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9 Systems of Equations and Inequalities

Throughout the preceding chapters we modeled real-world situations by equations. But many real-world situations involve too many variables to be modeled by a single equation. For example, weather depends on the relationships among many variables, including temperature, wind speed, air pressure, and humidity. So to model the weather (and forecast a snowstorm like the one pictured above), scientists use many equations, each having many variables. Such collections of equations, called systems of equations, *work together* to describe the weather. Systems of equations with hundreds of variables are used by airlines to establish consistent flight schedules and by telecommunications companies to find efficient routings for telephone calls. In this chapter we learn how to solve systems of equations that consist of several equations in several variables.

9.1 Systems of Linear Equations in Two Variables

■ Systems of Linear Equations and Their Solutions ■ Substitution Method ■ Elimination Method ■ Graphical Method ■ The Number of Solutions of a Linear System in Two Variables ■ Modeling with Linear Systems

■ Systems of Linear Equations and Their Solutions

A linear equation in two variables is an equation of the form

$$ax + by = c$$

The graph of a linear equation is a line (see Section 1.10).

A **system of equations** is a set of equations that involve the same variables. A **system of linear equations** is a system of equations in which each equation is linear. A **solution** of a system is an assignment of values for the variables that makes *each* equation in the system true. To **solve** a system means to find all solutions of the system.

Here is an example of a system of linear equations in two variables:

$$\begin{cases} 2x - y = 5 & \text{Equation 1} \\ x + 4y = 7 & \text{Equation 2} \end{cases}$$

We can check that $x = 3$ and $y = 1$ is a solution of this system.

Equation 1	**Equation 2**
$2x - y = 5$	$x + 4y = 7$
$2(3) - 1 = 5$ ✓	$3 + 4(1) = 7$ ✓

The solution can also be written as the ordered pair $(3, 1)$.

Note that the graphs of Equations 1 and 2 are lines (as shown in Figure 1). Since the solution $(3, 1)$ satisfies each equation, the point $(3, 1)$ lies on each line. So (3,1) is the point of intersection of the two lines.

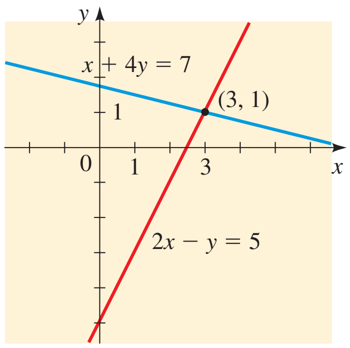


Figure 1

■ Substitution Method

To solve a system using the **substitution method**, we start with one equation in the system and solve for one variable in terms of the other variable.

Substitution Method

1. **Solve for One Variable.** Choose one equation, and solve for one variable in terms of the other variable.
2. **Substitute.** Substitute the expression you found in Step 1 into the other equation to get an equation in one variable, then solve for that variable.
3. **Back-Substitute.** Substitute the value you found in Step 2 back into the expression found in Step 1 to solve for the remaining variable.

Example 1 ■ Substitution Method

Find all solutions of the system.

$$\begin{cases} 2x + y = 1 & \text{Equation 1} \\ 3x + 4y = 14 & \text{Equation 2} \end{cases}$$

Solution **Solve for one variable.** We solve for y in the first equation.

$$y = 1 - 2x \qquad \text{Solve for } y \text{ in Equation 1}$$

Substitute. Now we substitute for y in the second equation and solve for x.

$$\begin{aligned} 3x + 4(1 - 2x) &= 14 && \text{Substitute } y = 1 - 2x \text{ into Equation 2} \\ 3x + 4 - 8x &= 14 && \text{Expand} \\ -5x + 4 &= 14 && \text{Simplify} \\ -5x &= 10 && \text{Subtract 4} \\ x &= -2 && \text{Solve for } x \end{aligned}$$

Back-substitute. Next we back-substitute $x = -2$ into the equation $y = 1 - 2x$.

$$y = 1 - 2(-2) = 5 \qquad \text{Back-substitute}$$

Thus $x = -2$ and $y = 5$, so the solution is the ordered pair $(-2, 5)$. Figure 2 shows that the graphs of the two equations intersect at the point $(-2, 5)$.

Check Your Answer

$x = -2, y = 5$:

$$\begin{cases} 2(-2) + 5 = 1 \\ 3(-2) + 4(5) = 14 \end{cases} \checkmark$$

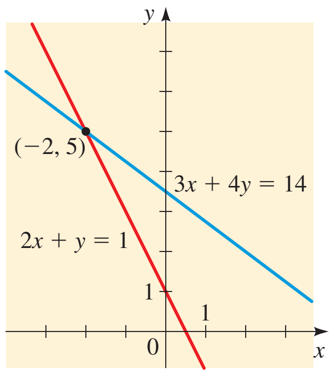


Figure 2

Now Try Exercise 5 ■

■ Elimination Method

To solve a system using the **elimination method**, we try to combine the equations using sums or differences so as to eliminate one of the variables.

Elimination Method

1. **Adjust the Coefficients.** Multiply one or more of the equations by appropriate numbers so that the coefficient of one variable in one equation is the negative of its coefficient in the other equation.
2. **Eliminate a Variable.** Add the two equations to eliminate one variable, then solve for the remaining variable.
3. **Back-Substitute.** Substitute the value that you found in Step 2 back into one of the original equations, and solve for the remaining variable.

Example 2 ■ Elimination Method

Find all solutions of the system.

$$\begin{cases} 3x + 5y = 11 & \text{Equation 1} \\ 2x - y = 3 & \text{Equation 2} \end{cases}$$

Solution **Adjust the coefficients.** We multiply the second equation by 5 to prepare for eliminating y from the equations. The second equation becomes $10x - 5y = 15$.

Eliminate a variable. We add the equations to eliminate y.

$$\begin{cases} 3x + 5y = 11 & \\ 10x - 5y = 15 & 5 \times \text{Equation 2} \end{cases}$$

$$\begin{aligned} 13x \quad &= 26 && \text{Add} \\ x \quad &= 2 && \text{Solve for } x \end{aligned}$$

Back-substitute. Now we back-substitute into the first equation and solve for y.

$$\begin{aligned} 3(2) + 5y &= 11 && \text{Back-substitute } x = 2 \\ 5y &= 5 && \text{Subtract 6} \\ y &= 1 && \text{Solve for } y \end{aligned}$$

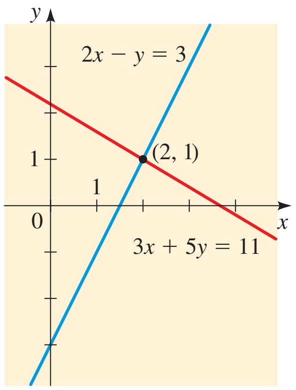


Figure 3

The solution is $(2, 1)$. Figure 3 shows that the graphs of the equations intersect at the point $(2, 1)$.

Now Try Exercise 9 ■

■ Graphical Method

In the **graphical method** we use a graphing device to solve the system of equations.

Graphical Method

1. **Graph Each Equation.** Use a graphing device to graph the equations on the same screen. To graph the equations using a graphing calculator, you may first need to solve for y as a function of x.
2. **Find Intersection Points.** The solutions are the x- and y-coordinates of the point(s) of intersection.

Example 3 ■ Graphical Method

Find all solutions of the system.

$$\begin{cases} 1.35x - 2.13y = -2.36 \\ 2.16x + 0.32y = 1.06 \end{cases}$$

Solution **Graph each equation.** To graph, we solve for y in each equation.

$$\begin{cases} y = 0.63x + 1.11 \\ y = -6.75x + 3.31 \end{cases}$$

where we have rounded the coefficients to two decimals.

Find intersection point. Figure 4 shows that the two lines intersect. From the graph we see that the solution is approximately $(0.30, 1.30)$.

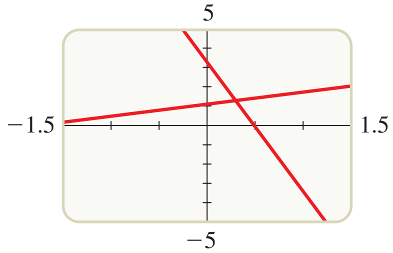


Figure 4

Now Try Exercises 13 and 51 ■

■ The Number of Solutions of a Linear System in Two Variables

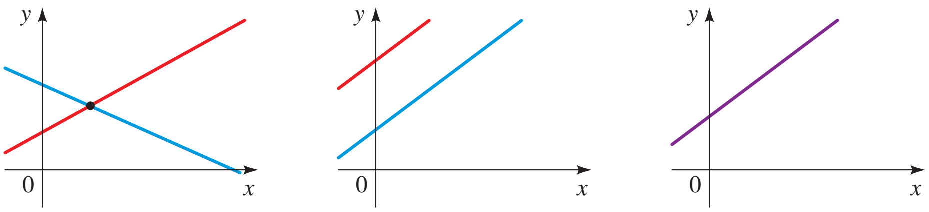
The graph of a linear system in two variables is a pair of lines, so to solve the system graphically, we must find the intersection point(s) of the lines. Two lines may intersect in a single point, they may be parallel, or they may coincide, as shown in Figure 5. So there are three possible outcomes in solving such a system.

Number of Solutions of a Linear System in Two Variables

For a system of linear equations in two variables, exactly one of the following is true. (See Figure 5.)

1. The system has exactly one solution.
2. The system has no solution.
3. The system has infinitely many solutions.

A system that has no solution is said to be **inconsistent**. A system with infinitely many solutions is called **dependent**.



(a) Lines intersect at a single point. The system has one solution.

(b) Lines are parallel and do not intersect. The system has no solution.

(c) Lines coincide—equations determine the same line. The system has infinitely many solutions.

Figure 5

Example 4 ■ A Linear System with One Solution

Solve the system and graph the lines.

$$\begin{cases} 3x - y = 0 & \text{Equation 1} \\ 5x + 2y = 22 & \text{Equation 2} \end{cases}$$

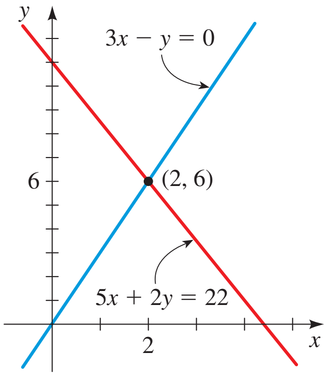


Figure 6 | The lines intersect

Solution We eliminate y from the equations and solve for x.

$$\begin{array}{rl} \begin{cases} 6x - 2y = 0 \\ 5x + 2y = 22 \end{cases} & 2 \times \text{Equation 1} \\ \hline 11x \qquad = 22 & \text{Add} \\ x = 2 & \text{Solve for } x \end{array}$$

Now we back-substitute into the first equation and solve for y:

$$\begin{aligned} 6(2) - 2y &= 0 && \text{Back-substitute } x = 2 \\ -2y &= -12 && \text{Subtract 12} \\ y &= 6 && \text{Solve for } y \end{aligned}$$

The solution of the system is the ordered pair (2, 6), that is,

$$x = 2 \qquad y = 6$$

The graph in Figure 6 shows that the lines in the system intersect at the point (2, 6).

Now Try Exercise 23 ■

Check Your Answer

$x = 2, y = 6$:

$$\begin{cases} 3(2) - (6) = 0 \\ 5(2) + 2(6) = 22 \end{cases}$$

Example 5 ■ A Linear System with No Solution

Solve the system.

$$\begin{cases} 8x - 2y = 5 & \text{Equation 1} \\ -12x + 3y = 7 & \text{Equation 2} \end{cases}$$

Solution This time we try to find a suitable combination of the two equations to eliminate the variable y. Multiplying the first equation by 3 and the second equation by 2 gives

$$\begin{array}{rl} \begin{cases} 24x - 6y = 15 \\ -24x + 6y = 14 \end{cases} & \begin{array}{l} 3 \times \text{Equation 1} \\ 2 \times \text{Equation 2} \end{array} \\ \hline 0 = 29 & \text{Add} \end{array}$$

Adding the two equations eliminates *both x and y* in this case, and we end up with $0 = 29$, which is obviously false. No matter what values we assign to x and y, we cannot make this statement true, so the system has *no solution*. Figure 7 shows that the lines in the system are parallel and so do not intersect. The system is inconsistent.

Now Try Exercise 37 ■

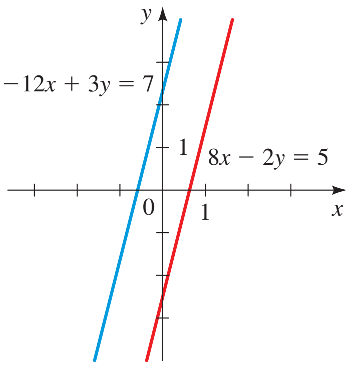


Figure 7 | The lines are parallel

Example 6 ■ A Linear System with Infinitely Many Solutions

Solve the system.

$$\begin{cases} 3x - 6y = 12 & \text{Equation 1} \\ 4x - 8y = 16 & \text{Equation 2} \end{cases}$$

Solution We multiply the first equation by 4 and the second equation by 3 to prepare for subtracting the equations to eliminate x. The new equations are

$$\begin{cases} 12x - 24y = 48 & 4 \times \text{Equation 1} \\ 12x - 24y = 48 & 3 \times \text{Equation 2} \end{cases}$$

We see that the two equations in the original system are simply different ways of expressing the equation of one single line. The coordinates of any point on this line

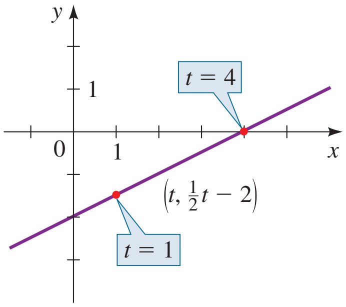


Figure 8 | The lines coincide

give a solution of the system. Writing the equation in slope-intercept form, we have $y = \frac{1}{2}x - 2$. So if we let t represent any real number, we can write the solution as

$$x = t$$
$$y = \tfrac{1}{2}t - 2$$

We can also write the solution in ordered-pair form as

$$\left(t, \tfrac{1}{2}t - 2\right)$$

where t is any real number. The system has infinitely many solutions (see Figure 8).

Now Try Exercise 39 ■

In Example 3, to get specific solutions we have to assign values to t. For instance, if $t = 1$, we get the solution $\left(1, -\frac{3}{2}\right)$. If $t = 4$, we get the solution $(4, 0)$. For every value of t we get a different solution. (See Figure 8.)

■ Modeling with Linear Systems

When we use equations to solve problems in the sciences or in other areas, we frequently obtain systems like the ones we've been considering. When modeling with systems of equations, we use the following guidelines, which are similar to those given in Section 1.7.

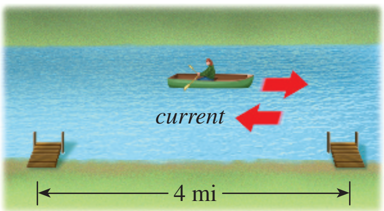
Guidelines for Modeling with Systems of Equations

1. **Identify the Variables.** Identify the quantities that the problem asks you to find. These are usually determined by a careful reading of the question posed at the end of the problem. Introduce notation for the variables (call them x and y, or some other letters).
2. **Express All Unknown Quantities in Terms of the Variables.** Read the problem again, and express all the quantities mentioned in the problem in terms of the variables you defined in Step 1.
3. **Set Up a System of Equations.** Find the crucial facts in the problem that give the relationships between the expressions you found in Step 2. Set up a system of equations (or a model) that expresses these relationships.
4. **Solve the System and Interpret the Results.** Solve the system you found in Step 3, check your solutions, and **state your answer** as a sentence that answers the question posed in the problem.

The next two examples illustrate how to model with systems of equations.

Example 7 ■ A Distance-Speed-Time Problem

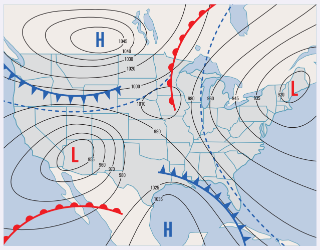
A boater rows a boat upstream from one point on a river to another point 4 miles away in $1\frac{1}{2}$ hours. The return trip, traveling with the current, takes only 45 minutes. What is the boater's rowing speed relative to the water, and at what speed is the current flowing?



Solution **Identify the variables.** We are asked to find the rowing speed and the speed of the current, so we let

$$x = \text{rowing speed (mi/h)}$$
$$y = \text{current speed (mi/h)}$$

Mathematics in the Modern World



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Weather Prediction

Today's meteorologists do much more than predict tomorrow's weather. They research long-term weather patterns, depletion of the ozone layer, global warming, and other effects of human activity on the weather. But daily weather prediction is still a major part of meteorology; its value is measured by the innumerable human lives that are saved each year through accurate prediction of hurricanes, blizzards, and other catastrophic weather phenomena. Early in the 20th century, mathematicians proposed to model weather with equations that used the current values of hundreds of atmospheric variables. Although this model worked in principle, it was impossible to predict future weather patterns with it because of the difficulty of measuring all the variables accurately and solving all the equations. Today, new mathematical models combined with high-speed computer simulations and better data have vastly improved weather prediction. As a result, many human as well as economic disasters have been averted. Mathematicians at the National Oceanographic and Atmospheric Administration (NOAA) are continually researching better methods of weather prediction.

Express unknown quantities in terms of the variable. When traveling upstream, the boat's speed is the rowing speed *minus* the speed of the current; when traveling downstream, the boat's speed is the rowing speed *plus* the speed of the current. Now we translate this information into the language of algebra.

In Words	In Algebra
Rowing speed	x
Current speed	y
Speed upstream	$x - y$
Speed downstream	$x + y$

Set up a system of equations. The distance upstream and downstream is 4 miles, so using the fact that speed × time = distance for both legs of the trip, we get

$$\boxed{\text{speed upstream}} \times \boxed{\text{time upstream}} = \boxed{\text{distance traveled}}$$

$$\boxed{\text{speed downstream}} \times \boxed{\text{time downstream}} = \boxed{\text{distance traveled}}$$

In algebraic notation this translates into the following equations.

$$(x - y)\tfrac{3}{2} = 4 \qquad \text{Equation 1}$$

$$(x + y)\tfrac{3}{4} = 4 \qquad \text{Equation 2}$$

(The times have been converted to hours, since we are expressing the speeds in miles per *hour*.)

Solve the system. We multiply the equations by 2 and 4, respectively, to clear the denominators.

$$\begin{cases} 3x - 3y = 8 & 2 \times \text{Equation 1} \\ 3x + 3y = 16 & 4 \times \text{Equation 2} \end{cases}$$

$$6x = 24 \qquad \text{Add}$$

$$x = 4 \qquad \text{Solve for } x$$

Back-substituting this value of x into the first equation (the second works just as well) and solving for y, we get

$$3(4) - 3y = 8 \qquad \text{Back-substitute } x = 4$$

$$-3y = 8 - 12 \qquad \text{Subtract 12}$$

$$y = \tfrac{4}{3} \qquad \text{Solve for } y$$

The boater's rowing speed relative to the water is 4 mi/h, and the current flows at $1\frac{1}{3}$ mi/h.

