P.2 Exponents and Radicals

What you should learn

- Use properties of exponents.
- Use scientific notation to represent real numbers.
- Use properties of radicals.
- Simplify and combine radicals.
- Rationalize denominators and numerators.
- Use properties of rational exponents.

Why you should learn it

Real numbers and algebraic expressions are often written with exponents and radicals. For instance, in Exercise 121 on page 27, you will use an expression involving rational exponents to find the times required for a funnel to empty for different water heights.

Integer Exponents

Repeated *multiplication* can be written in **exponential form.**

Repeated Multiplication	*Exponential Form*
$a \cdot a \cdot a \cdot a \cdot a$	a^5
$(-4)(-4)(-4)$	$(-4)^3$
$(2x)(2x)(2x)(2x)$	$(2x)^4$

Exponential Notation

If a is a real number and n is a positive integer, then

$$a^n = \underbrace{a \cdot a \cdot a \cdot \cdot \cdot a}_{n \text{ factors}}$$

where n is the **exponent** and a is the **base.** The expression a^n is read "a to the nth **power.**"

An exponent can also be negative. In Property 3 below, be sure you see how to use a negative exponent.

Properties of Exponents

Let a and b be real numbers, variables, or algebraic expressions, and let m and n be integers. (All denominators and bases are nonzero.)

Property	*Example*
1. $a^m a^n = a^{m+n}$	$3^2 \cdot 3^4 = 3^{2+4} = 3^6 = 729$
2. $\dfrac{a^m}{a^n} = a^{m-n}$	$\dfrac{x^7}{x^4} = x^{7-4} = x^3$
3. $a^{-n} = \dfrac{1}{a^n} = \left(\dfrac{1}{a}\right)^n$	$y^{-4} = \dfrac{1}{y^4} = \left(\dfrac{1}{y}\right)^4$
4. $a^0 = 1, \quad a \neq 0$	$(x^2 + 1)^0 = 1$
5. $(ab)^m = a^m b^m$	$(5x)^3 = 5^3x^3 = 125x^3$
6. $(a^m)^n = a^{mn}$	$(y^3)^{-4} = y^{3(-4)} = y^{-12} = \dfrac{1}{y^{12}}$
7. $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$	$\left(\dfrac{2}{x}\right)^3 = \dfrac{2^3}{x^3} = \dfrac{8}{x^3}$
8. $\lvert a^2 \rvert = \lvert a \rvert^2 = a^2$	$\lvert(-2)^2\rvert = \lvert-2\rvert^2 = (2)^2 = 4$

TECHNOLOGY

You can use a calculator to evaluate exponential expressions. When doing so, it is important to know when to use parentheses because the calculator follows the order of operations. For instance, evaluate $(-2)^4$ as follows.

Scientific:

(2 +/−) yˣ 4 =

Graphing:

((−) 2) ^ 4 ENTER

The display will be 16. If you omit the parentheses, the display will be -16.

It is important to recognize the difference between expressions such as $(-2)^4$ and -2^4. In $(-2)^4$, the parentheses indicate that the exponent applies to the negative sign as well as to the 2, but in $-2^4 = -(2^4)$, the exponent applies only to the 2. So, $(-2)^4 = 16$ and $-2^4 = -16$.

The properties of exponents listed on the preceding page apply to *all* integers m and n, not just to positive integers, as shown in the examples in this section.

Example 1 Evaluating Exponential Expressions

a. $(-5)^2 = (-5)(-5) = 25$ Negative sign is part of the base.

b. $-5^2 = -(5)(5) = -25$ Negative sign is *not* part of the base.

c. $2 \cdot 2^4 = 2^{1+4} = 2^5 = 32$ Property 1

d. $\dfrac{4^4}{4^6} = 4^{4-6} = 4^{-2} = \dfrac{1}{4^2} = \dfrac{1}{16}$ Properties 2 and 3

CHECK*Point* Now try Exercise 11.

Example 2 Evaluating Algebraic Expressions

Evaluate each algebraic expression when $x = 3$.

a. $5x^{-2}$ **b.** $\frac{1}{3}(-x)^3$

Solution

a. When $x = 3$, the expression $5x^{-2}$ has a value of

$$5x^{-2} = 5(3)^{-2} = \frac{5}{3^2} = \frac{5}{9}.$$

b. When $x = 3$, the expression $\frac{1}{3}(-x)^3$ has a value of

$$\frac{1}{3}(-x)^3 = \frac{1}{3}(-3)^3 = \frac{1}{3}(-27) = -9.$$

CHECK*Point* Now try Exercise 23.

Example 3 Using Properties of Exponents

Use the properties of exponents to simplify each expression.

a. $(-3ab^4)(4ab^{-3})$ **b.** $(2xy^2)^3$ **c.** $3a(-4a^2)^0$ **d.** $\left(\dfrac{5x^3}{y}\right)^2$

Solution

a. $(-3ab^4)(4ab^{-3}) = (-3)(4)(a)(a)(b^4)(b^{-3}) = -12a^2b$

b. $(2xy^2)^3 = 2^3(x)^3(y^2)^3 = 8x^3y^6$

c. $3a(-4a^2)^0 = 3a(1) = 3a, \quad a \neq 0$

d. $\left(\dfrac{5x^3}{y}\right)^2 = \dfrac{5^2(x^3)^2}{y^2} = \dfrac{25x^6}{y^2}$

CHECK*Point* Now try Exercise 31.

