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7 Analytic Trigonometry

In Chapters 5 and 6, we studied graphical and geometric properties of the trigonometric functions. In this chapter we study algebraic properties of these functions, that is, simplifying and factoring expressions and solving equations that involve trigonometric functions.

We have used the trigonometric functions to model different real-world phenomena, including periodic motion (such as the sound waves produced by the street musicians in the above photo). To obtain information from a model, we often need to solve equations. If the model involves trigonometric functions, we need to solve trigonometric equations. Solving trigonometric equations often requires using trigonometric identities. We've already encountered some basic trigonometric identities in the preceding chapters. We begin this chapter by finding many new identities.

7.1 Trigonometric Identities

■ Simplifying Trigonometric Expressions ■ Proving Trigonometric Identities

Recall that an **equation** is a statement that two mathematical expressions are equal. For example, the following are equations:

$$x + 2 = 5$$
$$(x + 1)^2 = x^2 + 2x + 1$$
$$\sin^2 t + \cos^2 t = 1$$

An **identity** is an equation that is true for all values of the variable(s) for which both sides of the equation are defined. The last two equations above are identities, but the first one is not, since it is not true for values of x other than 3.

A **trigonometric** identity is an identity involving trigonometric functions. We begin by listing some of the fundamental trigonometric identities. We studied most of these in Chapters 5 and 6; you are asked to prove the cofunction identities in Exercise 116.

Fundamental Trigonometric Identities

Reciprocal Identities

$$\csc x = \frac{1}{\sin x} \qquad \sec x = \frac{1}{\cos x} \qquad \cot x = \frac{1}{\tan x}$$

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1 \qquad \tan^2 x + 1 = \sec^2 x \qquad 1 + \cot^2 x = \csc^2 x$$

Even-Odd Identities

$$\sin(-x) = -\sin x \qquad \cos(-x) = \cos x \qquad \tan(-x) = -\tan x$$

Cofunction Identities

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x \qquad \tan\left(\frac{\pi}{2} - x\right) = \cot x \qquad \sec\left(\frac{\pi}{2} - x\right) = \csc x$$

$$\cos\left(\frac{\pi}{2} - x\right) = \sin x \qquad \cot\left(\frac{\pi}{2} - x\right) = \tan x \qquad \csc\left(\frac{\pi}{2} - x\right) = \sec x$$

■ Simplifying Trigonometric Expressions

We can use identities to write the same expression in different ways. It is often possible to rewrite a complicated-looking expression as a much simpler one. To simplify algebraic expressions, we used factoring, common denominators, and the Special Product Formulas. To simplify trigonometric expressions, we use these same techniques together with the fundamental trigonometric identities.

Example 1 ■ Simplifying a Trigonometric Expression

Simplify the expression $\cos t + \tan t \sin t$.

Solution We start by rewriting the expression in terms of sine and cosine.

$$\begin{aligned}
\cos t + \tan t \sin t &= \cos t + \left(\frac{\sin t}{\cos t}\right)\sin t && \text{Reciprocal identity} \\
&= \frac{\cos^2 t + \sin^2 t}{\cos t} && \text{Common denominator} \\
&= \frac{1}{\cos t} && \text{Pythagorean identity} \\
&= \sec t && \text{Reciprocal identity}
\end{aligned}$$

Now Try Exercise 3 ■

Example 2 ■ Simplifying by Combining Fractions

Simplify the expression $\dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{1 + \sin\theta}$.

Solution We combine the fractions by using a common denominator.

$$\begin{aligned}
\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{1 + \sin\theta} &= \frac{\sin\theta\,(1 + \sin\theta) + \cos^2\theta}{\cos\theta\,(1 + \sin\theta)} && \text{Common denominator} \\
&= \frac{\sin\theta + \sin^2\theta + \cos^2\theta}{\cos\theta\,(1 + \sin\theta)} && \text{Distribute } \sin\theta \\
&= \frac{\sin\theta + 1}{\cos\theta\,(1 + \sin\theta)} && \text{Pythagorean identity} \\
&= \frac{1}{\cos\theta} = \sec\theta && \text{Cancel, and use reciprocal identity}
\end{aligned}$$

Now Try Exercise 25 ■

■ Proving Trigonometric Identities

Many identities follow from the fundamental identities. In the examples that follow, we learn how to prove that a given trigonometric equation is an identity, and in the process we will see how to discover new identities.

First, to decide when a given equation is *not* an identity, all we need to do is show that the equation does not hold for some value of the variable (or variables). Thus the equation

$$\sin x + \cos x = 1$$

is not an identity, because when $x = \pi/4$, we have

$$\sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2} \neq 1$$

To verify that a trigonometric equation is an identity, we transform one side of the equation into the other side by a series of steps, each of which is itself an identity.

Guidelines for Proving Trigonometric Identities

1. **Start with One Side.** Choose one side of the equation, and write it down. Your goal is to transform it into the other side. It's usually easier to start with the more complicated side.
2. **Use Known Identities.** Use algebra and the identities you know to change the side you started with. Bring fractional expressions to a common denominator, factor, and use the fundamental identities to simplify expressions.
3. **Convert to Sines and Cosines.** If you are stuck, you may find it helpful to first rewrite all functions in terms of sines and cosines.

Warning To prove an identity, we do *not* just perform the same operations on both sides of the equation. For example, if we start with an equation that is not an identity, such as

$$\sin x = -\sin x$$

and square both sides, we get the equation

$$\sin^2 x = \sin^2 x$$

which is clearly an identity. Does this mean that the original equation is an identity? Of course not. The problem here is that the operation of squaring is not **reversible** in the sense that we cannot arrive back at the original equation by taking square roots (reversing the procedure). Only operations that are reversible will necessarily transform an identity into an identity.

Example 3 ■ Proving an Identity by Rewriting in Terms of Sine and Cosine

Consider the equation $\cos\theta\,(\sec\theta - \cos\theta) = \sin^2\theta$.

(a) Verify algebraically that the equation is an identity.

(b) Confirm graphically that the equation is an identity.

Solution

(a) The left-hand side looks more complicated, so we start with it and try to transform it into the right-hand side.

$$\begin{aligned} \text{LHS} &= \cos\theta\,(\sec\theta - \cos\theta) && \\ &= \cos\theta\left(\frac{1}{\cos\theta} - \cos\theta\right) && \text{Reciprocal identity} \\ &= 1 - \cos^2\theta && \text{Expand} \\ &= \sin^2\theta = \text{RHS} && \text{Pythagorean identity} \end{aligned}$$

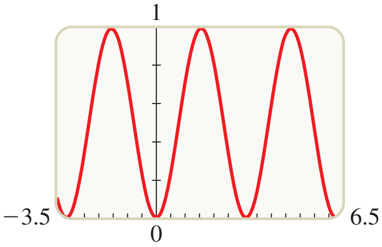


Figure 1

(b) We graph each side of the equation to see whether the graphs coincide. From Figure 1 we see that the graphs of $y = \cos\theta\,(\sec\theta - \cos\theta)$ and $y = \sin^2\theta$ are identical. This confirms that the equation is an identity.

Now Try Exercise 31 ■

Note In Example 3 it isn't easy to see how to change the right-hand side into the left-hand side, but it's definitely possible: notice that each step is reversible. In other words, if we start with the last expression in the proof and work backward through the steps, the right-hand side is transformed into the left-hand side. You will probably agree, however, that it's more difficult to prove the identity this way. That's why it's often better to change the more complicated side of the identity into the simpler side.

Example 4 ■ Proving an Identity by Combining Fractions

Verify the identity

$$2 \tan x \sec x = \frac{1}{1 - \sin x} - \frac{1}{1 + \sin x}$$

Solution Finding a common denominator and combining the fractions on the right-hand side of this equation, we get

$$\begin{aligned}
\text{RHS} &= \frac{1}{1 - \sin x} - \frac{1}{1 + \sin x} \\
&= \frac{(1 + \sin x) - (1 - \sin x)}{(1 - \sin x)(1 + \sin x)} && \text{Common denominator} \\
&= \frac{2 \sin x}{1 - \sin^2 x} && \text{Simplify} \\
&= \frac{2 \sin x}{\cos^2 x} && \text{Pythagorean identity} \\
&= 2\,\frac{\sin x}{\cos x}\left(\frac{1}{\cos x}\right) && \text{Factor} \\
&= 2 \tan x \sec x = \text{LHS} && \text{Reciprocal identities}
\end{aligned}$$

Now Try Exercise 67 ■

See the Prologue: *Principles of Problem Solving*

In Example 5 we "introduce something extra" to the problem: We multiply the numerator and the denominator by a trigonometric expression that we have chosen so that we can simplify the result.

Example 5 ■ Proving an Identity by Introducing Something Extra

Verify the identity $\dfrac{\cos u}{1 - \sin u} = \sec u + \tan u$.

Solution We start with the left-hand side and multiply the numerator and denominator by $1 + \sin u$.

We multiply by $1 + \sin u$ because we know by the difference of squares formula that

$$(1 - \sin u)(1 + \sin u) = 1 - \sin^2 u$$

and this is just $\cos^2 u$, a simpler expression.

$$\begin{aligned}
\text{LHS} &= \frac{\cos u}{1 - \sin u} \\
&= \frac{\cos u}{1 - \sin u} \cdot \frac{1 + \sin u}{1 + \sin u} && \text{Multiply numerator and denominator by } 1 + \sin u \\
&= \frac{\cos u\,(1 + \sin u)}{1 - \sin^2 u} && \text{Expand denominator} \\
&= \frac{\cos u\,(1 + \sin u)}{\cos^2 u} && \text{Pythagorean identity} \\
&= \frac{1 + \sin u}{\cos u} && \text{Cancel common factor} \\
&= \frac{1}{\cos u} + \frac{\sin u}{\cos u} && \text{Separate into two fractions} \\
&= \sec u + \tan u = \text{RHS} && \text{Reciprocal identities}
\end{aligned}$$

Now Try Exercise 79 ■

EUCLID (circa 300 B.C.) lived in Alexandria, Egypt. His book *Elements* is the most influential scientific book in history. For 2000 years it was the standard introduction to geometry in schools, and for many generations it was considered the best way to develop logical reasoning. Abraham Lincoln, for instance, studied the *Elements* as a way to sharpen his mind. The story is told that King Ptolemy once asked Euclid whether there was a faster way to learn geometry than through the *Elements*. Euclid replied that there is "no royal road to geometry"—meaning by this that mathematics does not respect wealth or social status. Euclid was revered in his own time and was referred to as "The Geometer" or "The Writer of the *Elements*." The greatness of the *Elements* stems from its precise, logical, and systematic treatment of geometry. For dealing with equality, Euclid lists the following rules, which he calls "common notions."

1. Things that are equal to the same thing are equal to each other.
2. If equals are added to equals, the sums are equal.
3. If equals are subtracted from equals, the remainders are equal.
4. Things that coincide with one another are equal.
5. The whole is greater than the part.

Here is another method for proving that an equation is an identity. If we can transform each side of the equation *separately*, by way of identities, to arrive at the same result, then the equation is an identity. Example 6 illustrates this procedure.

Example 6 ■ Proving an Identity by Working with Both Sides Separately

Verify the identity $\dfrac{1 + \cos\theta}{\cos\theta} = \dfrac{\tan^2\theta}{\sec\theta - 1}$.

Solution We prove the identity by changing each side separately into the same expression. (You should supply the reasons for each step.)

$$\text{LHS} = \frac{1 + \cos\theta}{\cos\theta} = \frac{1}{\cos\theta} + \frac{\cos\theta}{\cos\theta} = \sec\theta + 1$$

$$\text{RHS} = \frac{\tan^2\theta}{\sec\theta - 1} = \frac{\sec^2\theta - 1}{\sec\theta - 1} = \frac{(\sec\theta - 1)(\sec\theta + 1)}{\sec\theta - 1} = \sec\theta + 1$$

It follows that LHS = RHS, so the equation is an identity.

Now Try Exercise 85 ■

We conclude this section by describing the technique of *trigonometric substitution*, which we use to convert algebraic expressions to trigonometric ones. This is often useful in calculus, for instance, in finding the area of a circle or an ellipse.

Example 7 ■ Trigonometric Substitution

Substitute $\sin\theta$ for x in the expression $\sqrt{1 - x^2}$, and simplify. Assume that $0 \le \theta \le \pi/2$.

Solution Setting $x = \sin\theta$, we have

$$\begin{aligned}
\sqrt{1 - x^2} &= \sqrt{1 - \sin^2\theta} && \text{Substitute } x = \sin\theta \\
&= \sqrt{\cos^2\theta} && \text{Pythagorean identity} \\
&= \cos\theta && \text{Take square root}
\end{aligned}$$

The last equality is true because $\cos\theta \ge 0$ for the values of θ in question.

Now Try Exercise 91 ■

7.1 Exercises

■ Concepts

1. An equation is called an identity if it is valid for ________ values of the variable. The equation $2x = x + x$ is an algebraic identity, and the equation $\sin^2 x + \cos^2 x =$ ________ is a trigonometric identity.
2. For any x it is true that $\cos(-x)$ has the same value as $\cos x$. We express this fact as the identity ____________.

■ Skills

3–14 ■ Simplifying Trigonometric Expressions Write the trigonometric expression in terms of sine and cosine, and then simplify.

3. $\cos t \tan t$
4. $\cos t \csc t$
5. $\sin\theta \sec\theta$
6. $\tan\theta \csc\theta$
7. $\tan^2 x - \sec^2 x$
8. $\dfrac{\sec x}{\csc x}$

9. $\sin^2\left(\dfrac{\pi}{2} - y\right)\sec y$

10. $\tan\left(\dfrac{\pi}{2} - u\right)\sin u$

11. $\sin u + \cot u \cos u$

12. $\cos^2\theta\,(1 + \tan^2\theta)$

13. $\dfrac{\sec\theta - \cos\theta}{\sin\theta}$

14. $\dfrac{\cot\theta}{\csc\theta - \sin\theta}$

15–30 ■ Simplifying Trigonometric Expressions Simplify the trigonometric expression.

15. $\dfrac{\sin x \sec x}{\tan x}$

16. $\dfrac{\cos x \sec x}{\cot x}$

17. $\dfrac{\sin t + \tan t}{\tan t}$

18. $\dfrac{1 + \cot A}{\csc A}$

19. $\sin^3\left(\dfrac{\pi}{2} - x\right) + \sin^2 x \cos x$

20. $\sin^4\alpha - \cos^4\alpha + \cos^2\alpha$

21. $\dfrac{\sec^2 x - 1}{\sec^2 x}$

22. $\dfrac{\sec x - \cos x}{\tan x}$

23. $\dfrac{1 + \cos y}{1 + \sec y}$

24. $\dfrac{1 + \cos(y - \pi/2)}{1 + \csc y}$

25. $\dfrac{1 + \sin u}{\cos u} + \dfrac{\cos u}{1 + \sin u}$

26. $\dfrac{\sin t}{1 - \cos t} - \csc t$

27. $\dfrac{\cos(-x)}{\sec(-x) + \tan x}$

28. $\dfrac{\cot A - 1}{1 + \tan(-A)}$

29. $\dfrac{1}{1 - \sin\alpha} + \dfrac{1}{1 + \sin\alpha}$

30. $\dfrac{2 + \tan^2 x}{\sec^2 x} - 1$

31–32 ■ Proving an Identity Algebraically and Graphically Consider the given equation. **(a)** Verify algebraically that the equation is an identity. **(b)** Confirm graphically that the equation is an identity.

31. $\dfrac{\cos x}{\sec x \sin x} = \csc x - \sin x$

32. $\dfrac{\tan y}{\csc y} = \sec y - \cos y$

33–90 ■ Proving Identities Verify the identity using the fundamental trigonometric identities.

33. $\dfrac{\cos\alpha}{\sec\alpha} = \cos^2\alpha$

34. $\dfrac{\tan\beta}{\sin\beta} = \sec\beta$

35. $\dfrac{\cos u \sec u}{\tan u} = \cot u$

36. $\dfrac{\cot x \sec x}{\csc x} = 1$

37. $\cos^2\left(\dfrac{\pi}{2} - y\right)\csc y = \sin y$

38. $\tan\left(x - \dfrac{\pi}{2}\right)\sin x = \sin\left(x - \dfrac{\pi}{2}\right)$

39. $(\sin x + \cos x)^2 = 1 + 2\sin x \cos x$

40. $\tan\theta + \cot\theta = \sec\theta\csc\theta$

41. $\cos(-x) - \sin(-x) = \cos x + \sin x$

42. $\cot(-\alpha)\cos(-\alpha) + \sin(-\alpha) = -\csc\alpha$

43. $\dfrac{\sec A - 1}{\sec A + 1} = \dfrac{1 - \cos A}{1 + \cos A}$

44. $\dfrac{\cos^2 v}{\sin v} = \csc v - \sin v$

45. $(1 - \cos\beta)(1 + \cos\beta) = \dfrac{1}{\csc^2\beta}$

46. $\dfrac{\cos x}{\sec x} + \dfrac{\sin x}{\csc x} = 1$

47. $\dfrac{1}{1 - \sin^2 y} = 1 + \tan^2 y$

48. $\csc x - \sin x = \cos x \cot x$

49. $(\tan x + \cot x)^2 = \sec^2 x + \csc^2 x$

50. $\sin^2 y + \cos^2 y + \tan^2 y = \sec^2 y$

51. $(1 - \sin^2 t + \cos^2 t)^2 + 4\sin^2 t\cos^2 t = 4\cos^2 t$

52. $\dfrac{2\sin x \cos x}{(\sin x + \cos x)^2 - 1} = 1$

53. $\csc x \cos^2 x + \sin x = \csc x$

54. $\cot^2 t - \cos^2 t = \cot^2 t \cos^2 t$

55. $\dfrac{\sec t - \cos t}{\sec t} = \sin^2 t$

56. $(\cot x - \csc x)(\cos x + 1) = -\sin x$

57. $\cos^2 x - \sin^2 x = 2\cos^2 x - 1$

58. $2\cos^2 x - 1 = 1 - 2\sin^2 x$

59. $\sin^4\theta - \cos^4\theta = \sin^2\theta - \cos^2\theta$

60. $(1 - \cos^2 x)(1 + \cot^2 x) = 1$

61. $\dfrac{(\sin t + \cos t)^2}{\sin t \cos t} = 2 + \sec t \csc t$

62. $\sec t \csc t\,(\tan t + \cot t) = \sec^2 t + \csc^2 t$

63. $\dfrac{1 + \tan^2 u}{1 - \tan^2 u} = \dfrac{1}{\cos^2 u - \sin^2 u}$

64. $\dfrac{1 + \sec^2 x}{1 + \tan^2 x} = 1 + \cos^2 x$

65. $\dfrac{\sec x + \csc x}{\tan x + \cot x} = \sin x + \cos x$

66. $\dfrac{\sin x + \cos x}{\sec x + \csc x} = \sin x \cos x$

67. $\dfrac{1 - \cos x}{\sin x} + \dfrac{\sin x}{1 - \cos x} = 2\csc x$

68. $\dfrac{\csc^2 y - \cot^2 y}{\sec^2 y} = \cos^2 y$

69. $\tan^2 u - \sin^2 u = \tan^2 u \sin^2 u$

70. $\sec^4 x - \tan^4 x = \sec^2 x + \tan^2 x$

71. $\dfrac{1 + \tan x}{1 - \tan x} = \dfrac{\cos x + \sin x}{\cos x - \sin x}$

72. $\dfrac{\cos\theta}{1 - \sin\theta} = \dfrac{\sin\theta - \csc\theta}{\cos\theta - \cot\theta}$

73. $\dfrac{1}{\sec x + \tan x} + \dfrac{1}{\sec x - \tan x} = 2\sec x$

74. $\dfrac{\cos^2 t + \tan^2 t - 1}{\sin^2 t} = \tan^2 t$

75. $\dfrac{1 + \sin x}{1 - \sin x} - \dfrac{1 - \sin x}{1 + \sin x} = 4\tan x \sec x$

76. $\dfrac{\tan x + \tan y}{\cot x + \cot y} = \tan x \tan y$

77. $\dfrac{\sin^3 x + \cos^3 x}{\sin x + \cos x} = 1 - \sin x \cos x$

78. $\dfrac{\sin^3 y - \csc^3 y}{\sin y - \csc y} = \sin^2 y + \csc^2 y + 1$

79. $\dfrac{1 - \cos \alpha}{\sin \alpha} = \dfrac{\sin \alpha}{1 + \cos \alpha}$

80. $\dfrac{\sin x - 1}{\sin x + 1} = \dfrac{-\cos^2 x}{(\sin x + 1)^2}$

81. $\dfrac{\sin w}{\sin w + \cos w} = \dfrac{\tan w}{1 + \tan w}$

82. $\dfrac{\sin A}{1 - \cos A} - \cot A = \csc A$

83. $\dfrac{\sec x}{\sec x - \tan x} = \sec x (\sec x + \tan x)$

84. $\sec v - \tan v = \dfrac{1}{\sec v + \tan v}$

85. $\dfrac{(\sin x + \cos x)^2}{\sin^2 x - \cos^2 x} = \dfrac{\sin^2 x - \cos^2 x}{(\sin x - \cos x)^2}$

86. $\dfrac{\tan v \sin v}{\tan v + \sin v} = \dfrac{\tan v - \sin v}{\tan v \sin v}$

87. $\dfrac{1 - \sin x}{1 + \sin x} = (\sec x - \tan x)^2$

88. $\dfrac{1 + \sin x}{1 - \sin x} = (\tan x + \sec x)^2$

89. $\csc x - \cot x = \dfrac{1}{\csc x + \cot x}$

90. $\dfrac{\sec u - 1}{\sec u + 1} = \dfrac{\tan u - \sin u}{\tan u + \sin u}$

91–96 ■ Trigonometric Substitution Make the indicated trigonometric substitution in the given algebraic expression and simplify (see Example 7). Assume that $0 < \theta < \pi/2$.