Check Your Answer

Speed upstream is

$$\frac{\text{distance}}{\text{time}} = \frac{4 \text{ mi}}{1\frac{1}{2} \text{ h}} = 2\tfrac{2}{3} \text{ mi/h}$$

and this should equal

$$\text{rowing speed} - \text{current flow}$$
$$= 4 \text{ mi/h} - \tfrac{4}{3} \text{ mi/h} = 2\tfrac{2}{3} \text{ mi/h}$$

Speed downstream is

$$\frac{\text{distance}}{\text{time}} = \frac{4 \text{ mi}}{\frac{3}{4} \text{ h}} = 5\tfrac{1}{3} \text{ mi/h}$$

and this should equal

$$\text{rowing speed} + \text{current flow}$$
$$= 4 \text{ mi/h} + \tfrac{4}{3} \text{ mi/h} = 5\tfrac{1}{3} \text{ mi/h}$$

Now Try Exercise 65

Example 8 ■ A Mixture Problem

A vintner fortifies wine that contains 10% alcohol by adding a 70% alcohol solution to it. The resulting mixture has an alcoholic strength of 16% and fills 1000 one-liter bottles. How many liters (L) of the wine and of the alcohol solution does the vintner use?

Solution **Identify the variables.** Since we are asked for the amounts of wine and alcohol, we let

$$x = \text{amount of wine used (L)}$$
$$y = \text{amount of alcohol solution used (L)}$$

Express all unknown quantities in terms of the variable. From the fact that the wine contains 10% alcohol and the solution contains 70% alcohol, we get the following.

In Words	In Algebra
Amount of wine used (L)	x
Amount of alcohol solution used (L)	y
Amount of alcohol in wine (L)	$0.10x$
Amount of alcohol in solution (L)	$0.70y$

Set up a system of equations. The volume of the mixture must be the total of the two volumes the vintner is adding together, so

$$x + y = 1000$$

Also, the amount of alcohol in the mixture must be the total of the alcohol contributed by the wine and by the alcohol solution, that is,

$$0.10x + 0.70y = (0.16)1000$$
$$0.10x + 0.70y = 160 \qquad \text{Simplify}$$
$$x + 7y = 1600 \qquad \text{Multiply by 10 to clear decimals}$$

Thus we get the system

$$\begin{cases} x + y = 1000 & \text{Equation 1} \\ x + 7y = 1600 & \text{Equation 2} \end{cases}$$

Solve the system. Subtracting the first equation from the second eliminates the variable x, and we get

$$6y = 600 \qquad \text{Subtract Equation 1 from Equation 2}$$
$$y = 100 \qquad \text{Solve for } y$$

We now back-substitute $y = 100$ into the first equation and solve for x.

$$x + 100 = 1000 \qquad \text{Back-substitute } y = 100$$
$$x = 900 \qquad \text{Solve for } x$$

The vintner uses 900 L of wine and 100 L of the alcohol solution.

Now Try Exercise 67 ■

9.1 Exercises

Concepts

1. The system of equations

$$\begin{cases} 2x + 3y = 7 \\ 5x - y = 9 \end{cases}$$

is a system of two equations in the two variables ______ and ______. To determine whether $(5, -1)$ is a solution of this system, we check whether $x = 5$ and $y = -1$ satisfy each ______ in the system. Which of the following are solutions of this system?

$$(5, -1), \quad (-1, 3), \quad (2, 1)$$

2. A system of equations in two variables can be solved by the ______ method, the ______ method, or the ______ method. Solve the system in Exercise 1 by each of these methods. The solution is (___, ___).

3. A system of two linear equations in two variables can have one solution, ______ solution, or ______ ______ solutions.

4. The following is a system of two linear equations in two variables.

$$\begin{cases} x + y = 1 \\ 2x + 2y = 2 \end{cases}$$

The graph of the first equation is the same as the graph of the second equation, so the system has ______ ______ solutions. We express these solutions by writing

$$x = t \qquad y = \_\_\_\_\_\_$$

where t is any real number. Some of the solutions of this system are (1, ___), (−3, ___), and (5, ___).

Skills

5–8 ■ Substitution Method Use the substitution method to find all solutions of the system of equations.

5. $\begin{cases} x + y = 2 \\ 2x - 4y = 16 \end{cases}$ **6.** $\begin{cases} x - y = 8 \\ 5x - 4y = 35 \end{cases}$

7. $\begin{cases} x - 3y = 11 \\ 3x - 5y = 17 \end{cases}$ **8.** $\begin{cases} 2x + y = 7 \\ x + 2y = 2 \end{cases}$

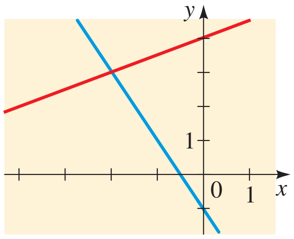
9–12 ■ Elimination Method Use the elimination method to find all solutions of the system of equations.

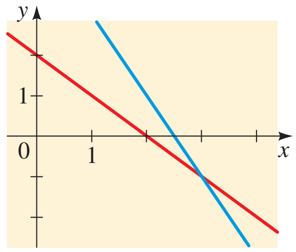
9. $\begin{cases} 2x - 3y = 7 \\ x - 5y = 0 \end{cases}$ **10.** $\begin{cases} 4x + y = 5 \\ 5x + 2y = 4 \end{cases}$

11. $\begin{cases} 3x - 2y = -13 \\ -6x + 5y = 28 \end{cases}$ **12.** $\begin{cases} 2x - 5y = -18 \\ 3x + 4y = 19 \end{cases}$

13–14 ■ Graphical Method Two equations and their graphs are given. Estimate the intersection point from the graph and check that the point is a solution to the system.

13. $\begin{cases} 2x + y = -1 \\ x - 2y = -8 \end{cases}$ **14.** $\begin{cases} x + y = 2 \\ 2x + y = 5 \end{cases}$





15–20 ■ Number of Solutions Determined Graphically Graph each linear system, either with or without a graphing device. Use the graph to determine whether the system has one solution, no solution, or infinitely many solutions. If there is exactly one solution, use the graph to find it.

15. $\begin{cases} x - y = 4 \\ 2x + y = 2 \end{cases}$ **16.** $\begin{cases} 2x - y = 4 \\ 3x + y = 6 \end{cases}$

17. $\begin{cases} 2x - 3y = 12 \\ -x + \frac{3}{2}y = 4 \end{cases}$ **18.** $\begin{cases} 2x + 6y = 0 \\ -3x - 9y = 18 \end{cases}$

19. $\begin{cases} -x + \frac{1}{2}y = -5 \\ 2x - y = 10 \end{cases}$ **20.** $\begin{cases} 12x + 15y = -18 \\ 2x + \frac{5}{2}y = -3 \end{cases}$

21–50 ■ Solving a System of Equations Solve the system, or show that it has no solution. If the system has infinitely many solutions, express them in the ordered-pair form given in Example 6.

21. $\begin{cases} 5x + 3y = 18 \\ 5x - 3y = 12 \end{cases}$ **22.** $\begin{cases} 2x + y = 10 \\ x - 3y = -16 \end{cases}$

23. $\begin{cases} 2x - 3y = 9 \\ 4x + 3y = 9 \end{cases}$ **24.** $\begin{cases} 3x + 2y = 0 \\ -x - 2y = 8 \end{cases}$

25. $\begin{cases} x + 3y = 5 \\ 2x - y = 3 \end{cases}$ **26.** $\begin{cases} x + y = 7 \\ 2x - 3y = -1 \end{cases}$

27. $\begin{cases} -x + y = 2 \\ 4x - 3y = -3 \end{cases}$ **28.** $\begin{cases} 4x - 3y = 28 \\ 9x - y = -6 \end{cases}$

29. $\begin{cases} x + 2y = 7 \\ 5x - y = 2 \end{cases}$ **30.** $\begin{cases} -4x + 12y = 0 \\ 12x + 4y = 160 \end{cases}$

31. $\begin{cases} -\frac{1}{3}x - \frac{1}{6}y = -1 \\ \frac{2}{3}x + \frac{1}{6}y = 3 \end{cases}$ **32.** $\begin{cases} \frac{3}{4}x + \frac{1}{2}y = 5 \\ -\frac{1}{4}x - \frac{3}{2}y = 1 \end{cases}$

33. $\begin{cases} \frac{1}{2}x + \frac{1}{3}y = 2 \\ \frac{1}{5}x - \frac{2}{3}y = 8 \end{cases}$ **34.** $\begin{cases} 0.2x - 0.2y = -1.8 \\ -0.3x + 0.5y = 3.3 \end{cases}$

35. $\begin{cases} 3x + 2y = 8 \\ x - 2y = 0 \end{cases}$ **36.** $\begin{cases} 4x + 2y = 16 \\ x - 5y = 70 \end{cases}$

37. $\begin{cases} x + 4y = 8 \\ 3x + 12y = 2 \end{cases}$

38. $\begin{cases} -3x + 5y = 2 \\ 9x - 15y = 6 \end{cases}$

39. $\begin{cases} 2x - 6y = 10 \\ -3x + 9y = -15 \end{cases}$

40. $\begin{cases} 2x - 3y = -8 \\ 14x - 21y = 3 \end{cases}$

41. $\begin{cases} 6x + 4y = 12 \\ 9x + 6y = 18 \end{cases}$

42. $\begin{cases} 25x - 75y = 100 \\ -10x + 30y = -40 \end{cases}$

43. $\begin{cases} 8s - 3t = -3 \\ 5s - 2t = -1 \end{cases}$

44. $\begin{cases} u - 30v = -5 \\ -3u + 80v = 5 \end{cases}$

45. $\begin{cases} \frac{1}{2}x + \frac{3}{5}y = 3 \\ \frac{5}{3}x + 2y = 10 \end{cases}$

46. $\begin{cases} \frac{3}{2}x - \frac{1}{3}y = \frac{1}{2} \\ 2x - \frac{1}{2}y = -\frac{1}{2} \end{cases}$

47. $\begin{cases} 0.4x + 1.2y = 14 \\ 12x - 5y = 10 \end{cases}$

48. $\begin{cases} 26x - 10y = -4 \\ -0.6x + 1.2y = 3 \end{cases}$

49. $\begin{cases} \frac{1}{3}x - \frac{1}{4}y = 2 \\ -8x + 6y = 10 \end{cases}$

50. $\begin{cases} -\frac{1}{10}x + \frac{1}{2}y = 4 \\ 2x - 10y = -80 \end{cases}$

51–54 ■ Solving a System of Equations Graphically Use a graphing device to graph both lines in the same viewing rectangle. Solve the system by finding (or approximating) the intersection point(s). Round your answers to two decimal places.

51. $\begin{cases} 0.21x + 3.17y = 9.51 \\ 2.35x - 1.17y = 5.89 \end{cases}$

52. $\begin{cases} 18.72x - 14.91y = 12.33 \\ 6.21x - 12.92y = 17.82 \end{cases}$

53. $\begin{cases} 2371x - 6552y = 13{,}591 \\ 9815x + 992y = 618{,}555 \end{cases}$

54. $\begin{cases} -435x + 912y = 0 \\ 132x + 455y = 994 \end{cases}$

Skills Plus

55–58 ■ Solving a General System of Equations Find x and y in terms of a and b.

55. $\begin{cases} x + y = 0 \\ x + ay = 1 \end{cases} \quad (a \neq 1)$

56. $\begin{cases} ax + by = 0 \\ x + y = 1 \end{cases} \quad (a \neq b)$

57. $\begin{cases} ax + by = 1 \\ bx + ay = 1 \end{cases} \quad (a^2 - b^2 \neq 0)$

58. $\begin{cases} ax + by = 0 \\ a^2x + b^2y = 1 \end{cases} \quad (a \neq 0, b \neq 0, a \neq b)$

Applications

59. Number Problem Find two numbers whose sum is 34 and whose difference is 10.

60. Number Problem The sum of two numbers is twice their difference. The larger number is 6 more than twice the smaller. Find the numbers.

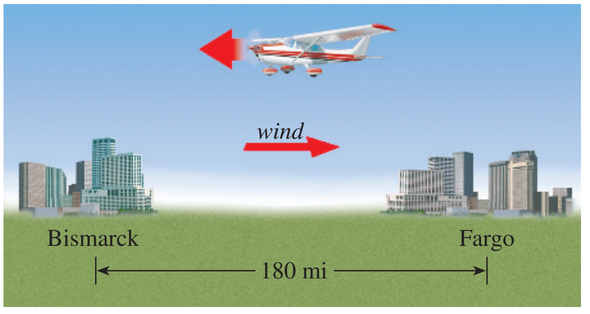
61. Value of Coins There are 14 coins, all of which are dimes and quarters. If the total value of the coins is \$2.75, how many dimes and how many quarters are there?

62. Admission Fees The admission fee at an amusement park is \$1.50 for children and \$4.00 for adults. On a certain day, 2200 people entered the park, and the admission fees that were collected totaled \$5050. How many children and how many adults were admitted?

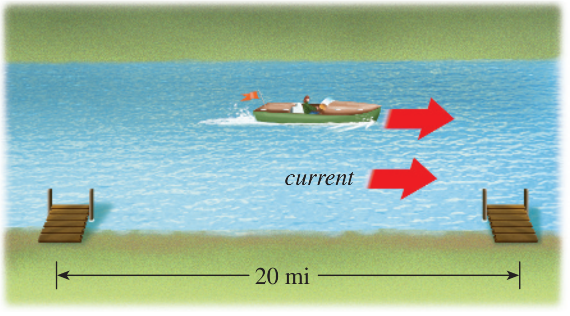
63. Gas Station A gas station sells regular gas for \$3.60 per gallon and premium gas for \$4.00 a gallon. At the end of a business day 185 gallons of gas had been sold, and receipts totaled \$690. How many gallons of each type of gas had been sold?

64. Fruit Stand A fruit stand sells two varieties of strawberries: standard and deluxe. A box of standard strawberries sells for \$7, and a box of deluxe strawberries sells for \$10. In one day the stand sold 135 boxes of strawberries for a total of \$1110. How many boxes of each type were sold?

65. Airplane Speed A pilot flies a small airplane from Fargo to Bismarck, North Dakota—a distance of 180 miles. Because the plane is flying into a headwind, the trip takes 2 hours. On the way back, the wind is still blowing at the same speed, so the return trip takes only 1 h 12 min. What is the speed of the airplane in still air, and how fast is the wind blowing?



66. Boat Speed A boat on a river travels downstream between two points, 20 miles apart, in 1 hour. The return trip against the current takes $2\frac{1}{2}$ hours. What is the boat's speed relative to the water, and how fast does the current in the river flow?



67. Nutrition A researcher performs an experiment to test a hypothesis that involves the nutrients niacin and retinol. In the experiment, one group of laboratory rats is fed a daily diet of precisely 32 units of niacin and 22,000 units of retinol, using two types of commercial pellets. Food A contains 0.12 units of niacin and 100 units of retinol per gram. Food B contains 0.20 units of niacin and 50 units of retinol per gram. How many grams of each food is fed to this group of rats each day?

68. Coffee Blends A coffee shop sells two types of coffee: Kenyan, costing \$7.00 a pound, and Sri Lankan, costing \$11.20 a pound. If 3 pounds of a blend of the two types of coffee costs \$23.10, how many pounds of each kind went into the mixture?

69. Mixture Problem A chemist has two large containers of sulfuric acid solution, with different concentrations of acid in each container. Blending 300 mL of the first solution and 600 mL of the second gives a mixture that is 15% acid, whereas blending 100 mL of the first with 500 mL of the second gives a $12\frac{1}{2}\%$ acid mixture. What are the concentrations of sulfuric acid in the original containers?

70. Mixture Problem A biologist has two brine solutions, one containing 5% salt and another containing 20% salt. How many milliliters of each solution should be mixed together to obtain 1 L of a solution that contains 14% salt?

71. Investments A total of \$20,000 is invested in two accounts, one paying 5% and the other paying 8% simple interest per year. The annual interest is \$1180. How much is invested at each rate?

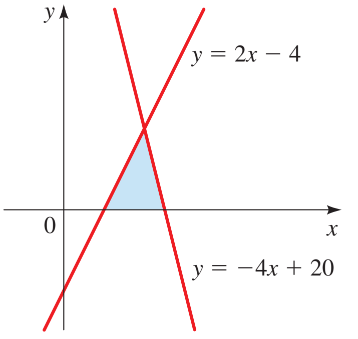
72. Investments A sum of money is invested in two accounts, one paying 6% and the other paying 10% simple interest per year. Twice as much is invested in the lower-yielding account because it is less risky. The annual interest is \$3520. How much is invested at each rate?

73. Distance, Speed, and Time A truck and an SUV leave a restaurant at the same time, going in opposite directions. The truck travels at 60 mi/h and travels 35 miles farther than the SUV, which travels at 40 mi/h. The SUV's trip takes 15 minutes longer than the truck's trip. For what length of time does each vehicle travel?

74. How Much Gold in the Crown? Archimedes was able to determine the amount of gold in a crown by first finding the crown's volume. (See the vignette *Archimedes* in Section 10.1.) Suppose a crown made of gold and silver weighs 235 grams and has a volume of 14 cubic centimeters (cm^3). Find the weight of the gold and the weight of the silver in the crown. (Use the fact that the density of a substance is its weight divided by its volume: the density of gold is 19.3 g/cm^3 and the density of silver is 10.5 g/cm^3.)

75. Number Problem The sum of the digits of a two-digit number is 7. When the digits are reversed, the number is increased by 27. Find the number.

76. Area of a Triangle Find the area of the triangle that lies in the first quadrant (with its base on the x-axis) and that is bounded by the lines $y = 2x - 4$ and $y = -4x + 20$.



Discuss ■ Discover ■ Prove ■ Write

77. Discuss: The Least Squares Line The *least squares* line or *regression* line is the line that best fits a set of points in the plane. We studied this line in the *Focus on Modeling* that follows Chapter 1. By using calculus, it can be shown that the line that best fits the n data points

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$$

is the line $y = ax + b$, where the coefficients a and b satisfy the following pair of linear equations. (The notation $\sum_{k=1}^{n} x_k$ stands for the sum of all the x's. See Section 11.1 for a complete description of sigma (Σ) notation.)

$$\left(\sum_{k=1}^{n} x_k\right) a + nb = \sum_{k=1}^{n} y_k$$

$$\left(\sum_{k=1}^{n} x_k^2\right) a + \left(\sum_{k=1}^{n} x_k\right) b = \sum_{k=1}^{n} x_k y_k$$

Use these equations to find the least squares line for the following data points.

$$(1, 3),\quad (2, 5),\quad (3, 6),\quad (5, 6),\quad (7, 9)$$

Sketch the points and your line to confirm that the line fits these points well. If your calculator computes regression lines, see whether it gives you the same line as the formulas.

9.2 Systems of Linear Equations in Several Variables

■ Solving a Linear System ■ The Number of Solutions of a Linear System ■ Modeling Using Linear Systems

A **linear equation in n variables** is an equation that can be put in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = c$$

where $a_1, a_2, \ldots, a_n$ and c are real numbers, and $x_1, x_2, \ldots, x_n$ are the variables. If we have only three or four variables, we generally use x, y, z, and w instead of x_1, x_2, x_3, and x_4. Such equations are called *linear* because if we have just two variables, the equation is $a_1x + a_2y = c$, which is the equation of a line. Here are some examples of equations in three variables that illustrate the difference between linear and nonlinear equations.

Linear equations	**Nonlinear equations**	
$6x_1 - 3x_2 + \sqrt{5}x_3 = 10$	$x^2 + 3y - \sqrt{z} = 5$	Not linear because it contains the square and the square root of a variable
$x + y + z = 2w - \frac{1}{2}$	$x_1x_2 + 6x_3 = -6$	Not linear because it contains a product of variables

In this section we study systems of linear equations in three or more variables.