Study Tip

Rarely in algebra is there only one way to solve a problem. Don't be concerned if the steps you use to solve a problem are not exactly the same as the steps presented in this text. The important thing is to use steps that you understand and, of course, steps that are justified by the rules of algebra. For instance, you might prefer the following steps for Example 4(d).

$$\left(\frac{3x^2}{y}\right)^{-2} = \left(\frac{y}{3x^2}\right)^2 = \frac{y^2}{9x^4}$$

Note how Property 3 is used in the first step of this solution. The fractional form of this property is

$$\left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^m.$$

Example 4 Rewriting with Positive Exponents

Rewrite each expression with positive exponents.

a. x^{-1} **b.** $\dfrac{1}{3x^{-2}}$ **c.** $\dfrac{12a^3b^{-4}}{4a^{-2}b}$ **d.** $\left(\dfrac{3x^2}{y}\right)^{-2}$

Solution

a. $x^{-1} = \dfrac{1}{x}$ — Property 3

b. $\dfrac{1}{3x^{-2}} = \dfrac{1(x^2)}{3} = \dfrac{x^2}{3}$ — The exponent -2 does not apply to 3.

c. $\dfrac{12a^3b^{-4}}{4a^{-2}b} = \dfrac{12a^3 \cdot a^2}{4b \cdot b^4}$ — Property 3

$= \dfrac{3a^5}{b^5}$ — Property 1

d. $\left(\dfrac{3x^2}{y}\right)^{-2} = \dfrac{3^{-2}(x^2)^{-2}}{y^{-2}}$ — Properties 5 and 7

$= \dfrac{3^{-2}x^{-4}}{y^{-2}}$ — Property 6

$= \dfrac{y^2}{3^2x^4}$ — Property 3

$= \dfrac{y^2}{9x^4}$ — Simplify.

CHECKPoint Now try Exercise 41.

Scientific Notation

Exponents provide an efficient way of writing and computing with very large (or very small) numbers. For instance, there are about 359 billion billion gallons of water on Earth—that is, 359 followed by 18 zeros.

359,000,000,000,000,000,000

It is convenient to write such numbers in **scientific notation.** This notation has the form $\pm c \times 10^n$, where $1 \le c < 10$ and n is an integer. So, the number of gallons of water on Earth can be written in scientific notation as

$$3.59 \times 100{,}000{,}000{,}000{,}000{,}000{,}000 = 3.59 \times 10^{20}.$$

The *positive* exponent 20 indicates that the number is *large* (10 or more) and that the decimal point has been moved 20 places. A *negative* exponent indicates that the number is *small* (less than 1). For instance, the mass (in grams) of one electron is approximately

$$9.0 \times 10^{-28} = 0.\underbrace{0000000000000000000000000009}_{\text{28 decimal places}}.$$

HISTORICAL NOTE

The French mathematician Nicolas Chuquet (ca. 1500) wrote *Triparty en la science des nombres*, in which a form of exponent notation was used. Our expressions $6x^3$ and $10x^2$ were written as $.6.^3$ and $.10.^2$. Zero and negative exponents were also represented, so x^0 would be written as $.1.^0$ and $3x^{-2}$ as $.3.^{2m}$. Chuquet wrote that $.72.^1$ divided by $.8.^3$ is $.9.^{2m}$. That is, $72x \div 8x^3 = 9x^{-2}$.

Example 5 Scientific Notation

Write each number in scientific notation.

a. 0.0000782 **b.** 836,100,000

Solution

a. $0.0000782 = 7.82 \times 10^{-5}$

b. $836{,}100{,}000 = 8.361 \times 10^{8}$

CHECK Point Now try Exercise 45.

Example 6 Decimal Notation

Write each number in decimal notation.

a. -9.36×10^{-6} **b.** 1.345×10^{2}

Solution

a. $-9.36 \times 10^{-6} = -0.00000936$ **b.** $1.345 \times 10^{2} = 134.5$

CHECK Point Now try Exercise 55.

TECHNOLOGY

Most calculators automatically switch to scientific notation when they are showing large (or small) numbers that exceed the display range.

To *enter* numbers in scientific notation, your calculator should have an exponential entry key labeled

$\boxed{\text{EE}}$ or $\boxed{\text{EXP}}$.

Consult the user's guide for your calculator for instructions on keystrokes and how numbers in scientific notation are displayed.

Example 7 Using Scientific Notation

Evaluate $\dfrac{(2{,}400{,}000{,}000)(0.0000045)}{(0.00003)(1500)}$.

Solution

Begin by rewriting each number in scientific notation and simplifying.

$$\begin{aligned}
\frac{(2{,}400{,}000{,}000)(0.0000045)}{(0.00003)(1500)} &= \frac{(2.4 \times 10^{9})(4.5 \times 10^{-6})}{(3.0 \times 10^{-5})(1.5 \times 10^{3})} \\
&= \frac{(2.4)(4.5)(10^{3})}{(4.5)(10^{-2})} \\
&= (2.4)(10^{5}) \\
&= 240{,}000
\end{aligned}$$

CHECK Point Now try Exercise 63(b).

Radicals and Their Properties

A **square root** of a number is one of its two equal factors. For example, 5 is a square root of 25 because 5 is one of the two equal factors of 25. In a similar way, a **cube root** of a number is one of its three equal factors, as in $125 = 5^3$.

