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1 Fundamentals

In this first chapter we review the real numbers, equations, and the coordinate plane. You are probably already familiar with these concepts, but it is useful to get a fresh look at how these ideas work together to help you solve problems and model (or describe) real-world situations.

In the *Focus on Modeling* at the end of the chapter we learn how to find linear trends in data and how to use these trends to make predictions about the object or process being modeled.

1.1 Real Numbers

■ Real Numbers ■ Properties of Real Numbers ■ Addition and Subtraction ■ Multiplication and Division ■ The Real Line ■ Sets and Intervals ■ Absolute Value and Distance

In the real world we use numbers to measure and compare quantities. For example, we measure temperature, length, height, weight, distance, speed, acceleration, energy, force, angles, pressure, cost, and so on. Figure 1 illustrates some situations in which numbers are used. Numbers also allow us to express relationships between different quantities—for example, relationships between the radius and volume of a ball, between miles driven and gas used, or between education level and starting salary.

Length Weight Speed Pressure

Figure 1 | Measuring with real numbers

■ Real Numbers

Let's review the types of numbers that make up the real number system. We start with the **natural numbers**:

$$1, 2, 3, 4, \ldots$$

The different types of real numbers were invented to meet specific needs. For example, natural numbers are needed for counting, negative numbers for describing debt or below-zero temperatures, rational numbers for concepts like "half a gallon of milk," and irrational numbers for measuring certain distances, like the diagonal of a square.

The **integers** consist of the natural numbers together with their negatives and 0:

$$\ldots, -3, -2, -1, 0, 1, 2, 3, 4, \ldots$$

We construct the **rational numbers** by taking ratios of integers. Thus any rational number r can be expressed as

$$r = \frac{m}{n}$$

where m and n are integers and $n \neq 0$. Examples are

$$\tfrac{1}{2} \qquad -\tfrac{3}{7} \qquad 46 = \tfrac{46}{1} \qquad 0.17 = \tfrac{17}{100}$$

(Recall that division by 0 is always ruled out, so expressions like $\frac{3}{0}$ and $\frac{0}{0}$ are undefined.) There are also real numbers, such as $\sqrt{2}$, that cannot be expressed as a ratio of integers and are therefore called **irrational numbers**. It can be shown, with varying degrees of difficulty, that these numbers are also irrational:

$$\sqrt{3} \qquad \sqrt{5} \qquad \sqrt[3]{2} \qquad \pi \qquad \frac{3}{\pi^2}$$

The set of all real numbers is usually denoted by the symbol $\mathbb{R}$. When we use the word *number* without qualification, we will mean "real number." Figure 2 is a diagram of the types of real numbers that we work with in this book.

Every real number has a decimal representation. If the number is rational, then its corresponding decimal is repeating. For example,

$$\tfrac{1}{2} = 0.5000\ldots = 0.5\overline{0} \qquad \tfrac{2}{3} = 0.66666\ldots = 0.\overline{6}$$

$$\tfrac{157}{495} = 0.3171717\ldots = 0.3\overline{17} \qquad \tfrac{9}{7} = 1.285714285714\ldots = 1.\overline{285714}$$

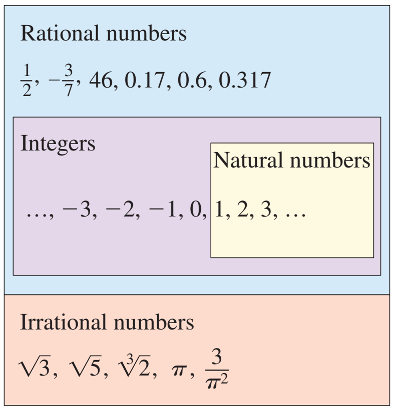


Figure 2 | The real number system

A repeating decimal such as

$$x = 3.5474747...$$

is a rational number. To convert it to a ratio of two integers, we write

$$\begin{aligned} 1000x &= 3547.47474747... \\ 10x &= 35.47474747... \\ \hline 990x &= 3512.0 \end{aligned}$$

Thus $x = \frac{3512}{990}$. (The idea is to multiply x by appropriate powers of 10 and then subtract to eliminate the repeating part.) See also Example 11.3.7 and Exercise 11.3.77.

(The bar indicates that the sequence of digits repeats indefinitely.) If a given number is irrational, its decimal representation is nonrepeating:

$$\sqrt{2} = 1.414213562373095... \qquad \pi = 3.141592653589793...$$

If we stop the decimal expansion of any number at a certain place, we get an approximation to the number. For instance, we can write

$$\pi \approx 3.14159265$$

where the symbol $\approx$ is read "is approximately equal to." The more decimal places we retain, the better our approximation.

■ Properties of Real Numbers

We all know that $2 + 3 = 3 + 2$, and $5 + 7 = 7 + 5$, and $513 + 87 = 87 + 513$, and so on. In algebra we express all these (infinitely many) facts by writing

$$a + b = b + a$$

where a and b stand for any two numbers. In other words, "$a + b = b + a$" is a concise way of saying that "when we add two numbers, the order of addition doesn't matter." This fact is called the *Commutative Property* of addition. From our experience with numbers we know that the properties in the following box are also valid.

Properties of Real Numbers

Property	**Example**	**Description**
Commutative Properties		
$a + b = b + a$	$7 + 3 = 3 + 7$	When we add two numbers, order doesn't matter.
$ab = ba$	$3 \cdot 5 = 5 \cdot 3$	When we multiply two numbers, order doesn't matter.
Associative Properties		
$(a + b) + c = a + (b + c)$	$(2 + 4) + 7 = 2 + (4 + 7)$	When we add three numbers, it doesn't matter which two we add first.
$(ab)c = a(bc)$	$(3 \cdot 7) \cdot 5 = 3 \cdot (7 \cdot 5)$	When we multiply three numbers, it doesn't matter which two we multiply first.
Distributive Property		
$a(b + c) = ab + ac$	$2 \cdot (3 + 5) = 2 \cdot 3 + 2 \cdot 5$	When we multiply a number by a sum of two numbers, we get the same result as we get if we multiply the number by each of the terms and then add the results.
$(b + c)a = ab + ac$	$(3 + 5) \cdot 2 = 2 \cdot 3 + 2 \cdot 5$	

The Distributive Property applies whenever we multiply a number by a sum. Figure 3 explains why this property works for the case in which all the numbers are positive integers, but the property is true for any real numbers a, b, and c.

The Distributive Property is crucial because it describes the way addition and multiplication interact with each other.

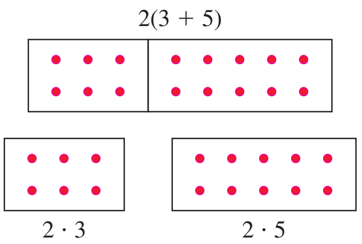


Figure 3 | The Distributive Property

Example 1 ■ Using the Distributive Property

(a) $2(x + 3) = 2 \cdot x + 2 \cdot 3$ — Distributive Property

$= 2x + 6$ — Simplify

(b) $(a + b)(x + y) = (a + b)x + (a + b)y$ — Distributive Property

$= (ax + bx) + (ay + by)$ — Distributive Property

$= ax + bx + ay + by$ — Associative Property of Addition

In the last step we removed all the parentheses because, according to the Associative Property, the order of addition doesn't matter.

Now Try Exercise 15 ■

Addition and Subtraction

Don't assume that $-a$ is a negative number. Whether $-a$ is negative or positive depends on the value of a. For example, if $a = 5$, then $-a = -5$, a negative number, but if $a = -5$, then $-a = -(-5) = 5$ (Property 2), a positive number.

The number 0 is special for addition; it is called the **additive identity** because $a + 0 = a$ for any real number a. Every real number a has a **negative**, $-a$, that satisfies $a + (-a) = 0$. **Subtraction** is the operation that undoes addition; to subtract a number from another, we simply add the negative of that number. By definition

$$a - b = a + (-b)$$

To combine real numbers involving negatives, we use the following properties.

Properties of Negatives

Property	Example
1. $(-1)a = -a$	$(-1)5 = -5$
2. $-(-a) = a$	$-(-5) = 5$
3. $(-a)b = a(-b) = -(ab)$	$(-5)7 = 5(-7) = -(5 \cdot 7)$
4. $(-a)(-b) = ab$	$(-4)(-3) = 4 \cdot 3$
5. $-(a + b) = -a - b$	$-(3 + 5) = -3 - 5$
6. $-(a - b) = b - a$	$-(5 - 8) = 8 - 5$

Properties 5 and 6 follow from the Distributive Property.

Property 6 states the intuitive fact that $a - b$ and $b - a$ are negatives of each other. Property 5 is often used with more than two terms:

$$-(a + b + c) = -a - b - c$$

Example 2 ■ Using Properties of Negatives

Let x, y, and z be real numbers.

(a) $-(x + 2) = -x - 2$ — Property 5: $-(a + b) = -a - b$

(b) $-(x + y - z) = -x - y - (-z)$ — Property 5: $-(a + b) = -a - b$

$= -x - y + z$ — Property 2: $-(-a) = a$

Now Try Exercise 23 ■

Multiplication and Division

The number 1 is special for multiplication; it is called the **multiplicative identity** because $a \cdot 1 = a$ for any real number a. Every nonzero real number a has an **inverse**, $1/a$, that satisfies $a \cdot (1/a) = 1$. **Division** is the operation that undoes multiplication; to divide by a number, we multiply by the inverse of that number. If $b \neq 0$, then, by definition,

$$a \div b = a \cdot \frac{1}{b}$$

We write $a \cdot (1/b)$ as simply a/b. We refer to a/b as the **quotient** of a and b or as the **fraction** a over b; a is the **numerator** and b is the **denominator** (or **divisor**). To combine real numbers using the operation of division, we use the following properties.

Properties of Fractions

Property	Example	Description
1. $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$	$\frac{2}{3} \cdot \frac{5}{7} = \frac{2 \cdot 5}{3 \cdot 7} = \frac{10}{21}$	When **multiplying fractions**, multiply numerators and denominators.
2. $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$	$\frac{2}{3} \div \frac{5}{7} = \frac{2}{3} \cdot \frac{7}{5} = \frac{14}{15}$	When **dividing fractions**, invert the divisor and multiply the numerators and denominators.
3. $\frac{a}{c} + \frac{b}{c} = \frac{a + b}{c}$	$\frac{2}{5} + \frac{7}{5} = \frac{2 + 7}{5} = \frac{9}{5}$	When **adding fractions** with the **same denominator**, add the numerators.
4. $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$	$\frac{2}{5} + \frac{3}{7} = \frac{2 \cdot 7 + 3 \cdot 5}{35} = \frac{29}{35}$	When **adding fractions** with **different denominators**, find a common denominator. Then add the numerators.
5. $\frac{ac}{bc} = \frac{a}{b}$	$\frac{2 \cdot 5}{3 \cdot 5} = \frac{2}{3}$	**Cancel** numbers that are **common factors** in numerator and denominator.
6. If $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$	$\frac{2}{3} = \frac{6}{9}$, so $2 \cdot 9 = 3 \cdot 6$	**Cross-multiply**.

When adding fractions with different denominators, we don't usually use Property 4. Instead we rewrite the fractions so that they have the smallest possible common denominator (often smaller than the product of the denominators), and then we use Property 3. This denominator is the **L**east **C**ommon **D**enominator (LCD) described in the next example.

Example 3 ■ Using the LCD to Add Fractions

Evaluate: $\frac{5}{36} + \frac{7}{120}$

Solution Factoring each denominator into prime factors gives

$$36 = 2^2 \cdot 3^2 \qquad \text{and} \qquad 120 = 2^3 \cdot 3 \cdot 5$$

We find the least common denominator (LCD) by forming the product of all the prime factors that occur in these factorizations, using the highest power of each prime factor. Thus the LCD is $2^3 \cdot 3^2 \cdot 5 = 360$. So

$$\frac{5}{36} + \frac{7}{120} = \frac{5 \cdot 10}{36 \cdot 10} + \frac{7 \cdot 3}{120 \cdot 3} \qquad \text{Use common denominator}$$

$$= \frac{50}{360} + \frac{21}{360} = \frac{71}{360} \qquad \text{Property 3: Adding fractions with the same denominator}$$

Now Try Exercise 29 ■

The Real Line

The real numbers can be represented by points on a line, as shown in Figure 4. The positive direction (toward the right) is indicated by an arrow. We choose an arbitrary reference point O, called the **origin**, which corresponds to the real number 0. Given any convenient unit of measurement, each positive number x is represented by the point on the line a distance of x units to the right of the origin, and the corresponding negative number $-x$ is represented by the point x units to the left of the origin. The number associated with the point P is called the coordinate of P, and the line is then called a **coordinate line**, or a **real number line**, or simply a **real line**. Often we identify the point with its coordinate and think of a number as being a point on the real line.

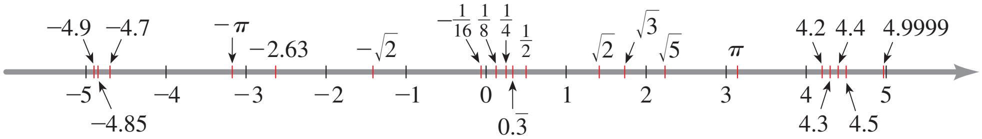


Figure 4 | The real line

The real numbers are *ordered*. We say that ***a* is less than *b*** and write $a < b$ if $b - a$ is a positive number. Geometrically, this means that a lies to the left of b on the number line. Equivalently, we can say that ***b* is greater than *a*** and write $b > a$. The symbol $a \le b$ (or $b \ge a$) means that either $a < b$ or $a = b$ and is read "a is less than or equal to b." For instance, the following are true inequalities (see Figure 4):

$$-5 < -4.9 \qquad -\pi < -3 \qquad \sqrt{2} < 2 \qquad 4 < 4.4 < 4.9999$$

Sets and Intervals

A **set** is a collection of objects, and these objects are called the **elements** of the set. If S is a set, the notation $a \in S$ means that a is an element of S, and $b \notin S$ means that b is not an element of S. For example, if Z represents the set of integers, then $-3 \in Z$ but $\pi \notin Z$.

Some sets can be described by listing their elements within braces. For instance, the set A that consists of all positive integers less than 7 can be written as

$$A = \{1, 2, 3, 4, 5, 6\}$$

We could also write A in **set-builder notation** as

$$A = \{x \mid x \text{ is an integer and } 0 < x < 7\}$$

which is read "A is the set of all x such that x is an integer and $0 < x < 7$."

If S and T are sets, then their **union** $S \cup T$ is the set that consists of all elements that are in S *or* T (or in both). The **intersection** of S and T is the set $S \cap T$ consisting of all

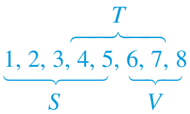
Discovery Project ■ **Real Numbers in the Real World**

Real-world measurements often involve units. For example, we usually measure distance in feet, miles, centimeters, or kilometers. Some measurements involve different types of units. For example, speed is measured in miles per hour or meters per second. We often need to convert a measurement from one type of unit to another. In this project we explore types of units used for different purposes and how to convert from one type of unit to another. You can find the project at **www.stewartmath.com**.

elements that are in both S *and* T. In other words, $S \cap T$ is the common part of S and T. The **empty set**, denoted by $\varnothing$, is the set that contains no element.

Example 4 ■ Union and Intersection of Sets

If $S = \{1, 2, 3, 4, 5\}$, $T = \{4, 5, 6, 7\}$, and $V = \{6, 7, 8\}$, find the sets $S \cup T$, $S \cap T$, and $S \cap V$.



Solution

$S \cup T = \{1, 2, 3, 4, 5, 6, 7\}$ All elements in S or T

$S \cap T = \{4, 5\}$ Elements common to both S and T

$S \cap V = \varnothing$ S and V have no element in common

Now Try Exercise 41 ■

Certain sets of real numbers, called **intervals**, occur frequently in calculus and correspond geometrically to line segments. If $a < b$, then the **open interval** from a to b consists of all numbers between a and b and is denoted (a, b). The **closed interval** from a to b includes the endpoints and is denoted $[a, b]$. Using set-builder notation, we can write

$$(a, b) = \{x \mid a < x < b\} \qquad [a, b] = \{x \mid a \le x \le b\}$$

Figure 5 | The open interval (a, b)

Figure 6 | The closed interval $[a, b]$

Note that parentheses ( ) in the interval notation and open circles on the graph in Figure 5 indicate that endpoints are *excluded* from the interval, whereas square brackets [ ] and solid circles in Figure 6 indicate that the endpoints are *included*. Intervals may also include one endpoint but not the other, or they may extend infinitely far in one direction or both directions. The following table lists the possible types of intervals.

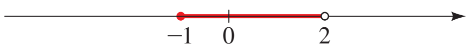
Notation	Set Description	Graph
(a, b)	$\{x \mid a < x < b\}$	
$[a, b]$	$\{x \mid a \le x \le b\}$	
$[a, b)$	$\{x \mid a \le x < b\}$	
$(a, b]$	$\{x \mid a < x \le b\}$	
(a, ∞)	$\{x \mid a < x\}$	
$[a, \infty)$	$\{x \mid a \le x\}$	
$(-\infty, b)$	$\{x \mid x < b\}$	
$(-\infty, b]$	$\{x \mid x \le b\}$	
$(-\infty, \infty)$	$\mathbb{R}$ (set of all real numbers)	

The symbol ∞ ("infinity") does not stand for a number. The notation (a, ∞), for instance, simply indicates that the interval has no endpoint on the right but extends infinitely far in the positive direction.

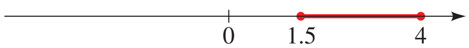
Example 5 ■ Graphing Intervals

Express each interval in terms of inequalities, and then graph the interval.

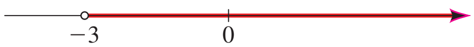
(a) $[-1, 2) = \{x \mid -1 \le x < 2\}$



(b) $[1.5, 4] = \{x \mid 1.5 \le x \le 4\}$



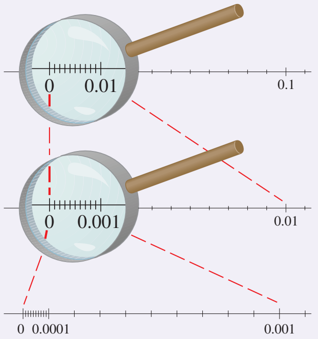
(c) $(-3, \infty) = \{x \mid -3 < x\}$



Now Try Exercise 47 ■

No Smallest or Largest Number in an Open Interval

Any interval contains infinitely many numbers—every point on the graph of an interval corresponds to a real number. In the closed interval $[0, 1]$, the smallest number is 0 and the largest is 1, but the open interval $(0, 1)$ contains no smallest or largest number. To see this, note that 0.01 is close to zero, but 0.001 is closer, 0.0001 is closer yet, and so on. We can always find a number in the interval $(0, 1)$ closer to zero than any given number. Since 0 itself is not in the interval, the interval contains no smallest number. Similarly, 0.99 is close to 1, but 0.999 is closer, 0.9999 closer yet, and so on. Since 1 itself is not in the interval, the interval has no largest number.



Example 6 ■ Finding Unions and Intersections of Intervals

Graph each set.

(a) $(1, 3) \cap [2, 7]$ **(b)** $(1, 3) \cup [2, 7]$

Solution

(a) The intersection of two intervals consists of the numbers that are in both intervals; geometrically, this is where the intervals overlap. Therefore

$$\begin{aligned}(1, 3) \cap [2, 7] &= \{x \mid 1 < x < 3 \text{ and } 2 \le x \le 7\}\\ &= \{x \mid 2 \le x < 3\} = [2, 3)\end{aligned}$$

This set is illustrated in Figure 7.

(b) The union of two intervals consists of the numbers that are in either one interval or the other (or both). Therefore

$$\begin{aligned}(1, 3) \cup [2, 7] &= \{x \mid 1 < x < 3 \text{ or } 2 \le x \le 7\}\\ &= \{x \mid 1 < x \le 7\} = (1, 7]\end{aligned}$$

This set is illustrated in Figure 8.

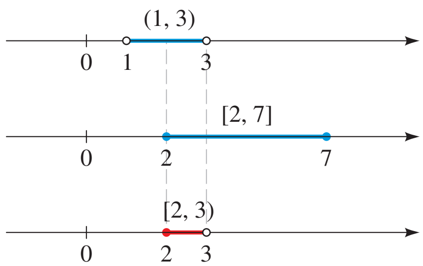


Figure 7 | $(1, 3) \cap [2, 7] = [2, 3)$

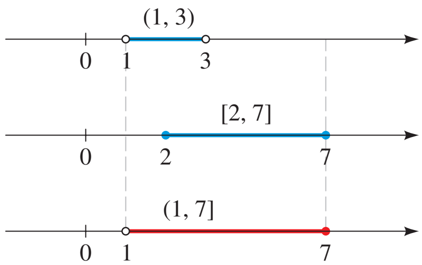


Figure 8 | $(1, 3) \cup [2, 7] = (1, 7]$

Now Try Exercise 61 ■

■ Absolute Value and Distance

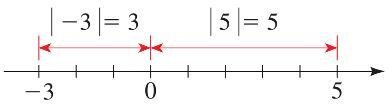


Figure 9

The **absolute value** of a number a, denoted by $|a|$, is the distance from a to 0 on the real number line (see Figure 9). Distance is always positive or zero, so we have $|a| \ge 0$ for every number a. Remembering that $-a$ is positive when a is negative, we have the following definition.

Definition of Absolute Value

If a is a real number, then the **absolute value** of a is

$$|a| = \begin{cases} a & \text{if } a \ge 0 \\ -a & \text{if } a < 0 \end{cases}$$

Example 7 ■ Evaluating Absolute Values of Numbers

(a) $|3| = 3$

(b) $|-3| = -(-3) = 3$

(c) $|0| = 0$

(d) $|3 - \pi| = -(3 - \pi) = \pi - 3 \qquad (\text{since } 3 < \pi \quad \Rightarrow \quad 3 - \pi < 0)$

Now Try Exercise 67 ■

When working with absolute values, we use the following properties.

Properties of Absolute Value

Property	Example	Description
1. $\lvert a \rvert \geq 0$	$\lvert -3 \rvert = 3 \geq 0$	The absolute value of a number is always positive or zero.
2. $\lvert a \rvert = \lvert -a \rvert$	$\lvert 5 \rvert = \lvert -5 \rvert$	A number and its negative have the same absolute value.
3. $\lvert ab \rvert = \lvert a \rvert \lvert b \rvert$	$\lvert -2 \cdot 5 \rvert = \lvert -2 \rvert \lvert 5 \rvert$	The absolute value of a product is the product of the absolute values.
4. $\left\lvert \dfrac{a}{b} \right\rvert = \dfrac{\lvert a \rvert}{\lvert b \rvert}$	$\left\lvert \dfrac{12}{-3} \right\rvert = \dfrac{\lvert 12 \rvert}{\lvert -3 \rvert}$	The absolute value of a quotient is the quotient of the absolute values.
5. $\lvert a + b \rvert \leq \lvert a \rvert + \lvert b \rvert$	$\lvert -3 + 5 \rvert \leq \lvert -3 \rvert + \lvert 5 \rvert$	Triangle Inequality

What is the distance on the real line between the numbers -2 and 11? From Figure 10 we see that the distance is 13. We arrive at this by finding either $\lvert 11 - (-2) \rvert = 13$ or $\lvert (-2) - 11 \rvert = 13$. From this observation we make the following definition.

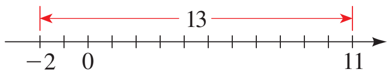
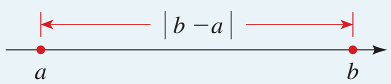


Figure 10

Distance between Points on the Real Line

If a and b are real numbers, then the **distance** between the points a and b on the real line is

$$d(a, b) = \lvert b - a \rvert$$



From Property 6 of negatives it follows that

$$\lvert b - a \rvert = \lvert a - b \rvert$$

This confirms that, as we would expect, the distance from a to b is the same as the distance from b to a.

Example 8 ■ Distance Between Points on the Real Line

The distance between the numbers 2 and -8 is

$$d(a, b) = \lvert -8 - 2 \rvert = \lvert -10 \rvert = 10$$

We can check this calculation geometrically, as shown in Figure 11.

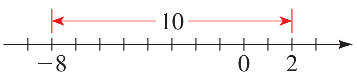


Figure 11

Now Try Exercise 75 ■

1.1 Exercises

Concepts

1. Give an example of each of the following:

(a) A natural number

(b) An integer that is not a natural number

(c) A rational number that is not an integer

(d) An irrational number

2. Complete each statement by using a property of real numbers and name the property you have used.

(a) $ab =$ ________; ________ Property

(b) $a + (b + c) =$ ________; ________ Property

(c) $a(b + c) =$ ________; ________ Property

3. Express the set of real numbers between but not including -3 and 5 as follows.

(a) In set-builder notation: ________________

(b) In interval notation: ________________

(c) As a graph: ________________

4. The symbol $|x|$ stands for the ________ of the number x. If x is not 0, then the sign of $|x|$ is always ________.

5. The distance between a and b on the real line is $d(a, b) =$ ________. So the distance between -5 and 2 is ________.

6–8 ■ *Yes or No*? If *No*, give a reason. Assume that a and b are nonzero real numbers.

6. (a) Does an interval always contain infinitely many numbers?

(b) Does the interval $(5, 6)$ contain a smallest element?

7. (a) Is $a - b$ equal to $b - a$?

(b) Is $-2(a - 5)$ equal to $-2a - 10$?

8. (a) Is the distance between any two different real numbers always positive?

(b) Is the distance between a and b the same as the distance between b and a?

Skills

9–10 ■ **Real Numbers** List the elements of the given set that are

(a) natural numbers

(b) integers

(c) rational numbers

(d) irrational numbers

9. $\{-1.5, 0, \frac{5}{2}, \sqrt{7}, 2.71, -\pi, 3.1\overline{4}, 100, -8\}$

10. $\{4.5, \frac{1}{3}, 1.6666\ldots, \sqrt{2}, 2, -\frac{100}{2}, \sqrt{9}, \sqrt{3.14}, 10\}$

11–18 ■ **Properties of Real Numbers** State the property of real numbers being used.

11. $3 + 7 = 7 + 3$

12. $4(2 + 3) = (2 + 3)4$

13. $(x + 2y) + 3z = x + (2y + 3z)$

14. $2(A + B) = 2A + 2B$

15. $(5x + 1)3 = 15x + 3$

16. $(x + a)(x + b) = (x + a)x + (x + a)b$

17. $2x(3 + y) = (3 + y)2x$

18. $7(a + b + c) = 7(a + b) + 7c$

19–22 ■ **Properties of Real Numbers** Rewrite the expression using the given property of real numbers.

19. Commutative Property of Addition, $x + 3 =$

20. Associative Property of Multiplication, $7(3x) =$

21. Distributive Property, $4(A + B) =$

22. Distributive Property, $5x + 5y =$

23–28 ■ **Properties of Real Numbers** Use properties of real numbers to write the expression without parentheses.

23. $-2(x + y)$

24. $(a - b)(-5)$

25. $5(2xy)$

26. $\frac{4}{3}(-6y)$

27. $-\frac{5}{2}(2x - 4y)$

28. $(3a)(b + c - 2d)$

29–32 ■ **Arithmetic Operations** Perform the indicated operations and simplify. Express your answer as a single fraction.

29. (a) $\frac{2}{3} + \frac{5}{7}$ **(b)** $\frac{5}{12} - \frac{3}{8}$

30. (a) $\frac{2}{5} - \frac{3}{8}$ **(b)** $\frac{3}{2} - \frac{5}{8} + \frac{1}{6}$

31. (a) $\frac{2}{3}\left(6 - \frac{3}{2}\right)$ **(b)** $\left(3 + \frac{1}{4}\right)\left(1 - \frac{4}{5}\right)$

32. (a) $\dfrac{2}{\frac{2}{3}} - \dfrac{\frac{2}{3}}{2}$ **(b)** $\dfrac{\frac{2}{5} + \frac{1}{2}}{\frac{1}{10} + \frac{3}{15}}$

33–34 ■ **Inequalities** Place the correct symbol ($<$, $>$, or $=$) in the blue box.

33. (a) $3 \;\square\; \frac{7}{2}$ **(b)** $-3 \;\square\; -\frac{7}{2}$ **(c)** $3.5 \;\square\; \frac{7}{2}$

34. (a) $\frac{2}{3} \;\square\; 0.67$ **(b)** $\frac{2}{3} \;\square\; -0.67$ **(c)** $|0.6| \;\square\; |-0.6|$

35–38 ■ **Inequalities** State whether each inequality is true or false.

35. (a) $-3 < -4$ **(b)** $3 < 4$

36. (a) $\sqrt{3} > 1.7325$ **(b)** $1.732 \geq \sqrt{3}$

37. (a) $\frac{10}{2} \geq 5$ **(b)** $\frac{6}{10} \geq \frac{5}{6}$

38. (a) $\frac{7}{11} \geq \frac{8}{13}$ **(b)** $-\frac{3}{5} > -\frac{3}{4}$

39–40 ■ **Inequalities** Write each statement in terms of inequalities.

39. (a) x is positive.

(b) t is less than 4.

(c) a is greater than or equal to π.

(d) x is less than $\frac{1}{3}$ and greater than -5.

(e) The distance from p to 3 is at most 5.

40. **(a)** y is negative.

(b) z is at least 3.

(c) b is at most 8.

(d) w is positive and less than or equal to 17.

(e) y is at least 2 units from π.

41–44 ■ Sets Find the indicated set if

$$A = \{1, 2, 3, 4, 5, 6, 7\} \qquad B = \{2, 4, 6, 8\}$$
$$C = \{7, 8, 9, 10\}$$

41. **(a)** $A \cup B$ **(b)** $A \cap B$

42. **(a)** $B \cup C$ **(b)** $B \cap C$

43. **(a)** $A \cup C$ **(b)** $A \cap C$

44. **(a)** $A \cup B \cup C$ **(b)** $A \cap B \cap C$

45–46 ■ Sets Find the indicated set if

$$A = \{x \mid x \geq -2\} \qquad B = \{x \mid x < 4\}$$
$$C = \{x \mid -1 < x \leq 5\}$$

45. **(a)** $B \cup C$ **(b)** $B \cap C$

46. **(a)** $A \cap C$ **(b)** $A \cap B$

47–52 ■ Intervals Express the interval in terms of inequalities, and then graph the interval.

47. $(-3, 0)$ **48.** $(2, 8]$

49. $[2, 8)$ **50.** $[-6, -\frac{1}{2}]$

51. $[2, \infty)$ **52.** $(-\infty, 1)$

53–58 ■ Intervals Express the inequality in interval notation, and then graph the corresponding interval.

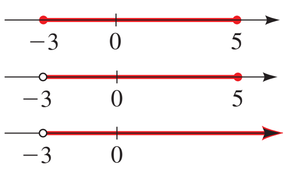
53. $x \leq 1$ **54.** $1 \leq x \leq 2$

55. $-2 < x \leq 1$ **56.** $x \geq -5$

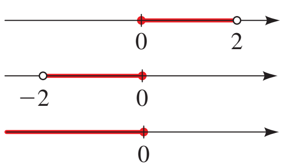
57. $x > -1$ **58.** $-5 < x < 2$

59–60 ■ Intervals Express each set in interval notation.

59. **(a)** **(b)** **(c)**



60. **(a)** **(b)** **(c)**



61–66 ■ Intervals Graph the set.

61. $(-2, 0) \cup (-1, 1)$ **62.** $(-2, 0) \cap (-1, 1)$

63. $[-4, 6] \cap [0, 8)$ **64.** $[-4, 6) \cup [0, 8)$

65. $(-\infty, -4) \cup (4, \infty)$ **66.** $(-\infty, 6] \cap (2, 10)$

67–72 ■ Absolute Value Evaluate each expression.

67. **(a)** $|50|$ **(b)** $|-13|$

68. **(a)** $|2 - 8|$ **(b)** $|8 - |-2||$

69. **(a)** $||-6| - |-4||$ **(b)** $\dfrac{-1}{|-1|}$

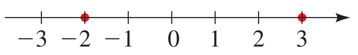
70. **(a)** $|2 - |-12||$ **(b)** $-1 - |1 - |-1||$

71. **(a)** $|(-2) \cdot 6|$ **(b)** $|(-\frac{1}{3})(-15)|$

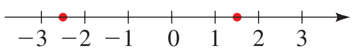
72. **(a)** $\left|\dfrac{-6}{24}\right|$ **(b)** $\left|\dfrac{7 - 12}{12 - 7}\right|$

73–76 ■ Distance Find the distance between the given numbers.

73.



74.



75. **(a)** 2 and 17 **(b)** -3 and 21 **(c)** $\frac{11}{8}$ and $-\frac{3}{10}$

76. **(a)** $\frac{7}{15}$ and $-\frac{1}{21}$ **(b)** -38 and -57 **(c)** -2.6 and -1.8

Skills Plus

77–78 ■ Repeating Decimal Express each repeating decimal as a fraction. (See the margin note in the subsection "Real Numbers.")

77. **(a)** $0.\overline{7}$ **(b)** $0.2\overline{8}$ **(c)** $0.\overline{57}$

78. **(a)** $5.\overline{23}$ **(b)** $1.3\overline{7}$ **(c)** $2.1\overline{35}$

79–82 ■ Simplifying Absolute Value Express the quantity without using absolute value.

79. $|\pi - 3|$ **80.** $|1 - \sqrt{2}|$

81. $|a - b|$, where $a < b$

82. $a + b + |a - b|$, where $a < b$

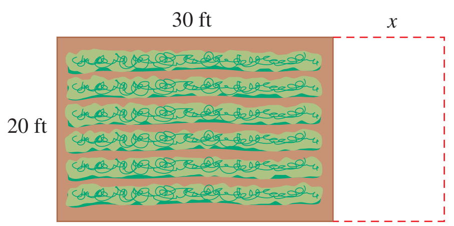
83–84 ■ Signs of Numbers Let a, b, and c be real numbers such that $a > 0$, $b < 0$, and $c < 0$. Find the sign of each expression.

83. **(a)** $-a$ **(b)** bc **(c)** $a - b$ **(d)** $ab + ac$

84. **(a)** $-b$ **(b)** $a + bc$ **(c)** $c - a$ **(d)** ab^2

Applications

85. Area of a Garden A backyard vegetable garden measures 20 ft by 30 ft, so its area is $20 \times 30 = 600$ ft^2. The garden needs to be made longer, as shown in the figure, so that the area increases to $A = 20(30 + x)$. Which property of real numbers tells us that the new area can also be written $A = 600 + 20x$?

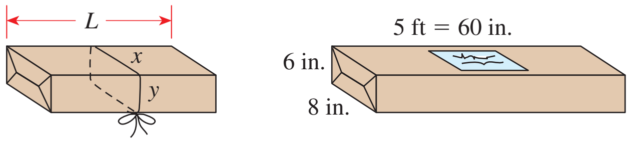


86. Mailing a Package The post office will accept only packages for which the length plus the "girth" (distance around) is no more than 108 in. Thus for the package in the figure, we must have

$$L + 2(x + y) \le 108$$

(a) Will the post office accept a package that is 6 in. wide, 8 in. deep, and 5 ft long? What about a package that measures 2 ft by 2 ft by 4 ft?

(b) What is the greatest acceptable length for a package that has a square base measuring 9 in. by 9 in.?



■ Discuss ■ Discover ■ Prove ■ Write

87. Discuss: Sums and Products of Rational and Irrational Numbers Explain why the sum, the difference, and the product of two rational numbers are rational numbers. Is the product of two irrational numbers necessarily irrational? What about the sum?

88. Discover ■ Prove: Combining Rational and Irrational Numbers Is $\frac{1}{2} + \sqrt{2}$ rational or irrational? Is $\frac{1}{2} \cdot \sqrt{2}$ rational or irrational? Experiment with sums and products of other rational and irrational numbers. Prove the following.

(a) The sum of a rational number r and an irrational number t is irrational.

(b) The product of a non-zero rational number r and an irrational number t is irrational.

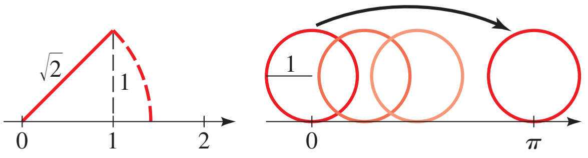
PS *Indirect reasoning.* For part (a), suppose that $r + t$ is a rational number q, that is, $r + t = q$. Show that this leads to a contradiction. Use similar reasoning for part (b).

89. Discover: Limiting Behavior of Reciprocals Complete the tables. What happens to the size of the fraction $1/x$ as x gets large? As x gets small?

x	$1/x$
1	
2	
10	
100	
1000	

x	$1/x$
1.0	
0.5	
0.1	
0.01	
0.001	

90. Discover: Locating Irrational Numbers on the Real Line Using the figures below, explain how to locate the point $\sqrt{2}$ on a number line. Can you locate $\sqrt{5}$ by a similar method? How can the circle shown in the figure help us locate π on a number line? List some other irrational numbers that you can locate on a number line.



91. Prove: Maximum and Minimum Formulas Let $\max(a, b)$ denote the maximum and $\min(a, b)$ denote the minimum of the real numbers a and b. For example, $\max(2, 5) = 5$ and $\min(-1, -2) = -2$.

(a) Prove that $\max(a, b) = \dfrac{a + b + |a - b|}{2}$.

(b) Prove that $\min(a, b) = \dfrac{a + b - |a - b|}{2}$.

PS *Take cases.* Write these expressions without absolute values. See Exercises 81 and 82.

92. Write: Real Numbers in the Real World Write a paragraph describing several real-world situations in which you would use natural numbers, integers, rational numbers, and irrational numbers. Give examples for each type of situation.

93. Discuss: Commutative and Noncommutative Operations We have learned that addition and multiplication are both commutative operations.

(a) Is subtraction commutative?

(b) Is division of nonzero real numbers commutative?

(c) Are the actions of putting on your socks and putting on your shoes commutative?

(d) Are the actions of putting on your hat and putting on your coat commutative?

(e) Are the actions of washing laundry and drying it commutative?

94. Prove: Triangle Inequality We prove Property 5 of absolute values, the Triangle Inequality:

$$|x + y| \le |x| + |y|$$

(a) Verify that the Triangle Inequality holds for $x = 2$ and $y = 3$, for $x = -2$ and $y = -3$, and for $x = -2$ and $y = 3$.

(b) Prove that the Triangle Inequality is true for all real numbers x and y.

PS *Take cases.* Consider the signs of x and y.

1.2 Exponents and Radicals

■ Integer Exponents ■ Rules for Working with Exponents ■ Scientific Notation
■ Radicals ■ Rational Exponents ■ Rationalizing the Denominator; Standard Form

In this section we give meaning to expressions such as $a^{m/n}$ in which the exponent m/n is a rational number. To do this, we need to recall some facts about integer exponents, radicals, and nth roots.

■ Integer Exponents

A product of identical numbers is usually written in exponential notation. For example, $5 \cdot 5 \cdot 5$ is written as 5^3. In general, we have the following definition.

Exponential Notation

If a is any real number and n is a positive integer, then the ***n*th power** of a is

$$a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ factors}}$$

The number a is called the **base**, and n is called the **exponent**.

Example 1 ■ Exponential Notation

Note the distinction between $(-3)^4$ and -3^4. In $(-3)^4$ the exponent applies to -3, but in -3^4 the exponent applies only to 3.

(a) $\left(\frac{1}{2}\right)^5 = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{32}$

(b) $(-3)^4 = (-3) \cdot (-3) \cdot (-3) \cdot (-3) = 81$

(c) $-3^4 = -(3 \cdot 3 \cdot 3 \cdot 3) = -81$

Now Try Exercise 9 ■

We can state several useful rules for working with exponential notation. To discover the rule for multiplication, we multiply 5^4 by 5^2:

$$5^4 \cdot 5^2 = \underbrace{(5 \cdot 5 \cdot 5 \cdot 5)}_{4 \text{ factors}}\underbrace{(5 \cdot 5)}_{2 \text{ factors}} = \underbrace{5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5}_{6 \text{ factors}} = 5^6 = 5^{4+2}$$

It appears that *to multiply two powers of the same base, we add their exponents*. In general, for any real number a and any positive integers m and n, we have

$$a^m a^n = \underbrace{(a \cdot a \cdot \cdots \cdot a)}_{m \text{ factors}}\underbrace{(a \cdot a \cdot \cdots \cdot a)}_{n \text{ factors}} = \underbrace{a \cdot a \cdot a \cdot \cdots \cdot a}_{m + n \text{ factors}} = a^{m+n}$$

Thus $a^m a^n = a^{m+n}$.

We would like this rule to be true even when m and n are 0 or negative integers. For instance, we must have

$$2^0 \cdot 2^3 = 2^{0+3} = 2^3$$

But this can happen only if $2^0 = 1$. Likewise, we want to have

$$5^4 \cdot 5^{-4} = 5^{4+(-4)} = 5^{4-4} = 5^0 = 1$$

and this will be true if $5^{-4} = 1/5^4$. These observations lead to the following definition.

Zero and Negative Exponents

If $a \neq 0$ is a real number and n is a positive integer, then

$$a^0 = 1 \qquad \text{and} \qquad a^{-n} = \frac{1}{a^n}$$

Example 2 ■ Zero and Negative Exponents

(a) $\left(\frac{4}{7}\right)^0 = 1$

(b) $x^{-1} = \dfrac{1}{x^1} = \dfrac{1}{x}$

(c) $(-2)^{-3} = \dfrac{1}{(-2)^3} = \dfrac{1}{-8} = -\dfrac{1}{8}$

Now Try Exercise 11 ■

■ Rules for Working with Exponents

Familiarity with the following rules is essential for our work with exponents and bases. In these laws the bases a and b are real numbers, and the exponents m and n are integers.

Laws of Exponents

Law	Example	Description
1. $a^m a^n = a^{m+n}$	$3^2 \cdot 3^5 = 3^{2+5} = 3^7$	To multiply two powers of the same number, add the exponents.
2. $\dfrac{a^m}{a^n} = a^{m-n}$	$\dfrac{3^5}{3^2} = 3^{5-2} = 3^3$	To divide two powers of the same number, subtract the exponents.
3. $(a^m)^n = a^{mn}$	$(3^2)^5 = 3^{2 \cdot 5} = 3^{10}$	To raise a power to a new power, multiply the exponents.
4. $(ab)^n = a^n b^n$	$(3 \cdot 4)^2 = 3^2 \cdot 4^2$	To raise a product to a power, raise each factor to the power.
5. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$	$\left(\dfrac{3}{4}\right)^2 = \dfrac{3^2}{4^2}$	To raise a quotient to a power, raise both numerator and denominator to the power.
6. $\left(\dfrac{a}{b}\right)^{-n} = \left(\dfrac{b}{a}\right)^n$	$\left(\dfrac{3}{4}\right)^{-2} = \left(\dfrac{4}{3}\right)^2$	To raise a fraction to a negative power, invert the fraction and change the sign of the exponent.
7. $\dfrac{a^{-n}}{b^{-m}} = \dfrac{b^m}{a^n}$	$\dfrac{3^{-2}}{4^{-5}} = \dfrac{4^5}{3^2}$	To move a number raised to a power from numerator to denominator or from denominator to numerator, change the sign of the exponent.

Proof of Law 3 If m and n are positive integers, we have

$$(a^m)^n = (\underbrace{a \cdot a \cdot \cdots \cdot a}_{m \text{ factors}})^n$$

$$= \underbrace{(\underbrace{a \cdot a \cdot \cdots \cdot a}_{m \text{ factors}})(\underbrace{a \cdot a \cdot \cdots \cdot a}_{m \text{ factors}}) \cdots (\underbrace{a \cdot a \cdot \cdots \cdot a}_{m \text{ factors}})}_{n \text{ groups of factors}}$$

$$= \underbrace{a \cdot a \cdot \cdots \cdot a}_{mn \text{ factors}} = a^{mn}$$

The cases for which $m \leq 0$ or $n \leq 0$ can be proved by using the definition of negative exponents. ■

Mathematics and Mathematicians

Mathematicians have been sought after in every civilization to use their craft in the service of society. Several brief biographies of notable mathematicians are highlighted in margin notes in this book, but keep in mind that their discoveries always depended on the insights of thousands of others who came before them. Isaac Newton put it this way: "If I have seen further than others it is by standing on the shoulders of giants." And David Hilbert noted that, "Mathematics knows no races or geographic boundaries; for mathematics, the cultural world is one country."

You will see in the biographies that some great mathematicians did not at first find mathematics interesting, but were later attracted to it for one reason or another. Pascal liked mathematics because of "the clarity of its reasoning," Hermann Hankel, because of the permanence of its theorems. Hankel said, "In most sciences one generation tears down what another has built. . . . In mathematics alone each generation adds a new story to the old structure." For example, the elements of Aristotle (ca. 300 BC)— earth, wind, fire, water—are now replaced by the Periodic Table, whereas the theorem of Pythagoras (ca. 500 BC) continues to be valid. Joseph Fourier's reason for being drawn to mathematics was that "Mathematics compares the most diverse phenomena and discovers the secret analogies that unite them." For example, we'll see that quadratic functions model both the path of a projectile and the relationship between rainfall and crop yield. Perhaps you too will find reasons for exploring mathematics.

Proof of Law 4 If n is a positive integer, we have

$$(ab)^n = \underbrace{(ab)(ab)\cdots(ab)}_{n\text{ factors}} = \underbrace{(a\cdot a\cdot\cdots\cdot a)}_{n\text{ factors}}\cdot\underbrace{(b\cdot b\cdot\cdots\cdot b)}_{n\text{ factors}} = a^n b^n$$

Here we have used the Commutative and Associative Properties repeatedly. If $n \le 0$, Law 4 can be proved by using the definition of negative exponents. ■

You are asked to prove Laws 2, 5, 6, and 7 in Exercises 106 and 107.

Example 3 ■ Using Laws of Exponents

(a) $x^4x^7 = x^{4+7} = x^{11}$ — Law 1: $a^m a^n = a^{m+n}$

(b) $y^4y^{-7} = y^{4-7} = y^{-3} = \dfrac{1}{y^3}$ — Law 1: $a^m a^n = a^{m+n}$

(c) $\dfrac{c^9}{c^5} = c^{9-5} = c^4$ — Law 2: $\dfrac{a^m}{a^n} = a^{m-n}$

(d) $(b^4)^5 = b^{4\cdot 5} = b^{20}$ — Law 3: $(a^m)^n = a^{mn}$

(e) $(3x)^3 = 3^3x^3 = 27x^3$ — Law 4: $(ab)^n = a^nb^n$

(f) $\left(\dfrac{x}{2}\right)^5 = \dfrac{x^5}{2^5} = \dfrac{x^5}{32}$ — Law 5: $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

Now Try Exercises 19, 21, and 23 ■

Example 4 ■ Simplifying Expressions with Exponents

Simplify:

(a) $(2a^3b^2)(3ab^4)^3$ **(b)** $\left(\dfrac{x}{y}\right)^3\left(\dfrac{y^2x}{z}\right)^4$

Solution

(a)

$$\begin{aligned}(2a^3b^2)(3ab^4)^3 &= (2a^3b^2)[3^3a^3(b^4)^3] && \text{Law 4: } (ab)^n = a^nb^n\\ &= (2a^3b^2)(27a^3b^{12}) && \text{Law 3: } (a^m)^n = a^{mn}\\ &= (2)(27)a^3a^3b^2b^{12} && \text{Group factors with the same base}\\ &= 54a^6b^{14} && \text{Law 1: } a^ma^n = a^{m+n}\end{aligned}$$

(b)

$$\begin{aligned}\left(\frac{x}{y}\right)^3\left(\frac{y^2x}{z}\right)^4 &= \frac{x^3}{y^3}\,\frac{(y^2)^4x^4}{z^4} && \text{Laws 5 and 4}\\ &= \frac{x^3}{y^3}\,\frac{y^8x^4}{z^4} && \text{Law 3}\\ &= (x^3x^4)\left(\frac{y^8}{y^3}\right)\frac{1}{z^4} && \text{Group factors with the same base}\\ &= \frac{x^7y^5}{z^4} && \text{Laws 1 and 2}\end{aligned}$$

Now Try Exercises 25 and 29 ■

When simplifying an expression, you will find that many different methods will lead to the same result; feel free to use any of the rules of exponents to arrive at your own method. In the next example we see how to simplify expressions with negative exponents.

Mathematics in the Modern World

Although we are often unaware of its presence, mathematics permeates nearly every aspect of our lives. With the advent of technology, mathematics plays an ever greater role. Today you were probably awakened by a digital alarm, sent a text, searched the Internet, watched a streaming video, listened to music on your cell phone, then slept in a room whose temperature is controlled by a digital thermostat. In each of these activities mathematics is crucially involved. In general, a property such as the intensity or frequency of sound, the colors in an image, or the temperature in your home is transformed into sequences of numbers by complex mathematical algorithms. These numerical data, which usually consist of many millions of bits (the digits 0 and 1), are then transmitted and reinterpreted. Dealing with such huge amounts of data was not feasible until the invention of computers, machines whose logical processes were invented by mathematicians.

The contributions of mathematics in the modern world are not limited to technological advances. The logical processes of mathematics are now used to analyze complex problems in the social, political, and life sciences in new and surprising ways.

In other *Mathematics in the Modern World*, we will describe in more detail how mathematics affects all of us in our everyday activities.

Example 5 ■ Simplifying Expressions with Negative Exponents

Eliminate negative exponents and simplify each expression.

(a) $\dfrac{6st^{-4}}{2s^{-2}t^2}$ **(b)** $\left(\dfrac{y}{3z^3}\right)^{-2}$

Solution

(a) We use Law 7, which allows us to move a number raised to a power from the numerator to the denominator (or vice versa) by changing the sign of the exponent.

t^{-4} moves to denominator and becomes t^4

$$\frac{6st^{-4}}{2s^{-2}t^2} = \frac{6ss^2}{2t^2t^4} \qquad \text{Law 7}$$

s^{-2} moves to numerator and becomes s^2

$$= \frac{3s^3}{t^6} \qquad \text{Law 1}$$

(b) We use Law 6, which allows us to change the sign of the exponent of a fraction by inverting the fraction.

$$\left(\frac{y}{3z^3}\right)^{-2} = \left(\frac{3z^3}{y}\right)^2 \qquad \text{Law 6}$$

$$= \frac{9z^6}{y^2} \qquad \text{Laws 5 and 4}$$

Now Try Exercise 31 ■

■ Scientific Notation

Scientists use exponential notation as a compact way of writing very large numbers and very small numbers. For example, the nearest star beyond the sun, Proxima Centauri, is approximately 40,000,000,000,000 km away. The mass of a hydrogen atom is about 0.000 000 000 000 000 000 000 001 66 g. Such numbers are difficult to read and to write, so scientists usually express them in *scientific notation*.

Scientific Notation

A positive number x is said to be written in **scientific notation** if it is expressed as follows:

$$x = a \times 10^n \qquad \text{where } 1 \le a < 10 \text{ and } n \text{ is an integer}$$

For instance, when we state that the distance to the star Proxima Centauri is 4×10^{13} km, the positive exponent 13 indicates that the decimal point should be moved 13 places to the *right*:

$$4 \times 10^{13} = 40{,}000{,}000{,}000{,}000$$

Move decimal point 13 places to the right

When we state that the mass of a hydrogen atom is 1.66×10^{-24} g, the exponent -24 indicates that the decimal point should be moved 24 places to the *left*:

$$1.66 \times 10^{-24} = 0.000\,000\,000\,000\,000\,000\,000\,001\,66$$

Move decimal point 24 places to the left

Example 6 ■ Changing from Decimal to Scientific Notation

Write each number in scientific notation.

(a) 56,920 **(b)** 0.000093

Solution

(a) $\underbrace{56{,}920}_{\text{4 places}} = 5.692 \times 10^4$ **(b)** $\underbrace{0.0000}_{\text{5 places}}93 = 9.3 \times 10^{-5}$

Now Try Exercise 81 ■

Example 7 ■ Changing from Scientific Notation to Decimal Notation

Write each number in decimal notation.

(a) 6.97×10^9 **(b)** 4.6271×10^{-6}

Solution

(a) $6.97 \times 10^9 = 6\underbrace{{,}970{,}000{,}000}_{\text{9 places}}$ Move decimal 9 places to the right

(b) $4.6271 \times 10^{-6} = 0\underbrace{.000004}_{\text{6 places}}6271$ Move decimal 6 places to the left

Now Try Exercise 83 ■

Note Scientific notation is often used on a graphing device or calculator to display a very large or very small number. For instance, if we use a calculator to square the number 1,111,111, the display panel may show the approximation

`1.234568 12` or `1.234568 E12`

Here the final digits indicate the power of 10, and we interpret the result as

$$1.234568 \times 10^{12}$$

Example 8 ■ Calculating with Scientific Notation

If $a \approx 0.00046$, $b \approx 1.697 \times 10^{22}$, and $c \approx 2.91 \times 10^{-18}$, use a calculator to approximate the quotient ab/c.

Solution We could enter the data using scientific notation, or we could use laws of exponents as follows:

$$\begin{aligned}\frac{ab}{c} &\approx \frac{(4.6 \times 10^{-4})(1.697 \times 10^{22})}{2.91 \times 10^{-18}} \\ &= \frac{(4.6)(1.697)}{2.91} \times 10^{-4+22+18} \\ &\approx 2.7 \times 10^{36}\end{aligned}$$

For guidelines on working with significant figures, see Appendix B, *Calculations and Significant Figures.* Go to **www.stewartmath.com**.

We state the answer rounded to two significant figures because the least accurate of the given numbers is stated to two significant figures.

Now Try Exercises 87 and 89 ■

■ Radicals

We know what 2^n means whenever n is an integer. To give meaning to a power, such as $2^{4/5}$, whose exponent is a rational number, we need to discuss radicals.

The symbol $\sqrt{\ }$ means "the positive square root of." Thus

It is true that the number 9 has two square roots, 3 and -3, but the notation $\sqrt{9}$ is reserved for the *positive* square root of 9 (sometimes called the *principal square root* of 9). If we want the negative root, we must write $-\sqrt{9}$, which is -3.

$$\sqrt{a} = b \qquad \text{means} \qquad b^2 = a \qquad \text{and} \qquad b \geq 0$$

Since $a = b^2 \geq 0$, the symbol $\sqrt{a}$ makes sense only when $a \geq 0$. For instance,

$$\sqrt{9} = 3 \qquad \text{because} \qquad 3^2 = 9 \qquad \text{and} \qquad 3 \geq 0$$

Square roots are special cases of nth roots. The nth root of x is the number that, when raised to the nth power, gives x.

Definition of nth Root

If n is any positive integer, then the **principal nth root** of a is defined as follows:

$$\sqrt[n]{a} = b \qquad \text{means} \qquad b^n = a$$

If n is even, then we must have $a \geq 0$ and $b \geq 0$.

For example,

$$\sqrt[4]{81} = 3 \qquad \text{because} \qquad 3^4 = 81 \qquad \text{and} \qquad 3 \geq 0$$

$$\sqrt[3]{-8} = -2 \qquad \text{because} \qquad (-2)^3 = -8$$

But $\sqrt{-8}$, $\sqrt[4]{-8}$, and $\sqrt[6]{-8}$ are not defined. (For instance, $\sqrt{-8}$ is not defined because the square of every real number is nonnegative.)

Notice that

$$\sqrt{4^2} = \sqrt{16} = 4 \qquad \text{but} \qquad \sqrt{(-4)^2} = \sqrt{16} = 4 = |-4|$$

So the equation $\sqrt{a^2} = a$ is not always true; it is true only when $a \geq 0$. However, we can always write $\sqrt{a^2} = |a|$. This last equation is true not only for square roots, but for any even root. This and other rules used in working with nth roots are listed in the following box. In each property we assume that all the given roots exist.

Properties of nth Roots

Property	Example
1. $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$	$\sqrt[3]{-8 \cdot 27} = \sqrt[3]{-8}\sqrt[3]{27} = (-2)(3) = -6$
2. $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$	$\sqrt[4]{\dfrac{16}{81}} = \dfrac{\sqrt[4]{16}}{\sqrt[4]{81}} = \dfrac{2}{3}$
3. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$	$\sqrt{\sqrt[3]{729}} = \sqrt[6]{729} = 3$
4. $\sqrt[n]{a^n} = a$ if n is odd	$\sqrt[3]{(-5)^3} = -5$, $\sqrt[5]{2^5} = 2$
5. $\sqrt[n]{a^n} = \vert a \vert$ if n is even	$\sqrt[4]{(-3)^4} = \vert -3 \vert = 3$

Example 9 ■ Simplifying Expressions Involving nth Roots

(a) $\sqrt[3]{x^4} = \sqrt[3]{x^3x}$ Factor out the largest cube

$= \sqrt[3]{x^3}\sqrt[3]{x}$ Property 1: $\sqrt[3]{ab} = \sqrt[3]{a}\sqrt[3]{b}$

$= x\sqrt[3]{x}$ Property 4: $\sqrt[3]{a^3} = a$

(b) $\sqrt[4]{81x^8y^4} = \sqrt[4]{81}\sqrt[4]{x^8}\sqrt[4]{y^4}$ Property 1: $\sqrt[4]{abc} = \sqrt[4]{a}\sqrt[4]{b}\sqrt[4]{c}$

$= 3\sqrt[4]{(x^2)^4}|y|$ Property 5: $\sqrt[4]{a^4} = |a|$

$= 3x^2|y|$ Property 5: $\sqrt[4]{a^4} = |a|, |x^2| = x^2$

Now Try Exercises 33 and 35 ■

It is frequently useful to combine like radicals in an expression such as $2\sqrt{3} + 5\sqrt{3}$. This can be done by using the Distributive Property. For example,

$$2\sqrt{3} + 5\sqrt{3} = (2 + 5)\sqrt{3} = 7\sqrt{3}$$

The next example further illustrates this process.

Avoid making the following error:

$$\sqrt{a+b} \cancel{=} \sqrt{a} + \sqrt{b}$$

For instance, if we let $a = 9$ and $b = 16$, then we see the error:

$$\sqrt{9+16} \stackrel{?}{=} \sqrt{9} + \sqrt{16}$$

$$\sqrt{25} \stackrel{?}{=} 3 + 4$$

$$5 \stackrel{?}{=} 7 \quad \text{Wrong!}$$

Example 10 ■ Combining Radicals

(a) $\sqrt{32} + \sqrt{200} = \sqrt{16 \cdot 2} + \sqrt{100 \cdot 2}$ Factor out the largest squares

$= \sqrt{16}\sqrt{2} + \sqrt{100}\sqrt{2}$ Property 1: $\sqrt{ab} = \sqrt{a}\sqrt{b}$

$= 4\sqrt{2} + 10\sqrt{2} = 14\sqrt{2}$ Distributive Property

(b) If $b > 0$, then

$\sqrt{25b} - \sqrt{b^3} = \sqrt{25}\sqrt{b} - \sqrt{b^2}\sqrt{b}$ Property 1: $\sqrt{ab} = \sqrt{a}\sqrt{b}$

$= 5\sqrt{b} - b\sqrt{b}$ Property 5, $b > 0$

$= (5 - b)\sqrt{b}$ Distributive Property

(c) $\sqrt{49x^2 + 49} = \sqrt{49(x^2 + 1)}$ Factor out the perfect square

$= 7\sqrt{x^2 + 1}$ Property 1: $\sqrt{ab} = \sqrt{a}\sqrt{b}$

Now Try Exercises 37, 39, and 41 ■

■ Rational Exponents

To define what is meant by a *rational exponent* or, equivalently, a *fractional exponent* such as $a^{1/3}$, we need to use radicals. To give meaning to the symbol $a^{1/n}$ in a way that is consistent with the Laws of Exponents, we would have to have

$$(a^{1/n})^n = a^{(1/n)n} = a^1 = a$$

So by the definition of nth root,

$$\boxed{a^{1/n} = \sqrt[n]{a}}$$

In general, we define rational exponents as follows.

Definition of Rational Exponents

For any rational exponent m/n in lowest terms, where m and n are integers and $n > 0$, we define

$$a^{m/n} = (\sqrt[n]{a})^m \qquad \text{or equivalently} \qquad a^{m/n} = \sqrt[n]{a^m}$$

If n is even, then we require that $a \geq 0$.

With this definition it can be proved that *the Laws of Exponents also hold for rational exponents*.

DIOPHANTUS lived in Alexandria about 250 A.D. His book *Arithmetica* is considered the first book on algebra. In it he gives methods for finding integer solutions of algebraic equations. *Arithmetica* was read and studied for more than a thousand years. Fermat (see Section 1.11) made some of his most important discoveries while studying this book. Diophantus's major contribution is the use of symbols to stand for the unknowns in a problem. Although his symbolism is not as simple as the symbols we use today, it was a major advance over writing everything in words. In Diophantus's notation the equation

$$x^5 - 7x^2 + 8x - 5 = 24$$

is written

$$\Delta K^{\gamma}\alpha\varsigma\eta \pitchfork \Delta^{\gamma}\zeta \overset{\circ}{M}\varepsilon\iota^{\sigma}\kappa\delta$$

Our modern algebraic notation did not come into common use until the 17th century.

Example 11 ■ Using the Definition of Rational Exponents

(a) $4^{1/2} = \sqrt{4} = 2$

(b) $8^{2/3} = (\sqrt[3]{8})^2 = 2^2 = 4$ Alternative solution: $8^{2/3} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$

(c) $125^{-1/3} = \dfrac{1}{125^{1/3}} = \dfrac{1}{\sqrt[3]{125}} = \dfrac{1}{5}$

Now Try Exercises 51 and 53 ■

Example 12 ■ Using the Laws of Exponents with Rational Exponents

(a) $a^{1/3}a^{7/3} = a^{8/3}$ Law 1: $a^m a^n = a^{m+n}$

(b) $\dfrac{a^{2/5}a^{7/5}}{a^{3/5}} = a^{2/5+7/5-3/5} = a^{6/5}$ Law 1, Law 2: $\dfrac{a^m}{a^n} = a^{m-n}$

(c)
$$\begin{aligned} (2a^3b^4)^{3/2} &= 2^{3/2}(a^3)^{3/2}(b^4)^{3/2} && \text{Law 4: } (abc)^n = a^nb^nc^n \\ &= (\sqrt{2})^3 a^{3(3/2)}b^{4(3/2)} && \text{Law 3: } (a^m)^n = a^{mn} \\ &= 2\sqrt{2}a^{9/2}b^6 \end{aligned}$$

(d)
$$\begin{aligned} \left(\frac{2x^{3/4}}{y^{1/3}}\right)^3 \left(\frac{y^4}{x^{-1/2}}\right) &= \frac{2^3(x^{3/4})^3}{(y^{1/3})^3} \cdot (y^4x^{1/2}) && \text{Laws 5, 4, and 7} \\ &= \frac{8x^{9/4}}{y} \cdot y^4x^{1/2} && \text{Law 3} \\ &= 8x^{11/4}y^3 && \text{Laws 1 and 2} \end{aligned}$$

Now Try Exercises 57, 59, 61, and 63 ■

Example 13 ■ Simplifying by Writing Radicals as Rational Exponents

(a) $\dfrac{1}{\sqrt[3]{x^4}} = \dfrac{1}{x^{4/3}}$ Definition of rational exponents

(b)
$$\begin{aligned} (2\sqrt{x})(3\sqrt[3]{x}) &= (2x^{1/2})(3x^{1/3}) && \text{Definition of rational exponents} \\ &= 6x^{1/2+1/3} = 6x^{5/6} && \text{Law 1} \end{aligned}$$

(c)
$$\begin{aligned} \sqrt{x\sqrt{x}} &= (xx^{1/2})^{1/2} && \text{Definition of rational exponents} \\ &= (x^{3/2})^{1/2} && \text{Law 1} \\ &= x^{3/4} && \text{Law 3} \end{aligned}$$

Now Try Exercises 67 and 71 ■

■ Rationalizing the Denominator; Standard Form

It is often useful to eliminate the radical in a denominator by multiplying both numerator and denominator by an appropriate expression. This procedure is called **rationalizing the denominator**. If the denominator is of the form $\sqrt{a}$, then we multiply numerator and denominator by $\sqrt{a}$. In doing this we multiply the given quantity by 1, so we do not change its value. For instance,

$$\frac{1}{\sqrt{a}} = \frac{1}{\sqrt{a}} \cdot 1 = \frac{1}{\sqrt{a}} \cdot \frac{\sqrt{a}}{\sqrt{a}} = \frac{\sqrt{a}}{a}$$

Note that the denominator in the last fraction contains no radical. In general, if the denominator is of the form $\sqrt[n]{a^m}$ with $m < n$, then multiplying the numerator and denominator by $\sqrt[n]{a^{n-m}}$ will rationalize the denominator, because (for $a > 0$)

$$\sqrt[n]{a^m}\sqrt[n]{a^{n-m}} = \sqrt[n]{a^{m+n-m}} = \sqrt[n]{a^n} = a$$

A fractional expression whose denominator contains no radicals is said to be in **standard form**.

Example 14 ■ Rationalizing Denominators

Put each fractional expression into standard form by rationalizing the denominator.