91. $\dfrac{x}{\sqrt{1 - x^2}}$, $x = \sin \theta$

92. $\sqrt{1 + x^2}$, $x = \tan \theta$

93. $\sqrt{x^2 - 1}$, $x = \sec \theta$

94. $\dfrac{1}{x^2\sqrt{4 + x^2}}$, $x = 2 \tan \theta$

95. $\sqrt{9 - x^2}$, $x = 3 \sin \theta$

96. $\dfrac{\sqrt{x^2 - 25}}{x}$, $x = 5 \sec \theta$

97–100 ■ Determining Identities Graphically Graph f and g in the same viewing rectangle. Do the graphs suggest that the equation $f(x) = g(x)$ is an identity? Prove your answer.

97. $f(x) = \cos^2 x - \sin^2 x$, $g(x) = 1 - 2 \sin^2 x$

98. $f(x) = \tan x (1 + \sin x)$, $g(x) = \dfrac{\sin x \cos x}{1 + \sin x}$

99. $f(x) = (\sin x + \cos x)^2$, $g(x) = 1$

100. $f(x) = \cos^4 x - \sin^4 x$, $g(x) = 2 \cos^2 x - 1$

Skills Plus

101–106 ■ Proving More Identities Verify the identity.

101. $(\tan x + \cot x)^2 = \sec^2 x + \csc^2 x$

102. $\dfrac{1 + \cos x + \sin x}{1 + \cos x - \sin x} = \dfrac{1 + \sin x}{\cos x}$

103. $(\sin \alpha - \tan \alpha)(\cos \alpha - \cot \alpha) = (\cos \alpha - 1)(\sin \alpha - 1)$

104. $\sin^6 \beta + \cos^6 \beta = 1 - 3 \sin^2 \beta \cos^2 \beta$

105. $\tan^6 y = \dfrac{\sin^2 y - \tan^2 y}{\cos^2 y - \cot^2 y}$

106. $\dfrac{1}{(\sin x + \cos x)^2} = \dfrac{\sec^2 x}{(1 + \tan x)^2}$

PS *Try to recognize something familiar.* One method is to expand the denominator of the LHS and then factor $\cos^2 x$ from the denominator.

107–110 ■ Proving Identities Involving Other Functions These identities involve trigonometric functions as well as other functions that we have studied. Verify the identity.

107. $\ln|\tan x \sin x| = 2 \ln|\sin x| + \ln|\sec x|$

108. $\ln|\tan x| + \ln|\cot x| = 0$

109. $e^{\sin^2 x} e^{\tan^2 x} = e^{\sec^2 x} e^{-\cos^2 x}$

110. $e^{x + 2 \ln|\sin x|} = e^x \sin^2 x$

111. An Identity Involving Three Variables Suppose $x = R \cos \theta \sin \phi$, $y = R \sin \theta \sin \phi$, and $z = R \cos \phi$. Verify the identity $x^2 + y^2 + z^2 = R^2$.

Discuss ■ Discover ■ Prove ■ Write

112. Discuss: Is the Equation an Identity? You have encountered many identities in this course. Which of the following equations do you recognize as identities? For those that you think are identities, test several values of the variables to confirm that the equation is true for those values.

(a) $(x + y)^2 = x^2 + 2xy + y^2$ **(b)** $x^2 + y^2 = 1$

(c) $x(y + z) = xy + xz$ **(d)** $e^{\sin^2 x} e^{\cos^2 x} = e$

(e) $xe^{\ln x^2} = x^3$ **(f)** $\sin t + \cos t = 1$

(g) $x^2 - \tan^2 x = 0$

(h) $t^2 - \cos^2 t = (t - \cos t)(t + \cos t)$

113. Discuss: Equations That Are Not Identities How can you tell if an equation is not an identity? Show that the following equations are not identities.

(a) $\sin 2x = 2 \sin x$

(b) $\sec^2 x + \csc^2 x = 1$

(c) $\sin(x + y) = \sin x + \sin y$

(d) $\dfrac{1}{\sin x + \cos x} = \csc x + \sec x$

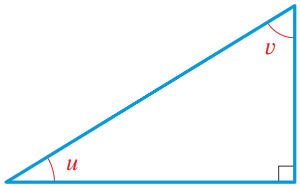
114. Discuss: Graphs and Identities Suppose you graph two functions, f and g, on a graphing device and their graphs appear identical in the viewing rectangle. Does this prove that the equation $f(x) = g(x)$ is an identity? Explain.

115. Discover: Making Up Your Own Identity If you start with a trigonometric expression and rewrite it or simplify it, then setting the original expression equal to the rewritten expression yields a trigonometric identity. For instance, from Example 1 we get the identity

$$\cos t + \tan t \sin t = \sec t$$

Use this technique to make up your own identity, then give it to a classmate to verify.

116. Discuss: Cofunction Identities Use the right triangle shown in the figure to explain why $v = (\pi/2) - u$. Explain how you can obtain all six cofunction identities from this triangle for $0 < u < \pi/2$.



Note that u and v are complementary angles. So the cofunction identities state that "a trigonometric function of an angle u is equal to the corresponding cofunction of the complementary angle v."

117. Discuss ■ Discover: Find the exact value of

$$\sin^{-1}\left(\frac{2}{3}\right) + \cos^{-1}\left(\frac{2}{3}\right) + \tan^{-1}\left(\frac{2}{3}\right) + \cot^{-1}\left(\frac{2}{3}\right)$$

PS *Draw a diagram.* Draw a triangle for each pair of cofunctions.

7.2 Addition and Subtraction Formulas

■ Addition and Subtraction Formulas ■ Expressions Involving Trigonometric Functions of a Sum ■ Expressions of the Form $A \sin x + B \cos x$

■ Addition and Subtraction Formulas

We now derive identities for trigonometric functions of sums and differences.

Addition and Subtraction Formulas

Formulas for Sine:

$$\sin(s + t) = \sin s \cos t + \cos s \sin t$$
$$\sin(s - t) = \sin s \cos t - \cos s \sin t$$

Formulas for Cosine:

$$\cos(s + t) = \cos s \cos t - \sin s \sin t$$
$$\cos(s - t) = \cos s \cos t + \sin s \sin t$$

Formulas for Tangent:

$$\tan(s + t) = \frac{\tan s + \tan t}{1 - \tan s \tan t}$$
$$\tan(s - t) = \frac{\tan s - \tan t}{1 + \tan s \tan t}$$

Proof of Addition Formula for Cosine To prove the formula

$$\cos(s + t) = \cos s \cos t - \sin s \sin t$$

we use Figure 1. In the figure, the distances t, $s + t$, and $-s$ have been marked on the unit circle, starting at $P_0(1, 0)$ and terminating at Q_1, P_1, and Q_0, respectively. The coordinates of these points are as follows:

$$P_0(1, 0) \qquad Q_0(\cos(-s), \sin(-s))$$
$$P_1(\cos(s + t), \sin(s + t)) \qquad Q_1(\cos t, \sin t)$$

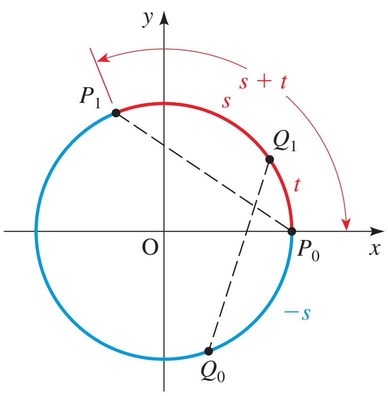


Figure 1

Since $\cos(-s) = \cos s$ and $\sin(-s) = -\sin s$, it follows that the point Q_0 has the coordinates $Q_0(\cos s, -\sin s)$. Notice that the distances between P_0 and P_1 and between Q_0 and Q_1 measured along the arc of the circle are equal. Since equal arcs are subtended by equal chords, it follows that $d(P_0, P_1) = d(Q_0, Q_1)$. Using the Distance Formula, we get

$$\sqrt{[\cos(s + t) - 1]^2 + [\sin(s + t) - 0]^2} = \sqrt{(\cos t - \cos s)^2 + (\sin t + \sin s)^2}$$


Stock Montage/Getty Images

JEAN BAPTISTE JOSEPH FOURIER (1768–1830) is responsible for the most powerful application of the trigonometric functions. He used sums of these functions to describe such physical phenomena as the transmission of sound and the flow of heat.

Orphaned as a young boy, Fourier was educated in a military school, where he became a mathematics teacher at the age of 20. He was later appointed professor at the École Polytechnique but resigned this position to accompany Napoleon on his expedition to Egypt, where Fourier served as governor. After returning to France, he began conducting experiments on heat. The French Academy refused to publish his early papers on this subject because of his lack of rigor. Fourier eventually became Secretary of the Academy and in this capacity had his papers published in their original form. Probably because of his study of heat and his years in the deserts of Egypt, Fourier became obsessed with keeping himself warm—he wore several layers of clothes, even in the summer, and kept his rooms at unbearably high temperatures. Evidently, these habits overburdened his heart and contributed to his death at the age of 62.

Squaring both sides and expanding, we have

These add to 1

$$\cos^2(s+t) - 2\cos(s+t) + 1 + \sin^2(s+t)$$
$$= \cos^2 t - 2\cos s\cos t + \cos^2 s + \sin^2 t + 2\sin s\sin t + \sin^2 s$$

These add to 1 These add to 1

Using the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$ three times gives

$$2 - 2\cos(s+t) = 2 - 2\cos s\cos t + 2\sin s\sin t$$

Finally, subtracting 2 from each side and dividing both sides by -2, we get

$$\cos(s+t) = \cos s\cos t - \sin s\sin t$$

which proves the Addition Formula for Cosine. ■

Proof of Subtraction Formula for Cosine Replacing t with $-t$ in the Addition Formula for Cosine, we obtain

$$\begin{aligned} \cos(s-t) &= \cos(s+(-t)) \\ &= \cos s\cos(-t) - \sin s\sin(-t) && \text{Addition Formula for Cosine} \\ &= \cos s\cos t + \sin s\sin t && \text{Even-odd identities} \end{aligned}$$

This proves the Subtraction Formula for Cosine. ■

See Exercises 77 and 78 for proofs of the other Addition Formulas.

Example 1 ■ Using the Addition and Subtraction Formulas

Find the exact value of each expression.

(a) $\cos 75°$ **(b)** $\cos\dfrac{\pi}{12}$

Solution

(a) Notice that $75° = 45° + 30°$. Since we know the exact values of sine and cosine at 45° and 30°, we use the Addition Formula for Cosine to get

$$\begin{aligned} \cos 75° &= \cos(45° + 30°) \\ &= \cos 45°\cos 30° - \sin 45°\sin 30° \\ &= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\frac{1}{2} = \frac{\sqrt{2}\sqrt{3}-\sqrt{2}}{4} = \frac{\sqrt{6}-\sqrt{2}}{4} \end{aligned}$$

(b) Since $\dfrac{\pi}{12} = \dfrac{\pi}{4} - \dfrac{\pi}{6}$, the Subtraction Formula for Cosine gives

$$\begin{aligned} \cos\frac{\pi}{12} &= \cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right) \\ &= \cos\frac{\pi}{4}\cos\frac{\pi}{6} + \sin\frac{\pi}{4}\sin\frac{\pi}{6} \\ &= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\frac{1}{2} = \frac{\sqrt{6}+\sqrt{2}}{4} \end{aligned}$$

Now Try Exercises 3 and 9 ■

Example 2 ■ Using the Addition Formula for Sine

Find the exact value of the expression $\sin 20° \cos 40° + \cos 20° \sin 40°$.

Solution We recognize the expression as the right-hand side of the Addition Formula for Sine with $s = 20°$ and $t = 40°$. So, we have

$$\sin 20° \cos 40° + \cos 20° \sin 40° = \sin(20° + 40°) = \sin 60° = \frac{\sqrt{3}}{2}$$

Now Try Exercise 15 ■

Example 3 ■ Proving a Cofunction Identity

Prove the cofunction identity $\cos\left(\frac{\pi}{2} - u\right) = \sin u$.

Solution By the Subtraction Formula for Cosine we have

$$\begin{aligned}\cos\left(\frac{\pi}{2} - u\right) &= \cos\frac{\pi}{2}\cos u + \sin\frac{\pi}{2}\sin u \\ &= 0 \cdot \cos u + 1 \cdot \sin u = \sin u\end{aligned}$$

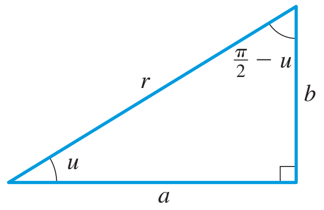


Figure 2 | $\cos\left(\frac{\pi}{2} - u\right) = \frac{b}{r} = \sin u$

Now Try Exercises 21 and 25 ■

For acute angles, the cofunction identity in Example 3, as well as the other cofunction identities, can also be derived from Figure 2.

Example 4 ■ Proving an Identity

Verify the identity $\dfrac{1 + \tan x}{1 - \tan x} = \tan\left(\dfrac{\pi}{4} + x\right)$.

Solution Starting with the right-hand side and using the Addition Formula for Tangent, we get

$$\begin{aligned}\text{RHS} = \tan\left(\frac{\pi}{4} + x\right) &= \frac{\tan\frac{\pi}{4} + \tan x}{1 - \tan\frac{\pi}{4}\tan x} \\ &= \frac{1 + \tan x}{1 - \tan x} = \text{LHS}\end{aligned}$$

Now Try Exercise 33 ■

The next example illustrates a typical use of the Addition and Subtraction Formulas in calculus.

Example 5 ■ An Identity from Calculus

If $f(x) = \sin x$, show that

$$\frac{f(x+h) - f(x)}{h} = \sin x \left(\frac{\cos h - 1}{h}\right) + \cos x \left(\frac{\sin h}{h}\right)$$

Solution

$$\begin{aligned}
\frac{f(x+h) - f(x)}{h} &= \frac{\sin(x+h) - \sin x}{h} && \text{Definition of } f \\
&= \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} && \text{Addition Formula for Sine} \\
&= \frac{\sin x (\cos h - 1) + \cos x \sin h}{h} && \text{Factor} \\
&= \sin x \left(\frac{\cos h - 1}{h}\right) + \cos x \left(\frac{\sin h}{h}\right) && \text{Separate the fraction}
\end{aligned}$$

Now Try Exercise 65 ■

■ Expressions Involving Trigonometric Functions of a Sum

Expressions involving trigonometric functions and their inverses arise in calculus. In the following examples we illustrate how to evaluate such expressions.

Example 6 ■ Simplifying an Expression Involving Inverse Trigonometric Functions

Write $\sin(\cos^{-1} x + \tan^{-1} y)$ as an algebraic expression in x and y, where $-1 \le x \le 1$ and y is any real number.

Solution Let $\theta = \cos^{-1} x$ and $\phi = \tan^{-1} y$. Using the methods of Section 6.4, we sketch triangles with angles θ and ϕ such that $\cos\theta = x$ and $\tan\phi = y$ (see Figure 3). From the triangles we have

$$\sin\theta = \sqrt{1 - x^2} \qquad \cos\phi = \frac{1}{\sqrt{1 + y^2}} \qquad \sin\phi = \frac{y}{\sqrt{1 + y^2}}$$

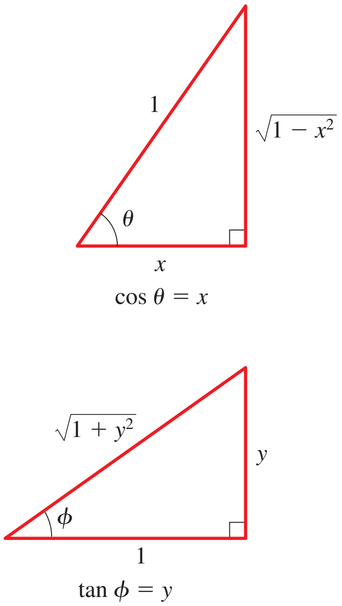


Figure 3

From the Addition Formula for Sine we have

$$\begin{aligned}
\sin(\cos^{-1} x + \tan^{-1} y) &= \sin(\theta + \phi) \\
&= \sin\theta \cos\phi + \cos\theta \sin\phi && \text{Addition Formula for Sine} \\
&= \sqrt{1 - x^2}\,\frac{1}{\sqrt{1 + y^2}} + x\,\frac{y}{\sqrt{1 + y^2}} && \text{From triangles} \\
&= \frac{1}{\sqrt{1 + y^2}}\left(\sqrt{1 - x^2} + xy\right) && \text{Factor } \frac{1}{\sqrt{1 + y^2}}
\end{aligned}$$

Now Try Exercises 47 and 51 ■

Example 7 ■ Evaluating an Expression Involving Trigonometric Functions

Evaluate $\sin(\theta + \phi)$, where $\sin\theta = \frac{12}{13}$ with θ in Quadrant II and $\tan\phi = \frac{3}{4}$ with ϕ in Quadrant III.

Solution We first sketch the angles θ and ϕ in standard position with terminal sides in the appropriate quadrants, as shown in Figure 4. Since $\sin\theta = y/r = \frac{12}{13}$, we can label a side and the hypotenuse in the triangle in Figure 4(a). To find the remaining side, we use the Pythagorean Theorem.

$$\begin{aligned} x^2 + y^2 &= r^2 && \text{Pythagorean Theorem} \\ x^2 + 12^2 &= 13^2 && y = 12, \quad r = 13 \\ x^2 &= 25 && \text{Solve for } x^2 \\ x &= -5 && \text{Because } x < 0 \text{ in Quadrant II} \end{aligned}$$

Similarly, since $\tan\phi = y/x = \frac{3}{4}$, we can label two sides of the triangle in Figure 4(b) and then use the Pythagorean Theorem to find the hypotenuse.

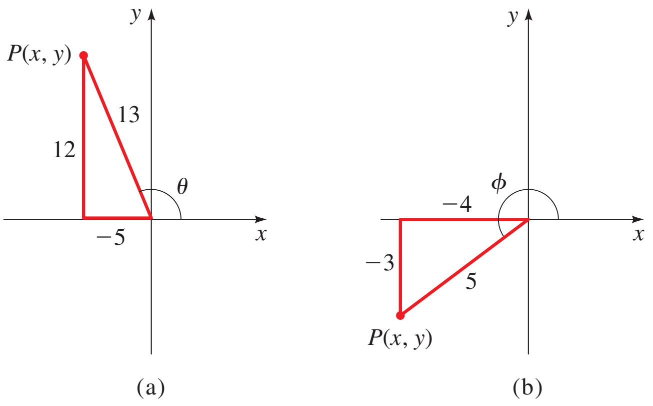


Figure 4

Now, to find $\sin(\theta + \phi)$, we use the Addition Formula for Sine and the triangles in Figure 4.

$$\begin{aligned} \sin(\theta + \phi) &= \sin\theta\cos\phi + \cos\theta\sin\phi && \text{Addition Formula} \\ &= \left(\tfrac{12}{13}\right)\left(-\tfrac{4}{5}\right) + \left(-\tfrac{5}{13}\right)\left(-\tfrac{3}{5}\right) && \text{From triangles} \\ &= -\tfrac{33}{65} && \text{Calculate} \end{aligned}$$

Now Try Exercise 55 ■

■ Expressions of the Form $A \sin x + B \cos x$

We can write expressions of the form $A \sin x + B \cos x$ in terms of a single trigonometric function using the Addition Formula for Sine. For example, consider the expression

$$\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x$$

If we set $\phi = \pi/3$, then $\cos\phi = \frac{1}{2}$ and $\sin\phi = \sqrt{3}/2$, and we can write

$$\begin{aligned} \frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x &= \cos\phi\sin x + \sin\phi\cos x \\ &= \sin(x + \phi) = \sin\left(x + \frac{\pi}{3}\right) \end{aligned}$$

We are able to do this because the coefficients $\frac{1}{2}$ and $\sqrt{3}/2$ are precisely the cosine and sine of a particular number, in this case, $\pi/3$. We can use this same idea in general to write $A \sin x + B \cos x$ in the form $k \sin(x + \phi)$. We start by multiplying the numerator and denominator by $\sqrt{A^2 + B^2}$ to get

$$A \sin x + B \cos x = \sqrt{A^2 + B^2}\left(\frac{A}{\sqrt{A^2 + B^2}} \sin x + \frac{B}{\sqrt{A^2 + B^2}} \cos x\right)$$

We need a number ϕ with the property that

$$\cos \phi = \frac{A}{\sqrt{A^2 + B^2}} \quad \text{and} \quad \sin \phi = \frac{B}{\sqrt{A^2 + B^2}}$$

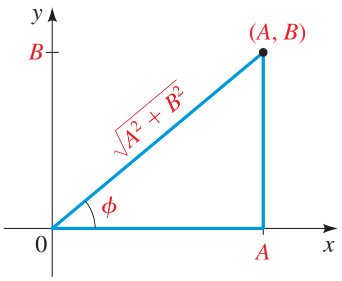


Figure 5

Figure 5 shows that the point (A, B) in the plane determines a number ϕ with precisely these properties. With this ϕ we have

$$\begin{aligned} A \sin x + B \cos x &= \sqrt{A^2 + B^2}(\cos \phi \sin x + \sin \phi \cos x) \\ &= \sqrt{A^2 + B^2} \sin(x + \phi) \end{aligned}$$

We have proved the following theorem.

Sums of Sines and Cosines

If A and B are real numbers, then

$$A \sin x + B \cos x = k \sin(x + \phi)$$

where $k = \sqrt{A^2 + B^2}$ and ϕ satisfies

$$\cos \phi = \frac{A}{\sqrt{A^2 + B^2}} \quad \text{and} \quad \sin \phi = \frac{B}{\sqrt{A^2 + B^2}}$$

Example 8 ■ A Sum of Sine and Cosine Terms

Express $3 \sin x + 4 \cos x$ in the form $k \sin(x + \phi)$.

