph of f in Figure 1, including the x- and y-intercepts found in part (b).

Obtaining the domain and range of a function from its graph is explained in Section 2.3.

(d) The domain of f is the set of all real numbers $(-\infty, \infty)$. From the graph we see that the range of f is $[-5, \infty)$.

Now Try Exercise 15 ■

■ Maximum and Minimum Values of Quadratic Functions

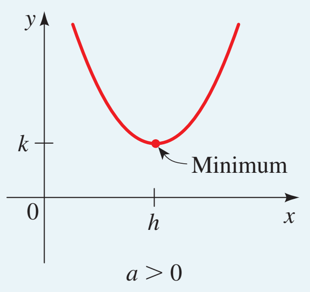
If a quadratic function has vertex (h, k), then the function has a minimum value at the vertex if its graph opens upward and a maximum value at the vertex if its graph opens downward. For example, the function graphed in Figure 1 has minimum value -5 when $x = 3$, since the vertex $(3, -5)$ is the lowest point on the graph.

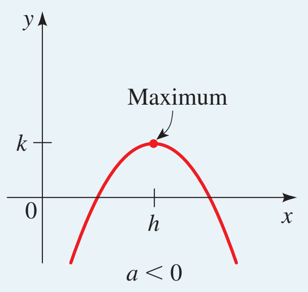
Maximum or Minimum Value of a Quadratic Function

Let $f(x) = a(x-h)^2 + k$ be a quadratic function in vertex form. The maximum or minimum value of f occurs at $x = h$.

If $a > 0$ then the **minimum value** of f is $f(h) = k$

If $a < 0$ then the **maximum value** of f is $f(h) = k$





Example 2 ■ Minimum Value of a Quadratic Function

Consider the quadratic function $f(x) = 5x^2 - 30x + 49$.

(a) Express f in vertex form.

(b) Sketch a graph of f.

(c) Find the minimum value of f.

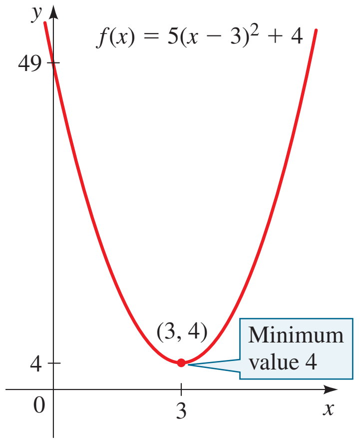


Figure 2

Solution

(a) To express this quadratic function in vertex form, we complete the square.

$$\begin{aligned} f(x) &= 5x^2 - 30x + 49 \\ &= 5(x^2 - 6x) + 49 && \text{Factor 5 from the } x\text{-terms} \\ &= 5(x^2 - 6x + 9) + 49 - 5 \cdot 9 && \text{Complete the square: Add 9 inside parentheses, subtract } 5 \cdot 9 \text{ outside} \\ &= 5(x - 3)^2 + 4 && \text{Factor and simplify} \end{aligned}$$

(b) The graph is a parabola that has its vertex at $(3, 4)$ and opens upward, as sketched in Figure 2.

(c) Since the coefficient of x^2 is positive, f has a minimum value. The minimum value is $f(3) = 4$.

Now Try Exercise 27 ■

Example 3 ■ Maximum Value of a Quadratic Function

Consider the quadratic function $f(x) = -x^2 + x + 2$.

(a) Express f in vertex form.

(b) Sketch a graph of f.

(c) Find the maximum value of f.

Solution

(a) To express this quadratic function in vertex form, we complete the square.

$$\begin{aligned} f(x) &= -x^2 + x + 2 \\ &= -(x^2 - x) + 2 && \text{Factor } -1 \text{ from the } x\text{-terms} \\ &= -\left(x^2 - x + \tfrac{1}{4}\right) + 2 - (-1)\tfrac{1}{4} && \text{Complete the square: Add } \tfrac{1}{4} \text{ inside parentheses, subtract } (-1)\tfrac{1}{4} \text{ outside} \\ &= -\left(x - \tfrac{1}{2}\right)^2 + \tfrac{9}{4} && \text{Factor and simplify} \end{aligned}$$

In Example 3 you can check that the x-intercepts of the parabola are -1 and 2. These are obtained by solving the equation $f(x) = 0$.

(b) From the vertex form we see that the graph is a parabola that opens downward and has vertex $\left(\frac{1}{2}, \frac{9}{4}\right)$. The graph of f is sketched in Figure 3.

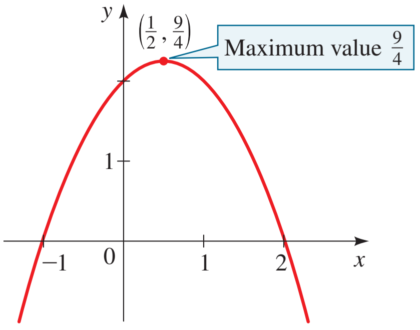


Figure 3 | Graph of $f(x) = -x^2 + x + 2$

(c) Since the coefficient of x^2 is negative, f has a maximum value, which is $f\left(\frac{1}{2}\right) = \frac{9}{4}$.

Now Try Exercise 29 ■

Expressing a quadratic function in vertex form helps us to sketch its graph as well as to find its maximum or minimum value. If we are interested only in finding the maximum or minimum value, then a formula is available for doing so. This formula is obtained by completing the square for the general quadratic function as follows.

$$\begin{aligned} f(x) &= ax^2 + bx + c \\ &= a\left(x^2 + \frac{b}{a}x\right) + c && \text{Factor } a \text{ from the } x\text{-terms} \\ &= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + c - a\left(\frac{b^2}{4a^2}\right) && \text{Complete the square: Add } \frac{b^2}{4a^2} \text{ inside parentheses, subtract } a\left(\frac{b^2}{4a^2}\right) \text{ outside} \\ &= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} && \text{Factor} \end{aligned}$$

This equation is in vertex form with $h = -b/(2a)$ and $k = c - b^2/(4a)$. Since the maximum or minimum value occurs at $x = h$, we have the following result.

Maximum or Minimum Value of a Quadratic Function

The maximum or minimum value of a quadratic function $f(x) = ax^2 + bx + c$ occurs at

$$x = -\frac{b}{2a}$$

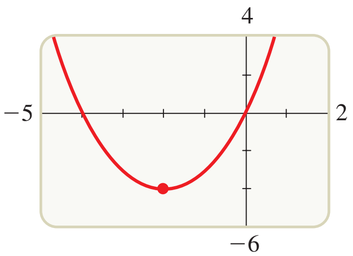
If $a > 0$, then the **minimum value** is $f\left(-\dfrac{b}{2a}\right)$.

If $a < 0$, then the **maximum value** is $f\left(-\dfrac{b}{2a}\right)$.

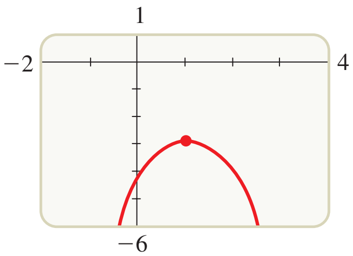
Example 4 ■ Finding Maximum and Minimum Values of Quadratic Functions

Find the maximum or minimum value of each quadratic function.

(a) $f(x) = x^2 + 4x$ **(b)** $g(x) = -2x^2 + 4x - 5$



The minimum value occurs at $x = -2$.



The maximum value occurs at $x = 1$.

Solution

(a) This is a quadratic function with $a = 1$ and $b = 4$. Thus the maximum or minimum value occurs at

$$x = -\frac{b}{2a} = -\frac{4}{2 \cdot 1} = -2$$

Since $a > 0$, the function has the *minimum* value

$$f(-2) = (-2)^2 + 4(-2) = -4$$

(b) This is a quadratic function with $a = -2$ and $b = 4$. Thus the maximum or minimum value occurs at

$$x = -\frac{b}{2a} = -\frac{4}{2 \cdot (-2)} = 1$$

Since $a < 0$, the function has the *maximum* value

$$f(1) = -2(1)^2 + 4(1) - 5 = -3$$

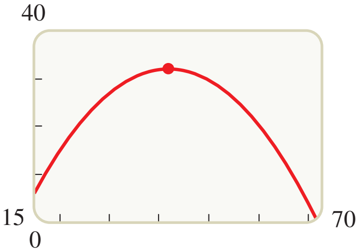
Now Try Exercises 35 and 37 ■

■ Modeling with Quadratic Functions

We now study some examples of real-world phenomena that are modeled by quadratic functions. These examples and the *Applications* exercises for this section show some of the variety of situations that are naturally modeled by quadratic functions.

Example 5 ■ Maximum Gas Mileage for a Car

The function that models gas mileage is a quadratic function because air resistance is proportional to the square of the speed. You can feel air resistance by putting your hand out the window of a moving car.



The maximum gas mileage occurs at 42 mi/h.

Most cars get their best gas mileage when traveling at relatively modest speeds. The gas mileage M for a certain new car is modeled by the function

$$M(s) = -\frac{1}{28}s^2 + 3s - 31 \qquad 15 \le s \le 70$$

where s is the speed in mi/h and M is measured in mi/gal. What is the car's best gas mileage, and at what speed is it attained?

Solution The function M is a quadratic function with $a = -\frac{1}{28}$ and $b = 3$. Thus its maximum value occurs when

$$s = -\frac{b}{2a} = -\frac{3}{2\left(-\frac{1}{28}\right)} = 42$$

The maximum value is $M(42) = -\frac{1}{28}(42)^2 + 3(42) - 31 = 32$. So the car's best gas mileage is 32 mi/gal when it is traveling at 42 mi/h.

Now Try Exercise 55 ■

Example 6 ■ Maximizing Revenue from Ticket Sales

A hockey team plays in an arena that has a seating capacity of 15,000 spectators. With the ticket price set at \$14, average attendance at recent games has been 9500. A market survey indicates that for each dollar the ticket price is lowered, the average attendance increases by 1000.

(a) Find a function that models the revenue in terms of ticket price.

(b) Find the price that maximizes revenue from ticket sales.

(c) What ticket price is so high that no one attends and so no revenue is generated?

Solution

(a) **Express the model in words.** The model that we want is a function that gives the revenue for any ticket price:

$$\boxed{\text{revenue}} = \boxed{\text{ticket price}} \times \boxed{\text{attendance}}$$

Discovery Project ■ Torricelli's Law

Evangelista Torricelli (1608–1647) is best known for his invention of the barometer. He also discovered that the speed at which a fluid leaks from the bottom of a tank is related to the height of the fluid in the tank (a principle now called Torricelli's Law). In this project we conduct a simple experiment to collect data on the speed of water leaking through a hole in the bottom of a large soft-drink bottle. We then find an algebraic expression for Torricelli's Law by fitting a quadratic function to the data we obtained. You can find the project at **www.stewartmath.com**.

Choose the variable. There are two varying quantities: ticket price and attendance. Since the function we want depends on price, we let

$$x = \text{ticket price}$$

Next, we express attendance in terms of x.

In Words	In Algebra
Ticket price	x
Amount ticket price is lowered	$14 - x$
Increase in attendance	$1000(14 - x)$
Attendance	$9500 + 1000(14 - x)$

Set up the model. The model that we want is the function R that gives the revenue for a given ticket price x.

$$\boxed{\text{revenue}} = \boxed{\text{ticket price}} \times \boxed{\text{attendance}}$$

$$R(x) = x \times [9500 + 1000(14 - x)]$$
$$R(x) = x(23{,}500 - 1000x)$$
$$R(x) = 23{,}500x - 1000x^2$$

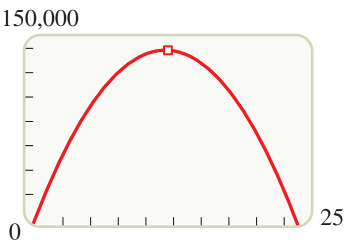
(b) **Use the model.** Since R is a quadratic function with $a = -1000$ and $b = 23{,}500$, the maximum occurs at

$$x = -\frac{b}{2a} = -\frac{23{,}500}{2(-1000)} = 11.75$$

So a ticket price of \$11.75 gives the maximum revenue.

(c) **Use the model.** We want to find the ticket price for which $R(x) = 0$.

$$\begin{aligned} 23{,}500x - 1000x^2 &= 0 && \text{Set } R(x) = 0 \\ 23.5x - x^2 &= 0 && \text{Divide by 1000} \\ x(23.5 - x) &= 0 && \text{Factor} \\ x = 0 \quad \text{or} \quad x &= 23.5 && \text{Solve for } x \end{aligned}$$



Maximum revenue occurs when ticket price is \$11.75.

So according to this model, a ticket price of \$23.50 is just too high; at that price no one attends to watch this team play. (Of course, revenue is also zero if the ticket price is zero.)

Now Try Exercise 65 ■

3.1 Exercises

Concepts

1. To put the quadratic function $f(x) = ax^2 + bx + c$ in vertex form, we complete the ________.

2. The quadratic function $f(x) = a(x - h)^2 + k$ is in vertex form.

(a) The graph of f is a parabola with vertex (____, ____).

(b) If $a > 0$, the graph of f opens ________. In this case $f(h) = k$ is the ________ value of f.

(c) If $a < 0$, the graph of f opens ________. In this case $f(h) = k$ is the ________ value of f.

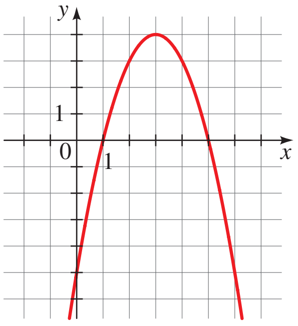
3. The graph of $f(x) = 3(x - 2)^2 - 6$ is a parabola that opens ________, with its vertex at (____, ____), and $f(2) =$ ________ is the (minimum/maximum) ________ value of f.

4. The graph of $f(x) = -3(x - 2)^2 - 6$ is a parabola that opens ________, with its vertex at (____, ____), and $f(2) =$ ________ is the (minimum/maximum) ________ value of f.

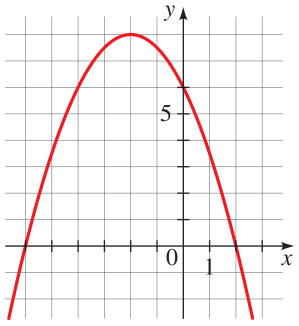
Skills

5–8 ■ Graphs of Quadratic Functions The graph of a quadratic function f is given. **(a)** Find the coordinates of the vertex and the x- and y-intercepts. **(b)** Find the maximum or minimum value of f. **(c)** Find the domain and range of f.

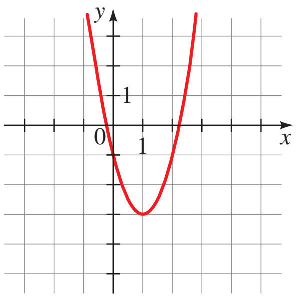
5. $f(x) = -x^2 + 6x - 5$



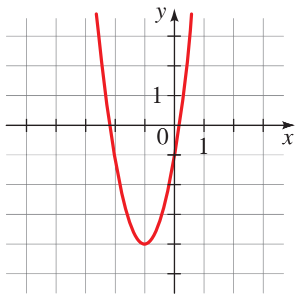
6. $f(x) = -\frac{1}{2}x^2 - 2x + 6$



7. $f(x) = 2x^2 - 4x - 1$



8. $f(x) = 3x^2 + 6x - 1$



9–24 ■ Graphing Quadratic Functions A quadratic function f is given. **(a)** Express f in vertex form. **(b)** Find the vertex and x- and y-intercepts of f. **(c)** Sketch a graph of f. **(d)** Find the domain and range of f.

9. $f(x) = x^2 - 4x + 9$ **10.** $f(x) = x^2 + 6x + 8$

11. $f(x) = x^2 - 6x$ **12.** $f(x) = x^2 + 8x$

13. $f(x) = 3x^2 + 6x$ **14.** $f(x) = -x^2 + 10x$

15. $f(x) = x^2 + 4x + 3$ **16.** $f(x) = x^2 - 2x + 2$

17. $f(x) = -x^2 - 10x - 15$ **18.** $f(x) = -x^2 + 12x - 11$

19. $f(x) = 3x^2 - 6x + 7$ **20.** $f(x) = -3x^2 + 6x - 2$

21. $f(x) = 0.5x^2 + 6x + 16$ **22.** $f(x) = 2x^2 + 12x + 10$

23. $f(x) = -4x^2 - 12x + 1$ **24.** $f(x) = 3x^2 + 2x - 2$

25–34 ■ Maximum and Minimum Values A quadratic function f is given. **(a)** Express f in vertex form. **(b)** Sketch a graph of f. **(c)** Find the maximum or minimum value of f.

25. $f(x) = x^2 + 2x - 1$ **26.** $f(x) = x^2 - 8x + 8$

27. $f(x) = 4x^2 - 8x - 1$ **28.** $f(x) = 2x^2 - 12x + 14$

29. $f(x) = -x^2 - 3x + 3$ **30.** $f(x) = 1 - 6x - x^2$

31. $f(x) = 3x^2 - 12x + 13$ **32.** $f(x) = 2x^2 + 12x + 20$

33. $f(x) = 1 - x - x^2$ **34.** $f(x) = 3 - 4x - 4x^2$

35–44 ■ Formula for Maximum and Minimum Values Find the maximum or minimum value of the function.

35. $f(x) = -7x^2 + 14x - 5$ **36.** $f(x) = 6x^2 + 48x + 1$

37. $f(t) = 4t^2 - 40t + 110$ **38.** $g(x) = -5x^2 + 60x - 200$

39. $f(s) = s^2 - 1.2s + 16$ **40.** $g(x) = 100x^2 - 1500x$

41. $h(x) = \frac{1}{2}x^2 + 2x - 6$ **42.** $f(x) = -\dfrac{x^2}{3} + 2x + 7$

43. $f(x) = 3 - x - \frac{1}{2}x^2$ **44.** $g(x) = 2x(x - 4) + 7$

45–46 ■ Maximum and Minimum Values A quadratic function f is given. **(a)** Use a graphing device to find the maximum or minimum value of f, rounded to two decimal places. **(b)** Find the exact maximum or minimum value of f, and compare it with your answer to part (a).

45. $f(x) = x^2 + 1.79x - 3.21$

46. $f(x) = 1 + x - \sqrt{2}x^2$

Skills Plus

47–48 ■ Finding Quadratic Functions Find a function f whose graph is a parabola that has the given vertex and passes through the indicated point.

47. Vertex $(2, -3)$; point $(3, 1)$

48. Vertex $(-1, 5)$; point $(-3, -7)$

49. Maximum of a Fourth-Degree Polynomial Find the maximum value of the function

$$f(x) = 3 + 4x^2 - x^4$$

[*Hint:* Let $t = x^2$.]

50. Minimum of a Sixth-Degree Polynomial Find the minimum value of the function

$$f(x) = 2 + 16x^3 + 4x^6$$

[*Hint:* Let $t = x^3$.]

Applications

51. Height of a Ball If a ball is thrown directly upward with a velocity of 40 ft/s, its height (in feet) after t seconds is given by $y = 40t - 16t^2$. What is the maximum height attained by the ball?

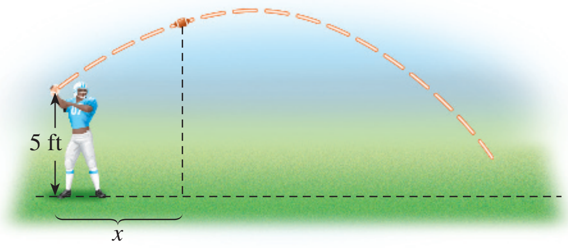
52. Path of a Ball A ball is thrown across a playing field from a height of 5 ft above the ground at an angle of 45° to the horizontal at a speed of 20 ft/s. (See the figure.) It can be deduced from physical principles that the path of the ball is modeled by the function

$$y = -\frac{32}{(20)^2}x^2 + x + 5$$

where x is the distance (in feet) that the ball has traveled horizontally.

(a) Find the maximum height attained by the ball.

(b) Find the horizontal distance the ball has traveled when it hits the ground.



53. Revenue A manufacturer finds that the revenue generated by selling x units of a certain commodity is given by the function $R(x) = 80x - 0.4x^2$, where the revenue $R(x)$ is measured in dollars. What is the maximum revenue, and how many units should be manufactured to obtain this maximum?

54. Sales A soft-drink vendor at a popular beach analyzes sales records and finds that if x cans of soda are sold in one day, then the profit (in dollars) from soda sales is given by

$$P(x) = -0.001x^2 + 3x - 1800$$

What is the maximum profit per day, and how many cans must be sold to produce this maximum profit?

55. Advertising The effectiveness of a YouTube commercial depends on how many times a viewer watches it. After some experiments an advertising agency found that if the effectiveness E is measured on a scale of 0 to 10, then

$$E(n) = \tfrac{2}{3}n - \tfrac{1}{90}n^2$$

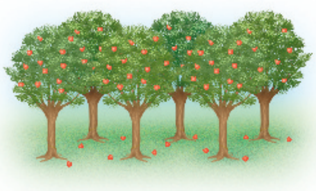
where n is the number of times a viewer watches a given commercial. For a commercial to have maximum effectiveness, how many times does a viewer need to watch it?

56. Pharmaceuticals When a certain drug is taken orally, the concentration of the drug in the patient's bloodstream after t minutes is given by $C(t) = 0.06t - 0.0002t^2$, where $0 \le t \le 240$ and the concentration is measured in mg/L. When is the maximum serum concentration reached, and what is that maximum concentration?

57. Agriculture The number of apples produced by each tree in an apple orchard depends on how densely the trees are planted. If n trees are planted on an acre of land, then each tree produces $900 - 9n$ apples. So the number of apples produced per acre is

$$A(n) = n(900 - 9n)$$

How many trees should be planted per acre to obtain the maximum yield of apples?



58. Agriculture At a certain vineyard each grape vine produces about 10 lb of grapes in a season when about 700 vines are planted per acre. For each additional vine that is planted, the production of each vine decreases by about 1 percent. So the number of pounds of grapes produced per acre is modeled by

$$A(n) = (700 + n)(10 - 0.01n)$$

where n is the number of additional vines planted. Find the number of vines that should be planted to maximize grape production.

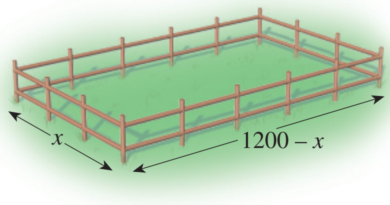
59–62 ■ Maximums and Minimums In the following problems from *Focus on Modeling: Modeling with Functions* at the end of Chapter 2, we found approximate maximum and minimum values graphically. Use the formulas of this section to find the exact maximum or minimum value for the indicated problem, and compare with the approximate value found graphically.

59. Problem 21
60. Problem 22
61. Problem 25
62. Problem 24

63. Fencing a Horse Corral A rancher has 2400 ft of fencing to fence in a rectangular horse corral.

(a) Find a function that models the area of the corral in terms of the width x of the corral.

(b) Find the dimensions of the rectangle that maximize the area of the corral.

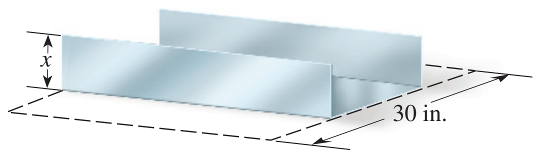


64. Making a Rain Gutter A rain gutter is formed by bending up the sides of a 30-inch-wide rectangular metal sheet as shown in the figure.

(a) Find a function that models the cross-sectional area of the gutter in terms of x.

(b) Find the value of x that maximizes the cross-sectional area of the gutter.

(c) What is the maximum cross-sectional area for the gutter?



65. Stadium Revenue A baseball team plays in a stadium that holds 55,000 spectators. With the ticket price set at \$10, the average attendance at recent games has been 27,000. A market survey indicates that for every dollar the ticket price is lowered, attendance increases by 3000.

(a) Find a function that models the revenue in terms of ticket price.

(b) Find the price that maximizes revenue from ticket sales.

(c) What ticket price is so high that no revenue is generated?

66. Maximizing Profit A community bird-watching society makes and sells simple bird feeders to raise money for conservation activities. The materials for each feeder cost \$6, and the society sells an average of 20 feeders per week at a price of \$10 each. The society has been considering raising the price, so it conducts a survey and finds that for every dollar increase, it will lose 2 sales per week.

(a) Find a function that models weekly profit in terms of price per feeder.

(b) What price should the society charge for each feeder to maximize profit? What is the maximum weekly profit?

■ Discuss ■ Discover ■ Prove ■ Write

67. Discover: Vertex and x-Intercepts We know that the graph of the quadratic function $f(x) = (x - m)(x - n)$ is a parabola. Sketch a rough graph of what such a parabola looks like. What are the x-intercepts of the graph of f? Can you tell from your graph the x-coordinate of the vertex in terms of m and n? (Use the symmetry of the parabola.) Confirm your answer by expanding and using the formulas of this section.

68. Discuss ■ Discover: Maximizing Revenue In Example 6 we found that a ticket price of \$11.75 would maximize revenue for a sports arena. What is the attendance and revenue at that ticket price? Now find the ticket price at which that arena would be filled to capacity and show that the revenue at that price is less than the maximum revenue. Discuss how the quadratic model we found in the example incorporates both ticket price and attendance in one equation and why, in general, models for profit and revenue tend to be quadratic functions (Exercises 53–54).

69. Discuss ■ Discover: Minimizing a Distance Explain why the distance between the point $P(3, -2)$ and any point $Q(x, y)$ on the line $y = 2x - 3$ is given by the function

$$g(x) = \sqrt{5x^2 - 10x + 10}$$

Find the minimum value of the function g and the value of x at which the minimum is achieved. Find the point on the line that is closest to P. (Don't use a graphing device.)

PS *Try to recognize something familiar.* We know how to find the value of x that minimizes the quadratic expression under the square root sign. Argue that the minimum value of g occurs at this same value of x.

3.2 Polynomial Functions and Their Graphs

■ Polynomial Functions ■ Graphing Basic Polynomial Functions ■ Graphs of Polynomial Functions: End Behavior ■ Using Zeros to Graph Polynomials ■ Shape of the Graph Near a Zero ■ Local Maxima and Minima of Polynomials

■ Polynomial Functions

In this section we study polynomial functions of any degree. But before we work with polynomial functions, we must agree on some terminology.

Polynomial Functions

A **polynomial function of degree n** is a function of the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where n is a nonnegative integer and $a_n \neq 0$.

The numbers $a_0, a_1, a_2, \ldots, a_n$ are called the **coefficients** of the polynomial.

The number a_0 is the **constant coefficient** or **constant term**.

The number a_n, the coefficient of the highest power, is the **leading coefficient**, and the term $a_n x^n$ is the **leading term**.

We often refer to polynomial functions simply as *polynomials*. The following polynomial has degree 5, leading coefficient 3, and constant term -6.

Leading coefficient 3

Degree 5

Constant term -6

$$3x^5 + 6x^4 - 2x^3 + x^2 + 7x - 6$$

Leading term $3x^5$

Coefficients 3, 6, -2, 1, 7, and -6

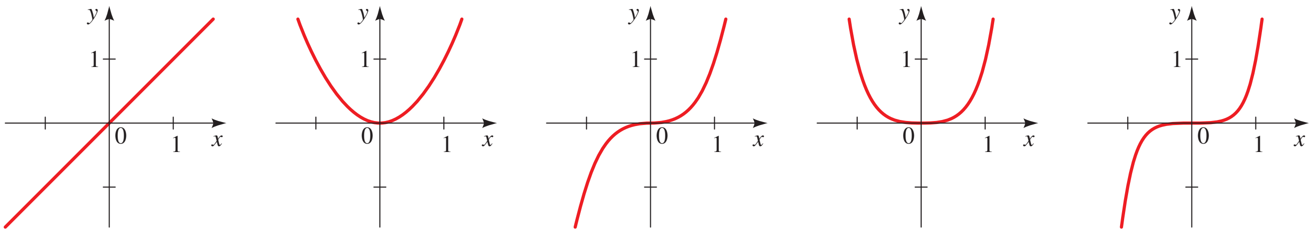
The table below lists some more examples of polynomials.

Polynomial	Degree	Leading Term	Constant Term
$P(x) = 4x - 7$	1	$4x$	-7
$P(x) = x^2 + x$	2	x^2	0
$P(x) = 2x^3 - 6x^2 + 10$	3	$2x^3$	10
$P(x) = -5x^4 + x - 2$	4	$-5x^4$	-2

If a polynomial consists of just a single term, then it is called a **monomial**. For example, $P(x) = x^3$ and $Q(x) = -6x^5$ are monomials.

■ Graphing Basic Polynomial Functions

The simplest polynomial functions are the monomials $P(x) = x^n$, whose graphs are shown in Figure 1. As the figure suggests, the graph of $P(x) = x^n$ has the same general shape as the graph of $y = x^2$ when n is even and the same general shape as the graph of $y = x^3$ when n is odd. However, as the degree n becomes larger, the graphs become flatter around the origin and steeper elsewhere.



(a) $y = x$ (b) $y = x^2$ (c) $y = x^3$ (d) $y = x^4$ (e) $y = x^5$

Figure 1 | Graphs of monomials

Example 1 ■ Transformations of Monomials

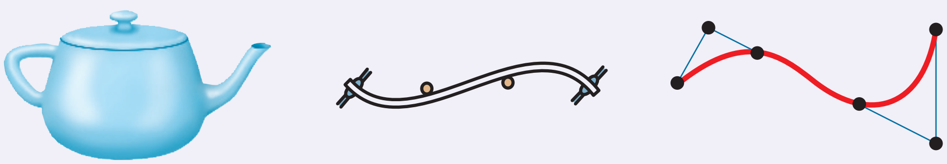
Sketch the graph of each function, and state its domain and range.

(a) $P(x) = -x^3$ **(b)** $Q(x) = (x - 2)^4$

(c) $R(x) = -2x^5 + 4$

Mathematics in the Modern World

Splines



A spline is a long strip of wood or metal that is curved while held fixed at certain points. In the past shipbuilders used splines to create the curved shape of a boat's hull. Splines are also used to make the curves of a piano, a violin, or the spout of a teapot.

Mathematicians discovered that the shapes of splines can be obtained by piecing together parts of polynomials. For example, the graph of a cubic polynomial can be made to fit specified points by adjusting the coefficients of the polynomial (see Example 10). Curves obtained in this way are called cubic splines.

In computer design programs, such as Adobe Illustrator, a curve can be drawn between fixed points, called anchor points. Moving the anchor points amounts to adjusting the coefficients of a cubic polynomial.

Solution We use the graphs in Figure 1 and transform them using the techniques of Section 2.6.

Obtaining the domain and range of a function from its graph is explained in Section 2.3.

(a) The graph of $P(x) = -x^3$ is the reflection of the graph of $y = x^3$ about the x-axis, as shown in Figure 2(a). The domain is all real numbers and from the graph we see that the range is all real numbers.

(b) The graph of $Q(x) = (x - 2)^4$ is the graph of $y = x^4$ shifted 2 units to the right, as shown in Figure 2(b). The domain is all real numbers and from the graph we see that the range is $[0, \infty)$.

(c) We begin with the graph of $y = x^5$. The graph of $y = -2x^5$ is obtained by stretching the graph vertically by a factor of 2 and reflecting it about the x-axis (see the dashed blue graph in Figure 2(c)). Finally, the graph of $R(x) = -2x^5 + 4$ is obtained by shifting upward 4 units (see the red graph in Figure 2(c)). The domain is all real numbers and from the graph we see that the range is all real numbers.

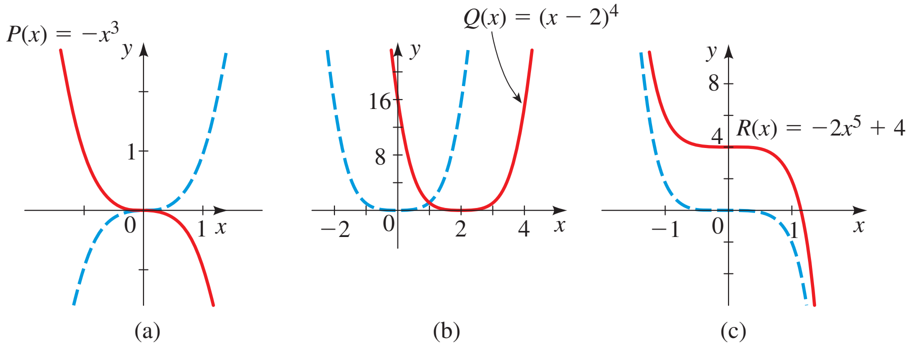
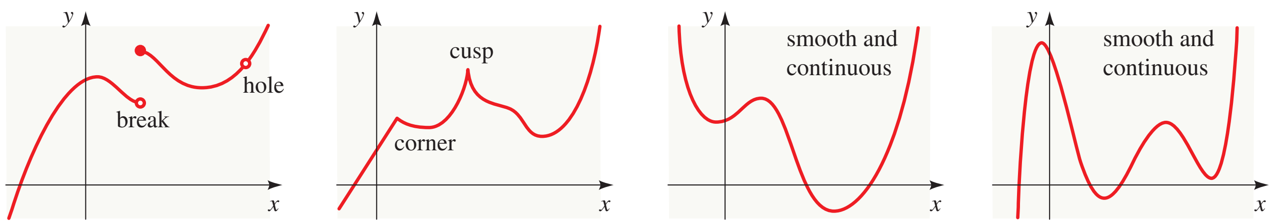


Figure 2

Now Try Exercise 5

Graphs of Polynomial Functions: End Behavior

The graphs of polynomials of degree 0 or 1 are lines (Sections 1.10 and 2.5), and the graphs of polynomials of degree 2 are parabolas (Section 3.1). The greater the degree of a polynomial, the more complicated its graph can be. However, the graph of a polynomial function is **continuous**. This means that the graph has no break or hole (see Figure 3). Moreover, the graph of a polynomial function is a smooth curve; that is, it has no corner or sharp point (cusp) as shown in Figure 3.



Not the graph of a polynomial function

Not the graph of a polynomial function

Graph of a polynomial function

Graph of a polynomial function

Figure 3

The domain of a polynomial function is the set of all real numbers, so we are able to sketch only a small portion of the graph. However, for values of x outside the portion of the graph we have drawn, we can describe the behavior of the graph.

The **end behavior** of a polynomial is a description of what happens as x becomes large in the positive or negative direction. To describe end behavior, we use the following **arrow notation**.

Symbol	Meaning
$x \to \infty$	x goes to infinity; that is, x increases without bound
$x \to -\infty$	x goes to negative infinity; that is, x decreases without bound

For example, the monomial $y = x^2$ in Figure 1(b) has the following end behavior.

$$y \to \infty \quad \text{as} \quad x \to \infty \qquad \text{and} \qquad y \to \infty \quad \text{as} \quad x \to -\infty$$

The monomial $y = x^3$ in Figure 1(c) has the following end behavior.

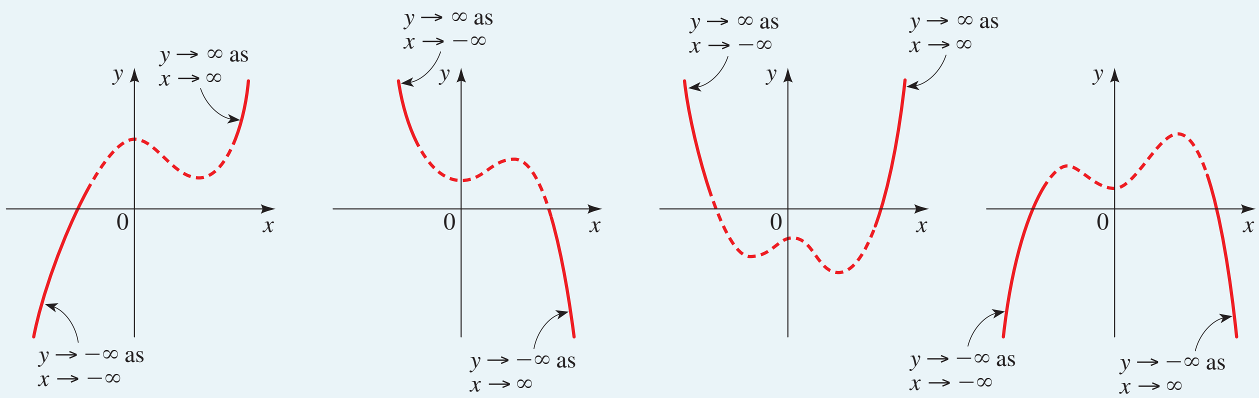
$$y \to \infty \quad \text{as} \quad x \to \infty \qquad \text{and} \qquad y \to -\infty \quad \text{as} \quad x \to -\infty$$

For any polynomial *the end behavior is determined by the term that contains the highest power of* x, because when x is large, the other terms are relatively insignificant in size. The following box shows the four possible types of end behavior, based on the highest power and the sign of its coefficient.