■ Solving a Linear System

The following are two examples of systems of linear equations in three variables. The second system is in **triangular form**; that is, the variable x doesn't appear in the second equation, and the variables x and y do not appear in the third equation.

A system of linear equations

$$\begin{cases} x - 2y - z = 1 \\ -x + 3y + 3z = 4 \\ 2x - 3y + z = 10 \end{cases}$$

A system in triangular form

$$\begin{cases} x - 2y - z = 1 \\ y + 2z = 5 \\ z = 3 \end{cases}$$

We can solve a system that is in triangular form by using back-substitution. So our goal in this section is to start with a system of linear equations and change it to a system in triangular form that has the same solutions as the original system. We begin by showing how to use back-substitution to solve a system that is already in triangular form.

Example 1 ■ Solving a Triangular System Using Back-Substitution

Solve the following system using back-substitution:

$$\begin{cases} x - 2y - z = 1 & \text{Equation 1} \\ y + 2z = 5 & \text{Equation 2} \\ z = 3 & \text{Equation 3} \end{cases}$$

Solution From the last equation we know that $z = 3$. We back-substitute this into the second equation and solve for y.

$$y + 2(3) = 5 \qquad \text{Back-substitute } z = 3 \text{ into Equation 2}$$

$$y = -1 \qquad \text{Solve for } y$$

Then we back-substitute $y = -1$ and $z = 3$ into the first equation and solve for x.

$$x - 2(-1) - (3) = 1 \qquad \text{Back-substitute } y = -1 \text{ and } z = 3 \text{ into Equation 1}$$

$$x = 2 \qquad \text{Solve for } x$$

The solution of the system is $x = 2$, $y = -1$, $z = 3$. We can also write the solution as the ordered triple $(2, -1, 3)$.

Now Try Exercise 7 ■

To change a system of linear equations to an **equivalent system** (that is, a system with the same solutions as the original system), we use the elimination method. This means that we can use the following operations.

Operations That Yield an Equivalent System

1. Add a nonzero multiple of one equation to another.
2. Multiply an equation by a nonzero constant.
3. Interchange the positions of two equations.

To solve a linear system, we use these operations to change the system to an equivalent triangular system. Then we use back-substitution as in Example 1. This process is called **Gaussian elimination**.

Example 2 ■ Solving a System of Three Equations in Three Variables

Solve the following system using Gaussian elimination:

$$\begin{cases} x - 2y + 3z = 1 & \text{Equation 1} \\ x + 2y - z = 13 & \text{Equation 2} \\ 3x + 2y - 5z = 3 & \text{Equation 3} \end{cases}$$

Solution We need to change this to a triangular system, so we begin by eliminating the x-term from the second equation.

$$\begin{array}{rl} x + 2y - z = 13 & \text{Equation 2} \\ x - 2y + 3z = 1 & \text{Equation 1} \\ \hline 4y - 4z = 12 & \text{Equation 2} + (-1) \times \text{Equation 1} = \text{new Equation 2} \end{array}$$

This gives us a new, equivalent system that is one step closer to triangular form.

$$\begin{cases} x - 2y + 3z = 1 & \text{Equation 1} \\ 4y - 4z = 12 & \text{Equation 2} \\ 3x + 2y - 5z = 3 & \text{Equation 3} \end{cases}$$

Now we eliminate the x-term from the third equation.

$$\begin{array}{r} 3x + 2y - 5z = 3 \\ -3x + 6y - 9z = -3 \\ \hline 8y - 14z = 0 \end{array}$$

$$\begin{cases} x - 2y + 3z = 1 \\ 4y - 4z = 12 \\ 8y - 14z = 0 & \text{Equation 3} + (-3) \times \text{Equation 1} = \text{new Equation 3} \end{cases}$$

Then we eliminate the y-term from the third equation.

$$\begin{array}{r} 8y - 14z = 0 \\ -8y + 8z = -24 \\ \hline -6z = -24 \end{array}$$

$$\begin{cases} x - 2y + 3z = 1 \\ 4y - 4z = 12 \\ -6z = -24 & \text{Equation 3} + (-2) \times \text{Equation 2} = \text{new Equation 3} \end{cases}$$

The system is now in triangular form, but it will be easier to work with if we divide the second and third equations by the common factors of each term.

$$\begin{cases} x - 2y + 3z = 1 \\ y - z = 3 & \frac{1}{4} \times \text{Equation 2} = \text{new Equation 2} \\ z = 4 & -\frac{1}{6} \times \text{Equation 3} = \text{new Equation 3} \end{cases}$$

Now we use back-substitution to solve the system. From the third equation we get $z = 4$. We back-substitute this into the second equation and solve for y.

$$y - (4) = 3 \qquad \text{Back-substitute } z = 4 \text{ into Equation 2}$$

$$y = 7 \qquad \text{Solve for } y$$

Now we back-substitute $y = 7$ and $z = 4$ into the first equation and solve for x.

$$x - 2(7) + 3(4) = 1 \qquad \text{Back-substitute } y = 7 \text{ and } z = 4 \text{ into Equation 1}$$

$$x = 3 \qquad \text{Solve for } x$$

The solution of the system is $x = 3$, $y = 7$, $z = 4$, which we can write as the ordered triple $(3, 7, 4)$.

Now Try Exercise 17 ■

Check Your Answer

$x = 3, y = 7, z = 4$:

$$(3) - 2(7) + 3(4) = 1$$
$$(3) + 2(7) - (4) = 13$$
$$3(3) + 2(7) - 5(4) = 3 \checkmark$$

The Number of Solutions of a Linear System

The graph of a linear equation in three variables is a plane in three-dimensional space. A system of three equations in three variables represents three planes in space. The solutions of the system are the points where all three planes intersect. Three planes may intersect in a point, in a line, or not at all, or all three planes may coincide. Figure 1 illustrates some of these possibilities. Checking these possibilities we see that there are three possible outcomes when solving such a system.

To learn more about three-dimensional space see the additional topic *Three-Dimensional Coordinate Geometry* at the book companion website **www.stewartmath.com.**

Number of Solutions of a Linear System

For a system of linear equations, exactly one of the following is true.

1. The system has exactly one solution.
2. The system has no solution.
3. The system has infinitely many solutions.

A system with no solution is said to be **inconsistent**, and a system with infinitely many solutions is said to be **dependent**. As we see in the next example, a linear system has no solution if we end up with a *false equation* after applying Gaussian elimination to the system.

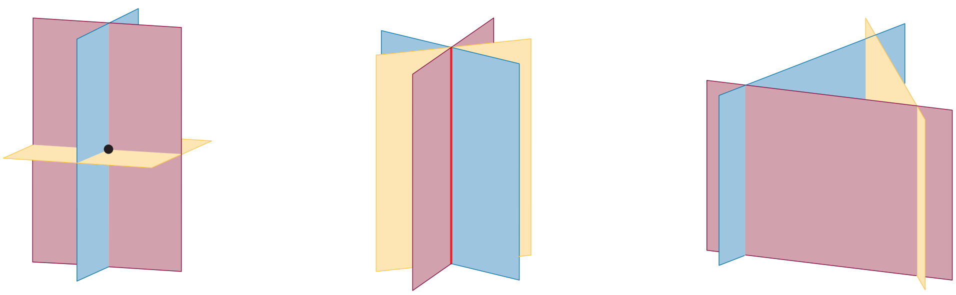


Figure 1

(a) The three planes intersect at a single point. The system has one solution.

(b) The three planes intersect along a line. The system has infinitely many solutions.

(c) The three planes have no point in common. The system has no solution.

Example 3 ■ A System with No Solution

Solve the following system:

$$\begin{cases} x + 2y - 2z = 1 & \text{Equation 1} \\ 2x + 2y - z = 6 & \text{Equation 2} \\ 3x + 4y - 3z = 5 & \text{Equation 3} \end{cases}$$

Solution To put this in triangular form, we begin by eliminating the x-terms from the second equation and the third equation.

$$\begin{cases} x + 2y - 2z = 1 \\ \quad -2y + 3z = 4 \\ 3x + 4y + 3z = 5 \end{cases} \qquad \text{Equation 2} + (-2) \times \text{Equation 1} = \text{new Equation 2}$$

$$\begin{cases} x + 2y - 2z = 1 \\ \quad -2y + 3z = 4 \\ \quad -2y + 3z = 2 \end{cases} \qquad \text{Equation 3} + (-3) \times \text{Equation 1} = \text{new Equation 3}$$

Now we eliminate the y-term from the third equation.

$$\begin{cases} x + 2y - 2z = 1 \\ \quad -2y + 3z = 4 \\ \qquad\qquad 0 = -2 \end{cases} \qquad \text{Equation 3} + (-1) \times \text{Equation 2} = \text{new Equation 3}$$

The system is now in triangular form, but the third equation says $0 = -2$, which is false. No matter what values we assign to x, y, and z, the third equation will never be true. This means that the system has *no solution*.

Now Try Exercise 29 ■

Example 4 ■ A System with Infinitely Many Solutions

Solve the following system:

$$\begin{cases} x - y + 5z = -2 & \text{Equation 1} \\ 2x + y + 4z = 2 & \text{Equation 2} \\ 2x + 4y - 2z = 8 & \text{Equation 3} \end{cases}$$

Solution To put this in triangular form, we begin by eliminating the x-terms from the second equation and the third equation.

$$\begin{cases} x - y + 5z = -2 \\ \quad 3y - 6z = 6 \\ 2x + 4y - 2z = 8 \end{cases} \qquad \text{Equation 2} + (-2) \times \text{Equation 1} = \text{new Equation 2}$$

$$\begin{cases} x - y + 5z = -2 \\ \quad 3y - 6z = 6 \\ \quad 6y - 12z = 12 \end{cases} \qquad \text{Equation 3} + (-2) \times \text{Equation 1} = \text{new Equation 3}$$

Discovery Project ■ Best Fit Versus Exact Fit

The law of gravity is precise. But when we obtain data on the distance an object falls in a given time, our measurements are not exact. We can, however, find the line (or parabola) that *best* fits our data. Not all of the data points will lie on the line (or parabola). But if we are given just two points, we can find a line of *exact* fit—that is, a line that passes through the two points. Similarly, we can find a parabola through three points. In this project we compare exact data with models of real-world data. You can find the project at the book companion website: **www.stewartmath.com**.

Now we eliminate the y-term from the third equation.

$$\begin{cases} x - y + 5z = -2 \\ \quad 3y - 6z = \ \ 6 \\ \qquad\quad\ 0 = \ \ 0 \end{cases} \qquad \text{Equation 3} + (-2) \times \text{Equation 2} = \text{new Equation 3}$$

The new third equation is true, but it gives us no new information, so we can drop it from the system. Only two equations are left. We can use them to solve for x and y in terms of z, but z can take on any value, so there are infinitely many solutions.

To find the complete solution of the system, we begin by solving for y in terms of z, using the new second equation.

$$\begin{aligned} 3y - 6z &= 6 && \text{Equation 2} \\ y - 2z &= 2 && \text{Multiply by } \tfrac{1}{3} \\ y &= 2z + 2 && \text{Solve for } y \end{aligned}$$

Then we solve for x in terms of z, using the first equation.

$$\begin{aligned} x - (2z + 2) + 5z &= -2 && \text{Substitute } y = 2z + 2 \text{ into Equation 1} \\ x + 3z - 2 &= -2 && \text{Simplify} \\ x &= -3z && \text{Solve for } x \end{aligned}$$

To describe the complete solution, we let z be any real number t. The solution is

$$\begin{aligned} x &= -3t \\ y &= 2t + 2 \\ z &= t \end{aligned}$$

We can also write this as the ordered triple $(-3t, 2t + 2, t)$.

Now Try Exercise 33 ■

In the solution of Example 4 the variable t is called a **parameter**. To get a specific solution, we give a specific value to the parameter t. For instance, if we set $t = 2$, we get

$$\begin{aligned} x &= -3(2) = -6 \\ y &= 2(2) + 2 = 6 \\ z &= 2 \end{aligned}$$

Thus $(-6, 6, 2)$ is a solution of the system. Here are some other solutions of the system obtained by substituting other values for the parameter t.

Parameter t	Solution $(-3t, 2t + 2, t)$
-1	$(3, 0, -1)$
0	$(0, 2, 0)$
3	$(-9, 8, 3)$
10	$(-30, 22, 10)$

You should check that these points satisfy the original equations. There are infinitely many choices for the parameter t, so the system has infinitely many solutions.

■ Modeling Using Linear Systems

Linear systems are used to model situations that involve several varying quantities. In the next example we consider an application of linear systems to finance.

Example 5 ■ Modeling a Financial Problem Using a Linear System

A student receives an inheritance of \$50,000. A financial advisor suggests that the money be invested in three mutual funds: a money-market fund, a blue-chip stock fund, and a high-tech stock fund. The advisor estimates that the money-market fund will return 5% over the next year, the blue-chip fund 9%, and the high-tech fund 16%. The student wants a total first-year return of \$4000. To avoid excessive risk, three times as much is invested in the money-market fund as in the high-tech stock fund. How much should be invested in each fund?

Solution

Let x = amount invested in the money-market fund

y = amount invested in the blue-chip stock fund

z = amount invested in the high-tech stock fund

We convert each fact given in the problem into an equation.

$$\begin{array}{rll} x + y + z = 50{,}000 & & \text{Total amount invested is \$50,000} \\ 0.05x + 0.09y + 0.16z = 4000 & & \text{Total investment return is \$4000} \\ x = 3z & & \text{Money-market amount is } 3 \times \text{high-tech amount} \end{array}$$

Multiplying the second equation by 100 and rewriting the third, we get the following system, which we solve using Gaussian elimination.

$$\begin{cases} x + y + z = 50{,}000 & \\ 5x + 9y + 16z = 400{,}000 & 100 \times \text{Equation 2} \\ x - 3z = 0 & \text{Subtract } 3z \end{cases}$$

$$\begin{cases} x + y + z = 50{,}000 & \\ 4y + 11z = 150{,}000 & \text{Equation 2} + (-5) \times \text{Equation 1} = \text{new Equation 2} \\ -y - 4z = -50{,}000 & \text{Equation 3} + (-1) \times \text{Equation 1} = \text{new Equation 3} \end{cases}$$

$$\begin{cases} x + y + z = 50{,}000 & \\ -5z = -50{,}000 & \text{Equation 2} + 4 \times \text{Equation 3} = \text{new Equation 2} \\ -y - 4z = -50{,}000 & \end{cases}$$

$$\begin{cases} x + y + z = 50{,}000 & \\ z = 10{,}000 & \left(-\frac{1}{5}\right) \times \text{Equation 2} \\ y + 4z = 50{,}000 & (-1) \times \text{Equation 3} \end{cases}$$

$$\begin{cases} x + y + z = 50{,}000 & \\ y + 4z = 50{,}000 & \text{Interchange Equations 2 and 3} \\ z = 10{,}000 & \end{cases}$$

Now that the system is in triangular form, we use back-substitution to find that $x = 30{,}000$, $y = 10{,}000$, and $z = 10{,}000$. This means that the student should invest

\$30,000 in the money-market fund

\$10,000 in the blue-chip stock fund

\$10,000 in the high-tech stock fund

Now Try Exercise 39 ■

9.2 Exercises

Concepts

1–2 ■ These exercises refer to the following system:

$$\begin{cases} x - y + z = 2 \\ -x + 2y + z = -3 \\ 3x + y - 2z = 2 \end{cases}$$

1. If we add 2 times the first equation to the second equation, the second equation becomes ______________ = _____.

2. To eliminate x from the third equation, we add ________ times the first equation to the third equation. The third equation becomes ______________ = _____.

Skills

3–6 ■ Is the System of Equations Linear? State whether the equation or system of equations is linear.

3. $6x - \sqrt{3}y + \frac{1}{2}z = 0$

4. $x^2 + y^2 + z^2 = 4$

5. $\begin{cases} xy - 3y + z = 5 \\ x - y^2 + 5z = 0 \\ 2x + yz = 3 \end{cases}$

6. $\begin{cases} x - 2y + 3z = 10 \\ 2x + 5y = 2 \\ y + 2z = 4 \end{cases}$

7–12 ■ Triangular Systems Use back-substitution to solve the triangular system.

7. $\begin{cases} x + 2y - z = -5 \\ y + z = 2 \\ z = 4 \end{cases}$

8. $\begin{cases} 3x - 3y + z = 0 \\ y + 4z = 10 \\ z = 3 \end{cases}$

9. $\begin{cases} x + 2y + z = 7 \\ -y + 3z = 9 \\ 2z = 6 \end{cases}$

10. $\begin{cases} x - 2y + 3z = 10 \\ 2y - z = 2 \\ 3z = 12 \end{cases}$

11. $\begin{cases} 2x - y + 6z = 5 \\ y + 4z = 0 \\ -2z = 1 \end{cases}$

12. $\begin{cases} 4x + 3z = 10 \\ 2y - z = -6 \\ \frac{1}{2}z = 4 \end{cases}$

13–16 ■ Eliminating a Variable Perform an operation on the given system that eliminates the indicated variable. Write the new equivalent system.

13. $\begin{cases} 3x + y + z = 4 \\ -x + y + 2z = 0 \\ x - 2y - z = -1 \end{cases}$

Eliminate the x-term from the second equation.

14. $\begin{cases} -5x + 2y - 3z = 3 \\ 10x - 3y + z = -20 \\ -x + 3y + z = 8 \end{cases}$

Eliminate the x-term from the second equation.

15. $\begin{cases} 2x + y - 3z = 5 \\ 2x + 3y + z = 13 \\ 6x - 5y - z = 7 \end{cases}$

Eliminate the x-term from the third equation.

16. $\begin{cases} x - 3y + 2z = -1 \\ y + z = -1 \\ 2y - z = 1 \end{cases}$

Eliminate the y-term from the third equation.

17–38 ■ Solving a System of Equations in Three Variables Find the complete solution of the linear system, or show that the system is inconsistent.