Definition of *n*th Root of a Number

Let a and b be real numbers and let $n \geq 2$ be a positive integer. If

$$a = b^n$$

then b is an ***n*th root of *a*.** If $n = 2$, the root is a **square root.** If $n = 3$, the root is a **cube root.**

Some numbers have more than one nth root. For example, both 5 and -5 are square roots of 25. The *principal square root* of 25, written as $\sqrt{25}$, is the positive root, 5. The **principal *n*th root** of a number is defined as follows.

Principal *n*th Root of a Number

Let a be a real number that has at least one nth root. The **principal *n*th root of *a*** is the nth root that has the same sign as a. It is denoted by a **radical symbol**

$$\sqrt[n]{a}. \qquad \text{Principal } n\text{th root}$$

The positive integer n is the **index** of the radical, and the number a is the **radicand.** If $n = 2$, omit the index and write $\sqrt{a}$ rather than $\sqrt[2]{a}$. (The plural of index is *indices*.)

A common misunderstanding is that the square root sign implies both negative and positive roots. This is not correct. The square root sign implies only a positive root. When a negative root is needed, you must use the negative sign with the square root sign.

Incorrect: $\cancel{\sqrt{4} = \pm 2}$ *Correct:* $-\sqrt{4} = -2$ *and* $\sqrt{4} = 2$

Example 8 Evaluating Expressions Involving Radicals

a. $\sqrt{36} = 6$ because $6^2 = 36$.

b. $-\sqrt{36} = -6$ because $-(\sqrt{36}) = -(\sqrt{6^2}) = -(6) = -6$.

c. $\sqrt[3]{\dfrac{125}{64}} = \dfrac{5}{4}$ because $\left(\dfrac{5}{4}\right)^3 = \dfrac{5^3}{4^3} = \dfrac{125}{64}$.

d. $\sqrt[5]{-32} = -2$ because $(-2)^5 = -32$.

e. $\sqrt[4]{-81}$ is not a real number because there is no real number that can be raised to the fourth power to produce -81.

CHECK*Point* Now try Exercise 65.

Here are some generalizations about the nth roots of real numbers.

Generalizations About nth Roots of Real Numbers

Real Number a	Integer n	Root(s) of a	Example
$a > 0$	$n > 0$, n is even.	$\sqrt[n]{a},\ -\sqrt[n]{a}$	$\sqrt[4]{81} = 3,\ -\sqrt[4]{81} = -3$
$a > 0$ or $a < 0$	n is odd.	$\sqrt[n]{a}$	$\sqrt[3]{-8} = -2$
$a < 0$	n is even.	No real roots	$\sqrt{-4}$ is not a real number.
$a = 0$	n is even or odd.	$\sqrt[n]{0} = 0$	$\sqrt[5]{0} = 0$

Integers such as 1, 4, 9, 16, 25, and 36 are called **perfect squares** because they have integer square roots. Similarly, integers such as 1, 8, 27, 64, and 125 are called **perfect cubes** because they have integer cube roots.

Properties of Radicals

Let a and b be real numbers, variables, or algebraic expressions such that the indicated roots are real numbers, and let m and n be positive integers.

Property	*Example*
1. $\sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$	$\sqrt[3]{8^2} = \left(\sqrt[3]{8}\right)^2 = (2)^2 = 4$
2. $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$	$\sqrt{5} \cdot \sqrt{7} = \sqrt{5 \cdot 7} = \sqrt{35}$
3. $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}},\quad b \neq 0$	$\dfrac{\sqrt[4]{27}}{\sqrt[4]{9}} = \sqrt[4]{\dfrac{27}{9}} = \sqrt[4]{3}$
4. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$	$\sqrt[3]{\sqrt{10}} = \sqrt[6]{10}$
5. $\left(\sqrt[n]{a}\right)^n = a$	$\left(\sqrt{3}\right)^2 = 3$
6. For n even, $\sqrt[n]{a^n} = \lvert a\rvert$.	$\sqrt{(-12)^2} = \lvert -12\rvert = 12$
For n odd, $\sqrt[n]{a^n} = a$.	$\sqrt[3]{(-12)^3} = -12$

A common special case of Property 6 is $\sqrt{a^2} = \lvert a\rvert$.

Example 9 Using Properties of Radicals

Use the properties of radicals to simplify each expression.

a. $\sqrt{8} \cdot \sqrt{2}$ **b.** $\left(\sqrt[3]{5}\right)^3$ **c.** $\sqrt[3]{x^3}$ **d.** $\sqrt[6]{y^6}$

Solution

a. $\sqrt{8} \cdot \sqrt{2} = \sqrt{8 \cdot 2} = \sqrt{16} = 4$

b. $\left(\sqrt[3]{5}\right)^3 = 5$

c. $\sqrt[3]{x^3} = x$

d. $\sqrt[6]{y^6} = \lvert y\rvert$

CHECK*Point* Now try Exercise 77.

Simplifying Radicals

An expression involving radicals is in **simplest form** when the following conditions are satisfied.

1. All possible factors have been removed from the radical.
2. All fractions have radical-free denominators (accomplished by a process called *rationalizing the denominator*).
3. The index of the radical is reduced.

To simplify a radical, factor the radicand into factors whose exponents are multiples of the index. The roots of these factors are written outside the radical, and the "leftover" factors make up the new radicand.

WARNING / CAUTION

When you simplify a radical, it is important that both expressions are defined for the same values of the variable. For instance, in Example 10(b), $\sqrt{75x^3}$ and $5x\sqrt{3x}$ are both defined only for nonnegative values of x. Similarly, in Example 10(c), $\sqrt[4]{(5x)^4}$ and $5|x|$ are both defined for all real values of x.