End Behavior of Polynomials

The end behavior of the polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is determined by the degree n and the sign of the leading coefficient a_n, as indicated in the following graphs.

***P* has odd degree** | ***P* has even degree**



Leading coefficient positive | Leading coefficient negative | Leading coefficient positive | Leading coefficient negative

Example 2 ■ End Behavior of a Polynomial

Determine the end behavior of the polynomial

$$P(x) = -2x^4 + 5x^3 + 4x - 7$$

Solution The polynomial P has degree 4 and leading coefficient -2. Thus P has *even* degree and *negative* leading coefficient, so it has the following end behavior.

$$y \to -\infty \quad \text{as} \quad x \to -\infty \qquad \text{and} \qquad y \to -\infty \quad \text{as} \quad x \to \infty$$

The graph in Figure 4 illustrates the end behavior of P.

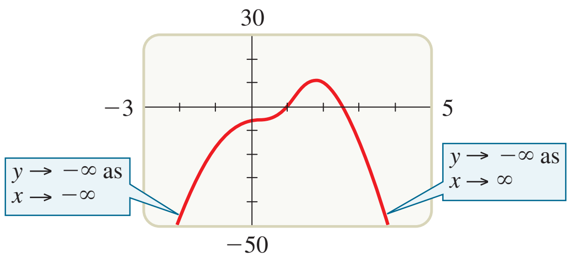


Figure 4 | $P(x) = -2x^4 + 5x^3 + 4x - 7$

Now Try Exercise 11 ■

Example 3 ■ End Behavior of a Polynomial

(a) Determine the end behavior of the polynomial $P(x) = 3x^5 - 5x^3 + 2x$.

(b) Confirm that P and its leading term $Q(x) = 3x^5$ have the same end behavior by graphing them together.

Solution

(a) Since P has odd degree and positive leading coefficient, it has the following end behavior.

$$y \to -\infty \quad \text{as} \quad x \to -\infty \qquad \text{and} \qquad y \to \infty \quad \text{as} \quad x \to \infty$$

(b) Figure 5 shows the graphs of P and Q in progressively larger viewing rectangles. The larger the viewing rectangle, the more the graphs look alike. This confirms that they have the same end behavior.

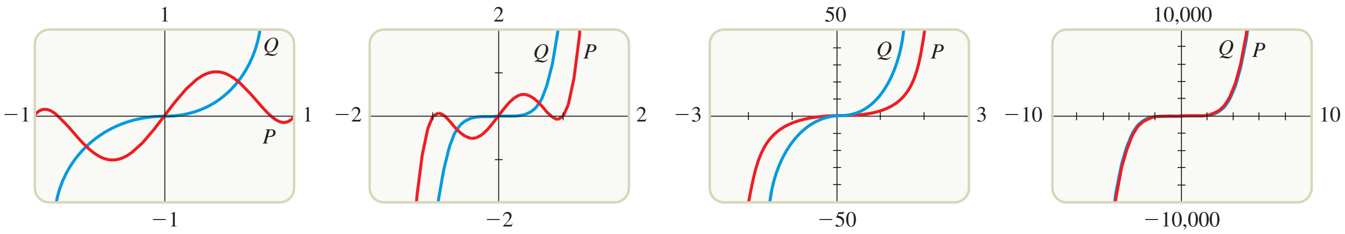


Figure 5 | $P(x) = 3x^5 - 5x^3 + 2x$
$Q(x) = 3x^5$

Now Try Exercise 45 ■

To see algebraically why P and Q in Example 3 have the same end behavior, factor P as follows and compare with Q.

$$P(x) = 3x^5\left(1 - \frac{5}{3x^2} + \frac{2}{3x^4}\right) \qquad Q(x) = 3x^5$$

When x is large, the terms $5/(3x^2)$ and $2/(3x^4)$ are close to 0. (See Exercise 1.1.89.) Thus for large x we have

$$P(x) \approx 3x^5(1 - 0 - 0) = 3x^5 = Q(x)$$

So when x is large, P and Q have approximately the same values. We can also see this numerically by making a table like the one shown below.

x	$P(x)$	$Q(x)$
15	2,261,280	2,278,125
30	72,765,060	72,900,000
50	936,875,100	937,500,000

By the same reasoning we can show that the end behavior of *any* polynomial is determined by its leading term.

■ Using Zeros to Graph Polynomials

If P is a polynomial function, then c is called a **zero** of P if $P(c) = 0$. In other words, the zeros of P are the solutions of the polynomial equation $P(x) = 0$. Note that if $P(c) = 0$, then the graph of P has an x-intercept at $x = c$, so the x-intercepts of the graph are the zeros of the function.

Real Zeros of Polynomials

If P is a polynomial and c is a real number, then the following are equivalent:

1. c is a zero of P.
2. $x = c$ is a solution of the equation $P(x) = 0$.
3. $x - c$ is a factor of $P(x)$.
4. c is an x-intercept of the graph of P.

To find the zeros of a polynomial P, we factor and then use the Zero-Product Property (see Section 1.5). For example, to find the zeros of $P(x) = x^2 + x - 6$, we factor P to get

$$P(x) = (x - 2)(x + 3)$$

From this factored form we see that

1. 2 is a zero of P.
2. $x = 2$ is a solution of the equation $x^2 + x - 6 = 0$.
3. $x - 2$ is a factor of $x^2 + x - 6$.
4. 2 is an x-intercept of the graph of P.

Similar facts are true for the other zero, -3.

The following theorem has many important consequences. (See, for instance, the *Discovery Project* referenced in Section 3.4.) Here we use it to help us graph polynomial functions.

Intermediate Value Theorem for Polynomials

If P is a polynomial function and $P(a)$ and $P(b)$ have opposite signs, then there exists at least one value c between a and b for which $P(c) = 0$.

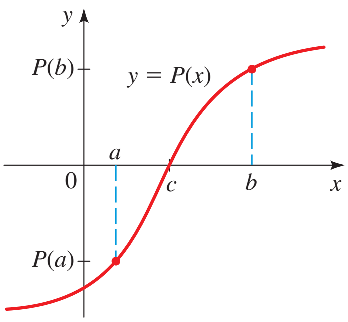


Figure 6

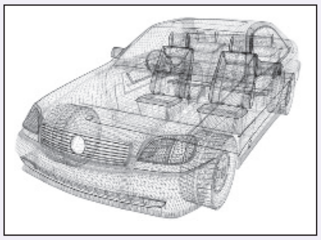
We will not prove this theorem, but Figure 6 shows why it is intuitively plausible.

One important consequence of this theorem is that between any two successive zeros the values of a polynomial are either all positive or all negative. That is, between two successive zeros the graph of a polynomial lies *entirely above* or *entirely below* the x-axis. To see why, suppose c_1 and c_2 are successive zeros of P. If P has both positive and negative values between c_1 and c_2, then by the Intermediate Value Theorem, P must have another zero between c_1 and c_2. But that's not possible because c_1 and c_2 are successive zeros. This observation allows us to use the following guidelines to graph polynomial functions.

Guidelines for Graphing Polynomial Functions

1. **Zeros.** Factor the polynomial to find all its real zeros: these are the x-intercepts of the graph.
2. **Test Points.** Make a table of values for the polynomial. Include test points to determine whether the graph of the polynomial lies above or below the x-axis on the intervals determined by the zeros. Include the y-intercept in the table.
3. **End Behavior.** Determine the end behavior of the polynomial.
4. **Graph.** Plot the intercepts and other points you found in the table. Sketch a smooth curve that passes through these points and exhibits the required end behavior.

Mathematics in the Modern World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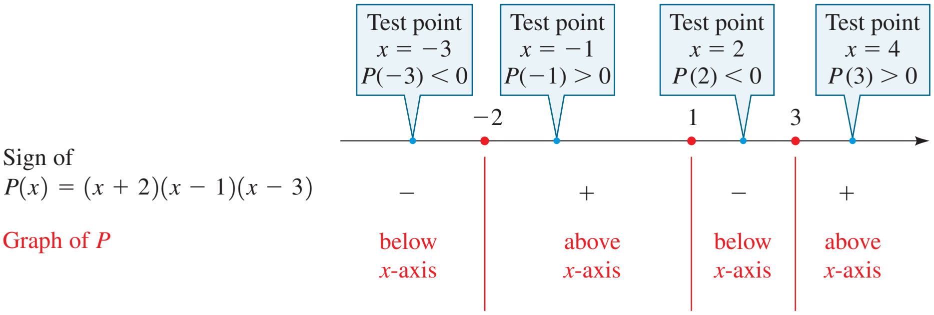
Automotive Design

Computer-aided design (CAD) has completely changed the way in which car companies design and manufacture cars. Before the 1980s automotive engineers would build a full-scale "nuts and bolts" model of a proposed new car; this was really the only way to tell whether the design was feasible. Today automotive engineers build a mathematical model, one that exists only in the memory of a computer. The model incorporates all the main design features of the car. Certain polynomial curves, called *splines* (see *Mathematics in the Modern World* earlier in this section), are used in shaping the body of the car. The resulting "mathematical car" can be tested for structural stability, handling, aerodynamics, suspension response, and more. All this testing is done before a prototype is built. As you can imagine, CAD saves car manufacturers millions of dollars each year. More importantly, CAD gives automotive engineers far more flexibility in design; desired changes can be created and tested within minutes. With the help of computer graphics, designers can see how good the "mathematical car" looks before they build the real one. Moreover, the mathematical car can be viewed from any perspective; it can be moved, rotated, or viewed from the inside. These manipulations of the car on the computer monitor translate mathematically into solving large systems of linear equations.

Example 4 ■ Using Zeros to Graph a Polynomial Function

Sketch the graph of the polynomial function $P(x) = (x + 2)(x - 1)(x - 3)$.

Solution The zeros are $x = -2$, 1, and 3. These determine the intervals $(-\infty, -2)$, $(-2, 1)$, $(1, 3)$, and $(3, \infty)$. Using test points in these intervals, we get the information in the following sign diagram (see Section 1.8).



Plotting a few additional points and connecting them with a smooth curve helps us to complete the graph in Figure 7.

	x	$P(x)$
Test point →	-3	-24
	-2	0
Test point →	-1	8
	0	6
	1	0
Test point →	2	-4
	3	0
Test point →	4	18

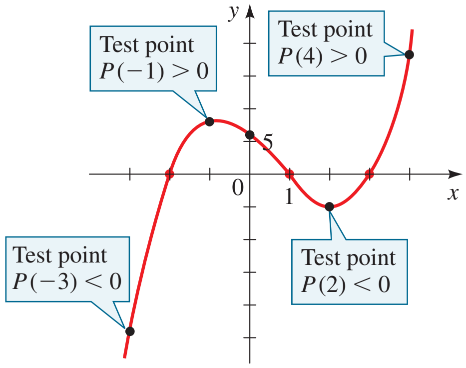


Figure 7 | $P(x) = (x + 2)(x - 1)(x - 3)$

Now Try Exercise 17 ■

Example 5 ■ Finding Zeros and Graphing a Polynomial Function

Let $P(x) = x^3 - 2x^2 - 3x$.

(a) Find the zeros of P. **(b)** Sketch a graph of P.

Solution

(a) To find the zeros, we factor completely.

$$\begin{aligned} P(x) &= x^3 - 2x^2 - 3x \\ &= x(x^2 - 2x - 3) && \text{Factor } x \\ &= x(x - 3)(x + 1) && \text{Factor quadratic} \end{aligned}$$

Thus the zeros are $x = 0$, $x = 3$, and $x = -1$.

(b) The x-intercepts are $x = 0$, $x = 3$, and $x = -1$. The y-intercept is $P(0) = 0$. We make a table of values of $P(x)$, making sure that we choose test points between (and to the right and left of) successive zeros.

Since P is of odd degree and its leading coefficient is positive, it has the following end behavior:

$$y \to -\infty \quad \text{as} \quad x \to -\infty \qquad \text{and} \qquad y \to \infty \quad \text{as} \quad x \to \infty$$

We plot the points in the table and connect them by a smooth curve to complete the graph, as shown in Figure 8.

	x	$P(x)$
Test point →	-2	-10
	-1	0
Test point →	$-\frac{1}{2}$	$\frac{7}{8}$
	0	0
Test point →	1	-4
	2	-6
	3	0
Test point →	4	20

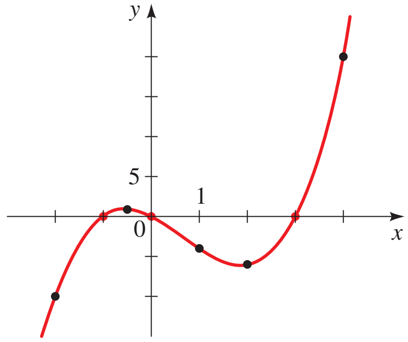


Figure 8 | $P(x) = x^3 - 2x^2 - 3x$

Now Try Exercise 31 ■

Example 6 ■ Finding Zeros and Graphing a Polynomial Function

Let $P(x) = -2x^4 - x^3 + 3x^2$.

(a) Find the zeros of P. **(b)** Sketch a graph of P.

Solution

(a) To find the zeros, we factor completely.

$$\begin{aligned} P(x) &= -2x^4 - x^3 + 3x^2 \\ &= -x^2(2x^2 + x - 3) && \text{Factor } -x^2 \\ &= -x^2(2x + 3)(x - 1) && \text{Factor quadratic} \end{aligned}$$

Thus the zeros are $x = 0$, $x = -\frac{3}{2}$, and $x = 1$.

(b) The x-intercepts are $x = 0$, $x = -\frac{3}{2}$, and $x = 1$. The y-intercept is $P(0) = 0$. We make a table of values of $P(x)$, making sure that we choose test points between (and to the right and left of) successive zeros.

Since P is of even degree and its leading coefficient is negative, it has the following end behavior.

$$y \to -\infty \quad \text{as} \quad x \to -\infty \qquad \text{and} \qquad y \to -\infty \quad \text{as} \quad x \to \infty$$

We plot the points from the table and connect the points by a smooth curve to complete the graph in Figure 9.

x	$P(x)$
-2	-12
-1.5	0
-1	2
-0.5	0.75
0	0
0.5	0.5
1	0
1.5	-6.75

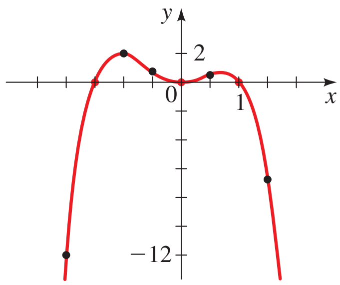


Figure 9 | $P(x) = -2x^4 - x^3 + 3x^2$

Now Try Exercise 35 ■

Example 7 ■ Finding Zeros and Graphing a Polynomial Function

Let $P(x) = x^3 - 2x^2 - 4x + 8$.

(a) Find the zeros of P. **(b)** Sketch a graph of P.

Solution

(a) To find the zeros, we factor completely.

$$
\begin{aligned}
P(x) &= x^3 - 2x^2 - 4x + 8 \\
&= x^2(x - 2) - 4(x - 2) && \text{Group and factor} \\
&= (x^2 - 4)(x - 2) && \text{Factor } x - 2 \\
&= (x + 2)(x - 2)(x - 2) && \text{Difference of squares} \\
&= (x + 2)(x - 2)^2 && \text{Simplify}
\end{aligned}
$$

Thus the zeros are $x = -2$ and $x = 2$.

(b) The x-intercepts are $x = -2$ and $x = 2$. The y-intercept is $P(0) = 8$. The table gives additional values of $P(x)$.

Since P is of odd degree and its leading coefficient is positive, it has the following end behavior.

$$y \to -\infty \quad \text{as} \quad x \to -\infty \qquad \text{and} \qquad y \to \infty \quad \text{as} \quad x \to \infty$$

We connect the points by a smooth curve to complete the graph in Figure 10.

x	$P(x)$
−3	−25
−2	0
−1	9
0	8
1	3
2	0
3	5

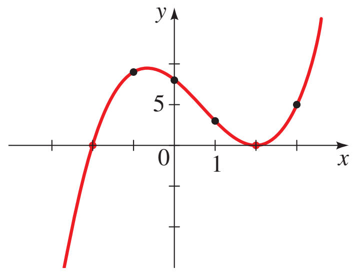


Figure 10 | $P(x) = x^3 - 2x^2 - 4x + 8$

Now Try Exercise 37 ■

■ Shape of the Graph Near a Zero

Although $x = 2$ is a zero of the polynomial in Example 7, the graph does not cross the x-axis at the x-intercept 2. This is because the factor $(x - 2)^2$ corresponding to that zero is raised to an even power, so it doesn't change sign as we test points on either side of 2. In the same way the graph does not cross the x-axis at $x = 0$ in Example 6.

In general, if c is a zero of P and the corresponding factor $x - c$ occurs exactly m times in the factorization of P, then we say that c is a **zero of multiplicity** $\boldsymbol{m}$. By

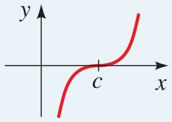
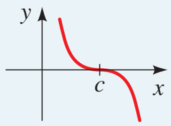
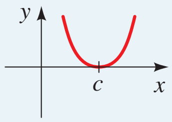
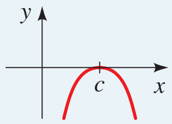
Discovery Project ■ **Bridge Science**

If you want to build a bridge, how can you be sure that your bridge design is strong enough to support the vehicles that will drive over it? In this project we perform a simple experiment using paper "bridges" to collect data on the weight our bridges can support. We model the data with linear and power functions to determine which model best fits the data. The model we obtain allows us to predict the strength of a large bridge *before* it is built. You can find the project at **www.stewartmath.com**.

considering test points on either side of the x-intercept c, we conclude that the graph crosses the x-axis at c if the multiplicity m is odd and does not cross the x-axis if m is even. Moreover, it can be shown by using calculus that near $x = c$ the graph has the same general shape as the graph of $y = A(x - c)^m$.

Shape of the Graph Near a Zero of Multiplicity m

If c is a zero of P of multiplicity m, then the shape of the graph of P near c is as follows.

Multiplicity of c	**Shape of the graph of P near the x-intercept c**
m odd, $m > 1$	 OR 
m even, $m > 1$	 OR 

Example 8 ■ Graphing a Polynomial Function Using Its Zeros

Graph the polynomial $P(x) = x^4(x - 2)^3(x + 1)^2$.

Solution The zeros of P are -1, 0, and 2 with multiplicities 2, 4, and 3, respectively:

0 is a zero of multiplicity 4

2 is a zero of multiplicity 3

–1 is a zero of multiplicity 2

$$P(x) = x^4(x - 2)^3(x + 1)^2$$

The zero 2 has *odd* multiplicity, so the graph crosses the x-axis at the x-intercept 2. But the zeros 0 and -1 have *even* multiplicities, so the graph does not cross the x-axis at the x-intercepts 0 and -1.

Since P is a polynomial of degree 9 and has positive leading coefficient, it has the following end behavior:

$$y \to -\infty \quad \text{as} \quad x \to -\infty \qquad \text{and} \qquad y \to \infty \quad \text{as} \quad x \to \infty$$

With this information and a table of values we sketch the graph in Figure 11.

x	$P(x)$
-1.3	-9.2
-1	0
-0.5	-0.2
0	0
1	-4
2	0
2.3	8.2

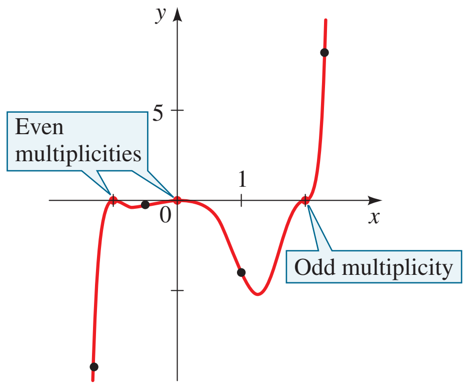


Figure 11 | $P(x) = x^4(x - 2)^3(x + 1)^2$

Now Try Exercise 29 ■

■ Local Maxima and Minima of Polynomials

Recall from Section 2.3 that if the point $(a, f(a))$ is the highest point on the graph of f within some viewing rectangle, then $f(a)$ is a local maximum value of f, and if $(b, f(b))$ is the lowest point on the graph of f within a viewing rectangle, then $f(b)$ is a local minimum value (see Figure 12). We say that such a point $(a, f(a))$ is a **local maximum point** on the graph and that $(b, f(b))$ is a **local minimum point**. The local maximum and minimum points on the graph of a function are called its **local extrema**.

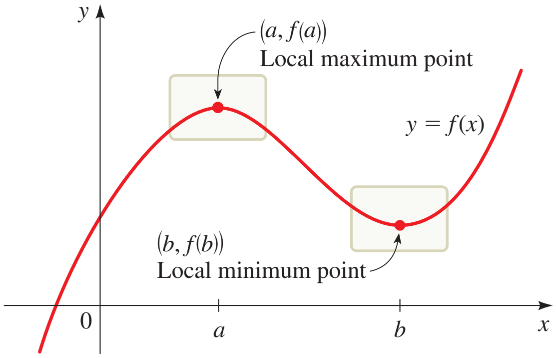


Figure 12

For a polynomial function the number of local extrema must be less than the degree, as the following principle indicates. (A proof of this principle requires calculus.)

Local Extrema of Polynomials

If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial of degree n, then the graph of P has at most $n - 1$ local extrema.

A polynomial of degree n may in fact have fewer than $n - 1$ local extrema. For example, $P(x) = x^5$ (graphed in Figure 1) has *no* local extrema, even though it is of degree 5. The preceding principle tells us only that a polynomial of degree n can have no more than $n - 1$ local extrema.

Example 9 ■ The Number of Local Extrema

Use a graphing device to graph the polynomial and determine how many local extrema it has.

(a) $P_1(x) = x^4 + x^3 - 16x^2 - 4x + 48$

(b) $P_2(x) = x^5 + 3x^4 - 5x^3 - 15x^2 + 4x - 15$

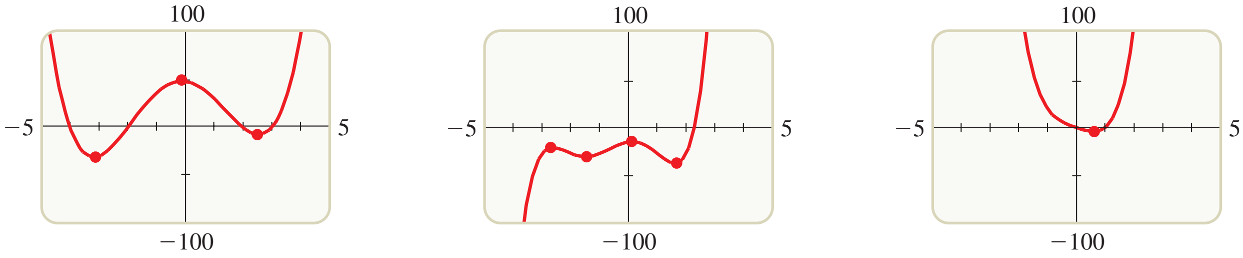
(c) $P_3(x) = 7x^4 + 3x^2 - 10x$

Solution The graphs are shown in Figure 13 on the next page.

(a) P_1 has two local minimum points and one local maximum point, for a total of three local extrema.

(b) P_2 has two local minimum points and two local maximum points, for a total of four local extrema.

(c) P_3 has just one local extremum, a local minimum.



(a) $P_1(x) = x^4 + x^3 - 16x^2 - 4x + 48$

(b) $P_2(x) = x^5 + 3x^4 - 5x^3 - 15x^2 + 4x - 15$

(c) $P_3(x) = 7x^4 + 3x^2 - 10x$

Figure 13

Now Try Exercises 65 and 67 ■

With a graphing device we can quickly draw the graphs of many functions at once, on the same viewing screen. This allows us to see how changing a value in the definition of the functions affects the shape of its graph. In the next example we apply this principle to a family of third-degree polynomials.

Example 10 ■ A Family of Polynomials

Sketch the family of polynomials $P(x) = x^3 - cx^2$ for $c = 0, 1, 2$, and 3. How does changing the value of c affect the graph?

Solution The polynomials

$$P_0(x) = x^3 \qquad P_1(x) = x^3 - x^2$$

$$P_2(x) = x^3 - 2x^2 \qquad P_3(x) = x^3 - 3x^2$$

are graphed in Figure 14. We see that increasing the value of c causes the graph to develop an increasingly deep "valley" to the right of the y-axis, creating a local maximum at the origin and a local minimum at a point in Quadrant IV. This local minimum moves lower and farther to the right as c increases. To see why this happens, factor $P(x) = x^2(x - c)$. The polynomial P has zeros at 0 and c, and the larger c gets, the farther to the right the minimum between 0 and c will be.

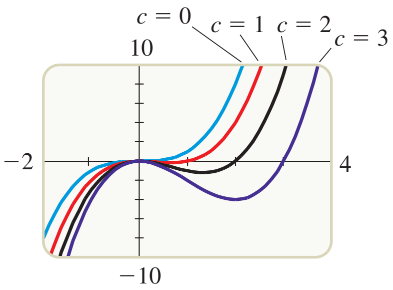


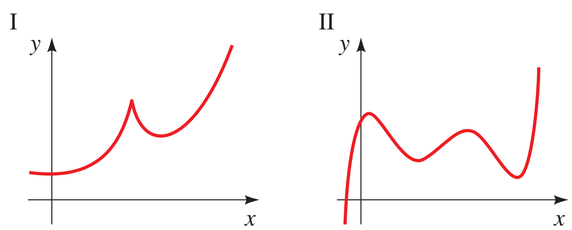
Figure 14 | A family of polynomials $P(x) = x^3 - cx^2$

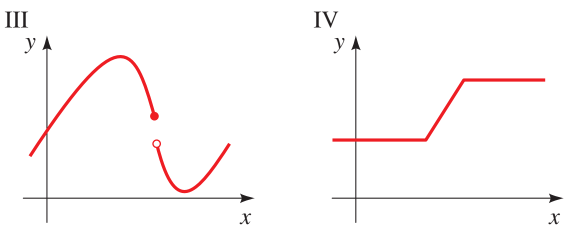
Now Try Exercise 75 ■

3.2 Exercises

Concepts

1. Only one of the following four graphs could be the graph of a polynomial function. Which one? Why are the others not graphs of polynomials?





2. Describe the end behavior of each polynomial.

(a) $y = x^3 - 8x^2 + 2x - 15$

End behavior: $y \to$ ________ as $x \to -\infty$

$y \to$ ________ as $x \to \infty$

(b) $y = -2x^4 + 12x + 100$

End behavior: $y \to$ ________ as $x \to -\infty$

$y \to$ ________ as $x \to \infty$

3. If c is a zero of the polynomial P, then

(a) $P(c) =$ ________.

(b) $x - c$ is a ________ of $P(x)$.

(c) c is a(n) ______ -intercept of the graph of P.

4. Which of the following statements couldn't possibly be true about the polynomial function P?

(a) P has degree 3, two local maxima, and two local minima.

(b) P has degree 3 and no local maxima or minima.

(c) P has degree 4, one local maximum, and no local minima.

Skills

5–8 ■ Transformations of Monomials Sketch the graph of each function by transforming the graph of an appropriate function of the form $y = x^n$ from Figure 1. Indicate all x- and y-intercepts on each graph, and state the domain and range.

5. (a) $P(x) = \frac{1}{2}x^2 - 2$ **(b)** $Q(x) = 2(x-3)^2$

(c) $R(x) = 4x^2 + 1$ **(d)** $S(x) = -(x+2)^2$

6. (a) $P(x) = -x^4 + 1$ **(b)** $Q(x) = (x+1)^4$

(c) $R(x) = 6x^4 - 6$ **(d)** $S(x) = \frac{1}{9}(x-3)^4$

7. (a) $P(x) = x^3 - 8$ **(b)** $Q(x) = -x^3 + 27$

(c) $R(x) = -(x+2)^3$ **(d)** $S(x) = \frac{1}{2}(x-1)^3 + 4$

8. (a) $P(x) = (x+3)^5$ **(b)** $Q(x) = 2(x+3)^5 - 64$

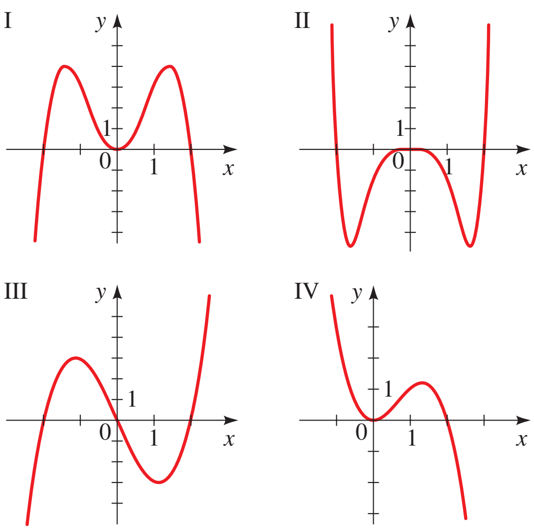
(c) $R(x) = -\frac{1}{2}(x-2)^5$ **(d)** $S(x) = -\frac{1}{2}(x-2)^5 + 16$

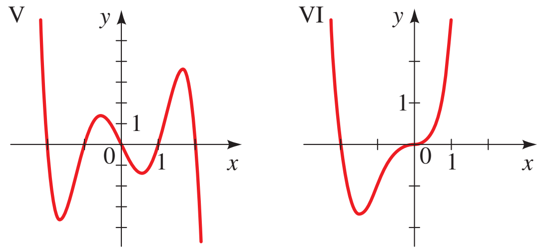
9–14 ■ End Behavior A polynomial function is given.
(a) Describe the end behavior of the polynomial function.
(b) Match the polynomial function with one of the graphs I–VI.

9. $P(x) = x(x^2 - 4)$ **10.** $Q(x) = -x^2(x^2 - 4)$

11. $R(x) = -x^5 + 5x^3 - 4x$ **12.** $S(x) = \frac{1}{2}x^6 - 2x^4$

13. $T(x) = x^4 + 2x^3$ **14.** $U(x) = -x^3 + 2x^2$





15–30 ■ Graphing Factored Polynomials Sketch the graph of the polynomial function. Make sure your graph shows all intercepts and exhibits the proper end behavior.

15. $P(x) = (x+2)(x-5)$

16. $P(x) = (4-x)(x+1)$

17. $P(x) = -x(x-2)(x+3)$

18. $P(x) = (x+3)(x-1)(x-4)$

19. $P(x) = -(2x-1)(x+1)(x+3)$

20. $P(x) = (x-3)(x+2)(3x-2)$

21. $P(x) = (x+2)(x+1)(x-2)(x-3)$

22. $P(x) = x(x+1)(x-1)(2-x)$

23. $P(x) = -2x(x-2)^2$

24. $P(x) = \frac{1}{5}x(x-5)^2$

25. $P(x) = (x+2)(x+1)^2(2x-3)$

26. $P(x) = -(x+2)^3(x-1)(x-3)^2$

27. $P(x) = \frac{1}{12}(x+2)^2(x-3)^2$

28. $P(x) = (x-1)^2(x+2)^3$

29. $P(x) = x(x-2)^2(x+2)^3$

30. $P(x) = \frac{1}{9}x^2(x-3)^2(x+3)^2$

31–44 ■ Graphing Polynomials Factor the polynomial and use the factored form to find the zeros. Then sketch the graph.

31. $P(x) = x^3 - x^2 - 6x$ **32.** $P(x) = x^3 + 2x^2 - 8x$

33. $P(x) = -x^3 + x^2 + 12x$ **34.** $P(x) = -2x^3 - x^2 + x$

35. $P(x) = x^4 - 3x^3 + 2x^2$ **36.** $P(x) = x^5 - 9x^3$

37. $P(x) = x^3 + x^2 - x - 1$

38. $P(x) = x^3 + 3x^2 - 4x - 12$

39. $P(x) = 2x^3 - x^2 - 18x + 9$

40. $P(x) = \frac{1}{8}(2x^4 + 3x^3 - 16x - 24)^2$

41. $P(x) = x^4 - 2x^3 - 8x + 16$

42. $P(x) = x^4 - 2x^3 + 8x - 16$

43. $P(x) = x^4 - 3x^2 - 4$ **44.** $P(x) = x^6 - 2x^3 + 1$

45–50 ■ End Behavior Determine the end behavior of P. Compare the graphs of P and Q in large and small viewing rectangles, as in Example 3(b).

45. $P(x) = 3x^3 - x^2 + 5x + 1$; $Q(x) = 3x^3$

46. $P(x) = -\frac{1}{8}x^3 + \frac{1}{4}x^2 + 12x$; $Q(x) = -\frac{1}{8}x^3$

47. $P(x) = x^4 - 7x^2 + 5x + 5$; $Q(x) = x^4$

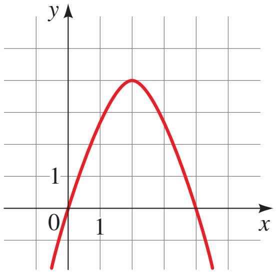
48. $P(x) = -x^5 + 2x^2 + x; \quad Q(x) = -x^5$

49. $P(x) = x^{11} - 9x^9; \quad Q(x) = x^{11}$

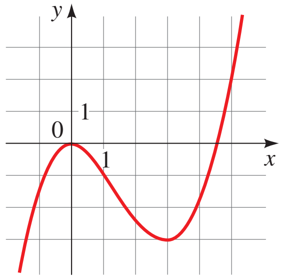
50. $P(x) = 2x^2 - x^{12}; \quad Q(x) = -x^{12}$

51–54 ■ Local Extrema The graph of a polynomial function is given. From the graph, find **(a)** the x- and y-intercepts, **(b)** the coordinates of all local extrema, and **(c)** the domain and range.

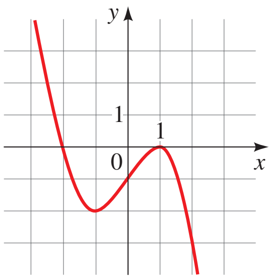
51. $P(x) = -x^2 + 4x$



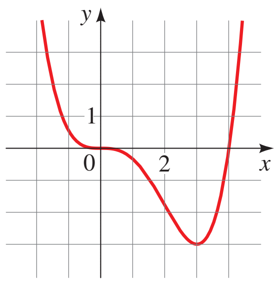
52. $P(x) = \frac{2}{9}x^3 - x^2$



53. $P(x) = -\frac{1}{2}x^3 + \frac{3}{2}x - 1$



54. $P(x) = \frac{1}{9}x^4 - \frac{4}{9}x^3$



55–62 ■ Local Extrema Graph the polynomial in the given viewing rectangle. Find the coordinates of all local extrema, rounded to two decimal places. State the domain and range.

55. $y = 10x - x^2, \quad [-2, 12]$ by $[-15, 30]$

56. $y = x^3 - 3x^2, \quad [-2, 5]$ by $[-10, 10]$

57. $y = x^3 - 12x + 9, \quad [-5, 5]$ by $[-30, 30]$

58. $y = 2x^3 - 3x^2 - 12x - 32, \quad [-5, 5]$ by $[-60, 30]$

59. $y = x^5 - 9x^3, \quad [-4, 4]$ by $[-50, 50]$

60. $y = x^4 - 18x^2 + 32, \quad [-5, 5]$ by $[-100, 100]$

61. $y = 3x^5 - 5x^3 + 3, \quad [-3, 3]$ by $[-5, 10]$

62. $y = x^5 - 5x^2 + 6, \quad [-3, 3]$ by $[-5, 10]$

63–72 ■ Number of Local Extrema Use a graph to determine the number of local maximums and minimums the polynomial has.

63. $y = -2x^2 + 3x + 5$

64. $y = x^3 + 12x$

65. $y = x^3 - x^2 - x$

66. $y = 6x^3 + 3x + 1$

67. $y = x^4 - 5x^2 + 4$

68. $y = 1.2x^5 + 3.75x^4 - 7x^3 - 15x^2 + 18x$

69. $y = (x - 2)^5 + 32$

70. $y = (x^2 - 2)^3$

71. $y = x^8 - 3x^4 + x$

72. $y = \frac{1}{3}x^7 - 17x^2 + 7$

73–78 ■ Families of Polynomials Graph the family of polynomials in the same viewing rectangle, using the given values of c. Explain how changing the value of c affects the graph.