17. $\begin{cases} x + 2y + z = 3 \\ y - z = -4 \\ -x - 2y + 3z = 9 \end{cases}$

18. $\begin{cases} x - 5y - 3z = 4 \\ 3y + 5z = 11 \\ x - 2y + z = 11 \end{cases}$

19. $\begin{cases} x - 3y - 2z = 5 \\ 3x - 2y + z = 8 \\ y - 3z = -1 \end{cases}$

20. $\begin{cases} x - 2y + 3z = -10 \\ 3y + z = 7 \\ x + y - z = 7 \end{cases}$

21. $\begin{cases} x + y + z = 4 \\ x + 3y + 3z = 10 \\ 2x + y - z = 3 \end{cases}$

22. $\begin{cases} x + y + z = 0 \\ -x + 2y + 5z = 3 \\ 3x - y = 6 \end{cases}$

23. $\begin{cases} x - 4z = 1 \\ 2x - y - 6z = 4 \\ 2x + 3y - 2z = 8 \end{cases}$

24. $\begin{cases} x - y + 2z = 2 \\ 3x + y + 5z = 8 \\ 2x - y - 2z = -7 \end{cases}$

25. $\begin{cases} 2x + 4y - z = 2 \\ x + 2y - 3z = -4 \\ 3x - y + z = 1 \end{cases}$

26. $\begin{cases} 2x + y - z = -8 \\ -x + y + z = 3 \\ -2x + 4z = 18 \end{cases}$

27. $\begin{cases} 2y + 4z = -1 \\ -2x + y + 2z = -1 \\ 4x - 2y = 0 \end{cases}$

28. $\begin{cases} y - z = -1 \\ 6x + 2y + z = 2 \\ -x - y - 3z = -2 \end{cases}$

29. $\begin{cases} x - 3y + z = 2 \\ 3x + 4y - 2z = 1 \\ -2x + 6y - 2z = 3 \end{cases}$

30. $\begin{cases} -x + 2y + 5z = 4 \\ x - 2z = 0 \\ 4x - 2y - 11z = 2 \end{cases}$

31. $\begin{cases} 2x + 3y - z = 1 \\ x + 2y = 3 \\ x + 3y + z = 4 \end{cases}$

32. $\begin{cases} x - 2y - 3z = 5 \\ 2x + y - z = 5 \\ 4x - 3y - 7z = 5 \end{cases}$

33. $\begin{cases} x + y - z = 0 \\ x + 2y - 3z = -3 \\ 2x + 3y - 4z = -3 \end{cases}$

34. $\begin{cases} x - 2y + z = 3 \\ 2x - 5y + 6z = 7 \\ 2x - 3y - 2z = 5 \end{cases}$

35. $\begin{cases} x + 3y - 2z = 0 \\ 2x + 4z = 4 \\ 4x + 6y = 4 \end{cases}$

36. $\begin{cases} 2x + 4y - z = 3 \\ x + 2y + 4z = 6 \\ x + 2y - 2z = 0 \end{cases}$

37. $\begin{cases} x + z + 2w = 6 \\ y - 2z = -3 \\ x + 2y - z = -2 \\ 2x + y + 3z - 2w = 0 \end{cases}$

38. $\begin{cases} x + y + z + w = 0 \\ x + y + 2z + 2w = 0 \\ 2x + 2y + 3z + 4w = 1 \\ 2x + 3y + 4z + 5w = 2 \end{cases}$

Applications

39. Financial Planning A financial planner invests $100,000 in three types of bonds: short-term, intermediate-term, and long-term. The short-term bonds pay 4%, the intermediate-term bonds pay 5%, and the long-term bonds pay 6% simple interest per year. The planner wishes to realize a total annual income of 5.1%, with equal amounts invested in short- and intermediate-term bonds. How much should be invested in each type of bond?

40. Financial Planning An amount of $50,000 is invested in three types of accounts: one paying 3%, one paying $5\frac{1}{2}$%, and one paying 9% simple interest per year. Twice as much is invested in the lowest-yielding, least-risky account as in the highest-yielding account. How much should be invested in each account to achieve a total annual return of $2540?

41. Agriculture A farmer has 1200 acres of land on which to grow corn, wheat, and soybeans. It costs $45 per acre to grow corn, $60 to grow wheat, and $50 to grow soybeans. Because of market demand, the farmer will grow twice as many acres of wheat as of corn. An amount of $63,750 is allocated for the cost of growing the three crops. How many acres of each crop should be planted?

42. Gas Station A gas station sells three types of gas: Regular for $3.00 a gallon, Performance Plus for $3.20 a gallon, and Premium for $3.30 a gallon. On a particular day 6500 gallons of gas were sold for a total of $20,050. Three times as many gallons of Regular as Premium gas were sold. How many gallons of each type of gas were sold that day?

43. Nutrition A biologist is performing an experiment on the effects of various combinations of vitamins. In the experiment, one group of laboratory rabbits is fed a daily diet of precisely 9 mg of niacin, 14 mg of thiamin, and 32 mg of riboflavin, using three types of commercial rabbit pellets; their vitamin content (per ounce) is given in the table. How many ounces of each type of food should each rabbit be given daily to satisfy the experiment requirements?

	Type A	Type B	Type C
Niacin (mg/oz)	2	3	1
Thiamin (mg/oz)	3	1	3
Riboflavin (mg/oz)	8	5	7

44. Diet Program A patient started a new diet that requires each meal to have 460 calories, 6 g of fiber, and 11 g of fat. The table shows the fiber, fat, and calorie content of one serving of each of three breakfast foods. How many servings of each food should the patient eat to follow this diet?

Food	Fiber (g)	Fat (g)	Calories
Toast	2	1	100
Cottage cheese	0	5	120
Fruit	2	0	60

45. Juice Blends The Juice Company offers three kinds of smoothies: Midnight Mango, Tropical Torrent, and Pineapple Power. Each smoothie contains the amounts of juices shown in the table.

On a particular day the Juice Company used 820 ounces of mango juice, 690 ounces of pineapple juice, and 450 ounces of orange juice. How many smoothies of each kind were sold that day?

Smoothie	Mango Juice (oz)	Pineapple Juice (oz)	Orange Juice (oz)
Midnight Mango	8	3	3
Tropical Torrent	6	5	3
Pineapple Power	2	8	4

46. Appliance Manufacturing Kitchen Korner produces refrigerators, dishwashers, and stoves at three different factories. The table gives the number of each type of product produced at each factory per day. Kitchen Korner receives an order for 110 refrigerators, 150 dishwashers, and 114 ovens. How many days should each plant be scheduled to fill this order?

Appliance	Factory A	Factory B	Factory C
Refrigerator	8	10	14
Dishwasher	16	12	10
Stove	10	18	6

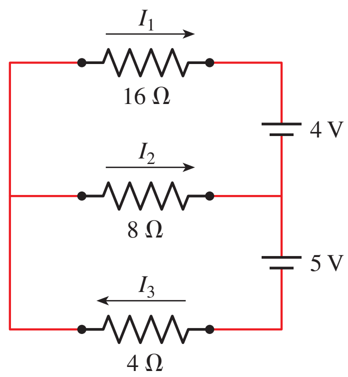
47. Stock Portfolio An investor owns three stocks: A, B, and C. The closing prices of the stocks on three successive trading days are given in the table.

Despite the volatility in the stock prices, the total value of the investor's stocks remained unchanged at $74,000 at the end of each of these three days. How many shares of each stock does the investor own?

	Stock A	Stock B	Stock C
Monday	$10	$25	$29
Tuesday	$12	$20	$32
Wednesday	$16	$15	$32

48. Electricity By using Kirchhoff's Laws, it can be shown that the currents I_1, I_2, and I_3 that pass through the three branches of the circuit in the figure satisfy the given linear system. Solve the system to find I_1, I_2, and I_3.

$$\begin{cases} I_1 + I_2 - I_3 = 0 \\ 16I_1 - 8I_2 = 4 \\ 8I_2 + 4I_3 = 5 \end{cases}$$



■ Discuss ■ Discover ■ Prove ■ Write

49. Prove: Can a Linear System Have Exactly Two Solutions?

(a) Suppose that (x_0, y_0, z_0) and (x_1, y_1, z_1) are solutions of the system

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$$

Show that

$$\left(\frac{x_0 + x_1}{2}, \frac{y_0 + y_1}{2}, \frac{z_0 + z_1}{2}\right)$$

is also a solution.

(b) Use the result of part (a) to prove that if the system has two different solutions, then it has infinitely many solutions.

9.3 Matrices and Systems of Linear Equations

■ Matrices ■ The Augmented Matrix of a Linear System ■ Elementary Row Operations ■ Gaussian Elimination ■ Gauss-Jordan Elimination ■ Inconsistent and Dependent Systems ■ Modeling with Linear Systems

The plural of *matrix* is *matrices*.

A *matrix* is simply a rectangular array of numbers. Matrices are used to organize information into categories that correspond to the rows and columns of the matrix. For example, a scientist might organize information on a population of endangered whales as follows:

$$\begin{array}{c} \\ \textbf{Male} \\ \textbf{Female} \end{array} \begin{array}{c} \begin{array}{ccc} \textbf{Immature} & \textbf{Juvenile} & \textbf{Adult} \end{array} \\ \begin{bmatrix} 12 & 52 & 18 \\ 15 & 42 & 11 \end{bmatrix} \end{array}$$

This is a compact way of saying that there are 12 immature males, 15 immature females, 18 adult males, and so on.

We represent a linear system by a matrix, called the *augmented matrix* of the system.

$$\begin{array}{ccc} \textbf{Linear system} & & \textbf{Augmented matrix} \\ \begin{cases} 2x - y = 5 \\ x + 4y = 7 \end{cases} & \begin{array}{c} \text{Equation 1} \\ \text{Equation 2} \end{array} & \begin{bmatrix} 2 & -1 & 5 \\ 1 & 4 & 7 \end{bmatrix} \\ & & \begin{array}{ccc} x & y & \end{array} \end{array}$$

The augmented matrix contains the same information as the system but in a simpler form. The operations we learned for solving systems of equations can now be performed on the augmented matrix.

■ Matrices

We begin by defining the various elements that make up a matrix.

Definition of Matrix

An $m \times n$ **matrix** is a rectangular array of numbers with m **rows** and n **columns**.

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}}_{n \text{ columns}} \left.\begin{array}{c} \leftarrow \\ \leftarrow \\ \leftarrow \\ \\ \leftarrow \end{array}\right\} m \text{ rows}$$

We say that the matrix has **dimension** $m \times n$. The numbers a_{ij} are the **entries** of the matrix. The subscript on the entry a_{ij} indicates that it is in the ith row and the jth column.

Here are some examples of matrices.

Matrix	Dimension	
$\begin{bmatrix} 1 & 3 & 0 \\ 2 & 4 & -1 \end{bmatrix}$	2×3	2 rows by 3 columns
$\begin{bmatrix} 6 & -5 & 0 & 1 \end{bmatrix}$	1×4	1 row by 4 columns

■ The Augmented Matrix of a Linear System

We can write a system of linear equations as a matrix, called the **augmented matrix** of the system, by writing only the coefficients and constants that appear in the equations. Here is an example.

Linear system

$$\begin{cases} 3x - 2y + z = 5 \\ x + 3y - z = 0 \\ -x \phantom{{}+3y} + 4z = 11 \end{cases}$$

Augmented matrix

$$\begin{bmatrix} 3 & -2 & 1 & 5 \\ 1 & 3 & -1 & 0 \\ -1 & 0 & 4 & 11 \end{bmatrix}$$

Notice that a missing variable in an equation corresponds to a 0 entry in the augmented matrix.

Example 1 ■ Finding the Augmented Matrix of a Linear System

Write the augmented matrix of the following system of equations:

$$\begin{cases} 6x - 2y - z = 4 \\ x + 3z = 1 \\ 7y + z = 5 \end{cases}$$

Solution First we write the linear system with the variables lined up in columns.

$$\begin{cases} 6x - 2y - z = 4 \\ x \phantom{{}-2y} + 3z = 1 \\ \phantom{6x -{}} 7y + z = 5 \end{cases}$$

The augmented matrix is the matrix whose entries are the coefficients and the constants in this system.

$$\begin{bmatrix} 6 & -2 & -1 & 4 \\ 1 & 0 & 3 & 1 \\ 0 & 7 & 1 & 5 \end{bmatrix}$$

Now Try Exercise 11 ■

■ Elementary Row Operations

The operations that we used in Section 9.2 to solve linear systems correspond to operations on the rows of the augmented matrix of the system. For example, adding a multiple of one equation to another corresponds to adding a multiple of one row to another.

Elementary Row Operations

1. Add a multiple of one row to another.
2. Multiply a row by a nonzero constant.
3. Interchange two rows.

Performing any of these operations on the augmented matrix of a system does not change its solution. We use the following notation to describe the elementary row operations:

Symbol	Description
$R_i + kR_j \to R_i$	Change the ith row by adding k times row j to it, and then put the result back in row i.
kR_i	Multiply the ith row by k.
$R_i \leftrightarrow R_j$	Interchange the ith and jth rows.

In the next example we compare the two ways of writing systems of linear equations.

Example 2 ■ Using Elementary Row Operations to Solve a Linear System

Solve the following system of linear equations:

$$\begin{cases} x - y + 3z = 4 \\ x + 2y - 2z = 10 \\ 3x - y + 5z = 14 \end{cases}$$

Solution Our goal is to eliminate the x-term from the second equation and the x- and y-terms from the third equation. For comparison we write both the system of equations and its augmented matrix for each step.

System | **Augmented matrix**

$$\begin{cases} x - y + 3z = 4 \\ x + 2y - 2z = 10 \\ 3x - y + 5z = 14 \end{cases} \qquad \begin{bmatrix} 1 & -1 & 3 & 4 \\ 1 & 2 & -2 & 10 \\ 3 & -1 & 5 & 14 \end{bmatrix}$$

Add $(-1) \times$ Equation 1 to Equation 2.
Add $(-3) \times$ Equation 1 to Equation 3.

$$\begin{cases} x - y + 3z = 4 \\ 3y - 5z = 6 \\ 2y - 4z = 2 \end{cases} \qquad \xrightarrow[R_3 - 3R_1 \to R_3]{R_2 - R_1 \to R_2} \begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 3 & -5 & 6 \\ 0 & 2 & -4 & 2 \end{bmatrix}$$

Multiply Equation 3 by $\frac{1}{2}$.

$$\begin{cases} x - y + 3z = 4 \\ 3y - 5z = 6 \\ y - 2z = 1 \end{cases} \qquad \xrightarrow{\frac{1}{2}R_3} \begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 3 & -5 & 6 \\ 0 & 1 & -2 & 1 \end{bmatrix}$$

Add $(-3) \times$ Equation 3 to Equation 2 (to eliminate y from Equation 2).

$$\begin{cases} x - y + 3z = 4 \\ z = 3 \\ y - 2z = 1 \end{cases} \qquad \xrightarrow{R_2 - 3R_3 \to R_2} \begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 1 & -2 & 1 \end{bmatrix}$$

Interchange Equations 2 and 3.

$$\begin{cases} x - y + 3z = 4 \\ y - 2z = 1 \\ z = 3 \end{cases} \qquad \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Now we use back-substitution to find that $x = 2$, $y = 7$, and $z = 3$. The solution is $(2, 7, 3)$.

Now Try Exercise 29 ■

■ Gaussian Elimination

In general, to solve a system of linear equations using its augmented matrix, we use elementary row operations to arrive at a matrix in a certain form. This form is described in the following box.

Row-Echelon Form and Reduced Row-Echelon Form of a Matrix

A matrix is in **row-echelon form** if it satisfies the following conditions.

1. The first nonzero number in each row (reading from left to right) is 1. This is called the **leading entry**.
2. The leading entry in each row is to the right of the leading entry in the row immediately above it.
3. All rows consisting entirely of zeros are at the bottom of the matrix.

A matrix is in **reduced row-echelon** form if it is in row-echelon form and also satisfies the following condition.

4. Every number above and below each leading entry is a 0.

In the following matrices the first one is not in row-echelon form. The second one *is* in row-echelon form, and the third one is in reduced row-echelon form. The entries in red are the leading entries.

Not in row-echelon form

$$\begin{bmatrix} 0 & 1 & -\frac{1}{2} & 0 & 6 \\ 1 & 0 & 3 & 4 & -5 \\ 0 & 0 & 0 & 1 & 0.4 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

Leading 1's do *not* shift to the right in successive rows

Row-echelon form

$$\begin{bmatrix} 1 & 3 & -6 & 10 & 0 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Leading 1's shift to the right in successive rows

Reduced row-echelon form

$$\begin{bmatrix} 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Leading 1's have 0's above and below them

Here is a systematic way to put a matrix into row-echelon form using elementary row operations:

- Start by obtaining 1 in the top left corner. Then obtain zeros below that 1 by adding appropriate multiples of the first row to the rows below it.
- Next, obtain a leading 1 in the next row, and then obtain zeros below that 1.
- At each stage make sure that every leading entry is to the right of the leading entry in the row above it—rearrange the rows if necessary.
- Continue this process until you arrive at a matrix in row-echelon form.

This is how the process might work for a 3×4 matrix:

$$\begin{bmatrix} 1 & \blacksquare & \blacksquare & \blacksquare \\ 0 & \blacksquare & \blacksquare & \blacksquare \\ 0 & \blacksquare & \blacksquare & \blacksquare \end{bmatrix} \qquad \begin{bmatrix} 1 & \blacksquare & \blacksquare & \blacksquare \\ 0 & 1 & \blacksquare & \blacksquare \\ 0 & 0 & \blacksquare & \blacksquare \end{bmatrix} \qquad \begin{bmatrix} 1 & \blacksquare & \blacksquare & \blacksquare \\ 0 & 1 & \blacksquare & \blacksquare \\ 0 & 0 & 1 & \blacksquare \end{bmatrix}$$

Once an augmented matrix is in row-echelon form, we can solve the corresponding linear system using back-substitution. This technique is called **Gaussian elimination**, in honor of its inventor, the German mathematician C. F. Gauss (see Section 3.5).

Solving a System Using Gaussian Elimination

1. **Augmented Matrix.** Write the augmented matrix of the system.
2. **Row-Echelon Form.** Use elementary row operations to change the augmented matrix to row-echelon form.
3. **Back-Substitution.** Write the new system of equations that corresponds to the row-echelon form of the augmented matrix and solve by back-substitution.

Example 3 ■ Solving a System Using Row-Echelon Form

Solve the following system of linear equations using Gaussian elimination:

$$\begin{cases} 4x + 8y - 4z = 4 \\ 3x + 8y + 5z = -11 \\ -2x + y + 12z = -17 \end{cases}$$

Solution We first write the augmented matrix of the system, and then we use elementary row operations to put it in row-echelon form.

Augmented matrix:

Need a 1 here

$$\begin{bmatrix} 4 & 8 & -4 & 4 \\ 3 & 8 & 5 & -11 \\ -2 & 1 & 12 & -17 \end{bmatrix}$$

Need 0's here

$$\xrightarrow{\frac{1}{4}R_1} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 3 & 8 & 5 & -11 \\ -2 & 1 & 12 & -17 \end{bmatrix}$$

Need a 1 here

$$\xrightarrow[R_3 + 2R_1 \to R_3]{R_2 - 3R_1 \to R_2} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 2 & 8 & -14 \\ 0 & 5 & 10 & -15 \end{bmatrix}$$

Need a 0 here

$$\xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 4 & -7 \\ 0 & 5 & 10 & -15 \end{bmatrix}$$

Need a 1 here

$$\xrightarrow{R_3 - 5R_2 \to R_3} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 4 & -7 \\ 0 & 0 & -10 & 20 \end{bmatrix}$$

Row-echelon form:

$$\xrightarrow{-\frac{1}{10}R_3} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

We now have an equivalent matrix in row-echelon form, and the corresponding system of equations is

$$\begin{cases} x + 2y - z = 1 \\ \quad y + 4z = -7 \\ \qquad z = -2 \end{cases}$$

Back-substitute: We use back-substitution to solve the system.

$$y + 4(-2) = -7 \qquad \text{Back-substitute } z = -2 \text{ into Equation 2}$$

$$y = 1 \qquad \text{Solve for } y$$

$$x + 2(1) - (-2) = 1 \qquad \text{Back-substitute } y = 1 \text{ and } z = -2 \text{ into Equation 1}$$

$$x = -3 \qquad \text{Solve for } x$$

So the solution of the system is $(-3, 1, -2)$.