Solution By the preceding theorem, $k = \sqrt{A^2 + B^2} = \sqrt{3^2 + 4^2} = 5$. The angle ϕ has the property that $\sin \phi = B/k = \frac{4}{5}$ and $\cos \phi = A/k = \frac{3}{5}$, with ϕ in Quadrant I (because $\sin \phi$ and $\cos \phi$ are both positive), so $\phi = \sin^{-1}\left(\frac{4}{5}\right)$. Using a calculator, we get $\phi \approx 53.1°$. Thus

$$3 \sin x + 4 \cos x \approx 5 \sin(x + 53.1°)$$

Now Try Exercise 59 ■

Example 9 ■ Graphing a Trigonometric Function

Write the function $f(x) = -\sin 2x + \sqrt{3} \cos 2x$ in the form $k \sin(2x + \phi)$, and use the new form to graph the function.

Solution Since $A = -1$ and $B = \sqrt{3}$, we have $k = \sqrt{A^2 + B^2} = \sqrt{1 + 3} = 2$. The angle ϕ satisfies $\cos \phi = -\frac{1}{2}$ and $\sin \phi = \sqrt{3}/2$. From the signs of these quantities we conclude that ϕ is in Quadrant II. Thus $\phi = 2\pi/3$. By the preceding theorem we can write

$$f(x) = -\sin 2x + \sqrt{3} \cos 2x = 2 \sin\left(2x + \frac{2\pi}{3}\right)$$

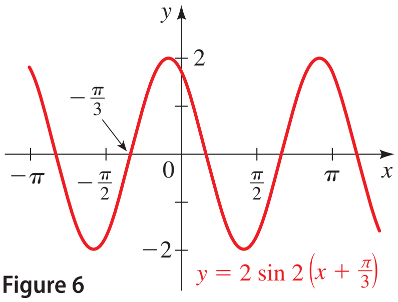


Figure 6

Using the form

$$f(x) = 2\sin 2\left(x + \frac{\pi}{3}\right)$$

we see that the graph is a sine curve with amplitude 2, period $2\pi/2 = \pi$, and horizontal shift $-\pi/3$. The graph is shown in Figure 6.

Now Try Exercise 63 ■

7.2 Exercises

Concepts

1. If we know the values of the sine and cosine of x and y, we can find the value of $\sin(x + y)$ by using the ________ Formula for Sine. State the formula: $\sin(x + y) =$ ________________.

2. If we know the values of the sine and cosine of x and y, we can find the value of $\cos(x - y)$ by using the ________ Formula for Cosine. State the formula: $\cos(x - y) =$ ________________.

Skills

3–14 ■ Values of Trigonometric Functions Use an Addition or Subtraction Formula to find the exact value of the expression, as demonstrated in Example 1.

3. $\sin 75°$ **4.** $\sin 15°$

5. $\cos 105°$ **6.** $\cos 195°$

7. $\tan 15°$ **8.** $\tan 165°$

9. $\sin \dfrac{19\pi}{12}$ **10.** $\cos \dfrac{17\pi}{12}$

11. $\tan\left(-\dfrac{\pi}{12}\right)$ **12.** $\sin\left(-\dfrac{5\pi}{12}\right)$

13. $\cos \dfrac{11\pi}{12}$ **14.** $\tan \dfrac{7\pi}{12}$

15–20 ■ Values of Trigonometric Functions Use an Addition or Subtraction Formula to write the expression as a trigonometric function of one number, and then find its exact value.

15. $\cos 23° \cos 67° - \sin 23° \sin 67°$

16. $\sin 35° \cos 25° + \cos 35° \sin 25°$

17. $\sin \dfrac{3\pi}{4} \cos \dfrac{\pi}{4} - \cos \dfrac{3\pi}{4} \sin \dfrac{\pi}{4}$

18. $\dfrac{\tan \dfrac{3\pi}{5} + \tan \dfrac{\pi}{15}}{1 - \tan \dfrac{3\pi}{5} \tan \dfrac{\pi}{15}}$

19. $\dfrac{\tan 55° - \tan 10°}{1 + \tan 55° \tan 10°}$

20. $\cos \dfrac{5\pi}{18} \cos \dfrac{\pi}{9} + \sin \dfrac{5\pi}{18} \sin \dfrac{\pi}{9}$

21–24 ■ Cofunction Identities Prove the cofunction identity using the Addition and Subtraction Formulas for sine and cosine.

21. $\tan\left(\dfrac{\pi}{2} - u\right) = \cot u$ **22.** $\cot\left(\dfrac{\pi}{2} - u\right) = \tan u$

23. $\sec\left(\dfrac{\pi}{2} - u\right) = \csc u$ **24.** $\csc\left(\dfrac{\pi}{2} - u\right) = \sec u$

25–46 ■ Proving Identities Prove the identity.

25. $\sin\left(x - \dfrac{\pi}{2}\right) = -\cos x$

26. $\cos\left(x - \dfrac{\pi}{2}\right) = \sin x$

27. $\sin(x - \pi) = -\sin x$

28. $\cos(x - \pi) = -\cos x$

29. $\tan(x - \pi) = \tan x$

30. $\cot\left(x - \dfrac{\pi}{2}\right) = -\tan x$

31. $\sin\left(\dfrac{\pi}{2} - x\right) = \sin\left(\dfrac{\pi}{2} + x\right)$

32. $\cos\left(x + \dfrac{\pi}{3}\right) + \sin\left(x - \dfrac{\pi}{6}\right) = 0$

33. $\tan\left(x + \dfrac{\pi}{3}\right) = \dfrac{\sqrt{3} + \tan x}{1 - \sqrt{3}\tan x}$

34. $\tan\left(x - \dfrac{\pi}{4}\right) = \dfrac{\tan x - 1}{\tan x + 1}$

35. $\sin(x + y) - \sin(x - y) = 2\cos x \sin y$

36. $\cos(x + y) + \cos(x - y) = 2\cos x \cos y$

37. $\cot(x - y) = \dfrac{\cot x \cot y + 1}{\cot y - \cot x}$

38. $\cot(x + y) = \dfrac{\cot x \cot y - 1}{\cot x + \cot y}$

39. $\tan x - \tan y = \dfrac{\sin(x - y)}{\cos x \cos y}$

40. $1 - \tan x \tan y = \dfrac{\cos(x + y)}{\cos x \cos y}$

41. $\dfrac{\tan x - \tan y}{1 - \tan x \tan y} = \dfrac{\sin(x - y)}{\cos(x + y)}$

42. $\dfrac{\sin(x + y) - \sin(x - y)}{\cos(x + y) + \cos(x - y)} = \tan y$

43. $\cos(x + y)\cos(x - y) = \cos^2 x - \sin^2 y$

44. $\cos(x + y)\cos y + \sin(x + y)\sin y = \cos x$

45. $\sin(x + y + z) = \sin x \cos y \cos z + \cos x \sin y \cos z + \cos x \cos y \sin z - \sin x \sin y \sin z$

46. $\tan(x - y) + \tan(y - z) + \tan(z - x) = \tan(x - y)\tan(y - z)\tan(z - x)$

47–50 ■ Expressions Involving Inverse Trigonometric Functions Write the given expression in terms of x and y only.

47. $\cos(\sin^{-1} x - \tan^{-1} y)$ **48.** $\tan(\sin^{-1} x + \cos^{-1} y)$

49. $\sin(\tan^{-1} x - \tan^{-1} y)$ **50.** $\sin(\sin^{-1} x + \cos^{-1} y)$

51–54 ■ Expressions Involving Inverse Trigonometric Functions Find the exact value of the expression.

51. $\sin\left(\cos^{-1}\left(\frac{1}{2}\right) + \tan^{-1} 1\right)$

52. $\cos\left(\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) + \cot^{-1}\sqrt{3}\right)$

53. $\tan\left(\sin^{-1}\left(\frac{3}{4}\right) - \cos^{-1}\left(\frac{1}{3}\right)\right)$

54. $\sin\left(\cos^{-1}\left(\frac{2}{3}\right) - \tan^{-1}\left(\frac{1}{2}\right)\right)$

55–58 ■ Evaluating Expressions Involving Trigonometric Functions Evaluate each expression under the given conditions.

55. $\sin(\theta - \phi)$; $\sin\theta = \frac{4}{5}$, θ in Quadrant II, $\tan\phi = \frac{1}{3}$, ϕ in Quadrant III

56. $\cos(\theta + \phi)$; $\tan\theta = \frac{4}{3}$, θ in Quadrant III, $\cos\phi = \frac{2}{3}$, ϕ in Quadrant IV

57. $\sin(\theta + \phi)$; $\sin\theta = \frac{5}{13}$, θ in Quadrant I, $\cos\phi = -2\sqrt{5}/5$, ϕ in Quadrant II

58. $\tan(\theta + \phi)$; $\cos\theta = -\frac{1}{3}$, θ in Quadrant III, $\sin\phi = \frac{1}{4}$, ϕ in Quadrant II

59–62 ■ Expressions in Terms of Sine Write the expression in terms of sine only.

59. $-\sqrt{3}\sin x + \cos x$

60. $\sin x - \cos x$

61. $5(\sin 2x - \cos 2x)$

62. $3\sin \pi x + 3\sqrt{3}\cos \pi x$

63–64 ■ Graphing a Trigonometric Function **(a)** Express the function in terms of sine only. **(b)** Graph the function.

63. $g(x) = \cos 2x + \sqrt{3}\sin 2x$

64. $f(x) = \sin x + \cos x$

Skills Plus

65–66 ■ Difference Quotient Let $f(x) = \cos x$ and $g(x) = \sin x$. Use Addition or Subtraction Formulas to show the following.

65. $\dfrac{f(x + h) - f(x)}{h} = -\cos x\left(\dfrac{1 - \cos h}{h}\right) - \sin x\left(\dfrac{\sin h}{h}\right)$

66. $\dfrac{g(x + h) - g(x)}{h} = \left(\dfrac{\sin h}{h}\right)\cos x - \sin x\left(\dfrac{1 - \cos h}{h}\right)$

67–68 ■ Discovering an Identity Graphically In these exercises we discover an identity graphically and then prove the identity. **(a)** Graph the function and make a conjecture, then **(b)** prove that your conjecture is true.

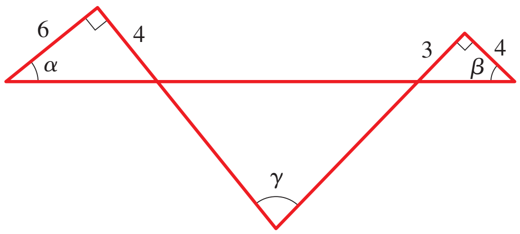
67. $y = \sin^2\left(x + \dfrac{\pi}{4}\right) + \sin^2\left(x - \dfrac{\pi}{4}\right)$

68. $y = -\frac{1}{2}[\cos(x + \pi) + \cos(x - \pi)]$

69. Difference of Two Angles Show that if $\beta - \alpha = \pi/2$, then

$$\sin(x + \alpha) + \cos(x + \beta) = 0$$

70. Sum of Two Angles Refer to the figure. Show that $\alpha + \beta = \gamma$, and find $\tan\gamma$.



71–72 ■ Identities Involving Inverse Trigonometric Functions Prove the identity.

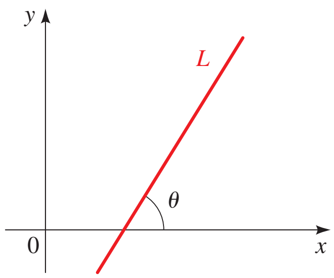
71. $\tan^{-1}\left(\dfrac{x + y}{1 - xy}\right) = \tan^{-1} x + \tan^{-1} y$

[*Hint:* Let $u = \tan^{-1} x$ and $v = \tan^{-1} y$, so that $x = \tan u$ and $y = \tan v$. Use an Addition Formula to find $\tan(u + v)$.]

72. $\tan^{-1} x + \tan^{-1}\left(\dfrac{1}{x}\right) = \dfrac{\pi}{2}$, $x > 0$ [*Hint:* Let $u = \tan^{-1} x$ and $v = \tan^{-1}\left(\dfrac{1}{x}\right)$, so that $x = \tan u$ and $\dfrac{1}{x} = \tan v$. Use an Addition Formula to find $\cot(u + v)$.]

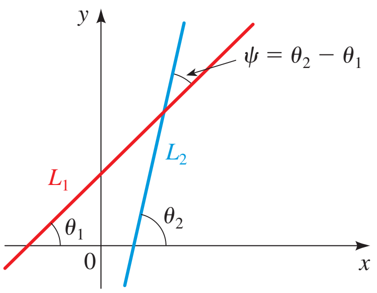
73. Angle Between Two Lines In this exercise we find a formula for the angle formed by two lines in a coordinate plane.

(a) If L is a line in the plane and θ is the angle formed by the line and the x-axis, as shown in the figure, show that the slope m of the line is given by $m = \tan\theta$.



(b) Let L_1 and L_2 be two nonparallel lines in the plane with slopes m_1 and m_2, respectively. Let ψ be the acute angle formed by the two lines (see the figure). Show that

$$\tan \psi = \frac{m_2 - m_1}{1 + m_1 m_2}$$

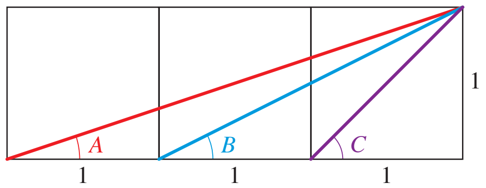


(c) Find the acute angle formed by the two lines

$$y = \tfrac{1}{3}x + 1 \qquad \text{and} \qquad y = \tfrac{1}{2}x - 3$$

(d) Show that if two lines are perpendicular, then the slope of one is the negative reciprocal of the slope of the other. [*Hint:* First find an expression for $\cot \psi$.]

74. A Sum of Three Angles Find $\angle A + \angle B + \angle C$ in the figure.



PS *Introduce something extra.* First find $\tan(A + B)$.

Applications

75. Adding an Echo A digital delay device echoes an input signal by repeating it a fixed length of time after it is received. If such a device receives the pure note $f_1(t) = 5 \sin t$ and echoes the pure note $f_2(t) = 5 \cos t$, then the combined sound is $f(t) = f_1(t) + f_2(t)$.

(a) Graph $y = f(t)$, and observe that the graph has the form of a sine curve $y = k \sin(t + \phi)$.

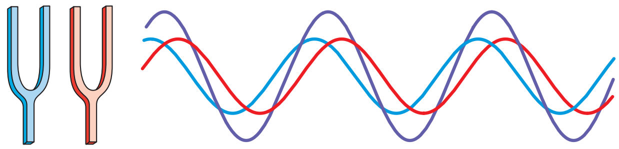
(b) Find k and ϕ.

76. Interference Two identical tuning forks are struck, one a fraction of a second after the other (see the figure). The sounds produced are modeled by $f_1(t) = C \sin \omega t$ and $f_2(t) = C \sin(\omega t + \alpha)$. The two sound waves interfere to produce a single sound modeled by the sum of these functions

$$f(t) = C \sin \omega t + C \sin(\omega t + \alpha)$$

(a) Use the Addition Formula for Sine to show that f can be written in the form $f(t) = A \sin \omega t + B \cos \omega t$, where A and B are constants that depend on α.

(b) Suppose that $C = 10$ and $\alpha = \pi/3$. Find constants k and ϕ so that $f(t) = k \sin(\omega t + \phi)$.



Discuss ■ Discover ■ Prove ■ Write

77. Prove: Addition Formula for Sine In the text we proved only the Addition and Subtraction Formulas for Cosine. Use these formulas and the cofunction identities for sine and cosine to prove the Addition Formula for Sine. [*Hint:* To get started, use the cofunction identity for sine to write

$$\sin(s + t) = \cos\left(\frac{\pi}{2} - (s + t)\right)$$
$$= \cos\left(\left(\frac{\pi}{2} - s\right) - t\right)$$

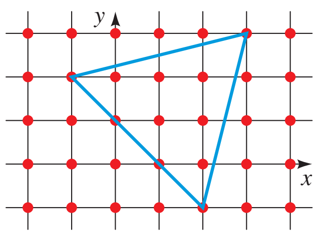
and use the Subtraction Formula for Cosine.]

78. Prove: Addition Formula for Tangent Use the Addition Formulas for Cosine and Sine to prove the Addition Formula for Tangent. [*Hint:* Use

$$\tan(s + t) = \frac{\sin(s + t)}{\cos(s + t)}$$

and divide the numerator and denominator by $\cos s \cos t$.]

79. Prove: An Impossible Triangle The points (m, n) in the coordinate plane, where m and n are integers, are called *lattice points*. Prove that a triangle with vertices at lattice points cannot be equilateral.



PS *Indirect reasoning.* Assume that such a triangle exists and use the formula in Exercise 73(b) to arrive at a contradiction.

80. Prove: Angles in an Acute Triangle Show that if ABC is an acute triangle, then

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C$$

PS *Try to recognize something familiar.* Use the familiar fact that the sum of the angles of a triangle is 180° and apply the Addition Formula for Tangent.

7.3 Double-Angle, Half-Angle, and Product-Sum Formulas

■ Double-Angle Formulas ■ Half-Angle Formulas ■ Expressions Involving Inverse Trigonometric Functions ■ Product-Sum Formulas

The identities we consider in this section are consequences of the addition formulas. The **Double-Angle Formulas** allow us to find the values of the trigonometric functions at $2x$ from their values at x. The **Half-Angle Formulas** relate the values of the trigonometric functions at $\frac{1}{2}x$ to their values at x. The **Product-Sum Formulas** relate products of sines and cosines to sums of sines and cosines.

■ Double-Angle Formulas

The following formulas are immediate consequences of the addition formulas, which we proved in Section 7.2.

Double-Angle Formulas

Formula for sine: $\sin 2x = 2 \sin x \cos x$

Formulas for cosine:
$$\begin{aligned}\cos 2x &= \cos^2 x - \sin^2 x\\ &= 1 - 2\sin^2 x\\ &= 2\cos^2 x - 1\end{aligned}$$

Formula for tangent: $\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$

The proofs for the formulas for cosine are given here. You are asked to prove the remaining formulas in Exercises 35 and 36.

Proof of Double-Angle Formulas for Cosine

$$\begin{aligned}\cos 2x &= \cos(x + x)\\ &= \cos x \cos x - \sin x \sin x && \text{Addition Formula}\\ &= \cos^2 x - \sin^2 x\end{aligned}$$

The second and third formulas for $\cos 2x$ are obtained from the formula we just proved and the Pythagorean identity. Substituting $\cos^2 x = 1 - \sin^2 x$ gives

$$\begin{aligned}\cos 2x &= \cos^2 x - \sin^2 x\\ &= (1 - \sin^2 x) - \sin^2 x\\ &= 1 - 2\sin^2 x\end{aligned}$$

The third formula is obtained in the same way, by substituting $\sin^2 x = 1 - \cos^2 x$. ■

Example 1 ■ Using the Double-Angle Formulas

If $\cos x = -\frac{2}{3}$ and x is in Quadrant II, find $\cos 2x$ and $\sin 2x$.

Solution Using one of the Double-Angle Formulas for Cosine, we get

$$\begin{aligned}\cos 2x &= 2\cos^2 x - 1 && \text{Double- Angle Formula}\\ &= 2\left(-\frac{2}{3}\right)^2 - 1 = \frac{8}{9} - 1 = -\frac{1}{9}\end{aligned}$$

To use the formula $\sin 2x = 2 \sin x \cos x$, we need to find $\sin x$ first. We have

$$\sin x = \sqrt{1 - \cos^2 x} = \sqrt{1 - \left(-\tfrac{2}{3}\right)^2} = \frac{\sqrt{5}}{3}$$

where we have used the positive square root because $\sin x$ is positive in Quadrant II. Thus

$$\begin{aligned} \sin 2x &= 2\sin x\cos x && \text{Double- Angle Formula} \\ &= 2\left(\frac{\sqrt{5}}{3}\right)\left(-\frac{2}{3}\right) = -\frac{4\sqrt{5}}{9} \end{aligned}$$

Now Try Exercises 3 and 51 ■

Example 2 ■ A Triple-Angle Formula

Write $\cos 3x$ in terms of $\cos x$.

Solution

$$\begin{aligned} \cos 3x &= \cos(2x + x) \\ &= \cos 2x\cos x - \sin 2x\sin x && \text{Addition formula} \\ &= (2\cos^2 x - 1)\cos x - (2\sin x\cos x)\sin x && \text{Double-Angle Formulas} \\ &= 2\cos^3 x - \cos x - 2\sin^2 x\cos x && \text{Expand} \\ &= 2\cos^3 x - \cos x - 2\cos x\,(1 - \cos^2 x) && \text{Pythagorean identity} \\ &= 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x && \text{Expand} \\ &= 4\cos^3 x - 3\cos x && \text{Simplify} \end{aligned}$$

Now Try Exercise 109 ■

Example 2 shows that $\cos 3x$ can be written as a polynomial of degree 3 in $\cos x$. The identity $\cos 2x = 2\cos^2 x - 1$ shows that $\cos 2x$ is a polynomial of degree 2 in $\cos x$. In fact, for any natural number n we can write $\cos nx$ as a polynomial in $\cos x$ of degree n (see the note in Exercise 109). The analogous result for $\sin nx$ is not true in general.

Example 3 ■ Proving an Identity

Prove the identity $\dfrac{\sin 3x}{\sin x\cos x} = 4\cos x - \sec x$.