73. $P(x) = cx^3; \quad c = 1, 2, 5, \frac{1}{2}$

74. $P(x) = (x - c)^4; \quad c = -1, 0, 1, 2$

75. $P(x) = x^4 + c; \quad c = -1, 0, 1, 2$

76. $P(x) = x^3 + cx; \quad c = 2, 0, -2, -4$

77. $P(x) = x^4 - cx; \quad c = 0, 1, 8, 27$

78. $P(x) = x^c; \quad c = 1, 3, 5, 7$

Skills Plus

79. Intersection Points of Two Polynomials

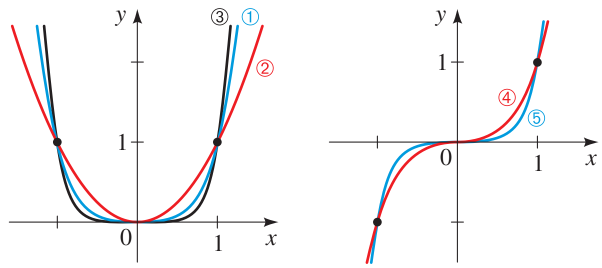
(a) On the same coordinate axes, sketch graphs of the given functions as accurately as possible:

$$y = x^3 - 2x^2 - x + 2 \qquad \text{and} \qquad y = -x^2 + 5x + 2$$

(b) On the basis of your graphs in part (a), at how many points do the two graphs appear to intersect?

(c) Find the coordinates of all intersection points.

80. Power Functions Portions of the graphs of $y = x^2$, $y = x^3$, $y = x^4$, $y = x^5$, and $y = x^6$ are plotted in the figures. Determine which function belongs to each graph.



81. Odd and Even Functions Recall that a function f is *odd* if $f(-x) = -f(x)$ or *even* if $f(-x) = f(x)$ for all real x.

(a) Show that a polynomial $P(x)$ that contains only odd powers of x is an odd function.

(b) Show that a polynomial $P(x)$ that contains only even powers of x is an even function.

(c) Show that if a polynomial $P(x)$ contains both odd and even powers of x, then it is neither an odd nor an even function.

(d) Express the function

$$P(x) = x^5 + 6x^3 - x^2 - 2x + 5$$

as the sum of an odd function and an even function.

82. Number of Intercepts and Local Extrema

(a) How many x-intercepts and how many local extrema does the polynomial $P(x) = x^3 - 4x$ have?

(b) How many x-intercepts and how many local extrema does the polynomial $Q(x) = x^3 + 4x$ have?

(c) If $a > 0$, how many x-intercepts and how many local extrema does each of the polynomials $P(x) = x^3 - ax$ and $Q(x) = x^3 + ax$ have? Explain your answer.

83–84 ■ Local Extrema These exercises involve local maximums and minimums of polynomial functions.

83. (a) Graph the function $P(x) = (x - 1)(x - 3)(x - 4)$ and find all local extrema, correct to the nearest tenth.

(b) Graph the function

$$Q(x) = (x - 1)(x - 3)(x - 4) + 5$$

and use your answers to part (a) to find all local extrema, correct to the nearest tenth.

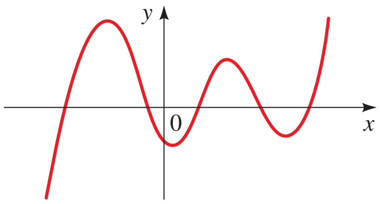
84. (a) Graph the function $P(x) = (x - 2)(x - 4)(x - 5)$ and determine how many local extrema it has.

(b) If $a < b < c$, explain why the function

$$P(x) = (x - a)(x - b)(x - c)$$

must have two local extrema.

85. Maximum Number of Local Extrema What is the smallest possible degree that the polynomial whose graph is shown can have? Explain.



86. A Family of Polynomials Graph the family of polynomials $P(x) = cx^4 - 2x^2$ for various values of c, including $c = -2, -1, 0, 1, 2$. How many extreme values does the polynomial have if $c < 0$? If $c > 0$? Why do you think $c = 0$ is called a "transitional value" for this family of polynomials?

Applications

87. Market Research A market analyst working for a small-appliance manufacturer finds that if the firm produces and sells x blenders annually, the total profit (in dollars) is

$$P(x) = 8x + 0.3x^2 - 0.0013x^3 - 372$$

Graph the function P in an appropriate viewing rectangle and use the graph to answer the following questions.

(a) When just a few blenders are manufactured, the firm loses money (profit is negative). (For example, $P(10) = -263.3$, so the firm loses \$263.30 if it produces and sells only 10 blenders.) How many blenders must the firm produce to break even?

(b) Does profit increase indefinitely as more blenders are produced and sold? If not, what is the largest possible profit the firm could have?

88. Population Change The rabbit population on a small island is observed to be given by the function

$$P(t) = 120t - 0.4t^4 + 1000$$

where t is the time (in months) since observations of the island began.

(a) What is the maximum population attained, and when is that maximum population?

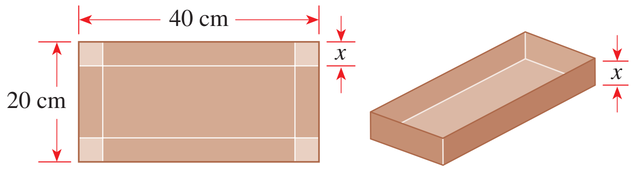
(b) When does the rabbit population disappear from the island?

89. Volume of a Box An open box is to be constructed from a piece of cardboard 20 cm by 40 cm by cutting squares of side length x from each corner and folding up the sides, as shown in the figure.

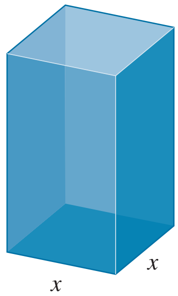
(a) Express the volume V of the box as a function of x.

(b) What is the domain of V? (Use the fact that length and volume must be positive.)

(c) Use a graph of the function V to estimate the maximum volume for such a box.



90. Volume of a Box A cardboard box has a square base, with each edge of the base having length x inches, as shown in the figure. The total length of all 12 edges of the box is 144 in.



(a) Show that the volume of the box is given by the function $V(x) = 2x^2(18 - x)$.

(b) What is the domain of V? (Use the fact that length and volume must be positive.)

(c) Draw a graph of the function V and use it to estimate the maximum volume for such a box.

Discuss ■ Discover ■ Prove ■ Write

91. Discover: Graphs of Large Powers Graph the functions $y = x^2$, $y = x^3$, $y = x^4$, and $y = x^5$, for $-1 \le x \le 1$, on the same coordinate axes. What do you think the graph of $y = x^{100}$ would look like on this same interval? What about $y = x^{101}$? Make a table of values to confirm your answers.

92. Discuss ■ Discover: Possible Number of Local Extrema Is it possible for a third-degree polynomial to have exactly one local extremum? Is it possible for any polynomial to have two local maximums and no local minimum? Explain. Give an example of a polynomial that has six local extrema.

93. Discover ■ Prove: Fixed Points A *fixed point* of a function f is a number x for which $f(x) = x$. (See Exercise 2.3.76.) Show that if f is a continuous function with domain $[0, 1]$ and range contained in $[0, 1]$, then f has a fixed point in $[0, 1]$.

PS *Draw a diagram.* Since f is continuous, its graph has no break or hole. Draw a graph of f and make a graphical argument.

3.3 Dividing Polynomials

■ Long Division of Polynomials ■ Synthetic Division ■ The Remainder and Factor Theorems

So far in this chapter we have been studying polynomial functions *graphically*. In this section we begin to study polynomials *algebraically*. Most of our work will be concerned with factoring polynomials, and to factor, we need to know how to divide polynomials.

Long Division of Polynomials

Dividing polynomials is much like the familiar process of dividing numbers. When we divide 38 by 7, the quotient is 5 and the remainder is 3. We write

$$\begin{array}{r} 5 \\ 7\overline{)38} \\ \underline{35} \\ 3 \end{array}$$

Dividend

Remainder

$$\frac{38}{7} = 5 + \frac{3}{7}$$

Divisor

Quotient

To divide polynomials, we use long division, as follows.

Division Algorithm

If $P(x)$ and $D(x)$ are polynomials, with $D(x) \neq 0$, then there exist unique polynomials $Q(x)$ and $R(x)$, where $R(x)$ is either 0 or of degree less than the degree of $D(x)$, such that

Dividend Quotient Remainder

$$\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$$

Divisor Divisor

or

Remainder

$$P(x) = D(x) \cdot Q(x) + R(x)$$

Dividend Divisor Quotient

The polynomials $P(x)$ and $D(x)$ are called the **dividend** and **divisor**, respectively; $Q(x)$ is the **quotient**, and $R(x)$ is the **remainder**.

Example 1 ■ Long Division of Polynomials

Divide $6x^2 - 26x + 12$ by $x - 4$. Express the result in each of the two forms shown in the preceding box.

Solution The *dividend* is $6x^2 - 26x + 12$, and the *divisor* is $x - 4$. We begin by arranging them as follows.

$$x - 4\overline{)6x^2 - 26x + 12}$$

Next we divide the leading term in the dividend by the leading term in the divisor to get the first term of the quotient: $6x^2/x = 6x$. Then we multiply the divisor by $6x$ and subtract the result from the dividend.

$$\begin{array}{rl} 6x\phantom{{}-26x+12} & \text{Divide leading terms: } \dfrac{6x^2}{x} = 6x \\ x - 4\overline{)6x^2 - 26x + 12} & \\ \underline{6x^2 - 24x}\phantom{{}+12} & \text{Multiply: } 6x(x-4) = 6x^2 - 24x \\ -2x + 12 & \text{Subtract and “bring down” 12} \end{array}$$

We repeat the process using the last line, $-2x + 12$, as the dividend.

$$
\begin{array}{r|l}
 & \;\; 6x \;\; - 2 \\
\cline{2-2}
x-4 & 6x^2 - 26x + 12 \\
 & \underline{6x^2 - 24x} \\
 & \qquad\quad -2x + 12 \\
 & \qquad\quad \underline{-2x + \;\;8} \\
 & \qquad\qquad\qquad 4
\end{array}
\qquad
\begin{array}{l}
\text{Divide leading terms: } \dfrac{-2x}{x} = -2 \\
\\
\\
\\
\text{Multiply: } -2(x-4) = -2x + 8 \\
\text{Subtract}
\end{array}
$$

The division process ends when the last line is of lower degree than the divisor. The last line then contains the *remainder*, and the top line contains the *quotient*. The result of the division can be interpreted in either of two ways:

$$
\frac{\overset{\text{Dividend}}{6x^2 - 26x + 12}}{\underset{\text{Divisor}}{x - 4}} = \overset{\text{Quotient}}{6x - 2} + \frac{\overset{\text{Remainder}}{4}}{\underset{\text{Divisor}}{x - 4}}
\qquad \text{or} \qquad
\underset{\text{Dividend}}{6x^2 - 26x + 12} = \underset{\text{Divisor}}{(x - 4)}\underset{\text{Quotient}}{(6x - 2)} + \overset{\text{Remainder}}{4}
$$

Now Try Exercises 3 and 9 ■

Example 2 ■ Long Division of Polynomials

Let $P(x) = 8x^4 + 6x^2 - 3x + 1$ and $D(x) = 2x^2 - x + 2$. Find polynomials $Q(x)$ and $R(x)$ such that $P(x) = D(x) \cdot Q(x) + R(x)$.

Solution We use long division after first inserting the term $0x^3$ into the dividend to ensure that the columns line up correctly.

$$
\begin{array}{r|l}
 & 4x^2 + 2x \\
\cline{2-2}
2x^2 - x + 2 & 8x^4 + 0x^3 + 6x^2 - 3x + 1 \\
 & \underline{8x^4 - 4x^3 + 8x^2} \\
 & \qquad\quad 4x^3 - 2x^2 - 3x \\
 & \qquad\quad \underline{4x^3 - 2x^2 + 4x} \\
 & \qquad\qquad\qquad\quad -7x + 1
\end{array}
\qquad
\begin{array}{l}
\\
\\
\text{Multiply divisor by } 4x^2 \\
\text{Subtract} \\
\text{Multiply divisor by } 2x \\
\text{Subtract}
\end{array}
$$

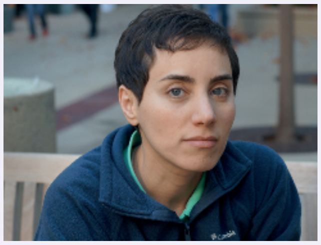
The process is complete at this point because $-7x + 1$ is of lower degree than the divisor, $2x^2 - x + 2$. From the long division above we see that $Q(x) = 4x^2 + 2x$ and $R(x) = -7x + 1$, so

$$8x^4 + 6x^2 - 3x + 1 = (2x^2 - x + 2)(4x^2 + 2x) + (-7x + 1)$$

Now Try Exercise 19 ■

■ Synthetic Division

Synthetic division is a quick method of dividing polynomials; it can be used when the divisor is of the form $x - c$. In synthetic division we write only the essential parts of the long division. Compare the following long and synthetic divisions, in which we divide $2x^3 - 7x^2 + 5$ by $x - 3$. (We'll explain how to perform the synthetic division in Example 3.)



Stanford News Service/ZUMA Press/
Stanford/California/U.S/NEWSCOM

MARYAM MIRZAKHANI (1977–2017) was one of the top mathematicians in the world until her untimely death in 2017. In 2014 she was awarded the Fields Medal for her work on the geometry of surfaces. The Fields Medal is the top prize in mathematics (it's sometimes called the "Nobel Prize" of Mathematics).

Maryam was born in Iran, where she attended an all-girls school. She said, "I did poorly in math for a couple of years in middle school; I was just not interested in thinking about it." But she later became fascinated by problem solving; she said, "The beauty of mathematics only shows itself to patient followers." Maryam and her childhood friend Roya Beheshti were chosen to represent their country at the International Mathematics Olympiad, a worldwide competition, for high school students; they received the gold and silver medals. Roya is now also a professor of mathematics.

Maryam was the first woman to be awarded the Fields Medal. In accepting the award, she said, "I hope that this award will inspire lots more girls and young women … to believe in their own abilities and aim to be the Fields Medalists of the future." The 2020 documentary film *Secrets of the Surface* features the life and mathematical vision of Maryam Mirzakhani.

Long Division

$$
\begin{array}{r l}
 & \underbrace{2x^2 - x - 3}_{\text{Quotient}} \\
x - 3 \,\big) & 2x^3 - 7x^2 + 0x + 5 \\
 & \underline{2x^3 - 6x^2} \\
 & \quad -x^2 + 0x \\
 & \quad \underline{-x^2 + 3x} \\
 & \qquad -3x + 5 \\
 & \qquad \underline{-3x + 9} \\
 & \qquad\qquad \underbrace{-4}_{\text{Remainder}}
\end{array}
$$

Synthetic Division

$$
\begin{array}{c|cccc}
3 & 2 & -7 & 0 & 5 \\
 & & 6 & -3 & -9 \\
\hline
 & \multicolumn{3}{c}{\underbrace{2 \quad -1 \quad -3}_{\text{Quotient}}} & \underbrace{-4}_{\text{Remainder}}
\end{array}
$$

Note that in synthetic division we abbreviate $2x^3 - 7x^2 + 5$ by writing only the coefficients: $2 \quad -7 \quad 0 \quad 5$, and instead of $x - 3$, we simply write 3. (Writing 3 instead of -3 allows us to add instead of subtract, but this changes the sign of all the numbers that appear in the gold boxes.)

The next example shows how synthetic division is performed.

Example 3 ■ Synthetic Division

Use synthetic division to divide $2x^3 - 7x^2 + 5$ by $x - 3$.

Solution We begin by writing the appropriate coefficients to represent the divisor and the dividend:

$$
\underbrace{3}_{\text{Divisor } x-3} \; \begin{array}{|cccc}
2 & -7 & 0 & 5 \\
\hline
\end{array} \; \underbrace{}_{\text{Dividend } 2x^3 - 7x^2 + 0x + 5}
$$

We bring down the 2, multiply $3 \cdot 2 = 6$, and write the result in the middle row. Then we add.

$$
\begin{array}{c|cccc}
3 & 2 & -7 & 0 & 5 \\
 & & 6 & & \\
\hline
 & \boxed{2} & -1 & &
\end{array}
\qquad
\begin{array}{l}
\text{Multiply: } 3 \cdot 2 = 6 \\
\\
\text{Add: } -7 + 6 = -1
\end{array}
$$

We repeat this process of multiplying and then adding until the table is complete.

$$
\begin{array}{c|cccc}
3 & 2 & -7 & 0 & 5 \\
 & & 6 & -3 & \\
\hline
 & 2 & \boxed{-1} & -3 &
\end{array}
\qquad
\begin{array}{l}
\text{Multiply: } 3(-1) = -3 \\
\\
\text{Add: } 0 + (-3) = -3
\end{array}
$$

$$
\begin{array}{c|cccc}
3 & 2 & -7 & 0 & 5 \\
 & & 6 & -3 & -9 \\
\hline
 & \multicolumn{3}{c}{\underbrace{2 \quad -1 \quad \boxed{-3}}_{\text{Quotient } 2x^2 - x - 3}} & \underbrace{-4}_{\text{Remainder } -4}
\end{array}
\qquad
\begin{array}{l}
\text{Multiply: } 3(-3) = -9 \\
\\
\text{Add: } 5 + (-9) = -4
\end{array}
$$

From the last line of the synthetic division we see that the quotient is $2x^2 - x - 3$ and the remainder is -4. Thus

$$2x^3 - 7x^2 + 5 = (x - 3)(2x^2 - x - 3) - 4$$

Now Try Exercise 31 ■

The Remainder and Factor Theorems

The following theorem shows how synthetic division can be used to evaluate polynomials easily.

Remainder Theorem

If the polynomial $P(x)$ is divided by $x - c$, then the remainder is the value $P(c)$.

Proof If the divisor in the Division Algorithm is of the form $x - c$ for some real number c, then the remainder must be a constant (since the degree of the remainder is less than the degree of the divisor). If we call this constant r, then

$$P(x) = (x - c) \cdot Q(x) + r$$

Replacing x by c in this equation, we get $P(c) = (c - c) \cdot Q(c) + r = 0 + r = r$, that is, $P(c)$ is the remainder r. ■

Example 4 ■ Using the Remainder Theorem to Find the Value of a Polynomial

Let $P(x) = 3x^5 + 5x^4 - 4x^3 + 7x + 3$.

(a) Find the quotient and remainder when $P(x)$ is divided by $x + 2$.

(b) Use the Remainder Theorem to find $P(-2)$.

Solution

(a) Since $x + 2 = x - (-2)$, the synthetic division for this problem takes the following form:

$$\begin{array}{r|rrrrrr} -2 & 3 & 5 & -4 & 0 & 7 & 3 \\ & & -6 & 2 & 4 & -8 & 2 \\ \hline & 3 & -1 & -2 & 4 & -1 & 5 \end{array}$$

Remainder is 5, so $P(-2) = 5$

The quotient is $3x^4 - x^3 - 2x^2 + 4x - 1$ and the remainder is 5.

(b) By the Remainder Theorem, $P(-2)$ is the remainder when $P(x)$ is divided by $x - (-2) = x + 2$. From part (a) the remainder is 5, so $P(-2) = 5$.

Now Try Exercise 39 ■

The next theorem says that *zeros* of polynomials correspond to *factors*. We used this fact in Section 3.2 to graph polynomials.

Factor Theorem

c is a zero of P if and only if $x - c$ is a factor of $P(x)$.

Proof If $P(x)$ factors as $P(x) = (x - c)Q(x)$, then

$$P(c) = (c - c)Q(c) = 0 \cdot Q(c) = 0$$

Conversely, if $P(c) = 0$, then by the Remainder Theorem

$$P(x) = (x - c)Q(x) + 0 = (x - c)Q(x)$$

so $x - c$ is a factor of $P(x)$. ■

Example 5 ■ Factoring a Polynomial Using the Factor Theorem

Let $P(x) = x^3 - 7x + 6$. Show that $P(1) = 0$, and use this fact to factor $P(x)$ completely.

Solution Substituting, we see that $P(1) = 1^3 - 7 \cdot 1 + 6 = 0$. By the Factor Theorem this means that $x - 1$ is a factor of $P(x)$. Using synthetic or long division (shown in the margin), we see that

$$
\begin{array}{r|rrrr}
1 & 1 & 0 & -7 & 6 \\
 & & 1 & 1 & -6 \\
\hline
 & 1 & 1 & -6 & 0
\end{array}
$$

$$
\begin{array}{r}
x^2 + x - 6 \\
x - 1 \overline{\smash{)}\, x^3 + 0x^2 - 7x + 6} \\
\underline{x^3 - x^2 } \\
x^2 - 7x \\
\underline{x^2 - x } \\
-6x + 6 \\
\underline{-6x + 6} \\
0
\end{array}
$$

$$
\begin{aligned}
P(x) &= x^3 - 7x + 6 && \text{Given polynomial} \\
&= (x - 1)(x^2 + x - 6) && \text{See margin} \\
&= (x - 1)(x - 2)(x + 3) && \text{Factor quadratic } x^2 + x - 6
\end{aligned}
$$

Now Try Exercises 53 and 57 ■

Example 6 ■ Finding a Polynomial with Specified Zeros

Find a polynomial of degree 4 that has zeros -3, 0, 1, and 5, and the coefficient of x^3 is -6.

Solution By the Factor Theorem, $x - (-3)$, $x - 0$, $x - 1$, and $x - 5$ must all be factors of the desired polynomial. Let

$$
\begin{aligned}
P(x) &= (x + 3)(x - 0)(x - 1)(x - 5) \\
&= x^4 - 3x^3 - 13x^2 + 15x && \text{Expand}
\end{aligned}
$$

The polynomial $P(x)$ is of degree 4 with the desired zeros, but the coefficient of x^3 is -3, not -6. Multiplication by a nonzero constant does not change the degree, so the desired polynomial is a constant multiple of $P(x)$. If we multiply $P(x)$ by the constant 2, we get

$$Q(x) = 2x^4 - 6x^3 - 26x^2 + 30x$$

which is a polynomial with all the desired properties. The polynomial Q is graphed in Figure 1. Note that the zeros of Q correspond to the x-intercepts of the graph.

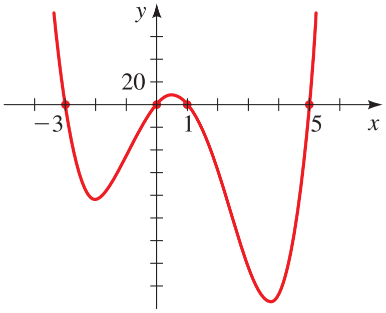


Figure 1 | $Q(x) = 2x(x + 3)(x - 1)(x - 5)$ has zeros -3, 0, 1, and 5, and the coefficient of x^3 is -6.

Now Try Exercises 63 and 67 ■

3.3 Exercises

Concepts

1. If we divide the polynomial P by the factor $x - c$ and we obtain the equation $P(x) = (x - c)Q(x) + R(x)$, then we say that $x - c$ is the divisor, $Q(x)$ is the ________, and $R(x)$ is the ________.

2. (a) If we divide the polynomial $P(x)$ by the factor $x - c$ and we obtain a remainder of 0, then we know that c is a ________ of P.

(b) If we divide the polynomial $P(x)$ by the factor $x - c$ and we obtain a remainder of k, then we know that $P(c) =$ ________.

Skills

3–8 ■ Division of Polynomials Two polynomials P and D are given. Use either synthetic division or long division to divide $P(x)$ by $D(x)$, and express the quotient $P(x)/D(x)$ in the form

$$\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$$

3. $P(x) = 3x^2 - 6x + 4$, $D(x) = x - 3$

4. $P(x) = 2x^2 + 10x - 7$, $D(x) = x + 5$

5. $P(x) = 12x^3 - 16x^2 - x - 1$, $D(x) = 3x + 2$

6. $P(x) = 10x^3 + 2x^2 - 4x + 1$, $D(x) = 5x + 1$

7. $P(x) = 2x^4 - x^3 + 9x^2, \quad D(x) = x^2 + 4$

8. $P(x) = 2x^5 + x^3 - 2x^2 + 3x - 5, \quad D(x) = x^2 - 3x + 1$

9–14 ■ Division of Polynomials Two polynomials P and D are given. Use either synthetic division or long division to divide $P(x)$ by $D(x)$, and express P in the form

$$P(x) = D(x) \cdot Q(x) + R(x)$$

9. $P(x) = 3x^3 + 5x^2 + 5, \quad D(x) = x + 5$

10. $P(x) = 5x^4 - 10x^2 + 3x + 2, \quad D(x) = x - 2$

11. $P(x) = 2x^3 - 3x^2 - 2x, \quad D(x) = 2x - 3$

12. $P(x) = 4x^3 + 7x + 9, \quad D(x) = 2x + 1$

13. $P(x) = 8x^4 + 4x^3 + 6x^2, \quad D(x) = 2x^2 + 1$

14. $P(x) = 27x^5 - 9x^4 + 3x^2 - 3, \quad D(x) = 3x^2 - 3x + 1$

15–24 ■ Long Division of Polynomials Find the quotient and remainder using long division.

15. $\dfrac{x^2 - 3x + 7}{x - 2}$

16. $\dfrac{x^3 + 2x^2 - x + 1}{x + 3}$

17. $\dfrac{9x^3 - 6x^2 + x + 1}{3x - 1}$

18. $\dfrac{8x^3 - 2x^2 - 2x + 3}{4x + 3}$

19. $\dfrac{4x^3 + 2x^2 - 3}{x^2 + x - 1}$

20. $\dfrac{x^4 - 4x^3 - x + 3}{x^2 - 3x + 2}$

21. $\dfrac{6x^3 + 2x^2 + 22x}{2x^2 + 5}$

22. $\dfrac{9x^2 - x + 5}{3x^2 - 7x}$

23. $\dfrac{x^6 + x^4 + x^2 + 1}{x^2 + 1}$

24. $\dfrac{2x^5 - 7x^4 - 13}{4x^2 - 6x + 8}$

25–38 ■ Synthetic Division of Polynomials Find the quotient and remainder using synthetic division.

25. $\dfrac{2x^2 - 5x + 3}{x - 3}$

26. $\dfrac{-x^2 + x - 4}{x + 1}$

27. $\dfrac{3x^2 + x}{x + 1}$

28. $\dfrac{4x^2 - 3}{x - 2}$

29. $\dfrac{3x^3 - 2x^2 + x - 5}{x - 2}$

30. $\dfrac{5x^3 + 20x^2 - 30x + 10}{x + 5}$

31. $\dfrac{x^3 - 10x + 13}{x + 4}$

32. $\dfrac{x^4 - x^3 - 10x}{x - 3}$

33. $\dfrac{x^5 + 3x^3 - 6}{x - 1}$

34. $\dfrac{x^3 - 9x^2 + 27x - 27}{x - 3}$

35. $\dfrac{2x^3 + 3x^2 - 2x + 1}{x - \frac{1}{2}}$

36. $\dfrac{6x^4 + 10x^3 + 5x^2 + x + 1}{x + \frac{2}{3}}$

37. $\dfrac{x^3 - 27}{x - 3}$

38. $\dfrac{x^4 - 16}{x + 2}$

39–51 ■ Remainder Theorem Use synthetic division and the Remainder Theorem to evaluate $P(c)$.

39. $P(x) = x^4 - x^2 + 5, \quad c = -2$

40. $P(x) = 2x^2 + 9x + 1, \quad c = \frac{1}{2}$

41. $P(x) = x^3 + 3x^2 - 7x + 6, \quad c = 2$

42. $P(x) = x^3 - x^2 + x + 5, \quad c = -1$

43. $P(x) = x^3 + 2x^2 - 7, \quad c = -2$

44. $P(x) = 2x^3 - 21x^2 + 9x - 200, \quad c = 11$

45. $P(x) = 5x^4 + 30x^3 - 40x^2 + 36x + 14, \quad c = -7$

46. $P(x) = 6x^5 + 10x^3 + x + 1, \quad c = -2$

47. $P(x) = x^7 - 3x^2 - 1, \quad c = 3$

48. $P(x) = -2x^6 + 7x^5 + 40x^4 - 7x^2 + 10x + 112, \quad c = -3$

49. $P(x) = 3x^3 + 4x^2 - 2x + 1, \quad c = \frac{2}{3}$

50. $P(x) = x^3 - x + 1, \quad c = \frac{1}{4}$

51. $P(x) = x^3 + 2x^2 - 3x - 8, \quad c = 0.1$

52. Remainder Theorem Let

$$P(x) = 6x^7 - 40x^6 + 16x^5 - 200x^4 - 60x^3 - 69x^2 + 13x - 139$$

Calculate $P(7)$ by **(a)** using synthetic division and **(b)** substituting $x = 7$ into the polynomial and evaluating directly.

53–56 ■ Factor Theorem Use the Factor Theorem to show that $x - c$ is a factor of $P(x)$ for the given value(s) of c.

53. $P(x) = x^3 - 3x^2 + 3x - 1, \quad c = 1$

54. $P(x) = x^3 + 2x^2 - 3x - 10, \quad c = 2$

55. $P(x) = 2x^3 + 7x^2 + 6x - 5, \quad c = \frac{1}{2}$

56. $P(x) = x^4 + 3x^3 - 16x^2 - 27x + 63, \quad c = 3, -3$

57–62 ■ Factor Theorem Show that the given value(s) of c are zeros of $P(x)$, and find all other zeros of $P(x)$.

57. $P(x) = x^3 - 3x^2 - 18x + 40, \quad c = -4$

58. $P(x) = x^3 - 5x^2 - 2x + 10, \quad c = 5$

59. $P(x) = x^3 - x^2 - 11x + 15, \quad c = 3$

60. $P(x) = 3x^4 - x^3 - 21x^2 - 11x + 6, \quad c = -2, \frac{1}{3}$

61. $P(x) = 3x^4 - 8x^3 - 14x^2 + 31x + 6, \quad c = -2, 3$

62. $P(x) = 2x^4 - 13x^3 + 7x^2 + 37x + 15, \quad c = -1, 3$

63–66 ■ Finding a Polynomial with Specified Zeros Find a polynomial of the specified degree that has the given zeros.

63. Degree 3; zeros $-1, 1, 3$

64. Degree 4; zeros $-2, 0, 2, 4$

65. Degree 4; zeros $-1, 1, 3, 5$

66. Degree 5; zeros $-2, -1, 0, 1, 2$

67–70 ■ Polynomials with Specified Zeros Find a polynomial of the specified degree that satisfies the given conditions.

67. Degree 4; zeros $-2, 0, 1, 3$; coefficient of x^3 is 4

68. Degree 4; zeros $-1, 0, 2, \frac{1}{2}$; coefficient of x^3 is 3

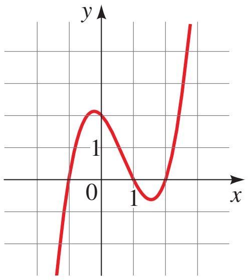
69. Degree 4; zeros $-1, 1, \sqrt{2}$; integer coefficients and constant term 6

70. Degree 5; zeros $-2, -1, 2, \sqrt{5}$; integer coefficients and constant term 40

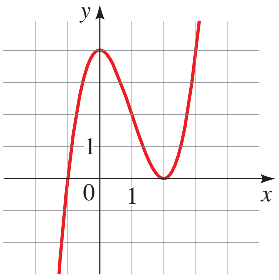
■ Skills Plus

71–74 ■ Finding a Polynomial from a Graph Find the polynomial of the specified degree whose graph is shown.

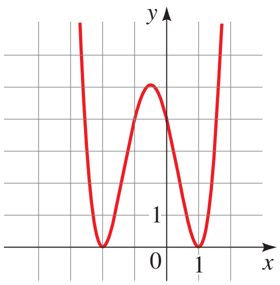
71. Degree 3



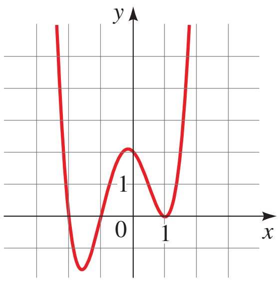
72. Degree 3



73. Degree 4



74. Degree 4



■ Discuss ■ Discover ■ Prove ■ Write

75. Discuss ■ Discover: Impossible Division? The following problems involve polynomials of very high degree. At first glance, it may seem impossible (or at least very time consuming) to solve them, but they can be solved quickly. Solve each problem.

(a) Find the remainder when $6x^{1000} - 17x^{562} + 12x + 26$ is divided by $x + 1$.

(b) Is $x - 1$ a factor of $x^{567} - 3x^{400} + x^9 + 1$?

PS *Try to recognize something familiar.* Use one or more of the theorems you have learned in this section to solve the problems without dividing.

76. Discover: Nested Form of a Polynomial Expand Q to prove that the polynomials P and Q are the same.

$$P(x) = 3x^4 - 5x^3 + x^2 - 3x + 5$$
$$Q(x) = (((3x - 5)x + 1)x - 3)x + 5$$

Try to evaluate $P(2)$ and $Q(2)$ in your head, using the forms given. Which is easier? Now write the polynomial

$$R(x) = x^5 - 2x^4 + 3x^3 - 2x^2 + 3x + 4$$

in "nested" form, like the polynomial Q. Use the nested form to find $R(3)$ in your head.

Do you see how calculating with the nested form follows the same arithmetic steps as calculating the value of a polynomial using synthetic division?

3.4 Real Zeros of Polynomials

■ Rational Zeros of Polynomials ■ Descartes's Rule of Signs ■ Upper and Lower Bounds Theorem ■ Using Algebra and Graphing Devices to Solve Polynomial Equations

The Factor Theorem tells us that finding the zeros of a polynomial is really the same thing as factoring it into linear factors. In this section we study some algebraic methods that help us find the real zeros of a polynomial and thereby factor the polynomial. We begin with the *rational* zeros of a polynomial.

■ Rational Zeros of Polynomials

To help us understand the next theorem, let's consider the polynomial

$$P(x) = (x - 2)(x - 3)(x + 4) \qquad \text{Factored form}$$
$$= x^3 - x^2 - 14x + 24 \qquad \text{Expanded form}$$

From the factored form we see that the zeros of P are 2, 3, and -4. When the polynomial is expanded, the constant 24 is obtained by multiplying $(-2) \times (-3) \times 4$. This means that the zeros of the polynomial are all factors of the constant term. The following theorem generalizes this observation.

We say that p/q is in *lowest terms* if p and q have no factor in common, other than 1.

Rational Zeros Theorem

If the polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ has integer coefficients (where $a_n \neq 0$ and $a_0 \neq 0$), then every rational zero of P is of the form

$$\frac{p}{q}$$

where p and q are integers with p/q in lowest terms, and

p is a factor of the constant coefficient a_0

q is a factor of the leading coefficient a_n

Proof If p/q is a rational zero, in lowest terms, of the polynomial P, then we have

$$a_n\left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \cdots + a_1\left(\frac{p}{q}\right) + a_0 = 0$$

$$a_n p^n + a_{n-1}p^{n-1}q + \cdots + a_1 p q^{n-1} + a_0 q^n = 0 \qquad \text{Multiply by } q^n$$

$$p(a_n p^{n-1} + a_{n-1}p^{n-2}q + \cdots + a_1 q^{n-1}) = -a_0 q^n \qquad \text{Subtract } a_0q^n \text{ and factor LHS}$$

Now p is a factor of the left side, so it must be a factor of the right side as well. Since p/q is in lowest terms, p and q have no factor in common, so p must be a factor of a_0. A similar proof shows that q is a factor of a_n. ■

We see from the Rational Zeros Theorem that if the leading coefficient is 1 or -1, then the rational zeros must be factors of the constant term.

Example 1 ■ Using the Rational Zeros Theorem

Find the rational zeros of $P(x) = x^3 - 3x + 2$.

Solution Since the leading coefficient is 1, any rational zero must be a divisor of the constant term 2. So the possible rational zeros are ± 1 and ± 2. We test each of these possibilities.

$$P(1) = (1)^3 - 3(1) + 2 = 0$$

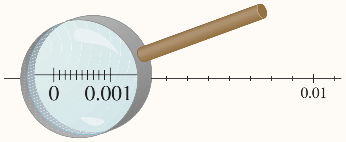
$$P(-1) = (-1)^3 - 3(-1) + 2 = 4$$

$$P(2) = (2)^3 - 3(2) + 2 = 4$$

$$P(-2) = (-2)^3 - 3(-2) + 2 = 0$$

The rational zeros of P are 1 and -2.

Now Try Exercise 15 ■



Discovery Project ■ Zeroing in on a Zero

We have learned how to find the zeros of a polynomial function algebraically and graphically. In this project we investigate a *numerical* method for finding the zeros of a polynomial. With this method we can approximate the zeros of a polynomial to any number of decimal places. The method involves finding smaller and smaller intervals that zoom in on a zero of a polynomial. You can find the project at **www.stewartmath.com**.

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EVARISTE GALOIS (1811–1832) is one of the very few mathematicians to have an entire theory named in their honor. Not yet 21 when he died, he completely settled the central problem in the theory of equations by describing a criterion that reveals whether a polynomial equation can be solved by algebraic operations. Galois recognized that he was one of the greatest mathematicians in the world at that time, although no one knew it. He repeatedly sent his work to the eminent mathematicians Cauchy and Poisson, who either lost his letters or did not understand his ideas. Galois wrote in a terse style and included few details, which probably played a role in his failure to pass the entrance exams at the Ecole Polytechnique in Paris. A political radical, Galois spent several months in prison for his revolutionary activities. His brief life came to a tragic end when he was killed in a duel over a love affair. The night before his duel, fearing that he would die, Galois wrote down the essence of his ideas and entrusted them to his friend Auguste Chevalier. He concluded by writing "there will, I hope, be people who will find it to their advantage to decipher all this mess." The mathematician Camille Jordan did just that, 14 years later.