Now Try Exercise 31 ■

Graphing devices are able to work with matrices. Matrices are stored in the memory of the device using names such as [A], [B], [C], ... On graphing devices the `ref` command gives the row-echelon form of a matrix. For the augmented matrix in Example 3 the `ref`

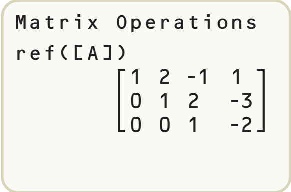


Figure 1

command gives the output shown in Figure 1. Notice that the row-echelon form in Figure 1 differs from the one we got in Example 3. This is because the graphing device used different row operations than we did. You should check that the row-echelon form that your device gives you leads to the same solution as ours.

Gauss-Jordan Elimination

If we put the augmented matrix of a linear system in *reduced* row-echelon form, then we don't need to back-substitute to solve the system. To put a matrix in reduced row-echelon form, we use the following steps.

- Use the elementary row operations to put the matrix in row-echelon form.
- Obtain zeros above each leading entry by adding multiples of the row containing that entry to the rows above it. Begin with the last leading entry and work upward.

Here is how the process works for a 3×4 matrix:

$$\begin{bmatrix} 1 & \blacksquare & \blacksquare & \blacksquare \\ 0 & 1 & \blacksquare & \blacksquare \\ 0 & 0 & 1 & \blacksquare \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \blacksquare & 0 & \blacksquare \\ 0 & 1 & 0 & \blacksquare \\ 0 & 0 & 1 & \blacksquare \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & \blacksquare \\ 0 & 1 & 0 & \blacksquare \\ 0 & 0 & 1 & \blacksquare \end{bmatrix}$$

Using the reduced row-echelon form to solve a system is called **Gauss-Jordan elimination**. The process is illustrated in the next example.

Example 4 ■ Solving a System Using Reduced Row-Echelon Form

Solve the following system of linear equations using Gauss-Jordan elimination:

$$\begin{cases} 4x + 8y - 4z = 4 \\ 3x + 8y + 5z = -11 \\ -2x + y + 12z = -17 \end{cases}$$

Solution In Example 3 we used Gaussian elimination on the augmented matrix of this system to arrive at an equivalent matrix in row-echelon form. We continue using elementary row operations on the last matrix in Example 3 to arrive at an equivalent matrix in reduced row-echelon form.

Need 0's here

$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

Need a 0 here

$$\xrightarrow[\mathrm{R}_1 + \mathrm{R}_3 \rightarrow \mathrm{R}_1]{\mathrm{R}_2 - 4\mathrm{R}_3 \rightarrow \mathrm{R}_2} \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$$\xrightarrow{\mathrm{R}_1 - 2\mathrm{R}_2 \rightarrow \mathrm{R}_1} \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

We now have an equivalent matrix in reduced row-echelon form, and the corresponding system of equations is

$$\begin{cases} x = -3 \\ y = 1 \\ z = -2 \end{cases}$$

Since the system is in reduced row-echelon form, back-subsitution is not required to get the solution.

So the solution is $(-3, 1, -2)$.

Now Try Exercise 33 ■

```
Matrix Operations
rref([A])
        [1 0 0 -3]
        [0 1 0  1]
        [0 0 1 -2]
```

Figure 2

Graphing devices also have a command that puts a matrix in reduced row-echelon form. (The command is usually `rref`.) For the augmented matrix in Example 4 the `rref` command gives the output shown in Figure 2. The device gives the same reduced row-echelon form as the one we got in Example 4. This is because every matrix has a *unique* reduced row-echelon form.

Inconsistent and Dependent Systems

The systems of linear equations that we considered in Examples 1–4 had exactly one solution. But as we know from Section 9.2, a linear system may have one solution, no solution, or infinitely many solutions. Fortunately, the row-echelon form of a system allows us to determine which of these cases applies, as described in the following box.

First we need some terminology: A **leading variable** in a linear system is one that corresponds to a leading entry in the row-echelon form of the augmented matrix of the system.

The Solutions of a Linear System in Row-Echelon Form

Suppose the augmented matrix of a system of linear equations has been transformed by Gaussian elimination into row-echelon form. Then exactly one of the following is true.

1. **No solution.** If the row-echelon form contains a row that represents the equation $0 = c$, where c is not zero, then the system has no solution. A system with no solution is called **inconsistent**.
2. **One solution.** If each variable in the row-echelon form is a leading variable, then the system has exactly one solution, which we find using back-substitution or Gauss-Jordan elimination.
3. **Infinitely many solutions.** If the variables in the row-echelon form are not all leading variables and if the system is not inconsistent, then it has infinitely many solutions. In this case the system is called **dependent**. We solve the system by putting the matrix in reduced row-echelon form and then expressing the leading variables in terms of the nonleading variables. The nonleading variables may take on any real numbers as their values.

The matrices below, all in row-echelon form, illustrate the three cases described in the above box.

No solution

$$\begin{bmatrix} 1 & 2 & 5 & 7 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Last equation says 0 = 1

One solution

$$\begin{bmatrix} 1 & 6 & -1 & 3 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 1 & 8 \end{bmatrix}$$

Each variable is a leading variable

Infinitely many solutions

$$\begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

z is not a leading variable

Example 5 ■ A System with No Solution

Solve the following system:

$$\begin{cases} x - 3y + 2z = 12 \\ 2x - 5y + 5z = 14 \\ x - 2y + 3z = 20 \end{cases}$$

Solution We transform the system into row-echelon form.

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & 12 \\ 2 & -5 & 5 & 14 \\ 1 & -2 & 3 & 20 \end{array}\right] \xrightarrow[R_3 - R_1 \to R_3]{R_2 - 2R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & -3 & 2 & 12 \\ 0 & 1 & 1 & -10 \\ 0 & 1 & 1 & 8 \end{array}\right]$$

$$\xrightarrow{R_3 - R_2 \to R_3} \left[\begin{array}{ccc|c} 1 & -3 & 2 & 12 \\ 0 & 1 & 1 & -10 \\ 0 & 0 & 0 & 18 \end{array}\right] \xrightarrow{\frac{1}{18}R_3} \left[\begin{array}{ccc|c} 1 & -3 & 2 & 12 \\ 0 & 1 & 1 & -10 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

This last matrix is in row-echelon form, so we can stop the Gaussian elimination process. Now if we translate the last row back into equation form, we get $0x + 0y + 0z = 1$, or $0 = 1$, which is false. No matter what values we choose for x, y, and z, the last equation will never be a true statement. This means that the system *has no solution.*

Now Try Exercise 39 ■

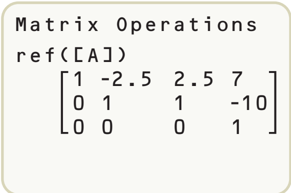


Figure 3

Figure 3 shows the row-echelon form produced by a graphing device for the augmented matrix in Example 5. You should check that the device result agrees with Example 5.

Example 6 ■ A System with Infinitely Many Solutions

Find the complete solution of the following system:

$$\begin{cases} -3x - 5y + 36z = 10 \\ -x + 7z = 5 \\ x + y - 10z = -4 \end{cases}$$

Solution We transform the system into reduced row-echelon form.

$$\left[\begin{array}{ccc|c} -3 & -5 & 36 & 10 \\ -1 & 0 & 7 & 5 \\ 1 & 1 & -10 & -4 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & -10 & -4 \\ -1 & 0 & 7 & 5 \\ -3 & -5 & 36 & 10 \end{array}\right]$$

$$\xrightarrow[R_3 + 3R_1 \to R_3]{R_2 + R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & -10 & -4 \\ 0 & 1 & -3 & 1 \\ 0 & -2 & 6 & -2 \end{array}\right] \xrightarrow{R_3 + 2R_2 \to R_3} \left[\begin{array}{ccc|c} 1 & 1 & -10 & -4 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 - R_2 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & -7 & -5 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The third row corresponds to the equation $0 = 0$. This equation is always true, no matter what values are used for x, y, and z. Since the equation adds no new information about the variables, we can drop it from the system. So the last matrix corresponds to the system

$$\begin{cases} x - 7z = -5 \\ y - 3z = 1 \end{cases} \qquad \begin{array}{l} \text{Equation 1} \\ \text{Equation 2} \end{array}$$

Leading variables

Now we solve for the leading variables x and y in terms of the nonleading variable z.

$$x = 7z - 5 \qquad \text{Solve for } x \text{ in Equation 1}$$

$$y = 3z + 1 \qquad \text{Solve for } y \text{ in Equation 2}$$

To obtain the complete solution, we let z be any real number t, and we express x, y, and z in terms of t.

$$x = 7t - 5$$
$$y = 3t + 1$$
$$z = t$$

We can also write the solution as the ordered triple $(7t - 5, 3t + 1, t)$, where t is any real number.

Now Try Exercise 41 ■

In Example 6, to get specific solutions, we give a specific value to t. For example, if $t = 1$, then

$$x = 7(1) - 5 = 2$$
$$y = 3(1) + 1 = 4$$
$$z = 1$$

Here are some other solutions of the system obtained by substituting other values for the parameter t.

Parameter t	**Solution $(7t - 5, 3t + 1, t)$**
-1	$(-12, -2, -1)$
0	$(-5, 1, 0)$
2	$(9, 7, 2)$
5	$(30, 16, 5)$

Example 7 ■ A System with Infinitely Many Solutions

Find the complete solution of the following system:

$$\begin{cases} x + 2y - 3z - 4w = 10 \\ x + 3y - 3z - 4w = 15 \\ 2x + 2y - 6z - 8w = 10 \end{cases}$$

Solution We transform the system into reduced row-echelon form.

$$\begin{bmatrix} 1 & 2 & -3 & -4 & 10 \\ 1 & 3 & -3 & -4 & 15 \\ 2 & 2 & -6 & -8 & 10 \end{bmatrix} \xrightarrow[R_3 - 2R_1 \to R_3]{R_2 - R_1 \to R_2} \begin{bmatrix} 1 & 2 & -3 & -4 & 10 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & -2 & 0 & 0 & -10 \end{bmatrix}$$

$$\xrightarrow{R_3 + 2R_2 \to R_3} \begin{bmatrix} 1 & 2 & -3 & -4 & 10 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 - 2R_2 \to R_1} \begin{bmatrix} 1 & 0 & -3 & -4 & 0 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The last matrix is in reduced row-echelon form. Since the last row represents the

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Discovery Project ■ Computer Graphics I

One significant use of matrices is in the digital representation of images. A camera or scanner converts an image into a matrix by dividing the image into a rectangular array of elements called pixels (like the pixelated image of a toucan shown here) and then assigning a value to each pixel. The value represents the color or brightness of that pixel. In this project we'll discover how changing the numbers in such a matrix can enhance an image by changing the contrast or brightness. You can find the project at **www.stewartmath.com**.

equation $0 = 0$, we may discard it. So the last matrix corresponds to the system

$$\begin{cases} x \quad -3z - 4w = 0 \\ \quad y \qquad\qquad = 5 \end{cases}$$

Leading variables

To obtain the complete solution, we solve for the leading variables x and y in terms of the nonleading variables z and w, and we let z and w be any real numbers s and t, respectively. Thus the complete solution is

$$\begin{aligned} x &= 3s + 4t \\ y &= 5 \\ z &= s \\ w &= t \end{aligned}$$

where s and t are any real numbers.

Now Try Exercise 61 ■

Note that s and t do *not* have to be the *same* real number in the solution for Example 7. We can choose arbitrary values for each if we wish to construct a specific solution to the system. For example, if we let $s = 1$ and $t = 2$, then we get the solution $(11, 5, 1, 2)$. You should check that this does indeed satisfy all three of the original equations in Example 7.

Examples 6 and 7 illustrate this general fact: If a system in row-echelon form has n nonzero equations in m variables $(m > n)$, then the complete solution will have $m - n$ nonleading variables. For instance, in Example 6 we arrived at *two* nonzero equations in the *three* variables x, y, and z, which gave us $3 - 2 = 1$ nonleading variable.

■ Modeling with Linear Systems

Linear equations, often containing hundreds or even thousands of variables, occur frequently in the applications of algebra to the sciences and to other fields. For now, let's consider an example that involves only three variables.

Example 8 ■ Nutritional Analysis Using a System of Linear Equations

A nutritionist is performing an experiment with student volunteers. A subject receives a daily diet that consists of a combination of three commercial diet foods: MiniCal, LiquiFast, and SlimQuick. For the experiment it is important that the subject consume exactly 500 mg of potassium, 75 g of protein, and 1150 units of vitamin D every day. The amounts of these nutrients in 1 oz of each food are given in the table. How many ounces of each food should the subject eat every day to satisfy the nutrient requirements exactly?

	MiniCal	LiquiFast	SlimQuick
Potassium (mg)	50	75	10
Protein (g)	5	10	3
Vitamin D (units)	90	100	50

Solution Let x, y, and z represent the number of ounces of MiniCal, LiquiFast, and SlimQuick, respectively, that the subject should eat every day. This means that the subject will get $50x$ mg of potassium from MiniCal, $75y$ mg from LiquiFast, and $10z$ mg from SlimQuick, for a total of $50x + 75y + 10z$ mg potassium in all. Since

```
Matrix Operations
rref([A])
        [1  0  0  5 ]
        [0  1  0  2 ]
        [0  0  1  10]
```

Figure 4

the potassium requirement is 500 mg, we get the first equation below. Similar reasoning for the protein and vitamin D requirements leads to the system

$$\begin{cases} 50x + 75y + 10z = 500 & \text{Potassium} \\ 5x + 10y + 3z = 75 & \text{Protein} \\ 90x + 100y + 50z = 1150 & \text{Vitamin D} \end{cases}$$

Dividing the first equation by 5 and the third one by 10 gives the system

$$\begin{cases} 10x + 15y + 2z = 100 \\ 5x + 10y + 3z = 75 \\ 9x + 10y + 5z = 115 \end{cases}$$

We can solve this system using Gaussian elimination, or we can use a graphing device to find the reduced row-echelon form of the augmented matrix of the system. Using the `rref` command on a graphing device, we get the output shown in Figure 4. From the reduced row-echelon form we see that $x = 5$, $y = 2$, $z = 10$. The subject should be served 5 oz of MiniCal, 2 oz of LiquiFast, and 10 oz of SlimQuick every day.

Check Your Answer

$x = 5, y = 2, z = 10$:

$$\begin{cases} 10(5) + 15(2) + 2(10) = 100 \\ 5(5) + 10(2) + 3(10) = 75 \\ 9(5) + 10(2) + 5(10) = 115 \end{cases} \checkmark$$

Now Try Exercise 69

A more practical application might involve dozens of foods and nutrients rather than just three. Such problems lead to systems with large numbers of variables and equations. Computers or graphing devices are essential for solving such large systems.

9.3 Exercises

Concepts

1. If a system of linear equations has infinitely many solutions, then the system is called ________. If a system of linear equations has no solution, then the system is called ________.

2. Write the augmented matrix of the following system of equations.

System

$$\begin{cases} x + y - z = 1 \\ x + 2z = -3 \\ 2y - z = 3 \end{cases}$$

Augmented matrix

$$\begin{bmatrix} \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \end{bmatrix}$$

3. The following matrix is the augmented matrix of a system of linear equations in the variables x, y, and z. (It is given in reduced row-echelon form.)

$$\begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) The leading variables are ________.

(b) Is the system inconsistent or dependent? ________

(c) The solution of the system is:

$x =$ ______, $y =$ ______, $z =$ ______

4. The augmented matrix of a system of linear equations is given in reduced row-echelon form. Find the solution of the system.

(a) $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$ $x =$ ______ $y =$ ______ $z =$ ______

(b) $\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $x =$ ______ $y =$ ______ $z =$ ______

(c) $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ $x =$ ______ $y =$ ______ $z =$ ______

Skills

5–10 ■ Dimension of a Matrix State the dimension of the matrix.

5. $\begin{bmatrix} 2 & 7 \\ 0 & -1 \\ 5 & -3 \end{bmatrix}$

6. $\begin{bmatrix} -1 & 5 & 4 & 0 \\ 0 & 2 & 11 & 3 \end{bmatrix}$

7. $\begin{bmatrix} 12 \\ 35 \end{bmatrix}$

8. $\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$

9. $\begin{bmatrix} 1 & 4 & 7 \end{bmatrix}$

10. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

11–12 ■ The Augmented Matrix Write the augmented matrix for the system of linear equations.

11. $\begin{cases} 3x + y - z = 2 \\ 2x - y = 1 \\ x - z = 3 \end{cases}$

12. $\begin{cases} -x + z = -1 \\ 3y - 2z = 7 \\ x - y + 3z = 3 \end{cases}$

13–20 ■ Form of a Matrix A matrix is given. **(a)** Determine whether the matrix is in row-echelon form. **(b)** Determine whether the matrix is in reduced row-echelon form. **(c)** Write the system of equations for which the given matrix is the augmented matrix.

13. $\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 5 \end{bmatrix}$

14. $\begin{bmatrix} 1 & 3 & -3 \\ 0 & 1 & 5 \end{bmatrix}$

15. $\begin{bmatrix} 1 & 2 & 8 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

16. $\begin{bmatrix} 1 & 0 & -7 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

17. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 5 & 1 \end{bmatrix}$

18. $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

19. $\begin{bmatrix} 1 & 3 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ **20.** $\begin{bmatrix} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$

21–24 ■ Elementary Row Operations Perform the indicated elementary row operation.

21. $\begin{bmatrix} -1 & 1 & 2 & 0 \\ 3 & 1 & 1 & 4 \\ 1 & -2 & -1 & -1 \end{bmatrix}$

Add 3 times Row 1 to Row 2.

22. $\begin{bmatrix} -5 & 2 & -3 & 3 \\ 10 & -3 & 1 & -20 \\ -1 & 3 & 1 & 8 \end{bmatrix}$

Add 2 times Row 1 to Row 2.

23. $\begin{bmatrix} 2 & 1 & -3 & 5 \\ 2 & 3 & 1 & 13 \\ 6 & -5 & -1 & 7 \end{bmatrix}$

Add -3 times Row 1 to Row 3.

24. $\begin{bmatrix} 1 & -3 & 2 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 2 & -1 & 1 \end{bmatrix}$

Add -2 times Row 2 to Row 3.

25–28 ■ Back-Substitution A matrix is given in row-echelon form. **(a)** Write the system of equations for which the given matrix is the augmented matrix. **(b)** Use back-substitution to solve the system.

25. $\begin{bmatrix} 1 & -2 & 4 & 3 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ **26.** $\begin{bmatrix} 1 & 1 & -3 & 8 \\ 0 & 1 & -3 & 5 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

27. $\begin{bmatrix} 1 & 2 & 3 & -1 & 7 \\ 0 & 1 & -2 & 0 & 5 \\ 0 & 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$

28. $\begin{bmatrix} 1 & 0 & -2 & 2 & 5 \\ 0 & 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$

29–38 ■ Linear Systems with One Solution The system of linear equations has a unique solution. Find the solution using Gaussian elimination or Gauss-Jordan elimination.