Example 10 Simplifying Even Roots

a. $\sqrt[4]{48} = \sqrt[4]{\overset{\text{Perfect 4th power}}{16} \cdot \overset{\text{Leftover factor}}{3}} = \sqrt[4]{2^4 \cdot 3} = 2\sqrt[4]{3}$

b. $\sqrt{75x^3} = \sqrt{\overset{\text{Perfect square}}{25x^2} \cdot \overset{\text{Leftover factor}}{3x}}$ Find largest square factor.

$= \sqrt{(5x)^2 \cdot 3x}$

$= 5x\sqrt{3x}$ Find root of perfect square.

c. $\sqrt[4]{(5x)^4} = |5x| = 5|x|$

CHECK*Point* Now try Exercise 79(a).

Example 11 Simplifying Odd Roots

a. $\sqrt[3]{24} = \sqrt[3]{\overset{\text{Perfect cube}}{8} \cdot \overset{\text{Leftover factor}}{3}} = \sqrt[3]{2^3 \cdot 3} = 2\sqrt[3]{3}$

b. $\sqrt[3]{24a^4} = \sqrt[3]{\overset{\text{Perfect cube}}{8a^3} \cdot \overset{\text{Leftover factor}}{3a}}$ Find largest cube factor.

$= \sqrt[3]{(2a)^3 \cdot 3a}$

$= 2a\sqrt[3]{3a}$ Find root of perfect cube.

c. $\sqrt[3]{-40x^6} = \sqrt[3]{(-8x^6) \cdot 5}$ Find largest cube factor.

$= \sqrt[3]{(-2x^2)^3 \cdot 5}$

$= -2x^2\sqrt[3]{5}$ Find root of perfect cube.

CHECK*Point* Now try Exercise 79(b).

Radical expressions can be combined (added or subtracted) if they are **like radicals**—that is, if they have the same index and radicand. For instance, $\sqrt{2}$, $3\sqrt{2}$, and $\frac{1}{2}\sqrt{2}$ are like radicals, but $\sqrt{3}$ and $\sqrt{2}$ are unlike radicals. To determine whether two radicals can be combined, you should first simplify each radical.

Example 12 Combining Radicals

a. $2\sqrt{48} - 3\sqrt{27} = 2\sqrt{16 \cdot 3} - 3\sqrt{9 \cdot 3}$ Find square factors.

$= 8\sqrt{3} - 9\sqrt{3}$ Find square roots and multiply by coefficients.

$= (8 - 9)\sqrt{3}$ Combine like terms.

$= -\sqrt{3}$ Simplify.

b. $\sqrt[3]{16x} - \sqrt[3]{54x^4} = \sqrt[3]{8 \cdot 2x} - \sqrt[3]{27 \cdot x^3 \cdot 2x}$ Find cube factors.

$= 2\sqrt[3]{2x} - 3x\sqrt[3]{2x}$ Find cube roots.

$= (2 - 3x)\sqrt[3]{2x}$ Combine like terms.

CHECK*Point* Now try Exercise 87.

Rationalizing Denominators and Numerators

To rationalize a denominator or numerator of the form $a - b\sqrt{m}$ or $a + b\sqrt{m}$, multiply both numerator and denominator by a **conjugate:** $a + b\sqrt{m}$ and $a - b\sqrt{m}$ are conjugates of each other. If $a = 0$, then the rationalizing factor for $\sqrt{m}$ is itself, $\sqrt{m}$. For cube roots, choose a rationalizing factor that generates a perfect cube.

Example 13 Rationalizing Single-Term Denominators

Rationalize the denominator of each expression.

a. $\dfrac{5}{2\sqrt{3}}$ **b.** $\dfrac{2}{\sqrt[3]{5}}$

Solution

a. $\dfrac{5}{2\sqrt{3}} = \dfrac{5}{2\sqrt{3}} \cdot \dfrac{\sqrt{3}}{\sqrt{3}}$ $\sqrt{3}$ is rationalizing factor.

$= \dfrac{5\sqrt{3}}{2(3)}$ Multiply.

$= \dfrac{5\sqrt{3}}{6}$ Simplify.

b. $\dfrac{2}{\sqrt[3]{5}} = \dfrac{2}{\sqrt[3]{5}} \cdot \dfrac{\sqrt[3]{5^2}}{\sqrt[3]{5^2}}$ $\sqrt[3]{5^2}$ is rationalizing factor.

$= \dfrac{2\sqrt[3]{5^2}}{\sqrt[3]{5^3}}$ Multiply.

$= \dfrac{2\sqrt[3]{25}}{5}$ Simplify.

CHECK*Point* Now try Exercise 95.

Example 14 Rationalizing a Denominator with Two Terms

$$\frac{2}{3+\sqrt{7}} = \frac{2}{3+\sqrt{7}} \cdot \frac{3-\sqrt{7}}{3-\sqrt{7}}$$ Multiply numerator and denominator by conjugate of denominator.

$$= \frac{2(3-\sqrt{7})}{3(3)+3(-\sqrt{7})+\sqrt{7}(3)-(\sqrt{7})(\sqrt{7})}$$ Use Distributive Property.

$$= \frac{2(3-\sqrt{7})}{(3)^2-(\sqrt{7})^2}$$ Simplify.

$$= \frac{2(3-\sqrt{7})}{9-7}$$ Square terms of denominator.

$$= \frac{2(3-\sqrt{7})}{2} = 3-\sqrt{7}$$ Simplify.