The following box explains how we use the Rational Zeros Theorem with synthetic division to factor a polynomial.

Finding the Rational Zeros of a Polynomial

1. **List Possible Zeros.** List all possible rational zeros, using the Rational Zeros Theorem.
2. **Divide.** Use synthetic division to evaluate the polynomial at candidates for the rational zeros that you found in Step 1, until you get a remainder 0. Note the quotient you have obtained when you get a remainder 0.
3. **Repeat.** Repeat Steps 1 and 2 for the quotient. Stop when you reach a quotient that is quadratic or factors easily, and use the quadratic formula or factor to find the remaining zeros.

Example 2 ■ Finding Rational Zeros

Write the polynomial $P(x) = 2x^3 + x^2 - 13x + 6$ in factored form, and find all its zeros.

Solution By the Rational Zeros Theorem the rational zeros of P are of the form

$$\text{possible rational zero of } P = \frac{\text{factor of constant term}}{\text{factor of leading coefficient}} = \frac{\text{factor of 6}}{\text{factor of 2}}$$

The factors of 6 are $\pm 1, \pm 2, \pm 3, \pm 6$, and the factors of 2 are $\pm 1, \pm 2$. Thus the possible rational zeros of P are

$$\pm\frac{1}{1}, \quad \pm\frac{2}{1}, \quad \pm\frac{3}{1}, \quad \pm\frac{6}{1}, \quad \pm\frac{1}{2}, \quad \pm\frac{2}{2}, \quad \pm\frac{3}{2}, \quad \pm\frac{6}{2}$$

Simplifying the fractions and eliminating duplicates, we get the following list of possible rational zeros:

$$\pm 1, \quad \pm 2, \quad \pm 3, \quad \pm 6, \quad \pm\frac{1}{2}, \quad \pm\frac{3}{2}$$

Now we use synthetic division to check these *possible* zeros until we find an *actual* zero (that is, when we get a remainder of 0).

Test whether 1 is a zero

$$\begin{array}{r|rrrr} 1 & 2 & 1 & -13 & 6 \\ & & 2 & 3 & -10 \\ \hline & 2 & 3 & -10 & -4 \end{array}$$

Remainder is *not* 0, so 1 is *not* a zero

Test whether 2 is a zero

$$\begin{array}{r|rrrr} 2 & 2 & 1 & -13 & 6 \\ & & 4 & 10 & -6 \\ \hline & 2 & 5 & -3 & 0 \end{array}$$

Remainder *is* 0, so 2 *is* a zero

From the last synthetic division we see that 2 is a zero of P and that P factors as

$$\begin{aligned} P(x) &= 2x^3 + x^2 - 13x + 6 && \text{Given polynomial} \\ &= (x - 2)(2x^2 + 5x - 3) && \text{From synthetic division} \\ &= (x - 2)(2x - 1)(x + 3) && \text{Factor } 2x^2 + 5x - 3 \end{aligned}$$

From the factored form we see that the zeros of P are 2, $\frac{1}{2}$, and –3.

Now Try Exercise 29 ■

Example 3 ■ Using the Rational Zeros Theorem and the Quadratic Formula

Let $P(x) = x^4 - 5x^3 - 5x^2 + 23x + 10$.

(a) Find the zeros of P. **(b)** Sketch a graph of P.

Solution

$$\begin{array}{r|rrrrr} 1 & 1 & -5 & -5 & 23 & 10 \\ & & 1 & -4 & -9 & 14 \\ \hline & 1 & -4 & -9 & 14 & 24 \end{array}$$

$$\begin{array}{r|rrrrr} 2 & 1 & -5 & -5 & 23 & 10 \\ & & 2 & -6 & -22 & 2 \\ \hline & 1 & -3 & -11 & 1 & 12 \end{array}$$

$$\begin{array}{r|rrrrr} 5 & 1 & -5 & 5 & 23 & 10 \\ & & 5 & 0 & -25 & -10 \\ \hline & 1 & 0 & -5 & -2 & 0 \end{array}$$

(a) The leading coefficient of P is 1, so all the rational zeros are integers: They are divisors of the constant term 10. Thus the possible candidates are

$$\pm 1, \quad \pm 2, \quad \pm 5, \quad \pm 10$$

Using synthetic division (see the margin), we find that 1 and 2 are not zeros but that 5 is a zero and that P factors as

$$x^4 - 5x^3 - 5x^2 + 23x + 10 = (x - 5)(x^3 - 5x - 2)$$

We now try to factor the quotient $x^3 - 5x - 2$. Its possible zeros are the divisors of -2, namely,

$$\pm 1, \quad \pm 2$$

$$\begin{array}{r|rrrr} -2 & 1 & 0 & -5 & -2 \\ & & -2 & 4 & 2 \\ \hline & 1 & -2 & -1 & 0 \end{array}$$

Since we already know that 1 and 2 are not zeros of the original polynomial P, we don't need to try them again. Checking the remaining candidates, -1 and -2, we see that -2 is a zero (see the margin), and P factors as

$$\begin{aligned} x^4 - 5x^3 - 5x^2 + 23x + 10 &= (x - 5)(x^3 - 5x - 2) \\ &= (x - 5)(x + 2)(x^2 - 2x - 1) \end{aligned}$$

Now we use the Quadratic Formula to solve $x^2 - 2x - 1 = 0$ and obtain the two remaining zeros of P:

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2} = 1 \pm \sqrt{2}$$

The zeros of P are 5, -2, $1 + \sqrt{2}$, and $1 - \sqrt{2}$.

(b) Now that we know the zeros of P, we can use the methods of Section 3.2 to sketch the graph. If we want to use a graphing device instead, then knowing the zeros allows us to choose an appropriate viewing rectangle—one that is wide enough to contain all the x-intercepts of P. The zeros of P are

$$5, \qquad -2, \qquad 1 + \sqrt{2} \approx 2.41, \qquad 1 - \sqrt{2} \approx -0.41$$

So we choose the viewing rectangle $[-3, 6]$ by $[-50, 50]$ and draw the graph shown in Figure 1.

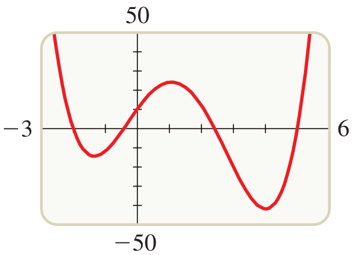


Figure 1 | $P(x) = x^4 - 5x^3 - 5x^2 + 23x + 10$

Now Try Exercises 45 and 55 ■

■ Descartes's Rule of Signs

In some cases, the following rule—discovered by the French philosopher and mathematician René Descartes around 1637 (see Section 2.6)—is helpful in eliminating candidates from lengthy lists of possible rational roots. To describe this rule, we need the concept of *variation in sign*. If $P(x)$ is a polynomial with real coefficients, written with descending powers of x (and omitting powers with coefficient 0), then a **variation in sign** occurs whenever adjacent coefficients have opposite signs. For example,

$$P(x) = 5x^7 - 3x^5 - x^4 + 2x^2 + x - 3$$

has three variations in sign.

Polynomial	Variations in Sign
$x^2 + 4x + 1$	0
$2x^3 + x - 6$	1
$x^4 - 3x^2 - x + 4$	2

Descartes's Rule of Signs

Let P be a polynomial with real coefficients.

1. The number of positive real zeros of $P(x)$ either is equal to the number of variations in sign in $P(x)$ or is less than that by an even whole number.
2. The number of negative real zeros of $P(x)$ either is equal to the number of variations in sign in $P(-x)$ or is less than that by an even whole number.

Multiplicity is discussed in Section 3.2.

In Descartes's Rule of Signs a zero with multiplicity m is counted m times. For example, the polynomial $P(x) = x^2 - 2x + 1$ has two sign changes and has the positive zero $x = 1$. But this zero is counted twice because it has multiplicity 2.

Example 4 ■ Using Descartes's Rule

Use Descartes's Rule of Signs to determine the possible number of positive and negative real zeros of the polynomial

$$P(x) = 3x^6 + 4x^5 + 3x^3 - x - 3$$

Solution The polynomial has one variation in sign, so it has one positive zero. Now

$$\begin{aligned} P(-x) &= 3(-x)^6 + 4(-x)^5 + 3(-x)^3 - (-x) - 3 \\ &= 3x^6 - 4x^5 - 3x^3 + x - 3 \end{aligned}$$

So $P(-x)$ has three variations in sign. Thus $P(x)$ has either three or one negative zero(s), making a total of either two or four real zeros.

Now Try Exercise 63 ■

■ Upper and Lower Bounds Theorem

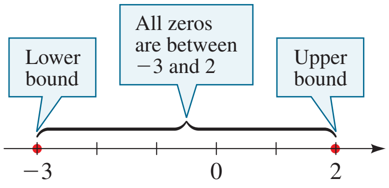
We say that a is a **lower bound** and b is an **upper bound** for the zeros of a polynomial if every real zero c of the polynomial satisfies $a \le c \le b$. The next theorem helps us find such bounds for the zeros of a polynomial.

The Upper and Lower Bounds Theorem

Let P be a polynomial with real coefficients.

1. If we divide $P(x)$ by $x - b$ (with $b > 0$) using synthetic division and if the row that contains the quotient and remainder has no negative entry, then b is an upper bound for the real zeros of P.
2. If we divide $P(x)$ by $x - a$ (with $a < 0$) using synthetic division and if the row that contains the quotient and remainder has entries that are alternately nonpositive and nonnegative, then a is a lower bound for the real zeros of P.

A proof of this theorem is suggested in Exercise 108. The phrase "alternately nonpositive and nonnegative" simply means that the signs of the numbers alternate, with 0 considered to be positive or negative as required.



Example 5 ■ Upper and Lower Bounds for the Zeros of a Polynomial

Show that all the real zeros of the polynomial $P(x) = x^4 - 3x^2 + 2x - 5$ lie between -3 and 2.

Solution We divide $P(x)$ by $x - 2$ and $x + 3$ using synthetic division:

2	1	0	−3	2	−5
		2	4	2	8
	1	2	1	4	3

All entries nonnegative

−3	1	0	−3	2	−5
		−3	9	−18	48
	1	−3	6	−16	43

Entries alternate in sign

By the Upper and Lower Bounds Theorem -3 is a lower bound and 2 is an upper bound for the zeros. Since neither -3 nor 2 is a zero (the remainders are not 0 in the division table), all the real zeros lie between these numbers.

Now Try Exercise 69 ■

Example 6 ■ A Lower Bound for the Zeros of a Polynomial

Show that all the real zeros of the polynomial $P(x) = x^4 + 4x^3 + 3x^2 + 7x - 5$ are greater than or equal to -4.

Solution We divide $P(x)$ by $x + 4$ using synthetic division:

−4	1	4	3	7	−5
		−4	0	−12	20
	1	0	3	−5	15

Alternately nonnegative and nonpositive

Since 0 can be considered either nonnegative or nonpositive, the entries alternate in sign. So -4 is a lower bound for the real zeros of P.

Now Try Exercise 73 ■

Example 7 ■ Factoring a Fifth-Degree Polynomial

Factor completely the polynomial

$$P(x) = 2x^5 + 5x^4 - 8x^3 - 14x^2 + 6x + 9$$

Solution The possible rational zeros of P are $\pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \pm 3, \pm\frac{9}{2}$, and ± 9. We check the positive candidates first, beginning with the smallest:

$\frac{1}{2}$	2	5	−8	−14	6	9
		1	3	$-\frac{5}{2}$	$-\frac{33}{4}$	$-\frac{9}{8}$
	2	6	−5	$-\frac{33}{2}$	$-\frac{9}{4}$	$\frac{63}{8}$

$\frac{1}{2}$ is not a zero

1	2	5	−8	−14	6	9
		2	7	−1	−15	−9
	2	7	−1	−15	−9	0

$P(1) = 0$

So 1 is a zero, and $P(x) = (x - 1)(2x^4 + 7x^3 - x^2 - 15x - 9)$. We continue by factoring the quotient. We still have the same list of possible zeros except that $\frac{1}{2}$ has been eliminated.

$$\begin{array}{r|rrrrr} 1 & 2 & 7 & -1 & -15 & -9 \\ & & 2 & 9 & 8 & -7 \\ \hline & 2 & 9 & 8 & -7 & -16 \end{array}$$

1 is not a zero

$$\begin{array}{r|rrrrr} \frac{3}{2} & 2 & 7 & -1 & -15 & -9 \\ & & 3 & 15 & 21 & 9 \\ \hline & 2 & 10 & 14 & 6 & 0 \end{array}$$

$P(\frac{3}{2}) = 0$, all entries nonnegative

We see that $\frac{3}{2}$ is both a zero and an upper bound for the zeros of $P(x)$, so we do not need to check any further for positive zeros, because all the remaining candidates are greater than $\frac{3}{2}$.

$$\begin{aligned} P(x) &= (x - 1)(x - \tfrac{3}{2})(2x^3 + 10x^2 + 14x + 6) && \text{From synthetic division} \\ &= (x - 1)(2x - 3)(x^3 + 5x^2 + 7x + 3) && \text{Factor 2 from last factor, multiply into second factor} \end{aligned}$$

By Descartes's Rule of Signs, $x^3 + 5x^2 + 7x + 3$ has no positive zero, so its only possible rational zeros are -1 and -3:

$$\begin{array}{r|rrrr} -1 & 1 & 5 & 7 & 3 \\ & & -1 & -4 & -3 \\ \hline & 1 & 4 & 3 & 0 \end{array}$$

$P(-1) = 0$

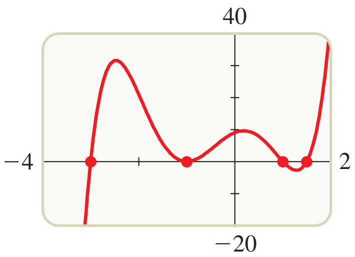


Figure 2 | $P(x) = 2x^5 + 5x^4 - 8x^3 - 14x^2 + 6x + 9$ $= (x - 1)(2x - 3)(x + 1)^2(x + 3)$

Therefore,

$$\begin{aligned} P(x) &= (x - 1)(2x - 3)(x + 1)(x^2 + 4x + 3) && \text{From synthetic division} \\ &= (x - 1)(2x - 3)(x + 1)^2(x + 3) && \text{Factor quadratic} \end{aligned}$$

This means that the zeros of P are 1, $\frac{3}{2}$, -1 (multiplicity 2), and -3. The graph of the polynomial is shown in Figure 2.

Now Try Exercise 81 ■

■ Using Algebra and Graphing Devices to Solve Polynomial Equations

In Section 1.11 we used graphing devices to solve equations graphically. When solving a polynomial equation graphically, we can use the algebraic techniques we've learned in this section to select an appropriate viewing rectangle (that is, a viewing rectangle that contains all the zeros of the polynomial).

Example 8 ■ Solving a Fourth-Degree Equation Graphically

Find all real solutions of the following equation, rounded to the nearest tenth:

$$3x^4 + 4x^3 - 7x^2 - 2x - 3 = 0$$

Solution To solve the equation graphically, we graph

$$P(x) = 3x^4 + 4x^3 - 7x^2 - 2x - 3$$

We use the Upper and Lower Bounds Theorem to see where the solutions can be found.

First we use the Upper and Lower Bounds Theorem to find two numbers between which all the solutions must lie. This allows us to choose a viewing rectangle that is certain to contain all the x-intercepts of P. We use synthetic division and proceed by trial and error.

To find an upper bound, we try the whole numbers, 1, 2, 3, . . . , as potential candidates. We see that 2 is an upper bound for the solutions:

$$\begin{array}{r|rrrrr} 2 & 3 & 4 & -7 & -2 & -3 \\ & & 6 & 20 & 26 & 48 \\ \hline & 3 & 10 & 13 & 24 & 45 \end{array}$$

All positive

Now we look for a lower bound, trying the numbers -1, -2, and $-3, \ldots,$ as potential candidates. We see that -3 is a lower bound for the solutions:

$$\begin{array}{r|rrrrr} -3 & 3 & 4 & -7 & -2 & -3 \\ & & -9 & 15 & -24 & 78 \\ \hline & 3 & -5 & 8 & -26 & 75 \end{array}$$

Entries alternate in sign

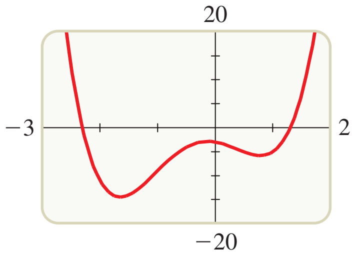


Figure 3 | $y = 3x^4 + 4x^3 - 7x^2 - 2x - 3$

Thus all the solutions lie between -3 and 2. So the viewing rectangle $[-3, 2]$ by $[-20, 20]$ contains all the x-intercepts of P. The graph in Figure 3 has two x-intercepts, one between -3 and -2 and the other between 1 and 2. Zooming in, we find that the solutions of the equation, to the nearest tenth, are -2.3 and 1.3.

Now Try Exercise 95 ■

Example 9 ■ Determining the Size of a Fuel Tank

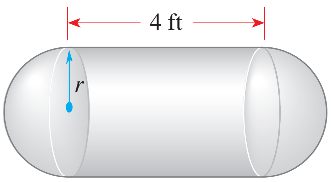


Figure 4

A fuel tank consists of a cylindrical center section that is 4 ft long and two hemispherical end sections, as shown in Figure 4. If the tank has a volume of 100 ft^3, what is the radius r (shown in the figure), rounded to the nearest hundredth of a foot?

Solution Using the volume formula listed on the inside front cover of this book, we see that the volume of the cylindrical section of the tank is

Volume of a cylinder: $V = \pi r^2 h$

$$\pi \cdot r^2 \cdot 4$$

The two hemispherical parts together form a complete sphere whose volume is

Volume of a sphere: $V = \frac{4}{3}\pi r^3$

$$\tfrac{4}{3}\pi r^3$$

Because the total volume of the tank is 100 ft^3, we get the following equation:

$$\tfrac{4}{3}\pi r^3 + 4\pi r^2 = 100$$

A negative solution for r would be meaningless in this physical situation, and by substitution we can verify that $r = 3$ leads to a tank that is over 226 ft^3 in volume, much larger than the required 100 ft^3. Thus we know the correct radius lies somewhere between 0 and 3 ft, so we use a viewing rectangle of $[0, 3]$ by $[50, 150]$ to graph the function $y = \frac{4}{3}\pi x^3 + 4\pi x^2$, as shown in Figure 5. Since we want the value of this function to be 100, we also graph the horizontal line $y = 100$ in the same viewing rectangle. The correct radius will be the x-coordinate of the point of intersection of the curve and the line. We see that at the point of intersection the x-coordinate is about $x \approx 2.15$. Thus the tank has a radius of about 2.15 ft.

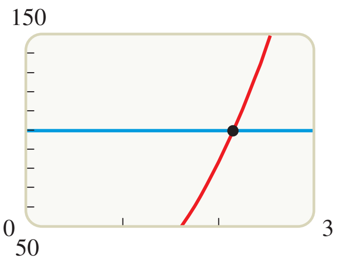


Figure 5 | $y = \frac{4}{3}\pi x^3 + 4\pi x^2$ and $y = 100$

Now Try Exercise 99 ■

Note that we also could have solved the equation in Example 9 by first writing it as

$$\tfrac{4}{3}\pi r^3 + 4\pi r^2 - 100 = 0$$

and then finding the x-intercept of the function $y = \frac{4}{3}\pi x^3 + 4\pi x^2 - 100$.

3.4 Exercises

Concepts

1. If the polynomial function

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

has integer coefficients, then the only numbers that could possibly be rational zeros of P are all of the form $\dfrac{p}{q}$, where p is a factor of ________ and q is a factor of ________. The possible rational zeros of $P(x) = 6x^3 + 5x^2 - 19x - 10$ are

________________.

2. Using Descartes's Rule of Signs, we can tell that the polynomial $P(x) = x^5 - 3x^4 + 2x^3 - x^2 + 8x - 8$ has

________, ________, or ________ positive real zeros and

________ negative real zeros.

3. *True or False*? If c is a real zero of the polynomial P, then all the other zeros of P are zeros of $P(x)/(x - c)$.

4. *True or False*? If a is an upper bound for the real zeros of the polynomial P, then $-a$ is necessarily a lower bound for the real zeros of P.

Skills

5–10 ■ Possible Rational Zeros List all possible rational zeros given by the Rational Zeros Theorem (but don't check to see which actually are zeros).

5. $P(x) = x^3 - 4x + 6$

6. $Q(x) = x^5 - 3x^2 + 5x + 10$

7. $R(x) = 3x^4 - 2x^3 + 8x^2 - 9$

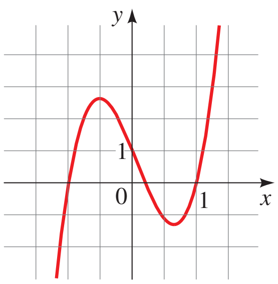
8. $S(x) = 5x^6 - 3x^4 + 20x^2 - 15$

9. $T(x) = 6x^5 - 8x^3 + 5$

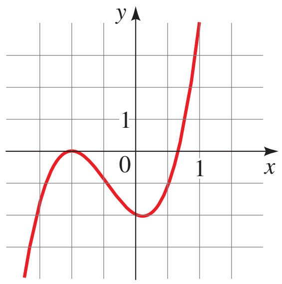
10. $U(x) = 12x^5 + 6x^3 - 2x - 8$

11–14 ■ Possible Rational Zeros A polynomial function P and its graph are given. **(a)** List all possible rational zeros of P given by the Rational Zeros Theorem. **(b)** From the graph, determine which of the possible rational zeros are actually zeros.

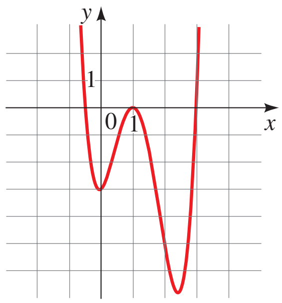
11. $P(x) = 5x^3 - x^2 - 5x + 1$



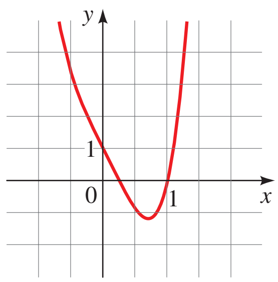
12. $P(x) = 3x^3 + 4x^2 - x - 2$



13. $P(x) = 2x^4 - 9x^3 + 9x^2 + x - 3$



14. $P(x) = 4x^4 - x^3 - 4x + 1$



15–28 ■ Integer Zeros All the real zeros of the given polynomial are integers. Find the zeros, and write the polynomial in factored form.

15. $P(x) = x^3 - 5x^2 - 8x + 12$

16. $P(x) = x^3 - 4x^2 - 19x - 14$

17. $P(x) = x^3 - 5x^2 + 3x + 9$

18. $P(x) = x^3 - 3x - 2$

19. $P(x) = x^3 - 6x^2 + 12x - 8$

20. $P(x) = x^3 + 12x^2 + 48x + 64$

21. $P(x) = x^3 - 27x + 54$

22. $P(x) = x^3 + 5x^2 - 9x - 45$

23. $P(x) = x^3 + 3x^2 - x - 3$

24. $P(x) = x^3 - 4x^2 - 11x + 30$

25. $P(x) = x^4 - 5x^2 + 4$

26. $P(x) = x^4 - 2x^3 - 3x^2 + 8x - 4$

27. $P(x) = x^4 + 6x^3 + 7x^2 - 6x - 8$

28. $P(x) = x^4 - x^3 - 23x^2 - 3x + 90$

29–44 ■ Rational Zeros Find all rational zeros of the polynomial, and write the polynomial in factored form.

29. $P(x) = 9x^4 - 82x^2 + 9$

30. $P(x) = 6x^4 - 23x^3 - 13x^2 + 32x + 16$

31. $P(x) = 6x^4 + 7x^3 - 9x^2 - 7x + 3$

32. $P(x) = 6x^3 + 37x^2 + 5x - 6$

33. $P(x) = 4x^3 + 4x^2 - x - 1$

34. $P(x) = 2x^3 - 3x^2 - 2x + 3$

35. $P(x) = 4x^3 - 7x + 3$

36. $P(x) = 12x^3 - 25x^2 + x + 2$

37. $P(x) = 24x^3 + 10x^2 - 13x - 6$

38. $P(x) = 12x^3 - 20x^2 + x + 3$

39. $P(x) = 6x^4 + 13x^3 - 32x^2 - 45x + 18$

40. $P(x) = 2x^4 + 11x^3 + 11x^2 - 15x - 9$

41. $P(x) = x^5 + 3x^4 - 9x^3 - 31x^2 + 36$

42. $P(x) = x^5 - 4x^4 - 3x^3 + 22x^2 - 4x - 24$

43. $P(x) = 3x^5 - 14x^4 - 14x^3 + 36x^2 + 43x + 10$

44. $P(x) = 2x^6 - 3x^5 - 13x^4 + 29x^3 - 27x^2 + 32x - 12$

45–54 ■ Real Zeros of a Polynomial Find all the real zeros of the polynomial. Use the Quadratic Formula if necessary, as in Example 3(a).

45. $P(x) = 3x^3 + 5x^2 - 2x - 4$

46. $P(x) = 3x^4 - 5x^3 - 16x^2 + 7x + 15$

47. $P(x) = x^4 - 6x^3 + 4x^2 + 15x + 4$

48. $P(x) = x^4 + 2x^3 - 2x^2 - 3x + 2$

49. $P(x) = x^4 - 7x^3 + 14x^2 - 3x - 9$

50. $P(x) = x^5 - 4x^4 - x^3 + 10x^2 + 2x - 4$

51. $P(x) = 4x^3 - 6x^2 + 1$

52. $P(x) = 3x^3 - 5x^2 - 8x - 2$

53. $P(x) = 2x^4 + 15x^3 + 17x^2 + 3x - 1$

54. $P(x) = 4x^5 - 18x^4 - 6x^3 + 91x^2 - 60x + 9$

55–62 ■ Real Zeros of a Polynomial A polynomial P is given. **(a)** Find all the real zeros of P. **(b)** Sketch a graph of P.

55. $P(x) = -x^3 - 3x^2 + x + 3$

56. $P(x) = x^3 + 3x^2 - 6x - 8$

57. $P(x) = 2x^3 - 7x^2 + 4x + 4$

58. $P(x) = 3x^3 + 17x^2 + 21x - 9$

59. $P(x) = x^4 - 5x^3 + 6x^2 + 4x - 8$

60. $P(x) = -x^4 + 10x^2 + 8x - 8$

61. $P(x) = x^5 - x^4 - 5x^3 + x^2 + 8x + 4$

62. $P(x) = x^5 + 2x^4 - 8x^3 - 16x^2 + 16x + 32$

63–68 ■ Descartes's Rule of Signs Use Descartes's Rule of Signs to determine how many positive and how many negative real zeros the polynomial can have. Then determine the possible total number of real zeros.

63. $P(x) = x^3 - x^2 - x - 3$

64. $P(x) = 2x^3 - x^2 + 4x - 7$

65. $P(x) = 2x^6 + 5x^4 - x^3 - 5x - 1$

66. $P(x) = x^4 + x^3 + x^2 + x + 12$

67. $P(x) = x^5 + 4x^3 - x^2 + 6x$

68. $P(x) = x^8 - x^5 + x^4 - x^3 + x^2 - x + 1$

69–76 ■ Upper and Lower Bounds Show that the given values for a and b are lower and upper bounds for the real zeros of the polynomial.

69. $P(x) = 2x^3 + 5x^2 + x - 2$; $a = -3, b = 1$

70. $P(x) = x^4 - 2x^3 - 9x^2 + 2x + 8$; $a = -3, b = 5$

71. $P(x) = 8x^3 + 10x^2 - 39x + 9$; $a = -3, b = 2$

72. $P(x) = 3x^4 - 17x^3 + 24x^2 - 9x + 1$; $a = 0, b = 6$

73. $P(x) = x^4 + 2x^3 + 3x^2 + 5x - 1$; $a = -2, b = 1$

74. $P(x) = x^4 + 3x^3 - 4x^2 - 2x - 7$; $a = -4, b = 2$

75. $P(x) = 2x^4 - 6x^3 + x^2 - 2x + 3$; $a = -1, b = 3$

76. $P(x) = 3x^4 - 5x^3 - 2x^2 + x - 1$; $a = -1, b = 2$

77–80 ■ Upper and Lower Bounds Find integers that are upper and lower bounds for the real zeros of the polynomial.

77. $P(x) = x^3 - 3x^2 + 4$

78. $P(x) = 2x^3 - 3x^2 - 8x + 12$

79. $P(x) = x^4 - 2x^3 + x^2 - 9x + 2$

80. $P(x) = x^5 - x^4 + 1$

81–86 ■ Zeros of a Polynomial Find all rational zeros of the polynomial, and then find the irrational zeros, if any. Whenever appropriate, use the Rational Zeros Theorem, the Upper and Lower Bounds Theorem, Descartes's Rule of Signs, the Quadratic Formula, or other factoring techniques.

81. $P(x) = 2x^4 + 3x^3 - 4x^2 - 3x + 2$

82. $P(x) = 2x^4 + 15x^3 + 31x^2 + 20x + 4$

83. $P(x) = 4x^4 - 21x^2 + 5$

84. $P(x) = 6x^4 - 7x^3 - 8x^2 + 5x$

85. $P(x) = x^5 - 7x^4 + 9x^3 + 23x^2 - 50x + 24$

86. $P(x) = 8x^5 - 14x^4 - 22x^3 + 57x^2 - 35x + 6$

87–90 ■ Polynomials With No Rational Zeros Show that the polynomial does not have any rational zeros.

87. $P(x) = x^3 - x - 2$

88. $P(x) = 2x^4 - x^3 + x + 2$

89. $P(x) = 3x^3 - x^2 - 6x + 12$

90. $P(x) = x^{50} - 5x^{25} + x^2 - 1$

91–94 ■ Verifying Zeros Using a Graphing Device The real solutions of the given equation are rational. Use a graph of the polynomial in the given viewing rectangle and the list of possible rational zeros (given by the Rational Zeros Theorem) to determine which values in the list are actually solutions. (All solutions can be seen in the given viewing rectangle.)

91. $x^3 - 3x^2 - 4x + 12 = 0$; $[-4, 4]$ by $[-15, 15]$

92. $x^4 - 5x^2 + 4 = 0$; $[-4, 4]$ by $[-30, 30]$

93. $2x^4 - 5x^3 - 14x^2 + 5x + 12 = 0$; $[-2, 5]$ by $[-40, 40]$

94. $3x^3 + 8x^2 + 5x + 2 = 0$; $[-3, 3]$ by $[-10, 10]$

95–98 ■ Finding Zeros Using a Graphing Device Use a graphing device to find all real solutions of the equation, rounded to two decimal places.

95. $x^4 - x - 4 = 0$

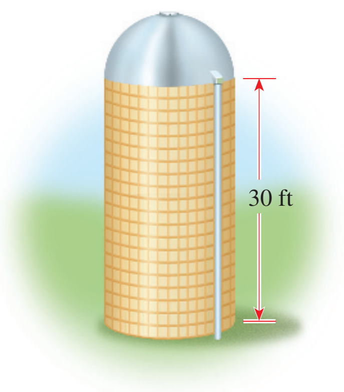
96. $2x^3 - 8x^2 + 9x - 9 = 0$

97. $4.00x^4 + 4.00x^3 - 10.96x^2 - 5.88x + 9.09 = 0$

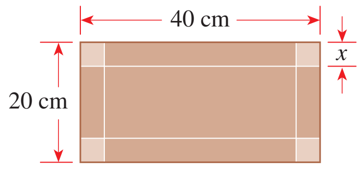
98. $x^5 + 2.00x^4 + 0.96x^3 + 5.00x^2 + 10.00x + 4.80 = 0$

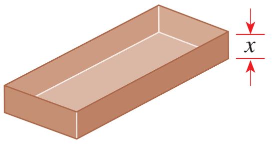
Applications

99. Volume of a Silo A grain silo consists of a cylindrical main section surmounted by a hemispherical roof. If the total volume of the silo (including the part inside the roof section) is 15,000 ft^3 and the cylindrical part is 30 ft tall, what is the radius of the silo, rounded to the nearest tenth of a foot?

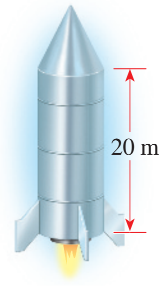


100. Volume of a Box An open box with a volume of 1500 cm^3 is to be constructed by taking a piece of cardboard 20 cm by 40 cm, cutting squares of side length x cm from each corner, and folding up the sides. Show that this can be done in two different ways, and find the exact dimensions of the box in each case.





101. Volume of a Rocket A rocket consists of a right circular cylinder of height 20 m surmounted by a cone whose height and diameter are equal and whose radius is the same as that of the cylindrical section. What should this radius be (rounded to two decimal places) if the total volume is to be $500\pi/3$ m^3?

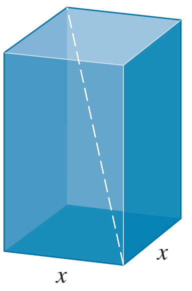


102. Volume of a Box A rectangular box with a volume of $2\sqrt{2}$ ft^3 has a square base as shown below. The diagonal of the box (between a pair of opposite corners) is 1 ft longer than each side of the base.

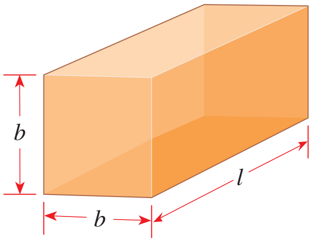
(a) If the base has sides of length x feet, show that

$$x^6 - 2x^5 - x^4 + 8 = 0$$

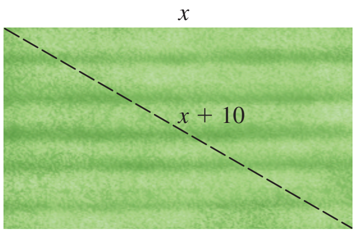
(b) Show that two different boxes satisfy the given conditions. Find the dimensions in each case, rounded to the nearest hundredth of a foot.



103. Girth of a Box A box with a square base has length plus girth of 108 in. (Girth is the distance "around" the box.) What is the length of the box if its volume is 2200 in^3?



104. Dimensions of a Lot A rectangular parcel of land has an area of 5000 ft^2. A diagonal between opposite corners is measured to be 10 ft longer than one side of the parcel. What are the dimensions of the land, rounded to the nearest foot?



■ Discuss ■ Discover ■ Prove ■ Write

105. Discuss ■ Discover: How Many Real Zeros Can a Polynomial Have? Give examples of polynomials that have the following properties, or explain why it is impossible to find such a polynomial.

(a) A polynomial of degree 3 that has no real zero

(b) A polynomial of degree 4 that has no real zero

(c) A polynomial of degree 3 that has three real zeros, only one of which is rational

(d) A polynomial of degree 4 that has four real zeros, none of which is rational

What must be true about the degree of a polynomial with integer coefficients if it has no real zero?

106. Discuss ■ Prove: Depressed Cubics The most general cubic (third-degree) equation with rational coefficients can be written as

$$x^3 + ax^2 + bx + c = 0$$

(a) Show that the substitution $x = u - a/3$ transforms the general cubic into a *depressed cubic* (that is, a cubic in the variable u with no u^2 term):

$$u^3 + pu + q = 0$$

(See Exercise 1.5.142.)

(b) Use the procedure described in part (a) to depress the equation $x^3 + 6x^2 + 9x + 4 = 0$.

107. Discuss: The Cubic Formula The Quadratic Formula can be used to solve any quadratic (second-degree) equation. You might have wondered whether similar formulas exist for cubic (third-degree), quartic (fourth-degree), and higher-degree equations. For the depressed cubic $x^3 + px + q = 0$, Cardano (Section 3.5) found the following formula for one solution:

$$x = \sqrt[3]{\frac{-q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{\frac{-q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

A formula for quartic equations was discovered by the Italian mathematician Ferrari in 1540. In 1824 the Norwegian mathematician Niels Henrik Abel proved that it is impossible to write a quintic formula, that is, a formula for fifth-degree equations. Finally, Galois (see the biography in this section) gave a criterion for determining which equations can be solved by a formula involving radicals.

Use the formula given above to find a solution for the following equations. Then solve the equations using the methods you learned in this section. Which method is easier?