Solution We start with the left-hand side.

$$\begin{aligned} \text{LHS} &= \frac{\sin 3x}{\sin x\cos x} = \frac{\sin(x + 2x)}{\sin x\cos x} \\ &= \frac{\sin x\cos 2x + \cos x\sin 2x}{\sin x\cos x} && \text{Addition Formula} \\ &= \frac{\sin x\,(2\cos^2 x - 1) + \cos x\,(2\sin x\cos x)}{\sin x\cos x} && \text{Double-Angle Formulas} \\ &= \frac{\sin x\,(2\cos^2 x - 1)}{\sin x\cos x} + \frac{\cos x\,(2\sin x\cos x)}{\sin x\cos x} && \text{Separate fraction} \\ &= \frac{2\cos^2 x - 1}{\cos x} + 2\cos x && \text{Cancel} \\ &= 2\cos x - \frac{1}{\cos x} + 2\cos x && \text{Separate fraction} \\ &= 4\cos x - \sec x = \text{RHS} && \text{Reciprocal identity} \end{aligned}$$

Now Try Exercise 87 ■

Half-Angle Formulas

The following formulas allow us to write any trigonometric expression involving even powers of sine and cosine in terms of the first power of cosine only. This technique is important in calculus. The Half-Angle Formulas are immediate consequences of these formulas.

Formulas for Lowering Powers

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\tan^2 x = \frac{1 - \cos 2x}{1 + \cos 2x}$$

Proof The first formula is obtained by solving for $\sin^2 x$ in the Double-Angle Formula $\cos 2x = 1 - 2\sin^2 x$. Similarly, the second formula is obtained by solving for $\cos^2 x$ in the Double-Angle Formula $\cos 2x = 2\cos^2 x - 1$.

The last formula follows from the first two and the reciprocal identities:

$$\tan^2 x = \frac{\sin^2 x}{\cos^2 x} = \frac{\dfrac{1 - \cos 2x}{2}}{\dfrac{1 + \cos 2x}{2}} = \frac{1 - \cos 2x}{1 + \cos 2x}$$

■

Example 4 ■ Lowering Powers in a Trigonometric Expression

Express $\sin^2 x \cos^2 x$ in terms of the first power of cosine.

Solution We use the formulas for lowering powers repeatedly.

$$\begin{aligned}
\sin^2 x \cos^2 x &= \left(\frac{1 - \cos 2x}{2}\right)\left(\frac{1 + \cos 2x}{2}\right) \\
&= \frac{1 - \cos^2 2x}{4} = \frac{1}{4} - \frac{1}{4}\cos^2 2x \\
&= \frac{1}{4} - \frac{1}{4}\left(\frac{1 + \cos(2 \cdot 2x)}{2}\right) = \frac{1}{4} - \frac{1}{8} - \frac{\cos 4x}{8} \\
&= \frac{1}{8} - \frac{1}{8}\cos 4x = \frac{1}{8}(1 - \cos 4x)
\end{aligned}$$

Another way to obtain this identity is to use the Double-Angle Formula for Sine in the form $\sin x \cos x = \frac{1}{2}\sin 2x$. Thus

$$\begin{aligned}
\sin^2 x \cos^2 x &= \frac{1}{4}\sin^2 2x = \frac{1}{4}\left(\frac{1 - \cos 4x}{2}\right) \\
&= \frac{1}{8}(1 - \cos 4x)
\end{aligned}$$

Now Try Exercise 11 ■

Half-Angle Formulas

$$\sin\frac{u}{2} = \pm\sqrt{\frac{1 - \cos u}{2}} \qquad \cos\frac{u}{2} = \pm\sqrt{\frac{1 + \cos u}{2}}$$

$$\tan\frac{u}{2} = \frac{1 - \cos u}{\sin u} = \frac{\sin u}{1 + \cos u}$$

The choice of the $+$ or $-$ sign depends on the quadrant in which $u/2$ lies.

Proof We substitute $x = u/2$ in the formulas for lowering powers and take the square root of each side. This gives the first two Half-Angle Formulas. In the case of the Half-Angle Formula for Tangent we get

$$\tan\frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{1+\cos u}}$$

$$= \pm\sqrt{\left(\frac{1-\cos u}{1+\cos u}\right)\left(\frac{1-\cos u}{1-\cos u}\right)} \qquad \text{Multiply numerator and denominator by } 1-\cos u$$

$$= \pm\sqrt{\frac{(1-\cos u)^2}{1-\cos^2 u}} \qquad \text{Simplify}$$

$$= \pm\frac{|1-\cos u|}{|\sin u|} \qquad \sqrt{A^2} = |A| \text{ and } 1-\cos^2 u = \sin^2 u$$

Now, $1 - \cos u$ is nonnegative for all values of u. It is also true that $\sin u$ and $\tan(u/2)$ always have the same sign. (Verify this.) It follows that

$$\tan\frac{u}{2} = \frac{1-\cos u}{\sin u}$$

The other Half-Angle Formula for Tangent is derived from this by multiplying the numerator and denominator by $1 + \cos u$. ■

Example 5 ■ Using a Half-Angle Formula

Find the exact value of $\sin 22.5°$.

Solution Since $22.5°$ is half of $45°$, we use the Half-Angle Formula for Sine with $u = 45°$. We choose the $+$ sign because $22.5°$ is in the first quadrant.

$$\sin\frac{45°}{2} = \sqrt{\frac{1-\cos 45°}{2}} \qquad \text{Half-Angle Formula}$$

$$= \sqrt{\frac{1-\sqrt{2}/2}{2}} \qquad \cos 45° = \sqrt{2}/2$$

$$= \sqrt{\frac{2-\sqrt{2}}{4}} \qquad \text{Common denominator}$$

$$= \tfrac{1}{2}\sqrt{2-\sqrt{2}} \qquad \text{Simplify}$$

Now Try Exercise 17 ■

Example 6 ■ Using a Half-Angle Formula

Find $\tan(u/2)$ if $\sin u = \frac{2}{5}$ and u is in Quadrant II.

Solution To use the Half-Angle Formula for Tangent, we first need to find $\cos u$. Since cosine is negative in Quadrant II, we have

$$\cos u = -\sqrt{1-\sin^2 u}$$

$$= -\sqrt{1-\left(\tfrac{2}{5}\right)^2} = -\frac{\sqrt{21}}{5}$$

Thus

$$\tan\frac{u}{2} = \frac{1-\cos u}{\sin u} \qquad \text{Half- Angle Formula}$$

$$= \frac{1+\sqrt{21}/5}{\frac{2}{5}} = \frac{5+\sqrt{21}}{2}$$

Now Try Exercise 37 ■

Expressions Involving Inverse Trigonometric Functions

Expressions involving trigonometric functions and their inverses arise in calculus. In the next examples we illustrate how to evaluate such expressions.

Example 7 ■ Evaluating an Expression Involving an Inverse Trigonometric Function

Evaluate $\sin\left(2\cos^{-1}\left(-\frac{2}{5}\right)\right)$.

Solution Let $\theta = \cos^{-1}\left(-\frac{2}{5}\right)$. By properties of $\cos^{-1}$, it follows that $\cos\theta = -\frac{2}{5}$ with $0 \le \theta \le \pi$. Thus θ is in Quadrant II. Let's sketch the angle θ in standard position with terminal side in Quadrant II, as shown in Figure 2. Since $\cos\theta = x/r = -\frac{2}{5}$, we can label a side and the hypotenuse of the triangle in Figure 1. To find the remaining side, we use the Pythagorean Theorem.

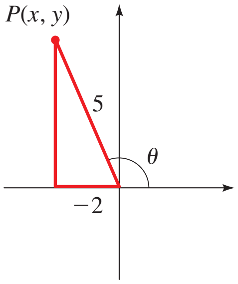


Figure 1

$$\begin{aligned} x^2 + y^2 &= r^2 && \text{Pythagorean Theorem} \\ (-2)^2 + y^2 &= 5^2 && x = -2, \quad r = 5 \\ y &= \pm\sqrt{21} && \text{Solve for } y^2 \\ y &= +\sqrt{21} && \text{Because } y > 0 \end{aligned}$$

We can now use the Double-Angle Formula for Sine.

$$\begin{aligned} \sin 2\theta &= 2\sin\theta\cos\theta && \text{Double-Angle Formula} \\ &= 2\left(\frac{\sqrt{21}}{5}\right)\left(-\frac{2}{5}\right) = -\frac{4\sqrt{21}}{25} && \text{From the triangle} \end{aligned}$$

Now Try Exercise 43 ■

Example 8 ■ Simplifying an Expression Involving an Inverse Trigonometric Function

Write $\sin(2\cos^{-1}x)$ as an algebraic expression in x only, where $-1 \le x \le 1$.

Solution Using the methods of Example 6.4.8, we let $\theta = \cos^{-1}x$, so $\cos\theta = x$, and sketch a triangle as in Figure 2. We need to find $\sin 2\theta$, but from the triangle we can find trigonometric functions of θ only, not 2θ. So we use the Double-Angle Formula for Sine.

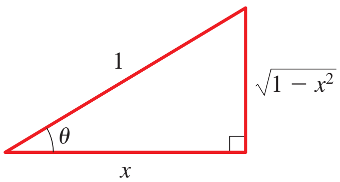


Figure 2

$$\begin{aligned} \sin(2\cos^{-1}x) &= \sin 2\theta && \cos^{-1}x = \theta \\ &= 2\sin\theta\cos\theta && \text{Double-Angle Formula} \\ &= 2x\sqrt{1 - x^2} && \text{From the triangle} \end{aligned}$$

Now Try Exercise 47 ■

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Discovery Project ■ Where to Sit at the Movies

To best view a painting or a movie requires that the viewing angle be as large as possible. If the painting or movie screen is at a height above eye level, then being too far away or too close results in a small viewing angle and hence a poor viewing experience. So what is the best distance from which to view a movie or a painting? In this project we use trigonometry to find the best location from which to view a painting or a movie. You can find the project at **www.stewartmath.com**.

Product-Sum Formulas

It is possible to write the product $\sin u \cos v$ as a sum of trigonometric functions. To see this, consider the Addition and Subtraction Formulas for Sine:

$$\sin(u+v) = \sin u \cos v + \cos u \sin v$$

$$\sin(u-v) = \sin u \cos v - \cos u \sin v$$

Adding the left- and right-hand sides of these formulas gives

$$\sin(u+v) + \sin(u-v) = 2 \sin u \cos v$$

Dividing by 2 gives the formula

$$\sin u \cos v = \tfrac{1}{2}[\sin(u+v) + \sin(u-v)]$$

The other three **Product-to-Sum Formulas** follow from the Addition Formulas in a similar way.

Product-to-Sum Formulas

$$\sin u \cos v = \tfrac{1}{2}[\sin(u+v) + \sin(u-v)]$$

$$\cos u \sin v = \tfrac{1}{2}[\sin(u+v) - \sin(u-v)]$$

$$\cos u \cos v = \tfrac{1}{2}[\cos(u+v) + \cos(u-v)]$$

$$\sin u \sin v = \tfrac{1}{2}[\cos(u-v) - \cos(u+v)]$$

Example 9 ■ Expressing a Trigonometric Product as a Sum

Express $\sin 3x \sin 5x$ as a sum of trigonometric functions.

Solution Using the fourth Product-to-Sum Formula with $u = 3x$ and $v = 5x$ and the fact that cosine is an even function, we get

$$\begin{aligned} \sin 3x \sin 5x &= \tfrac{1}{2}[\cos(3x - 5x) - \cos(3x + 5x)] \\ &= \tfrac{1}{2}\cos(-2x) - \tfrac{1}{2}\cos 8x \\ &= \tfrac{1}{2}\cos 2x - \tfrac{1}{2}\cos 8x \end{aligned}$$

Now Try Exercise 55

The Product-to-Sum Formulas can also be used as Sum-to-Product Formulas. This is possible because the right-hand side of each Product-to-Sum Formula is a sum and the left side is a product. For example, if we let

$$u = \frac{x+y}{2} \qquad \text{and} \qquad v = \frac{x-y}{2}$$

in the first Product-to-Sum Formula, we get

$$\sin\frac{x+y}{2}\cos\frac{x-y}{2} = \tfrac{1}{2}(\sin x + \sin y)$$

so

$$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$$

The remaining three of the following **Sum-to-Product Formulas** are obtained in a similar manner.

Sum-to-Product Formulas

$$\sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}$$

$$\sin x - \sin y = 2 \cos \frac{x+y}{2} \sin \frac{x-y}{2}$$

$$\cos x + \cos y = 2 \cos \frac{x+y}{2} \cos \frac{x-y}{2}$$

$$\cos x - \cos y = -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2}$$

Example 10 ■ Expressing a Trigonometric Sum as a Product

Write $\sin 7x + \sin 3x$ as a product.

Solution The first Sum-to-Product Formula gives

$$\begin{aligned}\sin 7x + \sin 3x &= 2 \sin \frac{7x+3x}{2} \cos \frac{7x-3x}{2} \\ &= 2 \sin 5x \cos 2x\end{aligned}$$

Now Try Exercise 61 ■

Example 11 ■ Proving an Identity

Verify the identity $\dfrac{\sin 3x - \sin x}{\cos 3x + \cos x} = \tan x$.

Solution We apply the second Sum-to-Product Formula to the numerator and the third formula to the denominator.

$$\begin{aligned}\text{LHS} = \frac{\sin 3x - \sin x}{\cos 3x + \cos x} &= \frac{2 \cos \frac{3x+x}{2} \sin \frac{3x-x}{2}}{2 \cos \frac{3x+x}{2} \cos \frac{3x-x}{2}} && \text{Sum-to-Product Formulas} \\ &= \frac{2 \cos 2x \sin x}{2 \cos 2x \cos x} && \text{Simplify} \\ &= \frac{\sin x}{\cos x} = \tan x = \text{RHS} && \text{Cancel}\end{aligned}$$

Now Try Exercise 93 ■

7.3 Exercises

Concepts

1. If we know the values of $\sin x$ and $\cos x$, we can find the value of $\sin 2x$ by using the ________ Formula for Sine. State the formula: $\sin 2x =$ ________________.

2. If we know the value of $\cos x$ and the quadrant in which $x/2$ lies, we can find the value of $\sin(x/2)$ by using the ________ Formula for Sine. State the formula: $\sin(x/2) =$ ________________.

Skills

3–10 ■ Double-Angle Formulas Find $\sin 2x$, $\cos 2x$, and $\tan 2x$ from the given information.

3. $\sin x = \frac{5}{13}$, x in Quadrant I

4. $\tan x = -\frac{4}{3}$, x in Quadrant II

5. $\cos x = \frac{4}{5}$, $\csc x < 0$

6. $\csc x = 4$, $\tan x < 0$

7. $\sin x = -\frac{3}{5}$, x in Quadrant III

8. $\sec x = 2$, x in Quadrant IV

9. $\tan x = -\frac{1}{3}$, $\cos x > 0$

10. $\cot x = \frac{2}{3}$, $\sin x > 0$

11–16 ■ Lowering Powers in a Trigonometric Expression Use the formulas for lowering powers to rewrite the expression in terms of the first power of cosine, as in Example 4.

11. $\sin^4 x$ **12.** $\cos^4 x$

13. $\cos^2 x \sin^4 x$ **14.** $\cos^4 x \sin^2 x$

15. $\cos^4 x \sin^4 x$ **16.** $\cos^6 x$

17–28 ■ Half-Angle Formulas Use an appropriate Half-Angle Formula to find the exact value of the expression.

17. $\sin 15°$ **18.** $\tan 15°$

19. $\tan 22.5°$ **20.** $\sin 75°$

21. $\cos 165°$ **22.** $\cos 112.5°$

23. $\tan \dfrac{5\pi}{8}$ **24.** $\cos \dfrac{3\pi}{8}$

25. $\cos \dfrac{\pi}{12}$ **26.** $\tan \dfrac{5\pi}{12}$

27. $\sin \dfrac{9\pi}{8}$ **28.** $\sin \dfrac{11\pi}{12}$

29–34 ■ Double- and Half-Angle Formulas Simplify the expression by using a Double-Angle Formula or a Half-Angle Formula.

29. (a) $2 \sin 16° \cos 16°$ **(b)** $2 \sin 4\theta \cos 4\theta$

30. (a) $\dfrac{2 \tan 5°}{1 - \tan^2 5°}$ **(b)** $\dfrac{2 \tan 5\theta}{1 - \tan^2 5\theta}$

31. (a) $\cos^2 21° - \sin^2 21°$ **(b)** $\cos^2 9\theta - \sin^2 9\theta$

32. (a) $\cos^2\left(\dfrac{\theta}{2}\right) - \sin^2\left(\dfrac{\theta}{2}\right)$ **(b)** $2 \sin\left(\dfrac{\theta}{2}\right) \cos\left(\dfrac{\theta}{2}\right)$

33. (a) $\dfrac{\sin 8°}{1 + \cos 8°}$ **(b)** $\dfrac{1 - \cos 4\theta}{\sin 4\theta}$

34. (a) $\sqrt{\dfrac{1 - \cos 30°}{2}}$ **(b)** $\sqrt{\dfrac{1 - \cos 8\theta}{2}}$

35. Proving a Double-Angle Formula Use the Addition Formula for Sine to prove the Double-Angle Formula for Sine.

36. Proving a Double-Angle Formula Use the Addition Formula for Tangent to prove the Double-Angle Formula for Tangent.

37–42 ■ Using a Half-Angle Formula Find $\sin \dfrac{x}{2}$, $\cos \dfrac{x}{2}$, and $\tan \dfrac{x}{2}$ from the given information.

37. $\sin x = \frac{3}{5}$, $0° < x < 90°$

38. $\cos x = -\frac{4}{5}$, $180° < x < 270°$

39. $\csc x = 3$, $90° < x < 180°$

40. $\tan x = 1$, $0° < x < 90°$

41. $\sec x = \frac{3}{2}$, $270° < x < 360°$

42. $\cot x = 5$, $180° < x < 270°$

43–46 ■ Expressions Involving Inverse Trigonometric Functions Find the exact value of the given expression.

43. $\sin\left(2 \cos^{-1}\left(\frac{7}{25}\right)\right)$ **44.** $\cos\left(2 \tan^{-1}\left(\frac{12}{5}\right)\right)$

45. $\sec\left(2 \sin^{-1}\left(\frac{1}{4}\right)\right)$ **46.** $\tan\left(\frac{1}{2} \cos^{-1}\left(\frac{2}{3}\right)\right)$

47–50 ■ Expressions Involving Inverse Trigonometric Functions Write the given expression as an algebraic expression in x.

47. $\sin(2 \tan^{-1} x)$ **48.** $\tan(2 \cos^{-1} x)$

49. $\sin\left(\frac{1}{2} \cos^{-1} x\right)$ **50.** $\cos(2 \sin^{-1} x)$

51–54 ■ Evaluating an Expression Involving Trigonometric Functions Evaluate each expression under the given conditions.

51. $\cos 2\theta$; $\sin \theta = -\frac{3}{5}$, θ in Quadrant III

52. $\sin(\theta/2)$; $\tan \theta = -\frac{5}{12}$, θ in Quadrant IV

53. $\sin 2\theta$; $\sin \theta = \frac{1}{7}$, θ in Quadrant II

54. $\tan 2\theta$; $\cos \theta = \frac{3}{5}$, θ in Quadrant I

55–60 ■ Product-to-Sum Formulas Write the product as a sum.

55. $\sin 5x \cos 4x$ **56.** $\sin 2x \sin 3x$

57. $\cos x \sin 4x$ **58.** $\cos 5x \cos 3x$

59. $3 \cos 4x \cos 7x$ **60.** $11 \sin \dfrac{x}{2} \cos \dfrac{x}{4}$

61–66 ■ Sum-to-Product Formulas Write the sum as a product.

61. $\sin 7x + \sin 5x$ **62.** $\sin 5x - \sin 4x$

63. $\cos 4x - \cos 6x$ **64.** $\cos 9x + \cos 2x$

65. $\sin 2x - \sin 7x$ **66.** $\sin 3x + \sin 4x$

67–72 ■ Value of a Product or Sum Find the exact value of the product or sum.

67. $2 \sin 52.5° \sin 97.5°$

68. $3 \cos 37.5° \cos 7.5°$

69. $\cos 37.5° \sin 7.5°$

70. $\sin 75° + \sin 15°$

71. $\cos 255° - \cos 195°$

72. $\cos \dfrac{\pi}{12} + \cos \dfrac{5\pi}{12}$

73–96 ■ Proving Identities Prove the identity.

73. $\cos^2 5x - \sin^2 5x = \cos 10x$

74. $\sin 8x = 2 \sin 4x \cos 4x$

75. $(\sin x + \cos x)^2 = 1 + \sin 2x$

76. $\cos^4 x - \sin^4 x = \cos 2x$

77. $\dfrac{2 \tan x}{1 + \tan^2 x} = \sin 2x$

78. $\dfrac{1 - \cos 2x}{\sin 2x} = \tan x$

79. $\tan \dfrac{x}{2} + \cos x \tan \dfrac{x}{2} = \sin x$

80. $\tan \dfrac{x}{2} + \csc x = \dfrac{2 - \cos x}{\sin x}$

81. $\dfrac{\sin 4x}{\sin x} = 4 \cos x \cos 2x$

82. $\dfrac{1 + \sin 2x}{\sin 2x} = 1 + \frac{1}{2} \sec x \csc x$

83. $\dfrac{\cos 2x}{1 + 2 \sin x \cos x} = \dfrac{1 - \tan x}{1 + \tan x}$

84. $\tan x = \dfrac{\sin 2x}{1 + \cos 2x}$

85. $\cot 2x = \dfrac{1 - \tan^2 x}{2 \tan x}$

86. $\sin^4 x + \cos^4 x = \frac{1}{2}(1 + \cos^2 2x)$

87. $\tan 3x = \dfrac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x}$

88. $\dfrac{2 \sin 2x + \sin 4x}{2 \cos x + 2 \cos 3x} = \tan 2x \cos x$

89. $\dfrac{\sin x + \sin 5x}{\cos x + \cos 5x} = \tan 3x$

90. $\dfrac{\sin 3x + \sin 7x}{\cos 3x - \cos 7x} = \cot 2x$

91. $\dfrac{\sin 10x}{\sin 9x + \sin x} = \dfrac{\cos 5x}{\cos 4x}$

92. $\dfrac{\sin x + \sin 3x + \sin 5x}{\cos x + \cos 3x + \cos 5x} = \tan 3x$

93. $\dfrac{\sin x + \sin y}{\cos x + \cos y} = \tan\left(\dfrac{x + y}{2}\right)$

94. $\tan y = \dfrac{\sin(x + y) - \sin(x - y)}{\cos(x + y) + \cos(x - y)}$

95. $\tan^2\left(\dfrac{x}{2} + \dfrac{\pi}{4}\right) = \dfrac{1 + \sin x}{1 - \sin x}$

96. $(1 - \cos 4x)(2 + \tan^2 x + \cot^2 x) = 8$

97–100 ■ Sum-to-Product Formulas Use a Sum-to-Product Formula to show the following.