CHECKPoint Now try Exercise 97.

Sometimes it is necessary to rationalize the numerator of an expression. For instance, in Section P.5 you will use the technique shown in the next example to rationalize the numerator of an expression from calculus.

WARNING / CAUTION

Do not confuse the expression $\sqrt{5}+\sqrt{7}$ with the expression $\sqrt{5+7}$. In general, $\sqrt{x+y}$ does not equal $\sqrt{x}+\sqrt{y}$. Similarly, $\sqrt{x^2+y^2}$ does not equal $x+y$.

Example 15 Rationalizing a Numerator

$$\frac{\sqrt{5}-\sqrt{7}}{2} = \frac{\sqrt{5}-\sqrt{7}}{2} \cdot \frac{\sqrt{5}+\sqrt{7}}{\sqrt{5}+\sqrt{7}}$$ Multiply numerator and denominator by conjugate of numerator.

$$= \frac{(\sqrt{5})^2-(\sqrt{7})^2}{2(\sqrt{5}+\sqrt{7})}$$ Simplify.

$$= \frac{5-7}{2(\sqrt{5}+\sqrt{7})}$$ Square terms of numerator.

$$= \frac{-2}{2(\sqrt{5}+\sqrt{7})} = \frac{-1}{\sqrt{5}+\sqrt{7}}$$ Simplify.

CHECKPoint Now try Exercise 101.

Rational Exponents

Definition of Rational Exponents

If a is a real number and n is a positive integer such that the principal nth root of a exists, then $a^{1/n}$ is defined as

$a^{1/n} = \sqrt[n]{a}$, where $1/n$ is the **rational exponent** of a.

Moreover, if m is a positive integer that has no common factor with n, then

$$a^{m/n} = (a^{1/n})^m = \left(\sqrt[n]{a}\right)^m \quad \text{and} \quad a^{m/n} = (a^m)^{1/n} = \sqrt[n]{a^m}.$$

The symbol ∫ indicates an example or exercise that highlights algebraic techniques specifically used in calculus.

WARNING / CAUTION

Rational exponents can be tricky, and you must remember that the expression $b^{m/n}$ is not defined unless $\sqrt[n]{b}$ is a real number. This restriction produces some unusual-looking results. For instance, the number $(-8)^{1/3}$ is defined because $\sqrt[3]{-8} = -2$, but the number $(-8)^{2/6}$ is undefined because $\sqrt[6]{-8}$ is not a real number.

The numerator of a rational exponent denotes the *power* to which the base is raised, and the denominator denotes the *index* or the *root* to be taken.

Power

Index

$$b^{m/n} = \left(\sqrt[n]{b}\right)^m = \sqrt[n]{b^m}$$

When you are working with rational exponents, the properties of integer exponents still apply. For instance, $2^{1/2}2^{1/3} = 2^{(1/2)+(1/3)} = 2^{5/6}$.

Example 16 Changing From Radical to Exponential Form

a. $\sqrt{3} = 3^{1/2}$

b. $\sqrt{(3xy)^5} = \sqrt[2]{(3xy)^5} = (3xy)^{5/2}$

c. $2x\sqrt[4]{x^3} = (2x)(x^{3/4}) = 2x^{1+(3/4)} = 2x^{7/4}$

CHECKPoint Now try Exercise 103.

Example 17 Changing From Exponential to Radical Form

a. $(x^2 + y^2)^{3/2} = \left(\sqrt{x^2 + y^2}\right)^3 = \sqrt{(x^2 + y^2)^3}$

b. $2y^{3/4}z^{1/4} = 2(y^3z)^{1/4} = 2\sqrt[4]{y^3z}$

c. $a^{-3/2} = \dfrac{1}{a^{3/2}} = \dfrac{1}{\sqrt{a^3}}$

d. $x^{0.2} = x^{1/5} = \sqrt[5]{x}$

CHECKPoint Now try Exercise 105.

TECHNOLOGY

There are four methods of evaluating radicals on most graphing calculators. For square roots, you can use the *square root key* $\boxed{\sqrt{\ }}$. For cube roots, you can use the *cube root key* $\boxed{\sqrt[3]{\ }}$. For other roots, you can first convert the radical to exponential form and then use the *exponential key* $\boxed{\wedge}$, or you can use the *x*th root key $\boxed{\sqrt[x]{\ }}$ (or menu choice). Consult the user's guide for your calculator for specific keystrokes.

Rational exponents are useful for evaluating roots of numbers on a calculator, for reducing the index of a radical, and for simplifying expressions in calculus.

Example 18 Simplifying with Rational Exponents

a. $(-32)^{-4/5} = \left(\sqrt[5]{-32}\right)^{-4} = (-2)^{-4} = \dfrac{1}{(-2)^4} = \dfrac{1}{16}$

b. $(-5x^{5/3})(3x^{-3/4}) = -15x^{(5/3)-(3/4)} = -15x^{11/12}, \quad x \neq 0$

c. $\sqrt[9]{a^3} = a^{3/9} = a^{1/3} = \sqrt[3]{a}$ Reduce index.

d. $\sqrt[3]{\sqrt{125}} = \sqrt[6]{125} = \sqrt[6]{(5)^3} = 5^{3/6} = 5^{1/2} = \sqrt{5}$

e. $(2x - 1)^{4/3}(2x - 1)^{-1/3} = (2x - 1)^{(4/3)-(1/3)}$

$$= 2x - 1, \qquad x \neq \frac{1}{2}$$

CHECKPoint Now try Exercise 115.