(a) $x^3 - 3x + 2 = 0$ **(b)** $x^3 + 3x + 4 = 0$

108. Prove: Upper and Lower Bounds Theorem Let $P(x)$ be a polynomial with real coefficients, and let $b > 0$. Use the Division Algorithm (Section 3.3) to write

$$P(x) = (x - b) \cdot Q(x) + r$$

Suppose that $r \geq 0$ and that all the coefficients in $Q(x)$ are nonnegative. Let $z > b$.

(a) Show that $P(z) > 0$.

(b) Prove the first part of the Upper and Lower Bounds Theorem.

(c) Use the first part of the Upper and Lower Bounds Theorem to prove the second part. [*Hint:* Show that if $P(x)$ satisfies the second part of the theorem, then $P(-x)$ satisfies the first part.]

109. Prove: Number of Rational and Irrational Solutions Show that the equation

$$x^5 - x^4 - x^3 - 5x^2 - 12x - 6 = 0$$

has exactly one rational solution, and then prove that it must have either two or four irrational solutions.

3.5 Complex Zeros and the Fundamental Theorem of Algebra

■ The Fundamental Theorem of Algebra and Complete Factorization ■ Zeros and Their Multiplicities ■ Complex Zeros Occur in Conjugate Pairs ■ Linear and Quadratic Factors

We have already seen that an nth-degree polynomial can have at most n real zeros. In the complex number system an nth-degree polynomial has exactly n zeros (counting multiplicity) and so can be factored into exactly n linear factors. This fact is a consequence of the Fundamental Theorem of Algebra, which was proved by the German mathematician C. F. Gauss in 1799 (see the biography in this section).

■ The Fundamental Theorem of Algebra and Complete Factorization

The following theorem is the basis for much of our work in factoring polynomials and solving polynomial equations.

Fundamental Theorem of Algebra

Every polynomial

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \qquad (n \geq 1,\ a_n \neq 0)$$

with complex coefficients has at least one complex zero.

Complex numbers are studied in Section 1.6.

Because any real number is also a complex number, the theorem applies to polynomials with real coefficients as well.

The Fundamental Theorem of Algebra and the Factor Theorem together show that a polynomial can be factored completely into linear factors, as we now prove.

Complete Factorization Theorem

If $P(x)$ is a polynomial of degree $n \geq 1$, then there exist complex numbers $a, c_1, c_2, \ldots, c_n$ (with $a \neq 0$) such that

$$P(x) = a(x - c_1)(x - c_2)\cdots(x - c_n)$$

Proof By the Fundamental Theorem of Algebra, P has at least one zero. Let's call it c_1. By the Factor Theorem (see Section 3.3), $P(x)$ can be factored as

$$P(x) = (x - c_1)Q_1(x)$$

where $Q_1(x)$ is of degree $n - 1$. Applying the Fundamental Theorem to the quotient $Q_1(x)$ gives us the factorization

$$P(x) = (x - c_1)(x - c_2)Q_2(x)$$

where $Q_2(x)$ is of degree $n - 2$ and c_2 is a zero of $Q_1(x)$. Continuing this process for n steps, we get a final quotient $Q_n(x)$ of degree 0, a nonzero constant that we will call a. This means that P has been factored as

$$P(x) = a(x - c_1)(x - c_2)\cdots(x - c_n)$$

■

To find the complex zeros of an nth-degree polynomial, we usually first factor as much as possible, then use the Quadratic Formula on parts that we can't factor further.

Example 1 ■ Factoring a Polynomial Completely

Let $P(x) = x^3 - 3x^2 + x - 3$.

(a) Find all the zeros of P.

(b) Find the complete factorization of P.

Solution

(a) We first factor P as follows.

$$\begin{aligned} P(x) &= x^3 - 3x^2 + x - 3 && \text{Given} \\ &= x^2(x - 3) + (x - 3) && \text{Group terms} \\ &= (x - 3)(x^2 + 1) && \text{Factor } x - 3 \end{aligned}$$

We find the zeros of P by setting each factor equal to 0:

$$P(x) = \underbrace{(x - 3)}_{\text{This factor is 0 when } x = 3}\underbrace{(x^2 + 1)}_{\text{This factor is 0 when } x = i \text{ or } -i}$$

Setting $x - 3 = 0$, we see that $x = 3$ is a zero. Setting $x^2 + 1 = 0$, we get $x^2 = -1$, so $x = \pm i$. So the zeros of P are 3, i, and $-i$.

(b) Since the zeros are 3, i, and $-i$, the complete factorization of P is

$$\begin{aligned} P(x) &= (x - 3)(x - i)[x - (-i)] \\ &= (x - 3)(x - i)(x + i) \end{aligned}$$

Now Try Exercise 7 ■

Example 2 ■ Factoring a Polynomial Completely

Let $P(x) = x^3 - 2x + 4$.

(a) Find all the zeros of P.

(b) Find the complete factorization of P.

Solution

$$\begin{array}{r|rrrr} -2 & 1 & 0 & -2 & 4 \\ & & -2 & 4 & -4 \\ \hline & 1 & -2 & -2 & 0 \end{array}$$

(a) The possible rational zeros are the factors of 4, which are ± 1, ± 2, ± 4. Using synthetic division (see the margin), we find that -2 is a zero, and the polynomial factors as

$$P(x) = \underbrace{(x + 2)}_{\text{This factor is 0 when } x = -2}\underbrace{(x^2 - 2x + 2)}_{\text{Use the Quadratic Formula to find when this factor is 0}}$$

To find the zeros, we set each factor equal to 0. Of course, $x + 2 = 0$ means that $x = -2$. We use the Quadratic Formula to find when the other factor is 0.

$$\begin{aligned} x^2 - 2x + 2 &= 0 && \text{Set factor equal to 0} \\ x &= \frac{2 \pm \sqrt{4 - 8}}{2} && \text{Quadratic Formula} \\ x &= \frac{2 \pm 2i}{2} && \text{Take square root} \\ x &= 1 \pm i && \text{Simplify} \end{aligned}$$

So the zeros of P are -2, $1 + i$, and $1 - i$.

(b) Since the zeros are -2, $1 + i$, and $1 - i$, the complete factorization of P is

$$\begin{aligned} P(x) &= [x - (-2)][x - (1 + i)][x - (1 - i)] \\ &= (x + 2)(x - 1 - i)(x - 1 + i) \end{aligned}$$

Now Try Exercise 9 ■

■ Zeros and Their Multiplicities

In the Complete Factorization Theorem the numbers $c_1, c_2, \ldots, c_n$ are the zeros of P. These zeros need not all be different. If the factor $x - c$ appears k times in the complete factorization of $P(x)$, then we say that c is a zero of **multiplicity** $\boldsymbol{k}$ (see Section 3.2). For example, the polynomial

$$P(x) = (x - 1)^3(x + 2)^2(x + 3)^5$$

has the following zeros:

$$1 \text{ (multiplicity 3)} \qquad -2 \text{ (multiplicity 2)} \qquad -3 \text{ (multiplicity 5)}$$

The polynomial P has the same number of zeros as its degree: It has degree 10 and has 10 zeros, provided that we count multiplicities. This is true for all polynomials, as we prove in the following theorem.

Zeros Theorem

Every polynomial of degree $n \geq 1$ has exactly n zeros, provided that a zero of multiplicity k is counted k times.

Proof Let P be a polynomial of degree n. By the Complete Factorization Theorem

$$P(x) = a(x - c_1)(x - c_2) \cdots (x - c_n)$$

Now suppose that c is any given zero of P. Then

$$P(c) = a(c - c_1)(c - c_2) \cdots (c - c_n) = 0$$

Thus by the Zero-Product Property, one of the factors $c - c_i$ must be 0, so $c = c_i$ for some i. It follows that P has exactly the n zeros $c_1, c_2, \ldots, c_n$. ■

Example 3 ■ Factoring a Polynomial with Complex Zeros

Find the complete factorization and the five zeros of the polynomial

$$P(x) = 3x^5 + 24x^3 + 48x$$

Solution Since $3x$ is a common factor, we have

$$\begin{aligned} P(x) &= 3x(x^4 + 8x^2 + 16) \\ &= 3x(x^2 + 4)^2 \end{aligned}$$

This factor is 0 when $x = 0$

This factor is 0 when $x = 2i$ or $x = -2i$

Historical/Corbis Historical/Getty Images

CARL FRIEDRICH GAUSS (1777–1855) is considered the greatest mathematician of modern times. His contemporaries called him the "Prince of Mathematics." He was born into a poor family; his father made a living as a mason. As a very small child, Gauss found a calculation error in his father's accounts, the first of many incidents that gave evidence of his mathematical precocity. (See also Section 11.2.) At 19, Gauss demonstrated that the regular 17-sided polygon can be constructed with straight-edge and compass alone. This was remarkable because, since the time of Euclid, it had been thought that the only regular polygons constructible in this way were the triangle, the square, and the pentagon. Because of this discovery Gauss decided to pursue a career in mathematics instead of languages, his other passion. In his doctoral dissertation, written at the age of 22, Gauss proved the Fundamental Theorem of Algebra: A polynomial of degree n with complex coefficients has n zeros. His other accomplishments range over every branch of mathematics as well as physics and astronomy.

To factor $x^2 + 4$, note that $2i$ and $-2i$ are zeros of this polynomial. Thus $x^2 + 4 = (x - 2i)(x + 2i)$, so

$$\begin{aligned} P(x) &= 3x[(x - 2i)(x + 2i)]^2 \\ &= 3x(x - 2i)^2(x + 2i)^2 \end{aligned}$$

0 is a zero of multiplicity 1

$2i$ is a zero of multiplicity 2

$-2i$ is a zero of multiplicity 2

The zeros of P are 0, $2i$, and $-2i$. Since the factors $x - 2i$ and $x + 2i$ each occur twice in the complete factorization of P, the zeros $2i$ and $-2i$ are each of multiplicity 2 (or *double* zeros). Thus we have found all five zeros.

Now Try Exercise 31

The following table gives further examples of polynomials with their complete factorizations and zeros.

Degree	Polynomial	Zero(s)	Number of Zeros
1	$P(x) = x - 4$	4	1
2	$P(x) = x^2 - 10x + 25$ $= (x - 5)(x - 5)$	5 (multiplicity 2)	2
3	$P(x) = x^3 + x$ $= x(x - i)(x + i)$	$0, i, -i$	3
4	$P(x) = x^4 + 18x^2 + 81$ $= (x - 3i)^2(x + 3i)^2$	$3i$ (multiplicity 2), $-3i$ (multiplicity 2)	4
5	$P(x) = x^5 - 2x^4 + x^3$ $= x^3(x - 1)^2$	0 (multiplicity 3), 1 (multiplicity 2)	5

Example 4 ■ Finding Polynomials with Specified Zeros

(a) Find a polynomial $P(x)$ of degree 4, with zeros i, $-i$, 2, and -2, and with $P(3) = 25$.

(b) Find a polynomial $Q(x)$ of degree 4, with zeros -2 and 0, where -2 is a zero of multiplicity 3.

Solution

(a) The required polynomial has the form

$$\begin{aligned} P(x) &= a(x - i)(x - (-i))(x - 2)(x - (-2)) \\ &= a(x^2 + 1)(x^2 - 4) && \text{Difference of squares} \\ &= a(x^4 - 3x^2 - 4) && \text{Multiply} \end{aligned}$$

We know that $P(3) = a(3^4 - 3 \cdot 3^2 - 4) = 50a = 25$, so $a = \frac{1}{2}$. Thus

$$P(x) = \tfrac{1}{2}x^4 - \tfrac{3}{2}x^2 - 2$$

(b) We require

$$\begin{aligned} Q(x) &= a[x - (-2)]^3(x - 0) \\ &= a(x + 2)^3x \\ &= a(x^3 + 6x^2 + 12x + 8)x && \text{Special Product Formula 4 (Section 1.3)} \\ &= a(x^4 + 6x^3 + 12x^2 + 8x) \end{aligned}$$

Since we are given no information about Q other than its zeros and their multiplicity, we can choose any number for a. If we use $a = 1$, we get

$$Q(x) = x^4 + 6x^3 + 12x^2 + 8x$$

Now Try Exercise 37 ■

Example 5 ■ Finding All the Zeros of a Polynomial

Find all four zeros of $P(x) = 3x^4 - 2x^3 - x^2 - 12x - 4$.

Solution Using the Rational Zeros Theorem from Section 3.4, we obtain the following list of possible rational zeros: $\pm 1, \pm 2, \pm 4, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{4}{3}$. Checking these using synthetic division, we find that 2 and $-\frac{1}{3}$ are zeros, and we get the following factorization.

$$\begin{aligned}
P(x) &= 3x^4 - 2x^3 - x^2 - 12x - 4 \\
&= (x - 2)(3x^3 + 4x^2 + 7x + 2) && \text{Factor } x - 2 \\
&= (x - 2)\left(x + \tfrac{1}{3}\right)(3x^2 + 3x + 6) && \text{Factor } x + \tfrac{1}{3} \\
&= 3(x - 2)\left(x + \tfrac{1}{3}\right)(x^2 + x + 2) && \text{Factor 3}
\end{aligned}$$

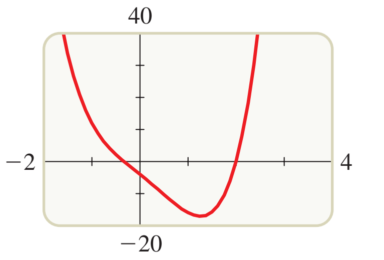


Figure 1 | $P(x) = 3x^4 - 2x^3 - x^2 - 12x - 4$

Figure 1 shows the graph of the polynomial P in Example 5. The x-intercepts correspond to the real zeros of P. The imaginary zeros cannot be determined from the graph.

The zeros of the quadratic factor are

$$x = \frac{-1 \pm \sqrt{1 - 8}}{2} = -\frac{1}{2} \pm \frac{\sqrt{7}}{2}i \qquad \text{Quadratic Formula}$$

so the zeros of $P(x)$ are

$$2, \quad -\frac{1}{3}, \quad -\frac{1}{2} + \frac{\sqrt{7}}{2}i, \quad \text{and} \quad -\frac{1}{2} - \frac{\sqrt{7}}{2}i$$

Now Try Exercise 47 ■

■ Complex Zeros Occur in Conjugate Pairs

As you may have noticed from the examples so far, the complex zeros of polynomials with real coefficients come in pairs. Whenever $a + bi$ is a zero, its complex conjugate $a - bi$ is also a zero.

Conjugate Zeros Theorem

If the polynomial P has real coefficients and if the complex number z is a zero of P, then its complex conjugate $\bar{z}$ is also a zero of P.

Proof Let

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where each coefficient is real. Suppose that $P(z) = 0$. We must prove that $P(\bar{z}) = 0$. We use the facts that the complex conjugate of a sum of two complex numbers is the sum of the conjugates and that the conjugate of a product is the product of the conjugates.

Properties of complex conjugates are stated in Exercises 1.6.77–84.

$$\begin{aligned}
P(\bar{z}) &= a_n(\bar{z})^n + a_{n-1}(\bar{z})^{n-1} + \cdots + a_1\bar{z} + a_0 \\
&= \overline{a_n}\,\overline{z^n} + \overline{a_{n-1}}\,\overline{z^{n-1}} + \cdots + \overline{a_1}\,\overline{z} + \overline{a_0} && \text{Because the coefficients are real} \\
&= \overline{a_n z^n} + \overline{a_{n-1}z^{n-1}} + \cdots + \overline{a_1 z} + \overline{a_0} \\
&= \overline{a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0} \\
&= \overline{P(z)} = \overline{0} = 0
\end{aligned}$$

This shows that $\bar{z}$ is also a zero of $P(x)$, which proves the theorem. ■

North Wind Picture Archives/Alamy Stock Photo

GEROLAMO CARDANO (1501–1576) is certainly one of the most colorful figures in the history of mathematics. He was the best-known physician in Europe in his day, yet throughout his life he was plagued by numerous maladies, including ruptures, hemorrhoids, and an irrational fear of encountering rabid dogs. He was a doting father, but his beloved sons broke his heart—his favorite was eventually beheaded for murdering his own wife. Cardano was also a compulsive gambler; indeed, this vice might have driven him to write the *Book on Games of Chance,* the first study of probability from a mathematical point of view.

In Cardano's major mathematical work, the *Ars Magna*, he detailed the solution of the general third- and fourth-degree polynomial equations. At the time of its publication, mathematicians were uncomfortable even with negative numbers, but Cardano's formulas paved the way for the acceptance not just of negative numbers, but also of imaginary numbers, because they occurred naturally in solving polynomial equations. For example, for the cubic equation

$$x^3 - 15x - 4 = 0$$

one of his formulas gives the solution

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$$

(See Exercise 3.4.107.) This value for x actually turns out to be the *integer* 4, yet to find it, Cardano had to use the imaginary number $\sqrt{-121} = 11i$.

Example 6 ■ A Polynomial With a Specified Complex Zero

Find a polynomial $P(x)$ of degree 3 that has integer coefficients and zeros $\frac{1}{2}$ and $3 - i$.

Solution Since $3 - i$ is a zero, then so is $3 + i$ by the Conjugate Zeros Theorem. This means that $P(x)$ must have the following form.

$$\begin{aligned}
P(x) &= a\left(x - \tfrac{1}{2}\right)[x - (3 - i)][x - (3 + i)] \\
&= a\left(x - \tfrac{1}{2}\right)[(x - 3) + i][(x - 3) - i] && \text{Regroup} \\
&= a\left(x - \tfrac{1}{2}\right)[(x - 3)^2 - i^2] && \text{Difference of Squares Formula} \\
&= a\left(x - \tfrac{1}{2}\right)(x^2 - 6x + 10) && \text{Expand} \\
&= a\left(x^3 - \tfrac{13}{2}x^2 + 13x - 5\right) && \text{Expand}
\end{aligned}$$

To make all coefficients integers, we set $a = 2$ and obtain

$$P(x) = 2x^3 - 13x^2 + 26x - 10$$

Any other polynomial that satisfies the given requirements must be an integer multiple of this one.

Now Try Exercise 41 ■

■ Linear and Quadratic Factors

We have seen that a polynomial factors completely into linear factors if we use complex numbers. If we don't use complex numbers, then a polynomial with real coefficients can always be factored into linear and quadratic factors. (We use this property in Section 9.7 when we study partial fractions.) A quadratic polynomial with no real zero is called **irreducible** over the real numbers. Such a polynomial cannot be factored without using complex numbers.

Linear and Quadratic Factors Theorem

Every polynomial with real coefficients can be factored into a product of linear and irreducible quadratic factors with real coefficients.

Proof We first observe that if $c = a + bi$ is a complex number, then

$$\begin{aligned}
(x - c)(x - \bar{c}) &= [x - (a + bi)][x - (a - bi)] \\
&= [(x - a) - bi][(x - a) + bi] \\
&= (x - a)^2 - (bi)^2 \\
&= x^2 - 2ax + (a^2 + b^2)
\end{aligned}$$

The last expression is a quadratic with *real* coefficients.

Now, if P is a polynomial with real coefficients, then by the Complete Factorization Theorem

$$P(x) = a(x - c_1)(x - c_2)\cdots(x - c_n)$$

Since the complex roots occur in conjugate pairs, we can multiply the factors corresponding to each such pair to get a quadratic factor with real coefficients. This results in P being factored into linear and irreducible quadratic factors. ■

Example 7 ■ Factoring a Polynomial into Linear and Quadratic Factors

Let $P(x) = x^4 + 2x^2 - 8$.

(a) Factor P into linear and irreducible quadratic factors with real coefficients.

(b) Factor P completely into linear factors with complex coefficients.

Solution

(a)

$$\begin{aligned} P(x) &= x^4 + 2x^2 - 8 \\ &= (x^2 - 2)(x^2 + 4) \\ &= (x - \sqrt{2})(x + \sqrt{2})(x^2 + 4) \end{aligned}$$

The quadratic factor $x^2 + 4$ is irreducible because it has no real zero.

(b) To get the complete factorization, we factor the remaining quadratic factor:

$$\begin{aligned} P(x) &= (x - \sqrt{2})(x + \sqrt{2})(x^2 + 4) \\ &= (x - \sqrt{2})(x + \sqrt{2})(x - 2i)(x + 2i) \end{aligned}$$

Now Try Exercise 67

3.5 Exercises

Concepts

1. The polynomial $P(x) = 5x^2(x - 4)^3(x + 7)$ has degree ________. It has zeros 0, 4, and ________. The zero 0 has multiplicity ________, and the zero 4 has multiplicity ________.

2. (a) If a is a zero of the polynomial P, then ________ must be a factor of $P(x)$.

(b) If a is a zero of multiplicity m of the polynomial P, then ________ must be a factor of $P(x)$ when we factor P completely.

3. A polynomial of degree $n \geq 1$ has exactly ________ zeros if a zero of multiplicity m is counted m times.

4. If the polynomial function P has real coefficients and if $a + bi$ is a zero of P, then ________ is also a zero of P. So if $3 + i$ is a zero of P, then ________ is also a zero of P.

5–6 ■ *True or False*? If *False*, give a reason.

5. Let $P(x) = x^4 + 1$.

(a) The polynomial P has four complex zeros.

(b) The polynomial P can be factored into linear factors with complex coefficients.

(c) Some of the zeros of P are real.

6. Let $P(x) = x^3 + x$.

(a) The polynomial P has three real zeros.

(b) The polynomial P has at least one real zero.

(c) The polynomial P can be factored into linear factors with real coefficients.

Skills

7–18 ■ **Complete Factorization** A polynomial P is given. **(a)** Find all zeros of P, real and complex. **(b)** Factor P completely.

7. $P(x) = x^4 + 4x^2$

8. $P(x) = x^5 + 9x^3$

9. $P(x) = x^3 - 2x^2 + 2x$

10. $P(x) = x^3 + x^2 + x$

11. $P(x) = x^4 + 2x^2 + 1$

12. $P(x) = x^4 - x^2 - 2$

13. $P(x) = x^4 - 16$

14. $P(x) = x^4 + 6x^2 + 9$

15. $P(x) = x^3 + 8$

16. $P(x) = x^3 - 8$

17. $P(x) = x^6 - 1$

18. $P(x) = x^6 - 7x^3 - 8$

19–36 ■ **Complete Factorization** Factor the polynomial completely, and find all its zeros. State the multiplicity of each zero.

19. $P(x) = x^4 + 16x^2$

20. $P(x) = 9x^6 + 16x^4$

21. $Q(x) = x^6 + 2x^5 + 2x^4$

22. $Q(x) = x^5 + x^4 + x^3$

23. $P(x) = x^3 + 4x$

24. $P(x) = x^3 - x^2 + x$

25. $Q(x) = x^4 - 1$

26. $Q(x) = x^4 - 625$

27. $P(x) = 16x^4 - 81$

28. $P(x) = x^3 - 64$

29. $P(x) = x^3 + x^2 + 9x + 9$

30. $P(x) = x^6 - 729$

31. $P(x) = x^6 + 10x^4 + 25x^2$

32. $P(x) = x^5 + 18x^3 + 81x$

33. $P(x) = x^4 + 3x^2 - 4$

34. $P(x) = x^5 + 7x^3$

35. $P(x) = x^5 + 6x^3 + 9x$

36. $P(x) = x^6 + 16x^3 + 64$

37–46 ■ Finding a Polynomial with Specified Zeros Find a polynomial with integer coefficients that satisfies the given conditions.

37. P has degree 2 and zeros $1 + i$ and $1 - i$.

38. P has degree 2 and zeros $1 + \sqrt{2}i$ and $1 - \sqrt{2}i$.

39. Q has degree 3 and zeros 3, $2i$, and $-2i$.

40. Q has degree 3 and zeros 0 and i.

41. P has degree 3 and zeros 2 and i.

42. Q has degree 3 and zeros -3 and $1 + i$.

43. R has degree 4 and zeros $1 - 2i$ and 1, with 1 a zero of multiplicity 2.

44. S has degree 4 and zeros $2i$ and $3i$.

45. T has degree 4, zeros i and $1 + i$, and constant term 12.

46. U has degree 5, zeros $\frac{1}{2}$, -1, and $-i$, and leading coefficient 4; the zero -1 has multiplicity 2.

47–64 ■ Finding Complex Zeros Find all zeros of the polynomial.

47. $P(x) = x^3 - 2x - 4$

48. $P(x) = x^3 - 7x^2 + 16x - 10$

49. $P(x) = x^3 - 2x^2 + 2x - 1$

50. $P(x) = x^3 + 7x^2 + 18x + 18$

51. $P(x) = x^3 - 3x^2 + 3x - 2$

52. $P(x) = x^3 - x - 6$

53. $P(x) = 2x^3 + 7x^2 + 12x + 9$

54. $P(x) = 2x^3 - 8x^2 + 9x - 9$

55. $P(x) = x^4 + x^3 + 7x^2 + 9x - 18$

56. $P(x) = x^4 - 2x^3 - 2x^2 - 2x - 3$

57. $P(x) = x^5 - x^4 + 7x^3 - 7x^2 + 12x - 12$ [*Hint:* Factor by grouping.]

58. $P(x) = x^5 + x^3 + 8x^2 + 8$ [*Hint:* Factor by grouping.]

59. $P(x) = x^4 - 6x^3 + 13x^2 - 24x + 36$

60. $P(x) = x^4 - x^2 + 2x + 2$

61. $P(x) = 4x^4 + 4x^3 + 5x^2 + 4x + 1$

62. $P(x) = 4x^4 + 2x^3 - 2x^2 - 3x - 1$

63. $P(x) = x^5 - 3x^4 + 12x^3 - 28x^2 + 27x - 9$

64. $P(x) = x^5 - 2x^4 + 2x^3 - 4x^2 + x - 2$

65–70 ■ Linear and Quadratic Factors A polynomial P is given. **(a)** Factor P into linear and irreducible quadratic factors with real coefficients. **(b)** Factor P completely into linear factors with complex coefficients.

65. $P(x) = x^3 - 5x^2 + 4x - 20$

66. $P(x) = x^3 - 2x - 4$

67. $P(x) = x^4 + 8x^2 - 9$

68. $P(x) = x^4 + 8x^2 + 16$

69. $P(x) = x^6 - 64$

70. $P(x) = x^5 - 16x$

Skills Plus

71. Number of Real and Nonreal Solutions By the Zeros Theorem, every nth-degree polynomial equation has exactly n solutions (including possibly some that are repeated). Some of these may be real, and some may be nonreal. Use a graphing device to determine how many real and nonreal solutions each equation has.

(a) $x^4 - 2x^3 - 11x^2 + 12x = 0$

(b) $x^4 - 2x^3 - 11x^2 + 12x - 5 = 0$

(c) $x^4 - 2x^3 - 11x^2 + 12x + 40 = 0$

72–74 ■ Real and Nonreal Coefficients So far, we have worked with polynomials that have only real coefficients. These exercises involve polynomials with real and imaginary coefficients.

72. Find all solutions of each equation.

(a) $2x + 4i = 1$ **(b)** $x^2 - ix = 0$

(c) $x^2 + 2ix - 1 = 0$ **(d)** $ix^2 - 2x + i = 0$

73. (a) Show that $2i$ and $1 - i$ are both solutions of the equation

$$x^2 - (1 + i)x + (2 + 2i) = 0$$

but that their complex conjugates $-2i$ and $1 + i$ are not.

(b) Explain why the result of part (a) does not violate the Conjugate Zeros Theorem.

74. (a) Find the polynomial with *real* coefficients of the smallest possible degree for which i and $1 + i$ are zeros and the coefficient of the highest power is 1.

(b) Find the polynomial with *complex* coefficients of the smallest possible degree for which i and $1 + i$ are zeros and the coefficient of the highest power is 1.

Discuss ■ Discover ■ Prove ■ Write

75. Discuss: Polynomials of Odd Degree The Conjugate Zeros Theorem says that the complex zeros of a polynomial with real coefficients occur in complex conjugate pairs. Explain how this fact proves that a polynomial with real coefficients and odd degree has at least one real zero.

76. Discuss ■ Discover: Factoring and Graphing Let's explore how a graph can help us factor a polynomial. Factor

$$P(x) = 2x^4 - 3x^3 + 6x^2 - 12x - 8$$

by first graphing the polynomial, then using the graph to identify the real zeroes of P. Use those zeros to factor the polynomial into linear and quadratic factors, and then find the complex zeroes. Finally, express the polynomial as a product of linear factors with complex coefficients.

3.6 Rational Functions

■ The Rational Function $f(x) = 1/x$ ■ Vertical and Horizontal Asymptotes ■ Finding Vertical and Horizontal Asymptotes of Rational Functions ■ Graphing Rational Functions ■ Common Factors in Numerator and Denominator ■ Slant Asymptotes and End Behavior ■ Applications

We studied rational expressions in Section 1.4. In this section we study functions that are defined by rational expressions.

Rational Functions

A **rational function** is a function of the form

$$r(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials. We assume that $P(x)$ and $Q(x)$ have no factor in common. The domain of r consists of all real numbers x except those values for which the denominator is zero. That is, the domain of r is $\{x \mid Q(x) \neq 0\}$.

Here are some examples of rational functions.

$$f(x) = \frac{1}{x} \qquad r(x) = \frac{2x}{x+3} \qquad s(x) = \frac{x+2}{x^2+5x+1} \qquad u(x) = \frac{x^3+3x^2-5}{4x^3+3x+4}$$

For instance, the function $f(x) = 1/x$ is a rational function because the numerator is the polynomial $P(x) = 1$ and the denominator is the polynomial $Q(x) = x$. Although rational functions are constructed from polynomials, their graphs can look quite different from the graphs of polynomials. We begin by graphing the rational function $f(x) = 1/x$ because its graph contains many of the main features of the graphs of rational functions.

■ The Rational Function $f(x) = 1/x$

When graphing a rational function, we must pay special attention to the behavior of the graph near those x-values for which the denominator is zero.

Example 1 ■ Graph of $f(x) = 1/x$

Graph the rational function $f(x) = 1/x$, and state the domain and range.

Solution The function f is not defined for $x = 0$. The following tables show that when x is close to zero, the value of $|f(x)|$ is large, and the closer x gets to zero, the larger $|f(x)|$ gets.

For positive real numbers,

$$\frac{1}{\text{BIG NUMBER}} = \text{small number}$$

$$\frac{1}{\text{small number}} = \text{BIG NUMBER}$$

x	$f(x)$
-0.1	-10
-0.01	-100
-0.00001	$-100{,}000$
Approaching 0^-	Approaching $-\infty$

x	$f(x)$
0.1	10
0.01	100
0.00001	$100{,}000$
Approaching 0^+	Approaching ∞

The first table shows that as x approaches 0 from the left, the values of $y = f(x)$ decrease without bound. We describe this behavior in symbols and in words as follows.

$$f(x) \to -\infty \quad \text{as} \quad x \to 0^-$$ "y approaches negative infinity as x approaches 0 from the left"

The second table on the previous page shows that as x approaches 0 from the right, the values of $f(x)$ increase without bound. In symbols,

$$f(x) \to \infty \quad \text{as} \quad x \to 0^+$$ "y approaches infinity as x approaches 0 from the right"

The next two tables show how $f(x)$ changes as $|x|$ becomes large.

x	$f(x)$
-10	-0.1
-100	-0.01
$-100{,}000$	-0.00001

Approaching $-\infty$ Approaching 0

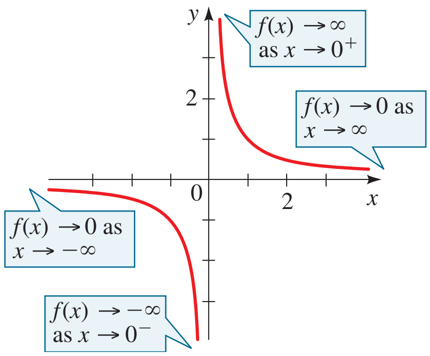
x	$f(x)$
10	0.1
100	0.01
100,000	0.00001

Approaching ∞ Approaching 0

These tables show that as $|x|$ becomes large, the value of $f(x)$ gets closer and closer to zero. We describe this situation in symbols by writing

$$f(x) \to 0 \quad \text{as} \quad x \to -\infty \qquad \text{and} \qquad f(x) \to 0 \quad \text{as} \quad x \to \infty$$

Using the information in the preceding tables and plotting a few additional points, we obtain the graph shown in Figure 1.



x	$f(x) = 1/x$
-2	$-\frac{1}{2}$
-1	-1
$-\frac{1}{2}$	-2
$\frac{1}{2}$	2
1	1
2	$\frac{1}{2}$

Figure 1 | $f(x) = 1/x$

Obtaining the domain and range of a function from its graph is explained in Section 2.3.

The function f is defined for all values of x other than 0, so the domain is $\{x \mid x \neq 0\}$. From the graph we see that the range is $\{y \mid y \neq 0\}$.

Now Try Exercise 11 ■

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Discovery Project ■ **Managing Traffic**

A highway engineer wants to determine the optimal safe driving speed for a road. The higher the speed limit, the more cars the road can accommodate, but safety requires a greater following distance at higher speeds. In this project we find a rational function that models the carrying capacity of a road at a given traffic speed. The model can be used to determine the speed limit at which the road has its maximum carrying capacity. You can find the project at **www.stewartmath.com**.

In Example 1 we used the following **arrow notation**.

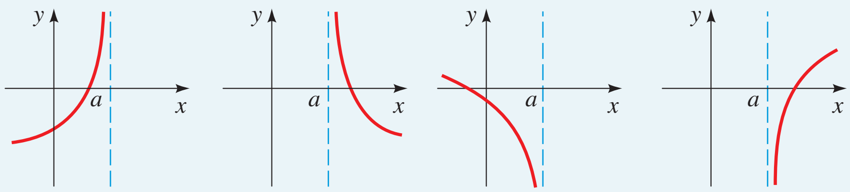
Symbol	Meaning
$x \to a^-$	x approaches a from the left
$x \to a^+$	x approaches a from the right
$x \to -\infty$	x goes to negative infinity; that is, x decreases without bound
$x \to \infty$	x goes to infinity; that is, x increases without bound

■ Vertical and Horizontal Asymptotes

Informally speaking, an *asymptote* of a function is a line that the graph of the function gets closer and closer to as one travels along that line. For instance, from Figure 1 we see that the line $x = 0$ is a *vertical asymptote* of the function $f(x) = 1/x$ and the line $y = 0$ is a *horizontal asymptote*. In general, we have the following definition.

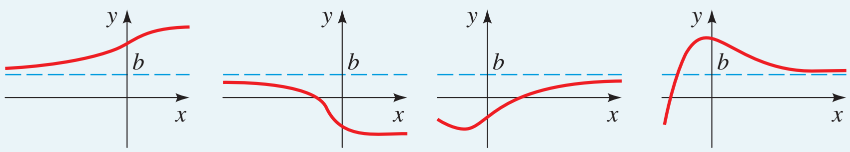
Definition of Vertical and Horizontal Asymptotes

1. **Vertical Asymptotes.** The line $x = a$ is a **vertical asymptote** of the function $y = f(x)$ if y approaches $\pm\infty$ as x approaches a from the left or right.



$y \to \infty$ as $x \to a^-$ $y \to \infty$ as $x \to a^+$ $y \to -\infty$ as $x \to a^-$ $y \to -\infty$ as $x \to a^+$

2. **Horizontal Asymptote.** The line $y = b$ is a **horizontal asymptote** of the function $y = f(x)$ if y approaches b as x approaches $\pm\infty$.



$y \to b$ as $x \to -\infty$ $y \to b$ as $x \to \infty$

Transformations of functions are studied in Section 2.6.

In the following example we find the vertical and horizontal asymptotes of certain transformations of $f(x) = 1/x$. In general, a rational function of the form

$$r(x) = \frac{ax + b}{cx + d}$$

can be graphed by shifting, stretching, and/or reflecting the graph of $f(x) = 1/x$. (Functions of this form are called *linear fractional transformations*.)

Example 2 ■ Using Transformations to Graph Rational Functions

Graph each rational function, and state the domain and range.

(a) $r(x) = \dfrac{2}{x - 3}$ **(b)** $s(x) = \dfrac{3x + 5}{x + 2}$

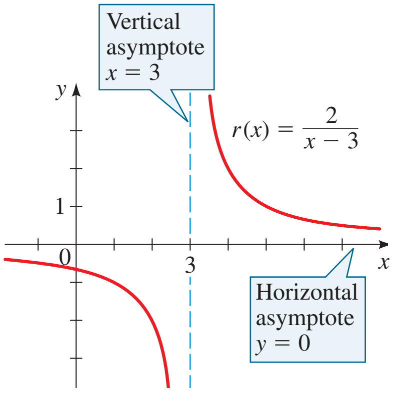


Figure 2

Solution

(a) Let $f(x) = 1/x$. Then we can express r in terms of f as follows:

$$\begin{aligned} r(x) &= \frac{2}{x - 3} \\ &= 2\left(\frac{1}{x - 3}\right) && \text{Factor 2} \\ &= 2(f(x - 3)) && \text{Because } f(x) = 1/x \end{aligned}$$

From this form we see that the graph of r is obtained from the graph of f by shifting 3 units to the right and stretching vertically by a factor of 2. Thus r has vertical asymptote $x = 3$ and horizontal asymptote $y = 0$. The graph of r is shown in Figure 2.