97. $\sin 130° - \sin 110° = -\sin 10°$

98. $\cos 100° - \cos 200° = \sin 50°$

99. $\sin 45° + \sin 15° = \sin 75°$

100. $\cos 87° + \cos 33° = \sin 63°$

Skills Plus

101. Proving an Identity Prove the identity

$$\frac{\sin x + \sin 2x + \sin 3x + \sin 4x + \sin 5x}{\cos x + \cos 2x + \cos 3x + \cos 4x + \cos 5x} = \tan 3x$$

102. Proving an Identity Use the identity

$$\sin 2x = 2 \sin x \cos x$$

n times to show that

$$\sin(2^n x) = 2^n \sin x \cos x \cos 2x \cos 4x \cdots \cos 2^{n-1} x$$

103–104 ■ Identities Involving Inverse Trigonometric Functions Prove the identity.

103. $2 \sin^{-1} x = \cos^{-1}(1 - 2x^2), \quad 0 \le x \le 1$
[*Hint:* Let $u = \sin^{-1} x$, so that $x = \sin u$. Use a Double-Angle Formula to show that $1 - 2x^2 = \cos 2u$.]

104. $2 \tan^{-1}\left(\dfrac{1}{x}\right) = \cos^{-1}\left(\dfrac{x^2 - 1}{x^2 + 1}\right)$
[*Hint:* Let $u = \tan^{-1}\left(\dfrac{1}{x}\right)$, so that $x = \dfrac{1}{\tan u} = \cot u$. Use a Double-Angle Formula to show that $\dfrac{x^2 - 1}{x^2 + 1} = \dfrac{\cot^2 u - 1}{\csc^2 u} = \cos 2u$.]

105–107 ■ Discovering an Identity Graphically In these exercises we discover an identity graphically and then prove the identity.

105. **(a)** Graph $f(x) = \dfrac{\sin 3x}{\sin x} - \dfrac{\cos 3x}{\cos x}$, and make a conjecture.
(b) Prove the conjecture you made in part (a).

106. **(a)** Graph $f(x) = \cos 2x + 2 \sin^2 x$, and make a conjecture.
(b) Prove the conjecture you made in part (a).

107. Let $f(x) = \sin 6x + \sin 7x$.
(a) Graph $y = f(x)$.
(b) Verify that $f(x) = 2 \cos \frac{1}{2}x \sin \frac{13}{2}x$.
(c) Graph $y = 2 \cos \frac{1}{2}x$ and $y = -2 \cos \frac{1}{2}x$, together with the graph in part (a), in the same viewing rectangle. How are these graphs related to the graph of f?

108. A Cubic Equation Let $3x = \pi/3$, and let $y = \cos x$. Use the result of Example 2 to show that y satisfies the equation

$$8y^3 - 6y - 1 = 0$$

[*Note:* This equation has solutions of a certain kind that are used to show that the angle $\pi/3$ cannot be trisected by using a straightedge and compass only.]

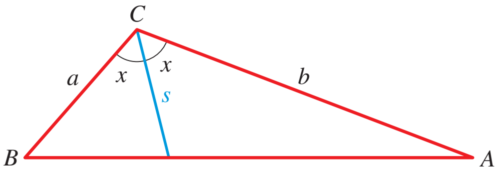
109. Tchebycheff Polynomials
(a) Show that there is a polynomial P of degree 4 such that $\cos 4x = P(\cos x)$ (see Example 2).
(b) Show that there is a polynomial Q of degree 5 such that $\cos 5x = Q(\cos x)$.

[*Note:* In general, there is a polynomial P_n of degree n such that $\cos nx = P_n(\cos x)$. These polynomials are called *Tchebycheff polynomials*, after the Russian mathematician P. L. Tchebycheff (1821–1894).]

110. Length of a Bisector In triangle ABC (see the figure) the line segment s bisects angle C. Show that the length of s is given by

$$s = \frac{2ab \cos x}{a + b}$$

[*Hint:* Use the Law of Sines.]



111. Angles of a Triangle If A, B, and C are the angles in a triangle, show that

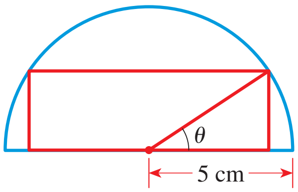
$$\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$$

112. Largest Area A rectangle is to be inscribed in a semicircle of radius 5 cm as shown in the figure.

(a) Show that the area of the rectangle is modeled by the function $A(\theta) = 25 \sin 2\theta$.

(b) Find the largest possible area for such an inscribed rectangle. [*Hint:* Use the fact that $\sin u$ achieves its maximum value at $u = \pi/2$.]

(c) Find the dimensions of the inscribed rectangle with the largest possible area.

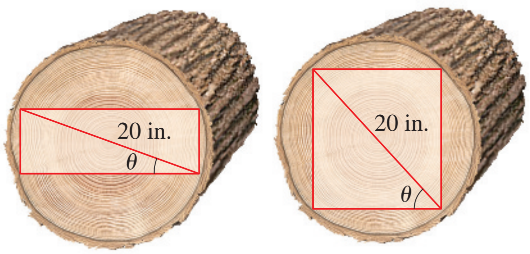


Applications

113. Sawing a Wooden Beam A rectangular beam is to be cut from a cylindrical log of diameter 20 in.

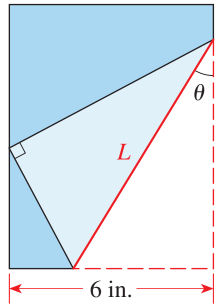
(a) Show that the cross-sectional area of the beam is modeled by the function $A(\theta) = 200 \sin 2\theta$ where θ is as shown in the figure.

(b) Show that the maximum cross-sectional area of such a beam is 200 in^2. [*Hint:* Use the fact that $\sin u$ achieves its maximum value at $u = \pi/2$.]



114. Length of a Fold The lower right-hand corner of a long piece of paper 6 in. wide is folded over to the left-hand edge as shown in the figure. The length L of the fold depends on the angle θ. Show that

$$L = \frac{3}{\sin \theta \cos^2 \theta}$$



115. Sound Beats When two pure notes that are close in frequency are played together, their sounds interfere to produce *beats*; that is, the loudness (or amplitude) of the sound alternately increases and decreases. If the two notes are given by

$$f_1(t) = \cos 11t \qquad \text{and} \qquad f_2(t) = \cos 13t$$

the resulting sound is $f(t) = f_1(t) + f_2(t)$.

(a) Graph the function $y = f(t)$.

(b) Verify that $f(t) = 2 \cos t \cos 12t$.

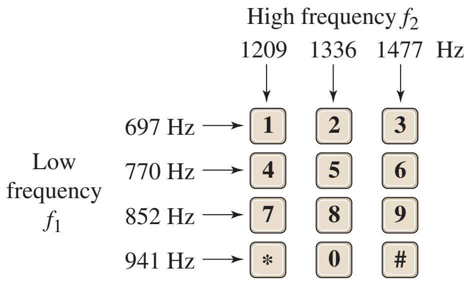
(c) Graph $y = 2 \cos t$ and $y = -2 \cos t$, together with the graph in part (a), in the same viewing rectangle. How do these graphs describe the variation in the loudness of the sound?

116. Phone Keypad Tones When a key is pressed on a phone, the keypad generates two pure tones, which combine to produce a sound that uniquely identifies the key. The figure shows the low frequency f_1 and the high frequency f_2 associated with each key. Pressing a key produces the sound wave $y = \sin(2\pi f_1 t) + \sin(2\pi f_2 t)$.

(a) Find the function that models the sound produced when the 4 key is pressed.

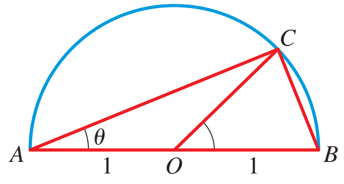
(b) Use a Sum-to-Product Formula to express the sound generated by the 4 key as a product of a sine and a cosine function.

(c) Graph the sound wave generated by the 4 key from $t = 0$ to $t = 0.006$ s.



■ Discuss ■ Discover ■ Prove ■ Write

117. Prove: Geometric Proof of a Double-Angle Formula Use the figure to prove that $\sin 2\theta = 2 \sin\theta \cos\theta$.



[*Hint:* Find the area of triangle ABC in two different ways. You will need the following facts from geometry. See Appendix A, *Geometry Review*:

An angle inscribed in a semicircle is a right angle, so $\angle ACB$ is a right angle.

The central angle subtended by the chord of a circle is twice the angle subtended by the chord on the circle, so in the figure $\angle BOC$ is 2θ.]

7.4 Basic Trigonometric Equations

■ Basic Trigonometric Equations ■ Solving Trigonometric Equations by Factoring

An equation that contains trigonometric functions is called a **trigonometric equation**. For example, the following are trigonometric equations:

$$\sin^2\theta + \cos^2\theta = 1 \qquad 2\sin\theta - 1 = 0$$

The first equation is an *identity*—that is, it is true for every value of the variable θ. The second equation is true only for certain values of θ. To solve a trigonometric equation, we find all the values of the variable that make the equation true.

■ Basic Trigonometric Equations

A **basic trigonometric equation** is an equation of the form $T(\theta) = c$, where T is a trigonometric function and c is a constant. For example,

$$\sin\theta = 0.5 \qquad \cos\theta = 0.2 \qquad \tan\theta = 10$$

are basic trigonometric equations. Each of these equations has infinitely many solutions. To find all of them, we use the following guidelines.

Solving Basic Trigonometric Equations

1. **Find the Solutions in One Period.** A basic trigonometric equation contains one trigonometric function. Find the solutions of the equation in one period of that trigonometric function.
2. **Find All Solutions.** Find all solutions by adding integer multiples of the period to the solutions you found in Step 1.

Example 1 ■ Solving a Basic Trigonometric Equation

Solve the equation $\sin\theta = \dfrac{1}{2}$.

Solution **Find the solutions in one period.** Because sine has period 2π, we first find the solutions in any interval of length 2π. To find these solutions, we look at the unit circle in Figure 1. We see that $\sin\theta = \frac{1}{2}$ in Quadrants I and II, so from the figure, the solutions in the interval $[0, 2\pi)$ are

$$\theta = \frac{\pi}{6} \qquad \theta = \frac{5\pi}{6}$$

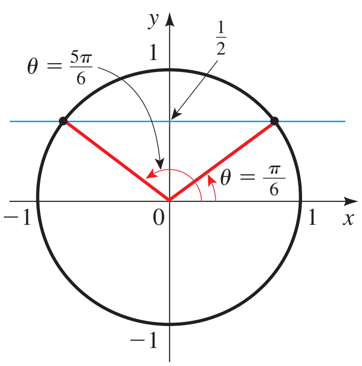


Figure 1

Find all solutions. Because the sine function repeats its values every 2π units, we get all solutions of the equation by adding integer multiples of 2π to these solutions:

$$\theta = \frac{\pi}{6} + 2k\pi \qquad \theta = \frac{5\pi}{6} + 2k\pi$$

where k is any integer. Figure 2 gives a graphical representation of the solutions.

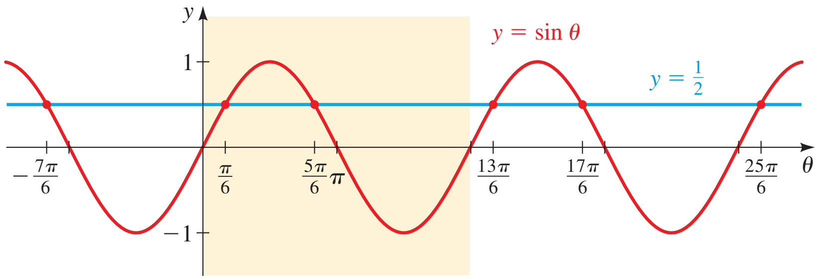


Figure 2

Now Try Exercise 5 ■

Example 2 ■ Solving a Basic Trigonometric Equation

Solve the equation $\cos\theta = -\dfrac{\sqrt{2}}{2}$, and list six specific solutions.

Solution **Find the solutions in one period.** Because cosine has period 2π, we first find the solutions in any interval of length 2π. From the unit circle in Figure 3 we see that $\cos\theta = -\sqrt{2}/2$ in Quadrants II and III, and so the solutions in the interval $[0, 2\pi)$ are

$$\theta = \frac{3\pi}{4} \qquad \theta = \frac{5\pi}{4}$$

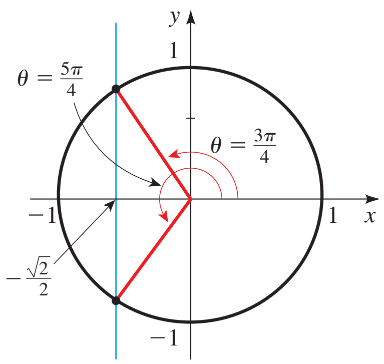


Figure 3

Find all solutions. Because the cosine function repeats its values every 2π units, we get all solutions of the equation by adding integer multiples of 2π to these solutions:

$$\theta = \frac{3\pi}{4} + 2k\pi \qquad \theta = \frac{5\pi}{4} + 2k\pi$$

where k is any integer. You can check that for $k = -1, 0, 1$ we get the following specific solutions:

$$\theta = \underbrace{-\frac{5\pi}{4}, -\frac{3\pi}{4}}_{k=-1}, \underbrace{\frac{3\pi}{4}, \frac{5\pi}{4}}_{k=0}, \underbrace{\frac{11\pi}{4}, \frac{13\pi}{4}}_{k=1}$$

Figure 4 gives a graphical representation of the solutions.

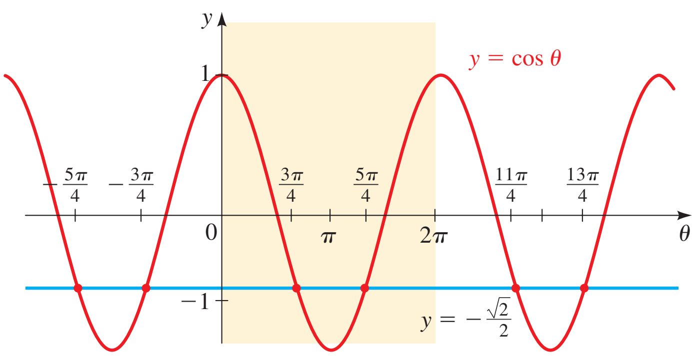


Figure 4

Now Try Exercise 17 ■

Example 3 ■ Solving a Basic Trigonometric Equation

Solve the equation $\cos\theta = 0.65$.

Solution **Find the solutions in one period.** We first find one solution by taking $\cos^{-1}$ of each side of the equation.

$$\cos\theta = 0.65 \qquad \text{Given equation}$$
$$\theta = \cos^{-1}(0.65) \qquad \text{Take } \cos^{-1} \text{ of each side}$$
$$\theta \approx 0.86 \qquad \text{Calculator (in radian mode)}$$

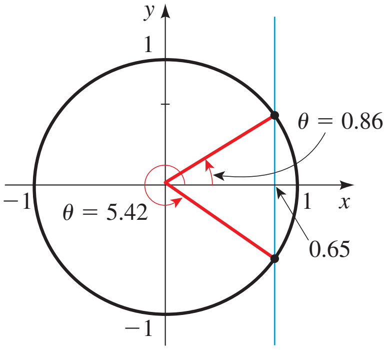


Figure 5

Because cosine has period 2π, we next find the solutions in any interval of length 2π. To find these solutions, we look at the unit circle in Figure 5. We see that $\cos\theta = 0.86$ in Quadrants I and IV, so the solutions are

$$\theta \approx 0.86 \qquad \theta \approx 2\pi - 0.86 \approx 5.42$$

Find all solutions. To get all solutions of the equation, we add integer multiples of 2π to these solutions:

$$\theta \approx 0.86 + 2k\pi \qquad \theta \approx 5.42 + 2k\pi$$

where k is any integer.

Now Try Exercise 21 ■

Example 4 ■ Solving a Basic Trigonometric Equation

Solve the equation $\tan\theta = 2$.

Solution **Find the solutions in one period.** We first find one solution by taking $\tan^{-1}$ of each side of the equation.

$$\tan\theta = 2 \qquad \text{Given equation}$$
$$\theta = \tan^{-1} 2 \qquad \text{Take } \tan^{-1} \text{ of each side}$$
$$\theta \approx 1.12 \qquad \text{Calculator (in radian mode)}$$

By the definition of $\tan^{-1}$ the solution that we obtained is the only solution in the interval $(-\pi/2, \pi/2)$, which is an interval of length π.

Find all solutions. Since tangent has period π, we get all solutions of the equation by adding integer multiples of π:

$$\theta \approx 1.12 + k\pi$$

where k is any integer. A graphical representation of the solutions is shown in Figure 6. You can check that the solutions shown in the graph correspond to $k = -1, 0, 1, 2, 3$.

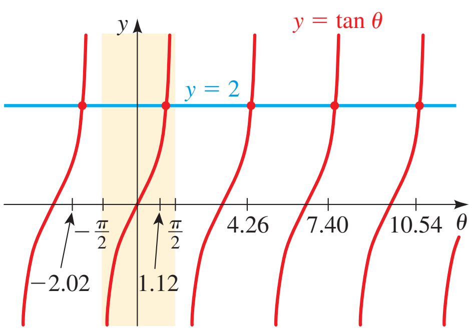


Figure 6

Now Try Exercise 23 ■

In the next example we solve trigonometric equations that are algebraically equivalent to basic trigonometric equations.

Example 5 ■ Solving Trigonometric Equations

Find all solutions of each equation.

(a) $2 \sin \theta - 1 = 0$ **(b)** $\tan^2 \theta - 3 = 0$

Solution

(a) We start by isolating $\sin \theta$.

$$2 \sin \theta - 1 = 0 \qquad \text{Given equation}$$
$$2 \sin \theta = 1 \qquad \text{Add 1}$$
$$\sin \theta = \frac{1}{2} \qquad \text{Divide by 2}$$

This last equation is the equation we solved in Example 1. The solutions are

$$\theta = \frac{\pi}{6} + 2k\pi \qquad \theta = \frac{5\pi}{6} + 2k\pi$$

where k is any integer.

(b) We start by isolating $\tan \theta$.

$$\tan^2 \theta - 3 = 0 \qquad \text{Given equation}$$
$$\tan^2 \theta = 3 \qquad \text{Add 3}$$
$$\tan \theta = \pm\sqrt{3} \qquad \text{Take the square root}$$

Because tangent has period π, we first find the solutions in any interval of length π. In the interval $(-\pi/2, \pi/2)$ the solutions are $\theta = \pi/3$ and $\theta = -\pi/3$. To get all solutions, we add integer multiples of π to these solutions:

$$\theta = \frac{\pi}{3} + k\pi \qquad \theta = -\frac{\pi}{3} + k\pi$$

where k is any integer.

Now Try Exercises 27 and 33 ■

■ Solving Trigonometric Equations by Factoring

Factoring is one of the most useful techniques for solving equations, including trigonometric equations. The idea is to move all terms to one side of the equation, factor, and then use the Zero-Product Property (see Section 1.5), thus reducing the problem to solving basic trigonometric equations.

Zero-Product Property

If $AB = 0$, then $A = 0$ or $B = 0$.

Example 6 ■ A Trigonometric Equation of Quadratic Type

Solve the equation $2 \cos^2 \theta - 7 \cos \theta + 3 = 0$.

Equation of Quadratic Type

$$2C^2 - 7C + 3 = 0$$
$$(2C - 1)(C - 3) = 0$$

Solution We factor the left-hand side of the equation.

$$2 \cos^2 \theta - 7 \cos \theta + 3 = 0 \qquad \text{Given equation}$$
$$(2 \cos \theta - 1)(\cos \theta - 3) = 0 \qquad \text{Factor}$$
$$2 \cos \theta - 1 = 0 \quad \text{or} \quad \cos \theta - 3 = 0 \qquad \text{Set each factor equal to 0}$$
$$\cos \theta = \frac{1}{2} \quad \text{or} \quad \cos \theta = 3 \qquad \text{Solve for } \cos \theta$$

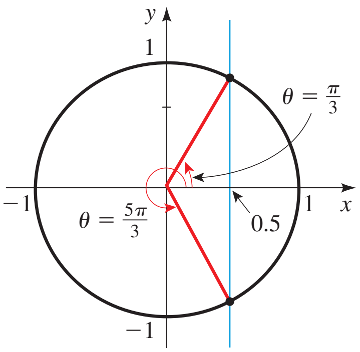


Figure 7

Because cosine has period 2π, we first find the solutions of these two equations in the interval $[0, 2\pi)$. For the first equation the solutions are $\theta = \pi/3$ and $\theta = 5\pi/3$. (See Figure 7.) The second equation has no solution because $\cos\theta$ is never greater than 1. Thus the solutions are

$$\theta = \frac{\pi}{3} + 2k\pi \qquad \theta = \frac{5\pi}{3} + 2k\pi$$

where k is any integer.

Now Try Exercise 41 ■

Example 7 ■ Solving a Trigonometric Equation by Factoring

Solve the equation $5\sin\theta\cos\theta + 4\cos\theta = 0$.