(a) $\dfrac{2}{\sqrt{3}}$ **(b)** $\dfrac{1}{\sqrt[3]{5}}$ **(c)** $\sqrt[7]{\dfrac{1}{a^2}}$

Solution

This equals 1

(a) $\dfrac{2}{\sqrt{3}} = \dfrac{2}{\sqrt{3}} \cdot \dfrac{\sqrt{3}}{\sqrt{3}}$ Multiply by $\dfrac{\sqrt{3}}{\sqrt{3}}$

$= \dfrac{2\sqrt{3}}{3}$ $\sqrt{3} \cdot \sqrt{3} = 3$

(b) $\dfrac{1}{\sqrt[3]{5}} = \dfrac{1}{\sqrt[3]{5}} \cdot \dfrac{\sqrt[3]{5^2}}{\sqrt[3]{5^2}}$ Multiply by $\dfrac{\sqrt[3]{5^2}}{\sqrt[3]{5^2}}$

$= \dfrac{\sqrt[3]{25}}{5}$ $\sqrt[3]{5} \cdot \sqrt[3]{5^2} = \sqrt[3]{5^3} = 5$

(c) $\sqrt[7]{\dfrac{1}{a^2}} = \dfrac{1}{\sqrt[7]{a^2}}$ Property 2: $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$

$= \dfrac{1}{\sqrt[7]{a^2}} \cdot \dfrac{\sqrt[7]{a^5}}{\sqrt[7]{a^5}}$ Multiply by $\dfrac{\sqrt[7]{a^5}}{\sqrt[7]{a^5}}$

$= \dfrac{\sqrt[7]{a^5}}{a}$ $\sqrt[7]{a^2} \cdot \sqrt[7]{a^5} = \sqrt[7]{a^7} = a$

Now Try Exercises 73 and 75 ■

1.2 Exercises

Concepts

1. (a) Using exponential notation, we can write the product $5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5$ as ________.

(b) In the expression 3^4 the number 3 is called the ________ and the number 4 is called the ________.

2. (a) When we multiply two powers with the same base, we ________ the exponents. So $3^4 \cdot 3^5 =$ ________.

(b) When we divide two powers with the same base, we ________ the exponents. So $\dfrac{3^5}{3^2} =$ ________.

3. To move a number raised to a power from numerator to denominator or from denominator to numerator, we change the sign of the ________. So $a^{-2} =$ ________, $\dfrac{1}{b^{-2}} =$ ________, $\dfrac{a^{-3}}{b^2} =$ ________, and $\dfrac{6a^2}{b^{-3}} =$ ________.

4. (a) Using exponential notation, we can write $\sqrt[3]{5}$ as ________.

(b) Using radicals, we can write $5^{1/2}$ as ________.

(c) Is there a difference between $\sqrt{5^2}$ and $(\sqrt{5})^2$? Explain.

5. Explain what $4^{3/2}$ means, then calculate $4^{3/2}$ in two different ways:

$(4^{1/2})^{\square} =$ ________ or $(4^3)^{\square} =$ ________

6. Explain how we rationalize a denominator, then complete the following steps to rationalize $\dfrac{1}{\sqrt{3}}$:

$$\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \cdot \frac{\square}{\square} = \frac{\square}{\square}$$

7. Find the missing power in the following calculation: $5^{1/3} \cdot 5^{\square} = 5$.

8. *Yes or No*? If *No*, give a reason.

(a) Is there a difference between $(-5)^4$ and -5^4?

(b) Is the expression $(x^2)^3$ equal to x^5?

(c) Is the expression $(2x^4)^3$ equal to $2x^{12}$?

(d) Is the expression $\sqrt{4a^2}$ equal to $2a$?

Skills

9–18 ■ Radicals and Exponents Evaluate each expression.

9. (a) -2^6 **(b)** $(-2)^6$ **(c)** $\left(\frac{1}{5}\right)^2 \cdot (-3)^3$

10. (a) $(-5)^3$ **(b)** -5^3 **(c)** $(-5)^2 \cdot \left(\frac{2}{5}\right)^2$

11. (a) $\left(\frac{5}{3}\right)^0 \cdot 2^{-1}$ **(b)** $\dfrac{2^{-3}}{3^0}$ **(c)** $\left(\frac{2}{3}\right)^{-2}$

12. **(a)** $-2^3 \cdot (-2)^0$ **(b)** $-2^{-3} \cdot (-2)^0$ **(c)** $\left(\dfrac{-3}{5}\right)^{-3}$

13. **(a)** $5^3 \cdot 5$ **(b)** $5^4 \cdot 5^{-2}$ **(c)** $(2^2)^3$

14. **(a)** $3^8 \cdot 3^5$ **(b)** $\dfrac{10^7}{10^4}$ **(c)** $(3^5)^4$

15. **(a)** $3\sqrt[3]{16}$ **(b)** $\dfrac{\sqrt{18}}{\sqrt{81}}$ **(c)** $\sqrt{\frac{27}{4}}$

16. **(a)** $2\sqrt[3]{81}$ **(b)** $\dfrac{\sqrt{18}}{\sqrt{25}}$ **(c)** $\sqrt{\frac{12}{49}}$

17. **(a)** $\sqrt{3}\sqrt{15}$ **(b)** $\dfrac{\sqrt{48}}{\sqrt{3}}$ **(c)** $\sqrt[3]{24}\sqrt[3]{18}$

18. **(a)** $\sqrt{10}\sqrt{32}$ **(b)** $\dfrac{\sqrt{54}}{\sqrt{6}}$ **(c)** $\sqrt[3]{15}\sqrt[3]{75}$

19–24 ■ Exponents Simplify each expression and eliminate any negative exponents.

19. **(a)** t^5t^2 **(b)** $(4z^3)^2$ **(c)** $x^{-3}x^5$

20. **(a)** a^4a^6 **(b)** $(-2b^{-3})^3$ **(c)** $-2y^{10}y^{-11}$

21. **(a)** $x^{-5} \cdot x^3$ **(b)** $w^{-2}w^{-4}w^5$ **(c)** $\dfrac{x^{16}}{x^{10}}$

22. **(a)** $y^2 \cdot y^{-5}$ **(b)** $z^5z^{-3}z^{-4}$ **(c)** $\dfrac{y^7y^0}{y^{10}}$

23. **(a)** $\dfrac{a^9a^{-2}}{a}$ **(b)** $(a^2a^4)^3$ **(c)** $\left(\dfrac{x}{2}\right)^3(5x^6)$

24. **(a)** $\dfrac{z^2z^4}{z^3z^{-1}}$ **(b)** $(2a^3a^2)^4$ **(c)** $(-3z^2)^3(2z^3)$

25–32 ■ Exponents Simplify each expression and eliminate any negative exponents.

25. **(a)** $(3x^3y^2)(2y^3)$ **(b)** $(5w^2z^{-2})^2(z^3)$

26. **(a)** $(8m^{-2}n^4)\left(\frac{1}{2}n^{-2}\right)$ **(b)** $(3a^4b^{-2})^3(a^2b^{-1})$

27. **(a)** $\dfrac{x^2y^{-1}}{x^{-5}}$ **(b)** $\left(\dfrac{a^3}{2b^2}\right)^3$

28. **(a)** $\dfrac{y^{-2}z^{-3}}{y^{-1}}$ **(b)** $\left(\dfrac{x^3y^{-2}}{x^{-3}y^2}\right)^{-2}$

29. **(a)** $\left(\dfrac{a^2}{b}\right)^5\left(\dfrac{a^3b^2}{c^3}\right)^3$ **(b)** $\dfrac{(u^{-1}v^2)^2}{(u^3v^{-2})^3}$

30. **(a)** $\left(\dfrac{x^4z^2}{4y^5}\right)\left(\dfrac{2x^3y^2}{z^3}\right)^2$ **(b)** $\dfrac{(rs^2)^3}{(r^{-3}s^2)^2}$

31. **(a)** $\dfrac{8a^3b^{-4}}{2a^{-5}b^5}$ **(b)** $\left(\dfrac{y}{5x^{-2}}\right)^{-3}$

32. **(a)** $\dfrac{5xy^{-2}}{x^{-1}y^{-3}}$ **(b)** $\left(\dfrac{2a^{-1}b}{a^2b^{-3}}\right)^{-3}$

33–36 ■ Radicals Simplify each expression. Remember to use Property 5 of nth roots where appropriate.

33. **(a)** $\sqrt[4]{x^4}$ **(b)** $\sqrt[4]{16x^8}$

34. **(a)** $\sqrt[5]{x^{10}}$ **(b)** $\sqrt[3]{x^3y^6}$

35. **(a)** $\sqrt[3]{8x^9y^3}$ **(b)** $\sqrt[4]{8x^6y^2}\sqrt[4]{2x^2y^2}$

36. **(a)** $\sqrt{16x^4y^6z^2}$ **(b)** $\sqrt[3]{\sqrt[3]{512x^9}}$

37–42 ■ Radical Expressions Simplify each expression. Assume that all letters denote positive real numbers.

37. **(a)** $\sqrt{32} + \sqrt{18}$ **(b)** $\sqrt{75} + \sqrt{48}$

38. **(a)** $\sqrt{125} + \sqrt{45}$ **(b)** $\sqrt[3]{54} - \sqrt[3]{16}$

39. **(a)** $\sqrt{9a^3} + \sqrt{a}$ **(b)** $\sqrt{16x} + \sqrt{x^5}$

40. **(a)** $\sqrt[3]{x^4} + \sqrt[3]{8x}$ **(b)** $4\sqrt{18rt^3} + 5\sqrt{32r^3t^5}$

41. **(a)** $\sqrt{36x^2 + 36x^4}$ **(b)** $\sqrt{81x^2 + 81y^2}$

42. **(a)** $\sqrt{27a^3 + 63a^2}$ **(b)** $\sqrt{25t^2 + 100t^2}$

43–50 ■ Radicals and Exponents Write each radical expression using exponents, and each exponential expression using radicals.

	Radical expression	Exponential expression
43.	$\sqrt{10}$	
44.	$\sqrt[5]{6}$	
45.		$7^{3/5}$
46.		$6^{-5/2}$
47.	$\dfrac{1}{\sqrt{5}}$	
48.		$5^{-3/4}$
49.		$y^{-1.5}$
50.	$\dfrac{1}{\sqrt[3]{x^2}}$	

51–56 ■ Rational Exponents Evaluate each expression, without using a calculator.

51. **(a)** $16^{1/4}$ **(b)** $-8^{1/3}$ **(c)** $9^{-1/2}$

52. **(a)** $27^{1/3}$ **(b)** $(-8)^{1/3}$ **(c)** $-\left(\frac{1}{8}\right)^{1/3}$

53. **(a)** $32^{2/5}$ **(b)** $\left(\frac{4}{9}\right)^{-1/2}$ **(c)** $\left(\frac{16}{81}\right)^{3/4}$

54. **(a)** $125^{2/3}$ **(b)** $\left(\frac{25}{64}\right)^{3/2}$ **(c)** $27^{-4/3}$

55. **(a)** $5^{2/3} \cdot 5^{1/3}$ **(b)** $\dfrac{3^{3/5}}{3^{2/5}}$ **(c)** $(\sqrt[3]{4})^3$

56. **(a)** $3^{2/7} \cdot 3^{12/7}$ **(b)** $\dfrac{7^{2/3}}{7^{5/3}}$ **(c)** $(\sqrt[5]{6})^{-10}$

57–64 ■ Rational Exponents Simplify each expression and eliminate any negative exponents. Assume that all letters denote positive numbers.

57. **(a)** $x^{3/4}x^{5/4}$ **(b)** $y^{2/3}y^{4/3}$

58. **(a)** $(4b)^{1/2}(8b^{1/4})$ **(b)** $(3a^{3/4})^2(5a^{1/2})$

59. **(a)** $\dfrac{w^{4/3}w^{2/3}}{w^{1/3}}$ **(b)** $(3x^{1/2}y^{1/3})^6$

60. **(a)** $(8y^3)^{-2/3}$ **(b)** $(u^4v^6)^{-1/3}$

61. **(a)** $(8a^6b^{3/2})^{2/3}$ **(b)** $(4a^{-4}b^6)^{3/2}$

62. **(a)** $(x^{-5}y^{1/3})^{-3/5}$ **(b)** $(u^3v^2)^{1/3}(16u^6v^{2/3})^{1/2}$

63. (a) $\left(\dfrac{3x^{1/4}}{y}\right)^2\left(\dfrac{x^{-1}}{y^4}\right)^{1/2}$ (b) $\left(\dfrac{3w^{-1/3}}{z^{-1}}\right)^{-4}\left(\dfrac{2w^{1/3}}{z^{1/2}}\right)^2$

64. (a) $\left(\dfrac{y^{2/3}}{x^{-1}}\right)^3\left(\dfrac{x}{y^{-2}}\right)^{-1}$ (b) $\left(\dfrac{4y^3z^{2/3}}{x^{1/2}}\right)^2\left(\dfrac{x^{-3}y^6}{8z^4}\right)^{1/3}$

65–72 ■ Radicals Write each expression using rational exponents and simplify. Eliminate any negative exponents. Assume that all letters denote positive numbers.

65. (a) $\sqrt{x^3}$ (b) $\sqrt[5]{x^6}$

66. (a) $\sqrt{x^5}$ (b) $\sqrt[4]{x^6}$

67. (a) $\sqrt[6]{y^5}\sqrt[3]{y^2}$ (b) $(5\sqrt[3]{x})(2\sqrt[4]{x})$

68. (a) $\sqrt[4]{b^3}\sqrt{b}$ (b) $(2\sqrt{a})(\sqrt[3]{a^2})$

69. (a) $\sqrt{4st^3}\sqrt[6]{s^3t^2}$ (b) $\dfrac{\sqrt[4]{x^7}}{\sqrt[4]{x^3}}$

70. (a) $\sqrt[5]{x^3y^2}\sqrt[10]{x^4y^{16}}$ (b) $\dfrac{\sqrt[3]{8x^2}}{\sqrt{x}}$

71. (a) $\sqrt[3]{y\sqrt{y}}$ (b) $\sqrt{\dfrac{18u^5v}{2u^3v^3}}$

72. (a) $\sqrt{s\sqrt{s^3}}$ (b) $\sqrt[3]{\dfrac{54x^2y^4}{2x^5y}}$

73–76 ■ Rationalize Put each fractional expression into standard form by rationalizing the denominator. Assume that all letters denote positive real numbers.

73. (a) $\dfrac{1}{\sqrt{6}}$ (b) $\sqrt{\frac{3}{2}}$ (c) $\dfrac{9}{\sqrt[4]{2}}$

74. (a) $\dfrac{12}{\sqrt{3}}$ (b) $\sqrt{\frac{12}{5}}$ (c) $\dfrac{8}{\sqrt[3]{5^2}}$

75. (a) $\dfrac{1}{\sqrt{5x}}$ (b) $\sqrt{\dfrac{x}{5}}$ (c) $\sqrt[5]{\dfrac{1}{x^3}}$

76. (a) $\sqrt{\dfrac{s}{3t}}$ (b) $\dfrac{a}{\sqrt[6]{b^2}}$ (c) $\dfrac{1}{c^{3/5}}$

77–80 ■ Putting It All Together Simplify the expression and eliminate any negative exponents. Assume that all letters denote positive numbers unless otherwise stated. Convert to rational exponents where it is helpful. If necessary, rationalize the denominator.

77. (a) $\sqrt[4]{\frac{1}{4}}\sqrt[4]{\frac{1}{64}}$ (b) $\dfrac{\sqrt{5}}{\sqrt{40}}$

78. (a) $\left(\dfrac{x^{3/2}}{y^{-1/2}}\right)^4$ (b) $(25u^2v^{-4})^{1/2}$, $(u < 0,\quad v > 0)$

79. (a) $\sqrt{y}\sqrt[4]{y^2}$ (b) $(81\sqrt[4]{w^8z^8})^{1/2}$, $(w > 0,\quad z < 0)$

80. (a) $\left(\dfrac{x^{3/2}}{y^{-1/2}}\right)^4\left(\dfrac{x^{-2}}{y^3}\right)$ (b) $\left(\dfrac{w^4}{\sqrt[3]{w^3z^6}}\right)^{1/2}$

81–82 ■ Scientific Notation Write each number in scientific notation.

81. (a) 69,300,000 (b) 7,200,000,000,000
(c) 0.000 028 536 (d) 0.000 121 3

82. (a) 129,540,000 (b) 7,259,000,000
(c) 0.000 000 001 4 (d) 0.000 702 9

83–84 ■ Decimal Notation Write each number in decimal notation.

83. (a) 3.19×10^5 (b) 2.721×10^8
(c) 2.670×10^{-8} (d) 9.999×10^{-9}

84. (a) 7.1×10^{14} (b) 6×10^{12}
(c) 8.55×10^{-3} (d) 6.257×10^{-10}

85–86 ■ Scientific Notation Write the number indicated in each statement in scientific notation.

85. (a) A light-year, the distance that light travels in one year, is about 5,900,000,000,000 mi.
(b) The diameter of an electron is about 0.000 000 000 000 4 cm.
(c) A drop of water contains more than 33 billion billion molecules.

86. (a) The distance from the earth to the sun is about 93 million miles.
(b) The mass of an oxygen molecule is about 0.000 000 000 000 000 000 000 053 g.
(c) The mass of the earth is about 5,970,000,000,000,000,000,000,000 kg.

87–92 ■ Scientific Notation Use scientific notation, the Laws of Exponents, and a calculator to perform the indicated operations. State your answer rounded to the number of significant digits indicated by the given data.

87. $(7.2 \times 10^{-9})(1.806 \times 10^{-12})$

88. $(1.062 \times 10^{24})(8.61 \times 10^{19})$

89. $\dfrac{1.295643 \times 10^9}{(3.610 \times 10^{-17})(2.511 \times 10^6)}$

90. $\dfrac{(73.1)(1.6341 \times 10^{28})}{0.0000000019}$

91. $\dfrac{(0.0000162)(0.01582)}{(594{,}621{,}000)(0.0058)}$ 92. $\dfrac{(3.542 \times 10^{-6})^9}{(5.05 \times 10^4)^{12}}$

Skills Plus

93. **Sign of an Expression** Let a, b, and c be real numbers with $a > 0$, $b < 0$, and $c < 0$. Determine the sign of each expression.
(a) b^5 (b) b^{10} (c) ab^2c^3
(d) $(b - a)^3$ (e) $(b - a)^4$ (f) $\dfrac{a^3c^3}{b^6c^6}$

94. **Comparing Roots** Without using a calculator, determine which number is larger in each pair.
(a) $2^{1/2}$ or $2^{1/3}$ (b) $\left(\frac{1}{2}\right)^{1/2}$ or $\left(\frac{1}{2}\right)^{1/3}$
(c) $7^{1/4}$ or $4^{1/3}$ (d) $\sqrt[3]{5}$ or $\sqrt{3}$

Applications

95. **Distance to the Nearest Star** Proxima Centauri, the star nearest to our solar system, is 4.3 light-years away. Use the information in Exercise 85(a) to express this distance in miles.

96. Speed of Light The speed of light is about 186,000 mi/s. Use the information in Exercise 86(a) to find how long it takes for a light ray from the sun to reach the earth.

97. Volume of the Oceans The average ocean depth is 3.7×10^3 m, and the surface area of the oceans is 3.6×10^{14} m^2. What is the total volume of the ocean in liters? (One cubic meter contains 1000 liters.)

98. National Debt In 2020, the population of the United States was 3.3145×10^8, and the national debt was 2.670×10^{13} dollars. How much was each person's share of the debt? [*Source:* U.S. Census Bureau and U.S. Department of the Treasury]

99. Number of Atoms in the Observable Universe The *Hubble Deep Field* is a long exposure image of a tiny region of the sky (about 2.6 minutes of arc). Each dot or smudge in the image is an entire galaxy; there are over ten thousand galaxies in this one image. Each one contains billions of stars, each of which consists almost entirely of hydrogen atoms. Use the following information to give an estimate of the number of atoms in the observable universe.

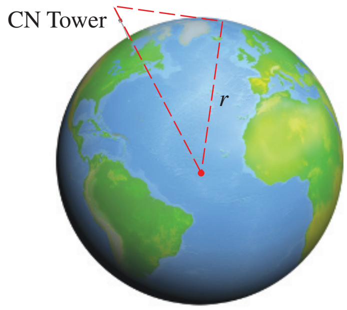
Mass of typical star: 1.77×10^{30} kg
Mass of hydrogen atom: 1.67×10^{-27} kg
Number of stars in typical galaxy: 2×10^{11}
Number of galaxies in observable universe: 1.5×10^{12}

NASA Image Collection/Alamy Stock Photo

100. How Far Can You See? Because of the curvature of the earth, the maximum distance D that you can see from the top of a tall building of height h is estimated by the formula

$$D = \sqrt{2rh + h^2}$$

where $r = 3960$ mi is the radius of the earth and D and h are also measured in miles. On a clear day, how far can you see from the observation deck of the Toronto CN Tower, 1135 ft above the ground?



101. Speed of a Skidding Car Police use the formula $s = \sqrt{30fd}$ to estimate the speed s (in mi/h) at which a car is traveling if it skids d feet after the brakes are applied suddenly. The number f is the coefficient of friction of the road, a measure of the "slipperiness" of the road. The table below gives some typical estimates for f.

	Tar	Concrete	Gravel
Dry	1.0	0.8	0.2
Wet	0.5	0.4	0.1

(a) If a car skids 65 ft on wet concrete, how fast was it moving when the brakes were applied?

(b) If a car is traveling at 50 mi/h, how far will it skid on wet tar?

102. Distance from the Earth to the Sun It follows from **Kepler's Third Law** of planetary motion that the average distance from a planet to the sun (in meters) is

$$d = \left(\frac{GM}{4\pi^2}\right)^{1/3} T^{2/3}$$

where $M = 1.99 \times 10^{30}$ kg is the mass of the sun, $G = 6.67 \times 10^{-11}$ N $\cdot$ m^2/kg^2 is the gravitational constant, and T is the period of the planet's orbit (in seconds). Use the fact that the period of the earth's orbit is about 365.25 days to find the distance from the earth to the sun.

■ Discuss ■ Discover ■ Prove ■ Write

103. Discuss: How Big is a Billion? If you had a million (10^6) dollars in a suitcase, and you spent a thousand (10^3) dollars each day, how many years would it take you to use all the money? Spending at the same rate, how many years would it take you to empty a suitcase filled with a *billion* (10^9) dollars?

104. Discover: Limiting Behavior of Powers Complete the following tables. What happens to the nth root of 2 as n gets large? What about the nth root of $\frac{1}{2}$?

n	$2^{1/n}$
1	
2	
5	
10	
100	

n	$\left(\frac{1}{2}\right)^{1/n}$
1	
2	
5	
10	
100	

Construct a similar table for $n^{1/n}$. What happens to the nth root of n as n gets large?

105. Discuss: Easy Powers that Look Hard Calculate these expressions in your head. Use the Laws of Exponents to help you.

(a) $\dfrac{18^5}{9^5}$ **(b)** $20^6 \cdot (0.5)^6$

106. Prove: Laws of Exponents Prove the following Laws of Exponents for the case in which m and n are positive integers and $m > n$.

(a) Law 2: $\dfrac{a^m}{a^n} = a^{m-n}$ **(b)** Law 5: $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

107. Prove: Laws of Exponents Prove the following Laws of Exponents.

(a) Law 6: $\left(\dfrac{a}{b}\right)^{-n} = \dfrac{b^n}{a^n}$ **(b)** Law 7: $\dfrac{a^{-n}}{b^{-m}} = \dfrac{b^m}{a^n}$

1.3 Algebraic Expressions

■ Adding and Subtracting Polynomials ■ Multiplying Algebraic Expressions ■ Special Product Formulas ■ Factoring Common Factors ■ Factoring Trinomials ■ Special Factoring Formulas ■ Factoring by Grouping Terms

A **variable** is a letter that can represent any number from a given set of numbers. If we start with variables, such as x, y, and z, and some real numbers and combine them using addition, subtraction, multiplication, division, powers, and roots, we obtain an **algebraic expression**. Here are some examples:

$$2x^2 - 3x + 4 \qquad \sqrt{x} + 10 \qquad \frac{y - 2z}{y^2 + 4}$$

In the first expression we say that 2 is the coefficient of x^2 and -3 is the coefficient of x. In general, a *coefficient* is any number that is multiplied with a variable.

A **monomial** is an expression of the form ax^k, where a is a real number and k is a nonnegative integer. A **binomial** is a sum of two monomials and a **trinomial** is a sum of three monomials. In general, a sum of monomials is called a *polynomial*. For example, the first expression listed above is a polynomial, but the other two are not.

Polynomials

A **polynomial** in the variable x is an expression of the form

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where $a_0, a_1, \ldots, a_n$ are real numbers, and n is a nonnegative integer. If $a_n \neq 0$, then the polynomial has **degree n**. The monomials $a_k x^k$ that make up the polynomial are called the **terms** of the polynomial.

Note that the degree of a polynomial is the highest power of the variable that appears in the polynomial.

Polynomial	Type	Terms	Degree
$2x^2 - 3x + 4$	trinomial	$2x^2, -3x, 4$	2
$x^8 + 5x$	binomial	$x^8, 5x$	8
$8 - x + x^2 - \frac{1}{2}x^3$	four terms	$-\frac{1}{2}x^3, x^2, -x, 8$	3
$5x + 1$	binomial	$5x, 1$	1
$9x^5$	monomial	$9x^5$	5
6	monomial	6	0

Adding and Subtracting Polynomials

We **add** and **subtract** polynomials using the properties of real numbers discussed in Section 1.1. The idea is to combine **like terms** (that is, terms with the same variables raised to the same powers) using the Distributive Property. For instance,

Distributive Property

$ac + bc = (a + b)c$

$$5x^7 + 3x^7 = (5 + 3)x^7 = 8x^7$$

In subtracting polynomials, we have to remember that if a minus sign precedes an expression in parentheses, then the sign of every term within the parentheses is changed when we remove the parentheses:

$$-(b + c) = -b - c$$

[This is simply a case of the Distributive Property, $a(b + c) = ab + ac$, with $a = -1$.]

Example 1 ■ Adding and Subtracting Polynomials

(a) Find the sum $(x^3 - 6x^2 + 2x + 4) + (x^3 + 5x^2 - 7x)$.

(b) Find the difference $(x^3 - 6x^2 + 2x + 4) - (x^3 + 5x^2 - 7x)$.

Solution

(a)
$$\begin{aligned}
&(x^3 - 6x^2 + 2x + 4) + (x^3 + 5x^2 - 7x) \\
&= (x^3 + x^3) + (-6x^2 + 5x^2) + (2x - 7x) + 4 && \text{Group like terms} \\
&= 2x^3 - x^2 - 5x + 4 && \text{Combine like terms}
\end{aligned}$$

(b)
$$\begin{aligned}
&(x^3 - 6x^2 + 2x + 4) - (x^3 + 5x^2 - 7x) \\
&= x^3 - 6x^2 + 2x + 4 - x^3 - 5x^2 + 7x && \text{Distributive Property} \\
&= (x^3 - x^3) + (-6x^2 - 5x^2) + (2x + 7x) + 4 && \text{Group like terms} \\
&= -11x^2 + 9x + 4 && \text{Combine like terms}
\end{aligned}$$

Now Try Exercises 17 and 19 ■

Multiplying Algebraic Expressions

To find the **product** of polynomials or other algebraic expressions, we need to use the Distributive Property repeatedly. In particular, using it three times on the product of two binomials, we get

$$(a + b)(c + d) = a(c + d) + b(c + d) = ac + ad + bc + bd$$

This says that we multiply the two factors by multiplying each term in one factor by each term in the other factor and adding these products. Schematically, we have

The acronym **FOIL** helps us remember that the product of two binomials is the sum of the products of the **F**irst terms, the **O**uter terms, the **I**nner terms, and the **L**ast terms.

$$(a + b)(c + d) = \underset{\substack{\uparrow \\ \textbf{F}}}{ac} + \underset{\substack{\uparrow \\ \textbf{O}}}{ad} + \underset{\substack{\uparrow \\ \textbf{I}}}{bc} + \underset{\substack{\uparrow \\ \textbf{L}}}{bd}$$

In general, we can multiply two algebraic expressions by using the Distributive Property and the Laws of Exponents.

Example 2 ■ Multiplying Binomials Using FOIL

$$\begin{aligned}
(2x + 1)(3x - 5) &= \underset{\substack{\uparrow \\ \textbf{F}}}{6x^2} - \underset{\substack{\uparrow \\ \textbf{O}}}{10x} + \underset{\substack{\uparrow \\ \textbf{I}}}{3x} - \underset{\substack{\uparrow \\ \textbf{L}}}{5} && \text{Distributive Property} \\
&= 6x^2 - 7x - 5 && \text{Combine like terms}
\end{aligned}$$

Now Try Exercise 27 ■

When we multiply trinomials or polynomials with more terms, we use the Distributive Property. It is also helpful to arrange our work in table form. The following example illustrates both methods.

Example 3 ■ Multiplying Polynomials

Find the product: $(2x + 3)(x^2 - 5x + 4)$

Solution 1: Using the Distributive Property

$$\begin{aligned}
(2x + 3)(x^2 - 5x + 4) &= 2x(x^2 - 5x + 4) + 3(x^2 - 5x + 4) && \text{Distributive Property}\\
&= (2x \cdot x^2 - 2x \cdot 5x + 2x \cdot 4) + (3 \cdot x^2 - 3 \cdot 5x + 3 \cdot 4) && \text{Distributive Property}\\
&= (2x^3 - 10x^2 + 8x) + (3x^2 - 15x + 12) && \text{Laws of Exponents}\\
&= 2x^3 - 7x^2 - 7x + 12 && \text{Combine like terms}
\end{aligned}$$

Solution 2: Using Table Form

$$\begin{array}{rrrrl}
 & x^2 - & 5x + & 4 & \\
 & & 2x + & 3 & \\
\hline
 & 3x^2 - & 15x + & 12 & \text{Multiply } x^2 - 5x + 4 \text{ by } 3\\
2x^3 - & 10x^2 + & 8x & & \text{Multiply } x^2 - 5x + 4 \text{ by } 2x\\
\hline
2x^3 - & 7x^2 - & 7x + & 12 & \text{Add like terms}
\end{array}$$

Now Try Exercise 47 ■

■ Special Product Formulas

Certain types of products occur so frequently that you should memorize them. You can verify the following formulas by performing the multiplications.

Special Product Formulas

If A and B are any real numbers or algebraic expressions, then

1. $(A + B)(A - B) = A^2 - B^2$ — Sum and difference of same terms
2. $(A + B)^2 = A^2 + 2AB + B^2$ — Square of a sum
3. $(A - B)^2 = A^2 - 2AB + B^2$ — Square of a difference
4. $(A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$ — Cube of a sum
5. $(A - B)^3 = A^3 - 3A^2B + 3AB^2 - B^3$ — Cube of a difference

The key idea in using these formulas (or any other formula in algebra) is the **Principle of Substitution**: We may substitute any algebraic expression for any letter in a formula. For example, to find $(x^2 + y^3)^2$ we use Product Formula 2, substituting x^2 for A and y^3 for B, to get

$$(x^2 + y^3)^2 = (x^2)^2 + 2(x^2)(y^3) + (y^3)^2$$

$$\boxed{(A + B)^2 = A^2 + 2AB + B^2}$$

Mathematics in the Modern World

Changing Words, Sound, and Pictures into Numbers

Pictures, sound, and text are routinely transmitted from one place to another via the Internet, fax machines, or modems. How can such things be transmitted electronically? The key to doing this is to change them into numbers or bits (the digits 0 or 1). It's easy to see how to change text to numbers. For example, we could use the correspondence A = 00000001, B = 00000010, C = 00000011, D = 00000100, E = 00000101, and so on. The word "BED" then becomes 000000100000010100000100. By reading the digits in groups of eight, it is possible to translate this number back to the word "BED."

Changing sound to bits is more complicated. A sound wave can be graphed on an oscilloscope or a computer. The graph is then broken down mathematically into simpler components corresponding to the different frequencies of the original sound. (A branch of mathematics called Fourier analysis is used here.) The intensity of each component is a number, and the original sound can be reconstructed from these numbers. For example, music is stored in a digital file as a sequence of bits; it may look like 1010100010100101001010101000001011110101000101011.... (One second of music requires 1.5 million bits!) The computer reconstructs the music from the numbers in the digital file.

Changing pictures into numbers involves expressing the color and brightness of each dot (or pixel) into a number. This is done very efficiently using a branch of mathematics called wavelet theory. The FBI uses wavelets as a compact way to store the millions of fingerprints they have on file.

Example 4 ■ Using the Special Product Formulas

Use the Special Product Formulas to find each product.

(a) $(3x + 5)^2$ **(b)** $(x^2 - 2)^3$

Solution

(a) Substituting $A = 3x$ and $B = 5$ in Product Formula 2, we get

$$(3x + 5)^2 = (3x)^2 + 2(3x)(5) + 5^2 = 9x^2 + 30x + 25$$

(b) Substituting $A = x^2$ and $B = 2$ in Product Formula 5, we get

$$\begin{aligned}(x^2 - 2)^3 &= (x^2)^3 - 3(x^2)^2(2) + 3(x^2)(2)^2 - 2^3\\ &= x^6 - 6x^4 + 12x^2 - 8\end{aligned}$$

Now Try Exercises 33 and 45 ■

Example 5 ■ Recognizing a Special Product Formula

Find each product.

(a) $(2x + \sqrt{y})(2x - \sqrt{y})$ **(b)** $(x + y - 1)(x + y + 1)$

Solution

(a) Substituting $A = 2x$ and $B = \sqrt{y}$ in Product Formula 1, we get

$$(2x + \sqrt{y})(2x - \sqrt{y}) = (2x)^2 - (\sqrt{y})^2 = 4x^2 - y$$

(b) If we group $x + y$ together and think of this as one algebraic expression, we can use Product Formula 1 with $A = x + y$ and $B = 1$.

$$\begin{aligned}(x + y - 1)(x + y + 1) &= [(x + y) - 1][(x + y) + 1]\\ &= (x + y)^2 - 1^2 && \text{Product Formula 1}\\ &= x^2 + 2xy + y^2 - 1 && \text{Product Formula 2}\end{aligned}$$

Now Try Exercises 59 and 63 ■

■ Factoring Common Factors

We use the Distributive Property to expand algebraic expressions. We sometimes need to reverse this process (again using the Distributive Property) by **factoring** an expression as a product of simpler ones. For example, we can write

EXPANDING

$$(x - 2)(x + 3) = x^2 + x - 6$$

FACTORING

We say that $x - 2$ and $x + 3$ are **factors** of $x^2 + x - 6$.

The easiest type of factoring occurs when the terms have a common factor.

Example 6 ■ Factoring Out Common Factors

Factor each expression.

(a) $3x^2 - 6x$ **(b)** $8x^4y^2 + 6x^3y^3 - 2xy^4$ **(c)** $(2x + 4)(x - 3) - 5(x - 3)$

Solution

(a) The greatest common factor of the terms $3x^2$ and $-6x$ is $3x$, so we have

$$3x^2 - 6x = 3x(x - 2)$$

Check Your Answer

Multiplying gives

$$3x(x - 2) = 3x^2 - 6x \checkmark$$

(b) We note that

$$8, 6, \text{ and } -2 \text{ have the greatest common factor } 2$$
$$x^4, x^3, \text{ and } x \text{ have the greatest common factor } x$$
$$y^2, y^3, \text{ and } y^4 \text{ have the greatest common factor } y^2$$

So the greatest common factor of the three terms in the polynomial is $2xy^2$, and we have

$$\begin{aligned} 8x^4y^2 + 6x^3y^3 - 2xy^4 &= (2xy^2)(4x^3) + (2xy^2)(3x^2y) + (2xy^2)(-y^2) \\ &= 2xy^2(4x^3 + 3x^2y - y^2) \end{aligned}$$

Check Your Answer

Multiplying gives

$$\begin{aligned} &2xy^2(4x^3 + 3x^2y - y^2) \\ &\quad = 8x^4y^2 + 6x^3y^3 - 2xy^4 \end{aligned}$$

(c) The two terms have the common factor $x - 3$.

$$\begin{aligned} (2x + 4)(x - 3) - 5(x - 3) &= [(2x + 4) - 5](x - 3) && \text{Distributive Property} \\ &= (2x - 1)(x - 3) && \text{Simplify} \end{aligned}$$

Now Try Exercises 65, 67, and 69 ■

Factoring Trinomials

To factor a trinomial of the form $x^2 + bx + c$, we note that

$$(x + r)(x + s) = x^2 + (r + s)x + rs$$

so we need to choose numbers r and s so that $r + s = b$ and $rs = c$.

Example 7 ■ Factoring $x^2 + bx + c$ by Trial and Error

Factor: $x^2 + 7x + 12$

Solution We need to find two integers whose product is 12 and whose sum is 7. By trial and error we find that the two integers are 3 and 4. Thus the factorization is

$$x^2 + 7x + 12 = (x + \underbrace{3)(x + 4}_{\text{factors of } 12})$$

Check Your Answer

Multiplying gives

$$(x + 3)(x + 4) = x^2 + 7x + 12$$

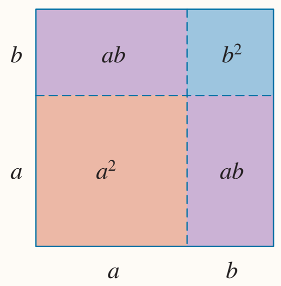
Now Try Exercise 73 ■

To factor a trinomial of the form $ax^2 + bx + c$ with $a \neq 1$, we look for factors of the form $px + r$ and $qx + s$:

$$ax^2 + bx + c = (px + r)(qx + s) = pqx^2 + (ps + qr)x + rs$$

$$ax^2 + bx + c = (\overset{\text{factors of } a}{\overbrace{px + r)(qx}} + s) \quad \text{factors of } c: r, s$$

Therefore we try to find numbers p, q, r, and s such that $pq = a$, $rs = c$, $ps + qr = b$. If these numbers are all integers, then we will have a limited number of possibilities to try for p, q, r, and s.



Discovery Project ■ **Visualizing a Formula**

Many of the Special Product Formulas in this section can be "seen" as geometrical facts about length, area, and volume. For example, the formula about the square of a sum can be interpreted to be about areas of squares and rectangles. The ancient Greeks always interpreted algebraic formulas in terms of geometric figures. Such figures give us special insight into how these formulas work. You can find the project at **www.stewartmath.com**.

Example 8 ■ Factoring $ax^2 + bx + c$ by Trial and Error

Factor: $6x^2 + 7x - 5$

Solution We can factor 6 as $6 \cdot 1$ or $3 \cdot 2$, and -5 as $-5 \cdot 1$ or $5 \cdot (-1)$. By trying these possibilities, we arrive at the factorization

$$6x^2 + 7x - 5 = (\overset{\text{factors of } 6}{3}x + \underset{\text{factors of } -5}{5})(\overset{}{2}x - \underset{}{1})$$

Check Your Answer

Multiplying gives

$(3x + 5)(2x - 1) = 6x^2 + 7x - 5$

Now Try Exercise 75 ■

Example 9 ■ Recognizing the Form of an Expression

Factor each expression.

(a) $x^2 - 2x - 3$ **(b)** $(5a + 1)^2 - 2(5a + 1) - 3$

Solution

(a) $x^2 - 2x - 3 = (x - 3)(x + 1)$ Trial and error

(b) This expression is of the form

$$\square^2 - 2\square - 3$$

where $\square$ represents $5a + 1$. This is the same form as the expression in part (a), so it will factor as $(\square - 3)(\square + 1)$.

$$(5a + 1)^2 - 2(5a + 1) - 3 = [(5a + 1) - 3][(5a + 1) + 1]$$
$$= (5a - 2)(5a + 2)$$

Now Try Exercises 79 and 131 ■

■ Special Factoring Formulas

Some special algebraic expressions can be factored by using the following formulas. The first three are simply Special Product Formulas written in reverse.

Special Factoring Formulas

Formula	Name
1. $A^2 - B^2 = (A + B)(A - B)$	Difference of squares
2. $A^2 + 2AB + B^2 = (A + B)^2$	Perfect square
3. $A^2 - 2AB + B^2 = (A - B)^2$	Perfect square
4. $A^3 - B^3 = (A - B)(A^2 + AB + B^2)$	Difference of cubes
5. $A^3 + B^3 = (A + B)(A^2 - AB + B^2)$	Sum of cubes

Example 10 ■ Recognizing a Difference of Squares

Factor each expression.

(a) $4x^2 - 25$ **(b)** $(x + y)^2 - z^2$

Terms and Factors

When we multiply two numbers together, each of the numbers is called a **factor** of the product. When we add two numbers together, each number is called a **term** of the sum.

$$\underbrace{2 \times 3}_{\text{Factors}} \qquad \underbrace{2 + 3}_{\text{Terms}}$$

If a factor is common to each term of an expression, we can factor it out. The following expression has two terms.

$$\underbrace{ax + 2ay}_{a \text{ is a factor of each term}}$$

Each term contains the factor a, so we can factor out a and write the expression as

$$ax + 2ay = a(x + 2y)$$

Solution

(a) Using the Difference of Squares Formula with $A = 2x$ and $B = 5$, we have

$$4x^2 - 25 = (2x)^2 - 5^2 = (2x + 5)(2x - 5)$$

$$A^2 - B^2 = (A + B)(A - B)$$

(b) We use the Difference of Squares Formula with $A = x + y$ and $B = z$.

$$(x + y)^2 - z^2 = (x + y - z)(x + y + z)$$

Now Try Exercises 81 and 115 ■

A trinomial is a perfect square if it is of the form

$$A^2 + 2AB + B^2 \qquad \text{or} \qquad A^2 - 2AB + B^2$$

So we **recognize a perfect square** if the middle term ($2AB$ or $-2AB$) is plus or minus twice the product of the square roots of the two outer terms.

Example 11 ■ Recognizing Perfect Squares

Factor each trinomial.

(a) $x^2 + 6x + 9$ **(b)** $4x^2 - 4xy + y^2$

Solution

(a) Here $A = x$ and $B = 3$, so $2AB = 2 \cdot x \cdot 3 = 6x$. Since the middle term is $6x$, the trinomial is a perfect square. By the Perfect Square Formula we have

$$x^2 + 6x + 9 = (x + 3)^2$$

(b) Here $A = 2x$ and $B = y$, so $2AB = 2 \cdot 2x \cdot y = 4xy$. Since the middle term is $-4xy$, the trinomial is a perfect square. By the Perfect Square Formula we have

$$4x^2 - 4xy + y^2 = (2x - y)^2$$

Now Try Exercises 111 and 113 ■

Example 12 ■ Factoring Differences and Sums of Cubes

Factor each polynomial.

(a) $27x^3 - 1$ **(b)** $x^6 + 8$

Solution

(a) Using the Difference of Cubes Formula with $A = 3x$ and $B = 1$, we get

$$\begin{aligned} 27x^3 - 1 &= (3x)^3 - 1^3 = (3x - 1)[(3x)^2 + (3x)(1) + 1^2] \\ &= (3x - 1)(9x^2 + 3x + 1) \end{aligned}$$

(b) Using the Sum of Cubes Formula with $A = x^2$ and $B = 2$, we have

$$x^6 + 8 = (x^2)^3 + 2^3 = (x^2 + 2)(x^4 - 2x^2 + 4)$$

Now Try Exercises 83 and 85 ■

When we factor an expression, the result can sometimes be factored further. In general, *we first factor out common factors*, then inspect the result to see whether it can be factored by any of the other methods of this section. We repeat this process until we have factored the expression completely.

Example 13 ■ Factoring an Expression Completely

Factor each expression completely.

(a) $2x^4 - 8x^2$ **(b)** $x^5y^2 - xy^6$

Solution

(a) We first factor out the power of x with the smallest exponent.

$$\begin{aligned} 2x^4 - 8x^2 &= 2x^2(x^2 - 4) && \text{Common factor is } 2x^2 \\ &= 2x^2(x + 2)(x - 2) && \text{Factor } x^2 - 4 \text{ as a difference of squares} \end{aligned}$$

(b) We first factor out the powers of x and y with the smallest exponents.

$$\begin{aligned} x^5y^2 - xy^6 &= xy^2(x^4 - y^4) && \text{Common factor is } xy^2 \\ &= xy^2(x^2 + y^2)(x^2 - y^2) && \text{Factor } x^4 - y^4 \text{ as a difference of squares} \\ &= xy^2(x^2 + y^2)(x + y)(x - y) && \text{Factor } x^2 - y^2 \text{ as a difference of squares} \end{aligned}$$

Now Try Exercises 121 and 123 ■

In the next example we factor out variables with fractional exponents. This type of factoring occurs in calculus.

Example 14 ■ Factoring Expressions with Fractional Exponents

Factor each expression.

(a) $3x^{3/2} - 9x^{1/2} + 6x^{-1/2}$ **(b)** $(2 + x)^{-2/3}x + (2 + x)^{1/3}$

Solution

(a) Factor out the power of x with the *smallest exponent*, that is, $x^{-1/2}$.

To factor out $x^{-1/2}$ from $x^{3/2}$, we *subtract* exponents:

$$\begin{aligned} x^{3/2} &= x^{-1/2}(x^{3/2-(-1/2)}) \\ &= x^{-1/2}(x^{3/2+1/2}) \\ &= x^{-1/2}(x^2) \end{aligned}$$

$$\begin{aligned} 3x^{3/2} - 9x^{1/2} + 6x^{-1/2} &= 3x^{-1/2}(x^2 - 3x + 2) && \text{Factor out } 3x^{-1/2} \\ &= 3x^{-1/2}(x - 1)(x - 2) && \text{Factor the quadratic } x^2 - 3x + 2 \end{aligned}$$

(b) Factor out the power of $2 + x$ with the *smallest exponent*, that is, $(2 + x)^{-2/3}$.

$$\begin{aligned} (2 + x)^{-2/3}x + (2 + x)^{1/3} &= (2 + x)^{-2/3}[x + (2 + x)] && \text{Factor out } (2 + x)^{-2/3} \\ &= (2 + x)^{-2/3}(2 + 2x) && \text{Simplify} \\ &= 2(2 + x)^{-2/3}(1 + x) && \text{Factor out 2} \end{aligned}$$

Check Your Answers

To see that you have factored correctly, multiply using the Laws of Exponents.

(a) $3x^{-1/2}(x^2 - 3x + 2)$
$= 3x^{3/2} - 9x^{1/2} + 6x^{-1/2}$ ✓

(b) $(2 + x)^{-2/3}[x + (2 + x)]$
$= (2 + x)^{-2/3}x + (2 + x)^{1/3}$ ✓

Now Try Exercises 97 and 99 ■

■ Factoring by Grouping Terms

Polynomials with at least four terms (quadrinomials) can sometimes be factored by grouping terms. The following example illustrates the idea.

Example 15 ■ Factoring by Grouping

Factor each polynomial.

(a) $x^3 + x^2 + 4x + 4$ **(b)** $x^3 - 2x^2 - 9x + 18$

Solution

(a) $x^3 + x^2 + 4x + 4 = (x^3 + x^2) + (4x + 4)$ — Group terms

$= x^2(x + 1) + 4(x + 1)$ — Factor out common factors

$= (x^2 + 4)(x + 1)$ — Factor $x + 1$ from each term

(b) $x^3 - 2x^2 - 9x + 18 = (x^3 - 2x^2) - (9x - 18)$ — Group terms

$= x^2(x - 2) - 9(x - 2)$ — Factor out common factors

$= (x^2 - 9)(x - 2)$ — Factor $x - 2$from each term

$= (x + 3)(x - 3)(x - 2)$ — Factor completely

Now Try Exercises 89 and 125 ■

1.3 Exercises

Concepts

1. The greatest common factor in the expression $18x^3 + 30x$ is ________, and the expression factors as ____ (____ + ____).

2. Consider the polynomial $2x^3 + 3x^2 + 10x$.

(a) How many terms does this polynomial have? ________

List the terms: ________________.

(b) What factor is common to all the terms? ________

Factor the polynomial: ____________________.

3. To factor the trinomial $x^2 + 8x + 12$, we look for two integers whose product is ________ and whose sum is ________. These integers are ________ and ________, so the trinomial factors as ____________.

4. The Special Product Formula for the "square of a sum" is $(A + B)^2 =$ ____________________.
So $(2x + 3)^2 =$ ____________________.

5. The Special Product Formula for the "product of the sum and difference of terms" is $(A + B)(A - B) =$ ________________.
So $(6 + x)(6 - x) =$ ________________.

6. The Special Factoring Formula for the "difference of squares" is $A^2 - B^2 =$ ________________. So $49x^2 - 9$ factors as ________________.

7. The Special Factoring Formula for a "perfect square" is $A^2 + 2AB + B^2 =$ ________________. So $x^2 + 10x + 25$ factors as ________________.

8. *Yes or No*? If *No*, give a reason.

(a) Is the expression $(x + 5)^2$ equal to $x^2 + 25$?

(b) When you expand $(x + a)^2$, where $a \neq 0$, do you get three terms?

(c) Is the expression $(x + 5)(x - 5)$ equal to $x^2 - 25$?

(d) When you expand $(x + a)(x - a)$, where $a \neq 0$, do you get three terms?

Skills

9–14 ■ Polynomials Complete the following table by stating whether the polynomial is a monomial, binomial, or trinomial; then list its terms and state its degree.

Polynomial	Type	Terms	Degree
9. $5x^3 + 6$			
10. $-2x^2 + 5x - 3$			
11. -8			
12. $\frac{1}{2}x^7$			
13. $x - x^2 + x^3 - x^4$			
14. $\sqrt{2}x - \sqrt{3}$			

15–26 ■ Polynomials Find the sum, difference, or product.

15. $(12x - 7) - (5x - 12)$ **16.** $(5 - 3x) + (2x - 8)$

17. $(-2x^2 - 3x + 1) + (3x^2 + 5x - 4)$

18. $(3x^2 + x + 1) - (2x^2 - 3x - 5)$

19. $(5x^3 + 4x^2 - 3x) - (x^2 + 7x + 2)$

20. $3(x - 1) + 4(x + 2)$

21. $8(2x + 5) - 7(x - 9)$

22. $4(x^2 - 3x + 5) - 3(x^2 - 2x + 1)$

23. $x^3(x^2 + 3x) - 2x(x^4 - 3x^2)$

24. $4x(1 - x^3) + 3x^3(x^3 - x)$

25. $4x(x - 2) - 2(x^2 - 4x) + 2x^2(x - 1)$

26. $6x^3(x^2 - 1) - 2x(2 + 3x^2) + 2(2x - 4)$

27–32 ■ Using FOIL Multiply the algebraic expressions using the FOIL method and simplify.

27. $(3t - 2)(7t - 4)$ **28.** $(4s - 1)(2s + 5)$

29. $(3x + 5)(2x - 1)$ **30.** $(7y - 3)(2y - 1)$

31. $(x + 3y)(2x - y)$ **32.** $(4x - 5y)(3x - y)$

33–46 ■ Using Special Product Formulas Multiply the algebraic expressions using a Special Product Formula and simplify.

33. $(4x+3)^2$ **34.** $(2-7y)^2$

35. $(y-3x)^2$ **36.** $(5x-y)^2$

37. $(2x+3y)^2$ **38.** $(r-2s)^2$

39. $(w+7)(w-7)$ **40.** $(5-y)(5+y)$

41. $(3x-4)(3x+4)$ **42.** $(2y+5)(2y-5)$

43. $(\sqrt{x}+2)(\sqrt{x}-2)$ **44.** $(\sqrt{y}+\sqrt{2})(\sqrt{y}-\sqrt{2})$

45. $(y+2)^3$ **46.** $(x-3)^3$

47–64 ■ Multiplying Algebraic Expressions Perform the indicated operations and simplify.

47. $(x+2)(x^2+2x+3)$ **48.** $(x+1)(2x^2-x+1)$

49. $(2x-5)(x^2-x+1)$ **50.** $(1+2x)(x^2-3x+1)$

51. $\sqrt{x}(x-\sqrt{x})$ **52.** $x^{3/2}\left(\sqrt{x}-\dfrac{1}{\sqrt{x}}\right)$

53. $y^{1/3}(y^{2/3}+y^{5/3})$ **54.** $x^{1/4}(2x^{3/4}-x^{1/4})$

55. $(x^{1/2}-y^{1/2})^2$ **56.** $\left(\sqrt{u}+\dfrac{1}{\sqrt{u}}\right)^2$

57. $(x^2-a^2)(x^2+a^2)$

58. $(x^{1/2}+y^{1/2})(x^{1/2}-y^{1/2})$

59. $(\sqrt{a}-b)(\sqrt{a}+b)$

60. $(\sqrt{h^2+1}+1)(\sqrt{h^2+1}-1)$

61. $[(x-1)+x^2][(x-1)-x^2]$

62. $[x+(2+x^2)][x-(2+x^2)]$

63. $(2x+y-3)(2x+y+3)$

64. $(x+y+z)(x-y-z)$

65–72 ■ Factoring a Common Factor Factor out the common factor.

65. $4x^4-x^2$ **66.** $3x^4-6x^3-x^2$

67. $y(y-6)+9(y-6)$ **68.** $(z+2)^2-5(z+2)$

69. $4x^3y^2-6xy^3+8x^2y^4$ **70.** $-7x^4y^2+14xy^3+21xy^4$

71. $(x+3)^5(x+2)^2-(x+3)^4(x+2)^3$

72. $3(2x-1)^3(x^2+1)^4-(2x-1)^4(x^2+1)^3$

73–80 ■ Factoring Trinomials Factor the trinomial.

73. $z^2-11z+18$ **74.** x^2+4x-5

75. $10x^2-19x+6$ **76.** $6y^2+11y-21$

77. $3x^2-16x+5$ **78.** $5x^2-7x-6$

79. $(3x+2)^2+8(3x+2)+12$

80. $2(a+b)^2+5(a+b)-3$

81–88 ■ Using Special Factoring Formulas Use a Special Factoring Formula to factor the expression.

81. $36a^2-49$ **82.** $(x+3)^2-4$

83. $27x^3+y^3$ **84.** a^3-b^6

85. $8s^3-125t^3$ **86.** $1+1000y^3$

87. $x^2+12x+36$ **88.** $16z^2-24z+9$

89–94 ■ Factoring by Grouping Factor the expression by grouping terms.

89. x^3+4x^2+x+4 **90.** $3x^3-x^2+6x-2$

91. $5x^3+x^2+5x+1$ **92.** $18x^3+9x^2+2x+1$

93. x^3+x^2+x+1 **94.** x^5+x^4+x+1

95–100 ■ Fractional Exponents Factor the expression by factoring out the lowest power of each common factor.

95. $x^{2/3}+3x^{5/3}$ **96.** $x^{3/4}-5x^{-1/4}$

97. $x^{-3/2}-x^{-1/2}+x^{1/2}$ **98.** $x^{5/3}+x^{2/3}+2x^{-1/3}$

99. $(x^2+1)^{1/2}+2(x^2+1)^{-1/2}$

100. $x^{-1/2}(x+1)^{1/2}+x^{1/2}(x+1)^{-1/2}$

101–134 ■ Factoring Completely Recognize the type of expression and use an appropriate method to factor it completely.

101. $2x+12x^3$ **102.** $12x^2+3x^3$

103. x^2-2x-8 **104.** $x^2-14x+48$

105. $2x^2+5x+3$ **106.** $2x^2+7x-4$

107. $9x^2-36x-45$ **108.** $8x^2+10x+3$

109. $49-4y^2$ **110.** $4t^2-9s^2$

111. t^2-6t+9 **112.** $x^2+10x+25$

113. $y^2-10yz+25z^2$ **114.** $r^2-6rs+9s^2$

115. $(a+b)^2-(a-b)^2$ **116.** $\left(1+\dfrac{1}{x}\right)^2-\left(1-\dfrac{1}{x}\right)^2$

117. $x^2(x^2-1)-9(x^2-1)$

118. $(a^2-1)(b-2)^2-4(a^2-1)$

119. $8x^3-125$ **120.** x^6+64

121. x^3+2x^2+x **122.** $3x^3-27x$

123. $x^4y^3-x^2y^5$ **124.** $18y^3x^2-2xy^4$

125. $3x^3-x^2-12x+4$ **126.** $9x^3+18x^2-x-2$

127. $x^{-3/2}+2x^{-1/2}+x^{1/2}$

128. $(x-1)^{7/2}-(x-1)^{3/2}$

129. $(x-1)(x+2)^2-(x-1)^2(x+2)$

130. $y^4(y+2)^3+y^5(y+2)^4$

131. $(a^2+1)^2-7(a^2+1)+10$

132. $(a^2+2a)^2-2(a^2+2a)-3$

133. $(x^2+3)^2(x-1)^3-(4x+1)^2(x-1)^3$

134. $(x+1)^2(x+2)-6(x+1)(x+2)+9(x+2)$

135–138 ■ Factoring Completely Factor the expression completely. (This type of expression arises in calculus when using the "Product Rule.")

135. $5(x^2+4)^4(2x)(x-2)^4+(x^2+4)^5(4)(x-2)^3$

136. $3(2x-1)^2(2)(x+3)^{1/2}+(2x-1)^3(\frac{1}{2})(x+3)^{-1/2}$

137. $(x^2+3)^{-1/3} - \frac{2}{3}x^2(x^2+3)^{-4/3}$

138. $\frac{1}{2}x^{-1/2}(3x+4)^{1/2} - \frac{3}{2}x^{1/2}(3x+4)^{-1/2}$

Skills Plus

139–140 ■ Verifying Formulas Show that the following formulas hold.

139. **(a)** $ab = \frac{1}{2}[(a+b)^2 - (a^2+b^2)]$

(b) $(a^2+b^2)^2 - (a^2-b^2)^2 = 4a^2b^2$

140. $(a^2+b^2)(c^2+d^2) = (ac+bd)^2 + (ad-bc)^2$

141. Factoring Completely Factor completely:

$$4a^2c^2 - (a^2 - b^2 + c^2)^2$$

142. Factoring $x^4 + ax^2 + b$ A trinomial of the form $x^4 + ax^2 + b$ can sometimes be factored easily. For example,

$$x^4 + 3x^2 - 4 = (x^2+4)(x^2-1)$$

But $x^4 + 3x^2 + 4$ cannot be factored in this way. Instead, we can use the following method.

$$\begin{aligned} x^4 + 3x^2 + 4 &= (x^4 + 4x^2 + 4) - x^2 && \text{Add and subtract } x^2 \\ &= (x^2+2)^2 - x^2 && \text{Factor perfect square} \\ &= [(x^2+2) - x][(x^2+2) + x] && \text{Difference of squares} \\ &= (x^2 - x + 2)(x^2 + x + 2) \end{aligned}$$

Factor each trinomial, using whichever method is appropriate.

(a) $x^4 + x^2 - 2$ **(b)** $x^4 + 2x^2 + 9$

(c) $x^4 + 4x^2 + 16$ **(d)** $x^4 + 2x^2 + 1$

Applications

143. Volume of Concrete A culvert is constructed out of large cylindrical shells cast in concrete, as shown in the figure. Using the formula for the volume of a cylinder given on the inside front cover of this book, explain why the volume of the cylindrical shell is

$$V = \pi R^2 h - \pi r^2 h$$

Factor to show that

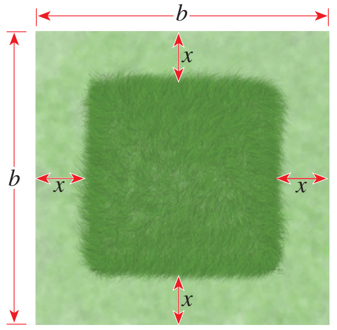
$$V = 2\pi \cdot \text{average radius} \cdot \text{height} \cdot \text{thickness}$$

Use the "unrolled" diagram to explain why this makes sense geometrically.

144. Mowing a Field A square field in a certain state park is mowed around the edges every week. The rest of the field is kept unmowed to serve as habitat for birds and small animals (see the figure). The field measures b feet by b feet, and the mowed strip is x feet wide.

(a) Explain why the area of the mowed portion is $b^2 - (b-2x)^2$.

(b) Factor the expression in part (a) to show that the area of the mowed portion is also $4x(b-x)$. Do you see that the region with mowed grass consists of four rectangles, each with area $x(b-x)$?



Discuss ■ Discover ■ Prove ■ Write

145. Discover: Degree of a Sum or Product of Polynomials Make up several pairs of polynomials, then calculate the sum and product of each pair. On the basis of your experiments and observations, answer the following questions.

(a) How is the degree of the product related to the degrees of the original polynomials?

(b) How is the degree of the sum related to the degrees of the original polynomials?

146. Discuss: The Power of Algebraic Formulas Use the formula $A^2 - B^2 = (A+B)(A-B)$ to evaluate the following without using a calculator.

(a) $528^2 - 527^2$

(b) $1020^2 - 1010^2$

(c) $501 \cdot 499$

(d) $1002 \cdot 998$

147. Discover: Factoring $A^n - 1$

(a) Verify the following formulas by expanding and simplifying the right-hand side.

$$\begin{aligned} A^2 - 1 &= (A-1)(A+1) \\ A^3 - 1 &= (A-1)(A^2+A+1) \\ A^4 - 1 &= (A-1)(A^3+A^2+A+1) \end{aligned}$$

(b) Try to recognize a pattern for the formulas listed in part (a). On the basis of your pattern, how do you think $A^5 - 1$ factors? Verify your conjecture. Now generalize the pattern you have observed to obtain a factoring formula for $A^n - 1$, where n is a positive integer.

148. Prove: Special Factoring Formulas Prove each of the following formulas by expanding the right-hand side.

(a) Difference of Cubes:
$A^3 - B^3 = (A-B)(A^2+AB+B^2)$

(b) Sum of Cubes:
$A^3 + B^3 = (A+B)(A^2-AB+B^2)$

1.4 Rational Expressions

■ The Domain of an Algebraic Expression ■ Simplifying Rational Expressions ■ Multiplying and Dividing Rational Expressions ■ Adding and Subtracting Rational Expressions ■ Compound Fractions ■ Rationalizing the Denominator or the Numerator ■ Avoiding Common Errors

A quotient of two algebraic expressions is called a **fractional expression**. Here are some examples:

$$\frac{2x}{x-1} \qquad \frac{y-2}{y^2+4} \qquad \frac{x^3-x}{x^2-5x+6} \qquad \frac{x}{\sqrt{x^2+1}}$$

A **rational expression** is a fractional expression in which both the numerator and the denominator are polynomials. For example, the first three expressions in the above list are rational expressions, but the fourth is not because its denominator contains a radical. In this section we learn how to perform algebraic operations on rational expressions.

■ The Domain of an Algebraic Expression

Expression	Domain
$\frac{1}{x}$	$\{x \mid x \neq 0\}$
$\sqrt{x}$	$\{x \mid x \geq 0\}$
$\frac{1}{\sqrt{x}}$	$\{x \mid x > 0\}$

In general, an algebraic expression may not be defined for all values of the variable. The **domain** of an algebraic expression is the set of real numbers that the variable is permitted to have. The table in the margin gives some basic expressions and their domains.

Example 1 ■ Finding the Domain of an Expression

Find the domain of each of the following expressions.

(a) $2x^2 + 3x - 1$ **(b)** $\dfrac{x}{x^2 - 5x + 6}$ **(c)** $\dfrac{\sqrt{x}}{x-5}$

Solution

(a) This polynomial is defined for every x. Thus the domain is the set $\mathbb{R}$ of real numbers.

(b) We first factor the denominator.

$$\frac{x}{x^2-5x+6} = \frac{x}{(x-2)(x-3)}$$

Denominator would be 0 if $x = 2$ or $x = 3$

Since the denominator is zero when $x = 2$ or 3, the expression is not defined for these numbers. The domain is $\{x \mid x \neq 2 \text{ and } x \neq 3\}$.

(c) For the numerator to be defined, we must have $x \geq 0$. Also, we cannot divide by zero, so $x \neq 5$.

Must have $x \geq 0$ to take square root

$$\frac{\sqrt{x}}{x-5}$$

Denominator would be 0 if $x = 5$

Thus the domain is $\{x \mid x \geq 0 \text{ and } x \neq 5\}$.

Now Try Exercise 13 ■

Simplifying Rational Expressions

To **simplify a rational expression**, we factor both numerator and denominator and use the following property of fractions:

$$\boxed{\frac{AC}{BC} = \frac{A}{B}}$$

This allows us to **cancel** common factors from the numerator and denominator.

Example 2 ■ Simplifying Rational Expressions by Cancellation

Simplify: $\dfrac{x^2 - 1}{x^2 + x - 2}$

Solution

We can't cancel the x^2 terms in $\dfrac{x^2 - 1}{x^2 + x - 2}$ because x^2 is not a factor.

$$\begin{aligned} \frac{x^2 - 1}{x^2 + x - 2} &= \frac{(x - 1)(x + 1)}{(x - 1)(x + 2)} && \text{Factor} \\ &= \frac{x + 1}{x + 2} && \text{Cancel common factors} \end{aligned}$$

Now Try Exercise 21 ■

Multiplying and Dividing Rational Expressions

To **multiply rational expressions**, we use the following property of fractions:

$$\boxed{\frac{A}{B} \cdot \frac{C}{D} = \frac{AC}{BD}}$$

This says that to multiply two fractions, we multiply their numerators and multiply their denominators.