The function r is defined for all x other than 3, so the domain is $\{x \mid x \neq 3\}$. From the graph we see that the range is $\{y \mid y \neq 0\}$.

$$\begin{array}{r} 3 \\ x + 2 \overline{)\, 3x + 5} \\ \underline{3x + 6} \\ -1 \end{array}$$

(b) Using long division we can express s in terms of f as follows:

$$\begin{aligned} s(x) &= 3 - \frac{1}{x + 2} && \text{Long division (see margin)} \\ &= -\frac{1}{x + 2} + 3 && \text{Rearrange terms} \\ &= -f(x + 2) + 3 && \text{Since } f(x) = 1/x \end{aligned}$$

From this form we see that the graph of s is obtained from the graph of f by shifting 2 units to the left, reflecting about the x-axis, and shifting upward 3 units. Thus s has vertical asymptote $x = -2$ and horizontal asymptote $y = 3$. The graph of s is shown in Figure 3.

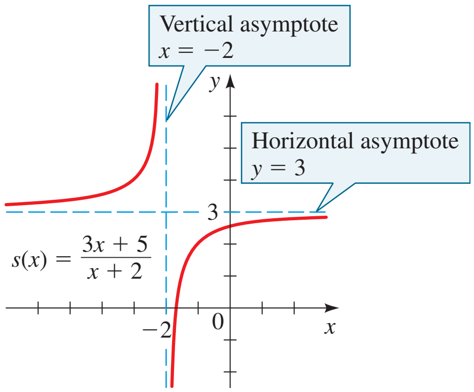


Figure 3

The function s is defined for all x other than -2, so the domain is $\{x \mid x \neq -2\}$. From the graph we see that the range is $\{y \mid y \neq 3\}$.

Now Try Exercises 17 and 19 ■

NOTE Example 2 shows that shifting the graph of a rational function vertically shifts the horizontal asymptotes; shifting the graph horizontally shifts the vertical asymptotes.

■ Finding Vertical and Horizontal Asymptotes of Rational Functions

In Example 2 we have found the asymptotes of a rational function by expressing the function in terms of transformations of the rational function $f(x) = 1/x$. But this method works only for rational functions for which both numerator and denominator are linear functions. For other rational functions we need to take a closer look at the behavior of the function near the values that make the denominator zero (for vertical asymptotes) as well as the behavior of the function as $x \to \pm\infty$ (for horizontal asymptotes).

Example 3 ■ Asymptotes of a Rational Function

Graph $r(x) = \dfrac{2x^2 - 4x + 5}{x^2 - 2x + 1}$, and state the domain and range.

Solution We first find the vertical and horizontal asymptotes of r.

Vertical asymptote. We first factor the denominator

$$r(x) = \frac{2x^2 - 4x + 5}{(x - 1)^2}$$

The line $x = 1$ is a vertical asymptote because the denominator of r is zero when $x = 1$.

To see what the graph of r looks like near the vertical asymptote, we make tables of values for x-values to the left and to the right of 1.

$x \to 1^-$

x	y
0	5
0.5	14
0.9	302
0.99	30,002

Approaching 1^- Approaching ∞

$x \to 1^+$

x	y
2	5
1.5	14
1.1	302
1.01	30,002

Approaching 1^+ Approaching ∞

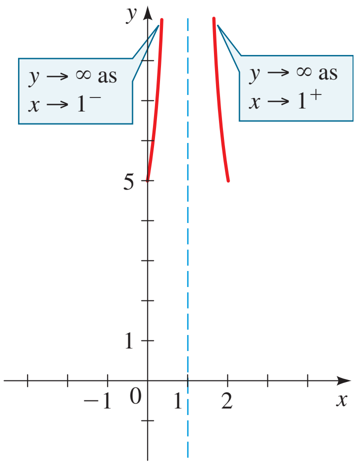


Figure 4

From the tables we see that

$$y \to \infty \quad \text{as} \quad x \to 1^- \qquad \text{and} \qquad y \to \infty \quad \text{as} \quad x \to 1^+$$

Thus near the vertical asymptote $x = 1$, the graph of r has the shape shown in Figure 4.

Horizontal asymptote. The horizontal asymptote is the value that y approaches as $x \to \pm\infty$. To help us find this value, we divide both numerator and denominator by x^2, the highest power of x that appears in the denominator:

$$y = \frac{2x^2 - 4x + 5}{x^2 - 2x + 1} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \frac{2 - \frac{4}{x} + \frac{5}{x^2}}{1 - \frac{2}{x} + \frac{1}{x^2}}$$

The fractional expressions $\frac{4}{x}, \frac{5}{x^2}, \frac{2}{x}$, and $\frac{1}{x^2}$ all approach 0 as $x \to \pm\infty$ (see Exercise 1.1.89). So as $x \to \pm\infty$, we have

These terms approach 0

$$y = \frac{2 - \frac{4}{x} + \frac{5}{x^2}}{1 - \frac{2}{x} + \frac{1}{x^2}} \longrightarrow \frac{2 - 0 + 0}{1 - 0 + 0} = 2$$

These terms approach 0

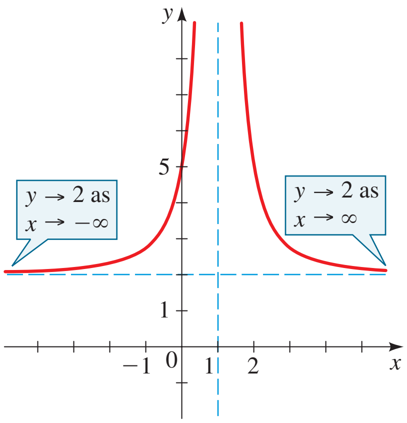


Figure 5 | $r(x) = \dfrac{2x^2 - 4x + 5}{x^2 - 2x + 1}$

Thus the horizontal asymptote is the line $y = 2$.

Since the graph must approach the horizontal asymptote, we can complete the graph as in Figure 5.

Domain and range. The function r is defined for all values of x other than 1, so the domain is $\{x \mid x \neq 1\}$. From the graph we see that the range is $\{y \mid y > 2\}$.

Now Try Exercise 47 ■

In Example 1 we saw that the vertical asymptotes are determined by the zeros of the denominator. In Example 3 we saw that the horizontal asymptote is determined by the leading coefficients of the numerator and denominator because, after dividing through by x^2 (the highest power of x), all other terms approach zero. In general, if $r(x) = P(x)/Q(x)$ and the degrees of P and Q are the same (both n, say), then dividing both numerator and denominator by x^n shows that the horizontal asymptote is

$$y = \frac{\text{leading coefficient of } P}{\text{leading coefficient of } Q}$$

Recall that for a rational function $R(x) = P(x)/Q(x)$ we assume that $P(x)$ and $Q(x)$ have no factor in common.

The following box summarizes the procedure for finding asymptotes.

Finding Asymptotes of Rational Functions

Let r be the rational function

$$r(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}$$

1. **Vertical Asymptotes** The vertical asymptotes of r are the lines $x = a$, where a is a zero of the denominator.
2. **Horizontal Asymptote** The horizontal asymptote of r is determined as follows:
   (a) If $n < m$, then r has horizontal asymptote $y = 0$.
   (b) If $n = m$, then r has horizontal asymptote $y = \dfrac{a_n}{b_m}$.
   (c) If $n > m$, then r has no horizontal asymptote.

Example 4 ■ Asymptotes of a Rational Function

Find the vertical and horizontal asymptotes of $r(x) = \dfrac{3x^2 - 2x - 1}{2x^2 + 3x - 2}$.

Solution

Vertical asymptotes. We first factor

$$r(x) = \frac{3x^2 - 2x - 1}{(2x - 1)(x + 2)}$$

This factor is 0 when $x = \frac{1}{2}$

This factor is 0 when $x = -2$

The vertical asymptotes are the lines $x = \frac{1}{2}$ and $x = -2$.

Horizontal asymptote. The degrees of the numerator and denominator are the same, and

$$\frac{\text{leading coefficient of numerator}}{\text{leading coefficient of denominator}} = \frac{3}{2}$$

Thus the horizontal asymptote is the line $y = \frac{3}{2}$.

To confirm our results, we graph r using a graphing device (see Figure 6).

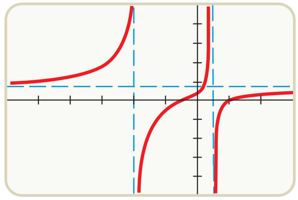


Figure 6 | $r(x) = \dfrac{3x^2 - 2x - 1}{2x^2 + 3x - 2}$

Now Try Exercises 35 and 37 ■

Graphing Rational Functions

In general, we use the following guidelines to graph rational functions.

Sketching Graphs of Rational Functions

1. **Factor.** Factor the numerator and denominator.
2. **Intercepts.** Find the x-intercepts by determining the zeros of the numerator and the y-intercept from the value of the function at $x = 0$.
3. **Vertical Asymptotes.** Find the vertical asymptotes by determining the zeros of the denominator, and then determine whether $y \to \infty$ or $y \to -\infty$ on each side of each vertical asymptote by using **test values**.
4. **Horizontal Asymptote.** Find the horizontal asymptote (if any), using the procedure described in the preceding box.
5. **Sketch the Graph.** Graph the information provided by steps 1–4. Then plot as many additional points as needed to fill in the rest of the graph of the function.

A fraction is 0 only if its numerator is 0 (and the denominator is not 0).

Example 5 ■ Graphing a Rational Function

Graph $r(x) = \dfrac{2x^2 + 7x - 4}{x^2 + x - 2}$, and state the domain and range.

Solution We factor the numerator and denominator, find the intercepts and asymptotes, and sketch the graph.

Factor. $y = \dfrac{(2x - 1)(x + 4)}{(x - 1)(x + 2)}$

x-Intercepts. The x-intercepts are the zeros of the numerator, $x = \frac{1}{2}$ and $x = -4$.

y-Intercept. To find the y-intercept, we substitute $x = 0$ into the original form of the function.

$$r(0) = \frac{2(0)^2 + 7(0) - 4}{(0)^2 + (0) - 2} = \frac{-4}{-2} = 2$$

The y-intercept is 2.

Vertical asymptotes. The vertical asymptotes occur where the denominator is 0, that is, where the function is undefined. From the factored form we see that the vertical asymptotes are the lines $x = 1$ and $x = -2$.

When choosing test values, we must make sure that there is no x-intercept between the test point and the vertical asymptote.

Behavior near vertical asymptotes. We need to know whether $y \to \infty$ or $y \to -\infty$ on each side of every vertical asymptote. To determine the sign of y for x-values near the vertical asymptotes, we use *test values*. For instance, as $x \to 1^-$, we use a test value close to and to the left of 1 ($x = 0.9$, say) to check whether y is positive or negative to the left of $x = 1$.

$$y = \frac{[2(0.9) - 1][(0.9) + 4]}{[(0.9) - 1][(0.9) + 2]} \qquad \text{whose sign is} \qquad \frac{(+)(+)}{(-)(+)} \quad \text{(negative)}$$

So $y \to -\infty$ as $x \to 1^-$. On the other hand, as $x \to 1^+$, we use a test value close to and to the right of 1 ($x = 1.1$, say), to get

$$y = \frac{[2(1.1) - 1][(1.1) + 4]}{[(1.1) - 1][(1.1) + 2]} \qquad \text{whose sign is} \qquad \frac{(+)(+)}{(+)(+)} \quad \text{(positive)}$$

So $y \to \infty$ as $x \to 1^+$. The other entries in the following table are calculated similarly.

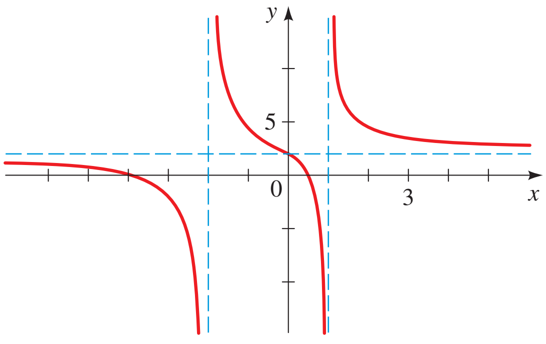
As $x \to$	-2^-	-2^+	1^-	1^+
the sign of $y = \dfrac{(2x-1)(x+4)}{(x-1)(x+2)}$ is	$\frac{(-)(+)}{(-)(-)}$	$\frac{(-)(+)}{(-)(+)}$	$\frac{(+)(+)}{(-)(+)}$	$\frac{(+)(+)}{(+)(+)}$
so $y \to$	$-\infty$	∞	$-\infty$	∞

Horizontal asymptote. The degree of the numerator and the degree of the denominator are the same, and

$$\frac{\text{leading coefficient of numerator}}{\text{leading coefficient of denominator}} = \frac{2}{1} = 2$$

Thus the horizontal asymptote is the line $y = 2$.

Graph. We use the information we have found, together with some additional values, to sketch the graph in Figure 7.



x	y
−6	0.93
−4	0
−3	−1.75
−1	4.50
0	2
0.5	0
1.5	6.29
2	4.50
6	2.75

Figure 7 | $r(x) = \dfrac{2x^2 + 7x - 4}{x^2 + x - 2}$

Domain and range. The domain is $\{x \mid x \neq 1, x \neq -2\}$. From the graph we see that the range is all real numbers.

Now Try Exercise 55

Example 6 ■ Graphing a Rational Function

Graph the rational function $r(x) = \dfrac{x^2 - 4}{2x^2 + 2x}$. State the domain and estimate the range from the graph.

Solution

Factor. $y = \dfrac{(x+2)(x-2)}{2x(x+1)}$

***x*-intercepts.** -2 and 2, from $x + 2 = 0$ and $x - 2 = 0$

***y*-intercept.** None, because $r(0)$ is undefined

Vertical asymptotes. $x = 0$ and $x = -1$, from the zeros of the denominator

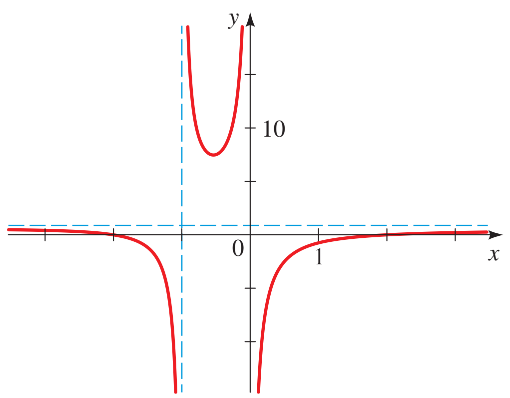
Behavior near vertical asymptote.

As $x \to$	-1^-	-1^+	0^-	0^+
the sign of $y = \dfrac{(x+2)(x-2)}{2x(x+1)}$ is	$\dfrac{(+)(-)}{(-)(-)}$	$\dfrac{(+)(-)}{(-)(+)}$	$\dfrac{(+)(-)}{(-)(+)}$	$\dfrac{(+)(-)}{(+)(+)}$
so $y \to$	$-\infty$	∞	∞	$-\infty$

Horizontal asymptote. $y = \frac{1}{2}$ because the degree of the numerator and the degree of the denominator are the same and

$$\frac{\text{leading coefficient of numerator}}{\text{leading coefficient of denominator}} = \frac{1}{2}$$

Graph. We use the information we have found, together with some additional values, to sketch the graph in Figure 8.



x	y
-2	0
-0.9	17.72
-0.5	7.50
-0.45	7.67
-0.4	8.00
-0.3	9.31
-0.1	22.17
2	0

Figure 8 | $r(x) = \dfrac{x^2 - 4}{2x^2 + 2x}$

Domain and range. The domain is $\{x \mid x \neq 0, x \neq -1\}$. From the graph we see that the range is approximately $\{y \mid y < \frac{1}{2} \text{ or } y > 7.5\}$.

Now Try Exercise 59

Mathematics in the Modern World

Unbreakable Codes

If you read spy novels, you know about secret codes and how the hero "breaks" the code. Today secret codes have a much more common use. Most of the information that is stored on computers is coded to prevent unauthorized use. For example, your banking records, medical records, and school records are coded. Many cellular phones code the signal carrying your voice so that no one can listen in. Fortunately, because of recent advances in mathematics, today's codes are "unbreakable."

Modern codes are based on a simple principle: Factoring is much harder than multiplying. For example, try multiplying 78 and 93; now try factoring 9991. It takes a long time to factor 9991 because it is a product of two primes 97 × 103, so to factor it, we have to find one of these primes. Now imagine trying to factor a number *N* that is the product of two primes *p* and *q*, each about 200 digits long. Even the fastest computers would take many many years to factor such a number! But the same computer would take less than a second to multiply two such numbers. This fact was used by Ron Rivest, Adi Shamir, and Leonard Adleman in the 1970s to devise the RSA code. Their code uses an extremely large number to encode a message but requires us to know its factors to decode it; such a code is practically unbreakable.

The RSA code is an example of a "public key encryption" code. In such codes, anyone can code a message using a publicly known procedure based on *N*, but to decode the message, they must know *p* and *q*, the factors of *N*. When the RSA code was developed, it was thought that a carefully selected 80-digit number would provide an unbreakable code. But interestingly, recent advances in the study of factoring have made much larger numbers necessary.

Example 7 ■ Graphing a Rational Function

Graph $r(x) = \dfrac{5x + 21}{x^2 + 10x + 25}$. State the domain and estimate the range from the graph.

Solution

Factor. $$y = \frac{5x + 21}{(x + 5)^2}$$

***x*-Intercept.** $-\dfrac{21}{5}$ from $5x + 21 = 0$

***y*-Intercept.** $\dfrac{21}{25}$ because
$$r(0) = \frac{5 \cdot 0 + 21}{0^2 + 10 \cdot 0 + 25} = \frac{21}{25}$$

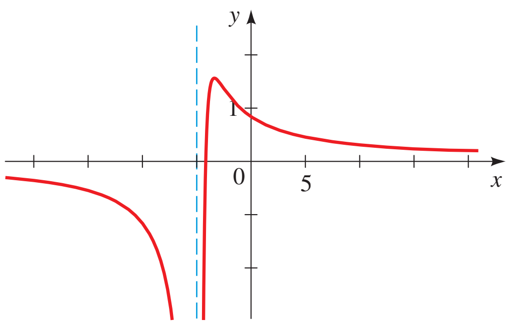
Vertical asymptote. $x = -5$, from the zeros of the denominator

Behavior near vertical asymptote.

As $x \to$	-5^-	-5^+
the sign of $y = \dfrac{5x + 21}{(x + 5)^2}$ is	$\dfrac{(-)}{(-)(-)}$	$\dfrac{(-)}{(+)(+)}$
so $y \to$	$-\infty$	$-\infty$

Horizontal asymptote. $y = 0$ because the degree of the numerator is less than the degree of the denominator

Graph. We use the information we have found, together with some additional values, to sketch the graph in Figure 9.



x	y
-15	-0.5
-4.2	0
-3	1.5
-1	1.0
0	0.84
3	0.6
10	0.3

Figure 9 | $r(x) = \dfrac{5x + 21}{x^2 + 10x + 25}$

Domain and range. The domain is $\{x \mid x \neq -5\}$. From the graph we see that the range is approximately the interval $(-\infty, 1.6]$.

Now Try Exercise 61 ■

From the graph in Figure 9 we see that, contrary to common misconception, a graph may cross a horizontal asymptote. The graph in Figure 9 crosses the x-axis (the horizontal asymptote) from below, reaches a maximum value near $x = -3$, and then approaches the x-axis from above as $x \to \infty$.

Common Factors in Numerator and Denominator

We have adopted the convention that the numerator and denominator of a rational function have no factor in common. If $s(x) = p(x)/q(x)$ and if p and q do have a factor in common, then we may cancel that factor, but only for those values of x for which that factor is *not zero* (because division by zero is not defined). Since s is not defined at those values of x, its graph has a "**hole**" at those points, as the following example illustrates.

Example 8 ■ Common Factor in Numerator and Denominator

Graph each of the following functions.

(a) $s(x) = \dfrac{x-3}{x^2-3x}$ **(b)** $t(x) = \dfrac{x^3-2x^2}{x-2}$

Solution

(a) We factor the numerator and denominator:

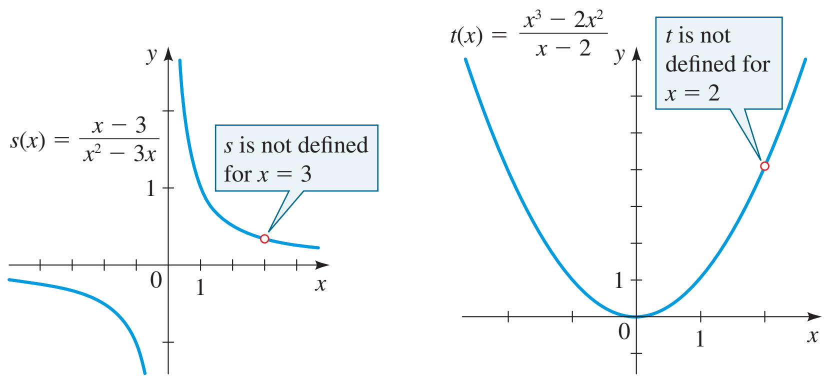
$$s(x) = \frac{x-3}{x^2-3x} = \frac{\cancel{(x-3)}}{x\cancel{(x-3)}} = \frac{1}{x} \quad \text{for } x \neq 3$$

So s has the same graph as the rational function $r(x) = 1/x$ but with a "hole" when x is 3 because the function s is not defined when $x = 3$ [see Figure 10(a)].

(b) We factor the numerator and denominator:

$$t(x) = \frac{x^3-2x^2}{x-2} = \frac{x^2\cancel{(x-2)}}{\cancel{x-2}} = x^2 \quad \text{for } x \neq 2$$

So the graph of t is the same as the graph of $f(x) = x^2$ but with a "hole" when x is 2 because the function t is not defined when $x = 2$ [see Figure 10(b)].



(a) $s(x) = 1/x$ for $x \neq 3$ (b) $t(x) = x^2$ for $x \neq 2$

Figure 10 | Graphs with "holes"

Now Try Exercise 65 ■

Slant Asymptotes and End Behavior

If $r(x) = P(x)/Q(x)$ is a rational function for which the degree of the numerator is one more than the degree of the denominator, we can use the Division Algorithm to express the function in the form

$$r(x) = ax + b + \frac{R(x)}{Q(x)}$$

where the degree of R is less than the degree of Q and $a \neq 0$. This means that as $x \to \pm\infty$, $R(x)/Q(x) \to 0$, so for large values of $|x|$ the graph of $y = r(x)$ approaches the graph of the line $y = ax + b$. In this situation we say that $y = ax + b$ is a **slant asymptote**, or an **oblique asymptote**.

Example 9 ■ A Rational Function with a Slant Asymptote

Graph the rational function $r(x) = \dfrac{x^2 - 4x - 5}{x - 3}$.

Solution

Factor. $y = \dfrac{(x+1)(x-5)}{x-3}$

x-Intercepts. -1 and 5, from $x + 1 = 0$ and $x - 5 = 0$

y-Intercept. $\frac{5}{3}$ because $r(0) = \dfrac{0^2 - 4 \cdot 0 - 5}{0 - 3} = \dfrac{5}{3}$

Vertical asymptote. $x = 3$, from the zero of the denominator

Behavior near vertical asymptote. $y \to \infty$ as $x \to 3^-$ and $y \to -\infty$ as $x \to 3^+$

Horizontal asymptote. None, because the degree of the numerator is greater than the degree of the denominator

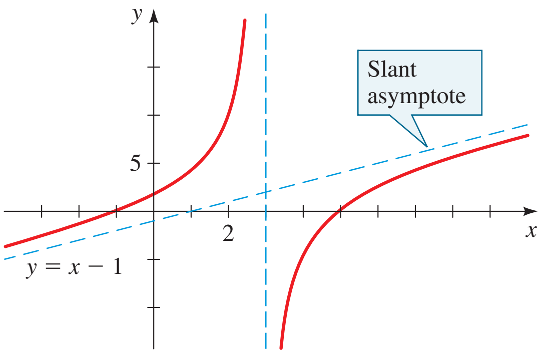
$$\begin{array}{r} x-1 \\ x-3\overline{)x^2-4x-5} \\ \underline{x^2-3x} \\ -x-5 \\ \underline{-x+3} \\ -8 \end{array}$$

Slant asymptote. Because the degree of the numerator is one more than the degree of the denominator, the function has a slant asymptote. Dividing (see the margin), we obtain

$$r(x) = x - 1 - \frac{8}{x-3}$$

Thus $y = x - 1$ is the slant asymptote.

Graph. We use the information we have found, together with some additional values, to sketch the graph in Figure 11.



x	y
-2	-1.4
-1	0
0	1.67
1	4
2	9
4	-5
5	0
6	2.33

Figure 11 | $r(x) = \dfrac{x^2 - 4x - 5}{x - 3}$

Now Try Exercise 71 ■

So far, we have considered only horizontal and slant asymptotes as end behaviors for rational functions. In the next example we graph a function whose end behavior is like that of a parabola.

Example 10 ■ End Behavior of a Rational Function

Graph the rational function

$$r(x) = \frac{x^3 - 2x^2 + 3}{x - 2}$$

and describe its end behavior.

Solution

Factor. $y = \dfrac{(x + 1)(x^2 - 3x + 3)}{x - 2}$

x-Intercept. -1 from $x + 1 = 0$ (The other factor in the numerator has no real zeros.)

y-Intercept. $-\frac{3}{2}$ because $r(0) = \dfrac{0^3 - 2 \cdot 0^2 + 3}{0 - 2} = -\dfrac{3}{2}$

Vertical asymptote. $x = 2$, from the zero of the denominator

Behavior near vertical asymptote. $y \to -\infty$ as $x \to 2^-$ and $y \to \infty$ as $x \to 2^+$

Horizontal asymptote. None, because the degree of the numerator is greater than the degree of the denominator

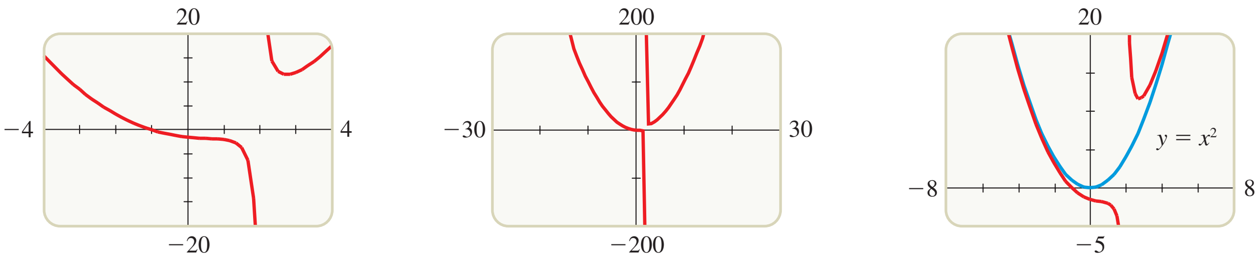
$$\begin{array}{r} x^2 \\ x - 2 \overline{) x^3 - 2x^2 + 0x + 3} \\ \underline{x^3 - 2x^2} \\ 3 \end{array}$$

End behavior. Dividing (see the margin), we obtain

$$r(x) = x^2 + \frac{3}{x - 2}$$

This shows that the end behavior of r is like that of the parabola $y = x^2$ because $3/(x - 2)$ is small when $|x|$ is large. That is, $3/(x - 2) \to 0$ as $x \to \pm\infty$. This means that the graph of r will be close to the graph of $y = x^2$ for large $|x|$.

Graph. In Figure 12(a) we graph r in a small viewing rectangle; we can see the intercepts, the vertical asymptotes, and the local minimum. In Figure 12(b) we graph r in a larger viewing rectangle; here the graph looks almost like the graph of a parabola. In Figure 12(c) we graph both $y = r(x)$ and $y = x^2$; these graphs are very close to each other except near the vertical asymptote.



(a) (b) (c)

Figure 12 | $r(x) = \dfrac{x^3 - 2x^2 + 3}{x - 2}$

Now Try Exercise 79 ■

■ Applications

Rational functions occur frequently in scientific applications of algebra. In the next example we analyze the graph of a function from the theory of electricity.

Example 11 ■ Electrical Resistance

When two resistors with resistances R_1 and R_2 are connected in parallel, their combined resistance R is given by the formula

$$R = \frac{R_1 R_2}{R_1 + R_2}$$

Suppose that a fixed 8-ohm resistor is connected in parallel with a variable resistor, as shown in Figure 13. If the resistance of the variable resistor is denoted by x, then the

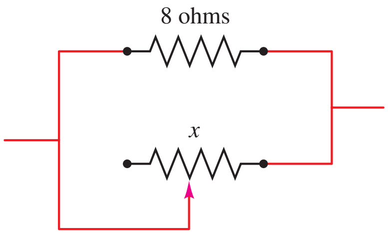


Figure 13

combined resistance R is a function of x. Graph R, and give a physical interpretation of the graph.

Solution Substituting $R_1 = 8$ and $R_2 = x$ into the formula gives the function

$$R(x) = \frac{8x}{8 + x}$$

Because resistance cannot be negative, this function has physical meaning only when $x > 0$. The function is graphed in Figure 14(a) using the viewing rectangle $[0, 20]$ by $[0, 10]$. The function has no vertical asymptote when x is restricted to positive values. The combined resistance R increases as the variable resistance x increases. If we widen the viewing rectangle to $[0, 100]$ by $[0, 10]$, we obtain the graph in Figure 14(b). For large x the combined resistance R levels off, getting closer and closer to the horizontal asymptote $R = 8$. No matter how large the variable resistance x, the combined resistance is never greater than 8 ohms.

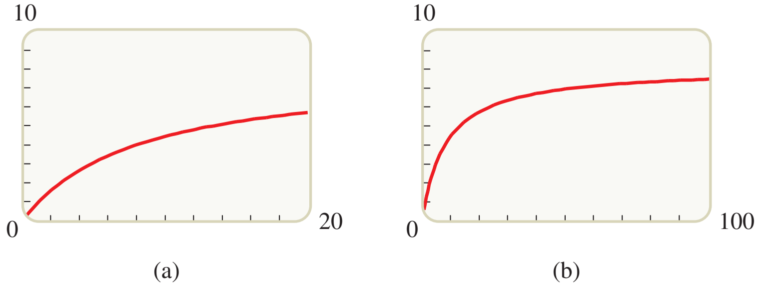


Figure 14 | $R(x) = \dfrac{8x}{8 + x}$

Now Try Exercise 93

3.6 Exercises

Concepts

1. If the rational function $y = r(x)$ has the vertical asymptote $x = 2$, then as $x \to 2^+$, either $y \to$ ______ or $y \to$ ______.

2. If the rational function $y = r(x)$ has the horizontal asymptote $y = 2$, then $y \to$ ______ as $x \to \pm\infty$.

3–6 ■ The following questions are about the rational function

$$r(x) = \frac{(x + 1)(x - 2)}{(x + 2)(x - 3)}$$

3. The function r has x-intercepts ______ and ______.

4. The function r has y-intercept ______.

5. The function r has vertical asymptotes $x =$ ______ and $x =$ ______.

6. The function r has horizontal asymptote $y =$ ______.

7–8 ■ Suppose that the graph of a rational function r has vertical asymptote $x = 2$ and horizontal asymptote $y = 4$. Determine the vertical and horizontal asymptotes for the graph of the given transformation of r.

7. $s(x) = r(x - 1)$

8. $t(x) = r(x) - 5$

9–10 ■ *True or False?*

9. Let $r(x) = \dfrac{x^2 + x}{(x + 1)(2x - 4)}$. The graph of r has

(a) vertical asymptote $x = 2$.

(b) vertical asymptote $x = -1$.

(c) horizontal asymptote $y = 1$.

(d) horizontal asymptote $y = \frac{1}{2}$.

10. The graph of a rational function may cross a horizontal asymptote.

Skills

11–14 ■ **Table of Values** A rational function is given. **(a)** Complete each table for the function. **(b)** Describe the behavior of the function near its vertical asymptote, based on Tables 1 and 2. **(c)** Determine the horizontal asymptote, based on Tables 3 and 4.

Table 1

x	$r(x)$
1.5	
1.9	
1.99	
1.999	

Table 2

x	$r(x)$
2.5	
2.1	
2.01	
2.001	

Table 3

x	$r(x)$
10	
50	
100	
1000	

Table 4

x	$r(x)$
-10	
-50	
-100	
-1000	

11. $r(x) = \dfrac{x}{x-2}$

12. $r(x) = \dfrac{4x+1}{x-2}$

13. $r(x) = \dfrac{3x-10}{(x-2)^2}$

14. $r(x) = \dfrac{3x^2+1}{(x-2)^2}$

15–22 ■ Graphing Rational Functions Using Transformations Use transformations of the graph of $y = 1/x$ to graph the rational function, and state the domain and range, as in Example 2.

15. $r(x) = \dfrac{4}{x-2}$

16. $r(x) = \dfrac{9}{x+3}$

17. $s(x) = -\dfrac{2}{x+1}$

18. $s(x) = -\dfrac{3}{x-4}$

19. $t(x) = \dfrac{2x-3}{x-2}$

20. $t(x) = \dfrac{3x-3}{x+2}$

21. $r(x) = \dfrac{x+2}{x+3}$

22. $r(x) = \dfrac{2x-9}{x-4}$

23–28 ■ Intercepts of Rational Functions Find the x- and y-intercepts of the rational function.

23. $r(x) = \dfrac{x-1}{x+4}$

24. $s(x) = \dfrac{3x}{x-5}$

25. $t(x) = \dfrac{x^2-x-2}{x-6}$

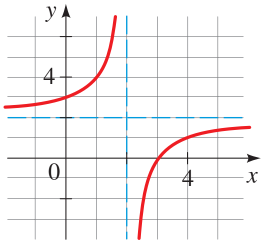
26. $r(x) = \dfrac{2}{x^2+3x-4}$

27. $r(x) = \dfrac{x^2-9}{x^2}$

28. $r(x) = \dfrac{x^3+8}{x^2+4}$

29–32 ■ Getting Information from a Graph From the graph, determine the x- and y-intercepts and the vertical and horizontal asymptotes.

29.



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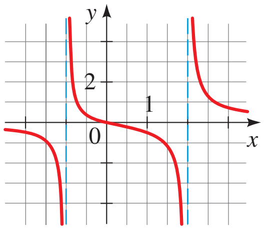

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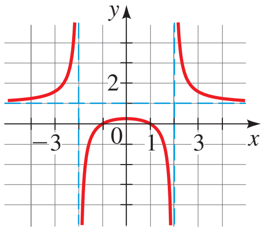

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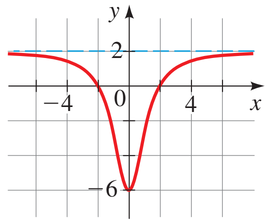


33–44 ■ Asymptotes Find all vertical and horizontal asymptotes (if any).

33. $r(x) = \dfrac{5}{x-2}$

34. $r(x) = \dfrac{2x-3}{x^2-1}$

35. $r(x) = \dfrac{3x+10}{x^2+5}$

36. $r(x) = \dfrac{2x^3+x^2}{x^4-16}$

37. $s(x) = \dfrac{10x^3-7}{x^3-x}$

38. $s(x) = \dfrac{18x^2+9}{9x^2+1}$

39. $r(x) = \dfrac{(x+1)(2x-3)}{(x-2)(4x+7)}$

40. $r(x) = \dfrac{(x-3)(x+2)}{(5x+1)(2x-3)}$

41. $r(x) = \dfrac{6x^3-2}{2x^3+5x^2+6x}$

42. $r(x) = \dfrac{5x^3}{x^3+2x^2+5x}$

43. $t(x) = \dfrac{x^2+2}{x-1}$

44. $r(x) = \dfrac{x^3+3x^2}{x^2-4}$

45–64 ■ Graphing Rational Functions Find the intercepts and asymptotes, and then sketch a graph of the rational function. State the domain and estimate the range from your graph.