29. $\begin{cases} x - 2y + z = 1 \\ y + 2z = 5 \\ x + y + 3z = 8 \end{cases}$ **30.** $\begin{cases} x + y + 6z = 3 \\ x + y + 3z = 3 \\ x + 2y + 4z = 7 \end{cases}$

31. $\begin{cases} x - 2y + z = 1 \\ 3x + y + 2z = 4 \\ -2x - 3y + z = -5 \end{cases}$ **32.** $\begin{cases} x + z = 7 \\ 2x - y - 2z = -5 \\ 5x + 3y - z = 2 \end{cases}$

33. $\begin{cases} x - y - 2z = 0 \\ 2x + 3y + z = -5 \\ 3x + y - 8z = -16 \end{cases}$ **34.** $\begin{cases} x + z = 3 \\ x + 2y + z = 3 \\ 2x - 3y - 2z = 2 \end{cases}$

35. $\begin{cases} x_1 + 2x_2 - x_3 = 9 \\ 2x_1 - x_3 = -2 \\ 3x_1 + 5x_2 + 2x_3 = 22 \end{cases}$ **36.** $\begin{cases} 2x_1 + x_2 = 7 \\ 2x_1 - x_2 + x_3 = 6 \\ 3x_1 - 2x_2 + 4x_3 = 11 \end{cases}$

37. $\begin{cases} 2x - 3y - z = 13 \\ -x + 2y - 5z = 6 \\ 5x - y - z = 49 \end{cases}$ **38.** $\begin{cases} 10x + 10y - 20z = 60 \\ 15x + 20y + 30z = -25 \\ -5x + 30y - 10z = 45 \end{cases}$

39–48 ■ Dependent or Inconsistent Linear Systems Determine whether the system of linear equations is inconsistent or dependent. If it is dependent, find the complete solution.

39. $\begin{cases} x + 2y - z = 3 \\ 3x + 7y + 2z = 5 \\ 2x + 3y - 7z = 4 \end{cases}$ **40.** $\begin{cases} x - 2y + 3z = 4 \\ 3x - z = -3 \\ x + 4y - 7z = 2 \end{cases}$

41. $\begin{cases} 2x - 3y - 9z = -5 \\ x + 3z = 2 \\ -3x + y - 4z = -3 \end{cases}$ **42.** $\begin{cases} x - 2y + 5z = 3 \\ -2x + 6y - 11z = 1 \\ 3x - 16y + 20z = -26 \end{cases}$

43. $\begin{cases} x - y + 3z = 3 \\ 4x - 8y + 32z = 24 \\ 2x - 3y + 11z = 4 \end{cases}$ **44.** $\begin{cases} -2x + 6y - 2z = -12 \\ x - 3y + 2z = 10 \\ -x + 3y + 2z = 6 \end{cases}$

45. $\begin{cases} x + 4y - 2z = -3 \\ 2x - y + 5z = 12 \\ 8x + 5y + 11z = 30 \end{cases}$ **46.** $\begin{cases} 3r + 2s - 3t = 10 \\ r - s - t = -5 \\ r + 4s - t = 20 \end{cases}$

47. $\begin{cases} 2x + y - 2z = 12 \\ -x - \frac{1}{2}y + z = -6 \\ 3x + \frac{3}{2}y - 3z = 18 \end{cases}$ **48.** $\begin{cases} y - 5z = 7 \\ 3x + 2y = 12 \\ 3x + 10z = 80 \end{cases}$

49–64 ■ Solving a Linear System Solve the system of linear equations.

49. $\begin{cases} 4x - 3y + z = -8 \\ -2x + y - 3z = -4 \\ x - y + 2z = 3 \end{cases}$ **50.** $\begin{cases} 2x - 3y + 5z = 14 \\ 4x - y - 2z = -17 \\ -x - y + z = 3 \end{cases}$

51. $\begin{cases} 3x - y + z = 3 \\ x - 2z = 4 \\ 2x + y - 11z = 1 \end{cases}$ **52.** $\begin{cases} x - 3y + 2z = 5 \\ 2x - 3y - 2z = -2 \\ -x + 4z = 7 \end{cases}$

53. $\begin{cases} x + 2y - 3z = -5 \\ -2x - 4y - 6z = 10 \\ 3x + 7y - 2z = -13 \end{cases}$ **54.** $\begin{cases} 3x + y = 2 \\ -4x + 3y + z = 4 \\ 2x + 5y + z = 0 \end{cases}$

55. $\begin{cases} x - y + 6z = 8 \\ x + z = 5 \\ x + 3y - 14z = -4 \end{cases}$ **56.** $\begin{cases} 3x - y + 2z = -1 \\ 4x - 2y + z = -7 \\ -x + 3y - 2z = -1 \end{cases}$

57. $\begin{cases} -x + 2y + z - 3w = 3 \\ 3x - 4y + z + w = 9 \\ -x - y + z + w = 0 \\ 2x + y + 4z - 2w = 3 \end{cases}$

58. $\begin{cases} x + y - z - w = 6 \\ 2x + z - 3w = 8 \\ x - y + 4w = -10 \\ 3x + 5y - z - w = 20 \end{cases}$

59. $\begin{cases} x + y + 2z - w = -2 \\ 3y + z + 2w = 2 \\ x + y + 3w = 2 \\ -3x + z + 2w = 5 \end{cases}$

60. $\begin{cases} x - 3y + 2z + w = -2 \\ x - 2y - 2w = -10 \\ z + 5w = 15 \\ 3x + 2z + w = -3 \end{cases}$

61. $\begin{cases} x - y \quad + w = 0 \\ 3x \quad - z + 2w = 0 \\ x - 4y + z + 2w = 0 \end{cases}$ **62.** $\begin{cases} 2x - y + 2z + w = 5 \\ -x + y + 4z - w = 3 \\ 3x - 2y - z \quad = 0 \end{cases}$

63. $\begin{cases} x \quad + z + w = 4 \\ y - z \quad = -4 \\ x - 2y + 3z + w = 12 \\ 2x \quad - 2z + 5w = -1 \end{cases}$

64. $\begin{cases} y - z + 2w = 0 \\ 3x + 2y \quad + w = 0 \\ 2x \quad + 4w = 12 \\ -2x \quad - 2z + 5w = 6 \end{cases}$

65–68 ■ Solving a Linear System Using a Graphing Device Solve the system of linear equations by using the `rref` command on a graphing device. State your answer rounded to two decimal places.

65. $\begin{cases} 0.75x - 3.75y + 2.95z = 4.0875 \\ 0.95x - 8.75y \quad = 3.375 \\ 1.25x - 0.15y + 2.75z = 3.6625 \end{cases}$

66. $\begin{cases} 1.31x + 2.72y - 3.71z = -13.9534 \\ -0.21x \quad + 3.73z = 13.4322 \\ 2.34y - 4.56z = -21.3984 \end{cases}$

67. $\begin{cases} 42x - 31y \quad - 42w = -0.4 \\ -6x \quad - 9w = 4.5 \\ 35x \quad - 67z + 32w = 348.8 \\ 31y + 48z - 52w = -76.6 \end{cases}$

68. $\begin{cases} 49x - 27y + 52z \quad = -145.0 \\ 27y \quad + 43w = -118.7 \\ -31y + 42z \quad = -72.1 \\ 73x - 54y \quad = -132.7 \end{cases}$

Applications

69. Nutrition A doctor recommends that a patient take 50 mg each of niacin, riboflavin, and thiamin daily to alleviate a vitamin deficiency. The patient has three brands of vitamin pills. The amounts of the relevant vitamins per pill are given in the table. How many pills of each type should be taken every day to get 50 mg of each vitamin?

	VitaMax	Vitron	VitaPlus
Niacin (mg)	5	10	15
Riboflavin (mg)	15	20	0
Thiamin (mg)	10	10	10

70. Mixtures A chemist has three acid solutions at various concentrations. The first is 10% acid, the second is 20%, and the third is 40%. How many milliliters of each solution should be used to make 100 mL of 18% solution, if four times as much of the 10% solution is used as the 40% solution?

71. Distance, Speed, and Time Athlete A, Athlete B, and Athlete C enter a race in which they have to run, swim, and cycle over a marked course. Their average speeds are given in the table. Athlete C finishes first with a total time of 1 h 45 min. Athlete A comes in second with a time of 2 h 30 min. Athlete B finishes last with a time of 3 h. Find the distance (in mi) for each part of the race.

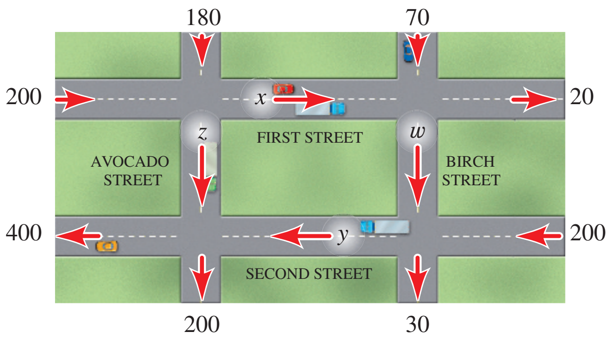
	Average Speed (mi/h)		
	Running	**Swimming**	**Cycling**
Athlete A	10	4	20
Athlete B	$7\frac{1}{2}$	6	15
Athlete C	15	3	40

72. Classroom Use A small school has 100 students who occupy three classrooms: A, B, and C. After the first period of the school day, half the students in room A move to room B, one-fifth of the students in room B move to room C, and one-third of the students in room C move to room A. Nevertheless, the total number of students in each room is the same for both periods. How many students occupy each room?

73. Manufacturing Furniture A furniture factory makes wooden tables, chairs, and armoires. Each piece of furniture requires three operations: cutting the wood, assembling, and finishing. Each operation requires the number of hours given in the table. The workers in the factory can provide 300 hours of cutting, 400 hours of assembling, and 590 hours of finishing each work week. How many tables, chairs, and armoires should be produced so that all available labor-hours are used? Or is this impossible?

	Table	Chair	Armoire
Cutting (h)	$\frac{1}{2}$	1	1
Assembling (h)	$\frac{1}{2}$	$1\frac{1}{2}$	1
Finishing (h)	1	$1\frac{1}{2}$	2

74. Traffic Flow A section of a city's street network is shown in the figure. The arrows indicate one-way streets, and the numbers show how many cars enter or leave this section of the city via the indicated street in a certain one-hour period. The variables x, y, z, and w represent the number of cars that travel along the portions of First, Second, Avocado, and Birch Streets during this period. Find x, y, z, and w, assuming that none of the cars stop or park on any of the streets shown.



■ Discuss ■ Discover ■ Prove ■ Write

75. Discuss: Polynomials Determined by a Set of Points Two points uniquely determine a line $y = ax + b$ in the coordinate plane. Similarly, three points uniquely determine a quadratic (second-degree) polynomial

$$y = ax^2 + bx + c$$

four points uniquely determine a cubic (third-degree) polynomial

$$y = ax^3 + bx^2 + cx + d$$

and so on. (Some exceptions to this rule occur if the three points actually lie on a line, or the four points lie on a quadratic or line, and so on.) For the following set of five points, find the line that contains the first two points, the quadratic that contains the first three points, the cubic that contains the first four points, and the fourth-degree polynomial that contains all five points.

$$(0, 0), \quad (1, 12), \quad (2, 40), \quad (3, 6), \quad (-1, -14)$$

Graph the points and functions in the same viewing rectangle using a graphing device.

9.4 The Algebra of Matrices

■ Equality of Matrices ■ Addition, Subtraction, and Scalar Multiplication of Matrices ■ Multiplication of Matrices ■ Properties of Matrix Multiplication ■ Applications of Matrix Multiplication

Thus far, we have used matrices simply for notational convenience when solving linear systems. Matrices have many other uses in mathematics and the sciences, and for most of these applications a knowledge of matrix algebra is essential. Like numbers, matrices can be added, subtracted, multiplied, and divided. In this section we learn how to perform these algebraic operations on matrices.

■ Equality of Matrices

Two matrices are equal if they have the same entries in the same positions.

Equal matrices

$$\begin{bmatrix} \sqrt{4} & 2^2 & e^0 \\ 0.5 & 1 & 1-1 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 1 \\ \frac{1}{2} & \frac{2}{2} & 0 \end{bmatrix}$$

Unequal matrices

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \neq \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

Equality of Matrices

The matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are **equal** if and only if they have the same dimension $m \times n$, and corresponding entries are equal, that is,

$$a_{ij} = b_{ij}$$

for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

Example 1 ■ Equal Matrices

Find a, b, c, and d if

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 5 & 2 \end{bmatrix}$$

Solution Since the two matrices are equal, corresponding entries must be the same. So we must have $a = 1$, $b = 3$, $c = 5$, and $d = 2$.

Now Try Exercises 5 and 7 ■

Courtesy UC Berkeley Office of Media Relations

JULIA ROBINSON (1919–1985) was born in St. Louis, Missouri, and grew up in Point Loma, California. Because of an illness, Robinson missed two years of school, but later, with the aid of a tutor, she completed fifth, sixth, seventh, and eighth grades, all in one year. Later, at San Diego State University, reading biographies of mathematicians in E. T. Bell's *Men of Mathematics* awakened in her what became a lifelong passion for mathematics. She said, "I cannot overemphasize the importance of such books . . . in the intellectual life of a student." Robinson is famous for her work on Hilbert's tenth problem (see Section 9.6), which asks for a general procedure for determining whether an equation has integer solutions. Her ideas led to a complete answer to the problem: the answer involved certain properties of the Fibonacci numbers (see Section 11.1) discovered by a 22-year-old Russian mathematician named Yuri Matijasevič. As a result of her brilliant work on Hilbert's tenth problem, Robinson was offered a professorship at the University of California, Berkeley, and became the first woman mathematician elected to the National Academy of Sciences. She also served as president of the American Mathematical Society.

Addition, Subtraction, and Scalar Multiplication of Matrices

Two matrices can be added or subtracted if they have the same dimension. (Otherwise, their sum or difference is undefined.) We add or subtract the matrices by adding or subtracting corresponding entries. To multiply a matrix by a number, we multiply every element of the matrix by that number. This is called the *scalar product*.

Sum, Difference, and Scalar Product of Matrices

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be matrices of the same dimension $m \times n$, and let c be any real number.

1. The **sum** $A + B$ is the $m \times n$ matrix obtained by adding corresponding entries of A and B.
$$A + B = [a_{ij} + b_{ij}]$$
2. The **difference** $A - B$ is the $m \times n$ matrix obtained by subtracting corresponding entries of A and B.
$$A - B = [a_{ij} - b_{ij}]$$
3. The **scalar product** cA is the $m \times n$ matrix obtained by multiplying each entry of A by c.
$$cA = [ca_{ij}]$$

Example 2 ■ Performing Algebraic Operations on Matrices

Let
$$A = \begin{bmatrix} 2 & -3 \\ 0 & 5 \\ 7 & -\frac{1}{2} \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ -3 & 1 \\ 2 & 2 \end{bmatrix}$$
$$C = \begin{bmatrix} 7 & -3 & 0 \\ 0 & 1 & 5 \end{bmatrix} \qquad D = \begin{bmatrix} 6 & 0 & -6 \\ 8 & 1 & 9 \end{bmatrix}$$

Carry out each indicated operation, or explain why it cannot be performed.

(a) $A + B$ **(b)** $C - D$ **(c)** $C + A$ **(d)** $5A$

Solution

(a) $$A + B = \begin{bmatrix} 2 & -3 \\ 0 & 5 \\ 7 & -\frac{1}{2} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -3 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ -3 & 6 \\ 9 & \frac{3}{2} \end{bmatrix}$$

(b) $$C - D = \begin{bmatrix} 7 & -3 & 0 \\ 0 & 1 & 5 \end{bmatrix} - \begin{bmatrix} 6 & 0 & -6 \\ 8 & 1 & 9 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 6 \\ -8 & 0 & -4 \end{bmatrix}$$

(c) $C + A$ is undefined because we can't add matrices of different dimensions.

(d) $$5A = 5\begin{bmatrix} 2 & -3 \\ 0 & 5 \\ 7 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 10 & -15 \\ 0 & 25 \\ 35 & -\frac{5}{2} \end{bmatrix}$$

Now Try Exercises 23 and 25 ■

The properties in the box follow from the definitions of matrix addition and scalar multiplication, together with the corresponding properties of real numbers.

Properties of Addition and Scalar Multiplication of Matrices

Let A, B, and C be $m \times n$ matrices and let c and d be scalars.

$A + B = B + A$	Commutative Property of Matrix Addition
$(A + B) + C = A + (B + C)$	Associative Property of Matrix Addition
$c(dA) = cdA$	Associative Property of Scalar Multiplication
$(c + d)A = cA + dA$ $c(A + B) = cA + cB$	Distributive Properties of Scalar Multiplication

Example 3 ■ Solving a Matrix Equation

Solve the matrix equation

$$2X - A = B$$

for the unknown matrix X, where

$$A = \begin{bmatrix} 2 & 3 \\ -5 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & -1 \\ 1 & 3 \end{bmatrix}$$

Solution We use the properties of matrices to solve for X.

$$2X - A = B \qquad \text{Given equation}$$

$$2X = B + A \qquad \text{Add the matrix } A \text{ to each side}$$

$$X = \tfrac{1}{2}(B + A) \qquad \text{Multiply each side by the scalar } \tfrac{1}{2}$$

So $$X = \frac{1}{2}\left(\begin{bmatrix} 4 & -1 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ -5 & 1 \end{bmatrix}\right) \qquad \text{Substitute the matrices } A \text{ and } B$$

$$= \frac{1}{2}\begin{bmatrix} 6 & 2 \\ -4 & 4 \end{bmatrix} \qquad \text{Add matrices}$$

$$= \begin{bmatrix} 3 & 1 \\ -2 & 2 \end{bmatrix} \qquad \text{Multiply by the scalar } \tfrac{1}{2}$$

Now Try Exercise 17 ■

■ Multiplication of Matrices

Multiplication of two matrices is more difficult to describe than other matrix operations. In later examples we will see why multiplying matrices involves a rather complex procedure, which we now describe.

First, the product AB (or $A \cdot B$) of two matrices A and B is defined only when the number of columns in A is equal to the number of rows in B. This means that if we write their dimensions side by side, the two inner numbers must match:

Matrices	A	B
Dimensions	$m \times n$	$n \times k$
	Columns in A	Rows in B

If we think of the row of A and the column of B as vectors, then their inner product is the same as their dot product (see Section 8.6).

If the dimensions of A and B match in this fashion, then the product AB is a matrix of dimension $m \times k$. Before describing how to obtain AB, we first define the **inner product** of a row of A and a column of B to be the number obtained by multiplying corresponding entries and adding the results as follows:

Row of A	**Column of B**	**Inner Product**
$\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$	$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$	$a_1b_1 + a_2b_2 + \cdots + a_nb_n$

For instance, consider the matrices D and B in Example 2. The inner product of the second row in D and the first column in B is

$$8 \cdot 1 + 1 \cdot (-3) + 9 \cdot 2 = 23$$

We now define the **product** AB of two matrices.

Matrix Multiplication

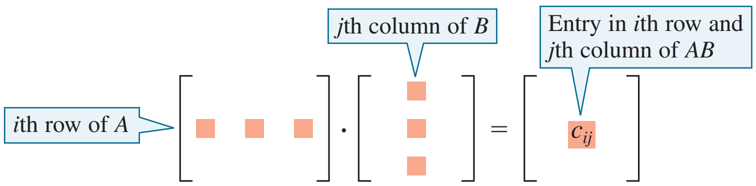
If $A = [a_{ij}]$ is an $m \times n$ matrix and $B = [b_{ij}]$ an $n \times k$ matrix, then their product is the $m \times k$ matrix

$$C = [c_{ij}]$$

where c_{ij} is the inner product of the ith row of A and the jth column of B. We write the product as

$$C = AB$$

This definition of matrix product says that each entry in the matrix AB is obtained from a *row* of A and a *column* of B as follows: The entry c_{ij} in the ith row and jth column of the matrix AB is obtained by multiplying the entries in the ith row of A with the corresponding entries in the jth column of B and adding the results.