Example 3 ■ Multiplying Rational Expressions

Perform the indicated multiplication and simplify: $\dfrac{x^2 + 2x - 3}{x^2 + 8x + 16} \cdot \dfrac{3x + 12}{x - 1}$

Solution We first factor.

$$\begin{aligned} \frac{x^2 + 2x - 3}{x^2 + 8x + 16} \cdot \frac{3x + 12}{x - 1} &= \frac{(x - 1)(x + 3)}{(x + 4)^2} \cdot \frac{3(x + 4)}{x - 1} && \text{Factor} \\ &= \frac{3(x - 1)(x + 3)(x + 4)}{(x - 1)(x + 4)^2} && \text{Property of fractions} \\ &= \frac{3(x + 3)}{x + 4} && \text{Cancel common factors} \end{aligned}$$

Now Try Exercise 29 ■

To **divide rational expressions**, we use the following property of fractions:

$$\frac{A}{B} \div \frac{C}{D} = \frac{A}{B} \cdot \frac{D}{C}$$

This says that to divide one fraction by another fraction, we invert the divisor and multiply.

Example 4 ■ Dividing Rational Expressions

Perform the indicated division and simplify: $\dfrac{x-4}{x^2-4} \div \dfrac{x^2-3x-4}{x^2+5x+6}$

Solution

$$\frac{x-4}{x^2-4} \div \frac{x^2-3x-4}{x^2+5x+6} = \frac{x-4}{x^2-4} \cdot \frac{x^2+5x+6}{x^2-3x-4} \qquad \text{Invert and multiply}$$

$$= \frac{(x-4)(x+2)(x+3)}{(x-2)(x+2)(x-4)(x+1)} \qquad \text{Factor}$$

$$= \frac{x+3}{(x-2)(x+1)} \qquad \text{Cancel common factors}$$

Now Try Exercise 35 ■

■ Adding and Subtracting Rational Expressions

Avoid making the following error:

$$\frac{A}{B+C} \cancel{=} \frac{A}{B} + \frac{A}{C}$$

For instance, if we let $A = 2$, $B = 1$, and $C = 1$, then we see the error:

$$\frac{2}{1+1} \stackrel{?}{=} \frac{2}{1} + \frac{2}{1}$$

$$\frac{2}{2} \stackrel{?}{=} 2 + 2$$

$$1 \stackrel{?}{=} 4 \qquad \text{Wrong!}$$

To **add or subtract rational expressions**, we first find a common denominator and then use the following property of fractions:

$$\frac{A}{C} + \frac{B}{C} = \frac{A+B}{C}$$

Although any common denominator will work, it is most efficient to use the **least common denominator** (LCD), as explained in Section 1.1. The LCD is found by factoring each denominator and taking the product of the distinct factors, using the highest power that appears in any of the factors.

Example 5 ■ Adding and Subtracting Rational Expressions

Perform the indicated operations and simplify.

(a) $\dfrac{3}{x-1} + \dfrac{x}{x+2}$ **(b)** $\dfrac{1}{x^2-1} - \dfrac{2}{(x+1)^2}$

Solution

(a) Here the LCD is simply the product $(x-1)(x+2)$.

$$\frac{3}{x-1} + \frac{x}{x+2} = \frac{3(x+2)}{(x-1)(x+2)} + \frac{x(x-1)}{(x-1)(x+2)} \qquad \text{Write fractions using LCD}$$

$$= \frac{3x+6+x^2-x}{(x-1)(x+2)} \qquad \text{Add fractions}$$

$$= \frac{x^2+2x+6}{(x-1)(x+2)} \qquad \text{Combine terms in numerator}$$

(b) The LCD of $x^2 - 1 = (x - 1)(x + 1)$ and $(x + 1)^2$ is $(x - 1)(x + 1)^2$.

$$\frac{1}{x^2 - 1} - \frac{2}{(x + 1)^2} = \frac{1}{(x - 1)(x + 1)} - \frac{2}{(x + 1)^2} \qquad \text{Factor}$$

$$= \frac{(x + 1) - 2(x - 1)}{(x - 1)(x + 1)^2} \qquad \text{Combine fractions using LCD}$$

$$= \frac{x + 1 - 2x + 2}{(x - 1)(x + 1)^2} \qquad \text{Distributive Property}$$

$$= \frac{3 - x}{(x - 1)(x + 1)^2} \qquad \text{Combine terms in numerator}$$

Now Try Exercises 45 and 47 ■

■ Compound Fractions

A **compound fraction** is a fraction in which the numerator, the denominator, or both, are themselves fractional expressions.

Example 6 ■ Simplifying a Compound Fraction

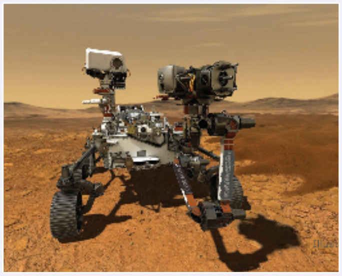
Simplify: $\dfrac{\dfrac{x}{y} + 1}{1 - \dfrac{y}{x}}$

Solution 1 We combine the terms in the numerator into a single fraction. We do the same in the denominator. Then we invert and multiply.

$$\frac{\dfrac{x}{y} + 1}{1 - \dfrac{y}{x}} = \frac{\dfrac{x + y}{y}}{\dfrac{x - y}{x}} = \frac{x + y}{y} \cdot \frac{x}{x - y}$$

$$= \frac{x(x + y)}{y(x - y)}$$

Mathematics in the Modern World



NASA

Error-Correcting Codes

The pictures sent back by the spacecraft rover *Perseverance* from the surface of Mars are astoundingly clear. But few viewing these pictures are aware of the complex mathematics used to accomplish this feat. The distance to Mars is enormous, and the background noise (or static) is many times stronger than the original signal emitted by the spacecraft. So when scientists receive the signal, it is full of errors. To get a clear picture, the errors must be found and corrected. This same problem of errors is routinely encountered in transmitting data over the Internet.

To understand how errors are found and corrected, we must first understand that to transmit pictures, sound, or text, we transform them into bits (the digits 0 or 1; see Mathematics in the Modern World, Section 1.3). To help the receiver recognize errors, the message is "coded" by inserting additional bits. For example, suppose you want to transmit the message "10100." A very simple-minded code is as follows: Send each digit a million times. The person receiving the message reads it in blocks of a million digits. If the first block is mostly 1's, he concludes that you are probably trying to transmit a 1, and so on. To say that this code is not efficient is a bit of an understatement; it requires sending a million times more data than the original message. Another method inserts "check digits." For example, for each block of eight digits insert a ninth digit; the inserted digit is 0 if there is an even number of 1's in the block and 1 if there is an odd number. So if a single digit is wrong (a 0 changed to a 1 or vice versa), the check digits allow us to recognize that an error has occurred. This method does not tell us where the error is, so we can't correct it. Modern error-correcting codes use mathematical algorithms that require inserting relatively few digits but that allow the receiver to not only recognize, but also correct, errors. The first error-correcting code was developed in the 1940s by Richard Hamming at MIT. The English language has a built-in error-correcting mechanism; to test it, try reading this error-filled sentence: Gve mo libty ox giv ne deth.

Solution 2 We find the LCD of all the fractions in the numerator and denominator of the expression, then multiply numerator and denominator by it. In this example the LCD of all the fractions is xy. Thus

$$\frac{\dfrac{x}{y}+1}{1-\dfrac{y}{x}} = \frac{\dfrac{x}{y}+1}{1-\dfrac{y}{x}} \cdot \frac{xy}{xy} \qquad \text{Multiply numerator and denominator by } xy$$

$$= \frac{x^2+xy}{xy-y^2} \qquad \text{Distribute and simplify}$$

$$= \frac{x(x+y)}{y(x-y)} \qquad \text{Factor}$$

Now Try Exercises 61 and 67 ■

The next two examples show situations in calculus that require the ability to work with fractional expressions.

Example 7 ■ Simplifying a Compound Fraction

Simplify: $\dfrac{\dfrac{1}{a+h}-\dfrac{1}{a}}{h}$

Solution We begin by combining the fractions in the numerator using a common denominator.

We can also simplify by multiplying the numerator and the denominator by $a(a+h)$.

$$\frac{\dfrac{1}{a+h}-\dfrac{1}{a}}{h} = \frac{\dfrac{a-(a+h)}{a(a+h)}}{h} \qquad \text{Combine fractions in the numerator}$$

$$= \frac{a-(a+h)}{a(a+h)} \cdot \frac{1}{h} \qquad \text{Property of fractions (invert divisor and multiply)}$$

$$= \frac{a-a-h}{a(a+h)} \cdot \frac{1}{h} \qquad \text{Distributive Property}$$

$$= \frac{-h}{a(a+h)} \cdot \frac{1}{h} \qquad \text{Simplify}$$

$$= \frac{-1}{a(a+h)} \qquad \text{Property of fractions (cancel common factors)}$$

Now Try Exercise 75 ■

Example 8 ■ Simplifying a Compound Fraction

Simplify: $\dfrac{(1+x^2)^{1/2}-x^2(1+x^2)^{-1/2}}{1+x^2}$

Solution 1 Factor $(1+x^2)^{-1/2}$ from the numerator.

Factor out the power of $1+x^2$ with the *smallest* exponent, in this case $(1+x^2)^{-1/2}$.

$$\frac{(1+x^2)^{1/2}-x^2(1+x^2)^{-1/2}}{1+x^2} = \frac{(1+x^2)^{-1/2}[(1+x^2)-x^2]}{1+x^2}$$

$$= \frac{(1+x^2)^{-1/2}}{1+x^2} = \frac{1}{(1+x^2)^{3/2}}$$

Solution 2 Since $(1 + x^2)^{-1/2} = 1/(1 + x^2)^{1/2}$ is a fraction, we can clear all fractions by multiplying numerator and denominator by $(1 + x^2)^{1/2}$.

$$\frac{(1 + x^2)^{1/2} - x^2(1 + x^2)^{-1/2}}{1 + x^2} = \frac{(1 + x^2)^{1/2} - x^2(1 + x^2)^{-1/2}}{1 + x^2} \cdot \frac{(1 + x^2)^{1/2}}{(1 + x^2)^{1/2}}$$

$$= \frac{(1 + x^2) - x^2}{(1 + x^2)^{3/2}} = \frac{1}{(1 + x^2)^{3/2}}$$

Now Try Exercise 83 ■

■ Rationalizing the Denominator or the Numerator

In Section 1.2 we learned how to rationalize a denominator of the form $\sqrt{C}$.

If a fraction has a denominator of the form $A + B\sqrt{C}$, we can rationalize the denominator by multiplying numerator and denominator by the **conjugate radical** $A - B\sqrt{C}$. This works because, by Special Product Formula 1 in Section 1.3, the product of the denominator and its conjugate radical does not contain a radical:

$$(A + B\sqrt{C})(A - B\sqrt{C}) = A^2 - B^2C$$

Example 9 ■ Rationalizing the Denominator

Rationalize the denominator: $\dfrac{1}{1 + \sqrt{2}}$

Solution We multiply both the numerator and the denominator by the conjugate radical of $1 + \sqrt{2}$, which is $1 - \sqrt{2}$.

$$\frac{1}{1 + \sqrt{2}} = \frac{1}{1 + \sqrt{2}} \cdot \frac{1 - \sqrt{2}}{1 - \sqrt{2}}$$ Multiply numerator and denominator by the conjugate radical

$$= \frac{1 - \sqrt{2}}{1^2 - (\sqrt{2})^2}$$ Special Product Formula 1

Special Product Formula 1
$(A + B)(A - B) = A^2 - B^2$

$$= \frac{1 - \sqrt{2}}{1 - 2} = \frac{1 - \sqrt{2}}{-1} = \sqrt{2} - 1$$

Now Try Exercise 87 ■

Example 10 ■ Rationalizing the Numerator

Rationalize the numerator: $\dfrac{\sqrt{4 + h} - 2}{h}$

Solution We multiply both numerator and denominator by the conjugate radical $\sqrt{4 + h} + 2$.

$$\frac{\sqrt{4 + h} - 2}{h} = \frac{\sqrt{4 + h} - 2}{h} \cdot \frac{\sqrt{4 + h} + 2}{\sqrt{4 + h} + 2}$$ Multiply numerator and denominator by the conjugate radical

$$= \frac{(\sqrt{4 + h})^2 - 2^2}{h(\sqrt{4 + h} + 2)}$$ Special Product Formula 1

Special Product Formula 1
$(A + B)(A - B) = A^2 - B^2$

$$= \frac{4 + h - 4}{h(\sqrt{4 + h} + 2)}$$

$$= \frac{h}{h(\sqrt{4 + h} + 2)} = \frac{1}{\sqrt{4 + h} + 2}$$ Property of fractions (cancel common factors)

Now Try Exercise 93 ■

Avoiding Common Errors

Don't make the mistake of applying properties of multiplication to the operation of addition. Many of the common errors in algebra involve doing just that. The following table states several properties of multiplication and illustrates the error in applying them to addition.

Multiplication Property	Common Error with Addition
$(a \cdot b)^2 = a^2 \cdot b^2$	$(a + b)^2 \cancel{=} a^2 + b^2$
$\sqrt{a \cdot b} = \sqrt{a}\sqrt{b} \quad (a, b \geq 0)$	$\sqrt{a + b} \cancel{=} \sqrt{a} + \sqrt{b}$
$\sqrt{a^2 \cdot b^2} = a \cdot b \quad (a, b \geq 0)$	$\sqrt{a^2 + b^2} \cancel{=} a + b$
$\dfrac{1}{a} \cdot \dfrac{1}{b} = \dfrac{1}{a \cdot b}$	$\dfrac{1}{a} + \dfrac{1}{b} \cancel{=} \dfrac{1}{a + b}$
$\dfrac{ab}{a} = b$	$\dfrac{a + b}{a} \cancel{=} b$
$(a \cdot b)^{-1} = a^{-1} \cdot b^{-1}$	$(a + b)^{-1} \cancel{=} a^{-1} + b^{-1}$

To verify that the equations in the right-hand column are wrong, simply substitute numbers for a and b and calculate each side. For example, if we take $a = 2$ and $b = 2$ in the fourth error, we get different values for the left- and right-hand sides:

$$\underbrace{\frac{1}{a} + \frac{1}{b} = \frac{1}{2} + \frac{1}{2} = 1}_{\text{Left-hand side}} \qquad \underbrace{\frac{1}{a + b} = \frac{1}{2 + 2} = \frac{1}{4}}_{\text{Right-hand side}}$$

Since $1 \neq \frac{1}{4}$, the stated equation is wrong. You should similarly convince yourself of the error in each of the other equations. (See Exercises 103 and 104.)

1.4 Exercises

Concepts

1. What is a rational expression? Which of the following are rational expressions?

(a) $\dfrac{3x}{x^2 - 1}$ **(b)** $\dfrac{\sqrt{x + 1}}{2x + 3}$ **(c)** $\dfrac{x(x^2 - 1)}{x + 3}$

2. To simplify a rational expression, we cancel *factors* that are common to the ________ and ________. So the expression

$$\frac{(x + 1)(x + 2)}{(x + 3)(x + 2)}$$

simplifies to ________.

3. To multiply two rational expressions, we multiply their ________ together and multiply their ________ together. So $\dfrac{2}{x + 1} \cdot \dfrac{x}{x + 3}$ is the same as ________.

4. Consider the expression $\dfrac{1}{x} - \dfrac{2}{x + 1} - \dfrac{x}{(x + 1)^2}$.

(a) How many terms does this expression have?

(b) Find the least common denominator of all the terms.

(c) Perform the addition and simplify.

5–6 ■ *Yes or No*? If *No*, give a reason. (Disregard any value that makes a denominator zero.)

5. (a) Is the expression $\dfrac{x(x + 1)}{(x + 1)^2}$ equal to $\dfrac{x}{x + 1}$?

(b) Is the expression $\sqrt{x^2 + 25}$ equal to $x + 5$?

6. (a) Is the expression $\dfrac{3 + a}{3}$ equal to $1 + \dfrac{a}{3}$?

(b) Is the expression $\dfrac{2}{4 + x}$ equal to $\dfrac{1}{2} + \dfrac{2}{x}$?

Skills

7–16 ■ **Domain** Find the domain of the expression.

7. $4x^2 - 10x + 3$

8. $-x^4 + x^3 + 9x$

9. $\dfrac{x^2 - 1}{x - 3}$

10. $\dfrac{2t^2 - 5}{3t + 6}$

11. $\sqrt{x + 3}$

12. $\dfrac{1}{\sqrt{x - 1}}$

13. $\dfrac{x^2 + 1}{x^2 - x - 2}$

14. $\dfrac{x}{x^2 - 4}$

15. $\dfrac{\sqrt{x - 2}}{x + 3}$

16. $\dfrac{\sqrt{x - 2}}{x^2 - 9}$

17–26 ■ Simplify Simplify the rational expression.

17. $\dfrac{(x-5)(x+5)}{2x-10}$ **18.** $\dfrac{x^3-2x}{x^2+x}$

19. $\dfrac{x-2}{x^2-4}$ **20.** $\dfrac{x^2-x-2}{x^2-1}$

21. $\dfrac{x^2-7x-8}{x^2-10x+16}$ **22.** $\dfrac{x^2-x-12}{x^2+5x+6}$

23. $\dfrac{y^2+y}{y^2-1}$ **24.** $\dfrac{y^3-5y^2-24y}{y^3-9y}$

25. $\dfrac{2x^3-x^2-6x}{2x^2-7x+6}$ **26.** $\dfrac{1-x^2}{x^3-1}$

27–40 ■ Multiply or Divide Perform the multiplication or division and simplify.

27. $\dfrac{4x}{x^2-4}\cdot\dfrac{x+2}{16x}$ **28.** $\dfrac{x^2-25}{x^2-16}\cdot\dfrac{x+4}{x+5}$

29. $\dfrac{x^2+2x-15}{x^2-25}\cdot\dfrac{x-5}{x+2}$ **30.** $\dfrac{x^2+2x-3}{x^2-2x-3}\cdot\dfrac{3-x}{3+x}$

31. $\dfrac{2t+3}{t^2+9}\cdot\dfrac{2t-3}{4t^2-9}$ **32.** $\dfrac{3y^2+9y}{y^3-9y}\cdot\dfrac{y^2-9}{2y^2+3y-9}$

33. $\dfrac{x^3-2x^2-8x}{x^2+8x+12}\cdot\dfrac{x^2+2x-24}{x^3-16x}$

34. $\dfrac{2x^2+xy-y^2}{x^2+xy-2y^2}\cdot\dfrac{x^2-2xy+y^2}{2x^2-3xy+y^2}$

35. $\dfrac{x+3}{4x^2-9}\div\dfrac{x^2+7x+12}{2x^2+7x-15}$

36. $\dfrac{2x+1}{2x^2+x-15}\div\dfrac{6x^2-x-2}{x+3}$

37. $\dfrac{\dfrac{x^3}{x+1}}{\dfrac{x}{x^2+2x+1}}$ **38.** $\dfrac{\dfrac{2x^2-3x-2}{x^2-1}}{\dfrac{2x^2+5x+2}{x^2+x-2}}$

39. $\dfrac{x/y}{z}$ **40.** $\dfrac{x}{y/z}$

41–60 ■ Add or Subtract Perform the addition or subtraction and simplify.

41. $1+\dfrac{1}{x+3}$ **42.** $\dfrac{3x-2}{x+1}-2$

43. $\dfrac{1}{x+5}+\dfrac{2}{x-3}$ **44.** $\dfrac{1}{x+1}+\dfrac{1}{x-1}$

45. $\dfrac{3}{x+1}-\dfrac{1}{x+2}$ **46.** $\dfrac{x}{x-4}-\dfrac{3}{x+6}$

47. $\dfrac{5}{2x-3}-\dfrac{3}{(2x-3)^2}$ **48.** $\dfrac{x}{(x+1)^2}+\dfrac{2}{x+1}$

49. $u+1+\dfrac{u}{u+1}$ **50.** $\dfrac{2}{a^2}-\dfrac{3}{ab}+\dfrac{4}{b^2}$

51. $\dfrac{1}{x^2}+\dfrac{1}{x^2+x}$ **52.** $\dfrac{1}{x}+\dfrac{1}{x^2}+\dfrac{1}{x^3}$

53. $\dfrac{2}{x+3}-\dfrac{1}{x^2+7x+12}$ **54.** $\dfrac{x}{x^2-4}+\dfrac{1}{x-2}$

55. $\dfrac{1}{x+3}+\dfrac{1}{x^2-9}$ **56.** $\dfrac{x}{x^2+x-2}-\dfrac{2}{x^2-5x+4}$

57. $\dfrac{2}{x}+\dfrac{3}{x-1}-\dfrac{4}{x^2-x}$ **58.** $\dfrac{1}{x^3y^3z}+\dfrac{1}{xy^3z^3}+\dfrac{1}{x^3yz^3}$

59. $\dfrac{1}{x^2(x+1)}+\dfrac{1}{x^2(x+1)^2}+\dfrac{1}{x^3(x+1)^2}$

60. $\dfrac{1}{x+1}-\dfrac{2}{(x+1)^2}+\dfrac{3}{x^2-1}$

61–74 ■ Compound Fractions Simplify the compound fractional expression.

61. $\dfrac{1+\dfrac{1}{x}}{\dfrac{1}{x}-2}$ **62.** $\dfrac{1-\dfrac{2}{y}}{\dfrac{3}{y}-1}$

63. $\dfrac{1+\dfrac{1}{x+2}}{1-\dfrac{1}{x+2}}$ **64.** $\dfrac{1+\dfrac{1}{c-1}}{1-\dfrac{1}{c-1}}$

65. $\dfrac{\dfrac{1}{x-1}+\dfrac{1}{x+3}}{x+1}$ **66.** $\dfrac{\dfrac{x-3}{x-4}-\dfrac{x+2}{x+1}}{x+3}$

67. $\dfrac{x-\dfrac{x}{y}}{y-\dfrac{y}{x}}$ **68.** $\dfrac{x+\dfrac{y}{x}}{y+\dfrac{x}{y}}$

69. $\dfrac{\dfrac{x}{y}-\dfrac{y}{x}}{\dfrac{1}{x^2}-\dfrac{1}{y^2}}$ **70.** $x-\dfrac{y}{\dfrac{x}{y}+\dfrac{y}{x}}$

71. $\dfrac{x^{-2}-y^{-2}}{x^{-1}+y^{-1}}$ **72.** $\dfrac{1}{1+u}+\dfrac{1}{1+\dfrac{1}{u}}$

73. $1-\dfrac{1}{1-\dfrac{1}{x}}$ **74.** $1+\dfrac{1}{1+\dfrac{1}{1+x}}$

75–80 ■ Expressions Found in Calculus Simplify the fractional expression.

75. $\dfrac{\dfrac{1}{1+x+h}-\dfrac{1}{1+x}}{h}$ **76.** $\dfrac{\dfrac{1}{\sqrt{x+h}}-\dfrac{1}{\sqrt{x}}}{h}$

77. $\dfrac{\dfrac{1}{(x+h)^2}-\dfrac{1}{x^2}}{h}$

78. $\dfrac{(x+h)^2+3(x+h)-x^2-3x}{h}$

79. $\sqrt{1+\left(\dfrac{x}{\sqrt{1-x^2}}\right)^2}$ **80.** $\sqrt{1+\left(x^3-\dfrac{1}{4x^3}\right)^2}$

81–86 ■ Expressions Found in Calculus Simplify the expression. (This type of expression arises in calculus when using the "quotient rule.")

81. $\dfrac{2(x-3)(x+5)^3 - 3(x+5)^2(x-3)^2}{(x+5)^6}$

82. $\dfrac{4x^3(1-x)^3 - 3(1-x)^2(-1)(x^4)}{(1-x)^6}$

83. $\dfrac{2(1+x)^{1/2} - x(1+x)^{-1/2}}{x+1}$ **84.** $\dfrac{(1-x^2)^{1/2} + x^2(1-x^2)^{-1/2}}{1-x^2}$

85. $\dfrac{3(1+x)^{1/3} - x(1+x)^{-2/3}}{(1+x)^{2/3}}$ **86.** $\dfrac{(7-3x)^{1/2} + \frac{3}{2}x(7-3x)^{-1/2}}{7-3x}$

87–92 ■ Rationalize Denominator Rationalize the denominator.

87. $\dfrac{1}{3+\sqrt{10}}$ **88.** $\dfrac{3}{2-\sqrt{5}}$

89. $\dfrac{2}{\sqrt{5}-\sqrt{3}}$ **90.** $\dfrac{1}{\sqrt{x}+1}$

91. $\dfrac{y}{\sqrt{3}+\sqrt{y}}$ **92.** $\dfrac{2(x-y)}{\sqrt{x}-\sqrt{y}}$

93–98 ■ Rationalize Numerator Rationalize the numerator.

93. $\dfrac{2-\sqrt{5}}{5}$ **94.** $\dfrac{\sqrt{3}+\sqrt{5}}{2}$

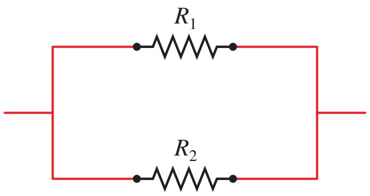
95. $\dfrac{\sqrt{r}+\sqrt{2}}{5}$ **96.** $\dfrac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}}$

97. $\sqrt{x^2+1} - x$ **98.** $\sqrt{x+1} - \sqrt{x}$

Applications

99. Electrical Resistance If two electrical resistors with resistances R_1 and R_2 are connected in parallel (see the figure), then the total resistance R is given by

$$R = \frac{1}{\dfrac{1}{R_1} + \dfrac{1}{R_2}}$$



(a) Simplify the expression for R.

(b) If $R_1 = 10$ ohms and $R_2 = 20$ ohms, what is the total resistance R?

100. Average Cost A clothing manufacturer finds that the cost of producing x shirts is $500 + 6x + 0.01x^2$ dollars.

(a) Explain why the average cost per shirt is given by the rational expression

$$A = \frac{500 + 6x + 0.01x^2}{x}$$

(b) Complete the table by calculating the average cost per shirt for the given values of x.

x	Average Cost
10	
20	
50	
100	
200	
500	
1000	

Discuss ■ Discover ■ Prove ■ Write

101. Discover: Limiting Behavior of a Rational Expression The rational expression

$$\frac{x^2-9}{x-3}$$

is not defined for $x = 3$. Complete the tables, and determine what value the expression approaches as x gets closer and closer to 3. Why is this reasonable? Factor the numerator of the expression and simplify to see why.

x	$\dfrac{x^2-9}{x-3}$
2.80	
2.90	
2.95	
2.99	
2.999	

x	$\dfrac{x^2-9}{x-3}$
3.20	
3.10	
3.05	
3.01	
3.001	

102. Discuss ■ Write: Is This Rationalization? In the expression $2/\sqrt{x}$ we would eliminate the radical if we were to square both numerator and denominator. Is this the same thing as rationalizing the denominator? Explain.

103. Discuss: Algebraic Errors The left-hand column of the table lists some common algebraic errors. In each case, give an example using numbers that shows that the formula is not valid. An example of this type, which shows that a statement is false, is called a *counterexample*.

Algebraic Errors	Counterexample
$\dfrac{1}{a} + \dfrac{1}{b} \cancel{=} \dfrac{1}{a+b}$	$\dfrac{1}{2} + \dfrac{1}{2} \neq \dfrac{1}{2+2}$
$(a+b)^2 \cancel{=} a^2 + b^2$	
$\sqrt{a^2+b^2} \cancel{=} a + b$	
$\dfrac{a+b}{a} \cancel{=} b$	
$\dfrac{a}{a+b} \cancel{=} \dfrac{1}{b}$	
$\dfrac{a^m}{a^n} \cancel{=} a^{m/n}$	

104. Discuss: Algebraic Errors Determine whether the given equation is true for all values of the variables. If not, give a counterexample. (Disregard any value that makes a denominator zero.)

(a) $\dfrac{5+a}{5} = 1 + \dfrac{a}{5}$ **(b)** $\dfrac{x+1}{y+1} = \dfrac{x}{y}$

(c) $\dfrac{x}{x+y} = \dfrac{1}{1+y}$ **(d)** $2\left(\dfrac{a}{b}\right) = \dfrac{2a}{2b}$

(e) $\dfrac{-a}{b} = -\dfrac{a}{b}$ **(f)** $\dfrac{1+x+x^2}{x} = \dfrac{1}{x} + 1 + x$

105. Discover ■ Prove: Values of a Rational Expression For $x > 0$, consider the expression

$$x + \frac{1}{x}$$

(a) Fill in the table, and try other values for x. What do you think is the smallest possible value for this expression?

x	1	3	$\frac{1}{2}$	$\frac{9}{10}$	$\frac{99}{100}$	
$x + \dfrac{1}{x}$						

(b) Prove that for $x > 0$,

$$x + \frac{1}{x} \geq 2$$

PS *Work backward.* Assume the inequality is valid; multiply by x, move terms to one side, and then factor to arrive at a true statement. Note that each step you made is reversible.

1.5 Equations

■ Solving Linear Equations ■ Formulas: Solving for One Variable in Terms of Others ■ Solving Quadratic Equations ■ Other Types of Equations

An equation is a statement that two mathematical expressions are equal. For example,

$$3 + 5 = 8$$

is an equation. Most equations that we study in algebra contain variables, which are symbols (usually letters) that stand for numbers. In the equation

$$4x + 7 = 19$$

the letter x is the variable. We think of x as the "unknown" in the equation, and our goal is to find the value of x that makes the equation true. The values of the unknown that make the equation true are called the **solutions** or **roots** of the equation, and the process of finding the solutions is called **solving the equation**.

$x = 3$ is a solution of the equation $4x + 7 = 19$ because substituting $x = 3$ makes the equation true:

$\boxed{x = 3}$

$4(3) + 7 = 19$ ✓

Two equations with exactly the same solutions are called **equivalent equations**. To solve an equation, we try to find a simpler, equivalent equation in which the variable stands alone on one side of the equal sign. Here are the properties that we use to solve an equation. (In these properties, A, B, and C stand for any algebraic expressions, and the symbol $\Leftrightarrow$ means "is equivalent to.")

Properties of Equality

Property	Description
1. $A = B \quad \Leftrightarrow \quad A + C = B + C$	Adding the same quantity to both sides of an equation gives an equivalent equation.
2. $A = B \quad \Leftrightarrow \quad CA = CB \quad (C \neq 0)$	Multiplying both sides of an equation by the same nonzero quantity gives an equivalent equation.

These properties require that you *perform the same operation on both sides of an equation* when solving it. Thus if we say "*add* -7" when solving an equation, that is just a short way of saying "*add* -7 to each side of the equation."

■ Solving Linear Equations

The simplest type of equation is a *linear equation*, or first-degree equation, which is an equation in which each term is either a constant or a nonzero multiple of the variable.

Linear Equations

A **linear equation** in one variable is an equation equivalent to one of the form

$$ax + b = 0$$

where a and b are real numbers and x is the variable.

Here are some examples that illustrate the difference between linear and nonlinear equations.

Linear equations	**Nonlinear equations**	
$4x - 5 = 3$	$x^2 + 2x = 8$	Not linear; contains the square of the variable
$2x = \frac{1}{2}x - 7$	$\sqrt{x} - 6x = 0$	Not linear; contains the square root of the variable
$x - 6 = \dfrac{x}{3}$	$\dfrac{3}{x} - 2x = 1$	Not linear; contains the reciprocal of the variable

Example 1 ■ Solving a Linear Equation

Solve the equation $7x - 4 = 3x + 8$.

Solution We solve this equation by changing it to an equivalent equation with all terms that have the variable x on one side and all constant terms on the other.

$$\begin{aligned} 7x - 4 &= 3x + 8 && \text{Given equation} \\ (7x - 4) + 4 &= (3x + 8) + 4 && \text{Add 4} \\ 7x &= 3x + 12 && \text{Simplify} \\ 7x - 3x &= (3x + 12) - 3x && \text{Subtract } 3x \\ 4x &= 12 && \text{Simplify} \\ \tfrac{1}{4}\cdot 4x &= \tfrac{1}{4}\cdot 12 && \text{Multiply by } \tfrac{1}{4} \\ x &= 3 && \text{Simplify} \end{aligned}$$

Check Your Answer

$x = 3$:

$\text{LHS} = 7(3) - 4 = 17$ ($x = 3$)

$\text{RHS} = 3(3) + 8 = 17$ ($x = 3$)

LHS = RHS ✓

Because it is important to CHECK YOUR ANSWER, we do this in many of our examples. In these checks, LHS stands for "left-hand side" and RHS stands for "right-hand side" of the original equation.

Now Try Exercise 17 ■

■ Formulas: Solving for One Variable in Terms of Others

In mathematics and the sciences, the term *formula* commonly refers to an equation that relates different variables. Examples include the formula $A = \pi r^2$, which relates the area of a circle to its radius, and the formula $PV = nRT$, which relates the pressure, volume, and temperature of an ideal gas. We can solve for any variable in a formula to find out how that variable relates to the other variables. For example, solving for r in the formula for the area of a circle gives $r = \sqrt{A/\pi}$ and solving for P in the ideal gas formula gives $P = nRT/V$. In the next example we solve for a variable in Newton's Law of Gravity.

Example 2 ■ Solving for One Variable in Terms of Others

This formula is Newton's Law of Gravity. It gives the gravitational force F between two masses m and M that are a distance r apart. The constant G is the universal gravitational constant.

Solve for the variable M in the equation

$$F = G\frac{mM}{r^2}$$

Solution Although this equation involves more than one variable, we solve it as usual by isolating M on one side and treating the other variables as we would numbers.

$$F = \left(\frac{Gm}{r^2}\right)M \qquad \text{Factor } M \text{ from RHS}$$

$$\left(\frac{r^2}{Gm}\right)F = \left(\frac{r^2}{Gm}\right)\left(\frac{Gm}{r^2}\right)M \qquad \text{Multiply by reciprocal of } \frac{Gm}{r^2}$$

$$\frac{r^2F}{Gm} = M \qquad \text{Simplify}$$

The solution is $M = \dfrac{r^2F}{Gm}$.

Now Try Exercise 29 ■

Example 3 ■ Solving for One Variable in Terms of Others

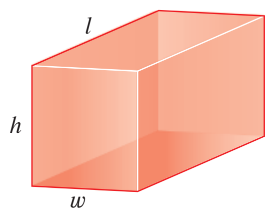


Figure 1 | A closed rectangular box

The surface area A of the closed rectangular box shown in Figure 1 can be calculated from the length l, the width w, and the height h according to the formula

$$A = 2lw + 2wh + 2lh$$

Solve for w in terms of the other variables in this equation.

Solution Although this equation involves more than one variable, we solve it as usual by isolating w on one side, treating the other variables as we would numbers.

$$A = (2lw + 2wh) + 2lh \qquad \text{Collect terms involving } w$$

$$A - 2lh = 2lw + 2wh \qquad \text{Subtract } 2lh$$

$$A - 2lh = (2l + 2h)w \qquad \text{Factor } w \text{ from RHS}$$

$$\frac{A - 2lh}{2l + 2h} = w \qquad \text{Divide by } 2l + 2h$$

The solution is $w = \dfrac{A - 2lh}{2l + 2h}$.

Now Try Exercise 31 ■

Discovery Project ■ **Weighing the Whole World**

Have you ever wondered how much the world weighs? In this project you will answer this question by using Newton's formula for gravitational force. That such questions can be answered by simply using formulas shows the remarkable power of formulas. Edsger Dijkstra, one of the founders of computer science, once said, "A picture may be worth a thousand words, a formula is worth a thousand pictures." You can find the project at **www.stewartmath.com**.

Solving Quadratic Equations

Linear equations are first-degree equations like $2x + 1 = 5$ or $4 - 3x = 2$. Quadratic equations are second-degree equations like $x^2 + 2x - 3 = 0$ or $2x^2 + 3 = 5x$.

Quadratic Equations

$x^2 - 2x - 8 = 0$

$3x + 10 = 4x^2$

$\frac{1}{2}x^2 + \frac{1}{3}x - \frac{1}{6} = 0$

Quadratic Equations

A **quadratic equation** is an equation of the form

$$ax^2 + bx + c = 0$$

where a, b, and c are real numbers with $a \neq 0$.

Some quadratic equations can be solved by factoring and using the following basic property of real numbers.

Zero-Product Property

$$AB = 0 \quad \text{if and only if} \quad A = 0 \quad \text{or} \quad B = 0$$

This means that if we can factor the left-hand side of a quadratic (or other) equation, then we can solve it by setting each factor equal to 0 in turn. This method works only when the right-hand side of the equation is 0.

Example 4 ■ Solving a Quadratic Equation by Factoring

Find all real solutions of the equation $x^2 + 5x = 24$.

Solution We must first rewrite the equation so that the right-hand side is 0.

$$x^2 + 5x = 24$$

$$x^2 + 5x - 24 = 0 \qquad \text{Subtract 24}$$

$$(x - 3)(x + 8) = 0 \qquad \text{Factor}$$

$$x - 3 = 0 \quad \text{or} \quad x + 8 = 0 \qquad \text{Zero-Product Property}$$

$$x = 3 \qquad\qquad x = -8 \qquad \text{Solve}$$

The solutions are $x = 3$ and $x = -8$.

Check Your Answers

$x = 3$:

$(3)^2 + 5(3) = 9 + 15 = 24$ ✔

$x = -8$:

$(-8)^2 + 5(-8) = 64 - 40 = 24$ ✔

Now Try Exercise 41 ■

Do you see why one side of the equation must be 0 in Example 4? Factoring the equation as $x(x + 5) = 24$ does not help us find the solutions because 24 can be factored in infinitely many ways, such as $6 \cdot 4$, $\frac{1}{2} \cdot 48$, $\left(-\frac{2}{5}\right) \cdot (-60)$, and so on.

A quadratic equation of the form $x^2 - c = 0$, where c is a positive constant, factors as $(x - \sqrt{c})(x + \sqrt{c}) = 0$, so the solutions are $x = \sqrt{c}$ and $x = -\sqrt{c}$. We often abbreviate this as $x = \pm\sqrt{c}$.

Solving a Simple Quadratic Equation

The solutions of the equation $x^2 = c$ are $x = \sqrt{c}$ and $x = -\sqrt{c}$.

Example 5 ■ Solving Simple Quadratic Equations

Find all real solutions of each equation.

(a) $x^2 = 5$ **(b)** $(x - 4)^2 = 5$

Solution

(a) From the principle in the preceding box we get $x = \pm\sqrt{5}$.

(b) We can take the square root of each side of this equation as well.

$$\begin{aligned} (x-4)^2 &= 5 \\ x - 4 &= \pm\sqrt{5} && \text{Take the square root} \\ x &= 4 \pm \sqrt{5} && \text{Add 4} \end{aligned}$$

The solutions are $x = 4 + \sqrt{5}$ and $x = 4 - \sqrt{5}$.

Now Try Exercises 47 and 49

See Section 1.3 for how to recognize when a quadratic expression is a perfect square.

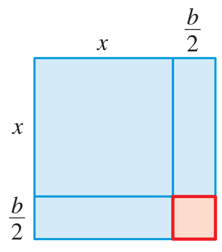
As we saw in Example 5, part (b), if a quadratic equation is of the form $(x \pm a)^2 = c$, then we can solve it by taking the square root of each side. In an equation of this form, the left-hand side is a *perfect square*: that is, the square of a linear expression in x. So if a quadratic equation does not factor readily, then we can solve it by using the technique of **completing the square**. This means that we add a constant to an expression to make it a perfect square. For example, to make $x^2 - 6x$ a perfect square, we must add 9 because $x^2 - 6x + 9 = (x - 3)^2$.

Completing the Square

The area of the blue region is

$$x^2 + 2\left(\frac{b}{2}\right)x = x^2 + bx$$

Add a small square of area $(b/2)^2$ to "complete" the square.



Completing the Square

To make $x^2 + bx$ a perfect square, add $\left(\frac{b}{2}\right)^2$, the square of half the coefficient of x. This gives the perfect square

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

Example 6 ■ Solving Quadratic Equations by Completing the Square

Find all real solutions of each equation.

(a) $x^2 - 8x + 13 = 0$ **(b)** $3x^2 - 12x + 6 = 0$

Solution

(a)
$$\begin{aligned} x^2 - 8x + 13 &= 0 && \text{Given equation} \\ x^2 - 8x &= -13 && \text{Subtract 13} \\ x^2 - 8x + 16 &= -13 + 16 && \text{Complete the square: add } \left(\frac{-8}{2}\right)^2 = 16 \\ (x-4)^2 &= 3 && \text{Perfect square} \\ x - 4 &= \pm\sqrt{3} && \text{Take square root} \\ x &= 4 \pm \sqrt{3} && \text{Add 4} \end{aligned}$$

(b) After subtracting 6 from each side of the equation, we must factor the coefficient of x^2 (the 3) from the left side to put the equation in the correct form for completing the square.

$$\begin{aligned} 3x^2 - 12x + 6 &= 0 && \text{Given equation} \\ 3x^2 - 12x &= -6 && \text{Subtract 6} \\ 3(x^2 - 4x) &= -6 && \text{Factor 3 from LHS} \end{aligned}$$

Now we complete the square by adding $(-2)^2 = 4$ *inside* the parentheses. Since everything inside the parentheses is multiplied by 3, this means that we are

When you complete the square, make sure the coefficient of x^2 is 1. If it isn't, then you must factor this coefficient from both terms that contain x:

$$ax^2 + bx = a\left(x^2 + \frac{b}{a}x\right)$$

Then complete the square inside the parentheses. Remember that the term added inside the parentheses is multiplied by a.

actually adding $3 \cdot 4 = 12$ to the left side of the equation. Thus we must add 12 to the right side as well.

$$
\begin{aligned}
3(x^2 - 4x + 4) &= -6 + 3 \cdot 4 && \text{Complete the square: add 4} \\
3(x - 2)^2 &= 6 && \text{Perfect square} \\
(x - 2)^2 &= 2 && \text{Divide by 3} \\
x - 2 &= \pm\sqrt{2} && \text{Take square root} \\
x &= 2 \pm \sqrt{2} && \text{Add 2}
\end{aligned}
$$

Now Try Exercises 53 and 57 ■

We can use the technique of completing the square to derive a formula for the solutions of the general quadratic equation $ax^2 + bx + c = 0$.

The Quadratic Formula

The solutions of the general quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Proof First, we divide each side of the equation by a and move the constant to the right side, giving

$$x^2 + \frac{b}{a}x = -\frac{c}{a} \qquad \text{Divide by } a$$

We now complete the square by adding $(b/2a)^2$ to each side of the equation:

$$
\begin{aligned}
x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 &= -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 && \text{Complete the square: Add } \left(\frac{b}{2a}\right)^2 \\
\left(x + \frac{b}{2a}\right)^2 &= \frac{-4ac + b^2}{4a^2} && \text{Perfect square} \\
x + \frac{b}{2a} &= \pm\frac{\sqrt{b^2 - 4ac}}{2a} && \text{Take square root} \\
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} && \text{Subtract } \frac{b}{2a}
\end{aligned}
$$
■

The Quadratic Formula can be used to solve any quadratic equation. You should confirm that the Quadratic Formula gives the same solutions when applied to the equations in Examples 4 and 6.

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FRANÇOIS VIÈTE (1540–1603) had a successful political career before taking up mathematics late in life. He became one of the most famous French mathematicians of the 16th century. Viète introduced a new level of abstraction in algebra by using letters to stand for *known* quantities in an equation. Before Viète's time, each equation had to be solved on its own. For instance, the quadratic equations

$$3x^2 + 2x + 8 = 0$$

$$5x^2 - 6x + 4 = 0$$

had to be solved separately by completing the square. Viète's idea was to consider all quadratic equations at once by writing

$$ax^2 + bx + c = 0$$

where a, b, and c are known quantities. Thus he made it possible to write a *formula* (in this case the Quadratic Formula) involving a, b, and c that can be used to solve all such equations in one fell swoop.

Viète's mathematical genius proved valuable during a war between France and Spain. To communicate with their troops, the Spaniards used a complicated code that Viète managed to decipher. Unaware of Viète's accomplishment, the Spanish king, Philip II, protested to the Pope, claiming that the French were using witchcraft to read his messages.

Example 7 ■ Using the Quadratic Formula

Find all real solutions of each equation.

(a) $3x^2 - 5x - 1 = 0$ **(b)** $4x^2 + 12x + 9 = 0$ **(c)** $x^2 + 2x = -2$

Solution

(a) In this quadratic equation $a = 3$, $b = -5$, and $c = -1$.

$$\underbrace{3}_{a\,=\,3}x^2 \overbrace{-\,5}^{b\,=\,-5}x \underbrace{-\,1}_{c\,=\,-1} = 0$$

By the Quadratic Formula,

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(3)(-1)}}{2(3)} = \frac{5 \pm \sqrt{37}}{6}$$

If approximations are desired, we can use a calculator to obtain

$$x = \frac{5 + \sqrt{37}}{6} \approx 1.8471 \qquad \text{and} \qquad x = \frac{5 - \sqrt{37}}{6} \approx -0.1805$$

Another Method

$$4x^2 + 12x + 9 = 0$$
$$(2x + 3)^2 = 0$$
$$2x + 3 = 0$$
$$x = -\tfrac{3}{2}$$

(b) Using the Quadratic Formula with $a = 4$, $b = 12$, and $c = 9$ gives

$$x = \frac{-12 \pm \sqrt{(12)^2 - 4 \cdot 4 \cdot 9}}{2 \cdot 4} = \frac{-12 \pm 0}{8} = -\frac{3}{2}$$

This equation has only one solution, $x = -\frac{3}{2}$.

(c) We first write the equation in the form $x^2 + 2x + 2 = 0$. Using the Quadratic Formula with $a = 1$, $b = 2$, and $c = 2$ gives

$$x = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 2}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2\sqrt{-1}}{2} = -1 \pm \sqrt{-1}$$

In Section 1.6 we study the complex number system, in which the square roots of negative numbers are defined.

Since the square of any real number is nonnegative, $\sqrt{-1}$ is undefined in the real number system. So this equation has no real solution.

Now Try Exercises 61, 67, and 71 ■

The quantity $b^2 - 4ac$ that appears under the square root sign in the Quadratic Formula is called the *discriminant* of the equation $ax^2 + bx + c = 0$ and is given the symbol D. If $D < 0$, then $\sqrt{b^2 - 4ac}$ is undefined, and the quadratic equation has no real solution, as in Example 7(c). If $D = 0$, then the equation has only one real solution, as in Example 7(b). Finally, if $D > 0$, then the equation has two distinct real solutions, as in Example 7(a). The following box summarizes these observations.

The Discriminant

The **discriminant** of the general quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$ is $D = b^2 - 4ac$.

1. If $D > 0$, then the equation has two distinct real solutions.
2. If $D = 0$, then the equation has exactly one real solution.
3. If $D < 0$, then the equation has no real solution.

Example 8 ■ Using the Discriminant

Use the discriminant to determine how many real solutions each equation has.

(a) $x^2 + 4x - 1 = 0$ **(b)** $4x^2 + 12x + 9 = 0$ **(c)** $\frac{1}{3}x^2 - 2x + 4 = 0$

Solution

(a) The discriminant is $D = 4^2 - 4(1)(-1) = 20 > 0$, so the equation has two distinct real solutions.

Compare part (b) with Example 7(b).

(b) The discriminant is $D = (12)^2 - 4 \cdot 4 \cdot 9 = 0$, so the equation has exactly one real solution.

(c) The discriminant is $D = (-2)^2 - 4\left(\frac{1}{3}\right)4 = -\frac{4}{3} < 0$, so the equation has no real solution.

Now Try Exercises 73, 75, and 77 ■

Now let's consider a real-life situation that can be modeled by a quadratic equation.

This formula depends on the fact that acceleration due to gravity is constant near the earth's surface. Here we neglect the effect of air resistance.

Example 9 ■ The Path of a Projectile

An object thrown or fired straight upward at an initial speed of v_0 ft/s will reach a height of h feet after t seconds, where h and t are related by the formula

$$h = -16t^2 + v_0 t$$

Suppose that a bullet is shot straight upward with an initial speed of 800 ft/s. Its path is shown in Figure 2.

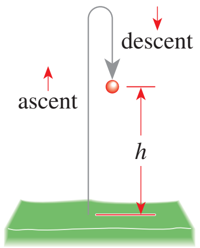


Figure 2

(a) When does the bullet fall back to ground level?

(b) When does the bullet reach a height of 6400 ft?

(c) When does the bullet reach a height of 2 mi?

(d) How high is the highest point the bullet reaches?

Solution

Since the initial speed in this case is $v_0 = 800$ ft/s, the formula is

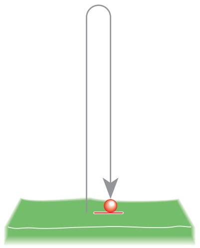
$$h = -16t^2 + 800t$$

(a) Ground level corresponds to $h = 0$, so we must solve the equation

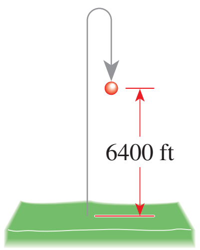
$$0 = -16t^2 + 800t \qquad \text{Set } h = 0$$

$$0 = -16t(t - 50) \qquad \text{Factor}$$

Thus $t = 0$ or $t = 50$. This means the bullet starts $(t = 0)$ at ground level and returns to ground level after 50 s.



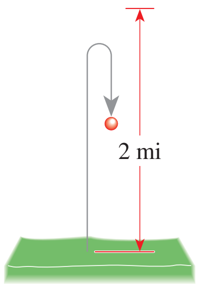
(b) Setting $h = 6400$ gives the equation



$$
\begin{aligned}
6400 &= -16t^2 + 800t && \text{Set } h = 6400 \\
16t^2 - 800t + 6400 &= 0 && \text{All terms to LHS} \\
t^2 - 50t + 400 &= 0 && \text{Divide by 16} \\
(t - 10)(t - 40) &= 0 && \text{Factor} \\
t = 10 \quad \text{or} \quad t &= 40 && \text{Solve}
\end{aligned}
$$

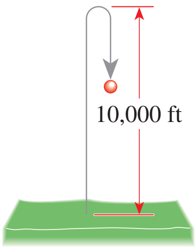
The bullet reaches 6400 ft after 10 s (on its ascent) and again after 40 s (on its descent).

(c) Two miles is $2 \times 5280 = 10{,}560$ ft.



$$
\begin{aligned}
10{,}560 &= -16t^2 + 800t && \text{Set } h = 10{,}560 \\
16t^2 - 800t + 10{,}560 &= 0 && \text{All terms to LHS} \\
t^2 - 50t + 660 &= 0 && \text{Divide by 16}
\end{aligned}
$$

The discriminant of this equation is $D = (-50)^2 - 4(660) = -140$, which is negative. Thus the equation has no real solution. The bullet never reaches a height of 2 mi.



(d) Each height the bullet reaches is attained twice, once on its ascent and once on its descent. The only exception is the highest point of its path, which is reached only once. This means that for the highest value of h, the following equation has only one solution for t:

$$h = -16t^2 + 800t$$

$$16t^2 - 800t + h = 0 \qquad \text{All terms to LHS}$$

This in turn means that the discriminant D of the equation is 0, so

$$D = (-800)^2 - 4(16)h = 0$$

$$640{,}000 - 64h = 0$$

$$h = 10{,}000$$

The maximum height reached is 10,000 ft.

Now Try Exercise 131 ■

■ Other Types of Equations

So far we have learned how to solve linear and quadratic equations. Now we study other types of equations, including those that involve higher powers, fractional expressions, and radicals.

When we solve an equation that involves fractional expressions or radicals, we must be especially careful to check our answers. The next two examples demonstrate why.

Example 10 ■ An Equation Involving Fractional Expressions

Solve the equation $\dfrac{3}{x} - \dfrac{2}{x-3} = \dfrac{-12}{x^2 - 9}$.

Solution We eliminate the denominators by multiplying each side by the lowest common denominator.

$$\left(\frac{3}{x} - \frac{2}{x-3}\right)x(x^2-9) = \frac{-12}{x^2-9}x(x^2-9) \qquad \text{Multiply by LCD, } x(x^2-9)$$

$$3(x^2-9) - 2x(x+3) = -12x \qquad \text{Simplify}$$

$$3x^2 - 27 - 2x^2 - 6x = -12x \qquad \text{Expand LHS}$$

$$x^2 - 6x - 27 = -12x \qquad \text{Add like terms on LHS}$$

$$x^2 + 6x - 27 = 0 \qquad \text{Add } 12x$$

$$(x-3)(x+9) = 0 \qquad \text{Factor}$$

$$x - 3 = 0 \quad \text{or} \quad x + 9 = 0 \qquad \text{Zero-Product Property}$$

$$x = 3 \qquad\qquad x = -9 \qquad \text{Solve}$$

Check Your Answers

$x = 3$:

$$\text{LHS} = \frac{3}{3} - \frac{2}{3-3} \text{ undefined}$$

$$\text{RHS} = \frac{-12}{3^2 - 9} \text{ undefined}$$

$x = -9$:

$$\text{LHS} = \frac{3}{-9} - \frac{2}{-9-3} = -\frac{1}{6}$$

$$\text{RHS} = \frac{-12}{(-9)^2 - 9} = -\frac{1}{6}$$

LHS = RHS

We must check our answer because multiplying by an expression that contains the variable can introduce extraneous solutions. From *Check Your Answers* we see that the only solution is $x = -9$.

Now Try Exercise 83 ■

Example 11 ■ An Equation Involving a Radical

Solve the equation $2x = 1 - \sqrt{2 - x}$.

Check Your Answers

$x = -\frac{1}{4}$:

$\text{LHS} = 2\left(-\frac{1}{4}\right) = -\frac{1}{2}$

$\text{RHS} = 1 - \sqrt{2 - \left(-\frac{1}{4}\right)}$

$= 1 - \sqrt{\frac{9}{4}}$

$= 1 - \frac{3}{2} = -\frac{1}{2}$

LHS = RHS ✓

$x = 1$:

$\text{LHS} = 2(1) = 2$

$\text{RHS} = 1 - \sqrt{2 - 1}$

$= 1 - 1 = 0$

LHS ≠ RHS ✗

Solution To eliminate the square root, we first isolate it on one side of the equal sign, then square.

$$\begin{aligned}
2x - 1 &= -\sqrt{2 - x} && \text{Subtract 1} \\
(2x - 1)^2 &= 2 - x && \text{Square each side} \\
4x^2 - 4x + 1 &= 2 - x && \text{Expand LHS} \\
4x^2 - 3x - 1 &= 0 && \text{Add } -2 + x \\
(4x + 1)(x - 1) &= 0 && \text{Factor} \\
4x + 1 = 0 \quad \text{or} &\quad x - 1 = 0 && \text{Zero-Product Property} \\
x = -\tfrac{1}{4} &\quad\quad x = 1 && \text{Solve}
\end{aligned}$$

The values $x = -\frac{1}{4}$ and $x = 1$ are only potential solutions. We must check them to see whether they satisfy the original equation. From *Check Your Answers* we see that $x = -\frac{1}{4}$ is a solution but $x = 1$ is not. The only solution is $x = -\frac{1}{4}$.

Now Try Exercise 89 ■

Note When we solve an equation, we may end up with one or more **extraneous solutions**, that is, potential solutions that do not satisfy the original equation. In Example 10 the value $x = 3$ is an extraneous solution, and in Example 11 the value $x = 1$ is an extraneous solution.

In the case of equations involving fractional expressions, potential solutions may be undefined in the original equation and hence are extraeous solutions. In the case of equations involving radicals, extraneous solutions may be introduced when we square each side of an equation because the operation of squaring can turn a false equation into a true one. For example, $-1 \neq 1$, but $(-1)^2 = 1^2$. Thus the squared equation may be true for more values of the variable than the original equation. That is why you must always check your answers to make sure that each satisfies the original equation.

An equation of the form $aW^2 + bW + c = 0$, where W is an algebraic expression, is an equation of **quadratic type**. We solve equations of quadratic type by substituting for the algebraic expression, as we see in the next two examples.

Example 12 ■ An Equation of Quadratic Type

Find all solutions of the equation $x^4 - 8x^2 + 8 = 0$.

Solution If we set $W = x^2$, then we get a quadratic equation in the new variable W.

$$\begin{aligned}
(x^2)^2 - 8x^2 + 8 &= 0 && \text{Write } x^4 \text{ as } (x^2)^2 \\
W^2 - 8W + 8 &= 0 && \text{Let } W = x^2 \\
W = \frac{-(-8) \pm \sqrt{(-8)^2 - 4 \cdot 8}}{2} &= 4 \pm 2\sqrt{2} && \text{Quadratic Formula} \\
x^2 = 4 \pm 2\sqrt{2} & && W = x^2 \\
x = \pm\sqrt{4 \pm 2\sqrt{2}} & && \text{Take square roots}
\end{aligned}$$

So there are four solutions:

$$\sqrt{4 + 2\sqrt{2}} \qquad \sqrt{4 - 2\sqrt{2}} \qquad -\sqrt{4 + 2\sqrt{2}} \qquad -\sqrt{4 - 2\sqrt{2}}$$

Using a calculator, we obtain the approximations $x \approx 2.61, 1.08, -2.61, -1.08$.

Now Try Exercise 91 ■

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AL-KHWARIZMI (780–850 CE) was a Persian mathematician, astronomer, and geographer; he was a scholar at the House of Wisdom in Baghdad. He is known today as the "father of algebra," because his book *Hisāb al-Jabr w'al-muqābala* was the first to deal with the rules of algebra. The title roughly translates to "Calculation by Completion and Balancing," which are the operations he used to solve algebraic equations. He described his book as containing "what is easiest and most useful in arithmetic." Among other things, the book contains the method of solving quadratic equations by completing the square. In Latin translations the title of the book was shortened to *Al-Jabr* from which we get the word *algebra*. Al Khwarizmi's name itself made its way into the English language in the word *algorithm*. On the 1200th anniversary of his birth, a stamp (shown here) was issued by the former USSR to celebrate his birth.

Example 13 ■ An Equation Involving Fractional Powers

Find all solutions of the equation $x^{1/3} + x^{1/6} - 2 = 0$.

Solution This equation is of quadratic type because if we let $W = x^{1/6}$, then $W^2 = (x^{1/6})^2 = x^{1/3}$.

$$\begin{aligned}
x^{1/3} + x^{1/6} - 2 &= 0 \\
(x^{1/6})^2 + x^{1/6} - 2 &= 0 && \text{Write } x^{1/3} \text{ as } (x^{1/6})^2 \\
W^2 + W - 2 &= 0 && \text{Let } W = x^{1/6} \\
(W-1)(W+2) &= 0 && \text{Factor} \\
W - 1 = 0 \quad \text{or} \quad W + 2 &= 0 && \text{Zero-Product Property} \\
W = 1 \qquad\qquad W &= -2 && \text{Solve} \\
x^{1/6} = 1 \qquad\qquad x^{1/6} &= -2 && W = x^{1/6} \\
x = 1^6 = 1 \qquad\qquad x &= (-2)^6 = 64 && \text{Take the 6th power}
\end{aligned}$$

From *Check Your Answers* we see that $x = 1$ is a solution but $x = 64$ is not. The only solution is $x = 1$.

Check Your Answers

$x = 1$:

$\text{LHS} = 1^{1/3} + 1^{1/6} - 2 = 0$

$\text{RHS} = 0$

$\text{LHS} = \text{RHS}$ ✓

$x = 64$:

$\text{LHS} = 64^{1/3} + 64^{1/6} - 2$

$= 4 + 2 - 2 = 4$

$\text{RHS} = 0$

$\text{LHS} \neq \text{RHS}$ ✗

Now Try Exercise 95 ■

When solving an absolute-value equation, we use the following property

$$|X| = C \quad \text{is equivalent to} \quad X = C \quad \text{or} \quad X = -C$$

where X is any algebraic expression. This property says that to solve an absolute-value equation, we must solve two separate equations.

Example 14 ■ An Absolute-Value Equation

Solve the equation $|2x - 5| = 3$.

Solution By the definition of absolute value, $|2x - 5| = 3$ is equivalent to

$$\begin{aligned}
2x - 5 = 3 \quad &\text{or} \quad 2x - 5 = -3 \\
2x = 8 \quad &\qquad\quad 2x = 2 \\
x = 4 \quad &\qquad\quad x = 1
\end{aligned}$$

The solutions are $x = 1$, $x = 4$.

Now Try Exercise 99 ■

1.5 Exercises

Concepts

1. *Yes or No*? If *No*, give a reason.

(a) When you add the same number to each side of an equation, do you always get an equivalent equation?

(b) When you multiply each side of an equation by the same nonzero number, do you always get an equivalent equation?

(c) When you square each side of an equation, do you always get an equivalent equation?

2. What is a logical first step in solving the equation?
(a) $(x+5)^2 = 64$ **(b)** $(x+5)^2 + 5 = 64$
(c) $x^2 + x = 2$

3. Explain how you would use each method to solve the equation $x^2 - 6x - 16 = 0$.
(a) By factoring: ________
(b) By completing the square: ________
(c) By using the Quadratic Formula: ________

4. **(a)** The Zero-Product Property says that if $a \cdot b = 0$, then either a or b must be ________.
(b) The solutions of the equation $x^2(x-4) = 0$ are ________.
(c) To solve the equation $x^3 - 4x^2 = 0$, we ________ the left side.

5. Solve the equation $\sqrt{2x} + x = 0$ by doing the following steps.
(a) Isolate the radical: ______________.
(b) Square both sides: ______________.
(c) The solutions of the resulting quadratic equation are ________.
(d) The solution(s) that satisfy the original equation are ________.

6. The equation $(x+1)^2 - 5(x+1) + 6 = 0$ is of ________ type. To solve the equation, we set $W =$ ________. The resulting quadratic equation is ________.

7. To eliminate the denominators in the equation $\dfrac{3}{x} + \dfrac{5}{x+2} = 2$, we multiply each side by the lowest common denominator ______________ to get the equivalent equation ______________.

8. To eliminate the square root in the equation $2x + 1 = \sqrt{x+1}$, we ______________ each side to get the equation ______________.

Skills

9–12 ■ Solution? Check whether the given value is a solution of the equation.

9. $4x + 7 = 9x - 3$
(a) $x = -2$ **(b)** $x = 2$

10. $1 - [2 - (3 - x)] = 4x - (6 + x)$
(a) $x = 2$ **(b)** $x = 4$

11. $\dfrac{1}{x} - \dfrac{1}{x-4} = 1$
(a) $x = 2$ **(b)** $x = 4$

12. $\dfrac{x^{3/2}}{x-6} = x - 8$
(a) $x = 4$ **(b)** $x = 8$

13–28 ■ Linear Equations The given equation is either linear or equivalent to a linear equation. Solve the equation.

13. $8x + 13 = 5$
14. $1 - x = 10$
15. $\frac{1}{2}x - 8 = 1$
16. $3 + \frac{1}{3}x = 5$
17. $-x + 3 = 4x$
18. $2x + 3 = 7 - 3x$
19. $\dfrac{x}{3} - 2 = \dfrac{5}{3}x + 7$
20. $\frac{2}{5}x - 1 = \frac{3}{10}x + 3$
21. $2(1 - x) = 3(1 + 2x) + 5$
22. $(3y + 2) - 5(2y - 1) = 2(3 - y) + 1$
23. $x - \frac{1}{3}x - \frac{1}{2}x - 5 = 0$
24. $2x - \dfrac{x}{2} + \dfrac{x+1}{4} = 6x$
25. $\dfrac{2x-1}{x+2} = \dfrac{4}{5}$
26. $\dfrac{3}{x-1} = \dfrac{5}{x+2} - \dfrac{1}{x-1}$
27. $\dfrac{1}{x} = \dfrac{4}{3x} + 1$
28. $\dfrac{3}{x+1} - \dfrac{1}{2} = \dfrac{1}{3x+3}$

29–40 ■ Solving for a Variable in a Formula Solve the equation for the indicated variable.

29. $E = \frac{1}{2}mv^2$; for m
30. $F = G\dfrac{mM}{r^2}$; for m
31. $P = 2l + 2w$; for w
32. $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2}$; for R_1
33. $\dfrac{ax+b}{cx+d} = 2$; for x
34. $a - 2[b - 3(c - x)] = 6$; for x
35. $a^2x + (a - 1) = (a + 1)x$; for x
36. $\dfrac{a+1}{b} = \dfrac{a-1}{b} + \dfrac{b+1}{a}$; for a
37. $V = \frac{1}{3}\pi r^2 h$; for r
38. $F = G\dfrac{mM}{r^2}$; for r
39. $a^2 + b^2 = c^2$; for b
40. $A = P\left(1 + \dfrac{i}{100}\right)^2$; for i

41–46 ■ Solving by Factoring Find all real solutions of the equation by factoring.