45. $r(x) = \dfrac{2x+2}{x-1}$

46. $r(x) = \dfrac{1-3x}{2x+4}$

47. $r(x) = \dfrac{3x^2-12x+13}{x^2-4x+4}$

48. $r(x) = \dfrac{-2x^2-8x-9}{x^2+4x+4}$

49. $r(x) = \dfrac{-x^2+8x-18}{x^2-8x+16}$

50. $r(x) = \dfrac{x^2+2x+3}{2x^2+4x+2}$

51. $s(x) = \dfrac{4x-8}{(x-4)(x+1)}$

52. $s(x) = \dfrac{9}{x^2-5x+4}$

53. $s(x) = \dfrac{9x-18}{x^2+x-2}$

54. $s(x) = \dfrac{x+2}{(x+3)(x-1)}$

55. $r(x) = \dfrac{(x-1)(x+2)}{(x+1)(x-3)}$

56. $r(x) = \dfrac{2x^2+10x-12}{x^2+x-6}$

57. $r(x) = \dfrac{x^2+2x-8}{x^2+2x}$

58. $r(x) = \dfrac{3x^2+6}{x^2-4x}$

59. $s(x) = \dfrac{x^2-2x+1}{x^3-3x^2}$

60. $r(x) = \dfrac{x^2-x-6}{x^2+3x}$

61. $r(x) = \dfrac{x^2-2x+1}{x^2+2x+1}$

62. $r(x) = \dfrac{9x^2}{4x^2+4x-8}$

63. $r(x) = \dfrac{5x^2+10x+5}{x^2+6x+9}$

64. $t(x) = \dfrac{x^3-x^2}{x^3-3x-2}$

65–70 ■ Rational Functions with Holes Find the factors that are common in the numerator and the denominator. Then find the intercepts and asymptotes, and sketch a graph of the rational function. State the domain and range of the function.

65. $r(x) = \dfrac{x^2 + 4x - 5}{x^2 + x - 2}$

66. $r(x) = \dfrac{x^2 + 3x - 10}{(x + 1)(x - 3)(x + 5)}$

67. $r(x) = \dfrac{x^2 - 2x - 3}{x + 1}$

68. $r(x) = \dfrac{x^3 - 2x^2 - 3x}{x - 3}$

69. $r(x) = \dfrac{x^3 - 5x^2 + 3x + 9}{x + 1}$

[*Hint:* Check that $x + 1$ is a factor of the numerator.]

70. $r(x) = \dfrac{x^2 + 4x - 5}{x^3 + 7x^2 + 10x}$

71–78 ■ Slant Asymptotes Find the slant asymptote and the vertical asymptotes, and sketch a graph of the function.

71. $r(x) = \dfrac{x^2}{x - 2}$

72. $r(x) = \dfrac{x^2 + 2x}{x - 1}$

73. $r(x) = \dfrac{x^2 - 2x - 8}{x}$

74. $r(x) = \dfrac{3x - x^2}{2x - 2}$

75. $r(x) = \dfrac{x^2 + 5x + 4}{x - 3}$

76. $r(x) = \dfrac{x^3 + 4}{2x^2 + x - 1}$

77. $r(x) = \dfrac{x^3 + x^2}{x^2 - 4}$

78. $r(x) = \dfrac{2x^3 + 2x}{x^2 - 1}$

Skills Plus

79–82 ■ End Behavior Graph the rational function f, and determine all vertical asymptotes from your graph. Then graph f and g in a sufficiently large viewing rectangle to show that they have the same end behavior.

79. $f(x) = \dfrac{2x^2 + 6x + 6}{x + 3}, \quad g(x) = 2x$

80. $f(x) = \dfrac{-x^3 + 6x^2 - 5}{x^2 - 2x}, \quad g(x) = -x + 4$

81. $f(x) = \dfrac{x^3 - 2x^2 + 16}{x - 2}, \quad g(x) = x^2$

82. $f(x) = \dfrac{-x^4 + 2x^3 - 2x}{(x - 1)^2}, \quad g(x) = 1 - x^2$

83–88 ■ End Behavior Graph the rational function, and find all vertical asymptotes, x- and y-intercepts, and local extrema, correct to the nearest tenth. Then use long division to find a polynomial that has the same end behavior as the rational function, and graph both functions in a sufficiently large viewing rectangle to verify that the end behaviors of the polynomial and the rational function are the same.

83. $y = \dfrac{2x^2 - 5x}{2x + 3}$

84. $y = \dfrac{x^4 - 3x^3 + x^2 - 3x + 3}{x^2 - 3x}$

85. $y = \dfrac{x^5}{x^3 - 1}$

86. $y = \dfrac{x^4}{x^2 - 2}$

87. $r(x) = \dfrac{x^4 - 3x^3 + 6}{x - 3}$

88. $r(x) = \dfrac{4 + x^2 - x^4}{x^2 - 1}$

89–92 ■ Families of Rational Functions Draw the family of rational functions in the same viewing rectangle using the given values of c. What properties do the members of the family share? How do they differ?

89. $r(x) = \dfrac{cx}{x^2 + 1}; \quad c = 1, 2, 3, 4$

90. $r(x) = \dfrac{x + 1}{x + c}; \quad c = 2, 3, 4, 5$

91. $r(x) = \dfrac{cx^2}{x^2 - 1}; \quad c = 1, 2, 3, 4$

92. $r(x) = \dfrac{x^2 - c}{x + 5}; \quad c = 1, 4, 9, 16$

Applications

93. Average Cost A manufacturer of leather purses finds that the cost (in dollars) of producing x purses is given by the function $C(x) = 750 + 45x + 0.03x^2$.

(a) Explain why the average cost per purse is given by the rational function

$$A(x) = \frac{C(x)}{x}$$

(b) Graph the function A and interpret the graph.

94. Population Growth Suppose that the rabbit population on a farm follows the formula

$$p(t) = \frac{3000t}{t + 1}$$

where $t \geq 0$ is the time (in months) since the beginning of the year.

(a) Draw a graph of the rabbit population.

(b) What eventually happens to the rabbit population?

95. Drug Concentration A drug is administered to a patient, and the concentration of the drug in the bloodstream is monitored. At time $t \geq 0$ (in hours since giving the drug) the concentration (in mg/L) is given by

$$c(t) = \frac{5t}{t^2 + 1}$$

Graph the function c with a graphing device.

(a) What is the highest concentration of drug that is reached in the patient's bloodstream?

(b) What happens to the drug concentration after a long period of time?

(c) How long does it take for the concentration to drop below 0.3 mg/L?

96. Flight of a Rocket Suppose a rocket is fired upward from the surface of the earth with an initial velocity v (measured in meters per second). Then the maximum height h (in meters) reached by the rocket is given by the function

$$h(v) = \frac{Rv^2}{2gR - v^2}$$

where $R = 6.4 \times 10^6$ m is the radius of the earth and $g = 9.8 \text{ m/s}^2$ is the acceleration due to gravity. Use a graphing device to draw a graph of the function h. (Note that h and v must both be positive, so the viewing rectangle need not contain negative values.) What does the vertical asymptote represent physically?

97. The Doppler Effect As a train moves toward an observer (see the figure), the pitch of its whistle sounds higher to the observer than it would if the train were at rest, because the crests of the sound waves are compressed closer together. This phenomenon is called the *Doppler effect*. The observed pitch P is a function of the speed v of the train and is given by

$$P(v) = P_0\left(\frac{s_0}{s_0 - v}\right)$$

where P_0 is the actual pitch of the whistle at the source and $s_0 = 332$ m/s is the speed of sound in air. Suppose that a train has a whistle pitched at $P_0 = 440$ Hz. Graph the function $y = P(v)$ using a graphing device. How can the vertical asymptote of this function be interpreted physically?

98. Focusing Distance For a camera with a lens of fixed focal length F to focus on an object located a distance x from the lens, the image sensor must be placed a distance y behind the lens, where F, x, and y are related by

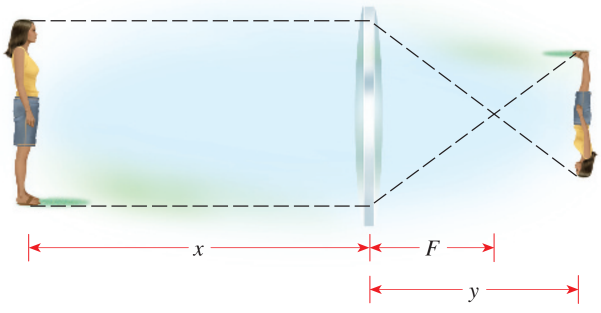
$$\frac{1}{x} + \frac{1}{y} = \frac{1}{F}$$

(See the figure.) Suppose the camera has a 55-mm lens ($F = 55$).

(a) Express y as a function of x, and graph the function.

(b) What happens to the focusing distance y as the object moves far away from the lens?

(c) What happens to the focusing distance y as the object moves close to the lens?



99. Salt Concentration A large tank is to be filled with brine. At time $t = 0$, the tank contains 100 gallons of water and 4 lb of salt. Water is being pumped into the tank at the rate of 50 gallons per minute and at the same time salt is poured into the tank at the rate of 5 pounds per minute.

(a) Express the concentration C of salt (in lb/gal) in the tank as a function of time t. Graph the function $C(t)$ for $t \geq 0$.

(b) What is the salt concentration in the tank after 10 minutes? What is the salt concentration when the tank has 1000 gal of water?

(c) If the process of adding water and salt can continue indefinitely (t approaches infinity), what would the salt concentration in the tank approach?

■ Discuss ■ Discover ■ Prove ■ Write

100. Discuss: Constructing a Rational Function From Its Asymptotes Give an example of a rational function that has vertical asymptote $x = 3$. Now give an example of one that has vertical asymptote $x = 3$ *and* horizontal asymptote $y = 2$. Now give an example of a rational function with vertical asymptotes $x = 1$ and $x = -1$, horizontal asymptote $y = 0$, and x-intercept 4.

101. Discuss: A Rational Function With No Asymptote Explain how you can tell (without graphing it) that the function

$$r(x) = \frac{x^6 + 10}{x^4 + 8x^2 + 15}$$

has no x-intercept and no horizontal, vertical, or slant asymptote. What is its end behavior?'

102. Discover: Transformations of $y = 1/x^2$ In Example 2 we saw that some simple rational functions can be graphed by shifting, stretching, or reflecting the graph of $y = 1/x$. In this exercise we consider rational functions that can be graphed by transforming the graph of $y = 1/x^2$.

(a) Graph the function

$$r(x) = \frac{1}{(x-2)^2}$$

by transforming the graph of $y = 1/x^2$.

(b) Use long division and factoring to show that the function

$$s(x) = \frac{2x^2 + 4x + 5}{x^2 + 2x + 1}$$

can be written as

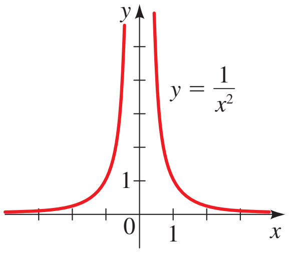
$$s(x) = 2 + \frac{3}{(x+1)^2}$$

Then graph s by transforming the graph of $y = 1/x^2$.

(c) One of the following functions can be graphed by transforming the graph of $y = 1/x^2$; the other cannot. Use transformations to graph the one that can be, and explain why this method doesn't work for the other one.

$$p(x) = \frac{2 - 3x^2}{x^2 - 4x + 4}$$

$$q(x) = \frac{12x - 3x^2}{x^2 - 4x + 4}$$



3.7 Polynomial and Rational Inequalities

■ Polynomial Inequalities ■ Rational Inequalities

In Section 1.8 we solved basic inequalities. In this section we solve polynomial and rational inequalities by using the methods we learned in Section 3.4 for factoring and graphing polynomials.

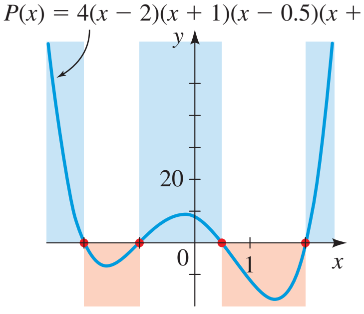


Figure 1 | $P(x) > 0$ or $P(x) < 0$ for x between successive zeros of P

■ Polynomial Inequalities

An important consequence of the Intermediate Value Theorem (Section 3.2) is that the values of a polynomial function P do not change sign between successive zeros. In other words, the values of P between successive zeros are either all positive or all negative. Graphically, this means that between successive x-intercepts, the graph of P is entirely above or entirely below the x-axis. Figure 1 illustrates this property of polynomials. This property allows us to solve **polynomial inequalities** like $P(x) \geq 0$ by finding the zeros of the polynomial and using test values between successive zeros to determine the intervals that satisfy the inequality. We use the following guidelines.

Solving Polynomial Inequalities

1. **Move All Terms to One Side.** Rewrite the inequality so that all nonzero terms appear on one side of the inequality symbol.
2. **Factor the Polynomial.** Factor the polynomial into irreducible factors, and find the **real zeros** of the polynomial.
3. **Find the Intervals.** List the intervals determined by the real zeros.
4. **Make a Table or Diagram.** Use **test values** to make a table or diagram of the signs of each factor in each interval. In the last row of the table determine the sign of the polynomial on that interval.
5. **Solve.** Determine the solutions of the inequality from the last row of the table. Check whether the **endpoints** of these intervals satisfy the inequality. (This may happen if the inequality involves $\leq$ or $\geq$.)

Example 1 ■ Solving a Polynomial Inequality

Solve the inequality $2x^3 + x^2 + 6 \geq 13x$.

Solution We follow the guidelines for solving polynomial inequalities.

Move all terms to one side. We move all terms to the left-hand side of the inequality to get

$$2x^3 + x^2 - 13x + 6 \geq 0$$

The left-hand side is a polynomial.

Factor the polynomial. This polynomial is factored in Example 3.4.2:

$$(x - 2)(2x - 1)(x + 3) \geq 0$$

The zeros of the polynomial are -3, $\frac{1}{2}$, and 2.

Find the intervals. The intervals determined by the zeros of the polynomial are

$$(-\infty, -3), \qquad \left(-3, \tfrac{1}{2}\right), \qquad \left(\tfrac{1}{2}, 2\right), \qquad (2, \infty)$$

Make a table or diagram. To determine the sign of each factor on each of the intervals we use test values. That is, we choose a number inside each interval and check the sign of each factor, as shown in the diagram.

See Section 1.8 for instructions on making a sign diagram.

	Test value $x = -4$	-3	Test value $x = 0$	$\frac{1}{2}$	Test value $x = 1$	2	Test value $x = 3$
Sign of $x - 2$	$-$		$-$		$-$		$+$
Sign of $2x - 1$	$-$		$-$		$+$		$+$
Sign of $x + 3$	$-$		$+$		$+$		$+$
Sign of $(x - 2)(2x - 1)(x + 3)$	$-$		$+$		$-$		$+$

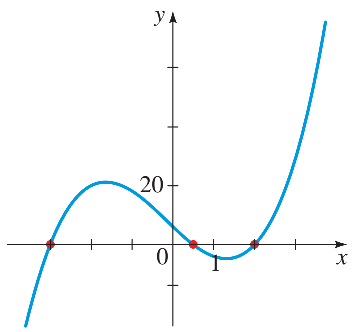


Figure 2

Solve. The last row of the diagram shows that the inequality

$$(x - 2)(2x - 1)(x + 3) \geq 0$$

is satisfied on the intervals $\left(-3, \frac{1}{2}\right)$ and $(2, \infty)$. Checking the endpoints, we see that -3, $\frac{1}{2}$, and 2 satisfy the inequality, so the solution is $\left[-3, \frac{1}{2}\right] \cup [2, \infty)$. The graph in Figure 2 confirms our solution.

Now Try Exercise 7 ■

Example 2 ■ Solving a Polynomial Inequality

Solve the inequality $3x^4 - x^2 - 4 < 2x^3 + 12x$.

Solution We follow the guidelines for solving polynomial inequalities.

Move all terms to one side. We move all terms to the left-hand side of the inequality to get

$$3x^4 - 2x^3 - x^2 - 12x - 4 < 0$$

The left-hand side is a polynomial.

Factor the polynomial. This polynomial is factored into linear and irreducible quadratic factors in Example 3.5.5:

$$(x - 2)(3x + 1)(x^2 + x + 2) < 0$$

From the first two factors we obtain the zeros 2 and $-\frac{1}{3}$. The third factor has no real zeros.

Find the intervals. The intervals determined by the zeros of the polynomial are

$$\left(-\infty, -\tfrac{1}{3}\right), \quad \left(-\tfrac{1}{3}, 2\right), \quad (2, \infty)$$

Make a table or diagram. We make a sign diagram by using test values.

		$-\frac{1}{3}$		2	
Sign of $x - 2$	$-$		$-$		$+$
Sign of $3x + 1$	$-$		$+$		$+$
Sign of $x^2 + x + 2$	$+$		$+$		$+$
Sign of $(x - 2)(3x + 1)(x^2 + x + 2)$	$+$		$-$		$+$

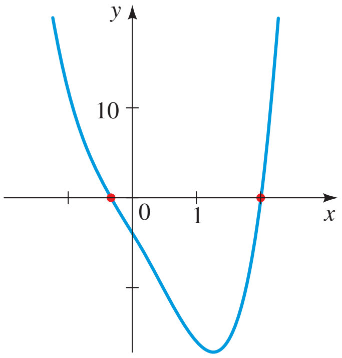


Figure 3

Solve. The inequality requires that the values of the polynomial be less than 0. From the last row of the sign diagram we see that the inequality is satisfied on the interval $\left(-\frac{1}{3}, 2\right)$. You can check that the two endpoints do not satisfy the inequality, so the solution is $\left(-\frac{1}{3}, 2\right)$. The graph in Figure 3 confirms our solution.

Now Try Exercise 11 ■

■ Rational Inequalities

Unlike polynomial functions, rational functions are not necessarily continuous. The vertical asymptotes of a rational function r break up the graph into separate "branches." So the intervals on which r does not change sign are determined by the vertical asymptotes as well as the zeros of r. This is the reason for the following definition: If $r(x) = P(x)/Q(x)$ is a rational function, the **cut points** of r are the values of x at which either $P(x) = 0$ or $Q(x) = 0$. In other words, the cut points of r are the zeros of the numerator and the zeros of the denominator (see Figure 4). So to solve a **rational inequality** like $r(x) \geq 0$, we use test points between successive cut points to determine the intervals that satisfy the inequality. We use the following guidelines.

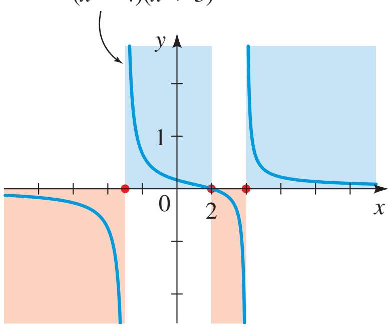


Figure 4 | $r(x) > 0$ or $r(x) < 0$ for x between successive cut points of r

Solving Rational Inequalities

1. **Move All Terms to One Side.** Rewrite the inequality so that all nonzero terms appear on one side of the inequality symbol. Bring all quotients to a common denominator.
2. **Factor Numerator and Denominator.** Factor the numerator and denominator into irreducible factors, and then find the **cut points**.
3. **Find the Intervals.** List the intervals determined by the cut points.
4. **Make a Table or Diagram.** Use **test values** to make a table or diagram of the sign of each factor in each interval. In the last row of the table determine the sign of the rational function on that interval.
5. **Solve.** Determine the solution of the inequality from the last row of the table. Check whether the **endpoints** of these intervals satisfy the inequality. (This may happen if the inequality involves $\leq$ or $\geq$.)

Example 3 ■ Solving a Rational Inequality

Solve the inequality

$$\frac{1 - 2x}{x^2 - 2x - 3} \geq 1$$

Solution We follow the guidelines for solving rational inequalities.

Move all terms to one side. We move all terms to the left-hand side of the inequality.

$$\frac{1 - 2x}{x^2 - 2x - 3} - 1 \geq 0 \qquad \text{Move terms to LHS}$$

$$\frac{(1 - 2x) - (x^2 - 2x - 3)}{x^2 - 2x - 3} \geq 0 \qquad \text{Common denominator}$$

$$\frac{4 - x^2}{x^2 - 2x - 3} \geq 0 \qquad \text{Simplify}$$

The left-hand side of the inequality is a rational function.

Factor numerator and denominator. Factoring the numerator and denominator, we get

$$\frac{(2 - x)(2 + x)}{(x - 3)(x + 1)} \geq 0$$

The zeros of the numerator are 2 and -2, and the zeros of the denominator are -1 and 3, so the cut points are -2, -1, 2, and 3.

Find the intervals. The intervals determined by the cut points are

$$(-\infty, -2), \quad (-2, -1), \quad (-1, 2), \quad (2, 3), \quad (3, \infty)$$

Make a table or diagram. We make a sign diagram by using test values.

		-2		-1		2		3	
Sign of $2 - x$	$+$		$+$		$+$		$-$		$-$
Sign of $2 + x$	$-$		$+$		$+$		$+$		$+$
Sign of $x - 3$	$-$		$-$		$-$		$-$		$+$
Sign of $x + 1$	$-$		$-$		$+$		$+$		$+$
Sign of $\dfrac{(2 - x)(2 + x)}{(x - 3)(x + 1)}$	$-$		$+$		$-$		$+$		$-$

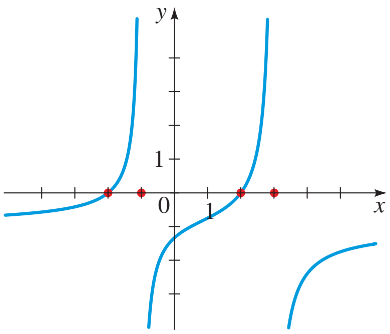


Figure 5

Solve. From the last row of the sign diagram we see that the inequality is satisfied on the intervals $(-2, -1)$ and $(2, 3)$. Checking the endpoints, we see that -2 and 2 satisfy the inequality, so the solution is $[-2, -1) \cup [2, 3)$. The graph in Figure 5 confirms our solution.

Now Try Exercises 19 and 27 ■

Example 4 ■ Solving a Rational Inequality

Solve the inequality

$$\frac{x^2 - 4x + 3}{x^2 - 4x - 5} \geq 0$$

Solution Since all nonzero terms are already on one side of the inequality symbol, we begin by factoring.

Factor numerator and denominator. Factoring the numerator and denominator, we get

$$\frac{(x - 3)(x - 1)}{(x - 5)(x + 1)} \geq 0$$

The cut points are -1, 1, 3, and 5.

Find the intervals. The intervals determined by the cut points are

$$(-\infty, -1), \qquad (-1, 1), \qquad (1, 3), \qquad (3, 5), \qquad (5, \infty)$$

Make a table or diagram. We make a sign diagram.

		-1		1		3		5	
Sign of $x - 5$	$-$		$-$		$-$		$-$		$+$
Sign of $x - 3$	$-$		$-$		$-$		$+$		$+$
Sign of $x - 1$	$-$		$-$		$+$		$+$		$+$
Sign of $x + 1$	$-$		$+$		$+$		$+$		$+$
Sign of $\dfrac{(x-3)(x-1)}{(x-5)(x+1)}$	$+$		$-$		$+$		$-$		$+$

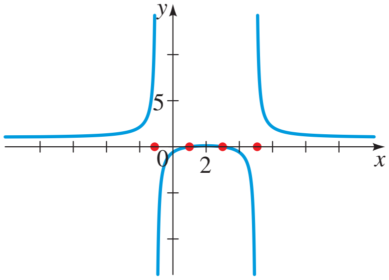


Figure 6

Solve. From the last row of the sign diagram we see that the inequality is satisfied on the intervals $(-\infty, -1)$, $(1, 3)$, and $(5, \infty)$. Checking the endpoints, we see that 1 and 3 satisfy the inequality, so the solution is $(-\infty, -1) \cup [1, 3] \cup (5, \infty)$. The graph in Figure 6 confirms our solution.

Now Try Exercises 23 and 29 ■

We can also solve polynomial and rational inequalities graphically (see Sections 1.11 and 2.3). In the next example we graph each side of the inequality and compare the values of left- and right-hand sides graphically.

Example 5 ■ Solving a Rational Inequality Graphically

Two light sources are 10 m apart. One is three times as intense as the other. The light intensity L (in lux) at a point x meters from the weaker source is given by

$$L(x) = \frac{10}{x^2} + \frac{30}{(10 - x)^2}$$

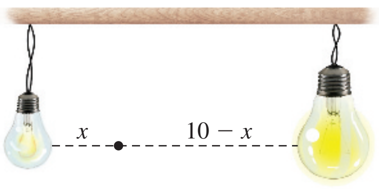


Figure 7

(See Figure 7.) Find the points at which the light intensity is 4 lux or less.

Solution We need to solve the inequality

$$\frac{10}{x^2} + \frac{30}{(10 - x)^2} \le 4$$

We solve the inequality graphically by graphing the two functions

$$y_1 = \frac{10}{x^2} + \frac{30}{(10 - x)^2} \qquad \text{and} \qquad y_2 = 4$$

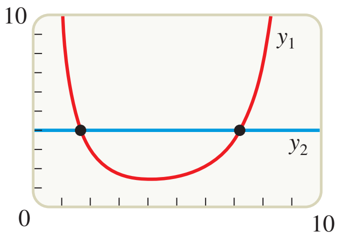


Figure 8

In this physical problem the possible values of x are between 0 and 10, so we graph the two functions in a viewing rectangle with x-values between 0 and 10, as shown in Figure 8. We want those values of x for which $y_1 \le y_2$. The graphs intersect at $x \approx 1.67$ and at $x \approx 7.19$, and between these x-values the graph of y_1 lies below the graph of y_2. So the solution of the inequality is the interval $(1.67, 7.19)$, rounded to two decimal places. Thus the light intensity is less than or equal to 4 lux when the distance from the weaker source is between 1.67 m and 7.19 m.

Now Try Exercises 49 and 59 ■

3.7 Exercises

Concepts

1. To solve a polynomial inequality, we factor the polynomial into irreducible factors and find all the real ________ of the polynomial. Then we find the intervals determined by the real ________ and use test points in each interval to find the sign of the polynomial on that interval. Let

$$P(x) = x(x + 2)(x - 1)$$

Fill in the diagram below to find the intervals on which $P(x) \geq 0$.

Sign of		-2		0		1	
x							
$x + 2$							
$x - 1$							
$x(x + 2)(x - 1)$							

From the diagram above we see that $P(x) \geq 0$ on the intervals ________ and ________.

2. To solve a rational inequality, we factor the numerator and the denominator into irreducible factors. The cut points are the real ________ of the numerator and the real ________ denominator. Then we find the intervals determined by the ________ ________, and we use test points to find the sign of the rational function on each interval. Let

$$r(x) = \frac{(x + 2)(x - 1)}{(x - 3)(x + 4)}$$

Fill in the diagram below to find the intervals on which $r(x) \geq 0$.

Sign of		-4		-2		1		3	
$x + 2$									
$x - 1$									
$x - 3$									
$x + 4$									
$\dfrac{(x + 2)(x - 1)}{(x - 3)(x + 4)}$									

From the diagram we see that $r(x) \geq 0$ on the intervals ________, ________, and ________.

Skills

3–16 ■ Polynomial Inequalities Solve the inequality.

3. $(x - 3)(x + 5)(2x + 5) < 0$

4. $(x - 1)(x + 2)(x - 3)(x + 4) \geq 0$

5. $(x + 5)^2(x + 3)(x - 1) > 0$

6. $(2x - 7)^4(x - 1)^3(x + 1) \leq 0$

7. $x^3 + 4x^2 \geq 4x + 16$

8. $2x^3 - 18x < x^2 - 9$

9. $2x^3 - x^2 < 9 - 18x$

10. $x^4 + 3x^3 < x + 3$

11. $x^4 - 7x^2 - 18 < 0$

12. $4x^4 - 25x^2 + 36 \leq 0$

13. $x^3 + x^2 - 17x + 15 \geq 0$

14. $x^4 + 3x^3 - 3x^2 + 3x - 4 < 0$

15. $x(1 - x^2)^3 > 7(1 - x^2)^3$

16. $x^2(7 - 6x) \leq 1$

17–40 ■ Rational Inequalities Solve the inequality.

17. $\dfrac{x - 1}{x - 10} < 0$

18. $\dfrac{3x - 7}{x + 2} \leq 0$

19. $\dfrac{x - 3}{2x + 5} \geq 1$

20. $\dfrac{x + 4}{x - 5} \leq 4$

21. $\dfrac{5x + 7}{4x + 10} \leq 1$

22. $\dfrac{4x - 6}{x + 7} > 2$

23. $\dfrac{2x + 5}{x^2 + 2x - 35} \geq 0$

24. $\dfrac{4x^2 - 25}{x^2 - 9} \leq 0$

25. $\dfrac{x^2}{x^2 + 3x - 10} \leq 0$

26. $\dfrac{x - 3}{x^2 + 6x + 9} \leq 0$

27. $\dfrac{x^2 + 3}{x + 1} > 2$

28. $\dfrac{4x - 3}{x^2 + 1} \leq 1$

29. $\dfrac{x^2 + 2x - 3}{3x^2 - 7x - 6} > 0$

30. $\dfrac{x - 1}{x^3 + 1} \geq 0$

31. $\dfrac{x^3 + 3x^2 - 9x - 27}{x + 4} \leq 0$

32. $\dfrac{x^2 - 16}{x^4 - 16} < 0$

33. $\dfrac{(x - 1)^2}{(x + 1)(x + 2)} > 0$

34. $\dfrac{x^2 - 2x + 1}{x^3 + 3x^2 + 3x + 1} \leq 0$

35. $\dfrac{x}{2} \geq \dfrac{5}{x + 1} + 4$

36. $\dfrac{x + 2}{x + 3} < \dfrac{x - 1}{x - 2}$

37. $\dfrac{6}{x - 1} - \dfrac{6}{x} \geq 1$

38. $\dfrac{1}{x - 3} + \dfrac{1}{x + 2} \geq \dfrac{2x}{x^2 + x - 2}$

39. $\dfrac{1}{x + 1} - \dfrac{1}{x + 2} \leq \dfrac{1}{(x + 2)^2}$

40. $\dfrac{1}{x} + \dfrac{1}{x + 1} < \dfrac{2}{x + 2}$

41–44 ■ Graphs of Two Functions Find all values of x for which the graph of f lies above the graph of g.

41. $f(x) = x^2;\quad g(x) = 3x + 10$

42. $f(x) = \dfrac{1}{x};\quad g(x) = \dfrac{1}{x - 1}$

43. $f(x) = 4x;\quad g(x) = \dfrac{1}{x}$

44. $f(x) = x^2 + x;\quad g(x) = \dfrac{2}{x}$

45–48 ■ Domain of a Function Find the domain of the given function.

45. $f(x) = \sqrt{6 + x - x^2}$

46. $g(x) = \sqrt{\dfrac{5 + x}{5 - x}}$

47. $h(x) = \sqrt[4]{x^4 - 1}$

48. $f(x) = \dfrac{1}{\sqrt{x^4 - 5x^2 + 4}}$

49–54 ■ Solving Inequalities Graphically Use a graphing device to solve the inequality, as in Example 5. Express your answer using interval notation, with the endpoints of the intervals rounded to two decimal places.

49. $x^3 - 2x^2 - 5x + 6 \geq 0$

50. $2x^3 + x^2 - 8x - 4 \leq 0$

51. $2x^3 - 3x + 1 < 0$

52. $x^4 - 4x^3 + 8x > 0$

53. $5x^4 < 8x^3$

54. $x^5 + x^3 \geq x^2 + 6x$

Skills Plus

55–56 ■ Rational Inequalities Solve the inequality. (These exercises involve expressions that arise in calculus.)

55. $\dfrac{(1-x)^2}{\sqrt{x}} \geq 4\sqrt{x}(x-1)$

56. $\frac{2}{3}x^{-1/3}(x+2)^{1/2} + \frac{1}{2}x^{2/3}(x+2)^{-1/2} < 0$

57. General Polynomial Inequality Solve the inequality

$$(x-a)(x-b)(x-c)(x-d) \geq 0$$

where $a < b < c < d$.

58. General Rational Inequality Solve the inequality

$$\frac{x^2 + (a-b)x - ab}{x+c} \leq 0$$

where $0 < a < b < c$.

Applications

59. Bonfire Temperature In the vicinity of a bonfire the temperature T (in °C) at a distance of x meters from the center of the fire is given by

$$T(x) = 25 + \frac{2500}{x^2 + 2}$$

At what range of distances from the fire's center is the temperature less than 300°C?

60. Stopping Distance For a certain model of car the distance d required to stop the vehicle if it is traveling at v mi/h is given by the function

$$d(t) = v + \frac{v^2}{25}$$

where d is measured in feet. What range of speeds ensure that the stopping distance of the car does not exceed 175 ft?

61. Managing Traffic A highway engineer develops a formula to estimate the number of cars that can safely travel a particular highway at a given speed x (in mi/h). The engineer finds that the number N of cars that can pass a given point per minute is modeled by the function

$$N(x) = \frac{88x}{17 + 17\left(\dfrac{x}{20}\right)^2}$$

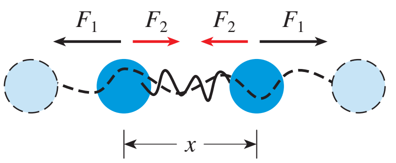
Graph the function in the viewing rectangle $[0, 100]$ by $[0, 60]$. If the number of cars that pass by the given point is greater than 40, at what range of speeds can the cars travel?

62. Two Forces Two balls experience a repulsive force F_1 that pushes them apart as well as an attractive force F_2 that pulls them together. The force F_1 is inversely proportional to the square of the distance between the two balls, whereas F_2 is inversely proportional to the cube of the distance between them. The constant of proportionality is -3 for force F_1 and 1 for F_2. The *net force* F between the balls is the sum of these two forces.

(a) Express F as a rational function of the distance x between the two balls and graph the function $F(x)$ for $x > 0$.

(b) From the graph, determine the distance x at which the net force is zero. For what distances is the net force attractive $[F(x) > 0]$? Repulsive $[F(x) < 0]$? At what distance does the net force achieve its greatest repulsive magnitude?

(c) Describe the net force F as the distance between the balls gets very small (approaches zero) or very large (approaches infinity). In each case state whether the net force is attractive or repulsive, and comment on the magnitude of the force.



Chapter 3 Review

Properties and Formulas

Quadratic Functions | Section 3.1

A **quadratic function** is a function of the form

$$f(x) = ax^2 + bx + c$$

It can be expressed in the **vertex form**

$$f(x) = a(x-h)^2 + k$$

by completing the square.

The graph of a quadratic function in the vertex form is a **parabola** with **vertex** (h, k).

If $a > 0$, then the quadratic function f has the **minimum value** k at $x = h = -b/(2a)$.

If $a < 0$, then the quadratic function f has the **maximum value** k at $x = h = -b/(2a)$.

Polynomial Functions | Section 3.2

A **polynomial function** of **degree** n is a function P of the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

(where $a_n \neq 0$). The numbers a_i are the **coefficients** of the polynomial; a_n is the **leading coefficient**, and a_0 is the **constant coefficient** (or **constant term**).

The graph of a polynomial function is a smooth, continuous curve.

Real Zeros of Polynomials | Section 3.2

A **zero** of a polynomial P is a number c for which $P(c) = 0$. The following are equivalent ways of describing real zeros of polynomials:

1. c is a real zero of P.
2. $x = c$ is a solution of the equation $P(x) = 0$.
3. $x - c$ is a factor of $P(x)$.
4. c is an x-intercept of the graph of P.

Multiplicity of a Zero | Section 3.2

A zero c of a polynomial P has multiplicity m if m is the highest power for which $(x - c)^m$ is a factor of $P(x)$.

Local Maximums and Minimums | Section 3.2

A polynomial function P of degree n has $n - 1$ or fewer **local extrema** (i.e., local maximums and minimums).

Division of Polynomials | Section 3.3

If P and D are any polynomials [with $D(x) \neq 0$], then we can divide P by D using either **long division** or **synthetic division**. The result of the division can be expressed in either of the following equivalent forms:

$$P(x) = D(x) \cdot Q(x) + R(x)$$

$$\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$$

In this division, P is the **dividend**, D is the **divisor**, Q is the **quotient**, and R is the **remainder**. When the division is continued to its completion, the degree of R is less than the degree of D [or $R(x) = 0$].

Remainder Theorem | Section 3.3

When $P(x)$ is divided by the linear divisor $D(x) = x - c$, the **remainder** is the constant $P(c)$. So one way to **evaluate** a polynomial function P at c is to use synthetic division to divide $P(x)$ by $x - c$ and observe the value of the remainder.

Rational Zeros of Polynomials | Section 3.4

If the polynomial P given by

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

has integer coefficients, then all the **rational zeros** of P have the form

$$x = \pm\frac{p}{q}$$

where p is a divisor of the constant term a_0 and q is a divisor of the leading coefficient a_n.

So to find all the rational zeros of a polynomial, we list all the *possible* rational zeros given by this principle and then check to see which are *actual* zeros by using synthetic division.

Descartes's Rule of Signs | Section 3.4

Let P be a polynomial with real coefficients. Then:

The number of positive real zeros of P either is the number of **changes of sign** in the coefficients of $P(x)$ or is less than that by an even number.

The number of negative real zeros of P either is the number of **changes of sign** in the coefficients of $P(-x)$ or is less than that by an even number.