Solution We factor the left-hand side of the equation.

$$5\sin\theta\cos\theta + 4\cos\theta = 0 \qquad \text{Given equation}$$

$$\cos\theta\,(5\sin\theta + 4) = 0 \qquad \text{Factor}$$

$$\cos\theta = 0 \quad \text{or} \quad 5\sin\theta + 4 = 0 \qquad \text{Set each factor equal to 0}$$

$$\sin\theta = -0.8 \qquad \text{Solve for } \sin\theta$$

Because sine and cosine have period 2π, we first find the solutions of these two equations in an interval of length 2π. For the first equation the solutions in the interval $[0, 2\pi)$ are $\theta = \pi/2$ and $\theta = 3\pi/2$. To solve the second equation, we take $\sin^{-1}$ of each side.

$$\sin\theta = -0.80 \qquad \text{Second equation}$$

$$\theta = \sin^{-1}(-0.80) \qquad \text{Take } \sin^{-1} \text{ of each side}$$

$$\theta \approx -0.93 \qquad \text{Calculator (in radian mode)}$$

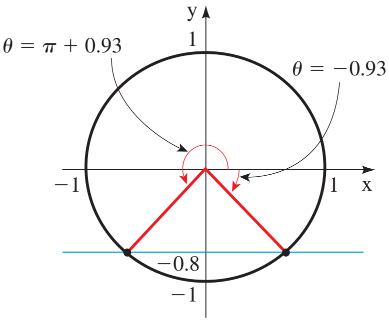


Figure 8

So the solutions in an interval of length 2π are $\theta = -0.93$ and $\theta = \pi + 0.93 \approx 4.07$. (See Figure 8.) We get all the solutions of the equation by adding integer multiples of 2π to these solutions.

$$\theta = \frac{\pi}{2} + 2k\pi \qquad \theta = \frac{3\pi}{2} + 2k\pi \qquad \theta \approx -0.93 + 2k\pi \qquad \theta \approx 4.07 + 2k\pi$$

where k is any integer.

Now Try Exercise 53 ■

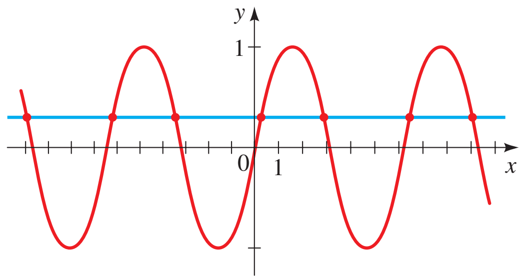
7.4 | Exercises

■ Concepts

1. Because the trigonometric functions are periodic, if a basic trigonometric equation has one solution, it has ________ (several/infinitely many) solutions.

2. The basic equation $\sin x = 2$ has ________ (no/one/infinitely many) solution(s), whereas the basic equation $\sin x = 0.3$ has ________ (no/one/infinitely many) solution(s).

3. We can find some of the solutions of $\sin x = 0.3$ graphically by graphing $y = \sin x$ and $y =$ ________. Use the graph below to estimate some of the solutions.



4. We can find the solutions of $\sin x = 0.3$ algebraically.

(a) First we find the solutions in the interval $[0, 2\pi)$. We get one such solution by taking $\sin^{-1}$ to get $x \approx$ ______. The other solution in this interval is $x \approx$ ______.

(b) We find all solutions by adding multiples of ______ to the solutions in $[0, 2\pi)$. The solutions are $x \approx$ ______ and $x \approx$ ______.

Skills

5–16 ■ Solving Basic Trigonometric Equations Solve the given equation. Give exact answers where possible.

5. $\sin\theta = \dfrac{\sqrt{3}}{2}$ **6.** $\sin\theta = -\dfrac{\sqrt{2}}{2}$

7. $\cos\theta = -1$ **8.** $\cos\theta = \dfrac{\sqrt{3}}{2}$

9. $\cos\theta = \frac{1}{4}$ **10.** $\sin\theta = -0.3$

11. $\sin\theta = -0.45$ **12.** $\cos\theta = 0.32$

13. $\tan\theta = -\sqrt{3}$ **14.** $\tan\theta = 1$

15. $\tan\theta = 5$ **16.** $\tan\theta = -\frac{1}{3}$

17–24 ■ Solving Basic Trigonometric Equations Solve the given equation, and list six specific solutions.

17. $\cos\theta = -\dfrac{\sqrt{3}}{2}$ **18.** $\cos\theta = \dfrac{1}{2}$

19. $\sin\theta = \dfrac{\sqrt{2}}{2}$ **20.** $\sin\theta = -\dfrac{\sqrt{3}}{2}$

21. $\cos\theta = 0.28$ **22.** $\tan\theta = 2.5$

23. $\tan\theta = -10$ **24.** $\sin\theta = -0.9$

25–38 ■ Solving Trigonometric Equations Find all solutions of the given equation.

25. $\cos\theta + 1 = 0$ **26.** $\sin\theta + 1 = 0$

27. $2\cos\theta - \sqrt{3} = 0$ **28.** $2\sin\theta + 1 = 0$

29. $3\cos\theta - 1 = 0$ **30.** $10\sin\theta + 3 = 0$

31. $3\tan^2\theta - 1 = 0$ **32.** $\cot\theta + 1 = 0$

33. $2\cos^2\theta - 1 = 0$ **34.** $4\sin^2\theta - 3 = 0$

35. $3\sin^2\theta - 1 = 0$ **36.** $\tan^2\theta - 9 = 0$

37. $\sec^2\theta - 2 = 0$ **38.** $\csc^2\theta - 4 = 0$

39–56 ■ Solving Trigonometric Equations by Factoring Solve the given equation.

39. $(\tan^2\theta - 4)(2\cos\theta + 1) = 0$

40. $(\tan\theta - 2)(16\sin^2\theta - 1) = 0$

41. $4\cos^2\theta - 4\cos\theta + 1 = 0$

42. $2\sin^2\theta - \sin\theta - 1 = 0$

43. $\tan^2\theta - \tan\theta - 6 = 0$

44. $3\cos^4\theta - 5\cos^2\theta + 2 = 0$

45. $2\cos^2\theta - 7\cos\theta + 3 = 0$

46. $\sin^2\theta - \sin\theta - 2 = 0$

47. $\cos^2\theta - \cos\theta - 6 = 0$

48. $2\sin^2\theta + 5\sin\theta - 12 = 0$

49. $\sin^2\theta = 2\sin\theta + 3$

50. $3\tan^3\theta = \tan\theta$

51. $\cos\theta\,(2\sin\theta + 1) = 0$

52. $\sec\theta\left(2\cos\theta - \sqrt{2}\right) = 0$

53. $\cos\theta\sin\theta - 2\cos\theta = 0$

54. $\tan\theta\sin\theta + \sin\theta = 0$

55. $3\tan\theta\sin\theta - 2\tan\theta = 0$

56. $4\cos\theta\sin\theta + 3\cos\theta = 0$

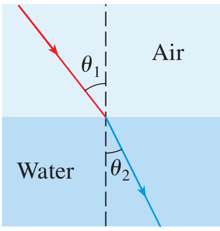
Applications

57. Refraction of Light It has been observed since ancient times that light refracts, or "bends," as it travels from one medium to another (from air to water, for example). If v_1 is the speed of light in one medium and v_2 is its speed in another medium, then according to **Snell's Law**,

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}$$

where θ_1 is the *angle of incidence* and θ_2 is the *angle of refraction* (see the figure). The number v_1/v_2 is called the *index of refraction*. The index of refraction for several substances is given in the table.

If a ray of light passes through the surface of a lake at an angle of incidence of 70°, what is the angle of refraction?



Substance	Refraction from Air to Substance
Water	1.33
Alcohol	1.36
Glass	1.52
Diamond	2.41

58. Total Internal Reflection When light passes from a more-dense to a less-dense medium—from glass to air, for example—the angle of refraction predicted by Snell's Law (see Exercise 57) can be 90° or larger. In this case the light beam is actually reflected back into the denser medium. This phenomenon, called *total internal reflection*, is the principle behind fiber optics. Set $\theta_2 = 90°$ in Snell's Law, and solve for θ_1 to determine the critical angle of incidence at which total internal reflection begins to occur when light passes from glass to air. (*Note:* The index of refraction from glass to air is the reciprocal of the index from air to glass.)

59. Phases of the Moon As the moon revolves around the earth, the side that faces the earth is usually just partially illuminated by the sun. The phases of the moon describe how much of the surface appears to be in sunlight. An astronomical measure of phase is given by the fraction F of the lunar disc that is lit. When the angle between the sun, earth, and moon is θ ($0 \le \theta < 360°$), then

$$F = \tfrac{1}{2}(1 - \cos\theta)$$

Determine the angles θ that correspond to the following phases:

(a) $F = 0$ (new moon)

(b) $F = 0.25$ (a crescent moon)

(c) $F = 0.5$ (first or last quarter)

(d) $F = 1$ (full moon)

■ Discuss ■ Discover ■ Prove ■ Write

60. Discuss ■ Write: Equations and Identities Which of the following statements is true?

A. Every identity is an equation.

B. Every equation is an identity.

Give examples to illustrate your answer. Write a short paragraph to explain the difference between an equation and an identity.

7.5 More Trigonometric Equations

■ Solving Trigonometric Equations by Using Identities ■ Equations with Trigonometric Functions of Multiples of Angles

In this section we solve trigonometric equations by first using identities to simplify the equation. We also solve trigonometric equations in which the terms contain multiples of angles.

■ Solving Trigonometric Equations by Using Identities

In the first two examples we use trigonometric identities to express a trigonometric equation in a form in which it can be factored.

Example 1 ■ Using a Trigonometric Identity

Solve the equation $1 + \sin\theta = 2\cos^2\theta$.

Solution We first need to rewrite this equation so that it contains only one trigonometric function. To do this, we use a trigonometric identity.

$$\begin{aligned}
1 + \sin\theta &= 2\cos^2\theta && \text{Given equation}\\
1 + \sin\theta &= 2(1 - \sin^2\theta) && \text{Pythagorean identity}\\
2\sin^2\theta + \sin\theta - 1 &= 0 && \text{Put all terms on one side}\\
(2\sin\theta - 1)(\sin\theta + 1) &= 0 && \text{Factor}
\end{aligned}$$

$$\begin{array}{rcrl}
2\sin\theta - 1 = 0 & \text{or} & \sin\theta + 1 = 0 & \text{Set each factor equal to 0}\\
\sin\theta = \dfrac{1}{2} & \text{or} & \sin\theta = -1 & \text{Solve for } \sin\theta\\
\theta = \dfrac{\pi}{6}, \dfrac{5\pi}{6} & \text{or} & \theta = \dfrac{3\pi}{2} & \text{Solve for } \theta \text{ in the interval } [0, 2\pi)
\end{array}$$

Because sine has period 2π, we get all the solutions of the equation by adding integer multiples of 2π to these solutions. Thus the solutions are

$$\theta = \frac{\pi}{6} + 2k\pi \qquad \theta = \frac{5\pi}{6} + 2k\pi \qquad \theta = \frac{3\pi}{2} + 2k\pi$$

where k is any integer.

Now Try Exercises 3 and 11 ■

Example 2 ■ Using a Trigonometric Identity

Solve the equation $\sin 2\theta - \cos\theta = 0$.

Solution The first term is a function of 2θ, and the second is a function of θ, so we begin by using a trigonometric identity to rewrite the first term as a function of θ only.

$$\sin 2\theta - \cos\theta = 0 \qquad \text{Given equation}$$
$$2\sin\theta\cos\theta - \cos\theta = 0 \qquad \text{Double-Angle Formula}$$
$$\cos\theta\,(2\sin\theta - 1) = 0 \qquad \text{Factor}$$
$$\cos\theta = 0 \quad \text{or} \quad 2\sin\theta - 1 = 0 \qquad \text{Set each factor equal to 0}$$
$$\sin\theta = \frac{1}{2} \qquad \text{Solve for } \sin\theta$$
$$\theta = \frac{\pi}{2}, \frac{3\pi}{2} \quad \text{or} \quad \theta = \frac{\pi}{6}, \frac{5\pi}{6} \qquad \text{Solve for } \theta \text{ in } [0, 2\pi)$$

Both sine and cosine have period 2π, so we get all the solutions of the equation by adding integer multiples of 2π to these solutions. Thus the solutions are

$$\theta = \frac{\pi}{2} + 2k\pi \qquad \theta = \frac{3\pi}{2} + 2k\pi \qquad \theta = \frac{\pi}{6} + 2k\pi \qquad \theta = \frac{5\pi}{6} + 2k\pi$$

where k is any integer.

Now Try Exercises 7 and 9 ■

Example 3 ■ Squaring an Equation and Using an Identity

Solve the equation $\cos\theta + 1 = \sin\theta$ in the interval $[0, 2\pi)$.

Solution To get an equation that involves either sine only or cosine only, we square both sides and use a Pythagorean identity.

$$\cos\theta + 1 = \sin\theta \qquad \text{Given equation}$$
$$\cos^2\theta + 2\cos\theta + 1 = \sin^2\theta \qquad \text{Square both sides}$$
$$\cos^2\theta + 2\cos\theta + 1 = 1 - \cos^2\theta \qquad \text{Pythagorean identity}$$
$$2\cos^2\theta + 2\cos\theta = 0 \qquad \text{Simplify}$$
$$2\cos\theta\,(\cos\theta + 1) = 0 \qquad \text{Factor}$$
$$2\cos\theta = 0 \quad \text{or} \quad \cos\theta + 1 = 0 \qquad \text{Set each factor equal to 0}$$
$$\cos\theta = 0 \quad \text{or} \quad \cos\theta = -1 \qquad \text{Solve for } \cos\theta$$
$$\theta = \frac{\pi}{2}, \frac{3\pi}{2} \quad \text{or} \quad \theta = \pi \qquad \text{Solve for } \theta \text{ in } [0, 2\pi)$$

Because we squared both sides, we need to check for extraneous solutions. From *Check Your Answers* we see that the solutions of the given equation are $\pi/2$ and π.

Check Your Answers

$\theta = \frac{\pi}{2}$	$\theta = \frac{3\pi}{2}$	$\theta = \pi$
$\cos\frac{\pi}{2} + 1 = \sin\frac{\pi}{2}$	$\cos\frac{3\pi}{2} + 1 = \sin\frac{3\pi}{2}$	$\cos\pi + 1 = \sin\pi$
$0 + 1 = 1$ ✓	$0 + 1 \stackrel{?}{=} -1$ ✗	$-1 + 1 = 0$ ✓

Now Try Exercise 13 ■

Example 4 ■ Finding Intersection Points

Find the values of x for which the graphs of $f(x) = \sin x$ and $g(x) = \cos x$ intersect.

Solution 1: Graphical

The graphs intersect where $f(x) = g(x)$. In Figure 1 we graph $y_1 = \sin x$ and $y_2 = \cos x$ on the same screen, for x between 0 and 2π. Using a graphing device, we see that the two points of intersection in this interval occur where $x \approx 0.785$ and $x \approx 3.927$. Since sine and cosine are periodic with period 2π, the intersection points occur where

$$x \approx 0.785 + 2k\pi \qquad \text{and} \qquad x \approx 3.927 + 2k\pi$$

where k is any integer.

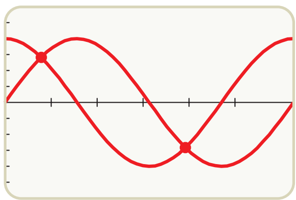


Figure 1

Solution 2: Algebraic

To find the exact solution, we set $f(x) = g(x)$ and solve the resulting equation algebraically:

$$\sin x = \cos x \qquad \text{Equate functions}$$

Since the numbers x for which $\cos x = 0$ are not solutions of the equation, we can divide both sides by $\cos x$:

$$\frac{\sin x}{\cos x} = 1 \qquad \text{Divide by } \cos x$$

$$\tan x = 1 \qquad \text{Reciprocal identity}$$

The only solution of this equation in the interval $(-\pi/2, \pi/2)$ is $x = \pi/4$. Since tangent has period π, we get all solutions of the equation by adding integer multiples of π:

$$x = \frac{\pi}{4} + k\pi$$

where k is any integer. The graphs intersect for these values of x. You should use your calculator to check that, rounded to three decimals, these are the same values that we obtained in Solution 1.

Now Try Exercise 35 ■

■ Equations with Trigonometric Functions of Multiples of Angles

When solving trigonometric equations that involve functions of multiples of angles, we first solve for the multiple of the angle, then divide to solve for the angle.

Example 5 ■ A Trigonometric Equation Involving a Multiple of an Angle

Consider the equation $2 \sin 3\theta - 1 = 0$.

(a) Find all solutions of the equation.

(b) List the solutions in the interval $[0, 2\pi)$.

Solution

(a) We first isolate $\sin 3\theta$ and then solve for the angle 3θ.

$$2 \sin 3\theta - 1 = 0 \qquad \text{Given equation}$$

$$2 \sin 3\theta = 1 \qquad \text{Add 1}$$

$$\sin 3\theta = \frac{1}{2} \qquad \text{Divide by 2}$$

$$3\theta = \frac{\pi}{6}, \frac{5\pi}{6} \qquad \text{Solve for } 3\theta \text{ in the interval } [0, 2\pi) \text{ (see Figure 2)}$$

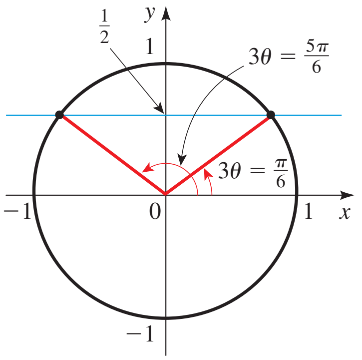


Figure 2

To get all solutions, we add integer multiples of 2π to these solutions. So the solutions are of the form

$$3\theta = \frac{\pi}{6} + 2k\pi \qquad 3\theta = \frac{5\pi}{6} + 2k\pi$$

To solve for θ, we divide by 3 to get the solutions

$$\theta = \frac{\pi}{18} + \frac{2k\pi}{3} \qquad \theta = \frac{5\pi}{18} + \frac{2k\pi}{3}$$

Compare to the solution of Example 7.4.5(a).

where k is any integer.

(b) The solutions from part (a) that are in the interval $[0, 2\pi)$ correspond to $k = 0, 1$, and 2. For all other values of k the corresponding values of θ lie outside this interval. So the solutions in the interval $[0, 2\pi)$ are

$$\theta = \underbrace{\frac{\pi}{18}, \frac{5\pi}{18}}_{k=0}, \underbrace{\frac{13\pi}{18}, \frac{17\pi}{18}}_{k=1}, \underbrace{\frac{25\pi}{18}, \frac{29\pi}{18}}_{k=2}$$

Now Try Exercise 17 ■

Example 6 ■ A Trigonometric Equation Involving a Half-Angle

Consider the equation $\sqrt{3} \tan \frac{\theta}{2} - 1 = 0$.

(a) Find all solutions of the equation.

(b) List the solutions in the interval $[0, 4\pi)$.

Solution

(a) We start by isolating $\tan \frac{\theta}{2}$.

$$\sqrt{3} \tan \frac{\theta}{2} - 1 = 0 \qquad \text{Given equation}$$

$$\sqrt{3} \tan \frac{\theta}{2} = 1 \qquad \text{Add 1}$$

$$\tan \frac{\theta}{2} = \frac{1}{\sqrt{3}} \qquad \text{Divide by } \sqrt{3}$$

$$\frac{\theta}{2} = \frac{\pi}{6} \qquad \text{Solve for } \frac{\theta}{2} \text{ in the interval } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

Since tangent has period π, to get all solutions we add integer multiples of π to this solution. So the solutions are of the form

$$\frac{\theta}{2} = \frac{\pi}{6} + k\pi$$

Multiplying by 2, we get the solutions

$$\theta = \frac{\pi}{3} + 2k\pi$$

where k is any integer.

(b) The solutions from part (a) that are in the interval $[0, 4\pi)$ correspond to $k = 0$ and $k = 1$. For all other values of k the corresponding values of x lie outside this interval. Thus the solutions in the interval $[0, 4\pi)$ are

$$x = \frac{\pi}{3}, \frac{7\pi}{3}$$

Now Try Exercise 23 ■

7.5 Exercises

Concepts

1–2 ■ We can use identities to help us solve trigonometric equations.

1. Using a Pythagorean identity we see that the equation $\sin x + \sin^2 x + \cos^2 x = 1$ is equivalent to the basic equation ________ whose solutions are $x =$ ________.

2. Using a Double-Angle Formula we see that the equation $\sin x + \sin 2x = 0$ is equivalent to the equation ________. Factoring, we see that solving this equation is equivalent to solving the two basic equations ________ and ________.

Skills

3–16 ■ Solving Trigonometric Equations by Using Identities Solve the given equation.

3. $2\cos^2\theta + \sin\theta = 1$

4. $\sin^2\theta = 4 - 2\cos^2\theta$

5. $\tan^2\theta - 2\sec\theta = 2$

6. $\csc^2\theta = \cot\theta + 3$

7. $\sin 2\theta - \sin\theta = 0$

8. $3\sin 2\theta - 2\sin\theta = 0$

9. $3\cos 2\theta - 2\cos^2\theta = 0$

10. $\cos 2\theta = \cos^2\theta - \frac{1}{2}$

11. $2\sin^2\theta - \cos\theta = 1$

12. $\tan\theta - 3\cot\theta = 0$

13. $\sin\theta - 1 = \cos\theta$

14. $\cos\theta - \sin\theta = 1$

15. $\tan\theta + 1 = \sec\theta$

16. $2\tan\theta + \sec^2\theta = 4$

17–30 ■ Solving Trigonometric Equations Involving a Multiple of an Angle An equation is given. **(a)** Find all solutions of the equation. **(b)** List the solutions in the interval $[0, 2\pi)$.