41. $x^2 + x = 12$
42. $x^2 + 9x = -20$
43. $x^2 + 13x - 30 = 0$
44. $x^2 - 13x + 30 = 0$
45. $4x^2 - 4x - 15 = 0$
46. $2y^2 + 7y + 3 = 0$

47–52 ■ Solving Simple Quadratics Find all real solutions of the equation.

47. $2x^2 = 8$
48. $3x^2 - 27 = 0$
49. $(2x - 5)^2 = 81$
50. $(5x + 1)^2 + 3 = 10$
51. $8(x + 5)^2 = 16$
52. $2(3x - 1)^2 - 3 = 11$

53–60 ■ Completing the Square Find all real solutions of the equation by completing the square.

53. $x^2 + 10x + 2 = 0$
54. $x^2 - 4x + 2 = 0$
55. $x^2 - 6x - 11 = 0$
56. $x^2 + 3x - \frac{7}{4} = 0$
57. $5x^2 + 10x = 1$
58. $3x^2 - 6x - 1 = 0$
59. $4x^2 - x = 0$
60. $x^2 = \frac{3}{4}x - \frac{1}{8}$

61–72 ■ Using the Quadratic Equation Find all real solutions of the quadratic equation.

61. $x^2 - 2x - 15 = 0$
62. $x^2 - 13x + 42 = 0$
63. $2x^2 + x - 3 = 0$
64. $3x^2 + 7x + 4 = 0$
65. $3x^2 + 6x - 5 = 0$
66. $x^2 - 6x + 1 = 0$
67. $9x^2 + 12x + 4 = 0$
68. $4x^2 - 4x + 1 = 0$
69. $4x^2 + 16x = 9$
70. $4x = x^2 + 1$
71. $7x^2 = 2x - 4$
72. $z(z - 3) = 5$

73–78 ■ Discriminant Use the discriminant to determine the number of real solutions of the equation. Do not solve the equation.

73. $x^2 - 6x + 1 = 0$

74. $3x^2 = 6x - 9$

75. $x^2 + 2.20x + 1.21 = 0$

76. $x^2 + 2.21x + 1.21 = 0$

77. $4x^2 + 5x + \frac{13}{8} = 0$

78. $x^2 + rx - s = 0 \quad (s > 0)$

79–102 ■ Other Equations Find all real solutions of the equation.

79. $\dfrac{x^2}{x + 100} = 50$

80. $\dfrac{1}{x - 1} - \dfrac{2}{x^2} = 0$

81. $\dfrac{y + 1}{y^2 + 1} = \dfrac{2}{y + 2}$

82. $\dfrac{3w + 1}{w} = \dfrac{w - 3}{w - 1}$

83. $\dfrac{1}{x - 1} + \dfrac{1}{x + 2} = \dfrac{5}{4}$

84. $\dfrac{x + 5}{x - 2} = \dfrac{5}{x + 2} + \dfrac{28}{x^2 - 4}$

85. $5 = \sqrt{4x - 3}$

86. $\sqrt{8x - 1} = 3$

87. $\sqrt{2x - 1} = \sqrt{3x - 5}$

88. $\sqrt{3 + x} = \sqrt{x^2 + 1}$

89. $\sqrt{2x + 1} + 1 = x$

90. $2x + \sqrt{x + 1} = 8$

91. $x^4 - 13x^2 + 40 = 0$

92. $x^6 - 2x^3 - 3 = 0$

93. $(x + 2)^4 - 5(x + 2)^2 + 4 = 0$

94. $2x^4 + 4x^2 + 1 = 0$

95. $x^{4/3} - 5x^{2/3} + 6 = 0$

96. $4(x + 1)^{1/2} - 5(x + 1)^{3/2} + (x + 1)^{5/2} = 0$

97. $x^{1/2} + 3x^{-1/2} = 10x^{-3/2}$

98. $x - 5\sqrt{x} + 6 = 0$

99. $|3x + 5| = 1$

100. $|2x| = 3$

101. $|x - 4| = 0.01$

102. $|x - 6| = -1$

103–118 ■ Putting It All Together Recognize the type of the equation and find all real solutions of the equation using an appropriate method.

103. $2x + \frac{1}{2} = \frac{1}{4}x - 3$

104. $x^2 - 7x + 12 = 0$

105. $\dfrac{4 - a}{3a - 2} = a$

106. $\dfrac{3x - 5}{x + 5} = 8$

107. $\dfrac{2}{3}y + \dfrac{1}{2}(y - 3) = \dfrac{y + 1}{4}$

108. $6x(x - 1) = 21 - x$

109. $x - \sqrt{9 - 3x} = 0$

110. $\sqrt{3}x + \sqrt{12} = \dfrac{x + 5}{\sqrt{3}}$

111. $\sqrt{x} - 3\sqrt[4]{x} - 4 = 0$

112. $x^6 - x^3 - 6 = 0$

113. $|3x - 10| = 29$

114. $x^{1/2} - 3x^{1/3} = 3x^{1/6} - 9$

115. $(z - 1)^2 - 8(z - 1) + 15 = 0$

116. $(t - 4)^2 = (t + 4)^2 + 32$

117. $|3x^2 - 1| - 4 = 7$

118. $\left(\dfrac{x}{x+1}\right)^2 - 6\left(\dfrac{x}{x+1}\right) + 8 = 0$

Skills Plus

119–124 ■ More on Solving Equations Find all real solutions of the equation.

119. $\dfrac{1}{x^3} + \dfrac{4}{x^2} + \dfrac{4}{x} = 0$

120. $4x^{-4} - 16x^{-2} + 4 = 0$

121. $\sqrt{\sqrt{x + 5} + x} = 5$

122. $\sqrt[3]{4x^2 - 4x} = x$

123. $x^2\sqrt{x + 3} = (x + 3)^{3/2}$

124. $\sqrt{11 - x^2} - \dfrac{2}{\sqrt{11 - x^2}} = 1$

125–128 ■ More on Solving Equations Solve the equation for the variable x. The constants a and b represent positive real numbers.

125. $x^4 - 5ax^2 + 4a^2 = 0$

126. $a^3x^3 + b^3 = 0$

127. $\sqrt{x + a} + \sqrt{x - a} = \sqrt{2}\sqrt{x + 6}$

128. $\sqrt{x} - a\sqrt[3]{x} + b\sqrt[6]{x} - ab = 0$

Applications

129–130 ■ Falling-Body Problems Suppose an object is dropped from a height h_0 above the ground. Then its height after t seconds is given by $h = -16t^2 + h_0$, where h is measured in feet. Use this information to solve the problem.

129. If a ball is dropped from 288 ft above the ground, how long does it take the ball to reach ground level?

130. A ball is dropped from the top of a building 96 ft tall.

(a) How long will it take the ball to fall half the distance to ground level?

(b) How long will it take the ball to fall to ground level?

131–132 ■ Falling-Body Problems Use the formula $h = -16t^2 + v_0t$ discussed in Example 9.

131. A ball is thrown straight upward at an initial speed of $v_0 = 40$ ft/s.

(a) When does the ball reach a height of 24 ft?

(b) When does the ball reach a height of 48 ft?

(c) What is the greatest height reached by the ball?

(d) When does the ball reach the highest point of its path?

(e) When does the ball hit the ground?

132. How fast would a ball have to be thrown upward to reach a maximum height of 100 ft? [*Hint:* Use the discriminant of the equation $16t^2 - v_0t + h = 0$.]

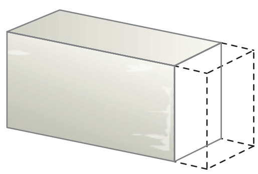
133. Shrinkage in Concrete Beams As concrete dries, it shrinks: the higher the water content, the greater the shrinkage. If a concrete beam has a water content of w kg/m^3, then it will shrink by a factor

$$S = \frac{0.032w - 2.5}{10{,}000}$$

where S is the fraction of the original beam length that disappears due to shrinkage.

(a) A beam 12.025 m long is cast in concrete that contains 250 kg/m^3 water. What is the shrinkage factor S? What length will the beam be when it has dried?

(b) A beam is 10.014 m long when wet. We want it to shrink to 10.009 m, so the shrinkage factor should be $S = 0.00050$. What water content will provide this amount of shrinkage?



134. The Lens Equation If F is the focal length of a convex lens and an object is placed at a distance x from the lens, then its image will be located a distance y from the lens, where F, x, and y are related by the *lens equation*

$$\frac{1}{F} = \frac{1}{x} + \frac{1}{y}$$

Suppose that a lens has a focal length of 4.8 cm and that the image of an object is 4 cm closer to the lens than the object itself. How far from the lens is the object?

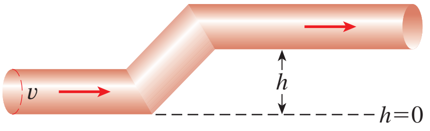
135. The Bernoulli Equation If a fluid with density $\rho\,(\mathrm{kg/m^3})$ is pumped with velocity v (m/s) through a given pipe, then the pressure P(Pa) in the pipe satisfies Bernoulli's equation

$$\tfrac{1}{2}\rho v^2 + \rho g h + P = C$$

where h is the height (m) above some fixed reference point, g is acceleration due to gravity ($9.81\ \mathrm{m/s^2}$), and C is a constant specific to the given pipe. Suppose water ($\rho = 1000\ \mathrm{kg/m^3}$) is flowing through a pipe at height 1m, with velocity 2 m/s and pressure 200,000 Pa.

(a) Find the constant C for the pipe.

(b) Solve for the variable P in Bernoulli's equation. If the height increases to 5 m, what is the new pressure? If the height returns to 1 m and the velocity increases to 4 m/s, what is the new pressure?

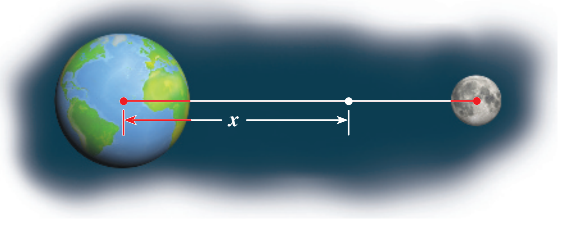


136. Profit A small-appliance manufacturer finds that the profit P (in dollars) generated by producing x microwave ovens per week is given by the formula $P = \frac{1}{10}x(300 - x)$, provided that $0 \le x \le 200$. How many ovens must be manufactured in a given week to generate a profit of \$1250?

137. Gravity If an imaginary line segment is drawn between the centers of the earth and the moon, then the net gravitational force F acting on an object situated on this line segment is

$$F = \frac{-K}{x^2} + \frac{0.012K}{(239 - x)^2} \qquad (0 < x < 239)$$

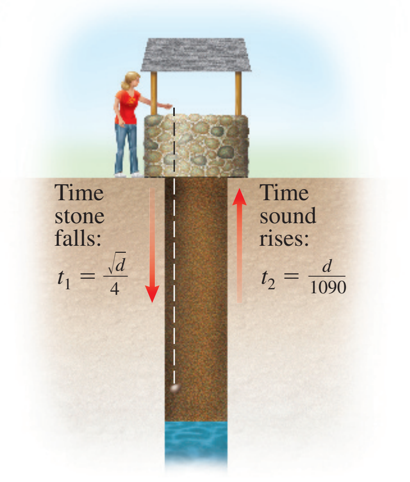
where $K > 0$ is a constant and x is the distance of the object from the center of the earth, measured in thousands of miles. How far from the center of the earth is the "dead spot" where no net gravitational force acts upon the object? (Express your answer to the nearest thousand miles.)



138. Depth of a Well One method for determining the depth of a well is to drop a stone into it and then measure the time it takes until the splash is heard. If d is the depth of the well (in feet) and t_1 the time (in seconds) it takes the stone to fall, then $d = 16t_1^2$, so $t_1 = \sqrt{d}/4$. Now if t_2 is the time it takes the sound to travel back up, then $d = 1090t_2$ because the speed of sound is 1090 ft/s. So $t_2 = d/1090$. Thus the total time elapsed between dropping the stone and hearing the splash is

$$t_1 + t_2 = \frac{\sqrt{d}}{4} + \frac{d}{1090}$$

How deep is the well if this total time is 3 seconds?



Discuss ■ Discover ■ Prove ■ Write

139. Discuss: A Family of Equations The equation

$$3x + k - 5 = kx - k + 1$$

is really a **family of equations**, because for each value of k, we get a different equation with the unknown x. The letter k is called a **parameter** for this family. What value should we pick for k to make the given value of x a solution of the resulting equation?

(a) $x = 0$ **(b)** $x = 1$ **(c)** $x = 2$

140. Discuss: Proof that 0 = 1? The following steps appear to give equivalent equations, which seem to prove that $1 = 0$. Find the error.

$$\begin{aligned}
x &= 1 && \text{Given}\\
x^2 &= x && \text{Multiply by } x\\
x^2 - x &= 0 && \text{Subtract } x\\
x(x-1) &= 0 && \text{Factor}\\
\frac{x(x-1)}{x-1} &= \frac{0}{x-1} && \text{Divide by } x - 1\\
x &= 0 && \text{Simplify}\\
1 &= 0 && \text{Given } x = 1
\end{aligned}$$

141. Discover ■ Prove: Relationship Between Solutions and Coefficients The Quadratic Formula gives us the solutions of a quadratic equation from its coefficients. We can also obtain the coefficients from the solutions.

(a) Find the solutions of the equation $x^2 - 9x + 20 = 0$, and show that the product of the solutions is the constant term 20 and the sum of the solutions is 9, the negative of the coefficient of x.

(b) Use the Quadratic Formula to prove that, in general, if the equation $x^2 + bx + c = 0$ has solutions r_1 and r_2, then $c = r_1 r_2$ and $b = -(r_1 + r_2)$.

142. Discover ■ Prove: Depressed Quadratics A quadratic equation is called *depressed* if it is missing the x-term. For example, $x^2 - 5 = 0$ is a depressed quadratic.

(a) For the quadratic equation $x^2 + bx + c = 0$, show that the substitution $x = u - b/2$ transforms it into a depressed quadratic in the variable u:

$$u^2 - \left(\frac{b^2}{4} - c\right) = 0$$

(b) Solve the equation $x^2 + 5x - 6 = 0$ by first transforming it into a depressed quadratic, as described in part (a).

143. Discuss: Solving an Equation in Different Ways We have learned several ways to solve an equation in this section. Some equations can be tackled by more than one method. Solve the following equations using both methods indicated, and show that you get the same final answers.

(a) $x - \sqrt{x} - 2 = 0$ quadratic type; solve for the radical, and square

(b) $\dfrac{12}{(x-3)^2} + \dfrac{10}{x-3} + 1 = 0$ quadratic type; multiply by LCD

1.6 Complex Numbers

■ Arithmetic Operations on Complex Numbers ■ Square Roots of Negative Numbers ■ Complex Solutions of Quadratic Equations

In Section 1.5 we saw that if the discriminant of a quadratic equation is negative, the equation has no real solution. For example, the equation

$$x^2 + 4 = 0$$

has no real solution. If we try to solve this equation, we get $x^2 = -4$, so

$$x = \pm\sqrt{-4}$$

But this is impossible, since the square of any real number is positive. [For example, $(-2)^2 = 4$, a positive number.] Thus negative numbers don't have real square roots.

See the note on Cardano in Section 3.5 for an example of how complex numbers are used to find real solutions of polynomial equations.

To make it possible to solve *all* quadratic equations, mathematicians invented an expanded number system, called the *complex number system*. First they defined the new number

$$i = \sqrt{-1}$$

This means that $i^2 = -1$. A complex number is then a number of the form $a + bi$, where a and b are real numbers.

Definition of Complex Numbers

A **complex number** is an expression of the form

$$a + bi$$

where a and b are real numbers and $i^2 = -1$. The **real part** of this complex number is a, and the **imaginary part** is b. Two complex numbers are **equal** if and only if their real parts are equal and their imaginary parts are equal.

Note that both the real and imaginary parts of a complex number are real numbers.

Example 1 ■ Complex Numbers

The following are examples of complex numbers.

$3 + 4i$	Real part 3, imaginary part 4
$\frac{1}{2} - \frac{2}{3}i$	Real part $\frac{1}{2}$, imaginary part $-\frac{2}{3}$
$6i$	Real part 0, imaginary part 6
-7	Real part -7, imaginary part 0

Now Try Exercises 7 and 11 ■

A number such as $6i$, which has real part 0, is called a **pure imaginary number**. A real number such as -7 can be thought of as a complex number with imaginary part 0.

In the complex number system every quadratic equation has solutions. The numbers $2i$ and $-2i$ are solutions of $x^2 = -4$ because

$$(2i)^2 = 2^2 i^2 = 4(-1) = -4 \qquad \text{and} \qquad (-2i)^2 = (-2)^2 i^2 = 4(-1) = -4$$

Note Although we use the term *imaginary* in this context, imaginary numbers should not be thought of as any less "real" (in the ordinary rather than the mathematical sense of that word) than negative numbers or irrational numbers. All numbers (except possibly the positive integers) are creations of the human mind—the numbers -1 and $\sqrt{2}$ as well as the number i. We study complex numbers because they complete, in a useful and elegant fashion, our study of the solutions of equations. In fact, imaginary numbers are useful not only in algebra and mathematics, but in the other sciences as well. For example, in electrical theory complex numbers are used in modeling electrical *impedance*.

■ Arithmetic Operations on Complex Numbers

Complex numbers are added, subtracted, multiplied, and divided just as we would any number of the form $a + b\sqrt{c}$. The only difference that we need to keep in mind is that $i^2 = -1$. Thus the following calculations are valid.

$$\begin{aligned}
(a + bi)(c + di) &= ac + (ad + bc)i + bdi^2 && \text{Multiply and collect like terms} \\
&= ac + (ad + bc)i + bd(-1) && i^2 = -1 \\
&= (ac - bd) + (ad + bc)i && \text{Combine real and imaginary parts}
\end{aligned}$$

We therefore define the sum, difference, and product of complex numbers as follows.

Adding, Subtracting, and Multiplying Complex Numbers

Definition	Description
Addition $(a + bi) + (c + di) = (a + c) + (b + d)i$	To add complex numbers, add the real parts and the imaginary parts.
Subtraction $(a + bi) - (c + di) = (a - c) + (b - d)i$	To subtract complex numbers, subtract the real parts and the imaginary parts.
Multiplication $(a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i$	Multiply complex numbers like binomials, using $i^2 = -1$.

Example 2 ■ Adding, Subtracting, and Multiplying Complex Numbers

When using FOIL to multiply complex numbers, the terms are real or imaginary as follows:

imaginary

FOIL

real

Express each of the following in the form $a + bi$.

(a) $(3 + 5i) + (4 - 2i)$ **(b)** $(3 + 5i) - (4 - 2i)$

(c) $(3 + 5i)(4 - 2i)$ **(d)** i^{23}

Solution

(a) According to the definition, we add the real parts and we add the imaginary parts:

$$(3 + 5i) + (4 - 2i) = (3 + 4) + (5 - 2)i = 7 + 3i$$

(b) $(3 + 5i) - (4 - 2i) = (3 - 4) + [5 - (-2)]i = -1 + 7i$

(c) $(3 + 5i)(4 - 2i) = [3 \cdot 4 - 5(-2)] + [3(-2) + 5 \cdot 4]i = 22 + 14i$

(d) $i^{23} = i^{22+1} = (i^2)^{11}i = (-1)^{11}i = (-1)i = -i$

Now Try Exercises 19, 23, 29, and 47 ■

Complex Conjugates

Number	Conjugate
$3 + 2i$	$3 - 2i$
$1 - i$	$1 + i$
$4i$	$-4i$
5	5

Division of complex numbers is much like rationalizing the denominator of a radical expression, which we considered in Section 1.2. For the complex number $z = a + bi$ we define its **complex conjugate** to be $\overline{z} = \overline{a + bi} = a - bi$. Note that

$$z \cdot \overline{z} = (a + bi)(a - bi) = a^2 + b^2$$

So the product of a complex number and its conjugate is always a nonnegative real number. We use this property to divide complex numbers.

Dividing Complex Numbers

To simplify the quotient $\dfrac{a + bi}{c + di}$, multiply the numerator and the denominator by the complex conjugate of the denominator:

$$\frac{a + bi}{c + di} = \left(\frac{a + bi}{c + di}\right)\left(\frac{c - di}{c - di}\right) = \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}$$

Rather than memorizing this entire formula, it is easier to just remember the first step and then multiply out the numerator and the denominator as usual.

Example 3 ■ Dividing Complex Numbers

Express each of the following in the form $a + bi$.

(a) $\dfrac{3 + 5i}{1 - 2i}$ **(b)** $\dfrac{7 + 3i}{4i}$

Solution

We multiply both the numerator and denominator by the complex conjugate of the denominator to make the new denominator a real number.

(a) The complex conjugate of $1 - 2i$ is $\overline{1 - 2i} = 1 + 2i$. Therefore

$$\frac{3 + 5i}{1 - 2i} = \left(\frac{3 + 5i}{1 - 2i}\right)\left(\frac{1 + 2i}{1 + 2i}\right) = \frac{-7 + 11i}{5} = -\frac{7}{5} + \frac{11}{5}i$$

(b) The complex conjugate of $4i$ is $-4i$. Therefore

$$\frac{7 + 3i}{4i} = \left(\frac{7 + 3i}{4i}\right)\left(\frac{-4i}{-4i}\right) = \frac{12 - 28i}{16} = \frac{3}{4} - \frac{7}{4}i$$

Now Try Exercises 39 and 43 ■

Square Roots of Negative Numbers

Just as every positive real number r has two square roots ($\sqrt{r}$ and $-\sqrt{r}$), every negative number has two square roots. If $-r$ is a negative number, then its square roots are $\pm\sqrt{r}\,i$, because $(\sqrt{r}\,i)^2 = ri^2 = -r$ and $(-\sqrt{r}\,i)^2 = r(-1)^2 i^2 = -r$.

Square Roots of Negative Numbers

If $-r$ is negative, then the **principal square root** of $-r$ is

$$\sqrt{-r} = i\sqrt{r}$$

The two square roots of $-r$ are $\sqrt{r}\,i$ and $-\sqrt{r}\,i$.

Example 4 ■ Square Roots of Negative Numbers

(a) $\sqrt{-1} = \sqrt{1}\,i = i$ **(b)** $\sqrt{-16} = \sqrt{16}\,i = 4i$ **(c)** $\sqrt{-3} = \sqrt{3}\,i$

Now Try Exercises 53 and 55 ■

Special care must be taken in performing calculations that involve square roots of negative numbers. Although $\sqrt{a}\cdot\sqrt{b} = \sqrt{ab}$ when a and b are positive, this is *not* true when both are negative. For example,

$$\sqrt{-2}\cdot\sqrt{-3} = \sqrt{2}\,i\cdot\sqrt{3}\,i = \sqrt{6}\,i^2 = -\sqrt{6}$$

but

$$\sqrt{(-2)(-3)} = \sqrt{6}$$

so

$$\sqrt{-2}\cdot\sqrt{-3} \neq \sqrt{(-2)(-3)}$$

When multiplying radicals of negative numbers, express them first in the form $\sqrt{r}\,i$ (where $r > 0$) to avoid errors of this type.

Example 5 ■ Using Square Roots of Negative Numbers

Evaluate $(\sqrt{12} - \sqrt{-3})(3 + \sqrt{-4})$ and express the result in the form $a + bi$.

Solution

$$\begin{aligned}(\sqrt{12} - \sqrt{-3})(3 + \sqrt{-4}) &= (\sqrt{12} - \sqrt{3}\,i)(3 + \sqrt{4}\,i)\\ &= (2\sqrt{3} - \sqrt{3}\,i)(3 + 2i)\\ &= (6\sqrt{3} + 2\sqrt{3}) + (2\cdot 2\sqrt{3} - 3\sqrt{3})i\\ &= 8\sqrt{3} + \sqrt{3}\,i\end{aligned}$$

Now Try Exercise 57 ■

Complex Solutions of Quadratic Equations

We have already seen that if $a \neq 0$, then the solutions of the quadratic equation $ax^2 + bx + c = 0$ are

The Quadratic Formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If $b^2 - 4ac < 0$, then the equation has no real solution. But in the complex number system this equation will always have solutions, because negative numbers have square roots in this expanded setting.

LEONHARD EULER (1707–1783) was born in Basel, Switzerland, the son of a pastor. When Euler was 13, his father sent him to the University at Basel to study theology, but Euler soon devoted himself to the sciences. Besides theology he studied mathematics, medicine, astronomy, physics, and Asian languages. It is said that Euler could calculate as effortlessly as "men breathe or as eagles fly." One hundred years before Euler, Fermat (see Section 1.11) had conjectured that $2^{2^n} + 1$ is a prime number for all n. The first five of these numbers are

5 17 257 65537 4,294,967,297

It is easy to show that the first four are prime. The fifth was also thought to be prime until Euler, with his phenomenal calculating ability, showed that it is the product $641 \times 6{,}700{,}417$ and so is not prime. Euler published more than any other mathematician in history. His collected works comprise 75 large volumes. Although he was blind for the last 17 years of his life, he continued to work and publish. In his writings he popularized the use of the symbols π, e, and i, which you will find in this textbook. One of Euler's enduring contributions is his development of complex numbers.

Example 6 ■ Quadratic Equations with Complex Solutions

Solve each equation.

(a) $x^2 + 9 = 0$ **(b)** $x^2 + 4x + 5 = 0$

Solution

(a) The equation $x^2 + 9 = 0$ means $x^2 = -9$, so

$$x = \pm\sqrt{-9} = \pm\sqrt{9}\,i = \pm 3i$$

The solutions are therefore $3i$ and $-3i$.

(b) By the Quadratic Formula, we have

$$\begin{aligned} x &= \frac{-4 \pm \sqrt{4^2 - 4\cdot 5}}{2} \\ &= \frac{-4 \pm \sqrt{-4}}{2} \\ &= \frac{-4 \pm 2i}{2} = -2 \pm i \end{aligned}$$

So the solutions are $-2 + i$ and $-2 - i$.

Now Try Exercises 61 and 63 ■

We see from Example 6 that if a quadratic equation with real coefficients has complex solutions, then these solutions are complex conjugates of each other. So if $a + bi$ is a solution, then $a - bi$ is also a solution.

Example 7 ■ Complex Conjugates as Solutions of a Quadratic

Show that the solutions of the equation

$$4x^2 - 24x + 41 = 0$$

are complex conjugates of each other.

Solution We use the Quadratic Formula to get

$$\begin{aligned} x &= \frac{-(-24) \pm \sqrt{(-24)^2 - 4(4)(41)}}{2(4)} \\ &= \frac{24 \pm \sqrt{-80}}{8} = \frac{24 \pm 4\sqrt{5}\,i}{8} = 3 \pm \frac{\sqrt{5}}{2}i \end{aligned}$$

So the solutions are $3 + \frac{\sqrt{5}}{2}i$ and $3 - \frac{\sqrt{5}}{2}i$, and these are complex conjugates.

Now Try Exercise 69 ■

1.6 Exercises

Concepts

1. The imaginary number i has the property that $i^2 =$ ________.

2. For the complex number $3 + 4i$ the real part is ________ and the imaginary part is ________.

3. (a) The complex conjugate of $3 + 4i$ is $\overline{3 + 4i} =$ ________.

(b) $(3 + 4i)(\overline{3 + 4i}) =$ ________.

4. If $3 + 4i$ is a solution of a quadratic equation with real coefficients, then ________ is also a solution of the equation.

5–6 ■ *Yes or No*? If *No*, give a reason.

5. Is every real number also a complex number?

6. Is the sum of a complex number and its complex conjugate a real number?

Skills

7–16 ■ Real and Imaginary Parts Find the real and imaginary parts of the complex number.

7. $3 - 8i$

8. $-(5 - i)$

9. $\dfrac{-2 - 5i}{3}$

10. $\dfrac{4 + 7i}{2}$

11. 3

12. $-\frac{1}{2}$

13. $-\frac{2}{3}i$

14. $\sqrt{3}i$

15. $\sqrt{3} + \sqrt{-4}$

16. $2 - \sqrt{-5}$

17–26 ■ Sums and Differences Evaluate the sum or difference and write the result in the form $a + bi$.

17. $(3 + 2i) + 5i$

18. $3i - (2 - 3i)$

19. $(5 - 3i) + (-4 - 7i)$

20. $(-3 + 4i) - (2 - 5i)$

21. $(-6 + 6i) + (9 - i)$

22. $(3 - 2i) + \left(-5 - \frac{1}{3}i\right)$

23. $\left(7 - \frac{1}{2}i\right) - \left(5 + \frac{3}{2}i\right)$

24. $(-4 + i) - (2 - 5i)$

25. $(-12 + 8i) - (7 + 4i)$

26. $6i - (4 - i)$

27–36 ■ Products Evaluate the product and write the result in the form $a + bi$.

27. $4(-1 + 2i)$

28. $-2(3 - 4i)$

29. $(3 - 4i)(2 + 5i)$

30. $(-5 + i)(6 - 2i)$

31. $(6 + 5i)(2 - 3i)$

32. $(-2 + i)(3 - 7i)$

33. $(3 - 2i)(3 + 2i)$

34. $(10 + i)(10 - i)$

35. $(3 - 2i)^2$

36. $(10 + i)^2$

37–46 ■ Quotients Evaluate the quotient and write the result in the form $a + bi$.

37. $\dfrac{1}{i}$

38. $\dfrac{1}{1 + i}$

39. $\dfrac{1 - 3i}{1 + 2i}$

40. $\dfrac{2 - i}{1 - 3i}$

41. $\dfrac{10i}{1 - 2i}$

42. $(2 - 3i)^{-1}$

43. $\dfrac{4 + 6i}{3i}$

44. $\dfrac{-3 + 5i}{15i}$

45. $\dfrac{1}{1 + i} - \dfrac{1}{1 - i}$

46. $\dfrac{(1 + 2i)(3 - i)}{2 + i}$

47–52 ■ Powers Evaluate the power, and write the result in the form $a + bi$.

47. i^3

48. i^{10}

49. $(3i)^5$

50. $(2i)^4$

51. i^{1000}

52. i^{1002}

53–60 ■ Radical Expressions Evaluate the radical expression and express the result in the form $a + bi$.

53. $\sqrt{-25}$

54. $\dfrac{\sqrt{-8}}{\sqrt{2}}$

55. $\sqrt{-4}\sqrt{-9}$

56. $\sqrt{\tfrac{1}{2}}\sqrt{-32}$

57. $(2 + \sqrt{-1})(3 - \sqrt{-3})$

58. $(\sqrt{3} - \sqrt{-4})(\sqrt{6} - \sqrt{-8})$

59. $\dfrac{2 + \sqrt{-8}}{1 + \sqrt{-2}}$

60. $\dfrac{\sqrt{-36}}{\sqrt{-2}\sqrt{-9}}$

61–72 ■ Quadratic Equations Find all solutions of the equation and express them in the form $a + bi$.

61. $x^2 + 25 = 0$

62. $2x^2 + 5 = 0$

63. $x^2 - 6x + 13 = 0$

64. $x^2 + 2x + 2 = 0$

65. $2x^2 - 2x + 5 = 0$

66. $8x^2 - 12x + 5 = 0$

67. $x^2 + x + 1 = 0$

68. $x^2 - 3x + 3 = 0$

69. $9x^2 - 4x + 4 = 0$

70. $t + 2 + \dfrac{6}{t} = 0$

71. $6x^2 + 12x + 7 = 0$

72. $x^2 + \frac{1}{2}x + 1 = 0$

Skills Plus

73–76 ■ Conjugates Evaluate the given expression for $z = 3 - 4i$ and $w = 5 + 2i$.

73. $\overline{z} + \overline{w}$

74. $\overline{z + w}$

75. $z \cdot \overline{z}$

76. $\overline{z} \cdot \overline{w}$

77–84 ■ Conjugates If $z = a + bi$ and $w = c + di$, show that each statement is true.

77. $\overline{z} + \overline{w} = \overline{z + w}$

78. $\overline{zw} = \overline{z} \cdot \overline{w}$

79. $(\overline{z})^2 = \overline{z^2}$

80. $\overline{\overline{z}} = z$

81. $z + \overline{z}$ is a real number.

82. $z - \overline{z}$ is a pure imaginary number.

83. $z \cdot \overline{z}$ is a real number.

84. $z = \overline{z}$ if and only if z is real.

Discuss ■ Discover ■ Prove ■ Write

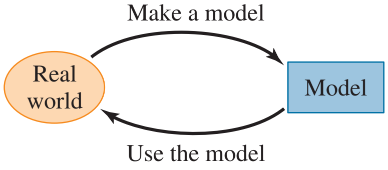
85. Prove: Complex Conjugate Solutions Suppose that the equation $ax^2 + bx + c = 0$ has real coefficients and complex solutions. Why must the solutions be complex conjugates of each other? [*Hint:* Think about how you would find the solutions using the Quadratic Formula.]

86. Discuss: Powers of *i* Explain how you would calculate any whole number power of i, and then calculate i^{4446}.

PS *Try to recognize a pattern.* Calculate several powers of i, that is, $i, i^2, i^3, i^4, \ldots$. Note the powers at which the values repeat.

1.7 Modeling with Equations

■ Making and Using Models ■ Problems About Interest ■ Problems About Area or Length ■ Problems About Mixtures ■ Problems About the Time Needed to Do a Job ■ Problems About Distance, Rate, and Time



In this section a **mathematical model** is an equation that describes a real-world object or process. Modeling is the process of finding such equations. Once the model or equation has been found, it is then used to obtain information about the thing being modeled. The process is described in the diagram in the margin. In this section we learn how to make and use models to solve real-world problems.

■ Making and Using Models

We will use the following guidelines to help us set up equations that model situations described in words. To show how the guidelines can help you set up equations, we refer to them as we work each example in this section.

Guidelines for Modeling with Equations

1. **Identify the Variable.** Identify the quantity that the problem asks you to find. This quantity can usually be determined by a careful reading of the question that is posed at the end of the problem. Then **introduce notation** for the variable (call it x or some other letter).
2. **Translate from Words to Algebra.** Read each sentence in the problem again, and express all the quantities mentioned in the problem in terms of the variable you defined in Step 1. To organize this information, it is sometimes helpful to **draw a diagram** or **make a table**.
3. **Set Up the Model.** Find the crucial fact in the problem that gives a relationship between the expressions you listed in Step 2. **Set up an equation** (or **model**) that expresses this relationship.
4. **Solve the Equation and Check Your Answer.** Solve the equation, check your answer, and **state your answer** as a sentence.

The following example illustrates how these guidelines are used to translate a "word problem" into the language of algebra.

Example 1 ■ Renting a Car

A car rental company charges \$30 a day and 15¢ a mile for renting a car. A tourist rents a car for two days, and the bill comes to \$108. How many miles was the car driven?

Solution **Identify the variable.** We are asked to find the number of miles driven. So we let

$$x = \text{number of miles driven}$$

Translate from words to algebra. Now we translate all the information given in the problem into the language of algebra.

In Words	In Algebra
Number of miles driven	x
Mileage cost (at \$0.15 per mile)	$0.15x$
Daily cost (at \$30 per day)	$2(30)$

Set up the model. Now we set up the model.

$$\boxed{\text{mileage cost}} + \boxed{\text{daily cost}} = \boxed{\text{total cost}}$$

$$0.15x + 2(30) = 108$$

Solve. Now we solve for x.

$$0.15x = 48 \qquad \text{Subtract 60}$$

$$x = \frac{48}{0.15} \qquad \text{Divide by 0.15}$$

$$x = 320 \qquad \text{Calculator}$$

The rental car was driven 320 miles.

Now Try Exercise 21

Check Your Answer

$$\text{total cost} = \text{mileage cost} + \text{daily cost}$$
$$= 0.15(320) + 2(30)$$
$$= 108 \checkmark$$

In the examples and exercises that follow, we construct equations that model problems in many different real-life situations.

Problems About Interest

When you borrow money from a bank or when a bank "borrows" your money by keeping it for you in a savings account, the borrower in each case must pay for the privilege of using the money. The fee that is paid is called **interest**. The most basic type of interest is **simple interest**, which is just an annual percentage of the total amount borrowed or deposited. The amount of a loan or deposit is called the **principal** P. The annual percentage paid for the use of this money is the **interest rate** r. We will use the variable t to stand for the number of years that the money is deposited (or borrowed) and the variable I to stand for the total interest earned (or paid). The following **simple interest formula** gives the amount of interest I when a principal P is deposited (or borrowed) for t years at an interest rate r.

Compound interest is studied in Section 4.1.

$$\boxed{I = Prt}$$

When using this formula, remember to convert r from a percentage to a decimal. For example, in decimal form, 5% is 0.05. So at an interest rate of 5%, the interest paid on a \$1000 deposit over a 3-year period is $I = Prt = 1000(0.05)(3) = \$150$.

Example 2 ■ Interest on an Investment

An amount of \$100,000 is invested in two certificates of deposit. One certificate pays 6% and the other pays $4\frac{1}{2}$% simple interest annually. If the total interest is \$5025 per year, how much money is invested at each rate?

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Discovery Project ■ Equations Through the Ages

Equations have always been important in solving real-world problems. Very old manuscripts from Babylon, Egypt, India, and China show that ancient peoples used equations to solve real-world problems that they encountered. In this project we discover that they also solved equations just for fun or for practice. You can find the project at **www.stewartmath.com**.

Solution **Identify the variable.** The problem asks for the amount invested at each rate. So we can let

$$x = \text{the amount invested at } 6\%$$

Translate from words to algebra. Since the total amount is \$100,000, it follows that $100{,}000 - x$ is invested at $4\frac{1}{2}\%$. We translate all the information given into the language of algebra.

In Words	In Algebra
Amount invested at 6%	x
Amount invested at $4\frac{1}{2}\%$	$100{,}000 - x$
Interest earned at 6%	$0.06x$
Interest earned at $4\frac{1}{2}\%$	$0.045(100{,}000 - x)$

Set up the model. We use the fact that the total interest earned is \$5025 to set up the model.

$$\boxed{\text{interest at } 6\%} + \boxed{\text{interest at } 4\tfrac{1}{2}\%} = \boxed{\text{total interest}}$$

$$0.06x + 0.045(100{,}000 - x) = 5025$$

Solve. Now we solve for x.

$$\begin{aligned} 0.06x + 4500 - 0.045x &= 5025 && \text{Distributive Property} \\ 0.015x + 4500 &= 5025 && \text{Combine the } x\text{-terms} \\ 0.015x &= 525 && \text{Subtract 4500} \\ x &= \frac{525}{0.015} = 35{,}000 && \text{Divide by 0.015} \end{aligned}$$

So \$35,000 is invested at 6% and the remaining \$65,000 at $4\frac{1}{2}\%$.

Check Your Answer

$$\begin{aligned} \text{total interest} &= 6\% \text{ of } \$35{,}000 + 4\tfrac{1}{2}\% \text{ of } \$65{,}000 \\ &= \$2100 + \$2925 = \$5025 \checkmark \end{aligned}$$

Now Try Exercise 25

■ Problems About Area or Length

When we use algebra to model a physical situation, we must sometimes use basic formulas from geometry. For example, we may need a formula for an area or a perimeter, or the formula that relates the sides of similar triangles, or the Pythagorean Theorem. The next two examples use these geometric formulas to solve some real-world problems.

You can find formulas for area and perimeter in the front endpapers of this book.

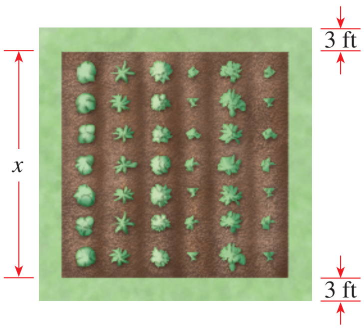


Figure 1

Example 3 ■ Dimensions of a Garden

A square garden has a walkway 3 ft wide around its outer edge, as shown in Figure 1. If the area of the entire garden, including the walkway, is 18,000 ft^2, what are the dimensions of the planted area?

Solution **Identify the variable.** We are asked to find the length and width of the planted area. So we let

$$x = \text{the length of the planted area}$$

Translate from words to algebra. Next, translate the information from Figure 1 into the language of algebra.

In Words	In Algebra
Length of planted area	x
Length of entire garden	$x + 6$
Area of entire garden	$(x + 6)^2$

Set up the model. We now set up the model.

$$\boxed{\text{area of entire garden}} = \boxed{18{,}000 \text{ ft}^2}$$

$$(x + 6)^2 = 18{,}000$$

Solve. Now we solve for x.

$$x + 6 = \sqrt{18{,}000} \qquad \text{Take square roots}$$

$$x = \sqrt{18{,}000} - 6 \qquad \text{Subtract 6}$$

$$x \approx 128$$

The planted area of the garden is about 128 ft by 128 ft.

Now Try Exercise 49 ■

Example 4 ■ Dimensions of a Building Lot

A rectangular building lot is 8 ft longer than it is wide and has an area of 2900 ft^2. Find the dimensions of the lot.

Solution **Identify the variable.** We are asked to find the width and length of the lot. So let

$$w = \text{width of lot}$$

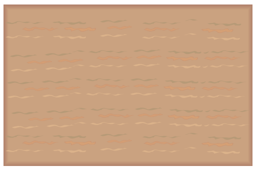


Figure 2

Translate from words to algebra. Then we translate the information given in the problem into the language of algebra (see Figure 2).

In Words	In Algebra
Width of lot	w
Length of lot	$w + 8$

Set up the model. Now we set up the model.

$$\boxed{\begin{matrix}\text{width}\\ \text{of lot}\end{matrix}} \cdot \boxed{\begin{matrix}\text{length}\\ \text{of lot}\end{matrix}} = \boxed{\begin{matrix}\text{area}\\ \text{of lot}\end{matrix}}$$

$$w(w + 8) = 2900$$

Solve. Now we solve for w.

$$w^2 + 8w = 2900 \qquad \text{Expand}$$

$$w^2 + 8w - 2900 = 0 \qquad \text{Subtract 2900}$$

$$(w - 50)(w + 58) = 0 \qquad \text{Factor}$$

$$w = 50 \quad \text{or} \quad w = -58 \qquad \text{Zero-Product Property}$$

Since the width of the lot must be a positive number, we conclude that $w = 50$ ft. The length of the lot is $w + 8 = 50 + 8 = 58$ ft.

Now Try Exercise 41 ■

Example 5 ■ Determining the Height of a Building Using Similar Triangles

A person needs to find the height of a certain four-story building and observes that the shadow of the building is 28 ft long. The person is 6 ft tall and has a shadow 3.5 ft long when standing next to the building. How tall is the building?

Solution **Identify the variable.** The problem asks for the height of the building, so let

$$h = \text{the height of the building}$$

Similar triangles are studied in Appendix A, *Geometry Review*.

Translate from words to algebra. We use the fact that the triangles in Figure 3 are similar. Recall that for any pair of similar triangles the ratios of corresponding sides are equal. Now we translate these observations into the language of algebra.

In Words	In Algebra
Height of building	h
Ratio of height to base in large triangle	$\frac{h}{28}$
Ratio of height to base in small triangle	$\frac{6}{3.5}$

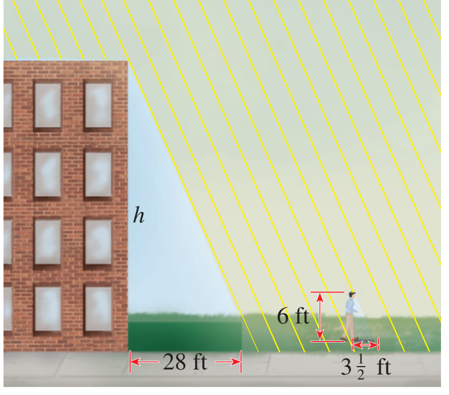


Figure 3

Set up the model. Since the large and small triangles are similar, we get the equation

$$\boxed{\text{ratio of height to base in large triangle}} = \boxed{\text{ratio of height to base in small triangle}}$$

$$\frac{h}{28} = \frac{6}{3.5}$$

Solve. Now we solve for h.

$$h = \frac{6 \cdot 28}{3.5} = 48 \qquad \text{Multiply by 28}$$

The building is 48 ft tall.

Now Try Exercise 53 ■

■ Problems About Mixtures

Many real-world problems involve mixing different types of substances. For example, construction workers may mix cement, gravel, and sand; fruit juice concentrate may involve a mixture of different types of juices. Problems involving mixtures and concentrations make use of the fact that if an amount x of a substance is dissolved in a solution with volume V, then the concentration C of the substance is given by

$$C = \frac{x}{V}$$

So if 10 g of sugar is dissolved in 5 L of water, then the resulting sugar concentration is $C = 10/5 = 2$ g/L. Solving a mixture problem usually requires us to analyze the amount x of the substance in the solution. When we solve for x in this equation, we see that $x = CV$. In many mixture problems the concentration C is expressed as a percentage, as in the next example.

Example 6 ■ Mixtures and Concentration

A manufacturer of soft drinks advertises their orange soda as "naturally flavored," although it contains only 5% orange juice. A new federal regulation stipulates that to be called "natural," a drink must contain at least 10% fruit juice. How much pure orange juice must this manufacturer add to 900 gal of orange soda to conform to the new regulation?

Solution **Identify the variable.** The problem asks for the amount of pure orange juice to be added. So let

$$x = \text{the amount (in gallons) of pure orange juice to be added}$$

Translate from words to algebra. In any problem of this type—in which two different substances are to be mixed—drawing a diagram helps us to organize the given information (see Figure 4). The information in the figure can be translated into the language of algebra, as follows.

In Words	In Algebra
Amount of orange juice to be added	x
Amount of the new mixture	$900 + x$
Amount of orange juice in the first vat	$0.05(900) = 45$
Amount of orange juice in the second vat	$1 \cdot x = x$
Amount of orange juice in the new mixture	$0.10(900 + x)$

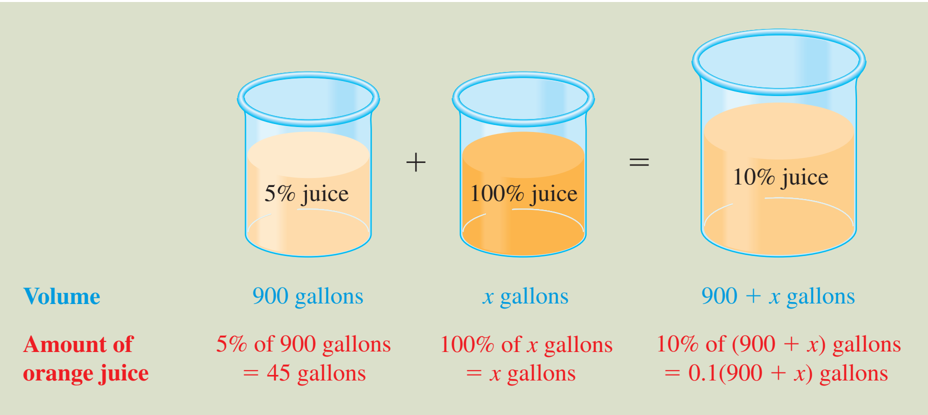


Figure 4

Set up the model. To set up the model, we use the fact that the total amount of orange juice in the new mixture is equal to the orange juice in the first two vats.

amount of orange juice in first vat	+	amount of orange juice in second vat	=	amount of orange juice in the new mixture

$$45 + x = 0.1(900 + x) \qquad \text{From Figure 4}$$

Solve. Now we solve for x.

$$45 + x = 90 + 0.1x \qquad \text{Distributive Property}$$
$$0.9x = 45 \qquad \text{Subtract } 0.1x \text{ and } 45$$
$$x = \frac{45}{0.9} = 50 \qquad \text{Divide by } 0.9$$

The manufacturer should add 50 gallons of pure orange juice to the soda.

Check Your Answer

$$\text{amount of juice before mixing} = 5\% \text{ of } 900 \text{ gal} + 50 \text{ gal pure juice}$$
$$= 45 \text{ gal} + 50 \text{ gal} = 95 \text{ gal}$$
$$\text{amount of juice after mixing} = 10\% \text{ of } 950 \text{ gal} = 95 \text{ gal}$$

Amounts are equal. ✓

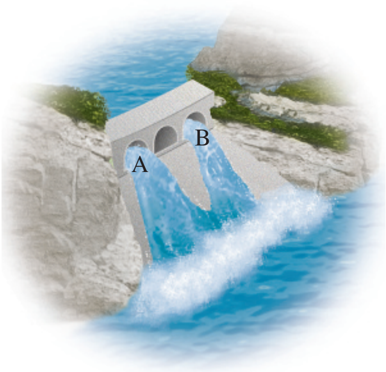
Now Try Exercise 55 ■

■ Problems About the Time Needed to Do a Job

When solving a problem that involves determining how long it takes several workers to complete a job, we use the fact that if a person or machine takes H time units to complete the task, then in one time unit the fraction of the task that has been completed is $1/H$. For example, if a worker takes 5 hours to mow a lawn, then in 1 hour the worker will mow $1/5$ of the lawn.

Example 7 ■ Time Needed to Do a Job

Because of an anticipated heavy rainstorm, the water level in a reservoir must be lowered by 1 ft. Opening spillway A (see the diagram in the margin) lowers the level by this amount in 4 hours, whereas opening the smaller spillway B does the job in 6 hours. How long will it take to lower the water level by 1 ft if both spillways are opened?



Solution **Identify the variable.** We are asked to find the time needed to lower the level by 1 ft if both spillways are opened. So let

$$x = \text{the time (in hours) it takes to lower the water level by 1 ft if both spillways are opened}$$

Translate from words to algebra. Finding an equation relating x to the other quantities in this problem is not easy. Certainly x is not simply $4 + 6$ because that would mean that together the two spillways require longer to lower the water level than either spillway alone. Instead, *we look at the fraction of the job that can be done in 1 hour by each spillway*.

In Words	In Algebra
Time it takes to lower level 1 ft with A and B together	x h
Distance A lowers level in 1 h	$\frac{1}{4}$ ft
Distance B lowers level in 1 h	$\frac{1}{6}$ ft
Distance A and B together lower levels in 1 h	$\frac{1}{x}$ ft

Set up the model. Now we set up the model.

$$\boxed{\text{fraction done by A}} + \boxed{\text{fraction done by B}} = \boxed{\text{fraction done by both}}$$

$$\frac{1}{4} + \frac{1}{6} = \frac{1}{x}$$

Solve. Now we solve for x.

$$3x + 2x = 12 \qquad \text{Multiply by the LCD, } 12x$$

$$5x = 12 \qquad \text{Add}$$

$$x = \frac{12}{5} \qquad \text{Divide by 5}$$

It will take $2\frac{2}{5}$ hours, or 2 h 24 min, to lower the water level by 1 ft if both spillways are opened.

Now Try Exercise 63 ■

■ Problems About Distance, Rate, and Time

The next example deals with distance, rate (speed), and time. The formula to keep in mind here is

$$\boxed{\text{distance} = \text{rate} \times \text{time}}$$

where the rate is either the constant speed or average speed of a moving object. For example, driving at 60 mi/h for 4 hours takes you a distance of $60 \cdot 4 = 240$ mi.

Example 8 ■ A Distance-Speed-Time Problem

A jet flew from New York to Los Angeles, a distance of 4200 km. The speed for the return trip was 100 km/h faster than the outbound speed. If the total trip took 13 hours of flying time, what was the jet's speed from New York to Los Angeles?

Solution **Identify the variable.** We are asked for the speed of the jet from New York to Los Angeles. So let

$$s = \text{speed from New York to Los Angeles}$$

Then $$s + 100 = \text{speed from Los Angeles to New York}$$

Translate from words to algebra. Now we organize the information in a table. We fill in the "Distance" column first, since we know that the cities are 4200 km apart. Then we fill in the "Speed" column, since we have expressed both speeds (rates) in terms of the variable s. Finally, we calculate the entries for the "Time" column, using

$$\text{time} = \frac{\text{distance}}{\text{rate}}$$

	Distance (km)	Speed (km/h)	Time (h)
N.Y. to L.A.	4200	s	$\frac{4200}{s}$
L.A. to N.Y.	4200	$s + 100$	$\frac{4200}{s + 100}$

Set up the model. The total trip took 13 hours, so we have the model

$$\boxed{\begin{array}{c}\text{time from}\\ \text{N.Y. to L.A.}\end{array}} + \boxed{\begin{array}{c}\text{time from}\\ \text{L.A. to N.Y.}\end{array}} = \boxed{\begin{array}{c}\text{total}\\ \text{time}\end{array}}$$

$$\frac{4200}{s} + \frac{4200}{s + 100} = 13$$

Solve. Multiplying by the common denominator, $s(s + 100)$, we get

$$\begin{aligned} 4200(s + 100) + 4200s &= 13s(s + 100) \\ 8400s + 420{,}000 &= 13s^2 + 1300s \\ 0 &= 13s^2 - 7100s - 420{,}000 \end{aligned}$$

Although this equation does factor, with numbers this large it is probably quicker to use the Quadratic Formula and a calculator.

$$\begin{aligned} s &= \frac{7100 \pm \sqrt{(-7100)^2 - 4(13)(-420{,}000)}}{2(13)} \\ &= \frac{7100 \pm 8500}{26} \end{aligned}$$

$$s = 600 \qquad \text{or} \qquad s = \frac{-1400}{26} \approx -53.8$$

Since s represents speed, we reject the negative answer and conclude that the jet's speed from New York to Los Angeles was 600 km/h.

Now Try Exercise 69 ■

Example 9 ■ Energy Expended in Bird Flight

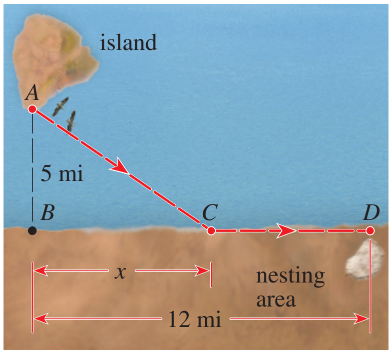


Figure 5

Ornithologists have determined that some species of birds tend to avoid flights over large bodies of water during daylight hours, because air generally rises over land and falls over water in the daytime, so flying over water requires more energy. A bird is released from point A on an island, 5 mi from B, the nearest point on a straight shoreline. The bird flies to a point C on the shoreline and then flies along the shoreline to its nesting area D, as shown in Figure 5. Suppose the bird has 170 kcal of energy in reserve. It uses 10 kcal/mi flying over land and 14 kcal/mi flying over water.

(a) Where should the point C be located so that the bird uses exactly 170 kcal of energy during its flight?

(b) Does the bird have enough energy in reserve to fly directly from A to D?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HYPATIA OF ALEXANDRIA (circa 350–370, died 415) was a Greek mathematician, astronomer, and philosopher. She is best known for her mathematical work on conic sections. She also edited the sevenvolume work *On Conics* by the Greek mathematician Apollonius of Perga (c. 240–c. 290 BC). Hypatia was the daughter of the prominent philosopher Theon, so she received an education rarely afforded girls in those days; she soon excelled as a mathematician and philosopher. Socrates Scholasticus, a church historian and contemporary of Hypatia, wrote that Hypatia "made such achievements in literature and science, as to far exceed all the philosophers of her own time." Hypatia was also a talented teacher; letters from her student Synesius (who later became the Bishop of Ptolemais) show his admiration for Hypatia's mathematical knowledge. Hypatia wrote about learning: "To understand the things that are at our door is the best preparation for understanding those that lie beyond." The 2009 film *Agora* is a historical drama about the life of Hypatia.

Solution

(a) **Identify the variable.** We are asked to find the location of C. So let

$$x = \text{distance from } B \text{ to } C$$

Translate from words to algebra. From the figure, and from the fact that

$$\text{energy used} = \text{energy per mile} \times \text{miles flown}$$

we determine the following:

In Words	In Algebra	
Distance from B to C	x	
Distance flown over water (from A to C)	$\sqrt{x^2 + 25}$	Pythagorean Theorem
Distance flown over land (from C to D)	$12 - x$	
Energy used over water	$14\sqrt{x^2 + 25}$	
Energy used over land	$10(12 - x)$	

Set up the model. Now we set up the model.

$$\boxed{\text{total energy used}} = \boxed{\text{energy used over water}} + \boxed{\text{energy used over land}}$$

$$170 = 14\sqrt{x^2 + 25} + 10(12 - x)$$

Solve. To solve this equation, we eliminate the square root by first bringing all other terms to the left of the equal sign and then squaring each side.

$$170 - 10(12 - x) = 14\sqrt{x^2 + 25} \qquad \text{Isolate square-root term on RHS}$$

$$50 + 10x = 14\sqrt{x^2 + 25} \qquad \text{Simplify LHS}$$

$$(50 + 10x)^2 = (14)^2(x^2 + 25) \qquad \text{Square each side}$$

$$2500 + 1000x + 100x^2 = 196x^2 + 4900 \qquad \text{Expand}$$

$$0 = 96x^2 - 1000x + 2400 \qquad \text{Move all terms to RHS}$$

This equation could be factored, but because the numbers are so large, it is easier to use the Quadratic Formula and a calculator.

$$x = \frac{1000 \pm \sqrt{(-1000)^2 - 4(96)(2400)}}{2(96)}$$

$$= \frac{1000 \pm 280}{192} = 6\tfrac{2}{3} \quad \text{or} \quad 3\tfrac{3}{4}$$

Point C should be either $6\frac{2}{3}$ mi or $3\frac{3}{4}$ mi from B so that the bird uses exactly 170 kcal of energy during its flight.

See Appendix A, *Geometry Review*, for the Pythagorean Theorem.

(b) By the Pythagorean Theorem the length of the route directly from A to D is $\sqrt{5^2 + 12^2} = 13$ mi, so the energy the bird requires for that route is $14 \times 13 = 182$ kcal. This is more energy than the bird has available, so it can't use this route.

Now Try Exercise 85 ■

1.7 Exercises

Concepts

1. Explain in your own words what it means for an equation to model a real-world situation, and give an example.

2. In the formula $I = Prt$ for simple interest, P stands for ________, r for ________, and t for ________.

3. Give a formula for the area of the geometric figure.

(a) A square of side x: $A =$ ________.

(b) A rectangle of length l and width w: $A =$ ________.

(c) A circle of radius r: $A =$ ________.

4. Balsamic vinegar contains 5% acetic acid, so a 32-oz bottle of balsamic vinegar contains ________ ounces of acetic acid.

5. A mason builds a wall in x hours, so the fraction of the wall that the mason builds in 1 hour is ________.

6. The formula $d = rt$ models the distance d traveled by an object moving at the constant rate r in time t. Find formulas for the following quantities.

$r =$ ________ $t =$ ________

Skills

7–20 ■ Using Variables Express the given quantity in terms of the indicated variable.

7. The sum of three consecutive integers; n = first integer of the three

8. The sum of three consecutive integers; n = middle integer of the three

9. The sum of three consecutive even integers; n = first integer of the three

10. The sum of the squares of two consecutive integers; n = first integer of the two

11. The average of three test scores if the first two scores are 78 and 82; s = third test score

12. The average of four quiz scores if each of the first three scores is 8; q = fourth quiz score

13. The interest obtained after 1 year on an investment at $2\frac{1}{2}\%$ simple interest per year; x = number of dollars invested

14. The total rent paid for an apartment if the rent is \$945 a month; n = number of months

15. The area (in ft^2) of a rectangle that is four times as long as it is wide; w = width of the rectangle (in ft)

16. The perimeter (in cm) of a rectangle that is 6 cm longer than it is wide; w = width of the rectangle (in cm)

17. The time (in hours) that it takes to travel a given distance at 55 mi/h; d = given distance (in mi)

18. The distance (in mi) that a car travels in 45 min; s = speed of the car (in mi/h)

19. The concentration (in oz/gal) of salt in a mixture of 3 gal of brine containing 25 oz of salt to which some pure water has been added; x = volume of pure water added (in gal)

20. The value (in cents) of the change in a purse that contains twice as many nickels as pennies, four more dimes than nickels, and as many quarters as dimes and nickels combined; p = number of pennies

Applications

21. Renting a Truck A rental company charges \$65 a day and 40 cents a mile for the rental of one of their trucks. A truck was rented for 3 days, and the total rental charge was \$283. How many miles was the truck driven?

22. Travel Cell Plan A cell phone company offers a travel plan for cell phone usage in countries outside the United States. The travel plan has a monthly fee of \$100 for the first 250 minutes of talk outside the United States, and \$0.25 for each additional minute (or part thereof). A tourist using this plan receives a bill of \$120.50 for the month of June. How many minutes of talk did the tourist use that month?

23. Average A student received scores of 82, 75, and 71 on their midterm algebra exams. If the final exam counts twice as much as a midterm, what score must the student make on their final exam to earn an average score of 80? (Assume that the maximum possible score on each test is 100.)

24. Average In a class of 25 students, the average score is 84. Six students in the class each received a maximum score of 100, and three students each received a score of 60. What is the average score of the remaining students?

25. Investments An amount of \$12,000 was invested in two accounts, each earning simple interest—one earned $2\frac{1}{2}\%$ per year and the other earned 3% per year. After one year the total interest earned on these investments was \$318. How much money was invested at each rate?

26. Investments If an amount of \$8000 is invested at a simple interest rate of $3\frac{1}{2}\%$ per year, how much additional money must be invested at a simple interest rate of 5% per year to ensure that the interest each year is 4% of the total amount invested?

27. Investments What annual rate of interest would you have to earn on an investment of \$3500 to ensure receiving \$262.50 interest after one year?

28. Investments A financial advisor invests \$3000 at a certain annual interest rate, and another \$5000 at an annual rate that is one-half percent higher. If the total interest earned in one year is \$265, at what interest rate is the \$3000 invested?

29. Salaries An executive in an engineering firm earns a monthly salary plus a Christmas bonus of \$7300. If the executive earns a total of \$180,100 per year, what is the monthly salary?

30. Salaries A factory foreman earns 15% more than their assistant. Together they make \$113,305 per year. What is the assistant's annual salary?

31. Overtime Pay A lab technician earns \$18.50 an hour and works 35 hours per week. For any hours worked more than this, the pay rate increases to $1\frac{1}{2}$ times the regular hourly rate for the overtime hours worked. If the technician's pay was \$814 for one week, how many overtime hours were worked that week?

32. Labor Cost A plumber and a cabinetmaker work together to remodel a kitchen. The plumber charges \$150 an hour for labor and the cabinetmaker charges \$80 an hour for labor. The cabinetmaker works nine times as long as the plumber on the remodeling job, and the labor charge on the final invoice is \$2610. How long did the cabinetmaker work on this job?

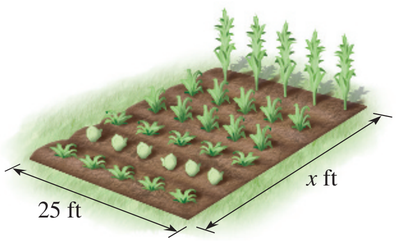
33. A Riddle A movie star, unwilling to reveal their age, posed the following riddle to a gossip columnist: "Seven years ago, I was eleven times as old as my daughter. Now I am four times as old as she is." How old is the movie star?

34. Career Home Runs During his major league career, Hank Aaron hit 41 more home runs than Babe Ruth hit during his career. Together they hit 1469 home runs. How many home runs did Babe Ruth hit?

35. Value of Coins A change purse contains an equal number of nickels, dimes, and quarters. The total value of the coins is \$2.80. How many coins of each type does the purse contain?

36. Value of Coins You have \$3.00 in nickels, dimes, and quarters. If you have twice as many dimes as quarters and five more nickels than dimes, how many coins of each type do you have?

37. Length of a Garden A rectangular garden is 25 ft wide. If its area is 1125 ft^2, what is the length of the garden?



38. Width of a Pasture A pasture is three times as long as it is wide. Its area is 132,300 ft^2. How wide is the pasture?

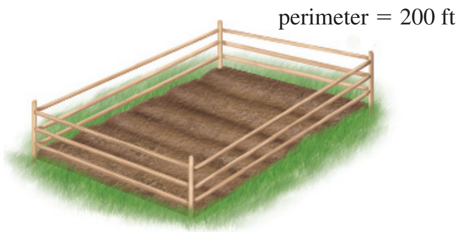
39. Dimensions of a Lot A half-acre building lot is five times as long as it is wide. What are its dimensions? [*Note:* 1 acre = 43,560 ft^2.]

40. Dimensions of a Lot A square plot of land has a building 60 ft long and 40 ft wide at one corner. The rest of the land outside the building forms a parking lot. If the parking lot has area 12,000 ft^2, what are the dimensions of the entire plot of land?

41. Dimensions of a Garden A rectangular garden is 30 ft longer than it is wide. Its area is 2800 ft^2. What are its dimensions?

42. Dimensions of a Room A rectangular bedroom is 5 ft longer than it is wide. Its area is 234 ft^2. What is the width of the room?

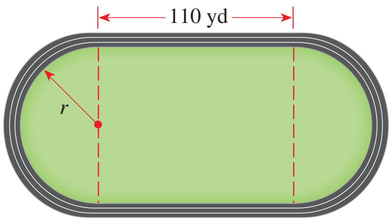
43. Dimensions of a Garden A farmer has a rectangular garden plot surrounded by 200 ft of fence. Find the length and width of the garden if its area is 2400 ft^2.