The expression in Example 18(e) is not defined when $x = \frac{1}{2}$ because

$$\left(2 \cdot \frac{1}{2} - 1\right)^{-1/3} = (0)^{-1/3}$$

is not a real number.

P.2 EXERCISES

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

VOCABULARY: Fill in the blanks.

1. In the exponential form a^n, n is the ________ and a is the ________.
2. A convenient way of writing very large or very small numbers is called ________ ________.
3. One of the two equal factors of a number is called a ________ ________ of the number.
4. The ________ ________ ________ of a number a is the nth root that has the same sign as a, and is denoted by $\sqrt[n]{a}$.
5. In the radical form $\sqrt[n]{a}$, the positive integer n is called the ________ of the radical and the number a is called the ________.
6. When an expression involving radicals has all possible factors removed, radical-free denominators, and a reduced index, it is in ________ ________.
7. Radical expressions can be combined (added or subtracted) if they are ________ ________.
8. The expressions $a + b\sqrt{m}$ and $a - b\sqrt{m}$ are ________ of each other.
9. The process used to create a radical-free denominator is known as ________ the denominator.
10. In the expression $b^{m/n}$, m denotes the ________ to which the base is raised and n denotes the ________ or root to be taken.

SKILLS AND APPLICATIONS

In Exercises 11–18, evaluate each expression.

11. (a) $3^2 \cdot 3$ (b) $3 \cdot 3^3$
12. (a) $\dfrac{5^5}{5^2}$ (b) $\dfrac{3^2}{3^4}$
13. (a) $(3^3)^0$ (b) -3^2
14. (a) $(2^3 \cdot 3^2)^2$ (b) $\left(-\frac{3}{5}\right)^3\left(\frac{5}{3}\right)^2$
15. (a) $\dfrac{3}{3^{-4}}$ (b) $48(-4)^{-3}$
16. (a) $\dfrac{4 \cdot 3^{-2}}{2^{-2} \cdot 3^{-1}}$ (b) $(-2)^0$
17. (a) $2^{-1} + 3^{-1}$ (b) $(2^{-1})^{-2}$
18. (a) $3^{-1} + 2^{-2}$ (b) $(3^{-2})^2$

In Exercises 19–22, use a calculator to evaluate the expression. (If necessary, round your answer to three decimal places.)

19. $(-4)^3(5^2)$
20. $(8^{-4})(10^3)$
21. $\dfrac{3^6}{7^3}$
22. $\dfrac{4^3}{3^{-4}}$

In Exercises 23–30, evaluate the expression for the given value of x.

23. $-3x^3$, $x = 2$
24. $7x^{-2}$, $x = 4$
25. $6x^0$, $x = 10$
26. $5(-x)^3$, $x = 3$
27. $2x^3$, $x = -3$
28. $-3x^4$, $x = -2$
29. $-20x^2$, $x = -\frac{1}{2}$
30. $12(-x)^3$, $x = -\frac{1}{3}$

In Exercises 31–38, simplify each expression.

31. (a) $(-5z)^3$ (b) $5x^4(x^2)$
32. (a) $(3x)^2$ (b) $(4x^3)^0$, $x \neq 0$
33. (a) $6y^2(2y^0)^2$ (b) $\dfrac{3x^5}{x^3}$
34. (a) $(-z)^3(3z^4)$ (b) $\dfrac{25y^8}{10y^4}$
35. (a) $\dfrac{7x^2}{x^3}$ (b) $\dfrac{12(x+y)^3}{9(x+y)}$
36. (a) $\dfrac{r^4}{r^6}$ (b) $\left(\dfrac{4}{y}\right)^3\left(\dfrac{3}{y}\right)^4$
37. (a) $[(x^2y^{-2})^{-1}]^{-1}$ (b) $\left(\dfrac{a^{-2}}{b^{-2}}\right)\left(\dfrac{b}{a}\right)^3$
38. (a) $(6x^7)^0$, $x \neq 0$ (b) $(5x^2z^6)^3(5x^2z^6)^{-3}$

In Exercises 39–44, rewrite each expression with positive exponents and simplify.

39. (a) $(x+5)^0$, $x \neq -5$ (b) $(2x^2)^{-2}$
40. (a) $(2x^5)^0$, $x \neq 0$ (b) $(z+2)^{-3}(z+2)^{-1}$
41. (a) $(-2x^2)^3(4x^3)^{-1}$ (b) $\left(\dfrac{x}{10}\right)^{-1}$
42. (a) $(4y^{-2})(8y^4)$ (b) $\left(\dfrac{x^{-3}y^4}{5}\right)^{-3}$
43. (a) $3^n \cdot 3^{2n}$ (b) $\left(\dfrac{a^{-2}}{b^{-2}}\right)\left(\dfrac{b}{a}\right)^3$
44. (a) $\dfrac{x^2 \cdot x^n}{x^3 \cdot x^n}$ (b) $\left(\dfrac{a^{-3}}{b^{-3}}\right)\left(\dfrac{a}{b}\right)^3$

In Exercises 45–52, write the number in scientific notation.