Example 4 ■ Multiplying Matrices

Let

$$A = \begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix}$$

Calculate, if possible, the products AB and BA.

Inner numbers match, so product is defined

$2 \times 2 \quad 2 \times 3$

Outer numbers give dimension of product: 2×3

Solution Since A has dimension 2×2 and B has dimension 2×3, the product AB is defined and has dimension 2×3. We can therefore write

$$AB = \begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix} = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \end{bmatrix}$$

where the question marks must be filled in using the rule defining the product of two matrices. If we define $C = AB = [c_{ij}]$, then the entry c_{11} is the inner product of the first row of A and the first column of B:

$$\begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix} \qquad 1\cdot(-1) + 3\cdot 0 = -1$$

Similarly, we calculate the remaining entries of the product as follows.

Entry	Inner product of:	Value	Product matrix
c_{12}	$\begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix}$	$1\cdot 5 + 3\cdot 4 = 17$	$\begin{bmatrix} -1 & 17 & \\ & & \end{bmatrix}$
c_{13}	$\begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix}$	$1\cdot 2 + 3\cdot 7 = 23$	$\begin{bmatrix} -1 & 17 & 23 \\ & & \end{bmatrix}$
c_{21}	$\begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix}$	$(-1)\cdot(-1) + 0\cdot 0 = 1$	$\begin{bmatrix} -1 & 17 & 23 \\ 1 & & \end{bmatrix}$
c_{22}	$\begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix}$	$(-1)\cdot 5 + 0\cdot 4 = -5$	$\begin{bmatrix} -1 & 17 & 23 \\ 1 & -5 & \end{bmatrix}$
c_{23}	$\begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix}$	$(-1)\cdot 2 + 0\cdot 7 = -2$	$\begin{bmatrix} -1 & 17 & 23 \\ 1 & -5 & -2 \end{bmatrix}$

Thus we have $\qquad AB = \begin{bmatrix} -1 & 17 & 23 \\ 1 & -5 & -2 \end{bmatrix}$

Not equal, so product is not defined

$2 \times 3 \quad 2 \times 2$

The product BA is not defined, however, because the dimensions of B and A are

$$2 \times 3 \qquad \text{and} \qquad 2 \times 2$$

The inner two numbers are not the same, so the rows and columns won't match up if we try to calculate the product.

Now Try Exercise 27

Graphing devices are capable of performing matrix algebra. For instance, if we enter the matrices in Example 4 into the matrix variables `[A]` and `[B]` on a graphing device, then the device finds their product, as shown in Figure 1.

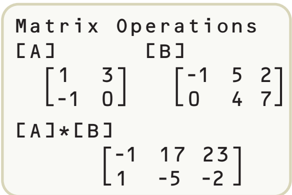


Figure 1

Properties of Matrix Multiplication

Although matrix multiplication is not commutative, it does obey the Associative and Distributive properties.

Properties of Matrix Multiplication

Let A, B, and C be matrices for which the following products are defined. Then

$A(BC) = (AB)C$	Associative Property
$A(B + C) = AB + AC$ $(B + C)A = BA + CA$	Distributive Property

The next example shows that even when both AB and BA are defined, they aren't necessarily equal. This proves that matrix multiplication is *not* commutative.

Example 5 ■ Matrix Multiplication Is Not Commutative

Let $$A = \begin{bmatrix} 5 & 7 \\ -3 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 \\ 9 & -1 \end{bmatrix}$$

Calculate the products AB and BA.

Solution Since both matrices A and B have dimension 2×2, both products AB and BA are defined, and each product is also a 2×2 matrix.

$$\begin{aligned} AB &= \begin{bmatrix} 5 & 7 \\ -3 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 9 & -1 \end{bmatrix} = \begin{bmatrix} 5\cdot 1 + 7\cdot 9 & 5\cdot 2 + 7\cdot(-1) \\ (-3)\cdot 1 + 0\cdot 9 & (-3)\cdot 2 + 0\cdot(-1) \end{bmatrix} \\ &= \begin{bmatrix} 68 & 3 \\ -3 & -6 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} BA &= \begin{bmatrix} 1 & 2 \\ 9 & -1 \end{bmatrix}\begin{bmatrix} 5 & 7 \\ -3 & 0 \end{bmatrix} = \begin{bmatrix} 1\cdot 5 + 2\cdot(-3) & 1\cdot 7 + 2\cdot 0 \\ 9\cdot 5 + (-1)\cdot(-3) & 9\cdot 7 + (-1)\cdot 0 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 7 \\ 48 & 63 \end{bmatrix} \end{aligned}$$

This shows that, in general, $AB \neq BA$. In fact, in this example AB and BA don't even have an entry in common.

Now Try Exercise 29 ■

■ Applications of Matrix Multiplication

We now consider some applied examples that give some indication of why mathematicians have chosen to define the matrix product in such an apparently bizarre fashion. Example 6 shows how our definition of matrix product allows us to express a system of linear equations as a single matrix equation.

Example 6 ■ Writing a Linear System as a Matrix Equation

Show that the following matrix equation is equivalent to the system of equations in Example 9.3.2.

Matrix equations like this one are described in more detail in Section 9.5.

$$\begin{bmatrix} 1 & -1 & 3 \\ 1 & 2 & -2 \\ 3 & -1 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ 14 \end{bmatrix}$$

Solution If we perform matrix multiplication on the left-hand side of the equation, we get

$$\begin{bmatrix} x - y + 3z \\ x + 2y - 2z \\ 3x - y + 5z \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ 14 \end{bmatrix}$$

Because two matrices are equal only if their corresponding entries are equal, we equate entries to get

$$\begin{cases} x - y + 3z = 4 \\ x + 2y - 2z = 10 \\ 3x - y + 5z = 14 \end{cases}$$

This is exactly the system of equations in Example 9.3.2.

Now Try Exercise 47 ■

Courtesy of the Archives, California Institute of Technology

OLGA TAUSSKY-TODD (1906–1995) was instrumental in developing applications of matrix theory. Described as "in love with anything matrices can do," she successfully applied matrices to aerodynamics, a field used in the design of airplanes and rockets. Taussky-Todd was also famous for her work in number theory, which deals with prime numbers and divisibility. Although number theory has often been called the least applicable branch of mathematics, it is now used in significant ways throughout the computer industry.

Taussky-Todd studied mathematics at a time when young women rarely aspired to be mathematicians. She said, "When I entered university I had no idea what it meant to study mathematics." One of the most respected mathematicians of her day, she was for many years a professor of mathematics at Caltech in Pasadena.

Example 7 ■ Representing Demographic Data by Matrices

In a certain city the proportions of voters in each age group who are registered as Democrats, Republicans, or Independents are given by the following matrix.

$$\begin{array}{l} \\ \\ \textbf{Democrat} \\ \textbf{Republican} \\ \textbf{Independent} \end{array} \begin{array}{c} \textbf{Age Group} \\ \begin{array}{ccc} \textbf{18–30} & \textbf{31–50} & \textbf{Over 50} \end{array} \\ \begin{bmatrix} 0.30 & 0.60 & 0.50 \\ 0.50 & 0.35 & 0.25 \\ 0.20 & 0.05 & 0.25 \end{bmatrix} \end{array} = A$$

The next matrix gives the distribution, by age group and sex, of the voting population of this city.

$$\textbf{Age Group} \quad \begin{array}{l} \\ \textbf{18–30} \\ \textbf{31–50} \\ \textbf{Over 50} \end{array} \begin{array}{c} \begin{array}{cc} \textbf{Male} & \textbf{Female} \end{array} \\ \begin{bmatrix} 5{,}000 & 6{,}000 \\ 10{,}000 & 12{,}000 \\ 12{,}000 & 15{,}000 \end{bmatrix} \end{array} = B$$

For this problem, let's make the (highly unrealistic) assumption that within each age group, political preference is not related to gender; that is, the percentage of Democratic males in the 18–30 group, for example, is the same as the percentage of Democratic females in this group.

(a) Calculate the product AB.

(b) How many males are registered as Democrats in this city?

(c) How many females are registered as Republicans?

Solution

(a) $$AB = \begin{bmatrix} 0.30 & 0.60 & 0.50 \\ 0.50 & 0.35 & 0.25 \\ 0.20 & 0.05 & 0.25 \end{bmatrix} \begin{bmatrix} 5{,}000 & 6{,}000 \\ 10{,}000 & 12{,}000 \\ 12{,}000 & 15{,}000 \end{bmatrix} = \begin{bmatrix} 13{,}500 & 16{,}500 \\ 9{,}000 & 10{,}950 \\ 4{,}500 & 5{,}550 \end{bmatrix}$$

(b) When we take the inner product of a row in A with a column in B, we are adding the number of people in each age group who belong to the category in question. For example, the entry c_{21} of AB (the 9000) is obtained by taking the inner product of the Republican row in A with the Male column in B. This number is therefore

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Discovery Project ■ Will the Species Survive?

To study how a species survives, scientists observe the stages in the life cycle of the species—for example, young, juvenile, adult. The proportion of the population at each stage and the proportion that survives to the next stage in each season are modeled by matrices. In this project we explore how matrix multiplication is used to predict the population proportions for the next season, the season after that, and so on, ultimately predicting the long-term prospects for the survival of the species. You can find the project at the book companion website: **www.stewartmath.com**.

the total number of male Republicans in this city. We can label the rows and columns of AB as follows.

$$\begin{array}{ll} & \begin{array}{cc} \textbf{Male} & \textbf{Female} \end{array} \\ \begin{array}{l} \textbf{Democrat} \\ \textbf{Republican} \\ \textbf{Independent} \end{array} & \begin{bmatrix} 13{,}500 & 16{,}500 \\ 9{,}000 & 10{,}950 \\ 4{,}500 & 5{,}550 \end{bmatrix} = AB \end{array}$$

Thus 13,500 males are registered as Democrats in this city.

(c) There are 10,950 females registered as Republicans.

Now Try Exercise 53 ■

In Example 7 the entries in each column of A add up to 1. (Can you see why this has to be true, given what the matrix describes?) A matrix with this property is called **stochastic**. Stochastic matrices are used extensively in statistics, where they arise frequently in situations like the one described here.

9.4 Exercises

Concepts

1. We can add (or subtract) two matrices only if they have the same ________________.

2. (a) We can multiply two matrices only if the number of ________ in the first matrix is the same as the number of ________ in the second matrix.

(b) If A is a 3×3 matrix and B is a 4×3 matrix, which of the following matrix multiplications are possible?

(i) AB (ii) BA (iii) AA (iv) BB

3. Which of the following operations can we perform for a matrix A of any dimension?

(i) $A + A$ (ii) $2A$ (iii) $A \cdot A$

4. Fill in the missing entries in the product matrix.

$$\begin{bmatrix} 3 & 1 & 2 \\ -1 & 2 & 0 \\ 1 & 3 & -2 \end{bmatrix}\begin{bmatrix} -1 & 3 & -2 \\ 3 & -2 & -1 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 4 & \square & -7 \\ 7 & -7 & \square \\ \square & -5 & -5 \end{bmatrix}$$

Skills

5–6 ■ Equality of Matrices Determine whether the matrices A and B are equal.

5. $A = \begin{bmatrix} 1 & -2 & 0 \\ \frac{1}{2} & 6 & 0 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & -2 \\ \frac{1}{2} & 6 \end{bmatrix}$

6. $A = \begin{bmatrix} \frac{1}{4} & \ln 1 \\ 2 & 3 \end{bmatrix}$ $\quad B = \begin{bmatrix} 0.25 & 0 \\ \sqrt{4} & \frac{6}{2} \end{bmatrix}$

7–8 ■ Equality of Matrices Find the values of a and b that make the matrices A and B equal.

7. $A = \begin{bmatrix} 3 & 4 \\ -1 & a \end{bmatrix}$ $\quad B = \begin{bmatrix} b & 4 \\ -1 & -5 \end{bmatrix}$

8. $A = \begin{bmatrix} 3 & 5 & 7 \\ -4 & a & 2 \end{bmatrix}$ $\quad B = \begin{bmatrix} 3 & 5 & b \\ -4 & -5 & 2 \end{bmatrix}$

9–16 ■ Matrix Operations Perform the matrix operation, or if it is impossible, explain why.

9. $\begin{bmatrix} 2 & 6 \\ -5 & 3 \end{bmatrix} + \begin{bmatrix} -1 & -3 \\ 6 & 2 \end{bmatrix}$

10. $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & -2 \end{bmatrix}$

11. $3\begin{bmatrix} 1 & 2 \\ 4 & -1 \\ 1 & 0 \end{bmatrix}$

12. $2\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}$

13. $\begin{bmatrix} 2 & 6 \\ 1 & 3 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 3 & 6 \\ -2 & 0 \end{bmatrix}$

14. $\begin{bmatrix} 2 & 1 & 2 \\ 6 & 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 3 & 6 \\ -2 & 0 \end{bmatrix}$

15. $\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 1 & -2 & 3 \\ 2 & 2 & -1 \end{bmatrix}$

16. $\begin{bmatrix} 2 & -3 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ 1 \end{bmatrix}$

17–22 ■ Matrix Equations Solve the matrix equation for the unknown matrix X, or explain why no solution exists.

$$A = \begin{bmatrix} 4 & 6 \\ 1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 3 \\ 1 & 0 \\ 0 & 2 \end{bmatrix} \qquad D = \begin{bmatrix} 10 & 20 \\ 30 & 20 \\ 10 & 0 \end{bmatrix}$$

17. $2X + A = B$ **18.** $3X - B = C$

19. $2(B - X) = D$ **20.** $5(X - C) = D$

21. $\frac{1}{5}(X + D) = C$ **22.** $2A = B - 3X$

23–36 ■ Matrix Operations The matrices A, B, C, D, E, F, G, and H are defined as follows.

$$A = \begin{bmatrix} 2 & -5 \\ 0 & 7 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & \frac{1}{2} & 5 \\ 1 & -1 & 3 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & -\frac{5}{2} & 0 \\ 0 & 2 & -3 \end{bmatrix}$$

$$D = \begin{bmatrix} 7 & 3 \end{bmatrix} \qquad E = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \qquad F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$G = \begin{bmatrix} 5 & -3 & 10 \\ 6 & 1 & 0 \\ -5 & 2 & 2 \end{bmatrix} \qquad H = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}$$

Carry out the indicated algebraic operation, or explain why it cannot be performed.

23. (a) $B + C$ **(b)** $B + F$

24. (a) $C - B$ **(b)** $2C - 6B$

25. (a) $5A$ **(b)** $C - 5A$

26. (a) $3B + 2C$ **(b)** $2H + D$

27. (a) AD **(b)** DA

28. (a) DH **(b)** HD

29. (a) AH **(b)** HA

30. (a) BC **(b)** BF

31. (a) GF **(b)** GE

32. (a) B^2 **(b)** F^2

33. (a) A^2 **(b)** A^3

34. (a) $(DA)B$ **(b)** $D(AB)$

35. (a) ABE **(b)** AHE

36. (a) $DB + DC$ **(b)** $BF + FE$

37–42 ■ Matrix Operations The matrices A, B, and C are defined as follows.

$$A = \begin{bmatrix} 0.3 & 1.1 & 2.4 \\ 0.9 & -0.1 & 0.4 \\ -0.7 & 0.3 & -0.5 \end{bmatrix} \qquad B = \begin{bmatrix} 1.2 & -0.1 \\ 0 & -0.5 \\ 0.5 & -2.1 \end{bmatrix}$$

$$C = \begin{bmatrix} -0.2 & 0.2 & 0.1 \\ 1.1 & 2.1 & -2.1 \end{bmatrix}$$

Use a graphing device to carry out the indicated algebraic operation, or explain why it cannot be performed.

37. AB **38.** BA **39.** BC

40. CB **41.** $B + C$ **42.** A^2

43–46 ■ Equality of Matrices Solve for x and y.

43. $\begin{bmatrix} x & 2y \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 2x & -6y \end{bmatrix}$

44. $3\begin{bmatrix} x & y \\ y & x \end{bmatrix} = \begin{bmatrix} 6 & -9 \\ -9 & 6 \end{bmatrix}$

45. $2\begin{bmatrix} x & y \\ x + y & x - y \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ -2 & 6 \end{bmatrix}$

46. $\begin{bmatrix} x & y \\ -y & x \end{bmatrix} - \begin{bmatrix} y & x \\ x & -y \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ -6 & 6 \end{bmatrix}$

47–50 ■ Linear Systems as Matrix Equations Write the system of equations as a matrix equation (see Example 6).

47. $\begin{cases} 2x - 5y = 7 \\ 3x + 2y = 4 \end{cases}$

48. $\begin{cases} 6x - y + z = 12 \\ 2x \phantom{{}-y} + z = 7 \\ \phantom{6x-{}} y - 2z = 4 \end{cases}$

49. $\begin{cases} 3x_1 + 2x_2 - x_3 + x_4 = 0 \\ x_1 \phantom{{}+2x_2} - x_3 \phantom{{}+x_4} = 5 \\ \phantom{3x_1+{}} 3x_2 + x_3 - x_4 = 4 \end{cases}$

50. $\begin{cases} x - y + z = 2 \\ 4x - 2y - z = 2 \\ x + y + 5z = 2 \\ -x - y - z = 2 \end{cases}$

Skills Plus

51. Products of Matrices The matrices A, B, and C are defined as follows.

$$A = \begin{bmatrix} 1 & 0 & 6 & -1 \\ 2 & \frac{1}{2} & 4 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 7 & -9 & 2 \end{bmatrix} \qquad C = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -2 \end{bmatrix}$$

Determine which of the following products are defined, and calculate the ones that are.

$$ABC \qquad ACB \qquad BAC$$
$$BCA \qquad CAB \qquad CBA$$

52. Expanding Matrix Bionomials

(a) Prove that if A and B are 2×2 matrices, then

$$(A + B)^2 = A^2 + AB + BA + B^2$$

(b) If A and B are 2×2 matrices, is it necessarily true that

$$(A + B)^2 \stackrel{?}{=} A^2 + 2AB + B^2$$

Applications

53. Education and Income A civic club takes a survey to determine the education and income of its members. Matrix A summarizes the proportions of members in various categories of income levels and years of postsecondary education. Matrix B shows the total number of members in each income category.

(a) Calculate the product matrix AB.

(b) Interpret the entries of the matrix AB.

$$\begin{array}{r} \\ \\ \textbf{None} \\ \textbf{1 to 4} \\ \textbf{More than 4} \end{array}\begin{array}{c} \textbf{Income Level} \\ \begin{array}{ccc} \textbf{Less than \$50,000} & \textbf{\$50,000 to \$100,000} & \textbf{\$100,000 or more} \end{array} \\ \begin{bmatrix} 0.75 & 0.10 & 0 \\ 0.25 & 0.70 & 0.70 \\ 0 & 0.20 & 0.30 \end{bmatrix} = A \end{array}$$

$$\begin{array}{r} \\ \textbf{Less than \$50,000} \\ \textbf{\$50,000 to \$100,000} \\ \textbf{\$100,000 or more} \end{array}\begin{array}{c} \textbf{Total} \\ \begin{bmatrix} 4 \\ 20 \\ 10 \end{bmatrix} = B \end{array}$$

54. Exam Scores A large physics class takes a survey of the number of hours the students slept before an exam and their exam scores. Matrix A summarizes the proportions of students in different categories of exam scores and hours of sleep. Matrix B shows the total number of students in three categories of exam scores.