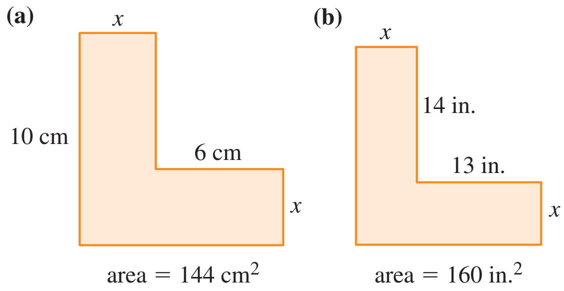
44. Dimensions of a Lot A parcel of land is 8 ft longer than it is wide. Each diagonal from one corner to the opposite corner is 232 ft long. What are the dimensions of the parcel?

45. Dimensions of a Lot A rectangular parcel of land is 50 ft wide. The length of a diagonal between opposite corners is 10 ft more than the length of the parcel. What is the length of the parcel?

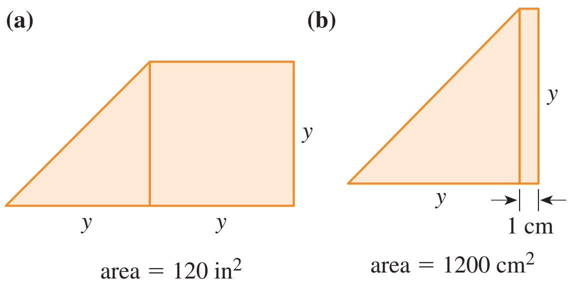
46. Dimensions of a Track A running track has the shape shown in the figure, with straight sides and semicircular ends. If the length of the track is 440 yd and the two straight parts are each 110 yd long, what is the radius of the semicircular parts (to the nearest yard)?



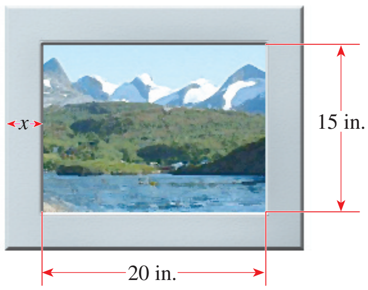
47. Length and Area Find the length x in each figure. The area of the shaded region is given.



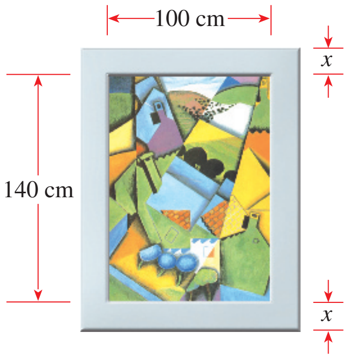
48. Length and Area Find the length y in each figure. The area of the shaded region is given.



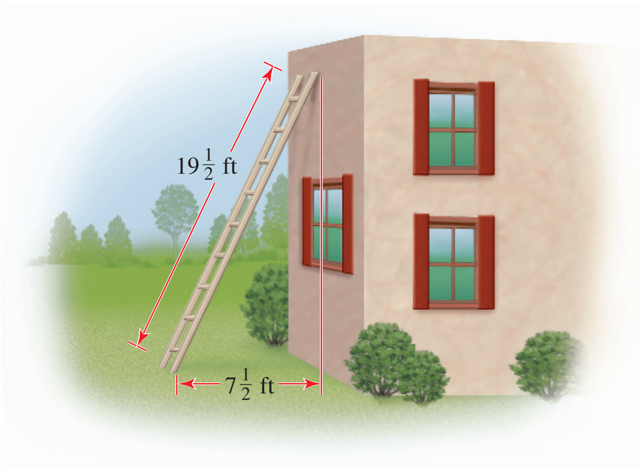
49. Framing a Painting An artist paints with watercolors on a sheet of paper 20 in. wide by 15 in. high. This sheet is then placed on a mat so that a uniformly wide strip of the mat shows all around the picture. The perimeter of the mat is 102 in. How wide is the strip of the mat showing around the picture?



50. Dimensions of a Poster A poster has a rectangular printed area 100 cm by 140 cm and a blank strip of uniform width around the edges. The perimeter of the poster is $1\frac{1}{2}$ times the perimeter of the printed area. What is the width of the blank strip?

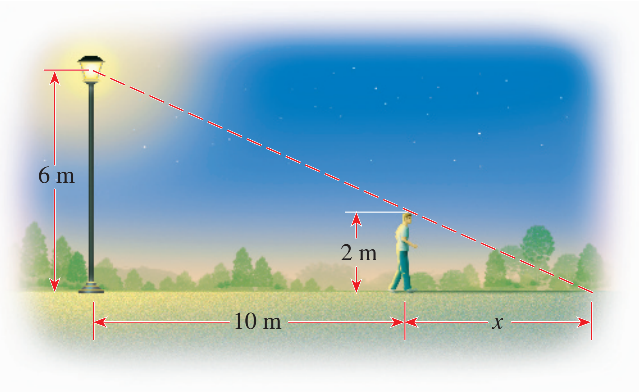


51. Reach of a Ladder A $19\frac{1}{2}$-foot ladder leans against a building. The base of the ladder is $7\frac{1}{2}$ ft from the building. How high up the building does the ladder reach?

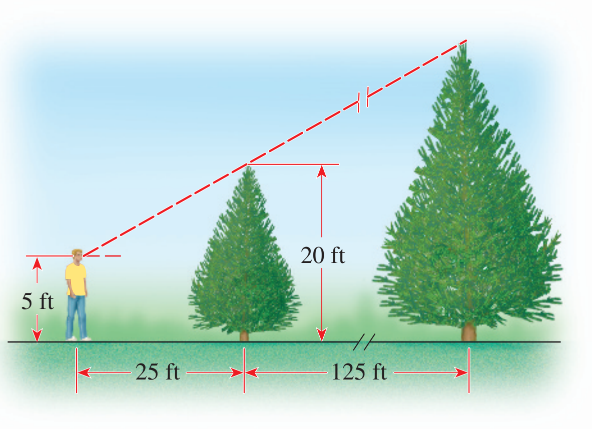


52. Height of a Flagpole A flagpole is secured on opposite sides by two guy wires, each of which is 5 ft longer than the pole. The distance between the points where the wires are fixed to the ground is equal to the length of one guy wire. How tall is the flagpole (to the nearest inch)?

53. Length of a Shadow A person is walking away from a lamppost with a light source 6 m above the ground. The person is 2 m tall. How long is the person's shadow when the person is 10 m from the lamppost? [*Hint:* Use similar triangles.]



54. Height of a Tree A woodcutter determines the height of a tall tree by first measuring the height of a smaller one, 125 ft away, then moving so that the tops of the two trees are in the same line of sight (see the figure). Suppose the small tree is 20 ft tall, the woodcutter is 25 ft from the small tree, and the woodcutter's eye level is 5 ft above the ground. How tall is the taller tree?



55. Mixture Problem What amount of a 60% acid solution must be mixed with a 30% solution to produce 300 mL of a 50% solution?

56. Mixture Problem What amount of pure acid must be added to 300 mL of a 50% acid solution to produce a 60% acid solution?

57. Mixture Problem A jeweler has five rings, each weighing 18 g, made of an alloy of 10% silver and 90% gold. The jeweler decides to melt down the rings and add enough silver to reduce the gold content to 75%. How much silver should be added?

58. Mixture Problem A pot contains 6 L of brine at a concentration of 120 g/L. How much of the water should be boiled off to increase the concentration to 200 g/L?

59. Mixture Problem The radiator in a car is filled with a solution of 60% antifreeze and 40% water. The manufacturer of the antifreeze suggests that for summer driving, optimal cooling of the engine is obtained with only 50% antifreeze. If the capacity of the radiator is 3.6 L, how much coolant should be drained and replaced with water to reduce the antifreeze concentration to the recommended level?

60. Mixture Problem A health clinic uses a solution of bleach to sterilize petri dishes in which cultures are grown. The sterilization tank contains 100 gal of a solution of 2% ordinary household bleach mixed with pure distilled water. New research indicates that the concentration of bleach should be 5% for complete sterilization. How much of the solution should be drained and replaced with bleach to increase the bleach content to the recommended level?

61. Mixture Problem A bottle contains 750 mL of fruit punch with a concentration of 50% pure fruit juice. A person drinks 100 mL of the punch and then refills the bottle with an equal amount of a cheaper brand of punch. If the concentration of juice in the bottle is now reduced to 48%, what was the concentration in the cheaper punch that was added?

62. Mixture Problem A merchant blends tea that sells for \$3.00 an ounce with tea that sells for \$2.75 an ounce to produce 80 oz of a mixture that sells for \$2.90 an ounce. How many ounces of each type of tea does the merchant use in the blend?

63. Sharing a Job Two friends work together to wash a car. If one takes 25 min to wash the car and the other takes 35 min, how long does it take the two friends when they work together?

64. Sharing a Job A landscaper and an assistant can mow a lawn in 10 min if they work together. If the landscaper works twice as fast as the assistant, how long does it take the assistant to mow the lawn alone?

65. Sharing a Job You and a friend have a summer job painting houses. Working together, you can paint a house in two-thirds the time it takes your friend to paint a house alone. If it takes you 7 h to paint a house alone, how long does it take your friend to paint a house alone?

66. Sharing a Job When a small-diameter hose and a large-diameter hose are used together to fill a swimming pool, it takes 16 h to fill the pool. The larger hose, used alone, takes 20% less time to fill the pool than the smaller hose used alone. How much time is required to fill the pool by each hose alone?

67. Sharing a Job You and your roommate can clean all the windows in your place in 1 h and 48 min. Working alone, it takes your roommate $1\frac{1}{2}$ h longer than it takes you to do the job. How long does it take each person working alone to wash all the windows?

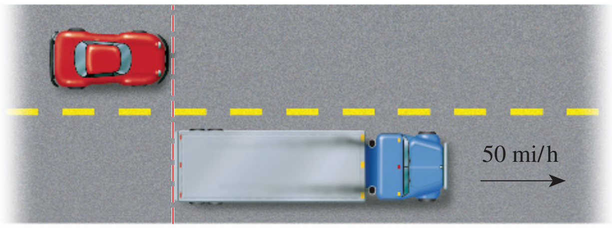
68. Sharing a Job You, your manager, and an assistant deliver advertising flyers in a small town. If you each work alone, it takes you 4 h to deliver all the flyers, and it takes the assistant 1 h longer than it takes your manager. Working together, you can deliver all the flyers in 40% of the time it takes your manager working alone. How long does it take your manager to deliver all the flyers alone?

69. Distance, Speed, and Time A commuter travels from Davenport to Omaha, a distance of 300 mi. The first part of the trip is traveled by bus, and the remainder is completed by train. The bus averages 40 mi/h and the train averages 60 mi/h. The entire trip takes $5\frac{1}{2}$ h. How long does the commuter spend on the train?

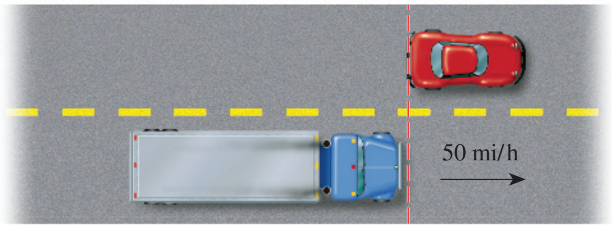
70. Distance, Speed, and Time Two cyclists, 90 mi apart, start riding toward each other at the same time. One cycles twice as fast as the other. If they meet 2 h later, what is the average speed at which each cyclist is traveling?

71. Distance, Speed, and Time A pilot flew a jet from Montreal to Los Angeles, a distance of 2500 mi. On the return trip, the average speed was 20% faster than the outbound speed. The round trip took 9 h 10 min. What was the speed from Montreal to Los Angeles?

72. Distance, Speed, and Time A 14-ft-long car is passing a 30-ft-long truck. The truck is traveling at 50 mi/h. How fast must the car be going so that it can pass the truck completely in 6 s, from the position shown in figure (a) to the position shown in figure (b)? [*Hint:* Use feet and seconds instead of miles and hours.]



(a)



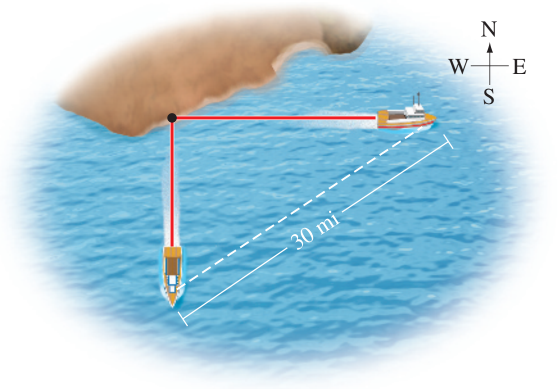
(b)

73. Distance, Speed, and Time A salesperson drives from Ajax to Berrington, a distance of 120 mi, and then from Berrington to Collins, a distance of 150 mi. The salesperson travels at a constant rate for the first leg of the trip and 10 mi/h faster for the second leg of the trip. If the second leg of the trip took 6 min longer than the first leg, how fast was the salesperson driving on the first leg of the trip?

74. Distance, Speed, and Time A trucker drove from Tortula to Dry Junction via Cactus. On the second leg of the trip, the trucker drove 10 mi/h faster than on the first leg. The distance from Tortula to Cactus is 250 mi and the distance from Cactus to Dry Junction is 360 mi. If the total trip took 11 h, what was the speed of the trucker from Tortula to Cactus?

75. Distance, Speed, and Time It took a crew 2 h 40 min to row 6 km upstream and back again. If the rate of flow of the stream was 3 km/h, what was the rowing speed of the crew in still water?

76. Speed of a Boat Two fishing boats depart a harbor at the same time, one traveling east, the other south. The eastbound boat travels at a speed 3 mi/h faster than the southbound boat. After 2 h the boats are 30 mi apart. Find the speed of the southbound boat.

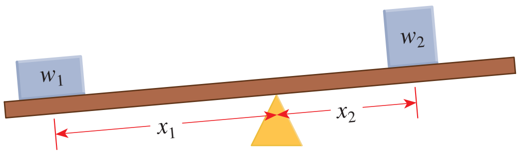


77. Law of the Lever The figure shows a lever system, similar to a seesaw that you might find in a children's playground. For the system to balance, the product of the weight and its distance from the fulcrum must be the same on each side; that is,

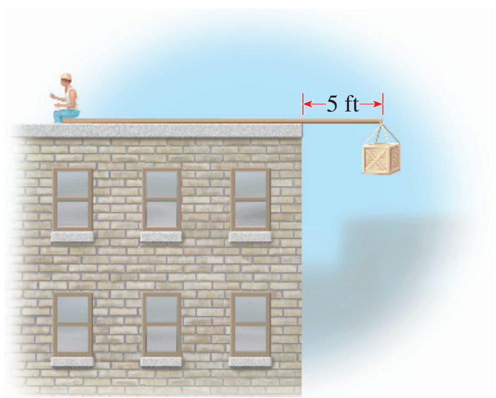
$$w_1x_1 = w_2x_2$$

This equation is called the **law of the lever** and was first discovered by Archimedes (see Section 10.1).

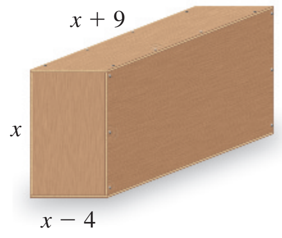
Two friends are playing on a seesaw. One weighs 100 pounds and the other 125 pounds. The 100-pound friend sits 8 ft from the fulcrum. If the see saw is to be balanced, where should the other friend sit?



78. Law of the Lever A plank 30 ft long rests on top of a flat-roofed building, with 5 ft of the plank projecting over the edge, as shown in the figure. A worker weighing 240 lb sits on one end of the plank. What is the largest weight that can be hung on the projecting end of the plank if it is to remain in balance? (Use the law of the lever stated in Exercise 77.)

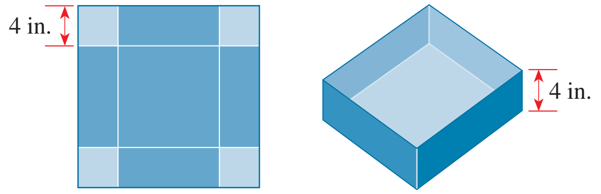


79. Dimensions of a Box A large plywood box has a volume of 180 ft^3. Its length is 9 ft greater than its height, and its width is 4 ft less than its height. What are the dimensions of the box?

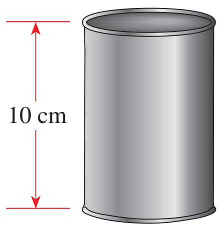


80. Radius of a Sphere A jeweler has three small solid spheres made of gold, of radius 2 mm, 3 mm, and 4 mm. The jeweler decides to melt these and make just one sphere out of them. What will the radius of this larger sphere be?

81. Dimensions of a Box A box with a square base and no top is to be made from a square piece of cardboard by cutting 4-inch squares from each corner and folding up the sides, as shown in the figure. The box is to hold 100 in^3. How big a piece of cardboard is needed?



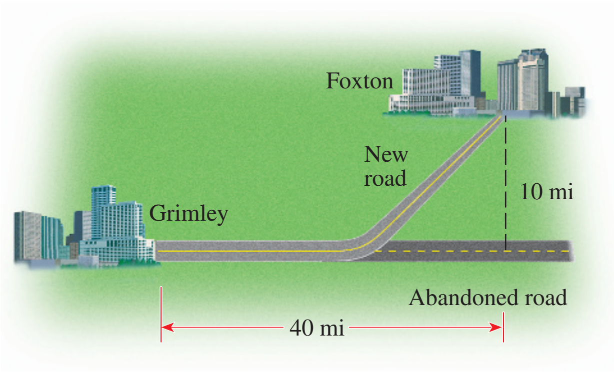
82. Dimensions of a Can A cylindrical can has a volume of 40π cm^3 and is 10 cm tall. What is its diameter? [*Hint:* Use the volume formula listed on the inside front cover of this book.]



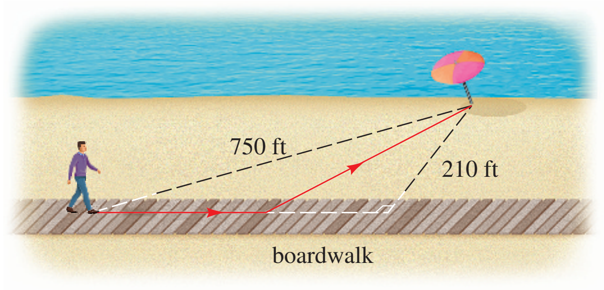
83. Radius of a Tank A spherical tank has a capacity of 750 gallons. Using the fact that one gallon is about 0.1337 ft^3, find the radius of the tank (to the nearest hundredth of a foot).

84. Dimensions of a Lot A city lot has the shape of a right triangle whose hypotenuse is 7 ft longer than one of the other sides. The perimeter of the lot is 392 ft. How long is each side of the lot?

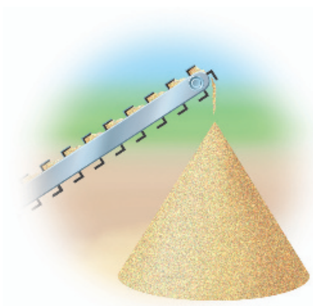
85. Construction Costs The town of Foxton lies 10 mi north of an abandoned east-west road that runs through Grimley, as shown in the figure. The point on the abandoned road closest to Foxton is 40 mi from Grimley. County officials are about to build a new road connecting the two towns. They have determined that restoring the old road would cost \$100,000 per mile, whereas building a new road would cost \$200,000 per mile. How much of the abandoned road should be used (as indicated in the figure) if the officials intend to spend exactly \$6.8 million? Would it cost less than this amount to build a new road connecting the towns directly?



86. Distance, Speed, and Time A boardwalk is parallel to and 210 ft inland from a straight shoreline. A sandy beach lies between the boardwalk and the shoreline. You are standing on the boardwalk, exactly 750 ft across the sand from your beach umbrella, which is right at the shoreline. You walk 4 ft/s on the boardwalk and 2 ft/s on the sand. How far should you walk on the boardwalk before veering off onto the sand in order to reach the umbrella in exactly 4 min 45 s?

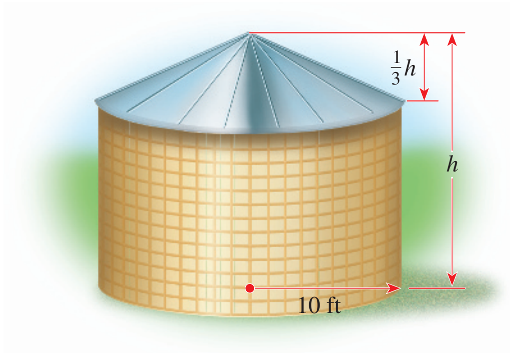


87. Volume of Grain Grain is spilling from a chute onto the ground, forming a conical pile whose diameter is always three times its height. How high is the pile (to the nearest hundredth of a foot) when it contains 1000 ft^3 of grain?

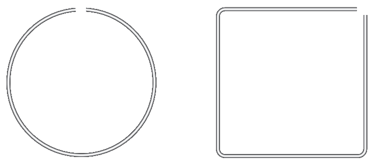


88. Computer Monitors Two computer monitors have the same screen height. One has a screen that is 7 in. wider than it is high. The other has a wider screen that is 1.8 times as wide as it is high. The diagonal measure of the wider screen is 3 in. more than the diagonal measure of the smaller screen. What is the height of the screens, correct to the nearest 0.1 in.?

89. Dimensions of a Structure A storage bin for corn consists of a cylindrical section made of wire mesh, surmounted by a conical tin roof, as shown in the figure. The height of the roof is one-third the height of the entire structure. If the total volume of the structure is 1400π ft^3 and its radius is 10 ft, what is its height? [*Hint:* Use the volume formulas listed on the inside front cover of this book.]



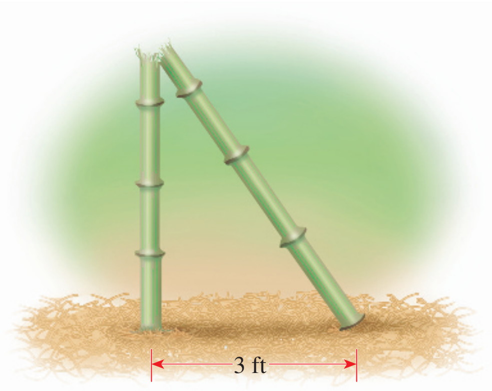
90. Comparing Areas A wire 360 in. long is cut into two pieces. One piece is formed into a square, and the other is formed into a circle. If the two figures have the same area, what are the lengths of the two pieces of wire (to the nearest tenth of an inch)?



91. An Ancient Chinese Problem This problem is taken from a Chinese mathematics textbook called *Chui-chang suan-shu*, or *Nine Chapters on the Mathematical Art*, which was written about 250 BC.

A 10-ft-long stem of bamboo is broken in such a way that its tip touches the ground 3 ft from the base of the stem, as shown in the figure. What is the height of the break?

[*Hint:* Use the Pythagorean Theorem.]



■ **Discuss** ■ **Discover** ■ **Prove** ■ **Write**

92. Write: Real-world Equations In this section we learned how to translate words into algebra. In this exercise we try to find real-world situations that could correspond to an algebraic equation. For instance, the equation $A = (x + y)/2$ could model the average amount of money in two bank accounts, where x represents the amount in one account and y the amount in the other. Write a story that could correspond to the given equation, stating what the variables represent.

(a) $C = 20{,}000 + 4.50x$

(b) $A = w(w + 10)$

(c) $C = 10.50x + 11.75y$

93. Discuss: A Babylonian Quadratic Equation The ancient Babylonians knew how to solve quadratic equations. Here is a problem from a cuneiform tablet found in a Babylonian school dating back to about 2000 BC.

> I have a reed, I know not its length. I broke from it one cubit, and it fit 60 times along the length of my field. I restored to the reed what I had broken off, and it fit 30 times along the width of my field. The area of my field is 375 square nindas. What was the original length of the reed?

Solve this problem. Use the fact that 1 ninda = 12 cubits.

1.8 Inequalities

■ Solving Linear Inequalities ■ Solving Nonlinear Inequalities ■ Absolute-Value Inequalities
■ Modeling with Inequalities

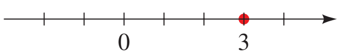
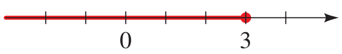
Some problems in algebra lead to **inequalities** instead of equations. An inequality looks just like an equation, except that in the place of the equal sign is one of the symbols $<$, $>$, $\le$, or $\ge$. Here is an example of an inequality:

$$4x + 7 \le 19$$

x	$4x + 7 \le 19$
1	$11 \le 19$ ✓
2	$15 \le 19$ ✓
3	$19 \le 19$ ✓
4	$23 \le 19$ ✗
5	$27 \le 19$ ✗

The table in the margin shows that some numbers satisfy the inequality and some numbers don't.

To **solve** an inequality that contains a variable means to find all values of the variable that make the inequality true. Unlike an equation, an inequality generally has infinitely many solutions, which form an interval or a union of intervals on the real line. The following illustration shows how an inequality differs from its corresponding equation:

		Solution	**Graph**
Equation:	$4x + 7 = 19$	$x = 3$	
Inequality:	$4x + 7 \le 19$	$x \le 3$	

To solve inequalities, we use the following rules to isolate the variable on one side of the inequality sign. These rules tell us when two inequalities are *equivalent* (the symbol $\Leftrightarrow$ means "is equivalent to"). In these rules the symbols A, B, and C stand for real numbers or algebraic expressions. Here we state the rules for inequalities involving the symbol $\le$, but they apply to all four inequality symbols.

Rules for Inequalities

Rule	Description
1. $A \le B \quad \Leftrightarrow \quad A + C \le B + C$	**Adding** the same quantity to each side of an inequality gives an equivalent inequality.
2. $A \le B \quad \Leftrightarrow \quad A - C \le B - C$	**Subtracting** the same quantity from each side of an inequality gives an equivalent inequality.
3. If $C > 0$, then $A \le B \quad \Leftrightarrow \quad CA \le CB$	**Multiplying** each side of an inequality by the same *positive* quantity gives an equivalent inequality.
4. If $C < 0$, then $A \le B \quad \Leftrightarrow \quad CA \ge CB$	**Multiplying** each side of an inequality by the same *negative* quantity *reverses the direction* of the inequality.
5. If $A > 0$ and $B > 0$, then $A \le B \quad \Leftrightarrow \quad \dfrac{1}{A} \ge \dfrac{1}{B}$	**Taking reciprocals** of each side of an inequality involving *positive* quantities *reverses the direction* of the inequality.
6. If $A \le B$ and $C \le D$, then $A + C \le B + D$	Inequalities can be added.
7. If $A \le B$ and $B \le C$, then $A \le C$	Inequality is transitive.

Pay special attention to Rules 3 and 4. Rule 3 says that we can multiply (or divide) each side of an inequality by a *positive* number, but Rule 4 says that if we multiply each side of an inequality by a *negative* number, then we reverse the direction of the inequality. For example, if we start with the inequality

$$3 < 5$$

and multiply by 2, we get

$$6 < 10$$

but if we multiply by -2, we get

$$-6 > -10$$

Solving Linear Inequalities

An inequality is **linear** if each term is constant or a multiple of the variable. To solve a linear inequality, we isolate the variable on one side of the inequality sign.

Example 1 ■ Solving a Linear Inequality

Solve the inequality $3x < 9x + 4$, and sketch the solution set.

Solution We isolate x on one side of the inequality sign.

$$
\begin{aligned}
3x &< 9x + 4 && \text{Given inequality}\\
3x - 9x &< 9x + 4 - 9x && \text{Subtract } 9x\\
-6x &< 4 && \text{Simplify}\\
\left(-\tfrac{1}{6}\right)(-6x) &> \left(-\tfrac{1}{6}\right)(4) && \text{Multiply by } -\tfrac{1}{6} \text{ and reverse inequality}\\
x &> -\tfrac{2}{3} && \text{Simplify}
\end{aligned}
$$

Multiplying by the negative number $-\frac{1}{6}$ *reverses* the direction of the inequality.

The solution set consists of all numbers greater than $-\frac{2}{3}$. In other words, the solution of the inequality is the interval $\left(-\frac{2}{3}, \infty\right)$. It is graphed in Figure 1.

Figure 1

Now Try Exercise 21 ■

Example 2 ■ Solving a Pair of Simultaneous Inequalities

Solve the inequalities $4 \leq 3x - 2 < 13$.

Solution The solution set consists of all values of x that satisfy both of the inequalities $4 \leq 3x - 2$ and $3x - 2 < 13$. Using Rules 1 and 3, we see that the following inequalities are equivalent:

$$
\begin{aligned}
4 &\leq 3x - 2 < 13 && \text{Given inequality} \\
6 &\leq 3x < 15 && \text{Add 2} \\
2 &\leq x < 5 && \text{Divide by 3}
\end{aligned}
$$

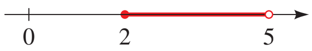


Figure 2

Therefore the solution set is $\{x \mid 2 \leq x < 5\} = [2, 5)$, as shown in Figure 2.

Now Try Exercise 31 ■

■ Solving Nonlinear Inequalities

To solve inequalities involving squares and other powers of the variable, we use factoring, together with the following principle.

The Sign of a Product or Quotient

- If a product or a quotient has an *even* number of *negative* factors, then its value is *positive*.
- If a product or a quotient has an *odd* number of *negative* factors, then its value is *negative*.

For example, to solve the inequality $x^2 \leq 5x - 6$, we first move all terms to the left-hand side and factor to get

$$(x - 2)(x - 3) \leq 0$$

This form of the inequality says that the product $(x - 2)(x - 3)$ must be negative or zero, so to solve the inequality, we must determine where each factor is negative or positive (because the sign of a product depends on the sign of the factors). The details are explained in Example 3, in which we use the following guidelines.

Guidelines for Solving Nonlinear Inequalities

1. **Move All Terms to One Side.** If necessary, rewrite the inequality so that all nonzero terms appear on one side of the inequality sign. If the nonzero side of the inequality involves quotients, bring them to a common denominator.
2. **Factor.** Factor the nonzero side of the inequality.
3. **Find the Intervals.** Determine the values for which each factor is zero. These numbers divide the real line into intervals. List the intervals that are determined by these numbers.
4. **Make a Table or Diagram.** Use **test values** to make a table or diagram of the signs of each factor on each interval. In the last row of the table determine the sign of the product (or quotient) of these factors.
5. **Solve.** Use the sign table to find the intervals on which the inequality is satisfied. Check whether the **endpoints** of these intervals satisfy the inequality. (This may happen if the inequality involves $\leq$ or $\geq$.)

The factoring technique that is described in these guidelines works only if all nonzero terms appear on one side of the inequality symbol. If the inequality is not written in this form, first rewrite it, as indicated in Step 1.

Example 3 ■ Solving a Quadratic Inequality

Solve the inequality $x^2 \leq 5x - 6$.

Solution We follow the preceding guidelines.

Move all terms to one side. We move all the terms to the left-hand side.

$$x^2 \leq 5x - 6 \qquad \text{Given inequality}$$

$$x^2 - 5x + 6 \leq 0 \qquad \text{Subtract } 5x \text{, add } 6$$

Factor. Factoring the left-hand side of the inequality, we get

$$(x - 2)(x - 3) \leq 0 \qquad \text{Factor}$$

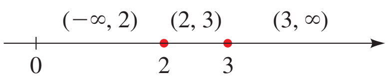


Figure 3

Find the intervals. The factors of the left-hand side are $x - 2$ and $x - 3$. These factors are zero when x is 2 and 3, respectively. As shown in Figure 3, the numbers 2 and 3 divide the real line into the three intervals

$$(-\infty, 2), \quad (2, 3), \quad (3, \infty)$$

The factors $x - 2$ and $x - 3$ change sign only at 2 and 3, respectively. So these factors maintain their sign on each of these three intervals.

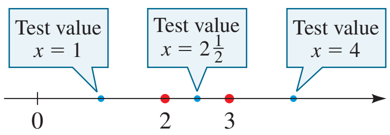


Figure 4

Make a table or diagram. To determine the sign of each factor on each of the intervals that we found, we use test values. We choose a number inside each interval and check the sign of the factors $x - 2$ and $x - 3$ at the number we have chosen. For the interval $(-\infty, 2)$, let's choose the test value 1 (see Figure 4). Substituting 1 for x in the factors $x - 2$ and $x - 3$, we get

$$x - 2 = 1 - 2 = -1 < 0$$

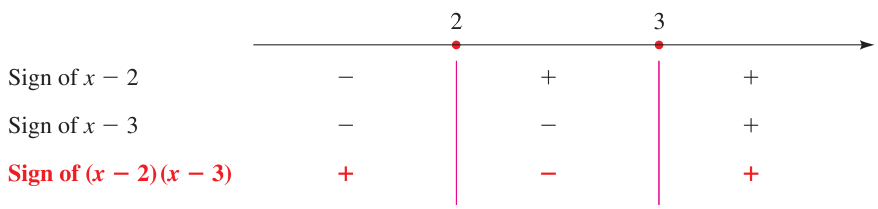
$$x - 3 = 1 - 3 = -2 < 0$$

So both factors are negative on this interval. Notice that we need to check only one test value for each interval because the factors $x - 2$ and $x - 3$ do not change sign on any of the three intervals we have found.

Using the test values $x = 2\frac{1}{2}$ and $x = 4$ for the intervals $(2, 3)$ and $(3, \infty)$ (see Figure 4), respectively, we construct the following sign table. The final row of the table is obtained from the fact that the expression in the last row is the product of the two factors.

Interval	$(-\infty, 2)$	$(2, 3)$	$(3, \infty)$
Sign of $x - 2$	$-$	$+$	$+$
Sign of $x - 3$	$-$	$-$	$+$
Sign of $(x - 2)(x - 3)$	$+$	$-$	$+$

If you prefer, you can represent this information on a real line, as in the following sign diagram. The vertical lines indicate the points at which the real line is divided into intervals:



Solve. We read from the table or the diagram that $(x-2)(x-3)$ is negative on the interval $(2, 3)$. You can check that the endpoints 2 and 3 satisfy the inequality, so the solution is

$$\{x \mid 2 \le x \le 3\} = [2, 3]$$

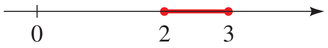


Figure 5

The solution is illustrated in Figure 5.

Now Try Exercise 43 ■

Example 4 ■ Solving an Inequality with Repeated Factors

Solve the inequality $x(x-1)^2(x-3) < 0$.

Solution All nonzero terms are already on one side of the inequality, and the nonzero side of the inequality is already factored. So we begin by finding the intervals for this inequality.

Find the intervals. The factors of the left-hand side are x, $(x-1)^2$, and $x-3$. These are zero when $x = 0, 1, 3$. These numbers divide the real line into the intervals

$$(-\infty, 0), \quad (0, 1), \quad (1, 3), \quad (3, \infty)$$

Make a diagram. We make the following diagram, using test points to determine the sign of each factor in each interval.

	$(-\infty, 0)$	$(0, 1)$	$(1, 3)$	$(3, \infty)$
Sign of x	$-$	$+$	$+$	$+$
Sign of $(x-1)^2$	$+$	$+$	$+$	$+$
Sign of $(x-3)$	$-$	$-$	$-$	$+$
Sign of $x(x-1)^2(x-3)$	**$+$**	**$-$**	**$-$**	**$+$**

Solve. From the diagram we see that the inequality is satisfied on the intervals $(0, 1)$ and $(1, 3)$. Since this inequality involves $<$, the endpoints of the intervals do not satisfy the inequality. So the solution set is the union of these two intervals:

$$\{x \mid 0 < x < 1 \quad \text{or} \quad 1 < x < 3\} = (0, 1) \cup (1, 3)$$

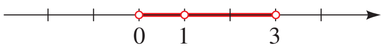


Figure 6

The solution set is graphed in Figure 6.

Now Try Exercise 55 ■

Example 5 ■ Solving an Inequality Involving a Quotient

Solve the inequality $\dfrac{1+x}{1-x} \ge 1$.

Solution **Move all terms to one side.** We move the terms to the left-hand side and simplify using a common denominator.

$$\frac{1+x}{1-x} \ge 1 \qquad \text{Given inequality}$$

$$\frac{1+x}{1-x} - 1 \ge 0 \qquad \text{Subtract 1}$$

$$\frac{1+x}{1-x} - \frac{1-x}{1-x} \ge 0 \qquad \text{Common denominator } 1-x$$

$$\frac{1+x-1+x}{1-x} \ge 0 \qquad \text{Combine the fractions}$$

$$\frac{2x}{1-x} \ge 0 \qquad \text{Simplify}$$

It is tempting to simply multiply both sides of the inequality by $1-x$ (as you would if this were an *equation*). But this doesn't work because we don't know whether $1-x$ is positive or negative, so we can't tell whether the inequality needs to be reversed. (See Exercise 135.)

Find the intervals. The factors of the left-hand side are $2x$ and $1 - x$. These are zero when x is 0 and 1. These numbers divide the real line into the intervals

$$(-\infty, 0), (0, 1), (1, \infty)$$

Make a diagram. We make the following diagram using test points to determine the sign of each factor in each interval.

		0		1	
Sign of $2x$	$-$		$+$		$+$
Sign of $1 - x$	$+$		$+$		$-$
Sign of $\dfrac{2x}{1-x}$	$-$		$+$		$-$

Figure 7

Solve. From the diagram we see that the inequality is satisfied on the interval $(0, 1)$. Checking the endpoints, we see that 0 satisfies the inequality but 1 does not (because the quotient in the inequality is not defined at 1). So the solution set is the interval $[0, 1)$. The solution set is graphed in Figure 7.

Now Try Exercise 63 ■

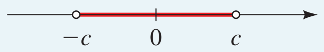
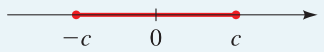
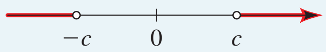
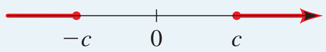
Example 5 shows that we should always check the endpoints of the solution set to see whether they satisfy the original inequality.

■ Absolute-Value Inequalities

The solutions of an absolute-value inequality like $|x| \le 5$ include both positive and negative numbers. For example, you can see that numbers like 0.5, -0.5, 1, -1, 3, -3 satisfy the inequality. In fact, the solution is the interval $[-5, 5]$. For the inequality $|x| \ge 5$, the solution contains numbers greater than 5 and their negatives (for example, 5.5, -5.5, 100, -100), so the solution is a union of intervals, $(-\infty, -5] \cup [5, \infty)$. In general we have the following properties.

These properties hold when x is replaced by any algebraic expression. (We assume that $c > 0$.)

Properties of Absolute-Value Inequalities

Inequality	Equivalent form	Graph
1. $\lvert x \rvert < c$	$-c < x < c$	
2. $\lvert x \rvert \le c$	$-c \le x \le c$	
3. $\lvert x \rvert > c$	$x < -c$ or $c < x$	
4. $\lvert x \rvert \ge c$	$x \le -c$ or $c \le x$	

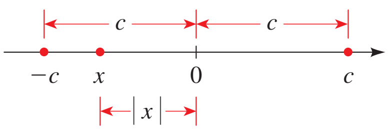


Figure 8

These properties can be proved using the definition of absolute-value. To prove Property 1, for example, note that the inequality $|x| < c$ says that the distance from x to 0 is less than c, and from Figure 8 you can see that this is true if and only if x is between $-c$ and c.

Example 6 ■ Solving an Absolute Value Inequality

Solve the inequality $|x - 5| < 2$.

Solution 1 The inequality $|x - 5| < 2$ is equivalent to

$$-2 < x - 5 < 2 \qquad \text{Property 1}$$

$$3 < x < 7 \qquad \text{Add 5}$$

The solution set is the open interval $(3, 7)$.

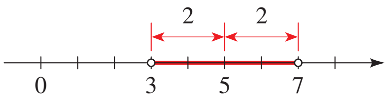


Figure 9

Solution 2 Geometrically, the solution set consists of all numbers x whose distance from 5 is less than 2. From Figure 9 we see that this is the interval $(3, 7)$.

Now Try Exercise 79 ■

Example 7 ■ Solving an Absolute Value Inequality

Solve the inequality $|3x + 2| \geq 4$.

Solution By Property 4 the inequality $|3x + 2| \geq 4$ is equivalent to

$$3x + 2 \geq 4 \quad \text{or} \quad 3x + 2 \leq -4$$
$$3x \geq 2 \qquad 3x \leq -6 \qquad \text{Subtract 2}$$
$$x \geq \tfrac{2}{3} \qquad x \leq -2 \qquad \text{Divide by 3}$$

So the solution set is

$$\{x \mid x \leq -2 \quad \text{or} \quad x \geq \tfrac{2}{3}\} = (-\infty, -2] \cup [\tfrac{2}{3}, \infty)$$

The set is graphed in Figure 10.

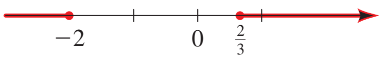


Figure 10

Now Try Exercise 81 ■

■ Modeling with Inequalities

Modeling real-life problems frequently leads to inequalities because we are often interested in determining when one quantity is more (or less) than another.

Example 8 ■ Truck Rental

Two truck rental companies offer the following pricing plans.

Company A: \$19 a day and \$0.40 per mile

Company B: \$68 a day and \$0.26 per mile

For a one-day rental, how many miles would you have to drive so that renting from Company B is less expensive than renting from Company A?

Solution **Identify the variable.** We are asked to find the number of miles you would have to drive so that renting from Company B is less expensive than renting from Company A. So let

$$x = \text{number of miles}$$

Translate from words to algebra. The information in the problem may be organized as follows:

In Words	In Algebra
Number of miles	x
Rental cost with Company A	$19 + 0.40x$
Rental cost with Company B	$68 + 0.26x$

Set up the model. Now we set up the model.

$$\boxed{\text{Rental cost with Company B}} < \boxed{\text{Rental cost with Company A}}$$

$$68 + 0.26x < 19 + 0.40x$$

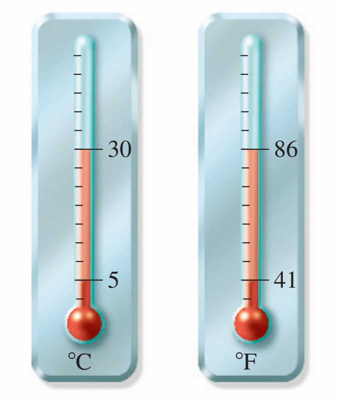
Solve. We solve for x.

$$49 + 0.26x < 0.40x \qquad \text{Subtract 19}$$
$$49 < 0.14x \qquad \text{Subtract } 0.26x$$
$$350 < x \qquad \text{Divide by } 0.14x$$

If you plan to drive *more than* 350 miles, renting from Company B would be less expensive.

Now Try Exercise 119 ■

Example 9 ■ Relationship Between Fahrenheit and Celsius Scales



The instructions on a bottle of medicine indicate that the bottle should be stored at a temperature between 5 °C and 30 °C. What range of temperatures does this correspond to on the Fahrenheit scale?

Solution The relationship between degrees Celsius (C) and degrees Fahrenheit (F) is given by the equation $C = \frac{5}{9}(F - 32)$. Expressing the statement on the bottle in terms of inequalities, we have

$$5 < C < 30$$

So the corresponding Fahrenheit temperatures satisfy the inequalities

$$5 < \tfrac{5}{9}(F - 32) < 30 \qquad \text{Substitute } C = \tfrac{5}{9}(F - 32)$$
$$\tfrac{9}{5}\cdot 5 < F - 32 < \tfrac{9}{5}\cdot 30 \qquad \text{Multiply by } \tfrac{9}{5}$$
$$9 < F - 32 < 54 \qquad \text{Simplify}$$
$$9 + 32 < F < 54 + 32 \qquad \text{Add 32}$$
$$41 < F < 86 \qquad \text{Simplify}$$

The medicine should be stored at a temperature between 41°F and 86 °F.

Now Try Exercise 117 ■

1.8 Exercises

Concepts

1. Fill in each blank with an appropriate inequality sign.

(a) If $x < 5$, then $x - 3$ ______ 2.

(b) If $x \le 5$, then $3x$ ______ 15.

(c) If $x \ge 2$, then $-3x$ ______ -6.

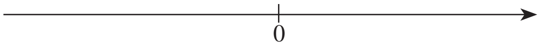
(d) If $x < -2$, then $-x$ ______ 2.

2. To solve the nonlinear inequality $\dfrac{x + 1}{x - 2} \le 0$, we first observe that the numbers ____ and ____ are zeros of the numerator and denominator. These numbers divide the real line into three intervals. Complete the table.

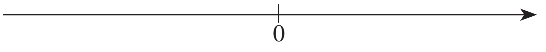
Interval			
Sign of $x + 1$ **Sign of $x - 2$**			
Sign of $(x + 1)/(x - 2)$			

Do any of the endpoints fail to satisfy the inequality? If so, which one(s)? ________. The solution of the inequality is ____________.

3. (a) Find three positive and three negative numbers that satisfy the inequality $|x| \le 3$, then express the solution as an interval: ______________. Graph the numbers you found and the solution:



(b) Find three positive and three negative numbers that satisfy the inequality $|x| \ge 3$, then express the solution as a union of intervals: ________ $\cup$ ________. Graph the numbers you found and the solution:



4. (a) The set of all points on the real line whose distance from zero is less than 3 can be described by the absolute value inequality $|x|$ ________.

(b) The set of all points on the real line whose distance from zero is greater than 3 can be described by the absolute value inequality $|x|$ ________.

5. *Yes or No*? If *No*, give an example.

(a) If $x(x + 1) > 0$, does it follow that x is positive?

(b) If $x(x + 1) > 5$, does it follow that $x > 5$?

6. What is a logical first step in solving the inequality?

(a) $3x \le 7$ **(b)** $5x - 2 \ge 1$ **(c)** $|3x + 2| \le 8$

Skills

7–12 ■ Solutions? Let $S = \{-5, -1, 0, \frac{2}{3}, \frac{5}{6}, 1, \sqrt{5}, 3, 5\}$. Determine which elements of S satisfy the inequality.

7. $-2 + 3x \ge \frac{1}{3}$ **8.** $1 - 2x \ge 5x$

9. $1 < 2x - 4 \le 7$ **10.** $-2 \le 3 - x < 2$

11. $\dfrac{1}{x} \le \dfrac{1}{2}$ **12.** $x^2 + 2 < 4$

13–36 ■ Linear Inequalities Solve the linear inequality. Express the solution using interval notation, and graph the solution set.

13. $2x \le 7$ **14.** $-4x \ge 10$

15. $2x - 5 > 3$ **16.** $3x + 11 < 5$

17. $7 - x \ge 5$ **18.** $5 - 3x \le -16$

19. $2x + 1 < 0$ **20.** $0 < 4x - 8$

21. $4x - 7 < 8 + 9x$ **22.** $5 - 3x \ge 8x - 7$

23. $x - \frac{3}{2} > \frac{1}{2}x$ **24.** $\frac{2}{5}x + 1 < \frac{1}{5} - 2x$

25. $\frac{1}{3}x + 2 < \frac{1}{6}x - 1$ **26.** $\frac{2}{3} - \frac{1}{2}x \ge \frac{1}{6} + x$

27. $4 - 3x \le -(1 + 8x)$ **28.** $2(7x - 3) \le 12x + 16$

29. $2 \le x + 5 < 4$ **30.** $-8 \le x - 3 \le 12$

31. $-1 < 2x - 5 < 7$ **32.** $1 < 3x + 4 \le 16$

33. $-2 < 8 - 2x \le -1$ **34.** $-3 \le 3x + 7 \le \frac{1}{2}$

35. $\dfrac{1}{6} < \dfrac{2x - 13}{12} \le \dfrac{2}{3}$ **36.** $-\dfrac{1}{2} \le \dfrac{4 - 3x}{5} \le \dfrac{1}{4}$

37–60 ■ Nonlinear Inequalities Solve the nonlinear inequality. Express the solution using interval notation, and graph the solution set.

37. $(x + 2)(x - 3) < 0$ **38.** $(x - 5)(x + 4) \ge 0$

39. $x(2x + 7) \ge 0$ **40.** $x(2 - 3x) \le 0$

41. $x^2 - 3x - 18 \le 0$ **42.** $x^2 - 8x + 7 > 0$

43. $3x^2 + 5x \ge 2$ **44.** $x^2 < x + 2$

45. $3x^2 - 3x < 2x^2 + 4$ **46.** $5x^2 + 3x \ge 3x^2 + 2$

47. $x^2 > 3(x + 6)$ **48.** $x^2 + 2x > 3$

49. $x^2 < 4$ **50.** $x^2 \ge 9$

51. $(x + 2)(x - 1)(x - 3) \le 0$

52. $(x - 5)(x - 2)(x + 1) > 0$

53. $(x - 4)(x + 2)^2 < 0$

54. $(x - 4)(x + 2)^2 > 0$

55. $(x + 3)^2(x - 2)(x + 5) \ge 0$

56. $4x^2(x^2 - 9) \le 0$

57. $x^3 - 4x > 0$ **58.** $9x \le x^3$

59. $x^4 > x^2$ **60.** $x^5 > x^2$

61–72 ■ Inequalities Involving Quotients Solve the nonlinear inequality. Express the solution using interval notation, and graph the solution set.

61. $\dfrac{x - 3}{x + 1} \ge 0$ **62.** $\dfrac{2x + 6}{x - 2} < 0$

63. $\dfrac{x}{x - 2} > 5$ **64.** $\dfrac{x - 4}{2x + 1} < 5$

65. $\dfrac{2x + 1}{x - 5} \le 3$ **66.** $\dfrac{3 + x}{3 - x} \ge 1$

67. $\dfrac{4}{x} < x$ **68.** $\dfrac{x}{x + 1} > 3x$

69. $1 + \dfrac{2}{x + 1} \le \dfrac{2}{x}$ **70.** $\dfrac{3}{x - 1} - \dfrac{4}{x} \ge 1$

71. $\dfrac{x + 2}{x + 3} < \dfrac{x - 1}{x - 2}$ **72.** $\dfrac{1}{x + 1} + \dfrac{1}{x + 2} \le 0$

73–88 ■ Absolute-Value Inequalities Solve the absolute-value inequality. Express the answer using interval notation, and graph the solution set.

73. $|5x| < 20$ **74.** $|16x| \le 8$

75. $|2x| > 7$ **76.** $\frac{1}{2}|x| \ge 1$

77. $|x - 3| \le 10$ **78.** $|x + 1| \ge 1$

79. $|3x + 2| < 4$ **80.** $|5x - 2| < 8$

81. $|3x - 2| \ge 5$ **82.** $|3x - 4| \ge 5$

83. $\left|\dfrac{x - 2}{3}\right| < 2$ **84.** $\left|\dfrac{x + 1}{2}\right| \ge 4$

85. $|x + 6| < 0.001$ **86.** $3 - |2x + 4| \le 1$

87. $8 - |2x - 1| \ge 6$ **88.** $7|x + 2| + 5 > 4$

89–98 ■ Putting It All Together Recognize the type of inequality and solve the inequality by an appropriate method. Express the answer using interval notation, and graph the solution set.

89. $0 < 5 - 2x$ **90.** $|x - 5| \le 3$

91. $16x \le x^3$ **92.** $\dfrac{6}{x - 1} - \dfrac{6}{x} \ge 1$

93. $x^2 + 5x + 6 > 0$ **94.** $5 \le 3x - 4 \le 14$

95. $(x + 3)^2(x + 1) > 0$ **96.** $\dfrac{x}{x + 1} > 3$

97. $1 \ge 3 - 2x \ge -5$ **98.** $|8x + 3| > 12$

99–102 ■ Absolute Value Inequalities A phrase describing a set of real numbers is given. Express the phrase as an inequality involving an absolute value.

99. All real numbers x less than 3 units from 0

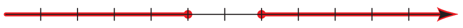
100. All real numbers x more than 2 units from 0

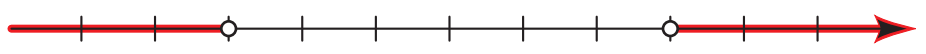
101. All real numbers x at least 5 units from 7

102. All real numbers x at most 4 units from 2

103–108 ■ Absolute Value Inequalities A set of real numbers is graphed. Find an inequality involving an absolute value that describes the set.

103. 

104. 

105. 

106. 

107. 

108. 

109–112 ■ Domain Determine the values of the variable for which the expression is defined as a real number.

109. $\sqrt{x^2 - 9}$

110. $\sqrt{x^2 - 5x - 50}$

111. $\left(\dfrac{1}{x^2 - 3x - 10}\right)^{1/2}$

112. $\sqrt[4]{\dfrac{1 - x}{2 + x}}$

Skills Plus

113–116 ■ Inequalities Solve the inequality for x. Assume that a, b, and c are positive constants.

113. $a(bx - c) \geq bc$

114. $a \leq bx + c < 2a$

115. $a|bx - c| + d \geq 4a$

116. $\left|\dfrac{bx + c}{a}\right| > 5a$

Applications

117. Temperature Scales Use the relationship between C and F given in Example 9 to find the interval on the Fahrenheit scale corresponding to the temperature range $20 \leq C \leq 30$.

118. Temperature Scales What interval on the Celsius scale corresponds to the temperature range $50 \leq F \leq 95$?

119. RV Rental Cost An RV rental company offers two plans for renting an RV.

Plan A: $95 per day and $0.40 per mile
Plan B: $135 per day with free unlimited mileage

For what range of miles will Plan B save you money?

120. International Plans A phone service provider offers two international plans.

Plan A: $25 per month and 5¢ per minute
Plan B: $5 per month and 12¢ per minute

For what range of minutes of international calls would Plan B be financially advantageous?

121. Driving Cost It is estimated that the annual cost of driving a certain new car is given by the formula

$$C = 0.35m + 2200$$

where m represents the number of miles driven per year and C is the cost in dollars. A teacher has purchased such a car and decides to budget between $6400 and $7100 for next year's driving costs. What is the corresponding range of miles that can be driven within this budget?

122. Temperature and Elevation As dry air moves upward, it expands and, in so doing, cools at a rate of about 1°C for each 100-m rise, up to about 12 km.

(a) If the ground temperature is 20°C, write a formula for the temperature at elevation h.

(b) What range of temperatures can be expected if a plane takes off and reaches an elevation of 5 km?

123. Airline Ticket Price A charter airline finds that on its Saturday flights from Philadelphia to London all 120 seats will be sold if the ticket price is $200. However, for each $3 increase in ticket price, the number of seats sold decreases by one.

(a) Find a formula for the number of seats sold if the ticket price is P dollars.

(b) Over a certain period the number of seats sold for this flight ranged between 90 and 115. What was the corresponding range of ticket prices?

124. Accuracy of a Scale A coffee merchant sells a customer 3 lb of Hawaiian Kona at $6.50 per pound. The merchant's scale is accurate to within ±0.03 lb. By how much could the customer have been overcharged or undercharged because of possible inaccuracy in the scale?

125. Gravity The gravitational force F exerted by the earth on an object having a mass of 100 kg is given by the equation

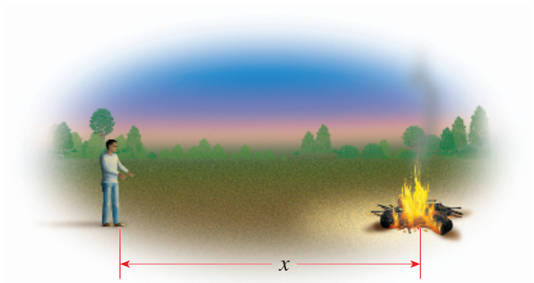
$$F = \frac{4{,}000{,}000}{d^2}$$

where d is the distance (in km) of the object from the center of the earth and the force F is measured in newtons (N). For what distances will the gravitational force exerted by the earth on this object be between 0.0004 N and 0.01 N?

126. Bonfire Temperature In the vicinity of a bonfire the temperature T in °C at a distance of x meters from the center of the fire was given by

$$T = \frac{600{,}000}{x^2 + 300}$$

At what range of distances from the fire's center was the temperature less than 500°C?



127. Quarter-Mile Time Performance cars are often compared using their quarter-mile time, that is, the time it takes to travel a quarter mile from a standing start. In a quarter-mile run the acceleration a is approximately constant, so the distance d the car travels in time t is

$$d = \tfrac{1}{2}at^2$$

where d is measured in feet, a in ft/s^2, and t in seconds.

(a) Find the values of acceleration that result in a quarter-mile time of less than 10 seconds. (One quarter mile equals 1320 ft.)

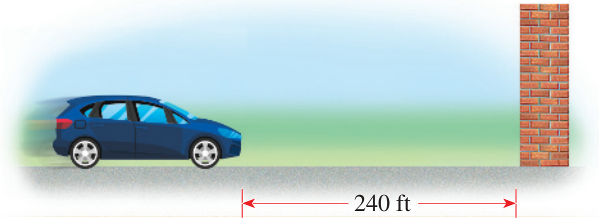
(b) Find the quarter-mile time for a car in free fall (acceleration $g = 32$ ft/s^2).

128. Gas Mileage The gas mileage g (measured in mi/gal) for a particular vehicle, driven at v mi/h, is given by the formula $g = 10 + 0.9v - 0.01v^2$, as long as v is between 10 mi/h and 75 mi/h. For what range of speeds is the vehicle's mileage 30 mi/gal or better?

129. Stopping Distance For a certain model of car the distance d required to stop the vehicle if it is traveling at v mi/h is given by the formula

$$d = v + \frac{v^2}{20}$$

where d is measured in feet. You want your stopping distance not to exceed 240 ft. At what range of speeds can you travel?



130. Manufacturer's Profit If a manufacturer sells x units of a certain product, revenue R and cost C (in dollars) are given by

$$R = 20x$$
$$C = 2000 + 8x + 0.0025x^2$$

Use the fact that

$$\text{profit} = \text{revenue} - \text{cost}$$

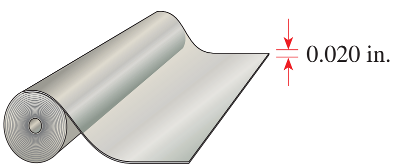
to determine how many units the manufacturer should sell to enjoy a profit of at least \$2400.

131. Fencing a Garden A determined gardener has 120 ft of deer-resistant fence to enclose a rectangular vegetable garden with area at least 800 ft^2. What range of values is possible for the length of the garden?

132. Thickness of a Laminate A company manufactures industrial laminates (thin nylon-based sheets) of thickness 0.020 in. with a tolerance of 0.003 in.

(a) Find an inequality involving absolute values that describes the range of possible thickness for the laminate.

(b) Solve the inequality you found in part (a).



133. Range of Height The average height of adult males is 68.2 in., and 95% of adult males have height h that satisfies the inequality

$$\left| \frac{h - 68.2}{2.9} \right| \le 2$$

Solve the inequality to find the range of heights.

Discuss ■ Discover ■ Prove ■ Write

134. Discuss ■ Discover: Do Powers Preserve Order? If $a < b$, is $a^2 < b^2$? (Check both positive and negative values for a and b.) If $a < b$, is $a^3 < b^3$? On the basis of your observations, state a general rule about the relationship between a^n and b^n when $a < b$ and n is a positive integer.

135. Discuss ■ Discover: What's Wrong Here? It is tempting to try to solve an inequality like an equation. For instance, we might try to solve $1 < 3/x$ by multiplying both sides by x to get $x < 3$, so the solution would be $(-\infty, 3)$. But that's wrong; for example, $x = -1$ lies in this interval but does not satisfy the original inequality. Explain why this method doesn't work (think about the *sign* of x). Then solve the inequality correctly.

136. Discuss ■ Discover: Using Distances to Solve Absolute Value Inequalities Recall that $|a - b|$ is the distance between a and b on the number line. For any number x, what do $|x - 1|$ and $|x - 3|$ represent? Use this interpretation to solve the inequality $|x - 1| < |x - 3|$ geometrically. In general, if $a < b$, what is the solution of the inequality $|x - a| < |x - b|$?

137–138 ■ Prove: Inequalities Use the rules for inequalities to prove the following.

137. Rule 6 for Inequalities: If a, b, c, and d are any real numbers such that $a < b$ and $c < d$, then $a + c < b + d$. [*Hint:* Use Rule 1 to show that $a + c < b + c$ and $b + c < b + d$. Use Rule 7.]

138. If a, b, c, and d are positive numbers such that $\frac{a}{b} < \frac{c}{d}$, then $\frac{a}{b} < \frac{a + c}{b + d} < \frac{c}{d}$.

PS *Establish subgoals.* First show that $\frac{ad}{b} + a < c + a$ and $a + c < \frac{cb}{d} + c$.

139. Prove: Arithmetic-Geometric Mean Inequality If $a_1, a_2, \ldots, a_n$ are nonnegative numbers, then their arithmetic mean is $\frac{a_1 + a_2 + \cdots + a_n}{n}$, and their geometric mean is $\sqrt[n]{a_1 a_2 \cdot \cdots \cdot a_n}$. The arithmetic-geometric mean inequality states that the geometric mean is always less than or equal to the arithmetic mean. In this problem we prove this in the case of two numbers x and y.

(a) Show that if x and y are nonnegative and $x \le y$, then $x^2 \le y^2$. [*Hint:* First use Rule 3 of Inequalities to show that $x^2 \le xy$ and $xy \le y^2$.]

(b) Prove the arithmetic-geometric mean inequality

$$\sqrt{xy} \le \frac{x + y}{2}$$

1.9 The Coordinate Plane; Graphs of Equations; Circles

■ The Coordinate Plane ■ The Distance and Midpoint Formulas ■ Graphs of Equations in Two Variables ■ Intercepts ■ Circles ■ Symmetry

The *coordinate plane* is the link between algebra and geometry. In the coordinate plane we can draw graphs of algebraic equations. The graphs, in turn, allow us to "see" the relationship between the variables in the equation.

■ The Coordinate Plane

The Cartesian plane is named in honor of the French mathematician René Descartes (1596–1650), although another Frenchman, Pierre Fermat (1607–1665), also invented the principles of coordinate geometry at the same time. (See their biographies in Section 1.11 and Section 2.6.)

Just as points on a line can be identified with real numbers to form the coordinate line, points in a plane can be identified with *ordered pairs* of numbers to form the **coordinate plane** or **Cartesian plane**. To do this, we draw two perpendicular real lines that intersect at 0 on each line. Usually, one line is horizontal with positive direction to the right and is called the ***x*-axis**; the other line is vertical with positive direction upward and is called the ***y*-axis**. The point of intersection of the x-axis and the y-axis is the **origin *O***, and the two axes divide the plane into four **quadrants**, labeled I, II, III, and IV in Figure 1. (The points that lie *on* the coordinate axes are not assigned to any quadrant.)

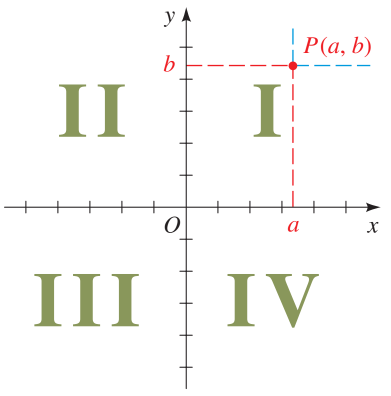


Figure 1

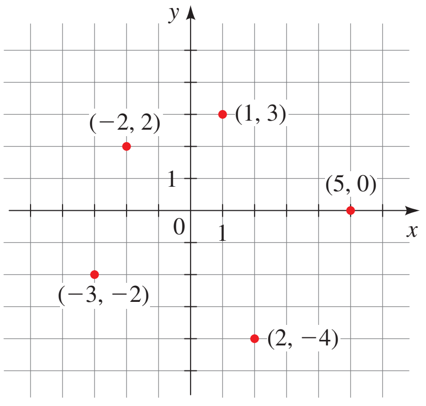


Figure 2

Any point P in the coordinate plane can be located by a unique **ordered pair** of numbers (a, b), as shown in Figure 1. The first number a is called the ***x*-coordinate** of P; the second number b is called the ***y*-coordinate** of P. Several points are labeled with their coordinates in Figure 2. We can think of the ordered pair (a, b) as the "address" of the point P because (a, b) specifies the location of the point P in the plane. We often refer to the ordered pair (a, b) as the point (a, b).

Although the notation for an ordered pair (a, b) is the same as the notation for an open interval (a, b), the context should make clear which meaning is intended.

Example 1 ■ Graphing Regions in the Coordinate Plane

Describe and sketch the regions given by each set.

(a) $\{(x, y) \mid x \geq 0\}$ **(b)** $\{(x, y) \mid y = 1\}$ **(c)** $\{(x, y) \mid |y| < 1\}$

Solution

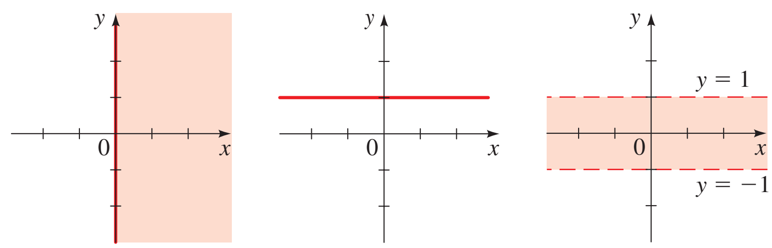
(a) The points whose x-coordinates are 0 or positive lie on the y-axis or to the right of it, as shown in Figure 3(a).