45. 10,250.4

46. $-7{,}280{,}000$

47. -0.000125

48. 0.00052

49. Land area of Earth: 57,300,000 square miles

50. Light year: 9,460,000,000,000 kilometers

51. Relative density of hydrogen: 0.0000899 gram per cubic centimeter

52. One micron (millionth of a meter): 0.00003937 inch

In Exercises 53–60, write the number in decimal notation.

53. 1.25×10^5

54. -1.801×10^5

55. -2.718×10^{-3}

56. 3.14×10^{-4}

57. Interior temperature of the sun: 1.5×10^7 degrees Celsius

58. Charge of an electron: 1.6022×10^{-19} coulomb

59. Width of a human hair: 9.0×10^{-5} meter

60. Gross domestic product of the United States in 2007: 1.3743021×10^{13} dollars (Source: U.S. Department of Commerce)

In Exercises 61 and 62, evaluate each expression without using a calculator.

61. (a) $(2.0 \times 10^9)(3.4 \times 10^{-4})$
(b) $(1.2 \times 10^7)(5.0 \times 10^{-3})$

62. (a) $\dfrac{6.0 \times 10^8}{3.0 \times 10^{-3}}$ (b) $\dfrac{2.5 \times 10^{-3}}{5.0 \times 10^2}$

In Exercises 63 and 64, use a calculator to evaluate each expression. (Round your answer to three decimal places.)

63. (a) $750\left(1 + \dfrac{0.11}{365}\right)^{800}$
(b) $\dfrac{67{,}000{,}000 + 93{,}000{,}000}{0.0052}$

64. (a) $(9.3 \times 10^6)^3(6.1 \times 10^{-4})$ (b) $\dfrac{(2.414 \times 10^4)^6}{(1.68 \times 10^5)^5}$

In Exercises 65–70, evaluate each expression without using a calculator.

65. (a) $\sqrt{9}$ (b) $\sqrt[3]{\frac{27}{8}}$

66. (a) $27^{1/3}$ (b) $36^{3/2}$

67. (a) $32^{-3/5}$ (b) $\left(\frac{16}{81}\right)^{-3/4}$

68. (a) $100^{-3/2}$ (b) $\left(\frac{9}{4}\right)^{-1/2}$

69. (a) $\left(-\dfrac{1}{64}\right)^{-1/3}$ (b) $\left(\dfrac{1}{\sqrt{32}}\right)^{-2/5}$

70. (a) $\left(-\dfrac{125}{27}\right)^{-1/3}$ (b) $-\left(\dfrac{1}{125}\right)^{-4/3}$

In Exercises 71–76, use a calculator to approximate the number. (Round your answer to three decimal places.)

71. (a) $\sqrt{57}$ (b) $\sqrt[5]{-27^3}$

72. (a) $\sqrt[3]{45^2}$ (b) $\sqrt[6]{125}$

73. (a) $(-12.4)^{-1.8}$ (b) $(5\sqrt{3})^{-2.5}$

74. (a) $\dfrac{7 - (4.1)^{-3.2}}{2}$ (b) $\left(\dfrac{13}{3}\right)^{-3/2} - \left(-\dfrac{3}{2}\right)^{13/3}$

75. (a) $\sqrt{4.5 \times 10^9}$ (b) $\sqrt[3]{6.3 \times 10^4}$

76. (a) $(2.65 \times 10^{-4})^{1/3}$ (b) $\sqrt{9 \times 10^{-4}}$

In Exercises 77 and 78, use the properties of radicals to simplify each expression.

77. (a) $(\sqrt[5]{2})^5$ (b) $\sqrt[5]{96x^5}$

78. (a) $\sqrt{12} \cdot \sqrt{3}$ (b) $\sqrt[4]{(3x^2)^4}$

In Exercises 79–90, simplify each radical expression.

79. (a) $\sqrt{20}$ (b) $\sqrt[3]{128}$

80. (a) $\sqrt[3]{\frac{16}{27}}$ (b) $\sqrt{\frac{75}{4}}$

81. (a) $\sqrt{72x^3}$ (b) $\sqrt{\dfrac{18^2}{z^3}}$

82. (a) $\sqrt{54xy^4}$ (b) $\sqrt{\dfrac{32a^4}{b^2}}$

83. (a) $\sqrt[3]{16x^5}$ (b) $\sqrt{75x^2y^{-4}}$

84. (a) $\sqrt[4]{3x^4y^2}$ (b) $\sqrt[5]{160x^8z^4}$

85. (a) $2\sqrt{50} + 12\sqrt{8}$ (b) $10\sqrt{32} - 6\sqrt{18}$

86. (a) $4\sqrt{27} - \sqrt{75}$ (b) $\sqrt[3]{16} + 3\sqrt[3]{54}$

87. (a) $5\sqrt{x} - 3\sqrt{x}$ (b) $-2\sqrt{9y} + 10\sqrt{y}$

88. (a) $8\sqrt{49x} - 14\sqrt{100x}$
(b) $-3\sqrt{48x^2} + 7\sqrt{75x^2}$

89. (a) $3\sqrt{x+1} + 10\sqrt{x+1}$
(b) $7\sqrt{80x} - 2\sqrt{125x}$

90. (a) $-\sqrt{x^3 - 7} + 5\sqrt{x^3 - 7}$
(b) $11\sqrt{245x^3} - 9\sqrt{45x^3}$

In Exercises 91–94, complete the statement with $<$, $=$, or $>$.