17. $2\cos 3\theta = 1$

18. $2\sin 2\theta = 1$

19. $2\cos 2\theta + 1 = 0$

20. $2\sin 3\theta + 1 = 0$

21. $\sqrt{3}\tan 3\theta + 1 = 0$

22. $\sec 4\theta - 2 = 0$

23. $\cos\frac{\theta}{2} - 1 = 0$

24. $\tan\frac{\theta}{4} + \sqrt{3} = 0$

25. $2\sin\frac{\theta}{3} + \sqrt{3} = 0$

26. $\sec\frac{\theta}{2} = \cos\frac{\theta}{2}$

27. $\sin 2\theta = 3\cos 2\theta$

28. $\csc 3\theta = 5\sin 3\theta$

29. $1 - 2\sin\theta = \cos 2\theta$

30. $\tan 3\theta + 1 = \sec 3\theta$

31–34 ■ Solving Trigonometric Equations by Factoring An equation is given. **(a)** Solve the equation by factoring. **(b)** List the solutions in the interval $[0, 2\pi)$.

31. $3\tan^3\theta - 3\tan^2\theta - \tan\theta + 1 = 0$

32. $4\sin\theta\cos\theta + 2\sin\theta - 2\cos\theta - 1 = 0$

33. $2\sin\theta\tan\theta - \tan\theta = 1 - 2\sin\theta$

34. $\sec\theta\tan\theta - \cos\theta\cot\theta = \sin\theta$

35–38 ■ Finding Intersection Points Graphically **(a)** Graph f and g in the given viewing rectangle and find the intersection points graphically, rounded to two decimal places. **(b)** Find the intersection points of f and g algebraically. Give exact answers.

35. $f(x) = 3\cos x + 1, \quad g(x) = \cos x - 1;$ $[-2\pi, 2\pi]$ by $[-2.5, 4.5]$

36. $f(x) = \sin 2x + 1, \quad g(x) = 2\sin 2x + 1;$ $[-2\pi, 2\pi]$ by $[-1.5, 3.5]$

37. $f(x) = \tan x, \quad g(x) = \sqrt{3};$ $[-\pi/2, \pi/2]$ by $[-10, 10]$

38. $f(x) = \sin x - 1, \quad g(x) = \cos x;$ $[-2\pi, 2\pi]$ by $[-2.5, 1.5]$

39–42 ■ Using Addition or Subtraction Formulas Use an Addition or Subtraction Formula to simplify the equation. Then find all solutions in the interval $[0, 2\pi)$.

39. $\cos\theta\cos 3\theta - \sin\theta\sin 3\theta = 0$

40. $\cos\theta\cos 2\theta + \sin\theta\sin 2\theta = \frac{1}{2}$

41. $\sin 2\theta\cos\theta - \cos 2\theta\sin\theta = \sqrt{3}/2$

42. $\sin 3\theta\cos\theta - \cos 3\theta\sin\theta = 0$

43–52 ■ Using Double- or Half-Angle Formulas Use a Double- or Half-Angle Formula to solve the equation in the interval $[0, 2\pi)$.

43. $\sin 2\theta + \cos\theta = 0$

44. $\tan\dfrac{\theta}{2} - \sin\theta = 0$

45. $\cos 2\theta + \cos\theta = 2$

46. $\tan\theta + \cot\theta = 4\sin 2\theta$

47. $\cos 2\theta - \cos^2\theta = 0$

48. $2\sin^2\theta = 2 + \cos 2\theta$

49. $\cos 2\theta - \cos 4\theta = 0$

50. $\sin 3\theta - \sin 6\theta = 0$

51. $\cos\theta - \sin\theta = \sqrt{2}\sin\dfrac{\theta}{2}$

52. $\sin\theta - \cos\theta = \frac{1}{2}$

53–56 ■ Using Sum-to-Product Formulas Solve the equation by first using a Sum-to-Product Formula.

53. $\sin\theta + \sin 3\theta = 0$

54. $\cos 5\theta - \cos 7\theta = 0$

55. $\cos 4\theta + \cos 2\theta = \cos\theta$

56. $\sin 5\theta - \sin 3\theta = \cos 4\theta$

57–62 ■ Solving Trigonometric Equations Graphically Use a graphing device to find the solutions of the equation, rounded to two decimal places.

57. $\sin 2x = x$

58. $\cos x = \dfrac{x}{3}$

59. $2^{\sin x} = x$

60. $\sin x = x^3$

61. $\dfrac{\cos x}{1 + x^2} = x^2$

62. $\cos x = \frac{1}{2}(e^x + e^{-x})$

Skills Plus

63–64 ■ Equations Involving Inverse Trigonometric Functions Solve the given equation for x.

63. $\tan^{-1}x + \tan^{-1}2x = \dfrac{\pi}{4}$ [*Hint:* Let $u = \tan^{-1}x$ and $v = \tan^{-1}2x$. Solve the equation $u + v = \dfrac{\pi}{4}$ by taking the tangent of each side.]

64. $2\sin^{-1}x + \cos^{-1}x = \pi$ [*Hint:* Take the cosine of each side.]

Applications

65. Range of a Projectile If a projectile is fired with velocity v_0 at an angle θ, then its *range*, the horizontal distance it travels (in ft), is modeled by the function

$$R(\theta) = \frac{v_0^2 \sin 2\theta}{32}$$

(See Focus on Modeling *The Path of a Projectile* following Chapter 8.) If $v_0 = 2200$ ft/s, what angle (in degrees) should be chosen for the projectile to hit a target on the ground 5000 ft away?

66. Damped Vibrations The displacement of a spring vibrating in damped harmonic motion is given by

$$y = 4e^{-3t}\sin 2\pi t$$

Find the times when the spring is at its equilibrium position $(y = 0)$.

67. Hours of Daylight In Philadelphia the number of hours of daylight on day t (where t is the number of days after January 1) is modeled by the function

$$L(t) = 12 + 2.83\sin\left(\frac{2\pi}{365}(t - 80)\right)$$

(a) Which days of the year have about 10 hours of daylight?

(b) How many days of the year have more than 10 hours of daylight?

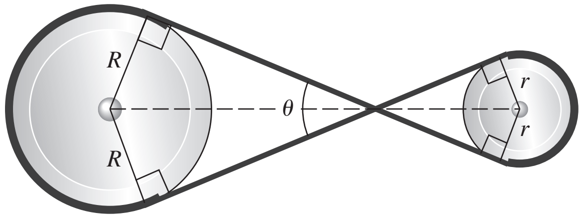
68. Belts and Pulleys A thin belt of length L surrounds two pulleys of radii R and r, as shown in the figure.

(a) Show that the angle θ (in rad) where the belt crosses itself satisfies the equation

$$\theta + 2\cot\frac{\theta}{2} = \frac{L}{R+r} - \pi$$

[*Hint:* Express L in terms of R, r, and θ by adding up the lengths of the curved and straight parts of the belt.]

(b) Suppose that $R = 2.42$ ft, $r = 1.21$ ft, and $L = 27.78$ ft. Find θ by solving the equation in part (a) graphically. Express your answer both in radians and in degrees.



■ Discuss ■ Discover ■ Prove ■ Write

69. Discuss: A Special Trigonometric Equation What makes the equation $\sin(\cos x) = 0$ different from all the other equations we've looked at in this section? Find all solutions of this equation.

Chapter 7 Review

Properties and Formulas

Fundamental Trigonometric Identities | Section 7.1

An **identity** is an equation that is true for all values of the variable(s). A **trigonometric identity** is an identity that involves trigonometric functions. The fundamental trigonometric identities are as follows.

Reciprocal Identities:

$$\csc x = \frac{1}{\sin x} \qquad \sec x = \frac{1}{\cos x} \qquad \cot x = \frac{1}{\tan x}$$

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

Pythagorean Identities:

$$\sin^2 x + \cos^2 x = 1$$
$$\tan^2 x + 1 = \sec^2 x$$
$$1 + \cot^2 x = \csc^2 x$$

Even-Odd Identities:

$$\sin(-x) = -\sin x$$
$$\cos(-x) = \cos x$$
$$\tan(-x) = -\tan x$$

Cofunction Identities:

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x \qquad \tan\left(\frac{\pi}{2} - x\right) = \cot x$$

$$\sec\left(\frac{\pi}{2} - x\right) = \csc x$$

$$\cos\left(\frac{\pi}{2} - x\right) = \sin x \qquad \cot\left(\frac{\pi}{2} - x\right) = \tan x$$

$$\csc\left(\frac{\pi}{2} - x\right) = \sec x$$

Proving Trigonometric Identities | Section 7.1

To prove that a trigonometric equation is an identity, we use the following guidelines.

1. **Start with one side.** Select one side of the equation.
2. **Use known identities.** Use algebra and known identities to change the side you started with into the other side.
3. **Convert to sines and cosines.** Sometimes it is helpful to convert all functions in the equation to sines and cosines.

Addition and Subtraction Formulas | Section 7.2

These identities involve the trigonometric functions of a sum or a difference.

Formulas for Sine:

$$\sin(s + t) = \sin s \cos t + \cos s \sin t$$
$$\sin(s - t) = \sin s \cos t - \cos s \sin t$$

Formulas for Cosine:

$$\cos(s + t) = \cos s \cos t - \sin s \sin t$$
$$\cos(s - t) = \cos s \cos t + \sin s \sin t$$

Formulas for Tangent:

$$\tan(s + t) = \frac{\tan s + \tan t}{1 - \tan s \tan t}$$

$$\tan(s - t) = \frac{\tan s - \tan t}{1 + \tan s \tan t}$$

Sums of Sines and Cosines | Section 7.2

If A and B are real numbers, then

$$A\sin x + B\cos x = k\sin(x + \phi)$$

where $k = \sqrt{A^2 + B^2}$ and ϕ satisfies

$$\cos\phi = \frac{A}{\sqrt{A^2 + B^2}} \qquad \sin\phi = \frac{B}{\sqrt{A^2 + B^2}}$$

Double-Angle Formulas | Section 7.3

These identities involve the trigonometric functions of twice the variable.

Formula for Sine:

$$\sin 2x = 2 \sin x \cos x$$

Formulas for Cosine:

$$\begin{aligned}\cos 2x &= \cos^2 x - \sin^2 x \\ &= 1 - 2\sin^2 x \\ &= 2\cos^2 x - 1\end{aligned}$$

Formula for Tangent:

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

Formulas for Lowering Powers | Section 7.3

These formulas allow us to write a trigonometric expression involving even powers of sine and cosine in terms of the first power of cosine only.

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\tan^2 x = \frac{1 - \cos 2x}{1 + \cos 2x}$$

Half-Angle Formulas | Section 7.3

These formulas involve trigonometric functions of half an angle.

$$\sin\frac{u}{2} = \pm\sqrt{\frac{1 - \cos u}{2}} \qquad \cos\frac{u}{2} = \pm\sqrt{\frac{1 + \cos u}{2}}$$

$$\tan\frac{u}{2} = \frac{1 - \cos u}{\sin u} = \frac{\sin u}{1 + \cos u}$$

Product-Sum Formulas | Section 7.3

These formulas involve products and sums of trigonometric functions.

Product-to-Sum Formulas:

$$\begin{aligned}\sin u \cos v &= \tfrac{1}{2}[\sin(u + v) + \sin(u - v)] \\ \cos u \sin v &= \tfrac{1}{2}[\sin(u + v) - \sin(u - v)] \\ \cos u \cos v &= \tfrac{1}{2}[\cos(u + v) + \cos(u - v)] \\ \sin u \sin v &= \tfrac{1}{2}[\cos(u - v) - \cos(u + v)]\end{aligned}$$

Sum-to-Product Formulas:

$$\begin{aligned}\sin x + \sin y &= 2\sin\frac{x + y}{2}\cos\frac{x - y}{2} \\ \sin x - \sin y &= 2\cos\frac{x + y}{2}\sin\frac{x - y}{2} \\ \cos x + \cos y &= 2\cos\frac{x + y}{2}\cos\frac{x - y}{2} \\ \cos x - \cos y &= -2\sin\frac{x + y}{2}\sin\frac{x - y}{2}\end{aligned}$$

Trigonometric Equations | Section 7.4

A **trigonometric equation** is an equation that contains trigonometric functions. A basic trigonometric equation is an equation of the form $T(\theta) = c$, where T is a trigonometric function and c is a constant. For example, $\sin\theta = 0.5$ and $\tan\theta = 2$ are basic trigonometric equations. Solving any trigonometric equation involves solving a basic trigonometric equation.

If a trigonometric equation has a solution, then it has infinitely many solutions.

To find all solutions, we first find the solutions in one period and then add integer multiples of the period.

We can sometimes use trigonometric identities to simplify a trigonometric equation.

Concept Check

1. What is an identity? What is a trigonometric identity?

2. (a) State the Pythagorean identities.

(b) Use a Pythagorean identity to express cosine in terms of sine.

3. (a) State the reciprocal identities for cosecant, secant, and cotangent.

(b) State the even-odd identities for sine and cosine.

(c) State the cofunction identities for sine, tangent, and secant.

(d) Suppose that $\cos(-x) = 0.4$; use the identities in parts (a) and (b) to find $\sec x$.

(e) Suppose that $\sin 10° = a$; use the identities in part (c) to find $\cos 80°$.

4. (a) How do you prove an identity?

(b) Prove the identity $\sin x(\csc x - \sin x) = \cos^2 x$

5. (a) State the Addition and Subtraction Formulas for Sine and Cosine.

(b) Use a formula from part (a) to find $\cos 75°$.

6. (a) State the formula for $A \sin x + B \cos x$.

(b) Express $4 \sin x + 3 \cos x$ as a function of sine only.

7. (a) State the Double-Angle Formula for Sine and the Double-Angle Formulas for Cosine.

(b) Prove the identity $\sec x \sin 2x = 2 \sin x$.

8. (a) State the formulas for lowering powers of sine and cosine.

(b) Prove the identity $4\sin^2 x \cos^2 x = \sin^2 2x$.

9. (a) State the Half-Angle Formulas for Sine and Cosine.

(b) Find $\cos 15°$.

10. (a) State the Product-to-Sum Formula for the product $\sin u \cos v$.

(b) Express $\sin 5x \cos 3x$ as a sum of trigonometric functions.

11. **(a)** State the Sum-to-Product Formula for the sum $\sin x + \sin y$.

(b) Express $\sin 5x + \sin 7x$ as a product of trigonometric functions.

12. What is a trigonometric equation? How do we solve a trigonometric equation?

(a) Solve the equation $\cos x = \frac{1}{2}$.

(b) Solve the equation $2 \sin x \cos x = \frac{1}{2}$.

Answers to the Concept Check can be found at the book companion website **stewartmath.com.**

Exercises

1–22 ■ Proving Identities Verify the identity.

1. $\sin\theta(\cot\theta + \tan\theta) = \sec\theta$

2. $(\sec\theta - 1)(\sec\theta + 1) = \tan^2\theta$

3. $\cos^2 x \csc x - \csc x = -\sin x$

4. $\dfrac{1}{1 - \sin^2 x} = 1 + \tan^2 x$

5. $\dfrac{\cos^2 x - \tan^2 x}{\sin^2 x} = \cot^2 x - \sec^2 x$

6. $\dfrac{1 + \sec x}{\sec x} = \dfrac{\sin^2 x}{1 - \cos x}$

7. $\dfrac{\cos^2 x}{1 - \sin x} = \dfrac{\cos x}{\sec x - \tan x}$

8. $(1 - \tan x)(1 - \cot x) = 2 - \sec x \csc x$

9. $\sin^2 x \cot^2 x + \cos^2 x \tan^2 x = 1$

10. $(\tan x + \cot x)^2 = \csc^2 x \sec^2 x$

11. $\dfrac{\sin 2x}{1 + \cos 2x} = \tan x$

12. $\dfrac{\cos(x + y)}{\cos x \sin y} = \cot y - \tan x$

13. $\csc x - \tan\dfrac{x}{2} = \cot x$

14. $1 + \tan x \tan\dfrac{x}{2} = \sec x$

15. $\dfrac{\sin 2x}{\sin x} - \dfrac{\cos 2x}{\cos x} = \sec x$

16. $\tan\left(x + \dfrac{\pi}{4}\right) = \dfrac{1 + \tan x}{1 - \tan x}$

17. $\dfrac{\sec x - 1}{\sin x \sec x} = \tan\dfrac{x}{2}$

18. $(\cos x + \cos y)^2 + (\sin x - \sin y)^2 = 2 + 2\cos(x + y)$

19. $\left(\cos\dfrac{x}{2} - \sin\dfrac{x}{2}\right)^2 = 1 - \sin x$

20. $\dfrac{\cos 3x - \cos 7x}{\sin 3x + \sin 7x} = \tan 2x$

21. $\dfrac{\sin(x + y) + \sin(x - y)}{\cos(x + y) + \cos(x - y)} = \tan x$

22. $\sin(x + y)\sin(x - y) = \sin^2 x - \sin^2 y$

23–26 ■ Checking Identities Graphically **(a)** Graph f and g. **(b)** Do the graphs suggest that the equation $f(x) = g(x)$ is an identity? Prove your answer.

23. $f(x) = 1 - \left(\cos\dfrac{x}{2} - \sin\dfrac{x}{2}\right)^2, \quad g(x) = \sin x$

24. $f(x) = \sin x + \cos x, \quad g(x) = \sqrt{\sin^2 x + \cos^2 x}$

25. $f(x) = \tan x \tan\dfrac{x}{2}, \quad g(x) = \dfrac{1}{\cos x}$

26. $f(x) = 1 - 8\sin^2 x + 8\sin^4 x, \quad g(x) = \cos 4x$

27–28 ■ Determining Identities Graphically **(a)** Graph the function(s) and make a conjecture, and **(b)** prove your conjecture.

27. $f(x) = 2\sin^2 3x + \cos 6x$

28. $f(x) = \sin x \cot\dfrac{x}{2}, \quad g(x) = \cos x$

29–46 ■ Solving Trigonometric Equations Solve the equation in the interval $[0, 2\pi)$.

29. $4\sin\theta - 3 = 0$

30. $5\cos\theta + 3 = 0$

31. $\cos x \sin x - \sin x = 0$

32. $\sin x - 2\sin^2 x = 0$

33. $2\sin^2 x - 5\sin x + 2 = 0$

34. $\sin x - \cos x - \tan x = -1$

35. $2\cos^2 x - 7\cos x + 3 = 0$

36. $4\sin^2 x + 2\cos^2 x = 3$

37. $\dfrac{1 - \cos x}{1 + \cos x} = 3$

38. $\sin x = \cos 2x$

39. $\tan^3 x + \tan^2 x - 3\tan x - 3 = 0$

40. $\cos 2x \csc^2 x = 2\cos 2x$

41. $\tan\frac{1}{2}x + 2\sin 2x = \csc x$

42. $\cos 3x + \cos 2x + \cos x = 0$

43. $\tan x + \sec x = \sqrt{3}$

44. $2\cos x - 3\tan x = 0$

45. $\cos x = x^2 - 1$

46. $e^{\sin x} = x$

47. Range of a Projectile If a projectile is fired with velocity v_0 at an angle θ, then the maximum height it reaches (in feet) is modeled by the function

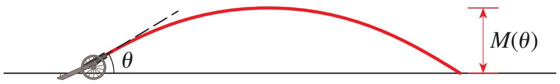
$$M(\theta) = \frac{v_0^2 \sin^2\theta}{64}$$

Suppose $v_0 = 400$ ft/s.

(a) At what angle θ should the projectile be fired so that the maximum height it reaches is 2000 ft?

(b) Is it possible for the projectile to reach a height of 3000 ft?

(c) Find the angle θ for which the projectile will travel highest.



48. Displacement of a Shock Absorber The displacement of an automobile shock absorber is modeled by the function

$$f(t) = 2^{-0.2t} \sin 4\pi t$$

Find the times when the shock absorber is at its equilibrium position [that is, when $f(t) = 0$]. [*Hint:* $2^x > 0$ for all real x.]

49–58 ■ Value of Expressions Find the exact value of the expression.

49. $\cos 15°$

50. $\sin \dfrac{5\pi}{12}$

51. $\tan \dfrac{\pi}{8}$

52. $2 \sin \dfrac{\pi}{12} \cos \dfrac{\pi}{12}$

53. $\sin 5° \cos 40° + \cos 5° \sin 40°$

54. $\dfrac{\tan 66° - \tan 6°}{1 + \tan 66° \tan 6°}$

55. $\cos^2\left(\dfrac{\pi}{8}\right) - \sin^2\left(\dfrac{\pi}{8}\right)$

56. $\dfrac{1}{2} \cos \dfrac{\pi}{12} + \dfrac{\sqrt{3}}{2} \sin \dfrac{\pi}{12}$

57. $\cos 37.5° \cos 7.5°$

58. $\cos 67.5° + \cos 22.5°$

59–64 ■ Evaluating Expressions Involving Trigonometric Functions Find the exact value of the expression given that $\sec x = \frac{3}{2}$, $\csc y = 3$, and x and y are in Quadrant I.

59. $\sin(x + y)$

60. $\cos(x - y)$

61. $\tan(x + y)$

62. $\sin 2x$

63. $\cos \dfrac{y}{2}$

64. $\tan \dfrac{y}{2}$

65–66 ■ Evaluating Expressions Involving Inverse Trigonometric Functions Find the exact value of the expression.

65. $\tan\left(2 \cos^{-1}\left(\frac{3}{7}\right)\right)$

66. $\sin\left(\tan^{-1}\left(\frac{3}{4}\right) + \cos^{-1}\left(\frac{5}{13}\right)\right)$

67–68 ■ Expressions Involving Inverse Trigonometric Functions Write the expression as an algebraic expression in the variable(s).

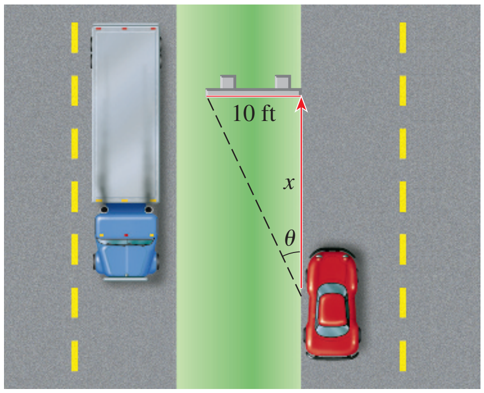
67. $\tan(2 \tan^{-1} x)$

68. $\cos(\sin^{-1} x + \cos^{-1} y)$

69. Viewing Angle of a Sign A 10-foot-wide highway sign is adjacent to a roadway, as shown in the figure. As a driver approaches the sign, the viewing angle θ changes.