(b) The set of all points with y-coordinate 1 is a horizontal line one unit above the x-axis, as shown in Figure 3(b).

(c) Recall from Section 1.8 that

$$|y| < 1 \qquad \text{if and only if} \qquad -1 < y < 1$$

So the given region consists of those points in the plane whose y-coordinates lie between -1 and 1. Thus the region consists of all points that lie between (but not on) the horizontal lines $y = 1$ and $y = -1$. These lines are shown as broken lines in Figure 3(c) to indicate that the points on these lines are not included in the set.



(a) $x \geq 0$ (b) $y = 1$ (c) $|y| < 1$

Figure 3

Now Try Exercises 15 and 17 ■

■ The Distance and Midpoint Formulas

We now find a formula for the distance $d(A, B)$ between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ in the plane. Recall from Section 1.1 that the distance between points a and b on a number line is $d(a, b) = |b - a|$. So from Figure 4 we see that the distance between the points $A(x_1, y_1)$ and $C(x_2, y_1)$ on a horizontal line must be $|x_2 - x_1|$, and the distance between $B(x_2, y_2)$ and $C(x_2, y_1)$ on a vertical line must be $|y_2 - y_1|$.

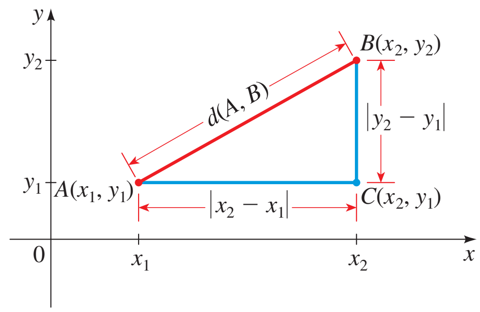


Figure 4

Since triangle ABC is a right triangle, the Pythagorean Theorem gives

$$d(A, B) = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Distance Formula

The distance between the points $A(x_1, y_1)$ and $B(x_2, y_2)$ in the plane is

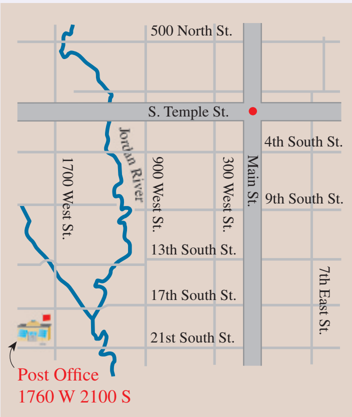
$$d(A, B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Coordinates as Addresses

The coordinates of a point in the xy-plane uniquely determine its location. We can think of the coordinates as the "address" of the point. In Salt Lake City, Utah, the addresses of most buildings are in fact expressed as coordinates. The city is divided into quadrants with Main Street as the vertical (North-South) axis and S. Temple Street as the horizontal (East-West) axis. An address such as

1760 W 2100 S

indicates a location 17.6 blocks west of Main Street and 21 blocks south of S. Temple Street. (This is the address of the main post office in Salt Lake City.) With this logical system it is possible for someone unfamiliar with the city to locate any address as easily as one locates a point in the coordinate plane.



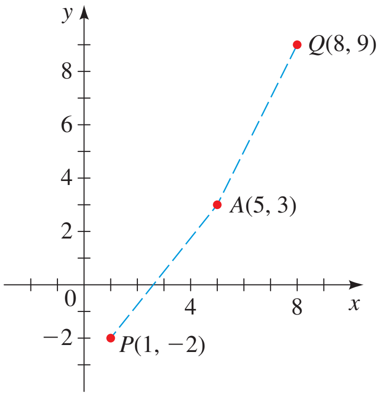


Figure 5

Example 2 ■ Applying the Distance Formula

Which of the points $P(1, -2)$ or $Q(8, 9)$ is closer to the point $A(5, 3)$?

Solution By the Distance Formula we have

$$d(P, A) = \sqrt{(5-1)^2 + [3-(-2)]^2} = \sqrt{4^2 + 5^2} = \sqrt{41}$$
$$d(Q, A) = \sqrt{(5-8)^2 + (3-9)^2} = \sqrt{(-3)^2 + (-6)^2} = \sqrt{45}$$

This shows that $d(P, A) < d(Q, A)$, so P is closer to A. (See Figure 5.)

Now Try Exercise 35 ■

Now let's find the coordinates (x, y) of the *midpoint* M of the line segment that joins the point $A(x_1, y_1)$ to the point $B(x_2, y_2)$. In Figure 6 notice that triangles APM and MQB are congruent because $d(A, M) = d(M, B)$ and the corresponding angles are equal. It follows that $d(A, P) = d(M, Q)$, so

$$x - x_1 = x_2 - x$$

Solving this equation for x, we get $2x = x_1 + x_2$, so

$$x = \frac{x_1 + x_2}{2} \qquad \text{Similarly,} \qquad y = \frac{y_1 + y_2}{2}$$

The x- and y-coordinates of the midpoint M are the averages of the x-coordinates of A and B and the y-coordinates of A and B, respectively.

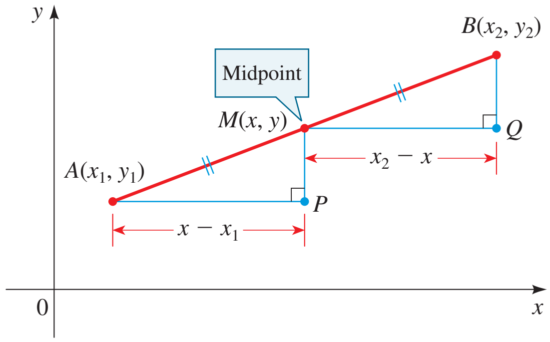


Figure 6

Midpoint Formula

The midpoint of the line segment from $A(x_1, y_1)$ to $B(x_2, y_2)$ is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Example 3 ■ Applying the Midpoint Formula

If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram. (See Appendix A, *Geometry Review*.)

Show that the quadrilateral with vertices $P(1, 2)$, $Q(4, 4)$, $R(5, 9)$, and $S(2, 7)$ is a parallelogram by proving that its two diagonals bisect each other.

Solution If the two diagonals have the same midpoint, then they must bisect each other. The midpoint of the diagonal PR is

$$\left(\frac{1+5}{2}, \frac{2+9}{2}\right) = \left(3, \frac{11}{2}\right)$$

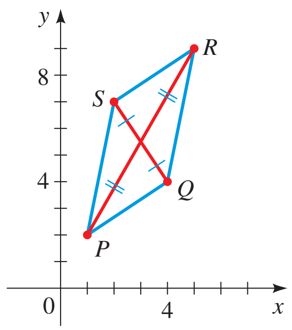


Figure 7

and the midpoint of the diagonal QS is

$$\left(\frac{4+2}{2}, \frac{4+7}{2}\right) = \left(3, \frac{11}{2}\right)$$

so each diagonal bisects the other, as shown in Figure 7.

Now Try Exercise 49 ■

■ Graphs of Equations in Two Variables

An **equation in two variables**, such as $y = x^2 + 1$, expresses a relationship between two quantities. A point (x, y) **satisfies** the equation if it makes the equation true when the values for x and y are substituted into the equation. For example, the point $(3, 10)$ satisfies the equation $y = x^2 + 1$ because $10 = 3^2 + 1$, but the point $(1, 3)$ does not, because $3 \neq 1^2 + 1$.

Fundamental Principle of Analytic Geometry

A point (x, y) lies on the graph of an equation if and only if its coordinates satisfy the equation.

The Graph of an Equation

The **graph** of an equation in x and y is the set of all points (x, y) in the coordinate plane that satisfy the equation.

The graph of an equation is a curve, so to graph an equation, we plot as many points as we can, then connect them by a smooth curve.

Example 4 ■ Sketching a Graph by Plotting Points

Sketch the graph of the equation $2x - y = 3$.

Solution We first solve the given equation for y to get

$$y = 2x - 3$$

This helps us calculate the y-coordinates in the following table.

x	$y = 2x - 3$	(x, y)
-1	-5	$(-1, -5)$
0	-3	$(0, -3)$
1	-1	$(1, -1)$
2	1	$(2, 1)$
3	3	$(3, 3)$
4	5	$(4, 5)$

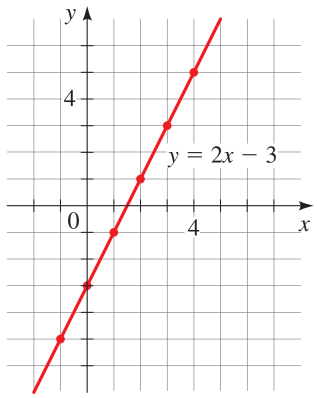


Figure 8

Of course, there are infinitely many points on the graph, and it is impossible to plot all of them. But the more points we plot, the better we can imagine what the graph represented by the equation looks like. In Figure 8 we plot the points corresponding to the ordered pairs we found in the table; they appear to lie on a line. So we complete the graph by joining the points with a line. (In Section 1.10 we verify that the graph of an equation of this type is indeed a line.)

Now Try Exercise 55 ■

Detailed discussions of parabolas and their geometric properties are presented in Sections 3.1 and 10.1.

Example 5 ■ Sketching a Graph by Plotting Points

Sketch the graph of the equation $y = x^2 - 2$.

Solution We find some of the ordered pairs (x, y) that satisfy the equation in the table on the next page. In Figure 9 we plot the points corresponding to these ordered

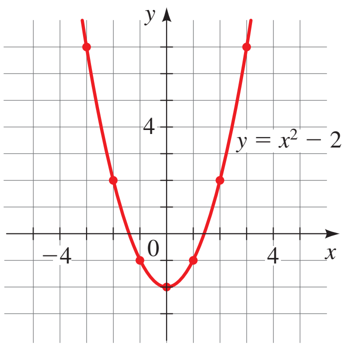


Figure 9

pairs and then connect them by a smooth curve. A curve with this shape is called a *parabola*.

x	$y = x^2 - 2$	(x, y)
-3	7	$(-3, 7)$
-2	2	$(-2, 2)$
-1	-1	$(-1, -1)$
0	-2	$(0, -2)$
1	-1	$(1, -1)$
2	2	$(2, 2)$
3	7	$(3, 7)$

Now Try Exercise 57 ■

Example 6 ■ Graphing an Absolute Value Equation

Sketch the graph of the equation $y = |x|$.

Solution We make a table of values:

x	$y = \|x\|$	(x, y)
-3	3	$(-3, 3)$
-2	2	$(-2, 2)$
-1	1	$(-1, 1)$
0	0	$(0, 0)$
1	1	$(1, 1)$
2	2	$(2, 2)$
3	3	$(3, 3)$

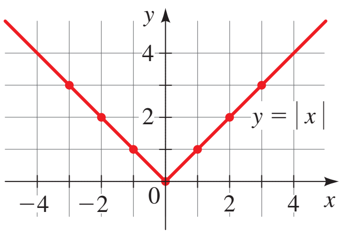


Figure 10

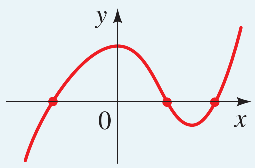
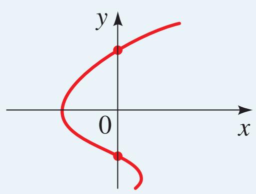
In Figure 10 we plot the points corresponding to the ordered pairs in the table and use them to sketch the graph of the equation.

Now Try Exercise 59 ■

■ Intercepts

The x-coordinates of the points where a graph intersects the x-axis are called the ***x*-intercepts** of the graph and are obtained by setting $y = 0$ in the equation of the graph. The y-coordinates of the points where a graph intersects the y-axis are called the ***y*-intercepts** of the graph and are obtained by setting $x = 0$ in the equation of the graph.

Definition of Intercepts

Intercepts	How to find them	Where they are on the graph
***x*-intercepts:** The x-coordinates of points where the graph of an equation intersects the x-axis	Set $y = 0$ and solve for x	
***y*-intercepts:** The y-coordinates of points where the graph of an equation intersects the y-axis	Set $x = 0$ and solve for y	

Example 7 ■ Finding Intercepts

Find the x- and y-intercepts of the graph of the equation $y = x^2 - 2$.

Solution To find the x-intercepts, we set $y = 0$ and solve for x. Thus

$$
\begin{aligned}
0 &= x^2 - 2 && \text{Set } y = 0 \\
x^2 &= 2 && \text{Add 2 to each side} \\
x &= \pm\sqrt{2} && \text{Take the square root}
\end{aligned}
$$

The x-intercepts are $\sqrt{2}$ and $-\sqrt{2}$.

To find the y-intercepts, we set $x = 0$ and solve for y. Thus

$$
\begin{aligned}
y &= 0^2 - 2 && \text{Set } x = 0 \\
y &= -2
\end{aligned}
$$

The y-intercept is -2.

The graph of this equation was sketched in Example 5. It is repeated in Figure 11 with the x- and y-intercepts labeled.

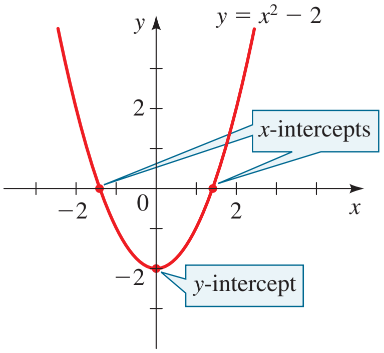


Figure 11

Now Try Exercise 67 ■

■ Circles

So far, we have discussed how to find the graph of an equation in x and y. The converse problem is to find an equation of a graph, that is, an equation that represents a given curve in the xy-plane. Such an equation is satisfied by the coordinates of the points on the curve and by no other point. This is the other half of the fundamental principle of analytic geometry as formulated by Descartes and Fermat. The idea is that if a geometric curve can be represented by an algebraic equation, then the rules of algebra can be used to analyze the curve.

As an example of this type of problem, let's find the equation of a circle with radius r and center (h, k). By definition the circle is the set of all points $P(x, y)$ whose distance from the center $C(h, k)$ is r. Thus P is on the circle if and only if $d(P, C) = r$. From the distance formula we have

$$
\begin{aligned}
\sqrt{(x-h)^2 + (y-k)^2} &= r \\
(x-h)^2 + (y-k)^2 &= r^2 && \text{Square each side}
\end{aligned}
$$

This is the desired equation.

Equation of a Circle

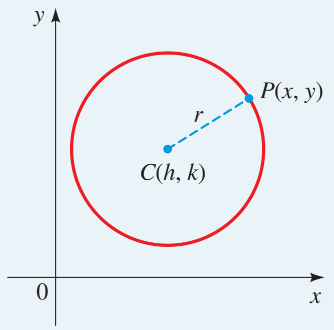
An equation of the circle with center (h, k) and radius $r > 0$ is

$$(x-h)^2 + (y-k)^2 = r^2$$

This is called the **standard form** for the equation of the circle.

If the center of the circle is the origin $(0, 0)$, then the equation is

$$x^2 + y^2 = r^2$$



Example 8 ■ Graphing a Circle

Graph each equation.

(a) $x^2 + y^2 = 25$ **(b)** $(x - 2)^2 + (y + 1)^2 = 25$

Solution

(a) Rewriting the equation as $x^2 + y^2 = 5^2$, we see that this is an equation of the circle of radius 5 centered at the origin. Its graph is shown in Figure 12.

(b) Rewriting the equation as $(x - 2)^2 + (y + 1)^2 = 5^2$, we see that this is an equation of the circle of radius 5 centered at $(2, -1)$. Its graph is shown in Figure 13.

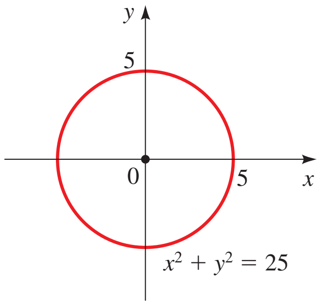


Figure 12

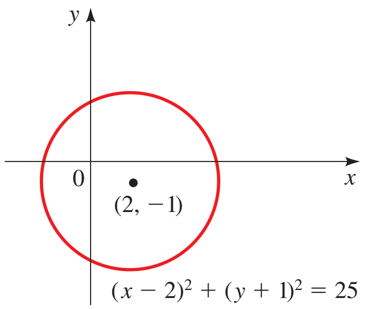


Figure 13

Now Try Exercises 75 and 77 ■

Example 9 ■ Finding an Equation of a Circle

(a) Find an equation of the circle with radius 3 and center $(2, -5)$.

(b) Find an equation of the circle that has the points $P(1, 8)$ and $Q(5, -6)$ as the endpoints of a diameter.

Solution

(a) Using the equation of a circle with $r = 3$, $h = 2$, and $k = -5$, we obtain

$$(x - 2)^2 + (y + 5)^2 = 9$$

The graph is shown in Figure 14.

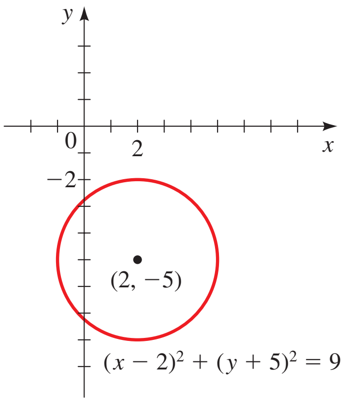


Figure 14

(b) We first observe that the center is the midpoint of the diameter PQ, so by the Midpoint Formula the center is

$$\left(\frac{1 + 5}{2}, \frac{8 - 6}{2}\right) = (3, 1)$$

The radius r is the distance from P to the center, so by the Distance Formula

$$r^2 = (3 - 1)^2 + (1 - 8)^2 = 2^2 + (-7)^2 = 53$$

Therefore the equation of the circle is

$$(x - 3)^2 + (y - 1)^2 = 53$$

The graph is shown in Figure 15.

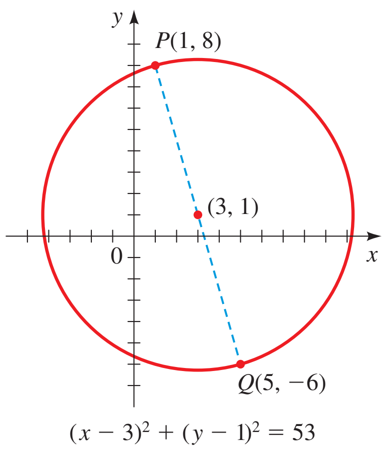


Figure 15

Now Try Exercises 81 and 85 ■

Let's expand the equation of the circle in the preceding example.

$$(x-3)^2+(y-1)^2=53 \qquad \text{Standard form}$$

$$x^2-6x+9+y^2-2y+1=53 \qquad \text{Expand the squares}$$

$$x^2-6x+y^2-2y=43 \qquad \text{Subtract 10 to get expanded form}$$

Completing the square is used in many contexts in algebra. In Section 1.5 we used completing the square to solve quadratic equations.

Suppose we are given the equation of a circle in expanded form. Then to find its center and radius, we must put the equation back into standard form. That means that we must reverse the steps in the preceding calculation, and to do that, we need to know what to add to an expression like x^2-6x to make it a perfect square—that is, we need to complete the square, as in the next example.

Example 10 ■ Identifying an Equation of a Circle

Show that the equation $x^2+y^2+2x-6y+7=0$ represents a circle, and find the center and radius of the circle.

Solution We need to put the equation in standard form, so we first group the x-terms and y-terms. Then we complete the square within each grouping. That is, we complete the square for x^2+2x by adding $\left(\frac{1}{2}\cdot 2\right)^2=1$, and we complete the square for y^2-6y by adding $\left[\frac{1}{2}\cdot(-6)\right]^2=9$.

We must add the same numbers to *each side* to maintain equality.

$$(x^2+2x\quad)+(y^2-6y\quad)=-7 \qquad \text{Group terms}$$

$$(x^2+2x+1)+(y^2-6y+9)=-7+1+9 \qquad \text{Complete the square by adding 1 and 9 to each side}$$

$$(x+1)^2+(y-3)^2=3 \qquad \text{Factor and simplify}$$

Comparing this equation with the standard equation of a circle, we see that $h=-1$, $k=3$, and $r=\sqrt{3}$, so the given equation represents a circle with center $(-1,3)$ and radius $\sqrt{3}$.

Now Try Exercise 91 ■

Note An equation such as $(x-h)^2+(y-k)^2=c$, where $c\le 0$, may look like the equation of a circle but its graph is not a circle. For example, the graph of $x^2+y^2=0$ is the single point $(0,0)$ and the graph of $x^2+y^2=-4$ is empty (because the equation has no solution). (See Exercise 117.)

■ Symmetry

Figure 16 shows the graph of $y=x^2$. Notice that the part of the graph to the left of the y-axis is the mirror image of the part to the right of the y-axis. The reason is that if the point (x,y) is on the graph, then so is $(-x,y)$, and these points are reflections of each other about the y-axis. In this situation we say that the graph is **symmetric with respect to the *y*-axis**. Similarly, we say that a graph is **symmetric with respect to the *x*-axis** if whenever the point (x,y) is on the graph, then so is $(x,-y)$. A graph is **symmetric with respect to the origin** if whenever (x,y) is on the graph, so is $(-x,-y)$. (We often say symmetric "about" instead of "with respect to.")

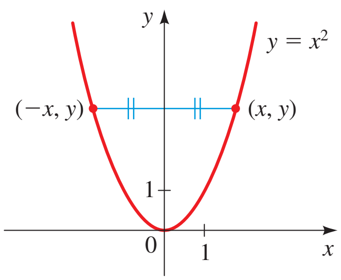
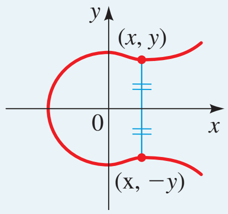
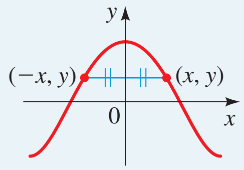
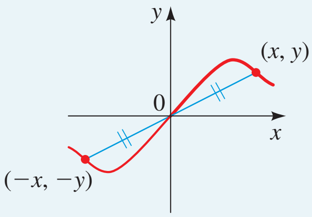


Figure 16

Types of Symmetry

Symmetry	Test	Graph	Property of Graph
With respect to the x-axis	Replace y by $-y$. The resulting equation is equivalent to the original one.		Graph is unchanged when reflected about the x-axis. See Figures 12 and 17.
With respect to the y-axis	Replace x by $-x$. The resulting equation is equivalent to the original one.		Graph is unchanged when reflected about the y-axis. See Figures 9, 10, 11, 12, and 16.
With respect to the origin	Replace x by $-x$ and y by $-y$. The resulting equation is equivalent to the original one.		Graph is unchanged when rotated 180° about the origin. See Figures 12 and 18.

The remaining examples in this section show how symmetry helps us sketch the graphs of equations.

Example 11 ■ Using Symmetry to Sketch a Graph

Test the equation $x = y^2$ for symmetry and sketch the graph.

Solution If y is replaced by $-y$ in the equation $x = y^2$, we get

$$x = (-y)^2 \qquad \text{Replace } y \text{ by } -y$$

$$x = y^2 \qquad \text{Simplify}$$

and so the equation is equivalent to the original one. Therefore the graph is symmetric about the x-axis. But changing x to $-x$ gives the equation $-x = y^2$, which is not equivalent to the original equation, so the graph is not symmetric about the y-axis.

We use the symmetry about the x-axis to sketch the graph by first plotting points just for $y > 0$ and then reflecting the graph about the x-axis, as shown in Figure 17.

y	$x = y^2$	(x, y)
0	0	(0, 0)
1	1	(1, 1)
2	4	(4, 2)
3	9	(9, 3)

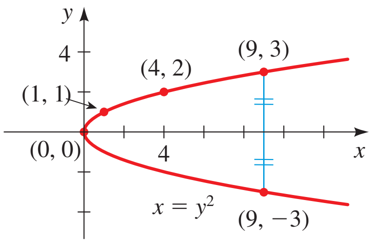


Figure 17

Now Try Exercises 97 and 103 ■

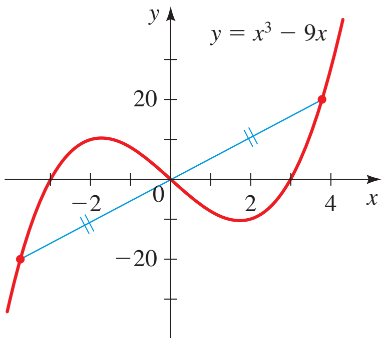


Figure 18

Example 12 ■ Testing an Equation for Symmetry

Test the equation $y = x^3 - 9x$ for symmetry.

Solution If we replace x by $-x$ and y by $-y$ in the equation, we get

$$-y = (-x)^3 - 9(-x) \qquad \text{Replace } x \text{ by } -x \text{ and } y \text{ by } -y$$

$$-y = -x^3 + 9x \qquad \text{Simplify}$$

$$y = x^3 - 9x \qquad \text{Multiply by } -1$$

and so the equation is equivalent to the original one. This means that the graph is symmetric with respect to the origin, as shown in Figure 18.

Now Try Exercise 99

1.9 Exercises

Concepts

1. (a) The point that is 3 units to the right of the y-axis and 5 units below the x-axis has coordinates (____, ____).

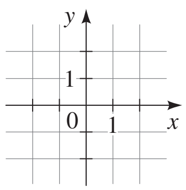
(b) Is the point $(2, 7)$ closer to the x-axis or to the y-axis?

2. The distance between the points (a, b) and (c, d) is ________. So the distance between $(1, 2)$ and $(7, 10)$ is ________.

3. The point midway between (a, b) and (c, d) is ________. So the point midway between $(1, 2)$ and $(7, 10)$ is ________.

4. If the point $(2, 3)$ is on the graph of an equation in x and y, then the equation is satisfied when we replace x by ________ and y by ________. Is the point $(2, 3)$ on the graph of the equation $2y = x + 1$? Complete the table, and sketch a graph.

x	y	(x, y)
-2		
-1		
0		
1		
2		



5. (a) To find the x-intercept(s) of the graph of an equation, we set ________ equal to 0 and solve for ________. So the x-intercept of $2y = x + 1$ is ________.

(b) To find the y-intercept(s) of the graph of an equation, we set ________ equal to 0 and solve for ________. So the y-intercept of $2y = x + 1$ is ________.

6. (a) The graph of the equation $(x - 1)^2 + (y - 2)^2 = 9$ is a circle with center (____, ____) and radius ________.

(b) Find the equation of the circle with center $(3, 4)$ that just touches (at one point) the y-axis.

7. (a) If a graph is symmetric with respect to the x-axis and (a, b) is on the graph, then (____, ____) is also on the graph.

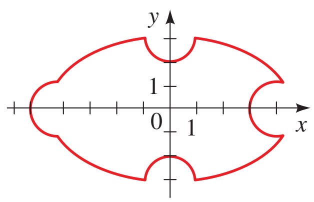
(b) If a graph is symmetric with respect to the y-axis and (a, b) is on the graph, then (____, ____) is also on the graph.

(c) If a graph is symmetric about the origin and (a, b) is on the graph, then (____, ____) is also on the graph.

8. The graph of an equation is shown below.

(a) The x-intercept(s) are ________, and the y-intercept(s) are ________.

(b) The graph is symmetric about the ________ (x-axis/y-axis/origin).



9–10 ■ *Yes or No*? If *No*, give a reason.

9. If the graph of an equation is symmetric with respect to both the x- and y-axes, is it necessarily symmetric with respect to the origin?

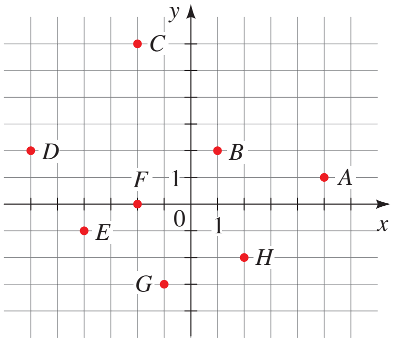
10. If the graph of an equation is symmetric with respect to the origin, is it necessarily symmetric with respect to the x- or y-axes?

Skills

11–12 ■ Points in a Coordinate Plane Refer to the figure below.

11. Find the coordinates of the points shown.

12. List the points that lie in either Quadrant I or Quadrant III.



13–14 ■ Points in a Coordinate Plane Plot the given points in a coordinate plane.

13. $(0, 5)$, $(-1, 0)$, $(-1, -2)$, $\left(\frac{1}{2}, \frac{2}{3}\right)$

14. $(-5, 0)$, $(2, 0)$, $(2.6, -1.3)$, $(-2.5, 3.5)$

15–20 ■ Sketching Regions Sketch the region given by each set.

15. (a) $\{(x, y) \mid x \geq 2\}$ **(b)** $\{(x, y) \mid y = 2\}$

16. (a) $\{(x, y) \mid y \leq -1\}$ **(b)** $\{(x, y) \mid x = 3\}$

17. (a) $\{(x, y) \mid -2 \leq x \leq 4\}$ **(b)** $\{(x, y) \mid |x| \leq 1\}$

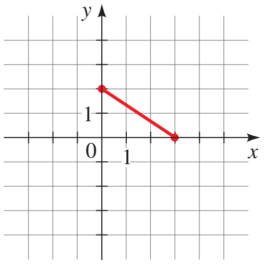
18. (a) $\{(x, y) \mid 0 \leq y \leq 2\}$ **(b)** $\{(x, y) \mid |y| > 2\}$

19. (a) $\{(x, y) \mid -1 < x < 1 \text{ and } y \leq 4\}$
(b) $\{(x, y) \mid xy < 0\}$

20. (a) $\{(x, y) \mid |x| < 3 \text{ and } |y| > 2\}$
(b) $\{(x, y) \mid xy > 0\}$

21–24 ■ Distance and Midpoint A pair of points is graphed. **(a)** Find the distance between them. **(b)** Find the midpoint of the segment that joins them.

21.



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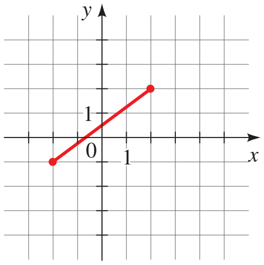

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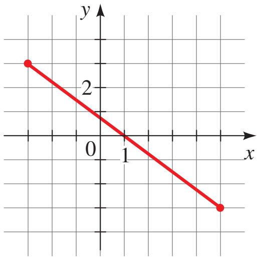

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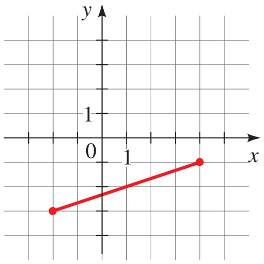


25–30 ■ Distance and Midpoint A pair of points is given. **(a)** Plot the points in a coordinate plane. **(b)** Find the distance between them. **(c)** Find the midpoint of the segment that joins them.

25. $(0, 8)$, $(6, 16)$ **26.** $(-2, 5)$, $(10, 0)$

27. $(3, -2)$, $(-4, 5)$ **28.** $(-1, 1)$, $(-6, -3)$

29. $(6, -2)$, $(-6, 2)$ **30.** $(0, -6)$, $(5, 0)$

31–34 ■ Area In these exercises we find the areas of plane figures.

31. Draw the rectangle with vertices $A(1, 3)$, $B(5, 3)$, $C(1, -3)$, and $D(5, -3)$ on a coordinate plane. Find the area of the rectangle.

32. Draw the parallelogram with vertices $A(1, 2)$, $B(5, 2)$, $C(3, 6)$, and $D(7, 6)$ on a coordinate plane. Find the area of the parallelogram.

33. Plot the points $A(1, 0)$, $B(5, 0)$, $C(4, 3)$, and $D(2, 3)$ on a coordinate plane. Draw the segments AB, BC, CD, and DA. What kind of quadrilateral is $ABCD$, and what is its area?

34. Plot the points $P(5, 1)$, $Q(0, 6)$, and $R(-5, 1)$ on a coordinate plane. Where must the point S be located so that the quadrilateral $PQRS$ is a square? Find the area of this square.

35–39 ■ Distance Formula In these exercises we use the Distance Formula.

35. Which of the points $A(6, 7)$ or $B(-5, 8)$ is closer to the origin?

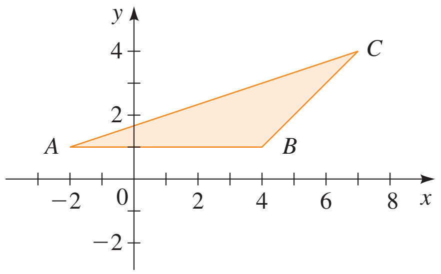
36. Which of the points $C(-6, 3)$ or $D(3, 0)$ is closer to the point $E(-2, 1)$?

37. Which of the points $P(3, 1)$ or $Q(-1, 3)$ is closer to the point $R(-1, -1)$?

38. (a) Show that the points $(7, 3)$ and $(3, 7)$ are the same distance from the origin.
(b) Show that the points (a, b) and (b, a) are the same distance from the origin.

39. Show that the triangle with vertices $A(0, 2)$, $B(-3, -1)$, and $C(-4, 3)$ is isosceles.

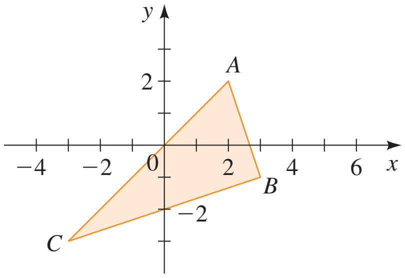
40. Area of Triangle Find the area of the triangle shown in the figure.



41–42 ■ Pythagorean Theorem In these exercises we use the converse of the Pythagorean Theorem (Appendix A) to show that the given triangle is a right triangle.

41. Refer to triangle ABC in the figure.
(a) Show that triangle ABC is a right triangle by using the converse of the Pythagorean Theorem.

(b) Find the area of triangle ABC.



42. Show that the triangle with vertices $A(6, -7)$, $B(11, -3)$, and $C(2, -2)$ is a right triangle by using the converse of the Pythagorean Theorem. Find the area of the triangle.

43–45 ■ Distance Formula In these exercises we use the Distance Formula.

43. Show that the points $A(-2, 9)$, $B(4, 6)$, $C(1, 0)$, and $D(-5, 3)$ are the vertices of a square.

44. Show that the points $A(-1, 3)$, $B(3, 11)$, and $C(5, 15)$ are collinear by showing that $d(A, B) + d(B, C) = d(A, C)$.

45. Find a point on the y-axis that is equidistant from the points $(5, -5)$ and $(1, 1)$.

46–50 ■ Distance and Midpoint Formulas In these exercises we use the Distance Formula and the Midpoint Formula.

46. Find the lengths of the medians of the triangle with vertices $A(1, 0)$, $B(3, 6)$, and $C(8, 2)$. (A *median* is a line segment from a vertex to the midpoint of the opposite side.)

47. Plot the points $P(-1, -4)$, $Q(1, 1)$, and $R(4, 2)$ on a coordinate plane. Where should the point S be located so that the figure $PQRS$ is a parallelogram?

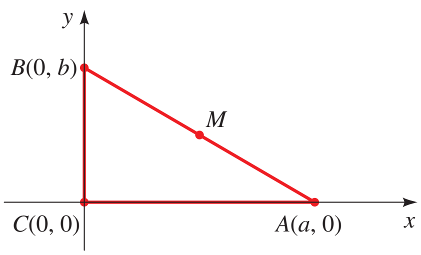
48. If $M(6, 8)$ is the midpoint of the line segment AB and if A has coordinates $(2, 3)$, find the coordinates of B.

49. (a) Sketch the parallelogram with vertices $A(-2, -1)$, $B(4, 2)$, $C(7, 7)$, and $D(1, 4)$.

(b) Find the midpoints of the diagonals of this parallelogram.

(c) From part (b) show that the diagonals bisect each other.

50. The point M in the figure is the midpoint of the line segment AB. Show that M is equidistant from the vertices of triangle ABC.



51–54 ■ Points on a Graph? Determine whether the given points are on the graph of the equation.

51. $3x - y + 5 = 0$; $(0, 5), (2, 1), (-2, -1)$

52. $y(x^2 + 1) = 1$; $(1, 1), \left(1, \frac{1}{2}\right), \left(-1, \frac{1}{2}\right)$

53. $x^2 + xy + y^2 = 4$; $(0, -2), (1, -2), (2, -2)$

54. $x^2 + y^2 = 1$; $(0, 1), \left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right), \left(\dfrac{\sqrt{3}}{2}, \dfrac{1}{2}\right)$

55–60 ■ Graphing Equations Make a table of values, and sketch the graph of the equation.

55. $4x + 5y = 40$ **56.** $2x - 3y = 12$

57. $y = x^2 - 3$ **58.** $y = 3 - x^2$

59. $y = |x| - 1$ **60.** $y = |x + 1|$

61–66 ■ Graphing Equations Make a table of values, and sketch the graph of the equation. Find the x- and y-intercepts, and test for symmetry.

61. (a) $2x - y = 6$ **(b)** $y = 2(x - 1)^2$

62. (a) $x - 4y = 8$ **(b)** $y = -x^2 + 4$

63. (a) $y = \sqrt{x} + 2$ **(b)** $y = -|x|$

64. (a) $y = \sqrt{x - 4}$ **(b)** $x = |y|$

65. (a) $y = \sqrt{4 - x^2}$ **(b)** $y = x^3 - 4x$

66. (a) $y = -\sqrt{4 - x^2}$ **(b)** $x = y^3$

67–70 ■ Intercepts Find the x- and y-intercepts of the graph of the equation.

67. (a) $y = x + 6$ **(b)** $y = x^2 - 5$

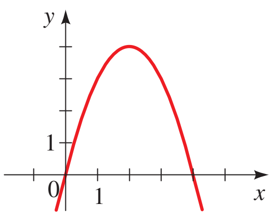
68. (a) $4x^2 + 25y^2 = 100$ **(b)** $x^2 - xy + 3y = 1$

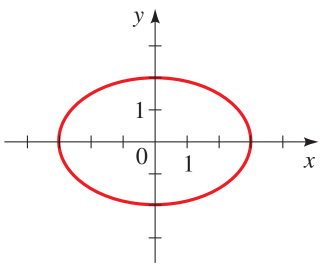
69. (a) $9x^2 - 4y^2 = 36$ **(b)** $y - 2xy + 4x = 1$

70. (a) $y = \sqrt{x^2 - 16}$ **(b)** $y = \sqrt{64 - x^3}$

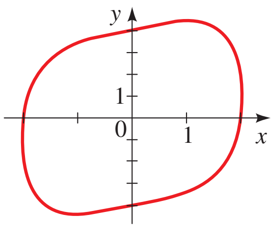
71–74 ■ Intercepts An equation and its graph are given. Find the x- and y-intercepts from the graph. Check that your answers satisfy the equation.

71. $y = 4x - x^2$ **72.** $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

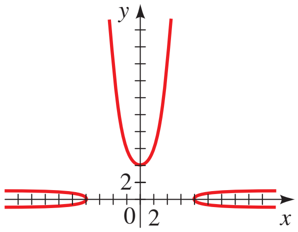




73. $x^4 + y^2 - xy = 16$



74. $x^2 + y^3 - x^2y^2 = 64$



75–80 ■ Graphing Circles Find the center and radius of the circle, and sketch its graph.

75. $x^2 + y^2 = 9$

76. $x^2 + y^2 = 5$

77. $(x - 2)^2 + y^2 = 9$

78. $x^2 + (y + 1)^2 = 4$

79. $(x + 3)^2 + (y - 4)^2 = 25$

80. $(x + 1)^2 + (y + 2)^2 = 36$

81–88 ■ Equations of Circles Find an equation of the circle that satisfies the given conditions.

81. Center $(-3, 1)$; radius 2

82. Center $(2, -5)$; radius 3

83. Center at the origin; passes through $(4, 7)$

84. Center $(-1, 5)$; passes through $(-4, -6)$

85. Endpoints of a diameter are $P(-1, 1)$ and $Q(5, 9)$

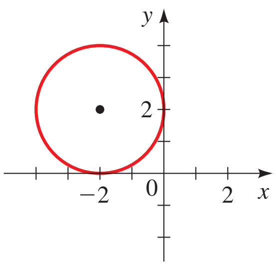
86. Endpoints of a diameter are $P(-1, 3)$ and $Q(7, -5)$

87. Center $(7, -3)$; tangent to the x-axis

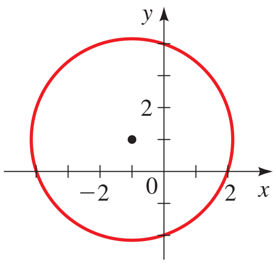
88. Circle lies in the first quadrant, tangent to both x- and y-axes; radius 5

89–90 ■ Equations of Circles Find the equation of the circle shown in the figure.

89.



90.



91–96 ■ Equations of Circles Show that the equation represents a circle, and find the center and radius of the circle.

91. $x^2 + y^2 + 4x - 6y + 12 = 0$

92. $x^2 + y^2 + 8x + 5 = 0$

93. $x^2 + y^2 - \frac{1}{2}x + \frac{1}{2}y = \frac{1}{8}$

94. $x^2 + y^2 + \frac{1}{2}x + 2y + \frac{1}{16} = 0$

95. $2x^2 + 2y^2 - 3x = 0$

96. $3x^2 + 3y^2 + 6x - y = 0$

97–102 ■ Symmetry Test the equation for symmetry.

97. $y = x^4 + x^2$

98. $x = y^4 - y^2$

99. $x^2y^4 + x^4y^2 = 2$

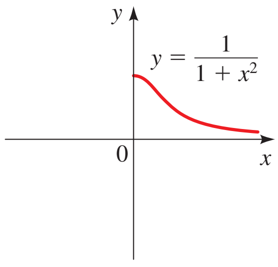
100. $x^3y + xy^3 = 1$

101. $y = x^3 + 10x$

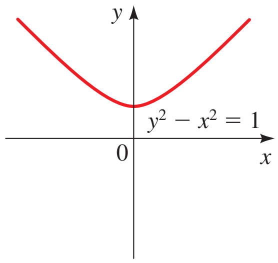
102. $y = x^2 + |x|$

103–106 ■ Symmetry Complete the graph using the given symmetry property.

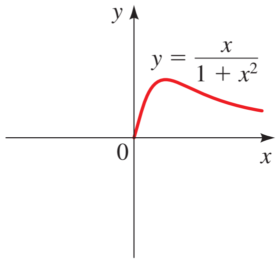
103. Symmetric with respect to the y-axis



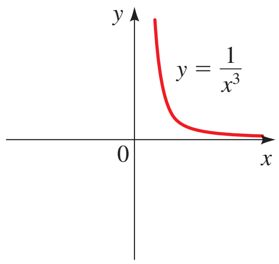
104. Symmetric with respect to the x-axis



105. Symmetric with respect to the origin



106. Symmetric with respect to the origin



Skills Plus

107–108 ■ Graphing Regions Sketch the region given by the set.

107. $\{(x, y) \mid x^2 + y^2 \le 1\}$

108. $\{(x, y) \mid x^2 + y^2 > 4\}$

109. Area of a Region Find the area of the region that lies outside the circle $x^2 + y^2 = 4$ but inside the circle

$$x^2 + y^2 - 4y - 12 = 0$$

110. Area of a Region Sketch the region in the coordinate plane that satisfies both the inequalities $x^2 + y^2 \le 9$ and $y \ge |x|$. What is the area of this region?

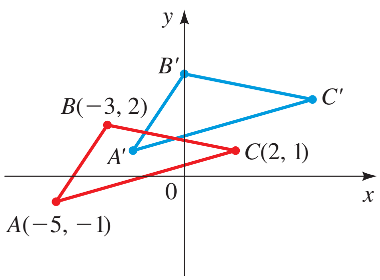
111. Shifting the Coordinate Plane Suppose that each point in the coordinate plane is shifted 3 units to the right and 2 units upward.

(a) The point $(5, 3)$ is shifted to what new point?

(b) The point (a, b) is shifted to what new point?

(c) What point is shifted to $(3, 4)$?

(d) Triangle ABC in the figure has been shifted to triangle $A'B'C'$. Find the coordinates of the points A', B', and C'.



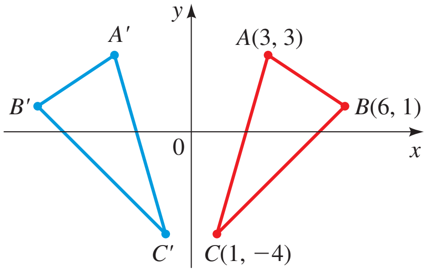
112. Reflecting in the Coordinate Plane Suppose that the y-axis acts as a mirror that reflects each point to the right of it into a point to the left of it.

(a) The point $(3, 7)$ is reflected to what point?

(b) The point (a, b) is reflected to what point?

(c) What point is reflected to $(-4, -1)$?

(d) Triangle ABC in the figure is reflected to triangle $A'B'C'$. Find the coordinates of the points A', B', and C'.

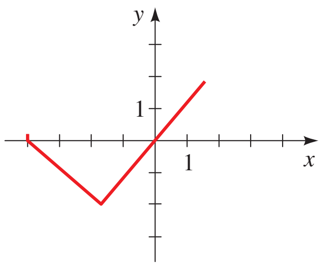


113. Making a Graph Symmetric The graph in the figure is not symmetric about the x-axis, the y-axis, or the origin. Add line segments to the graph so that it exhibits the indicated symmetry. In each case, add as little as possible.

(a) Symmetry about the x-axis

(b) Symmetry about the y-axis

(c) Symmetry about the origin



Applications

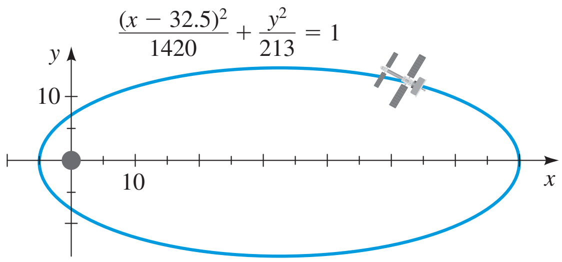
114. Lunar Gateway Orbit The *Lunar Gateway* (or simply *Gateway*) is a space station expected to be launched in 2024 to orbit the moon in a halo orbit. The station is intended to serve as a waypoint for missions to the moon and Mars. Its highly elliptical orbit forms a plane that is always visible from Earth. We can model the orbit by the equation.

$$\frac{(x - 32.5)^2}{1420} + \frac{y^2}{213} = 1$$

where x and y are measured in megameters (Mm).

(a) From the graph, estimate the closest and farthest distances of the satellite from the center of the moon. Use the equation to confirm your answers.

(b) There are two points in the orbit with y-coordinate 10. Find the x-coordinates of these points, and determine their distances from the center of the moon.

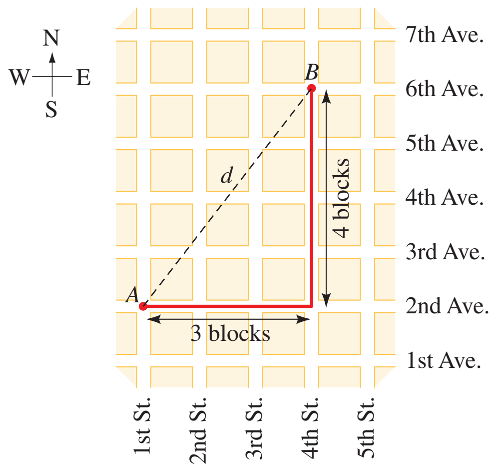


115. Distances in a City A city has streets that run north and south and avenues that run east and west, all equally spaced. Streets and avenues are numbered sequentially, as shown in the figure. The *walking* distance between points A and B is 7 blocks—that is, 3 blocks east and 4 blocks north. To find the *straight-line* distance d, we must use the Distance Formula.

(a) Find the straight-line distance (in blocks) between A and B.

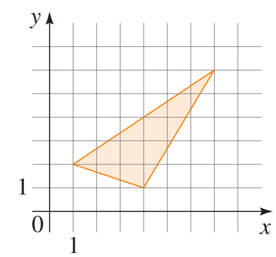
(b) Find the walking distance and the straight-line distance between the corner of 4th St. and 2nd Ave. and the corner of 11th St. and 26th Ave.

(c) What must be true about the points P and Q if the walking distance between P and Q equals the straight-line distance between P and Q?



Discuss ■ Discover ■ Prove ■ Write

116. Discuss ■ Discover: Area of a Triangle in the Coordinate Plane Find the area of the triangle in the figure.



PS *Introduce something extra.* Draw the rectangle (with sides parallel to the coordinate axes) that circumscribes the triangle. Note the new right triangles formed by adding the rectangle.

117. Discover: Circle, Point, or Empty Set? Complete the squares in the general equation

$$x^2 + ax + y^2 + by + c = 0$$

and simplify the result as much as possible. Under what conditions on the coefficients a, b, and c does this equation represent a circle? A single point? The empty set? In the case in which the equation does represent a circle, find its center and radius.

118. Prove: Coloring the plane Suppose that each point in the plane is colored either red or blue. Show that there must exist two points of the same color that are exactly one unit apart.

PS *Introduce something extra.* Imagine an equilateral triangle with side length 1 in the plane.

1.10 Lines

■ The Slope of a Line ■ Point-Slope Form of the Equation of a Line ■ Slope-Intercept Form of the Equation of a Line ■ Vertical and Horizontal Lines ■ General Equation of a Line ■ Parallel and Perpendicular Lines

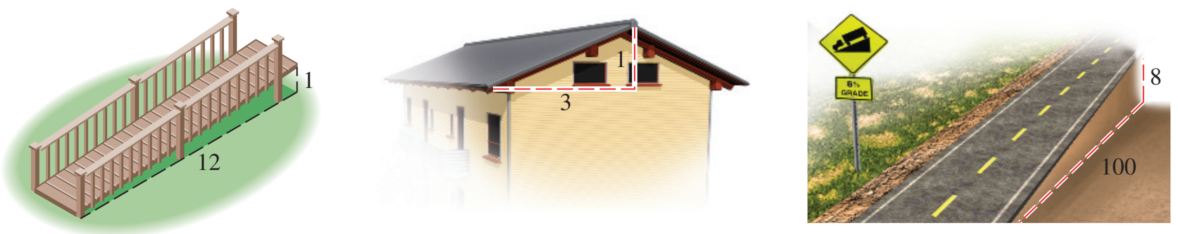
In this section we find equations for straight lines lying in a coordinate plane. The equations will depend on how the line is inclined, so we begin by discussing the concept of slope.

■ The Slope of a Line

We first need a way to measure the "steepness" of a line, or how quickly it rises (or falls) as we move from left to right. We define *run* to be the distance we move to the right and *rise* to be the corresponding distance that the line rises (or falls). The *slope* of a line is the ratio of rise to run:

$$\text{slope} = \frac{\text{rise}}{\text{run}}$$

Figure 1 shows situations in which slope is important. Carpenters use the term *pitch* for the slope of a roof or a staircase; the term *grade* is used for the slope of a road.



Slope of a ramp
Slope $= \frac{1}{12}$

Pitch of a roof
Slope $= \frac{1}{3}$

Grade of a road
Slope $= \frac{8}{100}$

Figure 1

If a line lies in a coordinate plane, then the **run** is the change in the x-coordinate and the **rise** is the corresponding change in the y-coordinate between any two points on the line (see Figure 2). This gives us the following definition of slope.

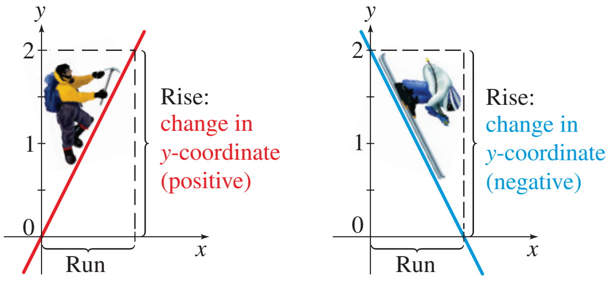


Figure 2

Slope of a Line

The **slope** m of a nonvertical line that passes through the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

The slope of a vertical line is not defined.

The slope is independent of which two points are chosen on the line. We can see that this is true from the similar triangles in Figure 3.

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_2' - y_1'}{x_2' - x_1'}$$

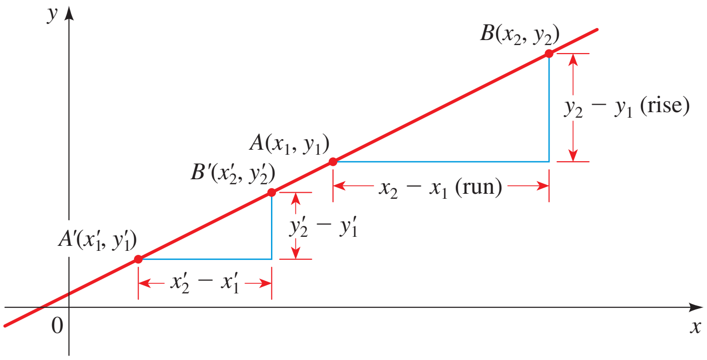
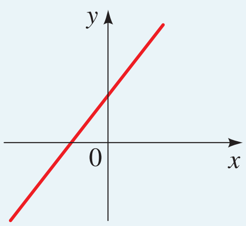


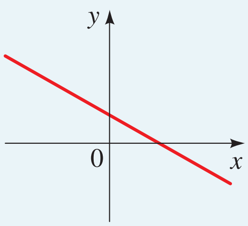
Figure 3

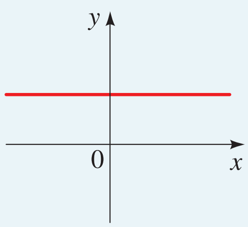
The figures in the box below show several lines labeled with their slopes. Notice that lines with positive slope slant upward to the right, whereas lines with negative slope slant downward to the right. The steepest lines are those for which the absolute value of the slope is the largest; a horizontal line has slope 0. The slope of a vertical line is undefined (it has a 0 denominator), so we say that a vertical line has no slope.

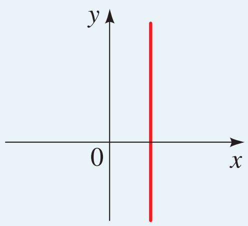
Slope of a Line

Positive Slope	Negative Slope	Zero Slope	No Slope (undefined)









Example 1 ■ Finding the Slope of a Line Through Two Points

Find the slope of the line that passes through the points $P(2, 1)$ and $Q(8, 5)$.

Solution Since any two different points determine a line, only one line passes through these two points. From the definition the slope is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - 1}{8 - 2} = \frac{4}{6} = \frac{2}{3}$$

This says that for every 3 units we move to the right, the line rises 2 units. The line is drawn in Figure 4.

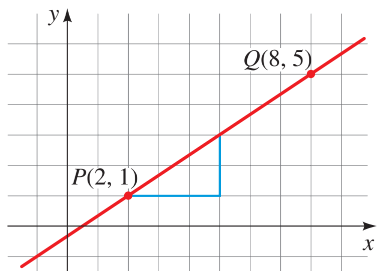


Figure 4

Now Try Exercise 9

Point-Slope Form of the Equation of a Line

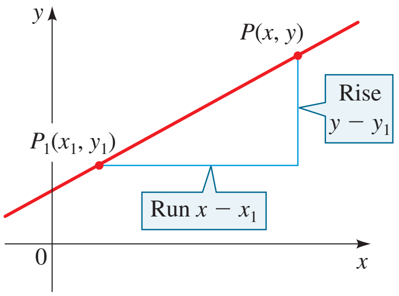


Figure 5

Now let's find an equation of the line that passes through a given point $P_1(x_1, y_1)$ and has slope m. A point $P(x, y)$ with $x \neq x_1$ lies on this line if and only if the slope of the line through P_1 and P is equal to m. (See Figure 5.) By the definition of slope,

$$\frac{y - y_1}{x - x_1} = m$$

This equation can be rewritten in the form $y - y_1 = m(x - x_1)$; note that the equation is also satisfied when $x = x_1$ and $y = y_1$. Therefore it is an equation of the given line.

Point-Slope Form of the Equation of a Line

An equation of the line that passes through the point (x_1, y_1) and has slope m is

$$y - y_1 = m(x - x_1)$$

Example 2 ■ Finding an Equation of a Line with Given Point and Slope

(a) Find an equation of the line through the point $(1, -3)$ with slope $-\frac{1}{2}$.

(b) Sketch the line.

Solution

(a) Using the point-slope form with $m = -\frac{1}{2}$, $x_1 = 1$, and $y_1 = -3$, we obtain an equation of the line as

$$\begin{aligned} y + 3 &= -\tfrac{1}{2}(x - 1) && \text{Slope } m = -\tfrac{1}{2}\text{, point } (1, -3) \\ 2y + 6 &= -x + 1 && \text{Multiply by 2} \\ x + 2y + 5 &= 0 && \text{Rearrange} \end{aligned}$$

(b) The fact that the slope is $-\frac{1}{2}$ tells us that when we move to the right 2 units, the line drops 1 unit. This enables us to sketch the line in Figure 6.

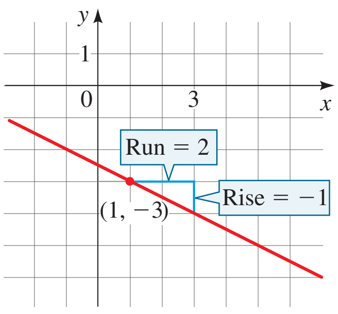


Figure 6

Now Try Exercise 25

Example 3 ■ Finding an Equation of a Line Through Two Given Points

Find an equation of the line through the points $(-1, 2)$ and $(3, -4)$.

Solution The slope of the line is

$$m = \frac{-4 - 2}{3 - (-1)} = -\frac{6}{4} = -\frac{3}{2}$$

We can use *either* point, $(-1, 2)$ *or* $(3, -4)$, in the point-slope equation. We will end up with the same final answer.

Using the point-slope form with $x_1 = -1$ and $y_1 = 2$, we obtain

$$\begin{aligned} y - 2 &= -\tfrac{3}{2}(x + 1) && \text{Slope } m = -\tfrac{3}{2}, \text{ point } (-1, 2) \\ 2y - 4 &= -3x - 3 && \text{Multiply by 2} \\ 3x + 2y - 1 &= 0 && \text{Rearrange} \end{aligned}$$

Now Try Exercise 29 ■

■ Slope-Intercept Form of the Equation of a Line

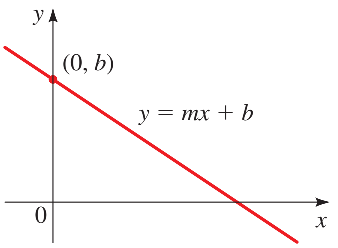


Figure 7

Suppose a nonvertical line has slope m and y-intercept b. (See Figure 7.) This means that the line intersects the y-axis at the point $(0, b)$, so the point-slope form of the equation of the line, with $x = 0$ and $y = b$, becomes

$$y - b = m(x - 0)$$

This simplifies to $y = mx + b$, which is called the **slope-intercept form** of the equation of a line.

Slope-Intercept Form of the Equation of a Line

An equation of the line that has slope m and y-intercept b is

$$y = mx + b$$

Example 4 ■ Lines in Slope-Intercept Form

(a) Find an equation of the line with slope 3 and y-intercept -2.

(b) Find the slope and y-intercept of the line $3y - 2x = 1$.

Solution

(a) Since $m = 3$ and $b = -2$, from the slope-intercept form of the equation of a line we get

$$y = 3x - 2$$

(b) We first write the equation in the form $y = mx + b$.

$$\begin{aligned} 3y - 2x &= 1 \\ 3y &= 2x + 1 && \text{Add } 2x \\ y &= \tfrac{2}{3}x + \tfrac{1}{3} && \text{Divide by 3} \end{aligned}$$

$$y = \overbrace{\tfrac{2}{3}}^{\text{Slope}}x + \overbrace{\tfrac{1}{3}}^{y\text{-intercept}}$$

From the slope-intercept form of the equation of a line, we see that the slope is $m = \frac{2}{3}$ and the y-intercept is $b = \frac{1}{3}$.

Now Try Exercises 23 and 61 ■

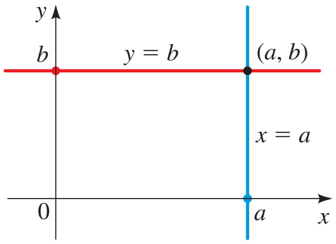


Figure 8

Vertical and Horizontal Lines

If a line is horizontal, its slope is $m = 0$, so its equation is $y = b$, where b is the y-intercept (see Figure 8). A vertical line does not have a slope, but we can write its equation as $x = a$, where a is the x-intercept, because the x-coordinate of every point on the line is a.

Vertical and Horizontal Lines

- An equation of the vertical line through (a, b) is $x = a$.
- An equation of the horizontal line through (a, b) is $y = b$.

Example 5 ■ Vertical and Horizontal Lines

(a) An equation for the vertical line through $(3, 2)$ is $x = 3$.

(b) The graph of the equation $x = 3$ is a vertical line with x-intercept 3.

(c) An equation for the horizontal line through $(4, -2)$ is $y = -2$.

(d) The graph of the equation $y = -2$ is a horizontal line with y-intercept -2.

The lines are graphed in Figure 9.

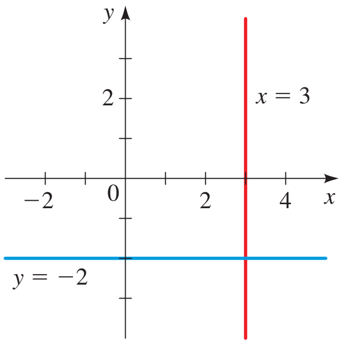


Figure 9

Now Try Exercises 35, 37, 63, and 65

General Equation of a Line

A **linear equation** in the variables x and y is an equation of the form

$$Ax + By + C = 0$$

where A, B, and C are constants and A and B are not both 0. An equation of a line is a linear equation:

- A nonvertical line has the equation $y = mx + b$ or $-mx + y - b = 0$, which is a linear equation with $A = -m$, $B = 1$, and $C = -b$.
- A vertical line has the equation $x = a$ or $x - a = 0$, which is a linear equation with $A = 1$, $B = 0$, and $C = -a$.

Conversely, the graph of a linear equation is a line:

- If $B \neq 0$, the equation becomes

$$y = -\frac{A}{B}x - \frac{C}{B} \qquad \text{Divide by } B$$

  and this is the slope-intercept form of the equation of a line (with $m = -A/B$ and $b = -C/B$).
- If $B = 0$, the equation becomes

$$Ax + C = 0 \qquad \text{Set } B = 0$$

  or $x = -C/A$, which represents a vertical line.

We have proved the following.

General Equation of a Line

The graph of every **linear equation**

$$Ax + By + C = 0 \qquad (A, B \text{ not both zero})$$

is a line. Conversely, every line is the graph of a linear equation.

Example 6 ■ Graphing a Linear Equation

Sketch the graph of the equation $2x - 3y - 12 = 0$.

Solution 1 Since the equation is linear, its graph is a line. To draw the graph, it is enough for us to find any two points on the line. The intercepts are the easiest points to find.

x-intercept: Substitute $y = 0$ to get $2x - 12 = 0$, so $x = 6$

y-intercept: Substitute $x = 0$ to get $-3y - 12 = 0$, so $y = -4$

With these points we sketch the graph in Figure 10.

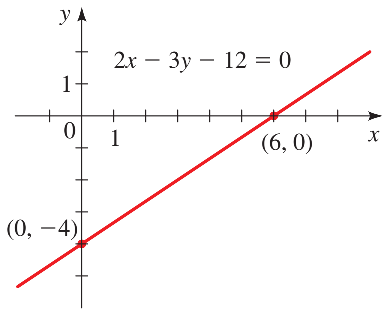


Figure 10

Solution 2 We write the equation in slope-intercept form.

$$2x - 3y - 12 = 0$$
$$2x - 3y = 12 \qquad \text{Add 12}$$
$$-3y = -2x + 12 \qquad \text{Subtract } 2x$$
$$y = \tfrac{2}{3}x - 4 \qquad \text{Divide by } -3$$

This equation is in the form $y = mx + b$, so the slope is $m = \frac{2}{3}$ and the *y*-intercept is $b = -4$. To sketch the graph, we plot the *y*-intercept and then move 3 units to the right and 2 units upward as shown in Figure 11.

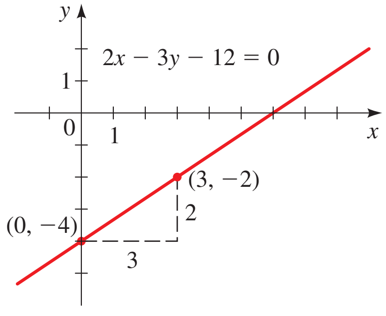


Figure 11

Now Try Exercise 67

Parallel and Perpendicular Lines

Since slope measures the steepness of a line, it seems reasonable that parallel lines should have the same slope. In fact, we can prove this.

Parallel Lines

Two nonvertical lines are parallel if and only if they have the same slope.