91. $\sqrt{5} + \sqrt{3} \ \boxed{}\ \sqrt{5+3}$

92. $\sqrt{\dfrac{3}{11}} \ \boxed{}\ \dfrac{\sqrt{3}}{\sqrt{11}}$

93. $5 \ \boxed{}\ \sqrt{3^2 + 2^2}$

94. $5 \ \boxed{}\ \sqrt{3^2 + 4^2}$

In Exercises 95–98, rationalize the denominator of the expression. Then simplify your answer.

95. $\dfrac{1}{\sqrt{3}}$

96. $\dfrac{8}{\sqrt[3]{2}}$

97. $\dfrac{5}{\sqrt{14} - 2}$

98. $\dfrac{3}{\sqrt{5} + \sqrt{6}}$

In Exercises 99–102, rationalize the numerator of the expression. Then simplify your answer.

99. $\dfrac{\sqrt{8}}{2}$ **100.** $\dfrac{\sqrt{2}}{3}$

101. $\dfrac{\sqrt{5}+\sqrt{3}}{3}$ **102.** $\dfrac{\sqrt{7}-3}{4}$

In Exercises 103–110, fill in the missing form of the expression.

	Radical Form	*Rational Exponent Form*
103.	$\sqrt{2.5}$	
104.	$\sqrt[3]{64}$	
105.		$81^{1/4}$
106.		$-(144^{1/2})$
107.	$\sqrt[3]{-216}$	
108.		$(-243)^{1/5}$
109.	$\left(\sqrt[4]{81}\right)^3$	
110.		$16^{5/4}$

In Exercises 111–114, perform the operations and simplify.

111. $\dfrac{(2x^2)^{3/2}}{2^{1/2}x^4}$ **112.** $\dfrac{x^{4/3}y^{2/3}}{(xy)^{1/3}}$

113. $\dfrac{x^{-3}\cdot x^{1/2}}{x^{3/2}\cdot x^{-1}}$ **114.** $\dfrac{5^{-1/2}\cdot 5x^{5/2}}{(5x)^{3/2}}$

In Exercises 115 and 116, reduce the index of each radical.

115. (a) $\sqrt[4]{3^2}$ (b) $\sqrt[6]{(x+1)^4}$

116. (a) $\sqrt[6]{x^3}$ (b) $\sqrt[4]{(3x^2)^4}$

In Exercises 117 and 118, write each expression as a single radical. Then simplify your answer.

117. (a) $\sqrt{\sqrt{32}}$ (b) $\sqrt{\sqrt[4]{2x}}$

118. (a) $\sqrt{\sqrt{243(x+1)}}$ (b) $\sqrt{\sqrt[3]{10a^7b}}$

119. PERIOD OF A PENDULUM The period T (in seconds) of a pendulum is $T = 2\pi\sqrt{L/32}$, where L is the length of the pendulum (in feet). Find the period of a pendulum whose length is 2 feet.

120. EROSION A stream of water moving at the rate of v feet per second can carry particles of size $0.03\sqrt{v}$ inches. Find the size of the largest particle that can be carried by a stream flowing at the rate of $\frac{3}{4}$ foot per second.

The symbol indicates an example or exercise that highlights algebraic techniques specifically used in calculus.

The symbol indicates an exercise or a part of an exercise in which you are instructed to use a graphing utility.

121. MATHEMATICAL MODELING A funnel is filled with water to a height of h centimeters. The formula

$$t = 0.03[12^{5/2} - (12-h)^{5/2}], \quad 0 \le h \le 12$$

represents the amount of time t (in seconds) that it will take for the funnel to empty.

(a) Use the *table* feature of a graphing utility to find the times required for the funnel to empty for water heights of $h = 0$, $h = 1$, $h = 2$, . . . , $h = 12$ centimeters.

(b) What value does t appear to be approaching as the height of the water becomes closer and closer to 12 centimeters?

122. SPEED OF LIGHT The speed of light is approximately 11,180,000 miles per minute. The distance from the sun to Earth is approximately 93,000,000 miles. Find the time for light to travel from the sun to Earth.

EXPLORATION

TRUE OR FALSE? In Exercises 123 and 124, determine whether the statement is true or false. Justify your answer.

123. $\dfrac{x^{k+1}}{x} = x^k$ **124.** $(a^n)^k = a^{n^k}$

125. Verify that $a^0 = 1$, $a \ne 0$. (*Hint:* Use the property of exponents $a^m/a^n = a^{m-n}$.)

126. Explain why each of the following pairs is not equal.

(a) $(3x)^{-1} \ne \dfrac{3}{x}$ (b) $y^3 \cdot y^2 \ne y^6$

(c) $(a^2b^3)^4 \ne a^6b^7$ (d) $(a+b)^2 \ne a^2 + b^2$

(e) $\sqrt{4x^2} \ne 2x$ (f) $\sqrt{2} + \sqrt{3} \ne \sqrt{5}$

127. THINK ABOUT IT Is 52.7×10^5 written in scientific notation? Why or why not?

128. List all possible digits that occur in the units place of the square of a positive integer. Use that list to determine whether $\sqrt{5233}$ is an integer.

129. THINK ABOUT IT Square the real number $5/\sqrt{3}$ and note that the radical is eliminated from the denominator. Is this equivalent to rationalizing the denominator? Why or why not?

130. CAPSTONE

(a) Explain how to simplify the expression $(3x^3y^{-2})^{-2}$.

(b) Is the expression $\sqrt{\dfrac{4}{x^3}}$ in simplest form? Why or why not?