(a) Express viewing angle θ as a function of the distance x between the driver and the sign.

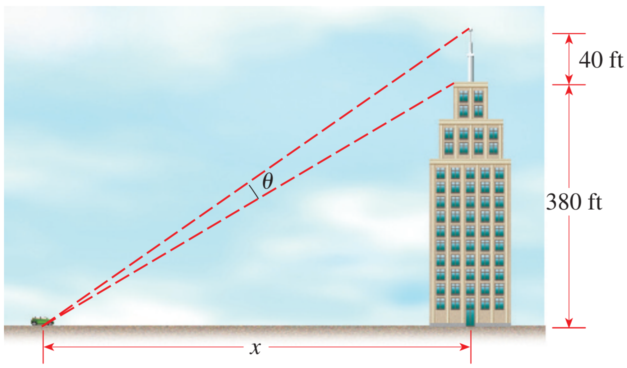
(b) The sign is legible when the viewing angle is 2° or greater. At what distance x does the sign first become legible?



70. Viewing Angle of a Tower A 380-foot-tall building supports a 40-ft communications tower (see the figure). As a driver approaches the building, the viewing angle θ of the tower changes.

(a) Express the viewing angle θ as a function of the distance x between the driver and the building.

(b) At what distance from the building is the viewing angle θ as large as possible?



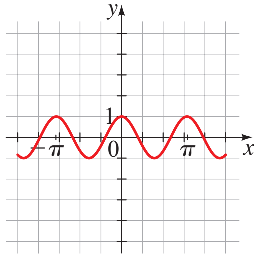
Matching

71. Equations and Their Graphs Match each equation with its graph and give reasons for your answers. Use identities to help recognize the graph. (Don't use a graphing device.)

(a) $y = \sin\left(\dfrac{\pi}{2} - x\right)$ **(b)** $y = 4 \sin x \cos x$ **(c)** $y = 1 - \cos 2x$ **(d)** $y = \dfrac{1 - \cos 2x}{1 + \cos 2x}$

(e) $y = \dfrac{1}{2} \sin x + \dfrac{\sqrt{3}}{2} \cos x$ **(f)** $y = 1 + \tan^2 x$ **(g)** $y = \cos^2 x - \sin^2 x$ **(h)** $y = \dfrac{\sin x}{1 + \cos x}$

I



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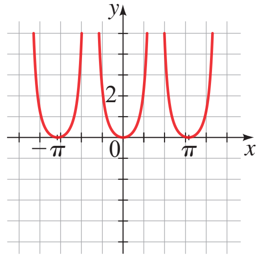

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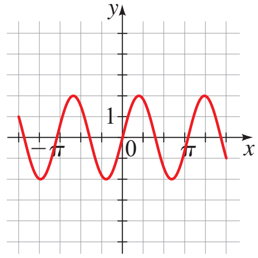

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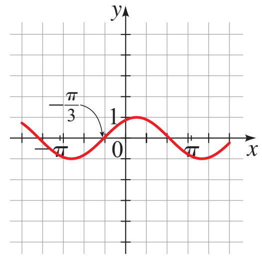

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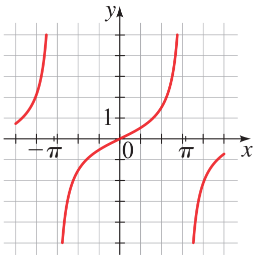

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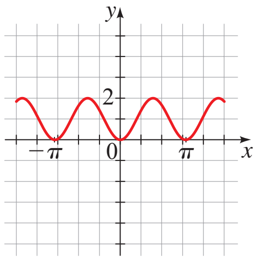
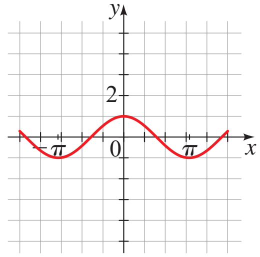
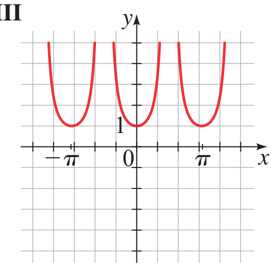


VII



VIII



Chapter 7 | Test

1–7 ■ Verify the identity.

1. $\tan\theta\sin\theta + \cos\theta = \sec\theta$

2. $\dfrac{\tan x}{1 - \cos x} = \csc x\,(1 + \sec x)$

3. $\dfrac{2\tan x}{1 + \tan^2 x} = \sin 2x$

4. $\sin x \tan\dfrac{x}{2} = 1 - \cos x$

5. $2\sin^2 3x = 1 - \cos 6x$

6. $\cos 4x = 1 - 8\sin^2 x + 8\sin^4 x$

7. $\left(\sin\dfrac{x}{2} + \cos\dfrac{x}{2}\right)^2 = 1 + \sin x$

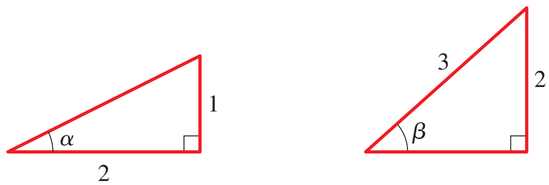
8. Let $x = 2\sin\theta$, $-\pi/2 < \theta < \pi/2$. Simplify the expression

$$\frac{x}{\sqrt{4 - x^2}}$$

9. Find the exact value of each expression.

(a) $\sin 8^\circ \cos 22^\circ + \cos 8^\circ \sin 22^\circ$ **(b)** $\sin 75^\circ$ **(c)** $\sin\dfrac{\pi}{12}$

10. For the angles α and β in the figures, find $\cos(\alpha + \beta)$.



11. Write $\sin 3x \cos 5x$ as a sum of trigonometric functions.

12. Write $\sin 2x - \sin 5x$ as a product of trigonometric functions.

13. If $\sin\theta = -\frac{4}{5}$ and θ is in Quadrant III, find $\tan(\theta/2)$.

14–20 ■ Solve the trigonometric equation in the interval $[0, 2\pi)$. Give the exact value, if possible; otherwise, round your answer to two decimal places.

14. $3\sin\theta - 1 = 0$

15. $(2\cos\theta - 1)(\sin\theta - 1) = 0$

16. $2\cos^2\theta + 5\cos\theta + 2 = 0$

17. $\sin 2\theta - \cos\theta = 0$

18. $5\cos 2\theta = 2$

19. $2\cos^2 x + \cos 2x = 0$

20. $2\tan\dfrac{x}{2} - \csc x = 0$

21. Find the exact value of $\cos\left(2\tan^{-1}\left(\frac{9}{40}\right)\right)$.

22. Rewrite the expression as an algebraic function of x and y: $\sin(\cos^{-1}x - \tan^{-1}y)$.

A Cumulative Review Test for Chapters 5, 6, and 7 can be found at the book companion website www.stewartmath.com.

We've learned that the position of a particle in simple harmonic motion is described by a function of the form $y = A \sin \omega t$ (see Section 5.6). For example, if a string is moved up and down as in Figure 1, then the red dot on the string moves up and down in simple harmonic motion. Of course, the same holds true for each point on the string.

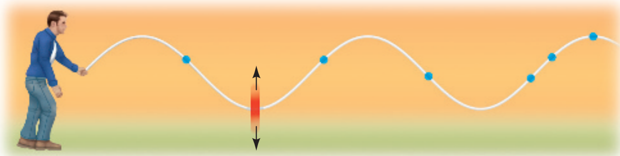


Figure 1

What function describes the shape of the whole string? If we fix an instant in time $(t = 0)$ and snap a photograph of the string, we get the shape in Figure 2, which is modeled by

$$y = A \sin kx$$

where y is the height of the string above the x-axis at the point x.

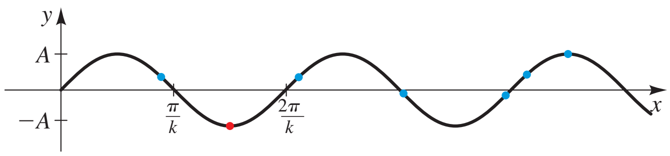


Figure 2 | $y = A \sin kx$

Traveling Waves

If we snap photographs of the string at other instants, as in Figure 3, it appears that the waves in the string "travel," or shift to the right.

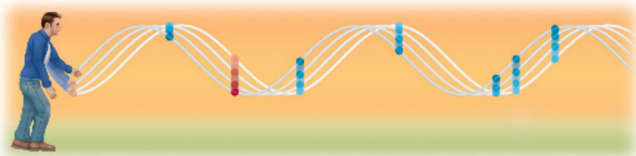


Figure 3

The **velocity** of the wave is the rate at which it moves to the right. If the wave has velocity v, then it moves to the right a distance vt in time t. So the graph of the shifted wave at time t is

$$y(x, t) = A \sin k(x - vt)$$

This function models the position of any point x on the string at any time t. We use the notation $y(x, t)$ to indicate that the function depends on the *two* variables x and t. Here is how this function models the motion of the string.

- **If we fix x**, then $y(x, t)$ is a function of t only, which gives the position of the fixed point x at time t.
- **If we fix t**, then $y(x, t)$ is a function of x only, whose graph is the shape of the string at the fixed time t.

Example 1 ■ A Traveling Wave

A traveling wave is described by the function

$$y(x, t) = 3 \sin\left(2x - \frac{\pi}{2} t\right) \qquad (x \geq 0)$$

(a) Find the function that models the position of the point $x = \pi/6$ at any time t. Observe that the point moves in simple harmonic motion.

(b) Sketch the shape of the wave when $t = 0$, 0.5, 1.0, 1.5, and 2.0. Does the wave appear to be traveling to the right?

(c) Find the velocity of the wave.

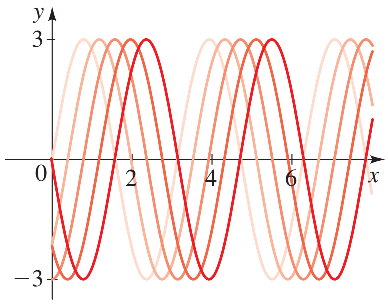


Figure 4 | Traveling wave

Solution

(a) Substituting $x = \pi/6$, we get

$$y\left(\frac{\pi}{6}, t\right) = 3 \sin\left(2 \cdot \frac{\pi}{6} - \frac{\pi}{2} t\right) = 3 \sin\left(\frac{\pi}{3} - \frac{\pi}{2} t\right)$$

The function $y = 3 \sin\left(\frac{\pi}{3} - \frac{\pi}{2} t\right)$ describes simple harmonic motion with amplitude 3 and period $2\pi/(\pi/2) = 4$.

(b) The graphs are shown in Figure 4. As t increases, the wave moves to the right.

(c) We express the given function in the standard form $y(x, t) = A \sin k(x - vt)$.

$$\begin{aligned} y(x, t) &= 3 \sin\left(2x - \frac{\pi}{2} t\right) && \text{Given} \\ &= 3 \sin 2\left(x - \frac{\pi}{4} t\right) && \text{Factor 2} \end{aligned}$$

Comparing this to the standard form, we see that the wave is moving with velocity $v = \pi/4$. ■

■ Standing Waves

If two waves are traveling along the same string, then the movement of the string is determined by the sum of the two waves. For example, if the string is attached to a wall, then the waves bounce back with the same amplitude and speed but in opposite directions. In this case, one wave is described by $y = A \sin k(x - vt)$, and the reflected wave is described by $y = A \sin k(x + vt)$. The resulting wave is

$$\begin{aligned} y(x, t) &= A \sin k(x - vt) + A \sin k(x + vt) && \text{Add the two waves} \\ &= 2A \sin kx \cos kvt && \text{Sum-to-Product Formula} \end{aligned}$$

The points where kx is a multiple of 2π are special because at these points $y = 0$ for any time t. In other words, these points never move. Such points are called **nodes**. Figure 5 shows the graph of the wave for several values of t. We see that the wave does not travel but simply vibrates up and down. Such a wave is called a **standing wave**.

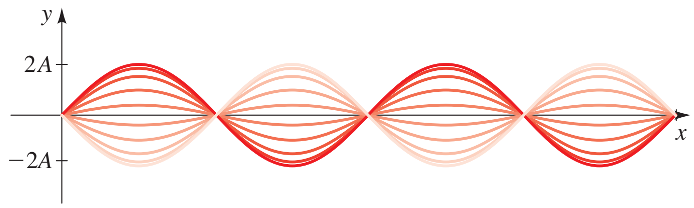
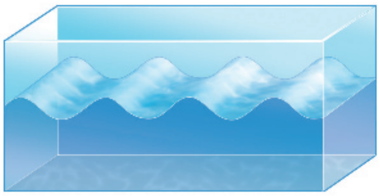


Figure 5 | A standing wave

Example 2 ■ A Standing Wave



Traveling waves are generated at each end of a wave tank 30 ft long, with equations

$$y = 1.5 \sin\left(\frac{\pi}{5}x - 3t\right)$$

and

$$y = 1.5 \sin\left(\frac{\pi}{5}x + 3t\right)$$

(a) Find the equation of the combined wave, and find the nodes.

(b) Sketch the graph for $t = 0, 0.17, 0.34, 0.51, 0.68, 0.85$, and 1.02. Is this a standing wave?

Solution

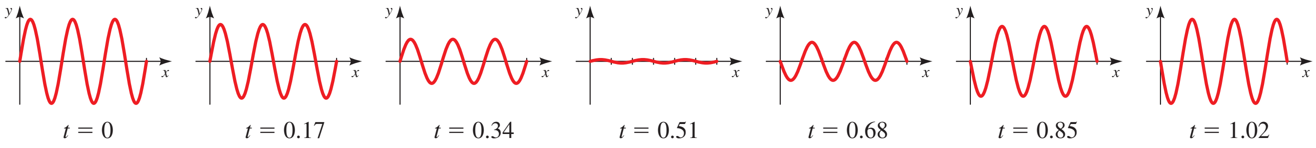
(a) The combined wave is obtained by adding the two equations.

$$\begin{aligned} y &= 1.5 \sin\left(\frac{\pi}{5}x - 3t\right) + 1.5 \sin\left(\frac{\pi}{5}x + 3t\right) && \text{Add the two waves} \\ &= 3 \sin\frac{\pi}{5}x \cos 3t && \text{Sum-to-Product Formula} \end{aligned}$$

The nodes occur at the values of x for which $\sin\frac{\pi}{5}x = 0$, that is, where $\frac{\pi}{5}x = k\pi$ (k an integer). Solving for x, we get $x = 5k$. So the nodes occur at

$$x = 0, 5, 10, 15, 20, 25, 30$$

(b) The graphs are shown in Figure 6. From the graphs we see that this is a standing wave.



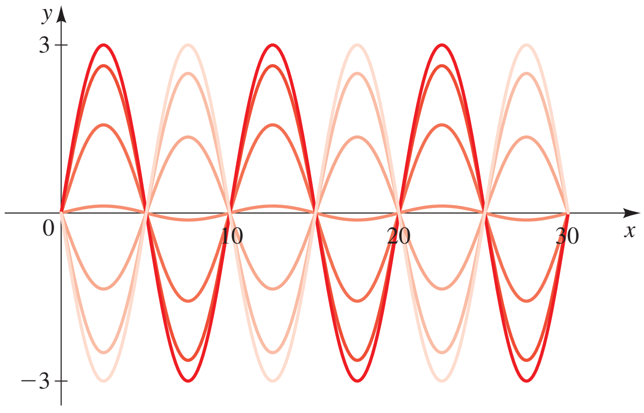
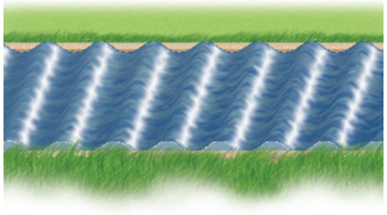


Figure 6 | $y(x, t) = 3 \sin\frac{\pi}{5}x \cos 3t$

■

Problems



1. **Wave on a Canal** A wave on the surface of a long canal is described by the function

$$y(x, t) = 5 \sin\left(4x - \frac{\pi}{8}t\right) \qquad (x \geq 0)$$

   (a) Find the function that models the position of the point $x = 0$ at any time t.

   (b) Sketch the shape of the wave when $t = 0, 1.6, 3.2, 4.8$, and 6.4. Is this a traveling wave?

   (c) Find the velocity of the wave.

2. **Wave in a Rope** Traveling waves are generated at each end of a tightly stretched rope 24 ft long, with equations

$$y = 0.2 \sin(1.047x - 0.524t)$$

$$y = 0.2 \sin(1.047x + 0.524t)$$

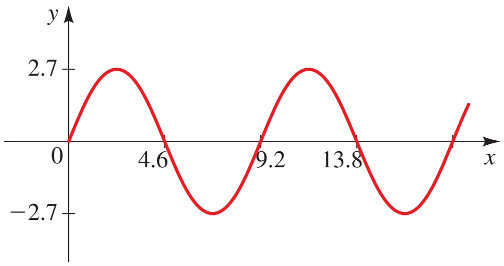
   (a) Find an equation of the combined wave, and find the nodes.

   (b) Sketch the graph for $t = 0, 1, 2, 3, 4, 5$, and 6. Is this a standing wave?

3. **Traveling Wave** A traveling wave is graphed at the instant $t = 0$. If it is moving to the right with velocity 6, find an equation of the form

$$y(x, t) = A \sin(kx - kvt)$$

   for this wave.



4. **Traveling Wave** A traveling wave has period $2\pi/3$, amplitude 5, and velocity 0.5.

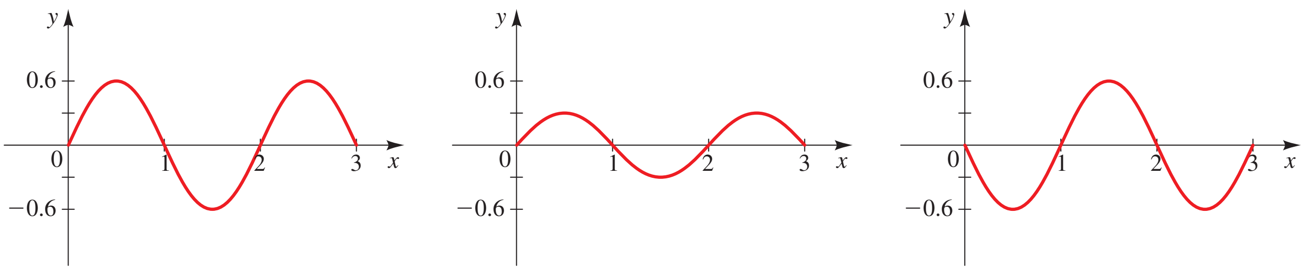
   (a) Find an equation of the wave.

   (b) Sketch the graph for $t = 0, 0.5, 1, 1.5$, and 2.

5. **Standing Wave** A standing wave with amplitude 0.6 is graphed at several times t, as shown in the figure. If the vibration has a frequency of 20 Hz, find an equation of the form

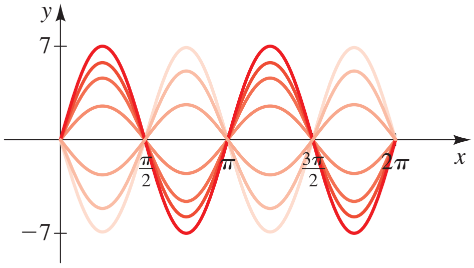
$$y(x, t) = A \sin \alpha x \cos \beta t$$

   that models this wave.



$t = 0$ s　　$t = 0.010$ s　　$t = 0.025$ s

6. Standing Wave A standing wave has maximum amplitude 7 and nodes at 0, $\pi/2$, π, $3\pi/2$, 2π, as shown in the figure. Each point that is not a node moves up and down with period 4π. Find a function of the form $y(x, t) = A \sin \alpha x \cos \beta t$ that models this wave.



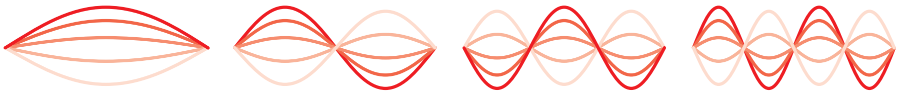
7. Vibrating String When a violin string vibrates, the sound produced results from a combination of standing waves that have evenly placed nodes. The figure illustrates some of the possible standing waves. Let's assume that the string has length π.

(a) For fixed t, the string has the shape of a sine curve $y = A \sin \alpha x$. Find the appropriate value of α for each of the standing waves illustrated.

(b) Do you notice a pattern in the values of α that you found in part (a)? What would the next two values of α be? Sketch rough graphs of the standing waves associated with these new values of α.

(c) Suppose that for fixed t, each point on the string that is not a node vibrates with frequency 440 Hz. Find the value of β for which an equation of the form $y = A \cos \beta t$ would model this motion.

(d) Combine your answers for parts (a) and (c) to find functions of the form $y(x, t) = A \sin \alpha x \cos \beta t$ that model each of the standing waves in the figure. (Assume that $A = 1$.)



8. Waves in a Tube Standing waves in a violin string must have nodes at the ends of the string because the string is fixed at its endpoints. But this need not be the case with sound waves in a tube (such as a flute or an organ pipe). The figure shows some possible standing waves in a tube.

Suppose that a standing wave in a tube 37.7 ft long is modeled by the function

$$y(x, t) = 0.3 \cos \tfrac{1}{2}x \cos 50\pi t$$

Here $y(x, t)$ represents the variation from normal air pressure at the point x feet from the left end of the tube, at time t seconds.

(a) At what points x are the nodes located? Are the endpoints of the tube nodes?

(b) At what frequency does the air vibrate at points that are not nodes?