Proof Let the lines l_1 and l_2 in Figure 12 have slopes m_1 and m_2. If the lines are parallel, then the right triangles ABC and DEF are similar, so

$$m_1 = \frac{d(B, C)}{d(A, C)} = \frac{d(E, F)}{d(D, F)} = m_2$$

Conversely, if the slopes are equal, then the triangles will be similar, so $\angle BAC = \angle EDF$ and hence the lines are parallel. ■

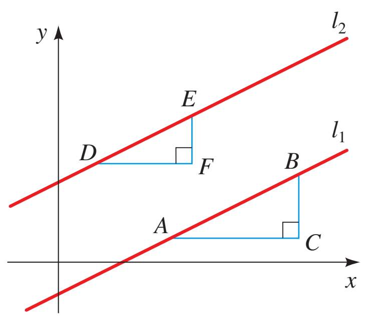


Figure 12

Example 7 ■ Finding an Equation of a Line Parallel to a Given Line

Find an equation of the line through the point $(5, 2)$ that is parallel to the line $4x + 6y + 5 = 0$.

Solution First we write the equation of the given line in slope-intercept form.

$$4x + 6y + 5 = 0$$
$$6y = -4x - 5 \qquad \text{Subtract } 4x + 5$$
$$y = -\tfrac{2}{3}x - \tfrac{5}{6} \qquad \text{Divide by 6}$$

So the line has slope $m = -\frac{2}{3}$. Since the required line is parallel to the given line, it also has slope $m = -\frac{2}{3}$. From the point-slope form of the equation of a line we get

$$\begin{aligned} y - 2 &= -\tfrac{2}{3}(x - 5) && \text{Slope } m = -\tfrac{2}{3}, \text{ point } (5, 2) \\ 3y - 6 &= -2x + 10 && \text{Multiply by 3} \\ 2x + 3y - 16 &= 0 && \text{Rearrange} \end{aligned}$$

Thus an equation of the required line is $2x + 3y - 16 = 0$.

Now Try Exercise 43 ■

The condition for perpendicular lines is not as obvious as that for parallel lines.

Perpendicular Lines

Two lines with slopes m_1 and m_2 are perpendicular if and only if $m_1 m_2 = -1$, that is, their slopes are negative reciprocals:

$$m_2 = -\frac{1}{m_1}$$

Also, a horizontal line (slope 0) is perpendicular to a vertical line (no slope).

Proof In Figure 13 we show two lines intersecting at the origin. (If the lines intersect at some other point, we consider lines parallel to these that intersect at the origin. These lines have the same slopes as the original lines.)

If the lines l_1 and l_2 have slopes m_1 and m_2, then their equations are $y = m_1 x$ and $y = m_2 x$. Notice that $A(1, m_1)$ lies on l_1 and $B(1, m_2)$ lies on l_2. By the Pythagorean Theorem and its converse (see Appendix A) $OA \perp OB$ if and only if

$$[d(O, A)]^2 + [d(O, B)]^2 = [d(A, B)]^2$$

By the Distance Formula this becomes

$$\begin{aligned} (1^2 + m_1^2) + (1^2 + m_2^2) &= (1 - 1)^2 + (m_2 - m_1)^2 \\ 2 + m_1^2 + m_2^2 &= m_2^2 - 2m_1 m_2 + m_1^2 \\ 2 &= -2m_1 m_2 \\ m_1 m_2 &= -1 \end{aligned}$$

■

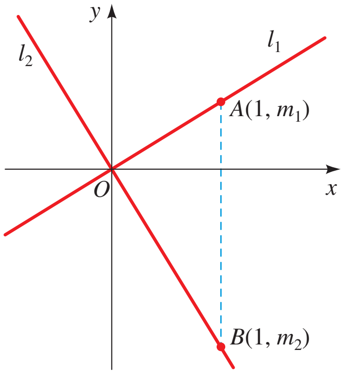


Figure 13

Example 8 ■ Perpendicular Lines

Show that the points $P(3, 3)$, $Q(8, 17)$, and $R(11, 5)$ are the vertices of a right triangle.

Solution The slopes of the lines containing PR and QR are, respectively,

$$m_1 = \frac{5 - 3}{11 - 3} = \frac{1}{4} \qquad \text{and} \qquad m_2 = \frac{5 - 17}{11 - 8} = -4$$

Since $m_1 m_2 = -1$, these lines are perpendicular, so PQR is a right triangle. It is sketched in Figure 14.

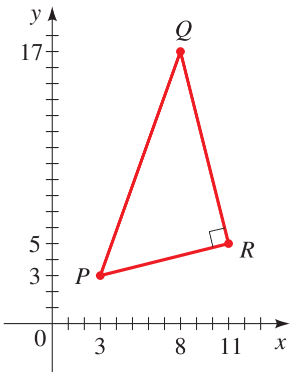


Figure 14

Now Try Exercise 81 ■

Example 9 ■ Finding an Equation of a Line Perpendicular to a Given Line

Find an equation of the line that is perpendicular to the line $4x + 6y + 5 = 0$ and passes through the origin.

Solution In Example 7 we found that the slope of the line $4x + 6y + 5 = 0$ is $-\frac{2}{3}$. Thus the slope of a perpendicular line is the negative reciprocal, that is, $\frac{3}{2}$. Since the required line passes through $(0, 0)$, the point-slope form gives

$$y - 0 = \tfrac{3}{2}(x - 0) \qquad \text{Slope } m = \tfrac{3}{2}, \text{ point } (0, 0)$$

$$y = \tfrac{3}{2}x \qquad \text{Simplify}$$

Now Try Exercise 47 ■

Example 10 ■ A Family of Lines

Graph the family of lines. What property do the lines share in common?

$$y = \tfrac{1}{2}x + b \qquad \text{for} \quad b = -2, -1, 0, 1, 2$$

Solution For $b = -2$ we get the line $y = \frac{1}{2}x - 2$, which has slope $\frac{1}{2}$ and y-intercept -2. In Figure 15, we graph this line along with the lines corresponding to $b = -1, 0, 1, 2$. From the equations we see that the lines all have slope $\frac{1}{2}$, so they are parallel. The graphs in Figure 15 confirm this observation.

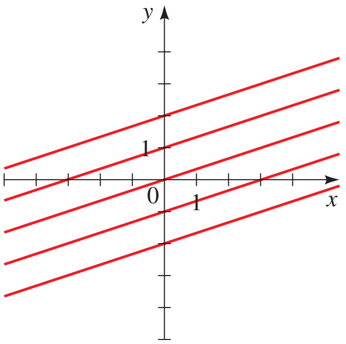


Figure 15 | $y = \frac{1}{2}x + b$

Now Try Exercise 53 ■

Example 11 ■ Application: Interpreting Slope

A hose is being used to fill a swimming pool. The water depth y (in feet) in the pool t hours after the hose is turned on is given by

$$y = 1.5t + 2$$

(a) Find the slope and y-intercept of the graph of this equation.

(b) What do the slope and y-intercept represent?

Solution

(a) This is the equation of a line with slope 1.5 and y-intercept 2.

(b) The slope represents an increase of 1.5 ft in water depth for every hour. The y-intercept indicates that the water depth was 2 ft at the time the hose was turned on.

Now Try Exercise 89 ■

1.10 Exercises

Concepts

1. We find the "steepness," or slope, of a line passing through two points by dividing the difference in the ____-coordinates of these points by the difference in the ____-coordinates. So the line passing through the points (0, 1) and (2, 5) has slope ________.

2. A line has the equation $y = 3x + 2$.

(a) This line has slope ________.

(b) Any line parallel to this line has slope ________.

(c) Any line perpendicular to this line has slope ________.

3. The point-slope form of the equation of the line with slope 3 passing through the point (1, 2) is ________.

4. For the linear equation $2x + 3y - 12 = 0$, the x-intercept is ________ and the y-intercept is ________. The equation in slope-intercept form is $y =$ ________________. The slope of the graph of this equation is ________.

5. The slope of a horizontal line is ________. The equation of the horizontal line passing through $(2, 3)$ is ________.

6. The slope of a vertical line is ________. The equation of the vertical line passing through $(2, 3)$ is ________.

7. *Yes or No*? If *No*, give a reason.

(a) Is the graph of $y = -3$ a horizontal line?

(b) Is the graph of $x = -3$ a vertical line?

(c) Does a line perpendicular to a horizontal line have slope 0?

(d) Does a line perpendicular to a vertical line have slope 0?

8. Sketch a graph of the lines $y = -3$ and $x = -3$. Are the lines perpendicular?

Skills

9–16 ■ Slope Find the slope of the line through P and Q.

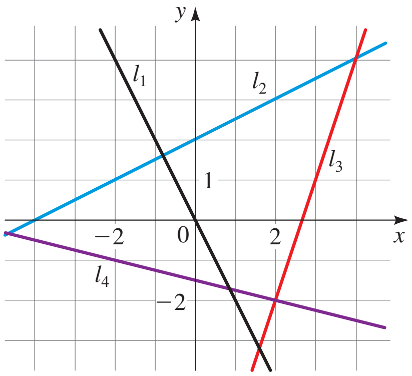
9. $P(-1, 2)$, $Q(0, 0)$ **10.** $P(0, 0)$, $Q(3, -1)$

11. $P(-3, 2)$, $Q(3, -3)$ **12.** $P(-5, 1)$, $Q(3, -2)$

13. $P(5, 4)$, $Q(0, 4)$ **14.** $P(4, -1)$, $Q(-2, -3)$

15. $P(8, 3)$, $Q(6, 5)$ **16.** $P(3, -2)$, $Q(6, -2)$

17. Slope Find the slopes of the lines l_1, l_2, l_3, and l_4 in the figure below.



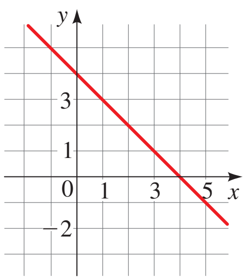
18. Slope

(a) Sketch lines through $(0, 0)$ with slopes $1, 0, \frac{1}{2}, 2$, and -1.

(b) Sketch lines through $(0, 0)$ with slopes $\frac{1}{3}, \frac{1}{2}, -\frac{1}{3}$, and 3.

19–22 ■ Equations of Lines Find an equation for the line whose graph is sketched.

19.



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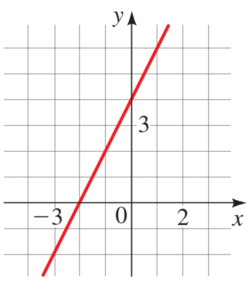

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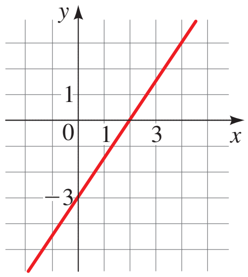

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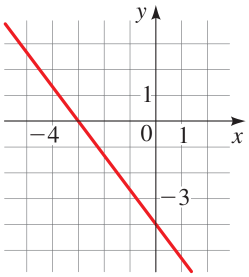


23–50 ■ Finding Equations of Lines Find an equation of the line that satisfies the given conditions.

23. Slope 3; y-intercept -2

24. Slope $\frac{2}{5}$; y-intercept 4

25. Through $(4, 1)$; slope 3

26. Through $(-2, 4)$; slope -1

27. Through $(1, 7)$; slope $\frac{2}{3}$

28. Through $(-3, -5)$; slope $-\frac{7}{2}$

29. Through $(2, 1)$ and $(1, 6)$

30. Through $(-1, -2)$ and $(4, 3)$

31. Through $(2, -5)$ and $(5, 1)$

32. Through $(1, 7)$ and $(4, 7)$

33. x-intercept 1; y-intercept -3

34. x-intercept -8; y-intercept 6

35. Through $(1, 3)$; slope 0

36. Through $(3, -2)$; slope undefined

37. Through $(2, -1)$; slope undefined

38. Through $(5, 1)$; slope 0

39. Through $(-1, 4)$; parallel to the line $y = 2x + 8$

40. Through $(-3, 2)$; perpendicular to the line $y = -\frac{1}{2}x + 7$

41. Through $(4, 5)$; parallel to the x-axis

42. Through $(4, 5)$; parallel to the y-axis

43. Through $(-3, -4)$; parallel to the line $3x + 2y = 4$

44. y-intercept 6; parallel to the line $2x + 3y + 4 = 0$

45. Through $(-1, 2)$; parallel to the line $x = 5$

46. Through $(2, 6)$; perpendicular to the line $y = 1$

47. Through $(-2, 1)$; perpendicular to the line $3x + 4y + 7 = 0$

48. Through $\left(\frac{1}{2}, -\frac{2}{3}\right)$; perpendicular to the line $4x - 8y = 1$

49. Through $(1, 7)$; parallel to the line passing through $(2, 5)$ and $(-2, 1)$

50. Through $(4, -10)$; perpendicular to the line passing through the points $(-3, 5)$ and $(6, 2)$

51. Finding Equations of Lines and Graphing

(a) Sketch the line with slope $\frac{3}{2}$ that passes through the point $(-2, 1)$.

(b) Find an equation for this line.

52. Finding Equations of Lines and Graphing

(a) Sketch the line with slope -2 that passes through the point $(4, -1)$.

(b) Find an equation for this line.

53–56 ■ Families of Lines Graph the given family of lines. What do the lines have in common?

53. $y = -2x + b$ for $b = 0, \pm 1, \pm 3, \pm 6$

54. $y = mx - 3$ for $m = 0, \pm\frac{1}{4}, \pm\frac{3}{4}, \pm\frac{3}{2}$

55. $y = m(x - 3)$ for $m = 0, \pm\frac{1}{4}, \pm\frac{3}{4}, \pm\frac{3}{2}$

56. $y = 2 + m(x + 3)$ for $m = 0, \pm\frac{1}{2}, \pm 1, \pm 2, \pm 6$

57–66 ■ Using Slopes and *y*-Intercepts to Graph Lines Find the slope and y-intercept of the line, and draw its graph.

57. $y = x - 4$

58. $y = -\frac{1}{2}x - 1$

59. $-2x + y = 7$

60. $2x - 5y = 0$

61. $4x + 5y = 10$

62. $3x - 4y = 12$

63. $y = 4$

64. $x = -5$

65. $x = 3$

66. $y = -2$

67–72 ■ Using *x*- and *y*-Intercepts to Graph Lines Find the x- and y-intercepts of the line, and draw its graph.

67. $3x - 2y - 6 = 0$

68. $6x - 7y - 42 = 0$

69. $\frac{1}{2}x - \frac{1}{3}y + 1 = 0$

70. $\frac{1}{3}x - \frac{1}{5}y - 2 = 0$

71. $y = 6x + 4$

72. $y = -4x - 10$

73–78 ■ Parallel and Perpendicular Lines The equations of two lines are given. Determine whether the lines are parallel, perpendicular, or neither.

73. $y = 2x + 3$; $2y - 4x - 5 = 0$

74. $y = \frac{1}{2}x + 4$; $2x + 4y = 1$

75. $2x - 5y = 8$; $10x + 4y = 1$

76. $15x - 9y = 2$; $3y - 5x = 5$

77. $7x - 3y = 2$; $9y + 21x = 1$

78. $6y - 2x = 5$; $2y + 6x = 1$

Skills Plus

79–82 ■ Using Slopes Verify the given geometric property.

79. Use slopes to show that $A(1, 1)$, $B(7, 4)$, $C(5, 10)$, and $D(-1, 7)$ are vertices of a parallelogram.

80. Use slopes to show that $A(-3, -1)$, $B(3, 3)$, and $C(-9, 8)$ are vertices of a right triangle.

81. Use slopes to show that $A(1, 1)$, $B(11, 3)$, $C(10, 8)$, and $D(0, 6)$ are vertices of a rectangle.

82. Use slopes to determine whether the given points are collinear (lie on a line).

(a) $(1, 1), (3, 9), (6, 21)$ **(b)** $(-1, 3), (1, 7), (4, 15)$

83. Perpendicular Bisector Find an equation of the perpendicular bisector of the line segment joining the points $A(1, 4)$ and $B(7, -2)$.

84. Area of a Triangle Find the area of the triangle formed by the coordinate axes and the line

$$2y + 3x - 6 = 0$$

85. Two-Intercept Form

(a) Show that if the x- and y-intercepts of a line are nonzero numbers a and b, then the equation of the line can be written in the form

$$\frac{x}{a} + \frac{y}{b} = 1$$

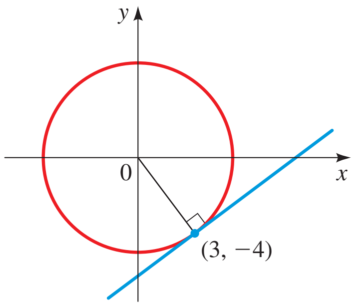
This is called the **two-intercept form** of the equation of a line.

(b) Use part (a) to find an equation of the line whose x-intercept is 6 and whose y-intercept is -8.

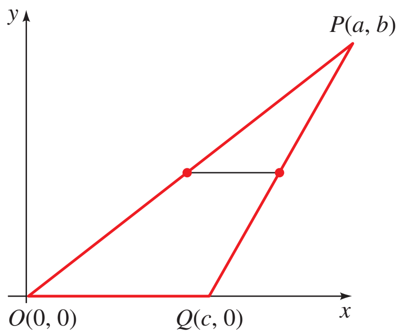
86. Tangent Line to a Circle

(a) Find an equation for the line tangent to the circle $x^2 + y^2 = 25$ at the point $(3, -4)$. (See the figure.)

(b) At what other point on the circle will a tangent line be parallel to the tangent line given in part (a)?



87. Triangle Midsegment Theorem Prove the following theorem from geometry: The line segment connecting the midpoints of two sides of a triangle is parallel to the third side and is half its length. [*Hint*: Place the triangle in a coordinate plane as shown in the figure, then use formulas from this section to verify the conclusions of the theorem.]



Applications

88. Global Warming Some scientists believe that the average surface temperature of the world has been rising steadily. The average surface temperature can be modeled by

$$T = 0.02t + 15.0$$

where T is temperature in °C and t is years since 1950.

(a) What do the slope and T-intercept represent?

(b) Use the equation to predict the average global surface temperature in 2050.

89. Drug Dosages If the recommended adult dosage for a drug is D (in mg), then to determine the appropriate dosage c for a child of age a, pharmacists use the equation

$$c = 0.0417D(a + 1)$$

Suppose the dosage for an adult is 200 mg.

(a) Find the slope. What does it represent?

(b) What is the dosage for a newborn?

90. Flea Market The manager of a flea market knows from past experience that if she charges x dollars for a rental space at the flea market, then the number y of spaces she can rent is given by the equation $y = 200 - 4x$.

(a) Sketch a graph of this linear equation. (Remember that the rental charge per space and the number of spaces rented must both be nonnegative quantities.)

(b) What do the slope, the y-intercept, and the x-intercept of the graph represent?

91. Production Cost A small-appliance manufacturer finds that if he produces x toaster ovens in a month, his production cost is given by the equation

$$y = 6x + 3000$$

where y is measured in dollars.

(a) Sketch a graph of this linear equation.

(b) What do the slope and y-intercept of the graph represent?

92. Temperature Scales The relationship between the Fahrenheit (F) and Celsius (C) scales is given by the equation $F = \frac{9}{5}C + 32$.

(a) Complete the following table to compare the two scales at the given values.

(b) Find the temperature at which the scales agree. [*Hint:* Suppose that a is the temperature at which the scales agree. Set $F = a$ and $C = a$. Then solve for a.]

C	F
$-30°$	
$-20°$	
$-10°$	
$0°$	
	$50°$
	$68°$
	$86°$

93. Crickets and Temperature Biologists have observed that the chirping rate of crickets of a certain species is related to temperature, and the relationship appears to be very nearly linear. A cricket produces 120 chirps per minute at 70°F and 168 chirps per minute at 80°F.

(a) Find the linear equation that relates the temperature t and the number of chirps per minute n.

(b) If the crickets are chirping at 150 chirps per minute, estimate the temperature.

94. Depreciation A small business buys a computer for \$4000. After 4 years the value of the computer is expected to be \$200. For accounting purposes the business uses *linear depreciation* to assess the value of the computer at a given time. This means that if V is the value of the computer at time t, then a linear equation is used to relate V and t.

(a) Find a linear equation that relates V and t.

(b) Sketch a graph of this linear equation.

(c) What do the slope and V-intercept of the graph represent?

(d) Find the depreciated value of the computer 3 years from the date of purchase.

95. Pressure and Depth At the surface of the ocean the water pressure is the same as the air pressure above the water, about 15 lb/in^2. Below the surface the water pressure increases by 4.34 lb/in^2 for every 10 ft of descent.

(a) Find an equation for the relationship between pressure P and depth d below the ocean surface.

(b) Sketch a graph of this linear equation.

(c) What do the slope and d-intercept of the graph represent?

(d) At what depth is the pressure 100 lb/in^2?

Discuss ■ Discover ■ Prove ■ Write

96. Discuss: What Does the Slope Mean? Suppose that the graph of the outdoor temperature over a certain period of time is a line. How is the weather changing if the slope of the line is positive? If it is negative? If it is zero?

97. Discuss: Collinear Points Suppose that you are given the coordinates of three points in the plane and you want to see whether they lie on the same line. How can you do this using slopes? Using the Distance Formula? Can you think of another method?

1.11 Solving Equations and Inequalities Graphically

■ Using Graphing Devices ■ Solving Equations Graphically ■ Solving Inequalities Graphically

If you are using a graphing device such as a math app on a computer or smartphone, be sure to familiarize yourself with how the app works. If you are using a graphing calculator, see Appendix C, *Graphing with a Graphing Calculator*, or Appendix D, *Using the TI-83/84 Graphing Calculator*. Go to **www.stewartmath.com**.

In the preceding two sections we studied the concept of the graph of an equation in two variables. We learned how to identify equations whose graphs are lines or circles and how to graph other equations by plotting points. **Graphing devices**, including math apps and graphing calculators, can perform the routine work of plotting the graph of an equation. In this section we use graphing devices to quickly draw graphs of equations and then use the graphs to obtain useful information about the equations. In particular, we'll solve equations and inequalities graphically.

■ Using Graphing Devices

Graphing devices draw the graph of an equation by plotting points, much as you would. They display a rectangular portion of the graph in a display window or viewing screen, which we call a **viewing rectangle**. If we choose the x-values to range over the interval $[a, b]$ and the y-values to range over the interval $[c, d]$ then the displayed portion of the graph lies in the rectangle

$$[a, b] \times [c, d] = \{(x, y) \mid a \le x \le b, c \le y \le d\}$$

as shown in Figure 1. We refer to this as the $[a, b]$ by $[c, d]$ viewing rectangle.

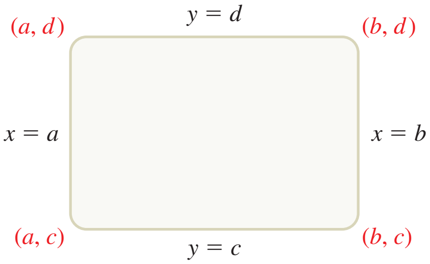


Figure 1 | The viewing rectangle $[a, b]$ by $[c, d]$

The device plots points of the form (x, y) for a certain number of values of x, equally spaced between a and b. If the equation is not defined for an x-value or if the corresponding y-value lies outside the viewing rectangle, the device ignores this value and moves on to the next x-value. The device connects each point to the preceding plotted point to form a representation of the graph of the equation.

Example 1 ■ Graphing an Equation with a Graphing Device

The graph in Figure 2 was obtained by using a graphing device to graph the equation

$$y = \frac{1}{1 + x^2}$$

in the viewing rectangle $[-5, 5]$ by $[-1, 2]$. Let's compare the graph with the equation. The graph has y-intercept 1, which is confirmed by the equation (because setting $x = 0$ in the equation gives $y = 1$). The graph appears to be symmetric about the y-axis and this is also confirmed by the equation (because replacing x by $-x$ in the equation leaves the equation unchanged). For points (x, y) on the graph (with $x > 0$), the larger the x-coordinate, the smaller the y-coordinate; this observation can be seen from the equation because larger values of x correspond to larger values of $1 + x^2$ and hence to smaller values of $y = 1/(1 + x^2)$. So we see that the graph gives visual confirmation of the algebraic properties of the equation.

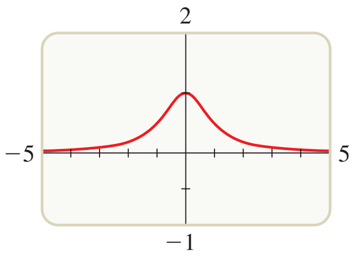


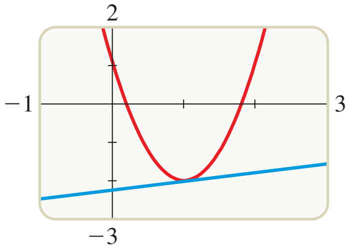
Figure 2 | Graph of $y = \dfrac{1}{1 + x^2}$

Now Try Exercise 5 ■

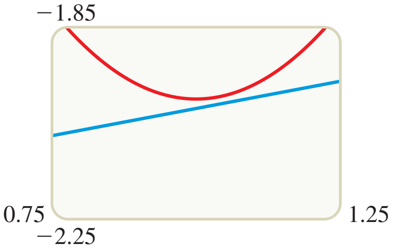
Example 2 ■ Two Graphs on the Same Screen

Graph the equations $y = 3x^2 - 6x + 1$ and $y = 0.23x - 2.25$ together in the viewing rectangle $[-1, 3]$ by $[-3, 2]$. Do the graphs intersect?

Solution Figure 3(a) shows the essential features of both graphs. One is a parabola and the other is a line. It looks as if the curves intersect near the point $(1, -2)$. However, if we zoom in on the area around this point as shown in Figure 3(b), we see that although the graphs almost touch, they do not actually intersect.



(a) [−1, 3] by [−3, 2]



(b) [0.75, 1.25] by [−2.25, −1.85]

Figure 3 | Graphs of the two equations in different viewing rectangles

Now Try Exercises 9 ■

■ Solving Equations Graphically

"Algebra is a merry science," Uncle Jakob would say. "We go hunting for a little animal whose name we don't know, so we call it *x*. When we bag our game we pounce on it and give it its right name."

ALBERT EINSTEIN

In Section 1.5 we solved equations *algebraically*. In this method, we view x as an *unknown* and then use the rules of algebra to "hunt it down" by isolating it on one side of the equation. Sometimes an equation may be difficult or impossible to solve algebraically. In this case we can solve it *graphically*. That is, we view x as a *variable*, sketch an appropriate graph, and get the solutions from the graph.

For example, to solve the one-variable equation $3x - 5 = 0$ graphically, we first draw a graph of the two-variable equation $y = 3x - 5$ that is obtained by setting the nonzero side of the equation equal to y. The solution(s) of the equation $y = 3x - 5$ are the values of x for which y is equal to zero. That is, the solutions are the x-intercepts of the graph.

Solving an Equation

Algebraic Method

Use the rules of algebra to isolate the unknown x on one side of the equation.

Example: $3x - 4 = 1$

$3x = 5$ Add 4

$x = \frac{5}{3}$ Divide by 3

The solution is $x = \frac{5}{3}$.

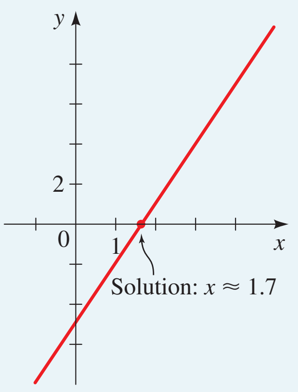
Graphical Method

Move all terms to one side, and set that side equal to y. Graph the resulting equation, and find the x-intercepts.

Example: $3x - 4 = 1$

$3x - 5 = 0$

Set $y = 3x - 5$ and graph. From the graph we see that the solution is $x \approx 1.7$



The advantage of the algebraic method is that it gives exact answers. Also, the process of unraveling the equation to arrive at the answer helps us understand the algebraic structure of the equation. On the other hand, for many equations it is difficult or impossible to isolate x.

The graphical method gives a numerical approximation to the answer—an advantage when a numerical answer is desired. (For example, an engineer might find an answer expressed as $x \approx 2.6$ more immediately useful than $x = \sqrt{7}$.) Also, graphing an equation helps us visualize how the solution is related to other values of the variable.

The *Discovery Project* referenced in Section 3.4 describes a numerical method for solving equations.

Example 3 ■ Solving a Quadratic Equation Algebraically and Graphically

Find all real solutions of each quadratic equation. Use both the algebraic method and the graphical method.

(a) $x^2 - 4x + 2 = 0$ **(b)** $x^2 - 4x + 4 = 0$ **(c)** $x^2 - 4x + 6 = 0$

Solution 1: Algebraic

The Quadratic Formula is discussed in Section 1.5.

You can check that the Quadratic Formula gives the following solutions.

(a) There are two real solutions, $x = 2 + \sqrt{2}$ and $x = 2 - \sqrt{2}$.

(b) There is one real solution, $x = 2$.

(c) There is no real solution. (The two complex solutions are $x = 2 + \sqrt{2}i$ and $x = 2 - \sqrt{2}i$.)

Solution 2: Graphical

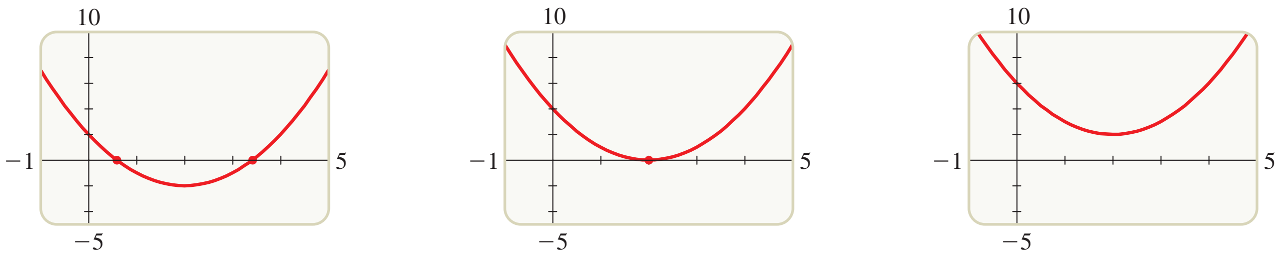
The x-intercepts are the x-values for which the corresponding y-values of the equation are equal to 0.

We use a graphing device to graph the equations $y = x^2 - 4x + 2$, $y = x^2 - 4x + 4$, and $y = x^2 - 4x + 6$ in Figure 4. By determining the x-intercepts of the graphs, we find the following solutions.

(a) The two x-intercepts give the two solutions $x \approx 0.6$ and $x \approx 3.4$.

(b) The one x-intercept gives the one solution $x = 2$.

(c) There is no x-intercept, so the equation has no real solution.



(a) $y = x^2 - 4x + 2$ (b) $y = x^2 - 4x + 4$ (c) $y = x^2 - 4x + 6$

Figure 4

Now Try Exercises 17, 19, and 23 ■

The graphs in Figure 4 show visually why a quadratic equation may have two solutions, one solution, or no real solution. We proved this fact algebraically in Section 1.5 when we studied the discriminant.

Bettmann/Getty Images

PIERRE DE FERMAT (1607–1665) was a French lawyer who became interested in mathematics at the age of 30. Because of his job as a magistrate, Fermat had little time to write complete proofs of his discoveries and often wrote them in the margin of whatever book he was reading at the time. After his death his copy of Diophantus's *Arithmetica* (see Section 1.2) was found to contain a particularly tantalizing comment. Where Diophantus discusses the solutions of $x^2 + y^2 = z^2$ (for example, $x = 3$, $y = 4$, and $z = 5$), Fermat states in the margin that for $n \geq 3$ there are no natural number solutions to the equation $x^n + y^n = z^n$. In other words, it's impossible for a cube to equal the sum of two cubes, a fourth power to equal the sum of two fourth powers, and so on. Fermat writes, "I have discovered a truly wonderful proof for this but the margin is too small to contain it." All the other margin comments in Fermat's copy of *Arithmetica* have been proved. This one, however, remained unproved, and it came to be known as "Fermat's Last Theorem."

In 1994, Andrew Wiles of Princeton University announced a proof of Fermat's Last Theorem, an astounding 350 years after it was conjectured. His proof is one of the most widely reported mathematical results in the popular press.

Example 4 ■ Another Graphical Method

Solve the equation algebraically and graphically: $5 - 3x = 8x - 20$

Solution 1: Algebraic

$$
\begin{aligned}
5 - 3x &= 8x - 20 && \text{Given equation} \\
-3x &= 8x - 25 && \text{Subtract 5} \\
-11x &= -25 && \text{Subtract } 8x \\
x &= \frac{-25}{-11} = 2\tfrac{3}{11} && \text{Divide by } -11 \text{ and simplify}
\end{aligned}
$$

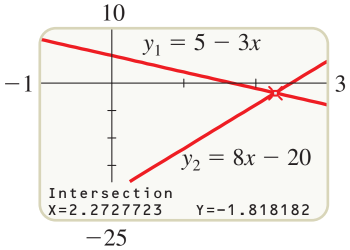


Figure 5

Solution 2: Graphical

We could move all terms to one side of the equal sign, set the result equal to y, and graph the resulting equation. Alternatively, we can graph the following two equations:

$$y_1 = 5 - 3x \qquad \text{and} \qquad y_2 = 8x - 20$$

The solution of the original equation will be the value of x that makes y_1 equal to y_2; that is, the solution is the x-coordinate of the intersection point of the two graphs. We see from the graph in Figure 5 that the solution is $x \approx 2.27$.

Now Try Exercise 13 ■

In the next example we use the graphical method to solve an equation that is very difficult to solve algebraically.

Example 5 ■ Solving an Equation in an Interval

Solve the equation $x^3 - 6x^2 + 9x = \sqrt{x}$ in the interval $[1, 6]$.

Solution We are asked to find all solutions x that satisfy $1 \le x \le 6$, so we use a graphing device to graph the equation in a viewing rectangle for which the x-values are restricted to this interval.

$$
\begin{aligned}
x^3 - 6x^2 + 9x &= \sqrt{x} && \text{Given equation} \\
x^3 - 6x^2 + 9x - \sqrt{x} &= 0 && \text{Subtract } \sqrt{x}
\end{aligned}
$$

Figure 6 shows the graph of the equation $y = x^3 - 6x^2 + 9x - \sqrt{x}$ in the viewing rectangle $[1, 6]$ by $[-5, 5]$. There are two x-intercepts in this viewing rectangle; zooming in, we see that the solutions are $x \approx 2.18$ and $x \approx 3.72$.

Graphing devices can accurately locate the intercepts of a graph and the points of intersection of two graphs.

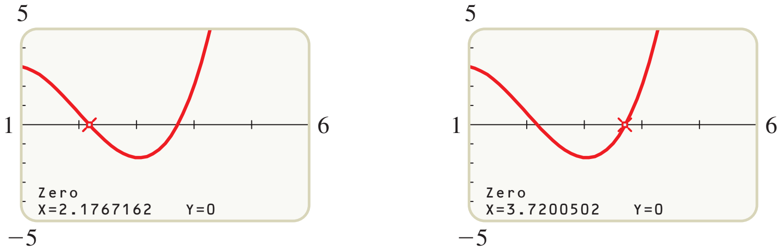


Figure 6

Now Try Exercise 25 ■

The equation in Example 5 actually has four solutions. You are asked to find the other two in Exercise 54.

Solving Inequalities Graphically

To solve a one-variable inequality such as $3x - 5 \geq 0$ graphically, we first draw a graph of the two-variable equation $y = 3x - 5$ that is obtained by setting the nonzero side of the inequality equal to a variable y. The solutions of the given inequality are the values of x for which y is greater than or equal to 0. That is, the solutions are the values of x for which the graph is above the x-axis.

Solving an Inequality

Algebraic Method

Use the rules of algebra to isolate the unknown x on one side of the inequality.

Example: $3x - 4 \geq 1$

$3x \geq 5$ Add 4

$x \geq \frac{5}{3}$ Divide by 3

The solution is $\left[\frac{5}{3}, \infty\right)$.

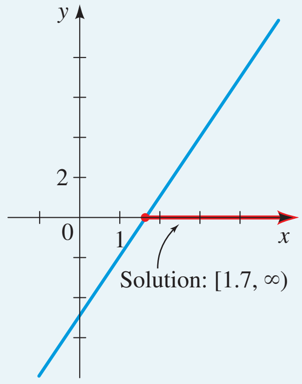
Graphical Method

Move all terms to one side, and set that side equal to y. Graph the resulting equation, and find the values of x where the graph is above or on the x-axis.

Example: $3x - 4 \geq 1$

$3x - 5 \geq 0$

Set $y = 3x - 5$ and graph. From the graph we see that the solution is approximately $[1.7, \infty)$.



Example 6 ■ Solving an Inequality Graphically

Solve the inequality $x^3 - 5x^2 \geq -8$.

Solution We write the inequality as

$$x^3 - 5x^2 + 8 \geq 0$$

We then use a graphing device to graph the equation

$$y = x^3 - 5x^2 + 8$$

and find the x-intercepts, as shown in Figure 7. The solution of the inequality consists of those intervals on which the graph lies on or above the x-axis. From the graph in Figure 7 we see that, rounded to one decimal place, the solution is $[-1.1, 1.5] \cup [4.6. \infty)$.

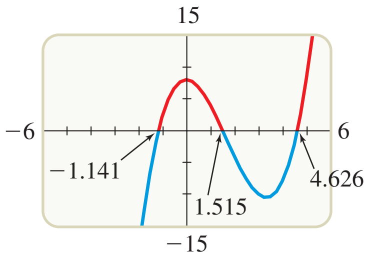


Figure 7 | $x^3 - 5x^2 + 8 \geq 0$

Now Try Exercise 41 ■

Example 7 ■ Another Graphical Method

Solve the inequality $3.7x^2 + 1.3x - 1.9 \leq 2.0 - 1.4x$.

Solution We use a graphing device to graph the equations

$$y_1 = 3.7x^2 + 1.3x - 1.9 \qquad \text{and} \qquad y_2 = 2.0 - 1.4x$$

The graphs are shown in Figure 8. We are interested in those values of x for which $y_1 \leq y_2$; these are points for which the graph of y_2 lies on or above the graph of y_1. To determine the appropriate interval, we look for the x-coordinates of points where the graphs intersect. We conclude that the solution is (approximately) the interval $[-1.45, 0.72]$.

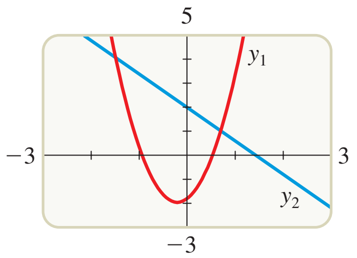


Figure 8 | $y_1 = 3.7x^2 + 1.3x - 1.9$
$y_2 = 2.0 - 1.4x$

Now Try Exercise 45 ■

1.11 Exercises

Concepts

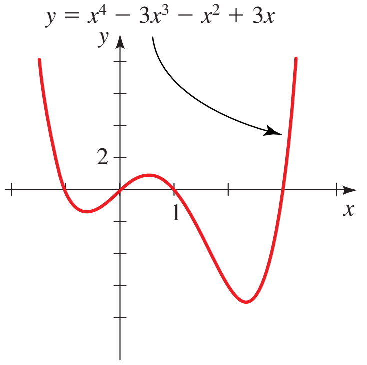
1. The solutions of the equation $x^2 - 2x - 3 = 0$ are the ____-intercepts of the graph of $y = x^2 - 2x - 3$.

2. The solutions of the inequality $x^2 - 2x - 3 > 0$ are the x-coordinates of the points on the graph of $y = x^2 - 2x - 3$ that lie ________ the x-axis.

3. The figure shows a graph of $y = x^4 - 3x^3 - x^2 + 3x$. Use the graph to find the solution(s) of each of the following.

(a) The equation $x^4 - 3x^3 - x^2 + 3x = 0$

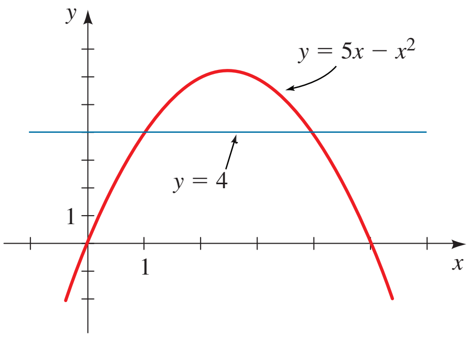
(b) The inequality $x^4 - 3x^3 - x^2 + 3x \le 0$



4. The figure shows the graphs of $y = 5x - x^2$ and $y = 4$. Use the graphs to find the solution(s) of each of the following.

(a) The equation $5x - x^2 = 4$

(b) The inequality $5x - x^2 > 4$



Skills

5–8 An equation is given.

(a) Use a graphing device to graph the equation in the given viewing rectangle.

(b) Find the x- and y-intercepts from the graph and confirm your answers algebraically (from the equation).

(c) If the graph appears to be symmetric, confirm that the equation satisfies the corresponding symmetry property.

5. $y = x^3 - x^2$; $[-2, 2]$ by $[-1, 1]$

6. $y = x^4 - 2x^3$; $[-2, 3]$ by $[-3, 3]$

7. $y = -\dfrac{2}{x^2 + 1}$; $[-5, 5]$ by $[-3, 1]$

8. $y = \sqrt[3]{1 - x^2}$; $[-5, 5]$ by $[-5, 3]$

9–12 Do the given graphs intersect in the indicated viewing rectangle? If so, how many points of intersection are there?

9. $y = -3x^2 + 6x - \frac{1}{2}, y = \sqrt{7 - \frac{7}{12}x^2}$; $[-4, 4]$ by $[-1, 3]$

10. $y = \sqrt{49 - x^2}, y = \frac{1}{5}(41 - 3x)$; $[-8, 8]$ by $[-1, 8]$

11. $y = 6 - 4x - x^2, y = 3x + 18$; $[-6, 2]$ by $[-5, 20]$

12. $y = x^3 - 4x, y = x + 5$; $[-4, 4]$ by $[-15, 15]$

13–24 ■ Equations Solve the equation both algebraically and graphically.

13. $3x + 2 = 5x - 4$

14. $\frac{2}{3}x + 11 = 1 - x$

15. $\dfrac{2}{x} + \dfrac{1}{2x} = 7$

16. $\dfrac{4}{x + 2} - \dfrac{6}{2x} = \dfrac{5}{2x + 4}$

17. $4x^2 - 8 = 0$

18. $x^3 + 10x^2 = 0$

19. $x^2 + 9 = 0$

20. $x^2 + 3 = 2x$

21. $81x^4 = 256$

22. $2x^5 - 243 = 0$

23. $(x - 5)^4 - 80 = 0$

24. $3(x + 5)^3 = 72$

25–32 ■ Equations Solve the equation graphically in the given interval. State each solution rounded to two decimals.

25. $x^2 - 11x + 30 = 0$; $[2, 8]$

26. $x^2 - 0.75x + 0.125 = 0$; $[-2, 2]$

27. $x^3 - 6x^2 + 11x - 6 = 0$; $[-1, 4]$

28. $16x^3 + 16x^2 = x + 1$; $[-2, 2]$

29. $x - \sqrt{x + 1} = 0$; $[-1, 5]$

30. $1 + \sqrt{x} = \sqrt{1 + x^2}$; $[-1, 5]$

31. $x^{1/3} - x = 0$; $[-3, 3]$

32. $x^{1/2} + x^{1/3} - x = 0$; $[-1, 5]$

33–36 ■ Equations Use the graphical method to solve the equation in the indicated exercise from Section 1.5.

33. Exercise 87

34. Exercise 88

35. Exercise 89

36. Exercise 90

37–40 ■ Equations Find all real solutions of the equation, and state each rounded to two decimals.

37. $x^3 - 2x^2 - x - 1 = 0$

38. $x^4 - 8x^2 + 2 = 0$

39. $x(x - 1)(x + 2) = \frac{1}{6}x$

40. $x^4 = 16 - x^3$

41–48 ■ Inequalities Find the solutions of the inequality by drawing appropriate graphs. State each solution rounded to two decimals.

41. $x^2 \le 3x + 10$

42. $0.5x^2 + 0.875x \le 0.25$

43. $x^3 + 11x \le 6x^2 + 6$

44. $16x^3 + 24x^2 > -9x - 1$

45. $x^{1/3} < x$

46. $\sqrt{0.5x^2 + 1} \le 2|x|$

47. $(x + 1)^2 < (x - 1)^2$

48. $(x + 1)^2 \le x^3$

49–52 ■ Inequalities Use the graphical method to solve the inequality in the indicated exercise from Section 1.8.

49. Exercise 45 **50.** Exercise 46

51. Exercise 55 **52.** Exercise 56

Skills Plus

53. Another Graphical Method In Example 4 we solved the equation $5 - 3x = 8x - 20$ by drawing graphs of two equations. Solve the equation by drawing a graph of only one equation. Compare your answer to the one obtained in Example 4.

54. Finding More Solutions In Example 5 we found two solutions of the equation $x^3 - 6x^2 + 9x = \sqrt{x}$ in the interval $[1, 6]$. Find two more solutions, and state each rounded to two decimals.

Applications

55. Estimating Profit An appliance manufacturer estimates that the profit y (in dollars) generated by producing x cooktops per month is given by the equation

$$y = 10x + 0.5x^2 - 0.001x^3 - 5000$$

where $0 \le x \le 450$.

(a) Graph the equation.

(b) How many cooktops must be produced to begin generating a profit?

(c) For what range of values of x is the company's profit greater than \$15,000?

56. How Far Can You See? If you stand on a ship in a calm sea, then your height x (in feet) above sea level is related to the farthest distance y (in miles) that you can see by the equation

$$y = \sqrt{1.5x + \left(\frac{x}{5280}\right)^2}$$

(a) Graph the equation for $0 \le x \le 100$.

(b) How high above sea level do you have to be standing to be able to see 10 miles?

Discuss ■ Discover ■ Prove ■ Write

57. Write: Algebraic and Graphical Solution Methods Write a short essay comparing the algebraic and graphical methods for solving equations. Make up your own examples to illustrate the advantages and disadvantages of each method.

58. Discuss: Enter Equations Carefully A student wishes to graph the equations

$$y = x^{1/3} \quad \text{and} \quad y = \frac{x}{x + 4}$$

on the same screen, so the student enters the following information into a graphing device:

```
Y1 = X^1/3    Y2 = X/X + 4
```

The device graphs two lines instead of the desired equations. What went wrong?

1.12 Modeling Variation

■ Direct Variation ■ Inverse Variation ■ Combining Different Types of Variation

When scientists talk about a *mathematical model* for a real-world phenomenon, they often mean an equation or formula that describes the relationship of one physical quantity to another. For instance, the model may be a formula that gives the relationship between the pressure and volume of a gas. In this section we study a kind of modeling—which occurs frequently in the sciences—called *variation.*

■ Direct Variation

One type of variation is called *direct variation*; it occurs when one quantity is a constant multiple of the other. We use a formula of the form $y = kx$ to model this relationship.

Direct Variation

If the quantities x and y are related by an equation

$$y = kx$$

for some constant $k \neq 0$, then we say that y **varies directly as** x, or y is **directly proportional to** x, or simply y **is proportional to** x. The constant k is called the **constant of proportionality**.

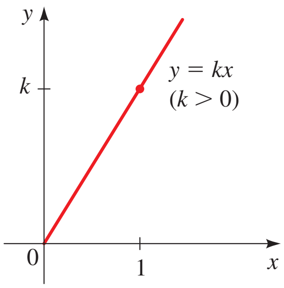


Figure 1

Recall that the graph of an equation of the form $y = mx + b$ is a line with slope m and y-intercept b. So the graph of an equation $y = kx$ that describes direct variation is a line with slope k and y-intercept 0. (See Figure 1.)

Example 1 ■ Direct Variation

During a thunderstorm you see the lightning before you hear the thunder because light travels much faster than sound. The distance between you and the storm varies directly as the time interval between the lightning and the thunder.

(a) Suppose that the thunder from a storm 5400 ft away takes 5 s to reach you. Determine the constant of proportionality, and write the equation for the variation.

(b) Sketch the graph of this equation. What does the constant of proportionality represent?

(c) If the time interval between the lightning and thunder is now 8 s, how far away is the storm?

Solution

(a) Let d be the distance from you to the storm, and let t be the length of the time interval. We are given that d varies directly as t, so

$$d = kt$$

where k is a constant. To find k, we use the fact that $t = 5$ when $d = 5400$. Substituting these values in the equation, we get

$$5400 = k(5) \qquad \text{Substitute}$$

$$k = \frac{5400}{5} = 1080 \qquad \text{Solve for } k$$

Substituting this value of k in the equation for d, we obtain

$$d = 1080t$$

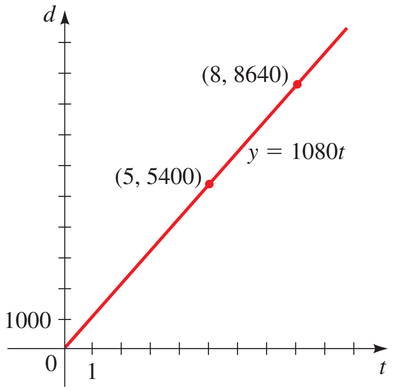


Figure 2

(b) The graph of the equation $d = 1080t$ is a line through the origin with slope 1080, as shown in Figure 2. The constant $k = 1080$ is the approximate speed of sound (in ft/s).

(c) When $t = 8$, we have

$$d = 1080 \cdot 8 = 8640$$

So the storm is 8640 ft $\approx$ 1.6 mi away.

Now Try Exercises 19 and 35

Inverse Variation

Another formula that is frequently used in mathematical modeling is $y = k/x$, where k is a constant.

Inverse Variation

If the quantities x and y are related by the equation

$$y = \frac{k}{x}$$

for some constant $k \neq 0$, then we say that y **varies inversely as** x or y **is inversely proportional to** x. The constant k is called the **constant of proportionality**.

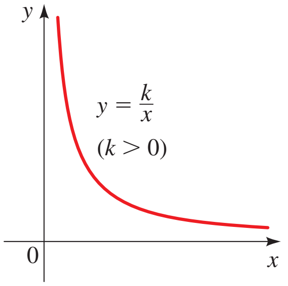


Figure 3 | Inverse variation

The graph of $y = k/x$ for $x > 0$ is shown in Figure 3 for the case $k > 0$. It gives a picture of what happens when y is inversely proportional to x.

Example 2 ■ Inverse Variation

Boyle's Law states that when a sample of gas is compressed at a constant temperature, the pressure of the gas is inversely proportional to the volume of the gas.

(a) Suppose the pressure of a sample of air that occupies 0.106 m^3 at 25°C is 50 kPa. Find the constant of proportionality, and write the equation that expresses the inverse proportionality. Sketch a graph of this equation.

(b) If the sample expands to a volume of 0.3 m^3, find the new pressure.

Solution

(a) Let P be the pressure of the sample of gas, and let V be its volume. Then, by the definition of inverse proportionality, we have

$$P = \frac{k}{V}$$

where k is a constant. To find k, we use the fact that $P = 50$ when $V = 0.106$. Substituting these values in the equation, we get

$$50 = \frac{k}{0.106} \qquad \text{Substitute}$$

$$k = (50)(0.106) = 5.3 \qquad \text{Solve for } k$$

Putting this value of k in the equation for P, we have

$$P = \frac{5.3}{V}$$

Since V represents volume (which is never negative), we sketch only the part of the graph for which $V > 0$. The graph is shown in Figure 4.

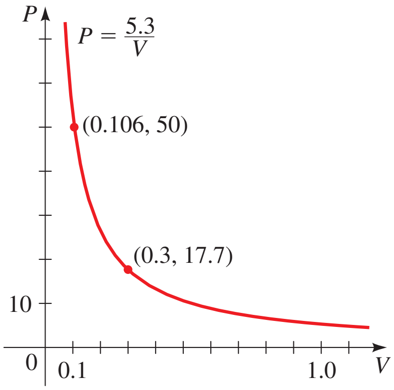


Figure 4

(b) When $V = 0.3$, we have

$$P = \frac{5.3}{0.3} \approx 17.7$$

so the new pressure is about 17.7 kPa.

Now Try Exercises 21 and 43

■ Combining Different Types of Variation

In the sciences, relationships between three or more variables are common, and any combination of the different types of proportionality that we have discussed is possible. For example, if the quantities x, y, and z are related by the equation

$$z = kxy$$

then we say that z is **proportional to the product** of x and y. We can also express this relationship by saying that z **varies jointly** as x and y or that z **is jointly proportional to** x and y. If the quantities x, y, and z are related by the equation

$$z = k\frac{x}{y}$$

we say that z **is proportional to** x **and inversely proportional to** y or that z **varies directly as** x **and inversely as** y.

Example 3 ■ Combining Variations

The apparent brightness B of a light source (measured in W/m^2) is directly proportional to the luminosity L (measured in watts, W) of the light source and inversely proportional to the square of the distance d from the light source (measured in meters).

(a) Write an equation that expresses this variation.

(b) If the distance is doubled, by what factor will the brightness change?

(c) If the distance is cut in half and the luminosity is tripled, by what factor will the brightness change?

Solution

(a) Since B is directly proportional to L and inversely proportional to d^2, we have

$$B = k\frac{L}{d^2} \qquad \text{Brightness at distance } d \text{ and luminosity } L$$

where k is a constant.

(b) To obtain the brightness at double the distance, we replace d by $2d$ in the equation we obtained in part (a).

$$B = k\frac{L}{(2d)^2} = \frac{1}{4}\left(k\frac{L}{d^2}\right) \qquad \text{Brightness at distance } 2d$$

Comparing this expression with that obtained in part (a), we see that the brightness will be $\frac{1}{4}$ the original brightness.

Discovery Project ■ Proportionality: Shape and Size

Many real-world quantities are related by proportionalities. In this project we use the proportionality symbol $\propto$ to relate proportionalities in the natural world. For example, for animals of the same shape, the skin area and volume are proportional, in different ways, to the length of the animal. In one situation we use proportionality to determine how a frog's size relates to its sensitivity to pollutants in the environment. You can find the project at **www.stewartmath.com**.

(c) To obtain the brightness at half the distance d and triple the luminosity L, we replace d by $d/2$ and L by $3L$ in the equation we obtained in part (a).

$$B = k\frac{3L}{\left(\frac{1}{2}d\right)^2} = \frac{3}{\frac{1}{4}}\left(k\frac{L}{d^2}\right) = 12\left(k\frac{L}{d^2}\right)$$ Brightness at distance $\frac{1}{2}d$ and luminosity $3L$

Comparing this expression with that obtained in part (a), we see that the brightness will be 12 times the original brightness.

Now Try Exercises 23 and 45 ■

The relationship between apparent brightness, actual brightness (or luminosity), and distance is used in estimating distances to stars (see Exercise 56).

Example 4 ■ Newton's Law of Gravity

Newton's Law of Gravity says that two objects with masses m_1 and m_2 attract each other with a force F that is jointly proportional to their masses and inversely proportional to the square of the distance r between the objects. Express Newton's Law of Gravity as an equation.

Solution Using the definitions of joint and inverse variation and the traditional notation G for the gravitational constant of proportionality, we have

$$F = G\frac{m_1 m_2}{r^2}$$

Now Try Exercises 31 and 47 ■

If m_1 and m_2 are fixed masses, then the gravitational force between them is $F = C/r^2$ (where $C = Gm_1m_2$ is a constant). Figure 5 shows the graph of this equation for $r > 0$ with $C = 1$. Observe how the gravitational attraction decreases with increasing distance.

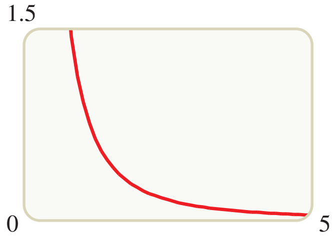


Figure 5 | Graph of $F = \dfrac{1}{r^2}$

Note Like the Law of Gravity, many laws of nature are *inverse square laws*. There is a geometric reason for this. Imagine a force or energy originating from a point source and spreading its influence equally in all directions, just like the light source in Example 3 or the gravitational force exerted by a planet in Example 4. The influence of the force or energy at a distance r from the source is spread out over the surface of a sphere of radius r, which has area $A = 4\pi r^2$. (See Figure 6.) So the intensity I at a distance r from the source is the source strength S divided by the area A of the sphere:

$$I = \frac{S}{4\pi r^2} = \frac{k}{r^2}$$

where k is the constant $S/(4\pi)$. Thus point sources of light, sound, gravity, electromagnetic fields, and radiation must all obey inverse square laws, simply because of the geometry of space.

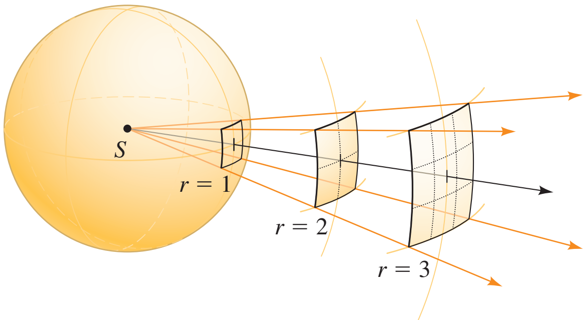


Figure 6 | Energy from a point source S

1.12 Exercises

Concepts

1. If the quantities x and y are related by the equation $y = 5x$, then we say that y is ________ ________ to x and the constant of ________ is 5.

2. If the quantities x and y are related by the equation $y = \dfrac{5}{x}$, then we say that y is ________ ________ to x and the constant of ________ is 5.

3. If the quantities x, y, and z are related by the equation $z = 5\dfrac{x}{y}$, then we say that z is ________ ________ to x and ________ ________ to y.

4. If z is directly proportional to the product of x and y and if z is 10 when x is 4 and y is 5, then x, y, and z are related by the equation $z =$ ________.

5–6 ■ In each equation, is y directly proportional, inversely proportional, or not proportional to x?

5. **(a)** $y = 3x$ **(b)** $y = 3x + 1$

6. **(a)** $y = \dfrac{3}{x+1}$ **(b)** $y = \dfrac{3}{x}$

Skills

7–18 ■ **Equations of Proportionality** Write an equation that expresses the statement.

7. T varies directly as x.

8. P is directly proportional to w.

9. v is inversely proportional to z.

10. w is proportional to the product of m and n.

11. y is proportional to s and inversely proportional to t.

12. P varies inversely as T.

13. z is proportional to the square root of y.

14. A is proportional to the square of x and inversely proportional to the cube of t.

15. V is proportional to the product of l, w, and h.

16. S is proportional to the product of the squares of r and θ.

17. R is proportional to the product of the squares of P and t and inversely proportional to the cube of b.

18. A is jointly proportional to the square roots of x and y.

19–30 ■ **Constants of Proportionality** Express the statement as an equation. Use the given information to find the constant of proportionality.

19. y is directly proportional to x. If $x = 8$, then $y = 32$.

20. w is inversely proportional to t. If $t = 8$, then $w = 3$.

21. A varies inversely as r. If $r = 5$, then $A = 15$.

22. P is directly proportional to T. If $T = 72$, then $P = 60$.

23. A is directly proportional to x and inversely proportional to t. If $x = 7$ and $t = 3$, then $A = 42$.

24. S is proportional to the product of p and q. If $p = 7$ and $q = 20$, then $S = 350$.

25. W is inversely proportional to the square of r. If $r = 3$, then $W = 24$.

26. t is proportional to the product of x and y and inversely proportional to r. If $x = 10$, $y = 15$, and $r = 12$, then $t = 125$.

27. C is jointly proportional to l, w, and h. If $l = w = h = 2$, then $C = 128$.

28. H is jointly proportional to the squares of l and w. If $l = 2$ and $w = \frac{1}{3}$, then $H = 36$.

29. R is inversely proportional to the square root of x. If $x = 121$, then $R = 2.5$.

30. M is jointly proportional to a, b, and c and inversely proportional to d. If a and d have the same value and if b and c are both 2, then $M = 128$.

31–34 ■ **Proportionality** A statement describing the relationship between the variables x, y, and z is given. **(a)** Express the statement as an equation of proportionality. **(b)** If x is tripled and y is doubled, by what factor does z change? (See Example 3.)

31. z varies directly as the cube of x and inversely as the square of y.

32. z is directly proportional to the square of x and inversely proportional to the fourth power of y.

33. z is jointly proportional to the cube of x and the fifth power of y.

34. z is inversely proportional to the square of x and the cube of y.

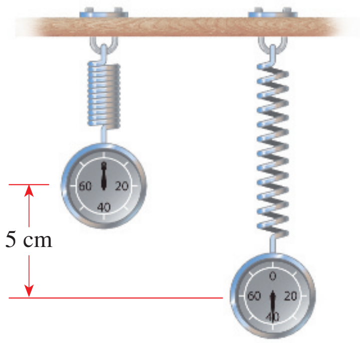
Applications

35. Hooke's Law Hooke's Law states that the force F needed to keep a spring stretched x units beyond its natural length is directly proportional to x. Here the constant of proportionality is called the **spring constant**.

(a) Write Hooke's Law as an equation.

(b) If a spring has a natural length of 5 cm and a force of 30 N is required to maintain the spring stretched to a length of 9 cm, find the spring constant.

(c) What force is needed to keep the spring stretched to a length of 11 cm?



36. Printing Costs The cost C of printing a magazine is jointly proportional to the number of pages p in the magazine and the number of magazines printed m.

(a) Write an equation that expresses this joint variation.

(b) Find the constant of proportionality if the printing cost is \$60,000 for 4000 copies of a 120-page magazine.

(c) How much would the printing cost be for 5000 copies of a 92-page magazine?

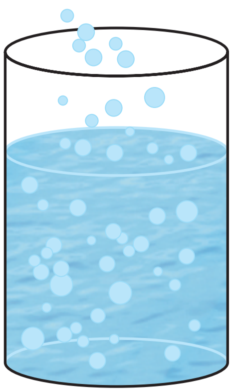
37. Power from a Windmill The power P that can be obtained from a windmill is directly proportional to the cube of the wind speed s.

(a) Write an equation that expresses this variation.

(b) Find the constant of proportionality for a windmill that produces 96 watts of power when the wind is blowing at 20 mi/h.

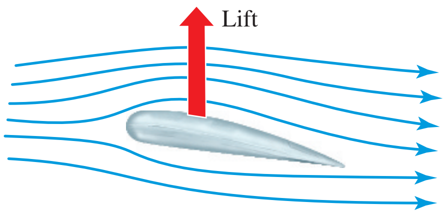
(c) How much power will this windmill produce if the wind speed increases to 30 mi/h?

38. Solubility of Carbon Dioxide The amount of carbon dioxide (CO_2) that can be dissolved in water is inversely proportional to the temperature of the water (in kelvins, K). An open glass of soda at a temperature of 273 K can dissolve about 3 g of CO_2. How much CO_2 can dissolve in the same bottle at the warmer temperature of 298 K?



39. Stopping Distance The stopping distance D of a car after the brakes have been applied varies directly as the square of the speed s. A certain car traveling at 40 mi/h can stop in 150 ft. What is the maximum speed it can be traveling if it needs to stop in 200 ft?

40. Aerodynamic Lift The lift L on an airplane wing at takeoff varies jointly as the square of the speed s of the plane and the area A of its wings. A plane with a wing area of 500 ft^2 traveling at 50 mi/h experiences a lift of 1700 lb. How much lift would a plane with a wing area of 600 ft^2 traveling at 40 mi/h experience?



41. Drag Force on a Boat The drag force F on a boat is jointly proportional to the wetted surface area A on the hull and the square of the speed s of the boat. A boat experiences a drag force of 220 lb when traveling at 5 mi/h with a wetted surface area of 40 ft^2. How fast must a boat be traveling if it has 28 ft^2 of wetted surface area and is experiencing a drag force of 175 lb?

42. Kepler's Third Law Kepler's Third Law of planetary motion states that the square of the period T of a planet (the time it takes for the planet to make a complete revolution about the Sun) is directly proportional to the cube of its average distance d from the Sun.

(a) Express Kepler's Third Law as an equation.

(b) Find the constant of proportionality by using the fact that for planet Earth the period is about 365 days and the average distance is about 93 million miles.

(c) The planet Neptune is about 2.79×10^9 mi from the Sun. Find the period of Neptune.

43. Ideal Gas Law The pressure P of a sample of gas is directly proportional to the temperature T and inversely proportional to the volume V.

(a) Write an equation that expresses this variation.

(b) Find the constant of proportionality if 100 L of gas exerts a pressure of 33.2 kPa at a temperature of 400 K (absolute temperature measured on the Kelvin scale).

(c) If the temperature is increased to 500 K and the volume is decreased to 80 L, what is the pressure of the gas?

44. Skidding in a Curve A car is traveling on a curve that forms a circular arc. The force F needed to keep the car from skidding is jointly proportional to the weight w of the car and the square of its speed s and is inversely proportional to the radius r of the curve.

(a) Write an equation that expresses this variation.

(b) A car weighing 1600 lb travels around a curve at 60 mi/h. The next car to round this curve weighs 2500 lb and requires the same force as the first car to keep from skidding. How fast is the second car traveling?

45. Loudness of Sound The loudness L of a sound (measured in decibels, dB) is inversely proportional to the square of the distance d from the source of the sound.