Upper and Lower Bounds Theorem | Section 3.4

Suppose we divide the polynomial P by the linear expression $x - c$ and arrive at the result

$$P(x) = (x - c) \cdot Q(x) + r$$

If $c > 0$ and the coefficients of Q, followed by r, are all nonnegative, then c is an **upper bound** for the zeros of P.

If $c < 0$ and the coefficients of Q, followed by r (including zero coefficients), are alternately nonnegative and nonpositive, then c is a **lower bound** for the zeros of P.

The Fundamental Theorem of Algebra, Complete Factorization, and the Zeros Theorem | Section 3.5

Every polynomial P of degree n with complex coefficients has exactly n complex zeros, provided that each zero of multiplicity m is counted m times. P factors into n linear factors as follows:

$$P(x) = a(x - c_1)(x - c_2) \cdots (x - c_n)$$

where a is the leading coefficient of P and $c_1, c_1, \ldots, c_n$ are the zeros of P.

Conjugate Zeros Theorem | Section 3.5

If the polynomial P has real coefficients and if $a + bi$ is a zero of P, then its complex conjugate $a - bi$ is also a zero of P.

Linear and Quadratic Factors Theorem | Section 3.5

Every polynomial with real coefficients can be factored into linear and irreducible quadratic factors with real coefficients.

Rational Functions | Section 3.6

A **rational function** r is a quotient of polynomial functions:

$$r(x) = \frac{P(x)}{Q(x)}$$

We generally assume that the polynomials P and Q have no factors in common.

Asymptotes | Section 3.6

The line $x = a$ is a **vertical asymptote** of the function $y = f(x)$ if

$$y \to \infty \quad \text{or} \quad y \to -\infty \quad \text{as} \quad x \to a^+ \quad \text{or} \quad x \to a^-$$

The line $y = b$ is a **horizontal asymptote** of the function $y = f(x)$ if

$$y \to b \quad \text{as} \quad x \to \infty \quad \text{or} \quad x \to -\infty$$

Asymptotes of Rational Functions | Section 3.6

Let $r(x) = \dfrac{P(x)}{Q(x)}$ be a rational function.

(continued)

The vertical asymptotes of r are the lines $x = a$ where a is a zero of Q.

If the degree of P is less than the degree of Q, then the horizontal asymptote of r is the line $y = 0$.

If the degrees of P and Q are the same, then the horizontal asymptote of r is the line $y = b$, where

$$b = \frac{\text{leading coefficient of } P}{\text{leading coefficient of } Q}$$

If the degree of P is greater than the degree of Q, then r has no horizontal asymptote.

Polynomial and Rational Inequalities | Section 3.7

A **polynomial inequality** is an inequality of the form $P(x) \geq 0$, where P is a polynomial. We solve $P(x) \geq 0$ by finding the zeros of P and using test values between successive zeros to determine the intervals that satisfy the inequality.

A **rational inequality** is an inequality of the form $r(x) \geq 0$, where

$$r(x) = \frac{P(x)}{Q(x)}$$

is a rational function. The cut points of r are the values of x at which either $P(x) = 0$ or $Q(x) = 0$. We solve $r(x) \geq 0$ by using test points between successive cut points to determine the intervals that satisfy the inequality.

Concept Check

1. (a) What is the degree of a quadratic function f? What is the vertex form of a quadratic function? How do you put a quadratic function into vertex form?

(b) The quadratic function $f(x) = a(x - h)^2 + k$ is in vertex form. The graph of f is a parabola. What is the vertex of the graph of f? How do you determine whether $f(h) = k$ is a minimum value or a maximum value?

(c) Express $f(x) = x^2 + 4x + 1$ in vertex form. Find the vertex of the graph and the maximum value or minimum value of f.

2. (a) Give the general form of polynomial function P of degree n.

(b) What does it mean to say that c is a zero of P? Give two equivalent conditions that tell us that c is a zero of P.

3. Sketch graphs showing the possible end behaviors of polynomials of odd degree and polynomials of even degree.

4. What steps do you follow to graph a polynomial function P?

5. (a) What is a local maximum point or local minimum point of a polynomial P?

(b) How many local extrema can a polynomial P of degree n have?

6. When we divide a polynomial $P(x)$ by a divisor $D(x)$, the Division Algorithm tells us that we can always obtain a quotient $Q(x)$ and a remainder $R(x)$. State the two forms in which the result of this division can be written.

7. (a) State the Remainder Theorem.

(b) State the Factor Theorem.

(c) State the Rational Zeros Theorem.

8. What steps would you take to find the rational zeros of a polynomial P?

9. Let $P(x) = 2x^4 - 3x^3 + x - 15$.

(a) Explain how Descartes's Rule of Signs is used to determine the possible number of positive and negative real zeros of P.

(b) What does it mean to say that a is a lower bound and b is an upper bound for the zeros of a polynomial?

(c) Explain how the Upper and Lower Bounds Theorem is used to show that all the real zeros of P lie between -3 and 3.

10. (a) State the Fundamental Theorem of Algebra.

(b) State the Complete Factorization Theorem.

(c) State the Zeros Theorem.

(d) State the Conjugate Zeros Theorem.

11. (a) What is a rational function?

(b) What does it mean to say that $x = a$ is a vertical asymptote of $y = f(x)$?

(c) What does it mean to say that $y = b$ is a horizontal asymptote of $y = f(x)$?

(d) Find the vertical and horizontal asymptotes of

$$f(x) = \frac{5x + 3}{x^2 - 4}.$$

12. Let s be the rational function

$$s(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}$$

(a) How do you find the vertical asymptotes of s?

(b) How do you find the horizontal asymptotes of s?

(c) Find the vertical and horizontal asymptotes of

$$r(x) = \frac{5x^3 + 3x^2 - x + 8}{(x - 3)(x - 1)(2x + 1)}$$

13. (a) Under what circumstances does a rational function have a slant asymptote?

(b) How do you determine the end behavior of a rational function?

14. (a) Explain how to solve a polynomial inequality.

(b) Solve the inequality $x^2 - 9 \leq 8x$.

15. (a) What are the cut points of a rational function? Explain how to solve a rational inequality.

(b) Solve the rational inequality $\dfrac{x}{x + 2} - \dfrac{1}{x} \leq 0$.

Answers to the Concept Check can be found at the book companion website **stewartmath.com**.

Exercises

1–4 ■ Graphs of Quadratic Functions A quadratic function is given. **(a)** Express the function in vertex form. **(b)** Graph the function.

1. $f(x) = x^2 + 6x + 2$ **2.** $f(x) = 2x^2 - 8x + 4$

3. $f(x) = 1 - 10x - x^2$ **4.** $g(x) = -2x^2 + 12x$

5–6 ■ Maximum and Minimum Values Find the maximum or minimum value of the quadratic function.

5. $f(x) = -x^2 + 3x - 1$ **6.** $f(x) = 3x^2 - 18x + 5$

7. Height of a Stone A stone is thrown upward from the top of a building. Its height (in feet) above the ground after t seconds is given by the function $h(t) = -16t^2 + 48t + 32$. What maximum height does the stone reach?

8. Profit The profit P (in dollars) generated by selling x units of a certain commodity is given by the function

$$P(x) = -1500 + 12x - 0.004x^2$$

What is the maximum profit, and how many units must be sold to generate it?

9–14 ■ Transformations of Monomials Graph the polynomial by transforming an appropriate graph of the form $y = x^n$. Show clearly all x- and y-intercepts, and state the domain and range.

9. $P(x) = -x^3 + 64$ **10.** $P(x) = 2x^3 - 16$

11. $P(x) = 2(x + 1)^4 - 32$ **12.** $P(x) = 81 - (x - 3)^4$

13. $P(x) = 32 + (x - 1)^5$ **14.** $P(x) = -3(x + 2)^5 + 96$

15–18 ■ Graphing Polynomials in Factored Form A polynomial function P is given. **(a)** Describe the end behavior. **(b)** Sketch a graph of P. Make sure your graph shows all intercepts.

15. $P(x) = (x - 3)(x + 1)(x - 5)$

16. $P(x) = -(x - 5)(x^2 - 9)(x + 2)$

17. $P(x) = -(x - 1)^2(x - 4)(x + 2)^2$

18. $P(x) = x^2(x^2 - 4)(x^2 - 9)$

19–20 ■ Graphing Polynomials A polynomial function P is given. **(a)** Find each zero of P and state its multiplicity. **(b)** Sketch a graph of P.

19. $P(x) = x^3(x - 2)^2$ **20.** $P(x) = x(x + 1)^3(x - 1)^2$

21–26 ■ Graphing Polynomials Use a graphing device to graph the polynomial. Find the x- and y-intercepts and the coordinates of all local extrema, rounded to one decimal place. Describe the end behavior of the polynomial.

21. $P(x) = -x^2 + 8x$

22. $P(x) = x^3 - 4x + 1$

23. $P(x) = -2x^3 + 6x^2 - 2$

24. $P(x) = x^4 + 4x^3$

25. $P(x) = 3x^4 - 4x^3 - 10x - 1$

26. $P(x) = x^5 + x^4 - 7x^3 - x^2 + 6x + 3$

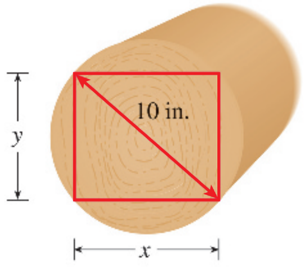
27. Strength of a Beam The strength S of a wooden beam of width x and depth y is given by the formula $S = 13.8xy^2$. A beam is to be cut from a log of diameter 10 in., as shown in the figure.

(a) Express the strength S of this beam as a function of x only.

(b) What is the domain of the function S?

(c) Draw a graph of S.

(d) What width will make the beam the strongest?

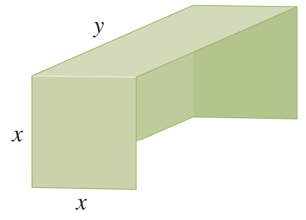


28. Volume A small shelter for delicate plants is to be constructed of thin plastic material. It will have square ends and a rectangular top and back, with an open bottom and front, as shown in the figure. The total area of the four plastic sides is to be 1200 in^2.

(a) Express the volume V of the shelter as a function of the depth x.

(b) Draw a graph of V.

(c) What dimensions will maximize the volume of the shelter?



29–36 ■ Division of Polynomials Find the quotient and remainder.

29. $\dfrac{x^2 - 5x + 2}{x - 3}$ **30.** $\dfrac{3x^2 + x - 5}{x + 2}$

31. $\dfrac{2x^3 - x^2 + 3x - 4}{x + 5}$ **32.** $\dfrac{-x^3 + 2x + 4}{x - 7}$

33. $\dfrac{x^4 - 8x^2 + 2x + 7}{x + 5}$ **34.** $\dfrac{2x^4 + 3x^3 - 12}{x + 4}$

35. $\dfrac{2x^3 + x^2 - 8x + 15}{x^2 + 2x - 1}$ **36.** $\dfrac{x^4 - 2x^2 + 7x}{x^2 - x + 3}$

37–40 ■ Remainder Theorem These exercises involve the Remainder Theorem.

37. If $P(x) = 2x^3 - 9x^2 - 7x + 13$, find $P(5)$.

38. If $Q(x) = x^4 + 4x^3 + 7x^2 + 10x + 15$, find $Q(-3)$.

39. What is the remainder when the polynomial $P(x) = x^{500} + 6x^{101} - x^2 - 2x + 4$ is divided by $x - 1$?

40. What is the remainder when the polynomial $Q(x) = x^{101} - x^4 + 2$ is divided by $x + 1$?

41–42 ■ Factor Theorem Use the Factor Theorem to show that the given statement is true.

41. Show that $\frac{1}{2}$ is a zero of the polynomial

$$P(x) = 2x^4 + x^3 - 5x^2 + 10x - 4$$

42. Show that $x + 4$ is a factor of the polynomial

$$P(x) = x^5 + 4x^4 - 7x^3 - 23x^2 + 23x + 12$$

43–46 ■ Number of Possible Zeros A polynomial P is given. **(a)** List all possible rational zeros (without testing to see whether they are actual zeros). **(b)** Determine the possible number of positive and negative real zeros using Descartes's Rule of Signs.

43. $P(x) = x^5 - 6x^3 - x^2 + 2x + 18$

44. $P(x) = 6x^4 + 3x^3 + x^2 + 3x + 4$

45. $P(x) = 3x^7 - x^5 + 5x^4 + x^3 + 8$

46. $P(x) = 6x^{10} - 2x^8 - 5x^3 + 2x^2 + 12$

47–54 ■ Finding Real Zeros and Graphing Polynomials A polynomial P is given. **(a)** Find all real zeros of P, and state their multiplicities. **(b)** Sketch the graph of P.

47. $P(x) = x^3 - 16x$

48. $P(x) = x^3 - 3x^2 - 4x$

49. $P(x) = x^4 + x^3 - 2x^2$

50. $P(x) = x^4 - 5x^2 + 4$

51. $P(x) = x^4 - 2x^3 - 7x^2 + 8x + 12$

52. $P(x) = x^4 - 2x^3 - 2x^2 + 8x - 8$

53. $P(x) = 2x^4 + x^3 + 2x^2 - 3x - 2$

54. $P(x) = 9x^5 - 21x^4 + 10x^3 + 6x^2 - 3x - 1$

55–56 ■ Polynomials with Specified Zeros Find a polynomial with real coefficients of the specified degree that satisfies the given conditions.

55. Degree 3; zeros $-\frac{1}{2}, 2, 3$; constant coefficient 12

56. Degree 4; zeros 4 (multiplicity 2) and $3i$; integer coefficients; coefficient of x^2 is -25

57. Complex Zeros of Polynomials Does there exist a polynomial of degree 4 with integer coefficients that has zeros i, $2i$, $3i$, and $4i$? If so, find it. If not, explain why.

58. Polynomial with no Real Solutions Prove that the equation $3x^4 + 5x^2 + 2 = 0$ has no real solution.

59–70 ■ Finding Real and Complex Zeros of Polynomials Find all rational, irrational, and complex zeros (and state their multiplicities). Use Descartes's Rule of Signs, the Upper and Lower Bounds Theorem, the Quadratic Formula, or other factoring techniques to help you wherever appropriate.

59. $P(x) = x^3 - x^2 + x - 1$

60. $P(x) = x^3 - 8$

61. $P(x) = x^3 - 3x^2 - 13x + 15$

62. $P(x) = 2x^3 + 5x^2 - 6x - 9$

63. $P(x) = x^4 + 6x^3 + 17x^2 + 28x + 20$

64. $P(x) = x^4 + 7x^3 + 9x^2 - 17x - 20$

65. $P(x) = x^5 - 3x^4 - x^3 + 11x^2 - 12x + 4$

66. $P(x) = x^4 - 81$

67. $P(x) = x^6 - 64$

68. $P(x) = 18x^3 + 3x^2 - 4x - 1$

69. $P(x) = 6x^4 - 18x^3 + 6x^2 - 30x + 36$

70. $P(x) = x^4 + 15x^2 + 54$

71–74 ■ Finding Zeros Graphically Use a graphing device to find all real solutions of the equation, rounded to two decimal places.

71. $2x^2 = 5x + 3$

72. $x^3 + x^2 - 14x - 24 = 0$

73. $x^4 - 3x^3 - 3x^2 - 9x - 2 = 0$

74. $x^5 = x + 3$

75–76 ■ Complete Factorization A polynomial function P is given. Find all the real zeros of P, and factor P completely into linear and irreducible quadratic factors with real coefficients.

75. $P(x) = x^3 - 2x - 4$

76. $P(x) = x^4 + 3x^2 - 4$

77–80 ■ Transformations of $y = 1/x$ A rational function is given. **(a)** Find all vertical and horizontal asymptotes, all x- and y-intercepts, and state the domain and range. **(b)** Use transformations of the graph of $y = 1/x$ to sketch a graph of the function.

77. $r(x) = \dfrac{3}{x + 4}$

78. $r(x) = \dfrac{-1}{x - 5}$

79. $r(x) = \dfrac{3x - 4}{x - 1}$

80. $r(x) = \dfrac{2x + 5}{x + 2}$

81–86 ■ Graphing Rational Functions Graph the rational function. Show clearly all x- and y-intercepts and asymptotes, and state the domain and range of r.

81. $r(x) = \dfrac{3x - 12}{x + 1}$

82. $r(x) = \dfrac{1}{(x + 2)^2}$

83. $r(x) = \dfrac{x - 2}{x^2 - 2x - 8}$

84. $r(x) = \dfrac{x^3 + 27}{x + 4}$

85. $r(x) = \dfrac{x^2 - 9}{2x^2 + 1}$

86. $r(x) = \dfrac{2x^2 - 6x - 7}{x - 4}$

87–90 ■ Rational Functions with Holes Find the common factors of the numerator and denominator of the given rational function. Then find the intercepts and asymptotes, and sketch a graph. State the domain and range.

87. $r(x) = \dfrac{x^2 + 5x - 14}{x - 2}$

88. $r(x) = \dfrac{x^3 - 3x^2 - 10x}{x + 2}$

89. $r(x) = \dfrac{x^2 + 3x - 18}{x^2 - 8x + 15}$

90. $r(x) = \dfrac{x^2 + 2x - 15}{x^3 + 4x^2 - 7x - 10}$

91–94 ■ Graphing Rational Functions Use a graphing device to analyze the graph of the rational function. Find all x- and y-intercepts and all vertical, horizontal, and slant asymptotes. If the function has no horizontal or slant asymptote, find a polynomial that has the same end behavior as the given function.

91. $r(x) = \dfrac{x - 3}{2x + 6}$ **92.** $r(x) = \dfrac{2x - 7}{x^2 + 9}$

93. $r(x) = \dfrac{x^3 + 8}{x^2 - x - 2}$ **94.** $r(x) = \dfrac{2x^3 - x^2}{x + 1}$

95–98 ■ Polynomial Inequalities Solve the inequality.

95. $2x^2 \geq x + 3$ **96.** $x^3 - 3x^2 - 4x + 12 \leq 0$

97. $x^4 - 7x^2 - 18 < 0$ **98.** $x^8 - 17x^4 + 16 > 0$

99–102 ■ Rational Inequalities Solve the inequality.

99. $\dfrac{5}{x^3 - x^2 - 4x + 4} < 0$ **100.** $\dfrac{3x + 1}{x + 2} \leq \dfrac{2}{3}$

101. $\dfrac{1}{x - 2} + \dfrac{2}{x + 3} \geq \dfrac{3}{x}$

102. $\dfrac{1}{x + 2} + \dfrac{3}{x - 3} \leq \dfrac{4}{x}$

103–104 ■ Domain of a Function Find the domain of the given function.

103. $f(x) = \sqrt{24 - x - 3x^2}$

104. $g(x) = \dfrac{1}{\sqrt[4]{x - x^4}}$

105–106 ■ Solving Inequalities Graphically Use a graphing device to solve the inequality. Express your answer using interval notation, with the endpoints of the intervals rounded to two decimal places.

105. $x^4 + x^3 \leq 5x^2 + 4x - 5$

106. $x^5 - 4x^4 + 7x^3 - 12x + 2 > 0$

107. Application of Descartes's Rule of Signs We use Descartes's Rule of Signs to show that a polynomial $Q(x) = 2x^3 + 3x^2 - 3x + 4$ has no positive real zero.

(a) Show that -1 is a zero of the polynomial $P(x) = 2x^4 + 5x^3 + x + 4$.

(b) Use the information from part (a) and Descartes's Rule of Signs to show that the polynomial $Q(x) = 2x^3 + 3x^2 - 3x + 4$ has no positive real zero. [*Hint:* Compare the coefficients of the polynomial Q to your synthetic division table from part (a).]

108. Points of Intersection Find the coordinates of all points of intersection of the graphs of

$$y = x^4 + x^2 + 24x \qquad \text{and} \qquad y = 6x^3 + 20$$

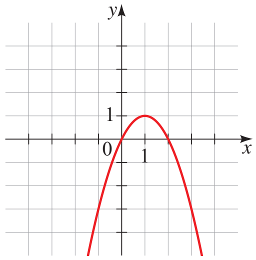
Matching

109. Equations and Their Graphs Match each equation with its graph. Give reasons for your answers. (Don't use a graphing device.)

(a) $y = \dfrac{x^2 + 4}{x(x^2 - 4)}$ **(b)** $y = \frac{1}{16}(x - 2)^2(x + 2)^3$ **(c)** $y = (x + 1)(x - 2)^2$ **(d)** $y = 2x - x^2$

(e) $y = x^2(2x + 3)(x - 1)$ **(f)** $x = 2y - y^2$ **(g)** $y = \dfrac{x^2}{x^2 - 1}$ **(h)** $y = \sqrt{9 - x^2}$

I



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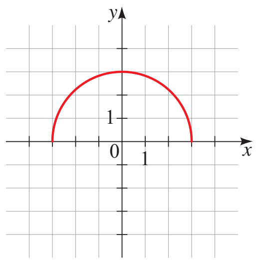

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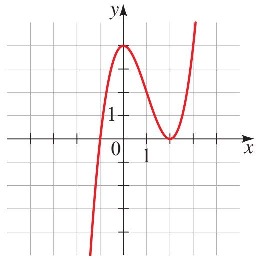

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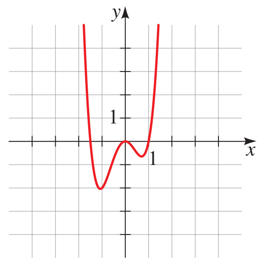

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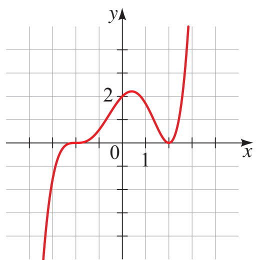

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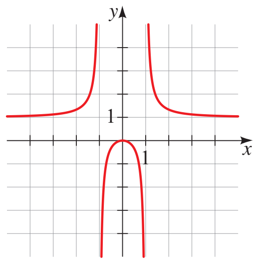

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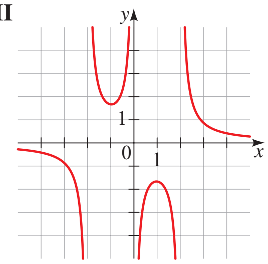

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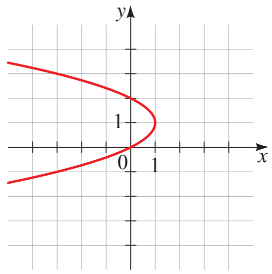
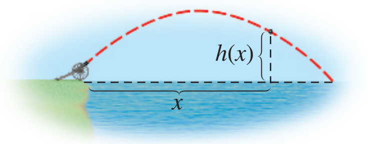


Chapter 3 | Test

1. Express the quadratic function $f(x) = x^2 - x - 6$ in vertex form, sketch its graph, and state the domain and range.

2. Find the maximum or minimum value of the quadratic function $g(x) = 2x^2 + 6x + 3$.

3. A cannonball fired out to sea from a shore battery follows a parabolic trajectory given by the graph of the function



$$h(x) = 10x - 0.01x^2$$

where $h(x)$ is the height of the cannonball above the water when it has traveled a horizontal distance of x feet.

(a) What is the maximum height that the cannonball reaches?

(b) How far does the cannonball travel horizontally before splashing into the water?

4. Graph the polynomial $P(x) = -(x + 2)^3 + 27$, showing clearly all x- and y-intercepts.

5. (a) Use synthetic division to find the quotient and remainder of $x^4 - 4x^2 + 2x + 5$ divided by $x - 2$.

(b) Use long division to find the quotient and remainder of $2x^5 + 4x^4 - x^3 - x^2 + 7$ divided by $2x^2 - 1$.

6. Let $P(x) = 2x^3 - 5x^2 - 4x + 3$.

(a) List all possible rational zeros of P.

(b) Find the complete factorization of P.

(c) Find the zeros of P.

(d) Sketch the graph of P.

7. Find all real and complex zeros of $P(x) = x^3 - x^2 - 4x - 6$.

8. Find the complete factorization of $P(x) = x^4 - 2x^3 + 5x^2 - 8x + 4$.

9. Find a fourth-degree polynomial with integer coefficients that has zeros $3i$ and -1, with -1 a zero of multiplicity 2.

10. Let $P(x) = 2x^4 - 7x^3 + x^2 - 18x + 3$.

(a) Use Descartes's Rule of Signs to determine how many positive and how many negative real zeros P can have.

(b) Show that 4 is an upper bound and -1 is a lower bound for the real zeros of P.

(c) Draw a graph of P, and use it to estimate the real zeros of P, rounded to two decimal places.

(d) Find the coordinates of all local extrema of P, rounded to two decimal places.

11. Match each property with one of the given polynomials.

(a) Has value 0 at $x = 1$

(b) Has remainder 0 when divided by $x + 1$

(c) Has a factor $x + 2$

(d) Has a zero 3 of multiplicity 3

(e) Has zeros 2 and $2i$

$$P(x) = x^3 + 2x^2 + x \qquad Q(x) = x^3 - 2x^2 - 4x + 8$$

$$R(x) = x^3 + 4x - 5 \qquad S(x) = x^3 - 2x^2 + 4x - 8$$

$$T(x) = x^3 - 9x^2 + 27x - 27$$

12. Consider the following rational functions:

$$r(x) = \frac{2x - 1}{x^2 - x - 2} \qquad s(x) = \frac{x^3 + 27}{x^2 + 4} \qquad t(x) = \frac{x^3 - 9x}{x + 2}$$

$$u(x) = \frac{x^2 + x - 6}{x^2 - 25} \qquad w(x) = \frac{x^3 + 6x^2 + 9x}{x + 3}$$

(a) Which of these rational functions has a horizontal asymptote?

(b) Which of these functions has a slant asymptote?

(c) Which of these functions has no vertical asymptote?

(d) Which of these functions has a "hole"?

(e) What are the asymptotes of the function $r(x)$?

(f) Graph $y = u(x)$, showing clearly any asymptotes and x- and y-intercepts the function may have.

(g) Use long division to find a polynomial P that has the same end behavior as t. Graph P and t on the same screen to verify that they have the same end behavior.

13. Solve the rational inequality $x \le \dfrac{6 - x}{2x - 5}$.

14. Find the domain of the function $f(x) = \dfrac{1}{\sqrt{4 - 2x - x^2}}$.

15. **(a)** Graph the following function in an appropriate viewing rectangle and find all x-intercepts and local extrema, rounded to two decimal places.

$$P(x) = x^4 - 4x^3 + 8x$$

(b) Use your graph from part (a) to solve the inequality

$$x^4 - 4x^3 + 8x \ge 0$$

Express your answer in interval form, with the endpoints rounded to two decimal places.

We have learned how to fit a line to data (see *Focus on Modeling* following Chapter 1). The line models the increasing or decreasing trend in the data. If the data exhibit more variability, such as an increase followed by a decrease, then to model the data, we need to use a curve rather than a line. Figure 1 shows a scatter plot with three possible models that appear to fit the data.

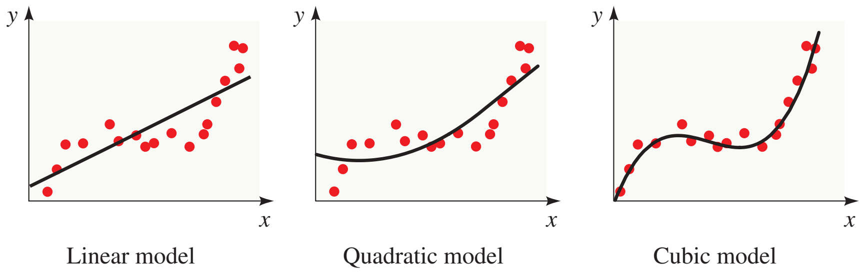


Figure 1

Although we can always find the linear model that best fits the data, it is clear from Figure 1 that sometimes a polynomial model of higher degree fits the data better. In general, it is important to choose an appropriate type of function for modeling real-world data; see the Discovery Project *The Art of Modeling* at **www.stewartmath.com**.

Polynomial Functions as Models

Polynomial functions are ideal for modeling data for which the scatter plot has peaks or valleys (that is, local maximums or minimums). Most graphing devices are programmed to find the **polynomial of best fit** of a specified degree. As is the case for lines, a polynomial of a given degree fits the data *best* if the sum of the squares of the vertical distances between the graph of the polynomial and the data points is minimized.

Dmac/Alamy Stock Photo

Example 1 ■ Rainfall and Crop Yield

Rain is essential for crops to grow, but too much rain can diminish crop yields. The data in the following table give rainfall and cotton yield per acre for 10 seasons in a certain county.

(a) Make a scatter plot of the data. What degree polynomial seems appropriate for modeling the data?

(b) Use a graphing device to find the polynomial of best fit. Graph the polynomial on the scatter plot.

(c) Use the model that you found to estimate the yield if there are 25 inches of rainfall.

Season	Rainfall (in.)	Yield (kg/acre)
1	23.3	5311
2	20.1	4382
3	18.1	3950
4	12.5	3137
5	30.9	5113
6	33.6	4814
7	35.8	3540
8	15.5	3850
9	27.6	5071
10	34.5	3881

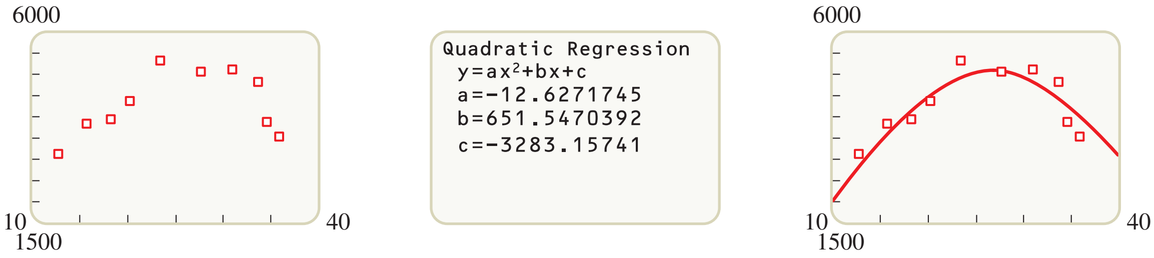
Solution

(a) The scatter plot is shown in Figure 2(a). The data appear to have a peak, so it is appropriate to model the data by a quadratic polynomial (degree 2).

(b) Using a graphing device [see Figure 2(b)], we find that the quadratic polynomial of best fit is approximately

$$y = -12.6x^2 + 651.5x - 3283.2$$

This quadratic model together with the scatter plot are graphed in Figure 2(c).



(a) Scatter plot (b) Quadratic model (c) Scatter plot and model

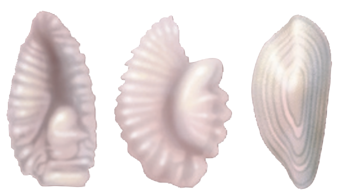
Figure 2

(c) Using the model with $x = 25$, we get

$$y = -12.6(25)^2 + 651.5(25) - 3283.2 \approx 5129.3$$

We estimate the yield to be about 5130 kg/acre. ■

Example 2 ■ Length-at-Age Data for Fish



Otoliths for several fish species

Otoliths ("earstones") are tiny structures that are found in the heads of fish. Microscopic growth rings on the otoliths—not unlike growth rings on a tree—record the age of a fish. The following table gives the lengths of rock bass caught at different ages, as determined by the otoliths. Scientists have proposed a cubic polynomial to model such data.

(a) Use a graphing device to make a scatter plot of the data and to find the cubic polynomial of best fit.

(b) Graph the polynomial from part (a) on the same screen as the scatter plot.

(c) A fisherman catches a rock bass 20 in. long. Use the model to estimate its age.

Using a length-at-age model, marine biologists can estimate the age of a fish from its length (without having to examine its otoliths).

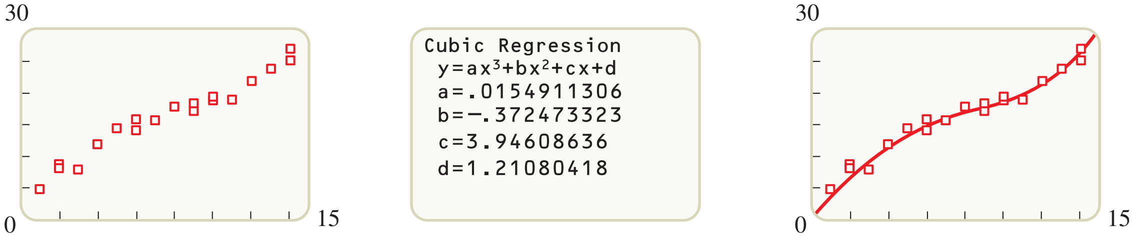
Age (yr)	Length (in.)	Age (yr)	Length (in.)
1	4.8	9	18.2
2	8.8	9	17.1
2	8.0	10	18.8
3	7.9	10	19.5
4	11.9	11	18.9
5	14.4	12	21.7
6	14.1	12	21.9
6	15.8	13	23.8
7	15.6	14	26.9
8	17.8	14	25.1

Solution

(a) Using a graphing device, we get the scatter plot in Figure 3(c), and we find the cubic polynomial of best fit [see Figure 3(b)]:

$$y = 0.0155x^3 - 0.372x^2 + 3.95x + 1.21$$

(b) The scatter plot of the data together with the cubic polynomial are graphed in Figure 3(c).



(a) Scatter plot (b) Cubic model (c) Scatter plot and model

Figure 3

(c) From the graph of the polynomial, we find that $y = 20$ when $x \approx 10.8$. Thus a 20-inch-long rock bass is about 11 years old. ■

Problems

Pressure (lb/in^2)	Tire Life (mi)
26	50,000
28	66,000
31	78,000
35	81,000
38	74,000
42	70,000
45	59,000

1. Tire Inflation and Tread Wear Car tires need to be inflated properly. Overinflation or underinflation can cause premature tread wear. The data in the margin show tire life for different inflation values for a certain type of tire.

(a) Find the quadratic polynomial that best fits the data.

(b) Draw a graph of the polynomial from part (a) together with a scatter plot of the data.

(c) Use your result from part (b) to estimate the pressure that gives the longest tire life.

2. Too Many Corn Plants per Acre? The more corn a farmer plants per acre, the greater the yield, but only up to a point. Too many plants per acre can cause overcrowding and decrease yield. The data give crop yield per acre for various densities of corn planting, as found by researchers at a university test farm.

(a) Find the quadratic polynomial that best fits the data.

(b) Draw a graph of the polynomial from part (a) together with a scatter plot of the data.

(c) Use your result from part (b) to estimate the yield for 37,000 plants per acre.

Density (plants/acre)	15,000	20,000	25,000	30,000	35,000	40,000	45,000	50,000
Crop Yield (bushels/acre)	43	98	118	140	142	122	93	67

3. How Fast Can You List Your Favorite Things? If you are asked to make a list of objects in a certain category, how fast you can list them follows a predictable pattern. For example, if you try to name as many vegetables as you can, you'll probably think of several right away—for example, carrots, peas, beans, corn, and so on. Then after a pause you might think of some that you eat less frequently—perhaps zucchini, eggplant, or asparagus. Finally, a few more exotic vegetables might come to mind—artichokes, jicama, bok choy, and the like. A psychologist performs this experiment on a number of subjects. The table on the next page gives the average number of vegetables that the subjects named in a given number of seconds.

(a) Find the cubic polynomial that best fits the data.

(b) Draw a graph of the polynomial from part (a) together with a scatter plot of the data.

(c) Use your result from part (b) to estimate the number of vegetables that subjects would be able to name in 40 seconds.

(d) According to the model, how long (to the nearest 0.1 second) would it take a person to name five vegetables?

Seconds	Number of Vegetables
1	2
2	6
5	10
10	12
15	14
20	15
25	18
30	21

4. Height of a Baseball A baseball is thrown upward, and its height is measured at 0.5-second intervals using a strobe light. The resulting data are given in the table.

(a) Draw a scatter plot of the data. What degree polynomial is appropriate for modeling the data?

(b) Find a polynomial model that best fits the data, and graph it on the scatter plot.

(c) Find the times when the ball is 20 ft above the ground.

(d) What is the maximum height attained by the ball?

Time (s)	Height (ft)
0	4.2
0.5	26.1
1.0	40.1
1.5	46.0
2.0	43.9
2.5	33.7
3.0	15.8

5. Torricelli's Law Water in a tank will flow out of a small hole in the bottom faster when the tank is nearly full than when it is nearly empty. According to Torricelli's Law, the height $h(t)$ of water remaining at time t is a quadratic function of t.

A certain tank is filled with water and allowed to drain. The height of the water is measured at different times as shown in the table.

(a) Find the quadratic polynomial that best fits the data.

(b) Draw a graph of the polynomial from part (a) together with a scatter plot of the data.

(c) Use your graph from part (b) to estimate how long it takes for the tank to drain completely.

Time (min)	Height (ft)
0	5.0
4	3.1
8	1.9
12	0.8
16	0.2