(a) Write an equation that expresses this variation.

(b) Find the constant of proportionality if a person 10 ft from a lawn mower experiences a sound level of 70 dB.

(c) If the distance in part (b) is doubled, by what factor is the loudness changed?

(d) If the distance in part (b) is cut in half, by what factor is the loudness changed?

46. A Jet of Water The power P of a jet of water is jointly proportional to the cross-sectional area A of the jet and to the cube of the velocity v.

(a) Write an equation that expresses this variation.

(b) If the velocity is doubled and the cross-sectional area is halved, by what factor is the power changed?

(c) If the velocity is halved and the cross-sectional area is tripled, by what factor is the power changed?

47. Electrical Resistance The resistance R of a wire varies directly as its length L and inversely as the square of its diameter d.

(a) Write an equation that expresses this joint variation.

(b) Find the constant of proportionality if a wire 1.2 m long and 0.005 m in diameter has a resistance of 140 ohms.

(c) Find the resistance of a wire made of the same material that is 3 m long and has a diameter of 0.008 m.

(d) If the diameter is doubled and the length is tripled, by what factor is the resistance changed?

48. Growing Cabbages In the short growing season of the Canadian arctic territory of Nunavut, some gardeners find it possible to grow gigantic cabbages in the midnight sun. Assume that the final size of a cabbage is proportional to the amount of nutrients it receives and inversely proportional to the number of other cabbages surrounding it. A cabbage that received 20 oz of nutrients and had 12 other cabbages around it grew to 30 lb. What size would it grow to if it received 10 oz of nutrients and had only 5 cabbage "neighbors"?

49. Radiation Energy The total radiation energy E emitted by a heated surface per unit area varies as the fourth power of its absolute temperature T. The temperature is 6000 K at the surface of the sun and 300 K at the surface of the earth.

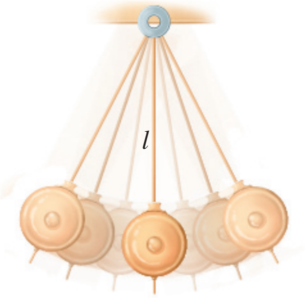
(a) How many times more radiation energy per unit area is produced by the sun than by the earth?

(b) The radius of the earth is 3960 mi, and the radius of the sun is 435,000 mi. How many times more total radiation does the sun emit than the earth?

50. Law of the Pendulum The period of a pendulum (the time elapsed during one complete swing of the pendulum) varies directly with the square root of the length of the pendulum.

(a) Express this relationship by writing an equation.

(b) To double the period, how would we have to change the length l?



51. Frequency of Vibration The frequency f of vibration of a violin string is inversely proportional to its length L. The constant of proportionality k is positive and depends on the tension and density of the string.

(a) Write an equation that represents this variation.

(b) What effect does doubling the length of the string have on the frequency of its vibration?

52. Spread of a Disease The rate r at which a disease spreads in a population of size P is jointly proportional to the number x of infected people and the number $P - x$ who are not infected. An infection erupts in a small town that has population $P = 5000$.

(a) Write an equation that expresses this variation.

(b) Compare the rate of spread of this infection when 10 people are infected to the rate of spread when 1000 people are infected. Which rate is larger? By what factor?

(c) Calculate the rate of spread when the entire population is infected. Why does this answer make intuitive sense?

53. Electric Vehicle The range R of an electric vehicle is the distance (in km) the vehicle can travel on one complete charge C (in kWh). The efficiency rating e (measured in kilowatt-hours per km, kWh/km) of the vehicle is the amount of electric charge it uses per kilometer, so $e = C/R$. Due to increased aerodynamic drag at higher speeds, the efficiency of a vehicle is directly proportional to the square of the speed v of the vehicle. So we have

$$R = \frac{C}{e} \qquad \text{and} \qquad e = kv^2$$

A particular electric vehicle has efficiency 0.20 kWh/km and a range of 500 km when traveling at a speed of 100 km/h.

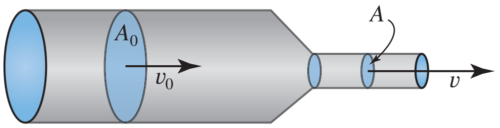
Estimate the range of the car when traveling at speeds of 130 km/h and 80 km/h.

Nerthuz/Shutterstock.com

54. Mass Flow Rate A fluid with density ρ is initially being pumped through a pipe with cross-sectional area A_0 at a constant velocity v_0. The *mass flow rate* of the fluid is $A_0 v_0 \rho$ and remains constant as the cross-sectional area of the pipe changes. (That is, if the cross-sectional area A of the pipe changes, the flow rate v also changes in such a way that the quantity $Av\rho$ remains unchanged.)

(a) Suppose that the cross-sectional area A_0 of a pipe is constricted to an area A, as shown in the figure. Use the fact that the mass flow rate is constant to show that the velocity v of the fluid becomes $v = (A_0 v_0)/A$ and conclude that the velocity of the fluid is inversely proportional to the cross-sectional area of the pipe.

(b) If a fluid is pumped with velocity 5 m/s through a pipe with radius 0.6 m, find the velocity of the fluid through a constricted part of the pipe with radius 0.2 m.



55–56 ■ Combining Variations Solve the problem using the relationship between brightness B, luminosity L, and distance d derived in Example 3. The proportionality constant is $k = 0.080$.

55. Brightness of a Star The luminosity of a star is given by $L = 2.5 \times 10^{26}$ W, and its distance from the earth is $d = 2.4 \times 10^{19}$ m. How bright does the star appear on the earth?

56. Distance to a Star The luminosity of a star is given by $L = 5.8 \times 10^{30}$ W, and its brightness as viewed from the earth is $B = 8.2 \times 10^{-16}$ W/m^2. Find the distance of the star from the earth.

■ Discuss ■ Discover ■ Prove ■ Write

57. Discuss: Is Proportionality Everything? A great many laws of physics and chemistry are expressible as proportionalities. Give at least one example of a relationship that occurs in the sciences that is *not* a proportionality.

Chapter 1 Review

Properties and Formulas

Properties of Real Numbers | Section 1.1

Commutative: $a + b = b + a$

$ab = ba$

Associative: $(a + b) + c = a + (b + c)$

$(ab)c = a(bc)$

Distributive: $a(b + c) = ab + ac$

Absolute Value | Section 1.1

$$|a| = \begin{cases} a & \text{if } a \ge 0 \\ -a & \text{if } a < 0 \end{cases}$$

$$|ab| = |a||b|$$

$$\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$$

Distance between a and b:

$$d(a, b) = |b - a|$$

Exponents | Section 1.2

$$a^m a^n = a^{m+n}$$

$$\frac{a^m}{a^n} = a^{m-n}$$

$$(a^m)^n = a^{mn}$$

$$(ab)^n = a^n b^n$$

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

Radicals | Section 1.2

$\sqrt[n]{a} = b$ means $b^n = a$

$$\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$$

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

$$\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$$

$$a^{m/n} = \sqrt[n]{a^m}$$

If n is odd, then $\sqrt[n]{a^n} = a$.

If n is even, then $\sqrt[n]{a^n} = |a|$.

Special Product Formulas | Section 1.3

Sum and difference of same terms:

$$(A + B)(A - B) = A^2 - B^2$$

Square of a sum or difference:

$$(A + B)^2 = A^2 + 2AB + B^2$$

$$(A - B)^2 = A^2 - 2AB + B^2$$

Cube of a sum or difference:

$$(A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$$

$$(A - B)^3 = A^3 - 3A^2B + 3AB^2 - B^3$$

Special Factoring Formulas | Section 1.3

Difference of squares:

$$A^2 - B^2 = (A + B)(A - B)$$

Perfect squares:

$$A^2 + 2AB + B^2 = (A + B)^2$$

$$A^2 - 2AB + B^2 = (A - B)^2$$

Sum or difference of cubes:

$$A^3 - B^3 = (A - B)(A^2 + AB + B^2)$$

$$A^3 + B^3 = (A + B)(A^2 - AB + B^2)$$

Rational Expressions | Section 1.4

We can cancel common factors:

$$\frac{AC}{BC} = \frac{A}{B}$$

To multiply two fractions, we multiply their numerators together and their denominators together:

$$\frac{A}{B} \times \frac{C}{D} = \frac{AC}{BD}$$

To divide fractions, we invert the divisor and multiply:

$$\frac{A}{B} \div \frac{C}{D} = \frac{A}{B} \times \frac{D}{C}$$

To add fractions, we find a common denominator:

$$\frac{A}{C} + \frac{B}{C} = \frac{A + B}{C}$$

Properties of Equality | Section 1.5

$A = B \quad \Leftrightarrow \quad A + C = B + C$

$A = B \quad \Leftrightarrow \quad CA = CB \quad (C \neq 0)$

Linear Equations | Section 1.5

A **linear equation** in one variable is an equation of the form $ax + b = 0$.

Zero-Product Property | Section 1.5

If $AB = 0$, then $A = 0$ or $B = 0$.

Completing the Square | Section 1.5

To make $x^2 + bx$ a perfect square, add $\left(\frac{b}{2}\right)^2$. This gives the perfect square

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

Quadratic Formula | Section 1.5

A **quadratic equation** is an equation of the form

$$ax^2 + bx + c = 0$$

Its solutions are given by the **Quadratic Formula**:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The **discriminant** is $D = b^2 - 4ac$.

If $D > 0$, the equation has two real solutions.

If $D = 0$, the equation has one solution.

If $D < 0$, the equation has two complex solutions.

Complex Numbers | Section 1.6

A **complex number** is a number of the form $a + bi$, where $i = \sqrt{-1}$.

The **complex conjugate** of $a + bi$ is

$$\overline{a + bi} = a - bi$$

To **multiply** complex numbers, treat them as binomials and use $i^2 = -1$ to simplify the result.

To **divide** complex numbers, multiply numerator and denominator by the complex conjugate of the denominator:

$$\frac{a + bi}{c + di} = \left(\frac{a + bi}{c + di}\right) \cdot \left(\frac{c - di}{c - di}\right) = \frac{(a + bi)(c - di)}{c^2 + d^2}$$

Inequalities | Section 1.8

Adding the same quantity to each side of an inequality gives an equivalent inequality:

$$A < B \quad \Leftrightarrow \quad A + C < B + C$$

Multiplying each side of an inequality by the same *positive* quantity gives an equivalent inequality. Multiplying each side by the same *negative* quantity reverses the direction of the inequality:

$$\text{If } C > 0, \text{ then } A < B \quad \Leftrightarrow \quad CA < CB$$

$$\text{If } C < 0, \text{ then } A < B \quad \Leftrightarrow \quad CA > CB$$

Absolute-Value Inequalities | Section 1.8

To solve absolute-value inequalities, we use

$$|x| < C \quad \Leftrightarrow \quad -C < x < C$$

$$|x| > C \quad \Leftrightarrow \quad x < -C \quad \text{or} \quad x > C$$

The Distance Formula | Section 1.9

The distance between the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$d(A, B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The Midpoint Formula | Section 1.9

The midpoint of the line segment from $A(x_1, y_1)$ to $B(x_2, y_2)$ is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Intercepts | Section 1.9

To find the ***x*-intercepts** of the graph of an equation, set $y = 0$ and solve for x.

To find the ***y*-intercepts** of the graph of an equation, set $x = 0$ and solve for y.

Circles | Section 1.9

The circle with center (0, 0) and radius r has equation

$$x^2 + y^2 = r^2$$

The circle with center (h, k) and radius r has equation

$$(x - h)^2 + (y - k)^2 = r^2$$

Symmetry | Section 1.9

The graph of an equation is **symmetric with respect to the *x*-axis** if the equation remains unchanged when y is replaced by $-y$.

The graph of an equation is **symmetric with respect to the *y*-axis** if the equation remains unchanged when x is replaced by $-x$.

The graph of an equation is **symmetric with respect to the origin** if the equation remains unchanged when x is replaced by $-x$ and y by $-y$.

Slope of a Line | Section 1.10

The slope of the nonvertical line that contains the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

Equations of Lines | Section 1.10

If a line has slope m, has y-intercept b, and contains the point (x_1, y_1), then:

the **point-slope form** of its equation is

$$y - y_1 = m(x - x_1)$$

the **slope-intercept form** of its equation is

$$y = mx + b$$

The equation of any line can be expressed in the **general form**

$$Ax + By + C = 0$$

where A and B can't both be 0.

Vertical and Horizontal Lines | Section 1.10

The **vertical** line containing the point (a, b) has the equation $x = a$.

The **horizontal** line containing the point (a, b) has the equation $y = b$.

Parallel and Perpendicular Lines | Section 1.10

Two lines with slopes m_1 and m_2 are

parallel if and only if $m_1 = m_2$

perpendicular if and only if $m_1 m_2 = -1$

Variation | Section 1.12

If y is **directly proportional** to x, then

$$y = kx$$

If y is **inversely proportional** to x, then

$$y = \frac{k}{x}$$

Concept Check

1. **(a)** What does the set of natural numbers consist of? What does the set of integers consist of? Give an example of an integer that is not a natural number.

(b) What does the set of rational numbers consist of? Give an example of a rational number that is not an integer.

(c) What does the set of irrational numbers consist of? Give an example of an irrational number.

(d) What does the set of real numbers consist of?

2. A property of real numbers is given. State the property and give an example in which the property is used.

(i) Commutative Property

(ii) Associative Property

(iii) Distributive Property

3. Explain the difference between the open interval (a, b) and the closed interval $[a, b]$. Give an example of an interval that is neither open nor closed.

4. Give the formula for finding the distance between two real numbers a and b. Use the formula to find the distance between 103 and -52.

5. Suppose $a \neq 0$ is any real number.

(a) In the expression a^n, which is the base and which is the exponent?

(b) What does a^n mean if n is a positive integer? What does 6^5 mean?

(c) What does a^{-n} mean if n is a positive integer? What does 3^{-2} mean?

(d) What does a^n mean if n is zero?

(e) If m and n are positive integers, what does $a^{m/n}$ mean? What does $4^{3/2}$ mean?

6. State the first five Laws of Exponents. Give examples in which you would use each law.

7. When you multiply two powers of the same number, what should you do with the exponents? When you raise a power to a new power, what should you do with the exponents?

8. **(a)** What does $\sqrt[n]{a} = b$ mean?

(b) Is it true that $\sqrt{a^2}$ is equal to $|a|$? Try values for a that are positive and negative.

(c) How many real nth roots does a positive real number have if n is even? If n is odd?

(d) Is $\sqrt[4]{-2}$ a real number? Is $\sqrt[3]{-2}$ a real number? Explain why or why not.

9. Explain the steps involved in rationalizing a denominator. What is the logical first step in rationalizing the denominator of the expression $\dfrac{5}{\sqrt{3}}$?

10. Explain the difference between expanding an expression and factoring an expression.

11. State the Special Product Formulas used for expanding each of the given expressions.

(a) $(a + b)^2$ **(b)** $(a - b)^2$ **(c)** $(a + b)^3$

(d) $(a - b)^3$ **(e)** $(a + b)(a - b)$

Use the appropriate formula to expand $(x + 5)^2$ and $(x + 5)(x - 5)$.

12. State the following Special Factoring Formulas.

(a) Difference of Squares

(b) Perfect Square

(c) Sum of Cubes

Use the appropriate formula to factor $x^2 - 9$.

13. If the numerator and the denominator of a rational expression have a common factor, how would you simplify the expression? Simplify the expression $\dfrac{x^2 + x}{x + 1}$.

14. Explain each of the following.

(a) How to multiply and divide rational expressions.

(b) How to add and subtract rational expressions.

(c) What LCD do we use to perform the addition in the expression $\dfrac{3}{x - 1} + \dfrac{5}{x + 2}$?

15. What is the logical first step in rationalizing the denominator of $\dfrac{3}{1 + \sqrt{x}}$?

16. What is the difference between an algebraic expression and an equation? Give examples.

17. Write the general form of each type of equation.

(a) Linear equation

(b) Quadratic equation

18. What are the three ways to solve a quadratic equation?

19. State the Zero-Product Property. Use the property to solve the equation $x(x - 1) = 0$.

20. What do you need to add to $ax^2 + bx$ to complete the square? Complete the square for the expression $x^2 + 6x$.

21. State the Quadratic Formula for the quadratic equation $ax^2 + bx + c = 0$, and use it to solve the equation $x^2 + 6x - 1 = 0$.

22. What is the discriminant of the quadratic equation $ax^2 + bx + c = 0$? Find the discriminant of the equation $2x^2 - 3x + 5 = 0$. How many real solutions does this equation have?

23. What is the logical first step in solving the equation $\sqrt{x - 1} = x - 3$? Why is it important to check your answers when solving equations of this type?

24. What is a complex number? Give an example of a complex number, and identify the real and imaginary parts.

25. What is the complex conjugate of a complex number $a + bi$?

26. **(a)** How do you add complex numbers?

(b) How do you multiply $(3 + 5i)(2 - i)$?

(c) Is $(3 - i)(3 + i)$ a real number?

(d) How do you simplify the quotient $\dfrac{3 + 5i}{3 - i}$?

27. State the guidelines for modeling with equations.

28. Explain how to solve the given type of problem.

(a) Linear inequality: $2x \geq 1$

(b) Nonlinear inequality: $(x - 1)(x - 4) < 0$

(c) Absolute-value equation: $|2x - 5| = 7$

(d) Absolute-value inequality: $|2x - 5| \leq 7$

29. **(a)** In the coordinate plane, what is the horizontal axis called and what is the vertical axis called?

(b) To graph an ordered pair of numbers (x, y), you need the coordinate plane. For the point $(2, 3)$, which is the x-coordinate and which is the y-coordinate?

(c) For an equation in the variables x and y, how do you determine whether a given point is on the graph? Is the point $(5, 3)$ on the graph of the equation $y = 2x - 1$?

30. **(a)** What is the formula for finding the distance between the points (x_1, y_1) and (x_2, y_2)?

(b) What is the formula for finding the midpoint between (x_1, y_1) and (x_2, y_2)?

31. How do you find x-intercepts and y-intercepts of a graph of an equation?

32. **(a)** Write an equation of the circle with center (h, k) and radius r.

(b) Find the equation of the circle with center $(2, -1)$ and radius 3.

33. **(a)** How do you test whether the graph an equation is symmetric with respect to the (i) x-axis, (ii) y-axis, and (iii) origin?

(b) What type of symmetry does the graph of the equation $xy^2 + y^2x^2 = 3x$ have?

34. **(a)** What is the slope of a line? How do you compute the slope of the line through the points $(-1, 4)$ and $(1, -2)$?

(b) How do you find the slope and y-intercept of the line $6x + 3y = 12$?

(c) How do you write the equation for a line that has slope 3 and passes through the point $(1, 2)$?

35. Give an equation of both a vertical line and a horizontal line that passes through the point $(2, 3)$.

36. State the general equation of a line.

37. Given lines with slopes m_1 and m_2, how you can tell whether the lines are (i) parallel? (ii) perpendicular?

38. How do you solve an equation (i) algebraically? (ii) graphically?

39. How do you solve an inequality (i) algebraically? (ii) graphically?

40. Write an equation that expresses each relationship.

(a) y is directly proportional to x.

(b) y is inversely proportional to x.

(c) z is jointly proportional to x and y.

Answers to the Concept Check can be found at the book companion website stewartmath.com.

Exercises

1–4 ■ Properties of Real Numbers State the property of real numbers being used.

1. $3x + 2y = 2y + 3x$

2. $(a + b)(a - b) = (a - b)(a + b)$

3. $4(a + b) = 4a + 4b$

4. $(A + 1)(x + y) = (A + 1)x + (A + 1)y$

5–6 ■ Intervals Express the interval in terms of inequalities, and then graph the interval.

5. $[-2, 6)$ **6.** $(-\infty, 4]$

7–8 ■ Intervals Express the inequality in interval notation, and then graph the corresponding interval.

7. $x \geq 5$ **8.** $-1 < x \leq 5$

9–16 ■ Evaluate Evaluate the expression.

9. $|1 - |-4||$ **10.** $5 - |10 - |-4||$

11. $2^{1/2}8^{1/2}$ **12.** $2^{-3} - 3^{-2}$

13. $216^{-1/3}$ **14.** $64^{2/3}$

15. $\dfrac{\sqrt{242}}{\sqrt{2}}$ **16.** $\sqrt{2}\sqrt{50}$

17–20 ■ Radicals and Exponents Simplify the expression.

17. **(a)** $(a^2)^{-3}(a^3b)^2(b^3)^4$ **(b)** $(3xy^2)^3\left(\frac{2}{3}x^{-1}y\right)^2$

18. **(a)** $\dfrac{x^4(3x)^2}{x^3}$ **(b)** $\left(\dfrac{r^2s^{4/3}}{r^{1/3}s}\right)^6$

19. **(a)** $\sqrt[3]{(x^3y)^2y^4}$ **(b)** $\sqrt{w^8z^{10}}$

20. **(a)** $\dfrac{8r^{1/2}s^{-3}}{2r^{-2}s^4}$ **(b)** $\left(\dfrac{ab^2c^{-3}}{2a^3b^{-4}}\right)^{-2}$

21–24 ■ Scientific Notation These exercises involve scientific notation.

21. Write the number 78,250,000,000 in scientific notation.

22. Write the number 2.08×10^{-8} in ordinary decimal notation.

23. If $a \approx 0.00000293$, $b \approx 1.582 \times 10^{-14}$, and $c \approx 2.8064 \times 10^{12}$, use a calculator to approximate the number ab/c.

24. If your heart beats 80 times per minute and you live to be 90 years old, estimate the number of times your heart beats during your lifetime. State your answer in scientific notation.

25–38 ■ Factoring Factor the expression completely.

25. $x^2 + 5x - 14$ **26.** $12x^2 + 10x - 8$

27. $x^4 - 2x^2 + 1$ **28.** $4y^2z^3 + 10y^3z - 12y^5z^2$

29. $16 - 4t^2$ **30.** $2y^6 - 32y^2$

31. $x^6 - 1$ **32.** $16a^4b^2 + 2ab^5$

33. $-3x^{-1/2} + 2x^{1/2} + 5x^{3/2}$ **34.** $7x^{-3/2} - 8x^{-1/2} + x^{1/2}$

35. $5x^3 + 15x^2 - x - 3$ **36.** $w^3 - 3w^2 - 4w + 12$

37. $(a + b)^2 - 3(a + b) - 10$

38. $(3x + 2)^2 - (3x + 2) - 6$

39–50 ■ Operations with Algebraic Expressions Perform the indicated operations and simplify.

39. $(2y - 7)(2y + 7)$

40. $(1 + x)(2 - x) - (3 - x)(3 + x)$

41. $x^2(x - 2) + x(x - 2)^2$

42. $\sqrt{x}(\sqrt{x} + 1)(2\sqrt{x} - 1)$

43. $\dfrac{x^2 - 4x - 5}{x^2 - 25} \cdot \dfrac{x^2 + 12x + 36}{x^2 + 7x + 6}$

44. $\dfrac{x^2 - 2x - 15}{x^2 - 6x + 5} \div \dfrac{x^2 - x - 12}{x^2 - 1}$

45. $\dfrac{2}{x} + \dfrac{1}{x - 2} + \dfrac{3}{(x - 2)^2}$

46. $\dfrac{1}{x + 2} + \dfrac{1}{x^2 - 4} - \dfrac{2}{x^2 - x - 2}$

47. $\dfrac{\dfrac{1}{x} - \dfrac{1}{2}}{x - 2}$ **48.** $\dfrac{\dfrac{1}{x} - \dfrac{1}{x + 1}}{\dfrac{1}{x} + \dfrac{1}{x + 1}}$

49. $\dfrac{\sqrt{x} - 2}{\sqrt{x} + 2}$ (rationalize the denominator)

50. $\dfrac{\sqrt{x + h} - \sqrt{x}}{h}$ (rationalize the numerator)

51–54 ■ Rationalizing Rationalize the denominator and simplify.

51. $\dfrac{1}{\sqrt{11}}$ **52.** $\dfrac{3}{\sqrt{6}}$

53. $\dfrac{5}{1 + \sqrt{2}}$ **54.** $\dfrac{\sqrt{6}}{\sqrt{3} + \sqrt{2}}$

55–72 ■ Solving Equations Find all real solutions of the equation.

55. $7x - 6 = 4x + 9$ **56.** $8 - 2x = 14 + x$

57. $\dfrac{3x + 6}{x + 3} = \dfrac{3x}{x + 1}$ **58.** $(x + 2)^2 = (x - 4)^2$

59. $x^2 - 9x + 14 = 0$ **60.** $x^2 + 24x + 144 = 0$

61. $2x^2 + x = 1$

62. $\sqrt{x-1} = \sqrt{x^2-3}$

63. $4x^3 - 25x = 0$

64. $x^3 - 2x^2 - 5x + 10 = 0$

65. $3x^2 + 4x - 1 = 0$

66. $\dfrac{1}{x} + \dfrac{2}{x-1} = 3$

67. $\dfrac{x}{x-2} + \dfrac{1}{x+2} = \dfrac{8}{x^2-4}$

68. $x^4 - 8x^2 - 9 = 0$

69. $(x-3)^2 - 3(x-3) - 4 = 0$

70. $9x^{-1/2} - 6x^{1/2} + x^{3/2} = 0$

71. $|x-7| = 4$

72. $|2x-5| = 9$

73–76 ■ Complex Numbers Evaluate the expression and write in the form $a + bi$.

73. (a) $(2-3i) + (1+4i)$ **(b)** $(2+i)(3-2i)$

74. (a) $(3-6i) - (6-4i)$ **(b)** $4i(2-\frac{1}{2}i)$

75. (a) $\dfrac{4+2i}{2-i}$ **(b)** $(1-\sqrt{-1})(1+\sqrt{-1})$

76. (a) $\dfrac{2-3i}{1+i}$ **(b)** $\sqrt{-10}\cdot\sqrt{-40}$

77–82 ■ Real and Complex Solutions Find all real and complex solutions of the equation.

77. $x^2 + 16 = 0$

78. $x^2 = -12$

79. $x^2 + 6x + 10 = 0$

80. $2x^2 - 3x + 2 = 0$

81. $x^4 - 256 = 0$

82. $x^3 - 2x^2 + 4x - 8 = 0$

83. Mixtures A bulk food store sells raisins for \$3.20 per pound and nuts for \$2.40 per pound. The store sells a 50-pound mix of raisins and nuts for \$2.72 per pound. What quantities of raisins and nuts are in the mixture?

84. Distance and Time A district supervisor leaves Kingstown at 2:00 P.M. and drives to Queensville, 160 mi distant, at 45 mi/h. At 2:15 P.M. a store manager leaves Queensville and drives to Kingstown at 40 mi/h. At what time do they pass each other on the road?

85. Distance and Time An athlete has a daily exercise program of cycling and running. Their cycling speed is 8 mi/h faster than their running speed. Every morning the athlete cycles 4 mi and runs $2\frac{1}{2}$ mi, for a total of one hour of exercise. How fast does the athlete run?

86. Geometry The hypotenuse of a right triangle has length 20 cm. The sum of the lengths of the other two sides is 28 cm. Find the lengths of the other two sides of the triangle.

87. Doing the Job An interior decorator paints twice as fast as the assistant and three times as fast as the apprentice. If it takes them 60 min to paint a room with all three working together, how long would it take the interior decorator to paint the room working alone?

88. Dimensions of a Garden A homeowner wishes to fence in three adjoining garden plots, as shown in the figure. If each plot is to be 80 ft^2 in area and there is 88 ft of fencing material at hand, what dimensions should each plot have?

89–96 ■ Inequalities Solve the inequality. Express the solution using interval notation, and graph the solution set on the real number line.

89. $3x - 2 > -11$

90. $-7 \le 3x - 1 \le -1$

91. $x^2 - 7x - 8 > 0$

92. $x^2 \le 1$

93. $\dfrac{x-4}{x^2-4} \le 0$

94. $\dfrac{5}{x^3 - x^2 - 4x + 4} < 0$

95. $|x-5| \le 3$

96. $|x-4| < 0.02$

97–98 ■ Coordinate Plane Two points P and Q are given. **(a)** Plot P and Q on a coordinate plane. **(b)** Find the distance from P to Q. **(c)** Find the midpoint of the segment PQ. **(d)** Sketch the line determined by P and Q, and find its equation in slope-intercept form. **(e)** Sketch the circle that passes through Q and has center P, and find the equation of this circle.

97. $P(2,0)$, $Q(-5,12)$

98. $P(7,-1)$, $Q(2,-11)$

99–100 ■ Graphing Regions Sketch the region given by the set.

99. $\{(x,y) \mid x \ge 4 \text{ or } y \ge 2\}$

100. $\{(x,y) \mid |x| < 1 \text{ and } |y| < 4\}$

101. Distance Formula Which of the points $A(4,4)$ or $B(5,3)$ is closer to the point $C(-1,-3)$?

102–104 ■ Circles In these exercises we find equations of circles.

102. Find an equation of the circle that has center $(-3,4)$ and radius $\sqrt{6}$.

103. Find an equation of the circle that has center $(-5,-1)$ and passes through the origin.

104. Find an equation of the circle that contains the points $P(2,3)$ and $Q(-1,8)$ and has the midpoint of the segment PQ as its center.

105–108 ■ Circles **(a)** Complete the square to determine whether the equation represents a circle or a point or has no graph. **(b)** If the equation is that of a circle, find its center and radius, and sketch its graph.

105. $x^2 + y^2 - 8x + 2y + 13 = 0$

106. $2x^2 + 2y^2 - 2x + 8y = \frac{1}{2}$

107. $x^2 + y^2 + 72 = 12x$

108. $x^2 + y^2 - 6x - 10y + 34 = 0$

109–114 ■ Graphing Equations Sketch the graph of the equation by making a table and plotting points.

109. $y = 2 - 3x$

110. $2x - y + 1 = 0$

111. $y = 16 - x^2$

112. $8x + y^2 = 0$

113. $x = \sqrt{y}$

114. $y = -\sqrt{1 - x^2}$

115–120 ■ Symmetry and Intercepts **(a)** Test the equation for symmetry with respect to the x-axis, the y-axis, and the origin. **(b)** Find the x- and y-intercepts of the graph of the equation.

115. $x = 16 - y^2$

116. $x^2 + 4y^2 = 9$

117. $x^2 - 9y = 9$

118. $(x + 1)^2 + y^2 = 4$

119. $x^2 + 4xy + y^2 = 1$

120. $x^3 + xy^2 = 5$

121–124 ■ Graphing Equations **(a)** Use a graphing device to graph the equation in an appropriate viewing rectangle. **(b)** Use the graph to find the x- and y-intercepts.

121. $y = x^2 - 6x$

122. $y = \sqrt{5 - x}$

123. $y = x^3 - 4x^2 - 5x$

124. $\dfrac{x^2}{4} + y^2 = 1$

125–132 ■ Lines A description of a line is given. **(a)** Find an equation for the line in slope-intercept form. **(b)** Find an equation for the line in general form. **(c)** Graph the line.

125. The line that has slope 2 and y-intercept 6

126. The line that has slope $-\frac{1}{2}$ and passes through the point $(6, -3)$

127. The line that passes through the points $(-3, -2)$ and $(1, 4)$

128. The line that has x-intercept 4 and y-intercept 12

129. The vertical line that passes through the point $(3, -2)$

130. The horizontal line with y-intercept 5

131. The line that passes through the origin and is parallel to the line containing $(2, 4)$ and $(4, -4)$

132. The line that passes through the point $(2, 3)$ and is perpendicular to the line $x - 4y + 7 = 0$

133. Stretching a Spring Hooke's Law states that if a weight w is attached to a hanging spring, then the stretched length s of the spring is linearly related to w. For a particular spring we have

$$s = 0.3w + 2.5$$

where s is measured in inches and w in pounds.

(a) What do the slope and s-intercept in this equation represent?

(b) How long is the spring when a 5-lb weight is attached?

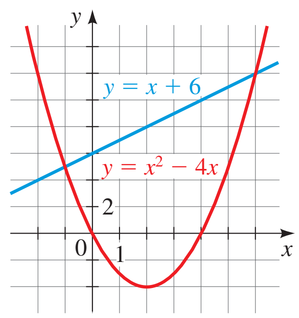
134. Annual Salary A tax accountant is hired by a firm at a salary of \$60,000 per year. Three years later the annual salary is increased to \$70,500. Assume that the accountant's salary increases linearly.

(a) Find an equation that relates the accountant's annual salary S and the number of years t that they have worked for the firm.

(b) What do the slope and S-intercept of the salary equation represent?

(c) What will be the accountant's annual salary after working 12 years with the firm?

135–140 ■ Equations and Inequalities Graphs of the equations $y = x^2 - 4x$ and $y = x + 6$ are given. Use the graphs to solve the equation or inequality.



135. $x^2 - 4x = x + 6$

136. $x^2 - 4x = 0$

137. $x^2 - 4x \le x + 6$

138. $x^2 - 4x \ge x + 6$

139. $x^2 - 4x \ge 0$

140. $x^2 - 4x \le 0$

141–144 ■ Equations Solve the equation graphically. State each answer rounded to two decimals.

141. $x^2 - 4x = 2x + 7$

142. $\sqrt{x + 4} = x^2 - 5$

143. $x^4 - 9x^2 = x - 9$

144. $\big||x + 3| - 5\big| = 2$

145–148 ■ Inequalities Solve the inequality graphically. State each answer rounded to two decimals.

145. $x^2 > 12 - 4x$

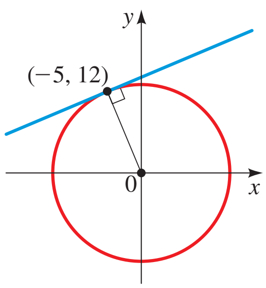
146. $x^3 - 4x^2 - 5x > 2$

147. $x^4 - 4x^2 < \frac{1}{2}x - 1$

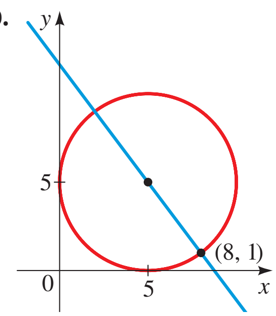
148. $|x^2 - 16| - 10 \ge 0$

149–150 ■ Circles and Lines Find an equation for both the circle and the line in the figure.

149.



150.



151. Variation Suppose that M varies directly as z, and $M = 120$ when $z = 15$. Write an equation that expresses this variation.

152. Variation Suppose that z is inversely proportional to y, and that $z = 12$ when $y = 16$. Write an equation that expresses z in terms of y.

153. Light Intensity The intensity of illumination I from a light varies inversely as the square of the distance d from the light.

(a) Write this statement as an equation.

(b) Determine the constant of proportionality if it is known that a lamp has an intensity of 1000 candles at a distance of 8 m.

(c) What is the intensity of this lamp at a distance of 20 m?

154. Vibrating String The frequency of a vibrating string under constant tension is inversely proportional to its length. If a violin string 12 inches long vibrates 440 times per second, to what length must it be shortened to vibrate 660 times per second?

155. Terminal Velocity The terminal velocity of a parachutist is directly proportional to the square root of their weight. A 160-lb parachutist attains a terminal velocity of 9 mi/h. What is the terminal velocity for a parachutist weighing 240 lb?

156. Range of a Projectile The maximum range of a projectile is directly proportional to the square of its velocity. A baseball pitcher throws a ball at 60 mi/h, with a maximum range of 242 ft. What is the maximum range if the pitcher throws the ball at 70 mi/h?

157. Speed of Sound The speed of sound in water is inversely proportional to the square root of the density of the water. The speed of sound in a freshwater lake with density 1 g/mL is 1480 m/s. Find the speed of sound in seawater with density 1.0273 g/mL.

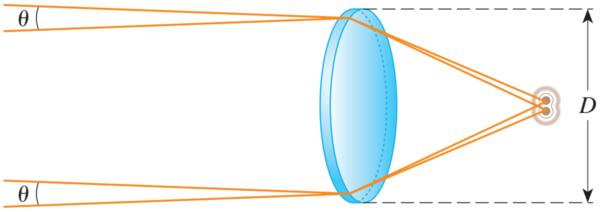
158. Resolution of a Telescope The *resolving power* of a telescope is its ability to distinguish (or resolve) objects that are separated by a small angular distance θ (the angle between the two objects as viewed from the telescope). The angle θ that a telescope can resolve is given by the formula

$$\theta = 1.22\frac{\lambda}{D}$$

where λ is the wavelength of light and D is the diameter of the telescope lens.

(a) For a fixed diameter, is the angular distance θ that can be resolved by a telescope smaller for shorter or for longer wavelengths of light?

(b) For a fixed wavelength, if the diameter D of the mirror is doubled how does the angular distance θ of the telescope change?



159. Hubble's Law The *redshift* z and *recessional* velocity v (km/s) of a galaxy are related by the formula

$$1 + z = \sqrt{\frac{c + v}{c - v}}$$

where $c = 3 \times 10^5$ km/s is the speed of light. Hubble's Law states that v varies directly with the distance D (in megalight-years, Mly) to the galaxy by the formula $v = H_0 D$, where $H_0 \approx 20.8$ (km/s)/Mly. For a galaxy with $z = 2$, find v and D.

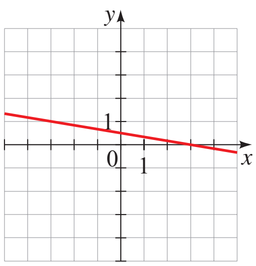
Matching

160. Equations and Their Graphs Match each equation with its graph. Give a reason for each answer. (Don't use a graphing device.)

(a) $y = 2|x| - 3$ **(b)** $2y - 3x = -2$ **(c)** $y = x^4$ **(d)** $(x + 1)^2 + (y + 1)^2 = 9$

(e) $6y + x = 3$ **(f)** $y = \dfrac{6x}{1 + x^4}$ **(g)** $x = y^3$ **(h)** $x^2 - 2x + y^2 - 4y + 1 = 0$

I



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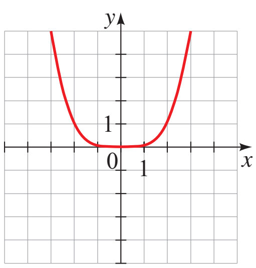

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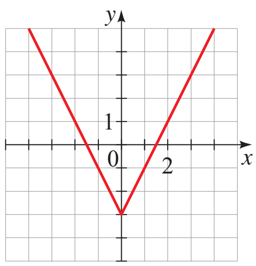

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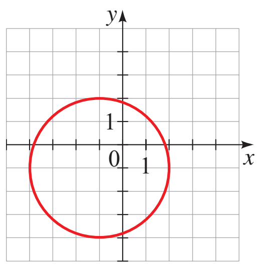

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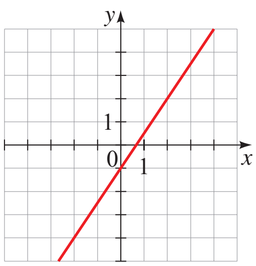

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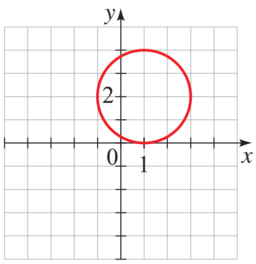
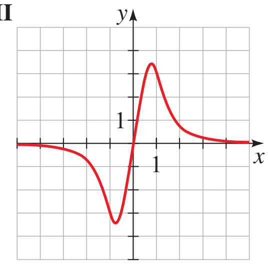
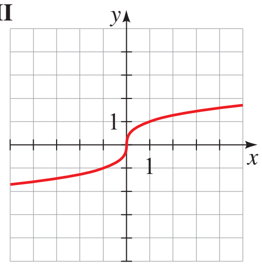


VII



VIII



Chapter 1 | Test

1. **(a)** Graph the intervals $(-5, 3]$ and $(2, \infty)$ on the real number line.

(b) Express the inequalities $x \le 3$ and $-1 \le x < 4$ in interval notation.

(c) Find the distance between -7 and 9 on the real number line.

2. Evaluate each expression.

(a) $(-3)^4$ **(b)** -3^4 **(c)** 3^{-4} **(d)** $\dfrac{3^{75}}{3^{72}}$ **(e)** $\left(\dfrac{2}{3}\right)^{-2}$ **(f)** $16^{-3/4}$

3. Write each number in scientific notation.

(a) 186,000,000,000 **(b)** 0.000 000 396 5

4. Simplify each expression. Write your final answer without negative exponents.

(a) $\sqrt{200} - \sqrt{32}$ **(b)** $(3a^3b^3)(4ab^2)^2$ **(c)** $\left(\dfrac{4x^9y^3}{xy^7}\right)^{-1/2}$

5. Perform the indicated operations and simplify.

(a) $z(4z - 3) + 2z(3 - 2z)$ **(b)** $(x + 3)(4x - 5)$ **(c)** $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})$

(d) $(2x + 3)^2$ **(e)** $(x + 2)^3$

6. Factor each expression completely.

(a) $4x^2 - 25$ **(b)** $2x^2 + 5x - 12$ **(c)** $x^3 - 3x^2 - 4x + 12$

(d) $x^4 + 27x$ **(e)** $2x^{3/2} + 8x^{1/2} - 10x^{-1/2}$ **(f)** $x^4y^2 - 9x^2y^2$

7. Simplify each expression.

(a) $\dfrac{w^2 + 4w + 3}{w^2 - 2w - 3}$ **(b)** $\dfrac{x^2}{x^2 - 4} - \dfrac{x + 1}{x + 2}$ **(c)** $\dfrac{\dfrac{y}{x} - \dfrac{x}{y}}{\dfrac{1}{y} - \dfrac{1}{x}}$

8. Rationalize the denominator and simplify: $\dfrac{\sqrt{2}}{4 - \sqrt{2}}$

9. Find all real solutions.

(a) $x + 5 = 14 - \frac{1}{2}x$ **(b)** $\dfrac{2x}{x + 1} = \dfrac{2x - 1}{x}$ **(c)** $x^2 - x - 12 = 0$

(d) $2x^2 + 4x + 1 = 0$ **(e)** $\sqrt{3 - \sqrt{x + 5}} = 2$ **(f)** $x^4 - 3x^2 + 2 = 0$

(g) $3|x - 4| = 10$

10. Perform the indicated operations and write the result in the form $a + bi$.

(a) $(3 - 2i) + (4 + 3i)$ **(b)** $(3 - 2i) - (4 + 3i)$

(c) $(3 - 2i)(4 + 3i)$ **(d)** $\dfrac{3 - 2i}{4 + 3i}$

(e) i^{48} **(f)** $(\sqrt{2} - \sqrt{-2})(\sqrt{8} + \sqrt{-2})$

11. Find all real and complex solutions of the equation $2x^2 + 4x + 3 = 0$.

12. A trucker drove from Amity to Belleville at a speed of 50 mi/h. On the way back, the trucker drove at 60 mi/h. The total trip took $4\frac{2}{5}$ h of driving time. Find the distance between these two cities.

13. A rectangular parcel of land is 70 ft longer than it is wide. Each diagonal between opposite corners is 130 ft. What are the dimensions of the parcel?

14. Solve each inequality. Write the answer using interval notation, and sketch the solution on the real number line.

(a) $-4 < 5 - 3x \le 17$ **(b)** $x(x - 1)(x + 2) > 0$

(c) $|x - 4| < 3$ **(d)** $\dfrac{3 - 2x}{2 - x} < x$

15. A bottle of medicine is to be stored at a temperature between 5°C and 10°C. What range does this correspond to on the Fahrenheit scale? [*Note:* Fahrenheit (F) and Celsius (C) temperatures satisfy the relation $C = \frac{5}{9}(F - 32)$.]

16. For what values of x is the expression $\sqrt{6x - x^2}$ defined as a real number?

17. **(a)** Plot the points $P(0, 3)$, $Q(3, 0)$, and $R(6, 3)$ in the coordinate plane. Where must the point S be located so that $PQRS$ is a square?

(b) Find the area of $PQRS$.

18. **(a)** Sketch the graph of $y = 4 - x^2$.

(b) Find the x- and y-intercepts of the graph.

(c) Is the graph symmetric about the x-axis, the y-axis, or the origin?

19. Let $P(-3, 1)$ and $Q(5, 6)$ be two points in the coordinate plane.

(a) Plot P and Q in the coordinate plane.

(b) Find the distance between P and Q.

(c) Find the midpoint of the segment PQ.

(d) Find an equation for the circle for which the segment PQ is a diameter.

20. Find the center and radius of each circle, and sketch its graph.

(a) $x^2 + y^2 = 5$ **(b)** $(x + 1)^2 + (y - 3)^2 = 4$ **(c)** $x^2 + y^2 - 10x + 16 = 0$

21. Write the linear equation $2x - 3y = 15$ in slope-intercept form, and sketch its graph. What are the slope and y-intercept?

22. Find an equation for the line with the given property.

(a) Passes through the points $(6, 7)$ and $(1, -3)$

(b) Passes through the point $(3, -6)$ and is parallel to the line $3x + y - 10 = 0$

(c) Has x-intercept 6 and y-intercept 4

23. A geologist measures the temperature T (in °C) of the soil at various depths below the surface and finds that at a depth of x centimeters, the temperature is given by $T = 0.08x - 4$.

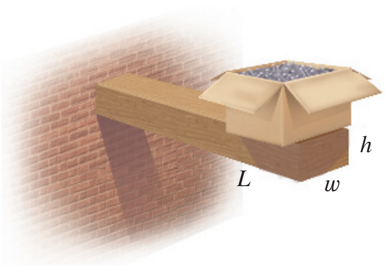
(a) What is the temperature at a depth of 1 m (100 cm)?

(b) Sketch a graph of the linear equation.

(c) What do the slope, the x-intercept, and T-intercept of the graph represent?

24. Solve each equation or inequality graphically. State your answer rounded to two decimal places.

(a) $x^3 - 9x - 1 = 0$ **(b)** $x^2 - 1 \le |x + 1|$



25. The maximum weight M that can be supported by a beam is jointly proportional to its width w and the square of its height h and inversely proportional to its length L.

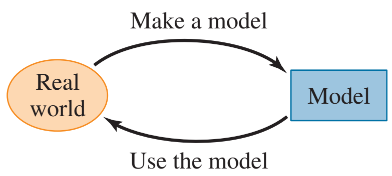
(a) Write an equation that expresses this proportionality.

(b) Determine the constant of proportionality if a beam 4 in. wide, 6 in. high, and 12 ft long can support a maximum weight of 4800 lb.

(c) If a 10-foot beam made of the same material is 3 in. wide and 10 in. high, what is the maximum weight it can support?

If you had difficulty with any of these problems, you may wish to review the section of this chapter indicated below.

Problem	Section	Problem	Section
1	Section 1.1	12, 13	Section 1.7
2, 3, 4	Section 1.2	14, 15, 16	Section 1.8
5, 6	Section 1.3	17, 18, 19, 20	Section 1.9
7, 8	Section 1.4	21, 22, 23	Section 1.10
9	Section 1.5	24	Section 1.11
10, 11	Section 1.6	25	Section 1.12



A model is a representation of an object or process. For example, a toy Ferrari is a model of the actual car; a road map is a model of the streets in a city. A **mathematical model** is a mathematical representation (usually an equation) of an object or process. Once a mathematical model has been made, it can be used to obtain useful information or make predictions about the thing being modeled. The process is described in the diagram in the margin. In these *Focus on Modeling* sections we explore different ways in which mathematics is used to model real-world phenomena.

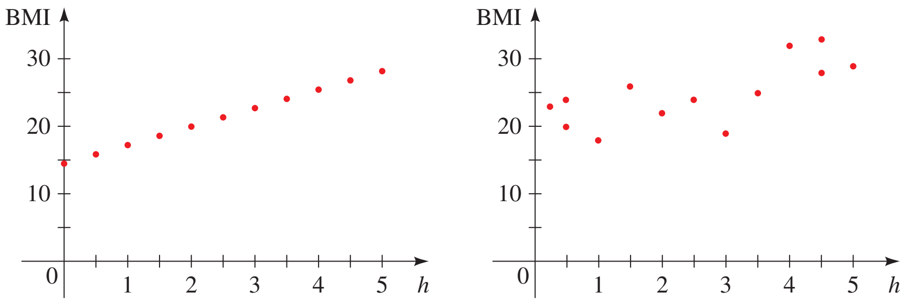
The Line That Best Fits the Data

The body-mass index (BMI) is defined as the body mass (weight) divided by the square of the height; the units are kg/m^2.

In Section 1.10 we used linear equations to model relationships between varying quantities. In practice, such relationships are often discovered by collecting and analyzing data. Real-world data seldom fall into a precise line. For example, the two-variable data in Table 1 are from a study on childhood health; the table gives average screen time (hours/day) and body mass index (BMI) for each of several adolescent subjects. A **scatter plot** of the data is shown in Figure 1(b). Of course, we would not expect the data to lie exactly on a line, as shown in Figure 1(a). But the scatter plot in Figure 1(b) shows a linear trend: the more hours of screen time, the correspondingly higher the BMI tends to be. (You can further explore how scatter plots can reveal hidden relationships in data in the *Discovery Project*: *Visualizing Data* at **stewartmath.com**.)

Table 1
Screen time-BMI

x (Hours)	y BMI
0.25	23
0.5	20
0.5	24
1	18
1.5	26
2	22
2.5	24
3	19
3.5	25
4	32
4.5	28
4.5	33
5	29

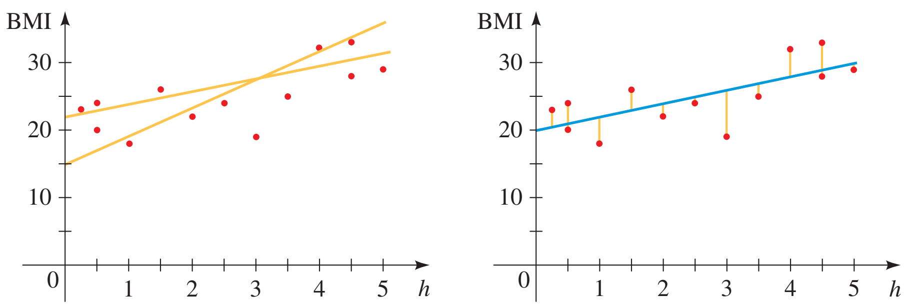


(a) Data fit exactly on a line

(b) Scatter plot of real-world data

Figure 1

The scatter plot in Figure 1(b) shows that the data lie roughly on a straight line. We can try to fit a line visually to approximate the data points, but since the data aren't exactly linear, there are many lines that may seem to fit the data. Figure 2(a) shows two such attempts at "eyeballing" a line to fit the data.



(a) Visually fitting a line to the data

(b) Distance from data points to line

Figure 2

Of all the lines that run through these data points, there is one that "best" fits the data. It seems reasonable that the line that best fits the data is the line that is as close as possible to all the data points. This is the line for which the sum of the vertical distances from the data points to the line is as small as possible, as shown in Figure 2(b). For technical reasons it is better to use the line for which the sum of the squares of these distances is smallest. This line is called the **regression line** (or the **least squares regression line**).

Examples of Linear Regression

The formula for the regression line is given in Exercise 9.1.77.

The formula for the regression line for a set of two-variable data is found using calculus. The formula is programmed into most graphing calculators; also, several Internet apps (such as Desmos or GeoGebra) can find the equation of the regression line for a given set of data.

Example 1 ■ Regression Line for the Screen Time-BMI Data

(a) Use a graphing device to draw a scatter plot of the data in Table 1 and find the regression line for the data. Graph the regression line and the scatter plot on the same screen. What does the y-intercept of the regression line represent?

(b) What does the model you found in part (a) predict about the BMI of an adolescent whose average screen time is 6 hours a day? 2.25 hours a day?

(c) Is a linear model reasonable for these data? What, if any, are the limitations of the model?

Solution

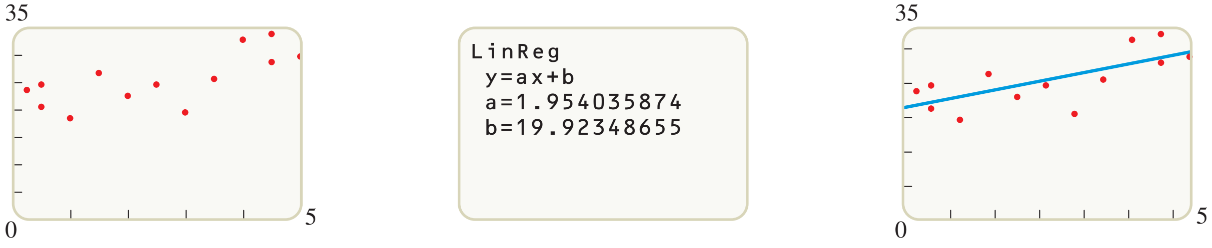
(a) A scatter plot of the data is shown in Figure 3(a). Using the `Linear Regression` command on a graphing device [see Figure 3(b)] we get the regression line $y = ax + b$, where $a = 1.954035874$ and $b = 19.92348655$. So, the equation of the regression line is

$$y = 1.95x + 19.92$$

A graph of the regression line, together with the scatter plot, is shown in Figure 3(c). The y-intercept is 19.92, which shows that according to this model, those children who spend no time in front of a screen would have a BMI of approximately 20.

(b) For an adolescent who averages 6 hours of screen time a day, $x = 6$. Substituting 6 for x in the equation of the regression line, we get $y = 1.95(6) + 19.92 \approx 31.6$. So, we would expect such an adolescent to have a BMI of about 32. For an average screen time of 2.25 the model predicts a BMI of $y = 1.95(2.25) + 19.92 \approx 24.3$.

(c) This model has some limitations. To begin, the model is based on too few data points; we need more data to increase the reliability of the model. Also, the model was based on data for screen times between 0 and 5 hours; the model should not be used to make predictions about screen times that are much greater than 5.



(a) Scatter plot (b) Regression line (c) Scatter plot and regression line

Figure 3

Extrapolation and **interpolation** refer to different ways we can use a model to predict values that are not included in the data. In Example 1, estimating the BMI associated with 6 hours of screen time is *extrapolation* (because 6 is “extra,” or greater than 5, the largest x-value in the data); estimating the BMI associated with 2.25 hours of screen time is *interpolation* (because 2.25 is “inter,” or between, data points).

Note that we can fit a line through any set of two-variable data, but a linear model is not always appropriate. In subsequent *Focus on Modeling* sections, we fit different types of curves to model data.

Table 2
Temperature-Energy Consumption

x (°C)	y (MWh/day)	x (°C)	y (MWh/day)
−5	24	8	19
−3	23	8	18
0	23	10	19
0	22	11	17
1	21	12	18
1	22	13	17
3	20	15	19
5	20	16	16
6	21	18	17

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Example 2 ■ Regression Line for Temperature-Energy Consumption Data

The data in Table 2 give several daily high temperatures (in °C) for a city in Austria and the amount of electricity (in MWh/day) the city residents used on each day.

(a) Use a graphing device to draw a scatter plot of the data in Table 2 and find the regression line for the data. Graph the regression line and the scatter plot on the same screen. What does the slope of the regression line tell us about the relationship between temperature and electricity usage?

(b) If a weather forecaster predicts a high temperature of -6°C for a particular day, how much demand for electricity (in MWh/day) should city officials prepare for?

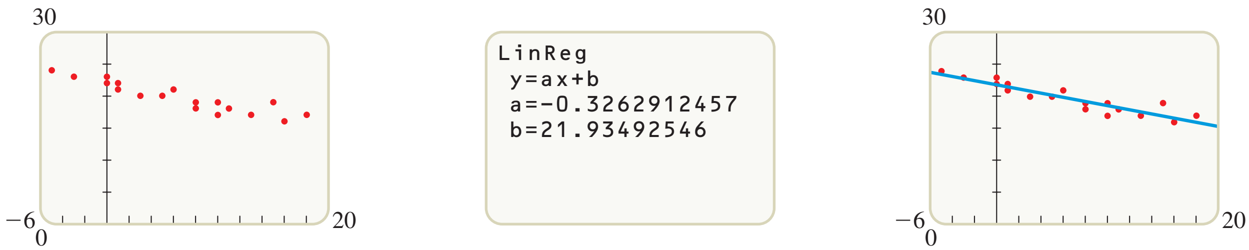
Solution

(a) A scatter plot of the data is shown in Figure 4(a). Using the `Linear Regression` command on a graphing device [see Figure 4(b)], we get the regression line $y = ax + b$, where $a = -0.3262912457$ and $b = 21.93492546$. So, the regression line is approximately

$$y = -0.33x + 21.93$$

A graph of the scatter plot and regression line is shown in Figure 4(c). The negative slope indicates that the warmer the temperature, the less the demand for electricity (because there is less need for heating).

(b) Substituting -6 for x gives $y = -0.33(-6) + 21.93 = 23.91$. So, city officials should prepare for a demand of about 24 MWh for that day.



(a) Scatter plot (b) Regression line (c) Scatter plot and regression line

Figure 4

■ How Good Is the Fit? The Correlation Coefficient

For *any* given set of two-variable data, the regression formula produces a regression line, even if the trend is not linear. So how closely do the data fall along a line? To answer this question, statisticians have invented the **correlation coefficient**, a number

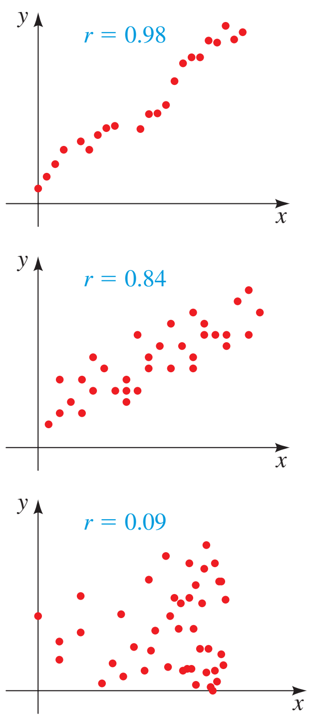


Figure 5

usually denoted by r that satisfies $-1 \leq r \leq 1$. The correlation coefficient measures how closely the data follow the regression line—in other words, how strongly the variables are *correlated*. Graphing devices usually give the value of r along with the equation of the regression line. If r is close to -1 or 1, then the variables are strongly correlated—that is, the scatter plot follows the regression line closely. If r is close to 0, then the variables are weakly correlated or not correlated at all. (The sign of r depends on the slope of the regression line.) Figure 5 shows the scatter plots of different data sets together with their correlation coefficients.

There are no hard and fast rules for deciding the values of r for which the correlation is "significant." The correlation coefficient as well as the number of data points serve as guides for deciding how well the regression line fits the data.

In Example 1 the correlation coefficient is 0.70, which shows that there is a good correlation between screen time and BMI. Does this mean that more screen time *causes* a higher BMI? In general, if two variables are correlated, it does not necessarily follow that a change in one variable causes a change in the other. The mathematician John Allen Paulos points out that shoe size is strongly correlated to mathematics scores among schoolchildren. Does this mean that big feet cause high math scores? Certainly not—both shoe size and math skills increase independently as children get older. So, it is important not to jump to conclusions: Correlation and causation are not the same thing. You can explore this topic further in the *Discovery Project: Correlation and Causation* at the book companion website **stewartmath.com**.

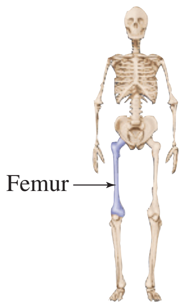
Correlation is a useful tool in bringing important cause-and-effect relationships to light; but to prove causation, we must explain the mechanism by which one variable affects the other. For example, the link between smoking and lung cancer was observed as a correlation long before science found the mechanism through which smoking can cause lung cancer.

Problems

1. Femur Length and Height Anthropologists use a linear model that relates femur length to height. The model allows an anthropologist to determine the height of an individual when only a partial skeleton (including the femur) is found. In this problem we find the model by analyzing the data on femur length and height for the eight males whose data is given in the table.

(a) Find the regression line for the data in the table, and graph the regression line and scatter plot on the same screen.

(b) An anthropologist finds a femur of length 58 cm. Use the model you found in part (a) to estimate how tall the person was.



Femur Length (cm)	Height (cm)
50.1	178.5
48.3	173.6
45.2	164.8
44.7	163.7
44.5	168.3
42.7	165.0
39.5	155.4
38.0	155.8

2. GDP and Carbon Dioxide Emissions A 2016 study examined the relationship between per capita GDP (gross domestic product) and per capita emissions of greenhouse gases (CO_2) for a one-year period. The table shows data collected in the study for various countries.

(a) Find the regression line for the data in the table, and graph the regression line and scatter plot on the same screen.

(b) What does the model that you found in part (a) predict about the per capita carbon emissions for a country with a per capita GDP of \$80,000? A country with per capita GDP of \$32,000?

(c) Does a linear model appear reasonable? What, if any, are the limitations of the model?

GDP per Capita (× \$1000)	Carbon Dioxide Emissions (tonnes per capita)	GDP per Capita (× \$1000)	Carbon Dioxide Emissions (tonnes per capita)
2	2	19	7
4	1	21	8
5	2	25	8
9	3	30	6
10	2	34	7
11	3	47	11
13	5	49	9
16	6	51	10

Source: The World Bank DataBank

3. Tree Diameter and Age To estimate ages of trees, forest rangers use a linear model that relates tree diameter to age. The model is useful because tree diameter is much easier to measure than tree age (which requires special tools for extracting a representative cross-section of the tree and counting the rings). To find the model, use the data in the table, which were collected for a certain variety of oaks.

(a) Find the regression line for the data in the table, and graph the regression line and scatter plot on the same screen.

(b) Use the model that you found in part (a) to estimate the age of an oak whose diameter is 18 in.

Diameter (in.)	Age (yr)
2.5	15
4.0	24
6.0	32
8.0	56
9.0	49
9.5	76
12.5	90
15.5	89

4. Temperature and Chirping Crickets Biologists have observed that the chirping rate of crickets of a certain species appears to be related to temperature. The table in the margin shows the chirping rates for various temperatures.

(a) Find the regression line for the data in the table, and graph the regression line and scatter plot on the same screen.

(b) Use the linear model that you found in part (a) to estimate the chirping rate at 100°F.

Temperature (°F)	Chirping Rate (chirps/min)
50	20
55	46
60	79
65	91
70	113
75	140
80	173
85	198
90	211

5. Extent of Arctic Sea Ice The National Snow and Ice Data Center monitors the amount of ice in the Arctic year round. The table below gives approximate values for the sea ice extent in millions of square kilometers from 1994 to 2020, in two-year intervals.

(a) Find the regression line for the data in the table, and graph the regression line and scatter plot on the same screen.

(b) Use the linear model that you found in part (a) to estimate the sea ice extent in 2019. Compare your answer with the actual value of 4.4 that was measured for 2019.

(c) What limitations do you think this model has? Can this model be used to predict sea ice extent for many years in the future?

Year	Ice Extent (million km^2)	Year	Ice Extent (million km^2)
1994	7.2	2008	4.7
1996	7.9	2010	4.9
1998	6.6	2012	3.6
2000	6.3	2014	5.2
2002	6.0	2016	4.5
2004	6.0	2018	4.8
2006	5.9	2020	3.9

Source: National Snow and Ice Data Center

Flow Rate (%)	Mosquito Positive Rate (%)
0	22
10	16
40	12
60	11
90	6
100	2

6. Mosquito Prevalence The table in the margin lists the relative abundance of mosquitoes (as measured by the mosquito positive rate) versus the flow rate (measured as a percentage of maximum flow) of canal networks in Saga City, Japan.

(a) Find the regression line for the data in the table, and graph the regression line and scatter plot on the same screen.

(b) Use the linear model that you found in part (a) to estimate the mosquito positive rate if the canal flow is 70% of maximum.

7. Noise and Intelligibility Audiologists study the intelligibility of spoken sentences under different noise levels. Intelligibility, the MRT score, is measured as the percent of a spoken sentence that the listener can decipher at a certain noise level (in decibels, dB). The table shows the results of one such test.

(a) Find the regression line for the data in the table, and graph the regression line and scatter plot on the same screen.

(b) Find the correlation coefficient. Do you think a linear model is appropriate?

(c) Use the linear model that you found in part (a) to estimate the intelligibility of a sentence at a 94-dB noise level.

Noise Level (dB)	MRT Score (%)
80	99
84	91
88	84
92	70
96	47
100	23
104	11

Would you buy a candy bar from a vending machine if the price were as indicated?

Price	Yes or No
$1.00	
$1.25	
$1.50	
$1.75	
$2.00	
$2.50	
$3.00	

8. Shoe Size and Height Do you think that shoe size and height are correlated? Find out by surveying the shoe sizes and heights of at least ten students in your class. Find the correlation coefficient.

9. Demand for Candy Bars In this problem you will determine a linear demand equation that describes the demand for candy bars in your class. Survey your classmates to determine what price they would be willing to pay for a candy bar. Your survey form might look like the sample shown at the left.

(a) Make a table of the number of respondents who answered "yes" at each price level.

(b) Make a scatter plot of your data.

(c) Find and graph the regression line $y = mp + b$, which gives the number of respondents y who would buy a candy bar if the price were p cents. This is the *demand equation*. Why is the slope m negative?

(d) What is the p-intercept of the demand equation? What does this intercept tell you about pricing candy bars?