(a) Calculate the product matrix AB.

(b) Interpret the entries of the matrix AB.

$$\begin{array}{r} \\ \\ \textbf{Less than 4} \\ \textbf{4 to 7} \\ \textbf{More than 7} \end{array}\begin{array}{c} \textbf{Exam Score} \\ \begin{array}{ccc} \textbf{Below 60} & \textbf{60 to 80} & \textbf{Above 80} \end{array} \\ \begin{bmatrix} 0.75 & 0.20 & 0.05 \\ 0.60 & 0.30 & 0.10 \\ 0.40 & 0.30 & 0.30 \end{bmatrix} = A \end{array}$$

$$\begin{array}{r} \\ \textbf{Below 60} \\ \textbf{60 to 80} \\ \textbf{Above 80} \end{array}\begin{array}{c} \textbf{Total} \\ \begin{bmatrix} 80 \\ 170 \\ 40 \end{bmatrix} = B \end{array}$$

55. Frozen-Food Revenue Some of the frozen foods that Joe's Specialty Foods sells are pesto pizza, spinach ravioli, and macaroni and cheese. The sales distribution for these products is tabulated in matrix A. The retail price (in dollars) for each item is tabulated in matrix B.

(a) Calculate the product matrix AB.

(b) What is the total revenue for Monday?

(c) What is the total revenue from all three days?

$$\begin{array}{r} \\ \\ \textbf{Monday} \\ \textbf{Tuesday} \\ \textbf{Wednesday} \end{array}\begin{array}{c} \textbf{Specialty Food} \\ \begin{array}{ccc} \textbf{Pizza} & \textbf{Ravioli} & \textbf{Mac \& Cheese} \end{array} \\ \begin{bmatrix} 50 & 20 & 15 \\ 40 & 75 & 20 \\ 35 & 60 & 100 \end{bmatrix} = A \end{array}$$

$$\begin{array}{r} \\ \textbf{Pizza} \\ \textbf{Ravioli} \\ \textbf{Mac \& Cheese} \end{array}\begin{array}{c} \textbf{Price (\$)} \\ \begin{bmatrix} 3.50 \\ 5.75 \\ 4.25 \end{bmatrix} = B \end{array}$$

56. Fast-Food Sales A small fast-food chain with restaurants in Santa Monica, Long Beach, and Anaheim sells only hamburgers, hot dogs, and milkshakes. On a certain day, sales were distributed according to the following matrix.

$$\begin{array}{r} \\ \\ \textbf{Hamburgers} \\ \textbf{Hot dogs} \\ \textbf{Milkshakes} \end{array}\begin{array}{c} \textbf{Number of Items Sold} \\ \begin{array}{ccc} \textbf{Santa Monica} & \textbf{Long Beach} & \textbf{Anaheim} \end{array} \\ \begin{bmatrix} 4000 & 1000 & 3500 \\ 400 & 300 & 200 \\ 700 & 500 & 9000 \end{bmatrix} = A \end{array}$$

The price of each item is given by the following matrix.

$$\begin{array}{c} \begin{array}{ccc} \textbf{Hamburger} & \textbf{Hot Dog} & \textbf{Milkshake} \end{array} \\ \begin{bmatrix} \$0.90 & \$0.80 & \$1.10 \end{bmatrix} = B \end{array}$$

(a) Calculate the product BA.

(b) Interpret the entries in the product matrix BA.

57. Car-Manufacturing Profits A specialty-car manufacturer has plants in Auburn, Biloxi, and Chattanooga. Three models are produced, with daily production given in the following matrix.

$$\begin{array}{r} \\ \\ \textbf{Auburn} \\ \textbf{Biloxi} \\ \textbf{Chattanooga} \end{array}\begin{array}{c} \textbf{Cars Produced Each Day} \\ \begin{array}{ccc} \textbf{Model K} & \textbf{Model R} & \textbf{Model W} \end{array} \\ \begin{bmatrix} 12 & 10 & 0 \\ 4 & 4 & 20 \\ 8 & 9 & 12 \end{bmatrix} = A \end{array}$$

Because of a wage increase, February profits are lower than January profits. The profit per car is tabulated by model in the following matrix.

$$\begin{array}{r} \\ \textbf{Model K} \\ \textbf{Model R} \\ \textbf{Model W} \end{array}\begin{array}{c} \begin{array}{cc} \textbf{January} & \textbf{February} \end{array} \\ \begin{bmatrix} \$1000 & \$500 \\ \$2000 & \$1200 \\ \$1500 & \$1000 \end{bmatrix} = B \end{array}$$

(a) Calculate the product AB.

(b) Assuming that all cars produced were sold, what was the daily profit in January from the Biloxi plant?

(c) What was the total daily profit (from all three plants) in February?

58. Canning Tomato Products Jaeger Foods produces tomato sauce and tomato paste, canned in small, medium, large, and giant-sized cans. The matrix A gives the size (in ounces) of each container.

$$\begin{array}{r} \\ \textbf{Ounces} \end{array}\begin{array}{c} \begin{array}{cccc} \textbf{Small} & \textbf{Medium} & \textbf{Large} & \textbf{Giant} \end{array} \\ \begin{bmatrix} 6 & 10 & 14 & 28 \end{bmatrix} = A \end{array}$$

The matrix B tabulates one day's production of tomato sauce and tomato paste.

$$\begin{array}{r} \\ \textbf{Small} \\ \textbf{Medium} \\ \textbf{Large} \\ \textbf{Giant} \end{array}\begin{array}{c} \begin{array}{cc} \textbf{Cans of Sauce} & \textbf{Cans of Paste} \end{array} \\ \begin{bmatrix} 2000 & 2500 \\ 3000 & 1500 \\ 2500 & 1000 \\ 1000 & 500 \end{bmatrix} = B \end{array}$$

(a) Calculate the product AB.

(b) Interpret the entries in the product matrix AB.

59. Produce Sales A farmer's three children, Ashton, Bryn, and Cimeron, run three roadside produce stands during the summer months. One weekend they all sell watermelons, yellow squash, and tomatoes. The matrices A and B tabulate the number of pounds of each product sold by each sibling on Saturday and Sunday.

Saturday

$$\begin{array}{r} \\ \textbf{Ashton} \\ \textbf{Bryn} \\ \textbf{Cimeron} \end{array} \begin{array}{c} \begin{array}{ccc} \textbf{Melons} & \textbf{Squash} & \textbf{Tomatoes} \end{array} \\ \begin{bmatrix} 120 & 50 & 60 \\ 40 & 25 & 30 \\ 60 & 30 & 20 \end{bmatrix} \end{array} = A$$

Sunday

$$\begin{array}{r} \\ \textbf{Ashton} \\ \textbf{Bryn} \\ \textbf{Cimeron} \end{array} \begin{array}{c} \begin{array}{ccc} \textbf{Melons} & \textbf{Squash} & \textbf{Tomatoes} \end{array} \\ \begin{bmatrix} 100 & 60 & 30 \\ 35 & 20 & 20 \\ 60 & 25 & 30 \end{bmatrix} \end{array} = B$$

The matrix C gives the price per pound (in dollars) for each type of produce that they sell.

Price per Pound

$$\begin{array}{r} \textbf{Melons} \\ \textbf{Squash} \\ \textbf{Tomatoes} \end{array} \begin{bmatrix} 0.10 \\ 0.50 \\ 1.00 \end{bmatrix} = C$$

Perform each of the following matrix operations, and interpret the entries in each result.

(a) AC **(b)** BC **(c)** $A + B$ **(d)** $(A + B)C$

■ Discuss ■ Discover ■ Prove ■ Write

60. Discuss: When Are Both Products Defined? What must be true about the dimensions of the matrices A and B if both products AB and BA are defined?

61. Discover: Powers of a Matrix For the given matrix A, find a formula for A^n, the product of the matrix A with itself n times.

(a) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ **(b)** $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

PS *Try to recognize a pattern.* Calculate $A^2, A^3, A^4, \ldots$ until you recognize a pattern.

62. Discuss: Square Roots of Matrices A **square root** of a matrix B is a matrix A with the property that $A^2 = B$. (This is the same definition as for a square root of a number.) Find as many square roots as you can of each matrix:

$$\begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix} \qquad \begin{bmatrix} 1 & 5 \\ 0 & 9 \end{bmatrix}$$

[*Hint:* If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, write the equations that a, b, c, and d would have to satisfy if A is the square root of the given matrix.]

9.5 Inverses of Matrices and Matrix Equations

■ The Inverse of a Matrix ■ Finding the Inverse of a 2 × 2 Matrix ■ Finding the Inverse of an $n \times n$ Matrix ■ Matrix Equations ■ Modeling with Matrix Equations

In Section 9.4 we saw that when the dimensions are appropriate, matrices can be added, subtracted, and multiplied. In this section we investigate division of matrices. With this operation we can solve equations that involve matrices.

■ The Inverse of a Matrix

First, we define *identity matrices*, which play the same role for matrix multiplication as the number 1 does for ordinary multiplication of numbers; that is, $1 \cdot a = a \cdot 1 = a$ for all numbers a. A **square matrix** is one that has the same number of rows as columns. The **main diagonal** of a square matrix consists of the entries whose row and column numbers are the same. These entries stretch diagonally down the matrix, from top left to bottom right.

Identity Matrix

The **identity matrix** I_n is the $n \times n$ matrix for which each main diagonal entry is a 1 and for which all other entries are 0.

Thus the 2×2, 3×3, and 4×4 identity matrices are

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Identity matrices behave like the number 1 in the sense that

$$A \cdot I_n = A \qquad \text{and} \qquad I_n \cdot B = B$$

whenever these products are defined.

Example 1 ■ Identity Matrices

The following matrix products show how multiplying a matrix by an identity matrix of the appropriate dimension leaves the matrix unchanged.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 5 & 6 \\ -1 & 2 & 7 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 6 \\ -1 & 2 & 7 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 7 & \frac{1}{2} \\ 12 & 1 & 3 \\ -2 & 0 & 7 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 7 & \frac{1}{2} \\ 12 & 1 & 3 \\ -2 & 0 & 7 \end{bmatrix}$$

Now Try Exercise 1(a), (b) ■

If A and B are $n \times n$ matrices, and if $AB = BA = I_n$, then we say that B is the *inverse* of A, and we write $B = A^{-1}$. The concept of the inverse of a matrix is analogous to that of the reciprocal of a real number.

Inverse of a Matrix

Let A be a square $n \times n$ matrix. If there exists an $n \times n$ matrix A^{-1} with the property that

$$AA^{-1} = A^{-1}A = I_n$$

then we say that A^{-1} is the **inverse** of A. If A has an inverse, then we say that A is **invertible**.

Example 2 ■ Verifying That a Matrix Is an Inverse

Verify that B is the inverse of A, where

$$A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$

Solution We perform the matrix multiplications to show that $AB = I$ and $BA = I$.

$$\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 2 \cdot 3 + 1(-5) & 2(-1) + 1 \cdot 2 \\ 5 \cdot 3 + 3(-5) & 5(-1) + 3 \cdot 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 3 \cdot 2 + (-1)5 & 3 \cdot 1 + (-1)3 \\ (-5)2 + 2 \cdot 5 & (-5)1 + 2 \cdot 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now Try Exercise 3 ■

■ Finding the Inverse of a 2 × 2 Matrix

The following rule provides a simple way to find the inverse of a 2×2 matrix, when it exists. For larger matrices there is a more general procedure for finding inverses, which we consider later in this section.

Inverse of a 2 × 2 Matrix

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, then A has no inverse.

Example 3 ■ Finding the Inverse of a 2 × 2 Matrix

Let

$$A = \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}$$

Find A^{-1}, and verify that $AA^{-1} = A^{-1}A = I_2$.

Solution Using the rule for the inverse of a 2×2 matrix, we get

$$A^{-1} = \frac{1}{4 \cdot 3 - 5 \cdot 2}\begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & -\frac{5}{2} \\ -1 & 2 \end{bmatrix}$$

To verify that this is indeed the inverse of A, we calculate AA^{-1} and $A^{-1}A$:

$$AA^{-1} = \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} \frac{3}{2} & -\frac{5}{2} \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 4 \cdot \frac{3}{2} + 5(-1) & 4\left(-\frac{5}{2}\right) + 5 \cdot 2 \\ 2 \cdot \frac{3}{2} + 3(-1) & 2\left(-\frac{5}{2}\right) + 3 \cdot 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} \frac{3}{2} & -\frac{5}{2} \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \cdot 4 + \left(-\frac{5}{2}\right)2 & \frac{3}{2} \cdot 5 + \left(-\frac{5}{2}\right)3 \\ (-1)4 + 2 \cdot 2 & (-1)5 + 2 \cdot 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now Try Exercise 7 ■

The quantity $ad - bc$ that appears in the rule for calculating the inverse of a 2×2 matrix is called the **determinant** of the matrix. If the determinant is 0, then the matrix does not have an inverse (since we cannot divide by 0).

■ Finding the Inverse of an $n \times n$ Matrix

For 3×3 and larger square matrices the following technique provides the most efficient way to calculate the inverse. If A is an $n \times n$ matrix, we first construct the $n \times 2n$ matrix that has the entries of A on the left and of the identity matrix I_n on the right:

$$\left[\begin{array}{cccc|cccc} a_{11} & a_{12} & \cdots & a_{1n} & 1 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 & 0 & \cdots & 1 \end{array}\right]$$

We then use the elementary row operations on this new large matrix to change the left side into the identity matrix. (This means that we are changing the large matrix to reduced row-echelon form.) The right side is transformed automatically into A^{-1}. (We omit the proof of this fact.)

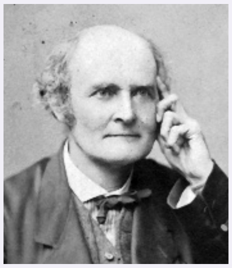
Example 4 ■ Finding the Inverse of a 3 × 3 Matrix

Let A be the matrix

$$A = \begin{bmatrix} 1 & -2 & -4 \\ 2 & -3 & -6 \\ -3 & 6 & 15 \end{bmatrix}$$

(a) Find A^{-1}.

(b) Verify that $AA^{-1} = A^{-1}A = I_3$.



Bygone Collection/Alamy Stock Photo

ARTHUR CAYLEY (1821–1895) was an English mathematician who was instrumental in developing the theory of matrices. He was the first to use a single symbol such as *A* to represent a matrix, thereby introducing the idea that a matrix is a single entity rather than just a collection of numbers. Cayley practiced law until the age of 42, but his primary interest from adolescence was mathematics, and he published almost 200 articles on the subject in his spare time. In 1863 he accepted a professorship in mathematics at Cambridge, where he taught until his death. Cayley's work on matrices was of purely theoretical interest in his day, but in the 20th century many of his results found application in physics, the social sciences, business, and other fields. One of the most common uses of matrices today is in computers, where matrices are employed for data storage, error correction, image manipulation, and many other purposes.

Solution

(a) We begin with the 3×6 matrix whose left half is A and whose right half is the identity matrix I_3.

$$\left[\begin{array}{rrr|rrr} 1 & -2 & -4 & 1 & 0 & 0 \\ 2 & -3 & -6 & 0 & 1 & 0 \\ -3 & 6 & 15 & 0 & 0 & 1 \end{array}\right]$$

We then transform the left half of this new matrix into the identity matrix by performing the following sequence of elementary row operations on the *entire* new matrix.

$$\xrightarrow[R_3 + 3R_1 \to R_3]{R_2 - 2R_1 \to R_2} \left[\begin{array}{rrr|rrr} 1 & -2 & -4 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 3 & 3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{3}R_3} \left[\begin{array}{rrr|rrr} 1 & -2 & -4 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & \frac{1}{3} \end{array}\right]$$

$$\xrightarrow{R_1 + 2R_2 \to R_1} \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -3 & 2 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & \frac{1}{3} \end{array}\right]$$

$$\xrightarrow{R_2 - 2R_3 \to R_2} \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -3 & 2 & 0 \\ 0 & 1 & 0 & -4 & 1 & -\frac{2}{3} \\ 0 & 0 & 1 & 1 & 0 & \frac{1}{3} \end{array}\right]$$

We have now transformed the left half of this matrix into an identity matrix. (This means that we have put the entire matrix in reduced row-echelon form.) Note that to do this in as systematic a fashion as possible, we first changed the elements below the main diagonal to zeros, just as we would if we were using Gaussian elimination. We then changed each main diagonal element to a 1 by multiplying by the appropriate constant(s). Finally, we completed the process by changing the entries above the main diagonal to zeros.

The right half is now A^{-1}.

$$A^{-1} = \begin{bmatrix} -3 & 2 & 0 \\ -4 & 1 & -\frac{2}{3} \\ 1 & 0 & \frac{1}{3} \end{bmatrix}$$

(b) We calculate AA^{-1} and $A^{-1}A$ and verify that both products give the identity matrix I_3.

$$AA^{-1} = \begin{bmatrix} 1 & -2 & -4 \\ 2 & -3 & -6 \\ -3 & 6 & 15 \end{bmatrix} \begin{bmatrix} -3 & 2 & 0 \\ -4 & 1 & -\frac{2}{3} \\ 1 & 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} -3 & 2 & 0 \\ -4 & 1 & -\frac{2}{3} \\ 1 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & -2 & -4 \\ 2 & -3 & -6 \\ -3 & 6 & 15 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now Try Exercises 19 ■

Graphing devices are also able to calculate matrix inverses. On a graphing calculator, to find the inverse of `[A]`, we key in

`[A]` `X⁻¹` `ENTER`

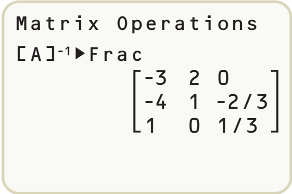


Figure 1

For the matrix of Example 4 this results in the output shown in Figure 1, where we use the ▸Frac command (or $\frac{\square}{\square}$ command) to display the output in fraction form rather than in decimal form.

The next example shows that not every square matrix has an inverse.

Example 5 ■ A Matrix That Does Not Have an Inverse

Find the inverse of the matrix

$$\begin{bmatrix} 2 & -3 & -7 \\ 1 & 2 & 7 \\ 1 & 1 & 4 \end{bmatrix}$$

Solution We proceed as follows.

$$\left[\begin{array}{ccc|ccc} 2 & -3 & -7 & 1 & 0 & 0 \\ 1 & 2 & 7 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 7 & 0 & 1 & 0 \\ 2 & -3 & -7 & 1 & 0 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 - R_1 \to R_3]{R_2 - 2R_1 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 7 & 0 & 1 & 0 \\ 0 & -7 & -21 & 1 & -2 & 0 \\ 0 & -1 & -3 & 0 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{7}R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 7 & 0 & 1 & 0 \\ 0 & 1 & 3 & -\frac{1}{7} & \frac{2}{7} & 0 \\ 0 & -1 & -3 & 0 & -1 & 1 \end{array}\right]$$

$$\xrightarrow[R_1 - 2R_2 \to R_1]{R_3 + R_2 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & \frac{2}{7} & \frac{3}{7} & 0 \\ 0 & 1 & 3 & -\frac{1}{7} & \frac{2}{7} & 0 \\ 0 & 0 & 0 & -\frac{1}{7} & -\frac{5}{7} & 1 \end{array}\right]$$

At this point we would like to change the 0 in the (3, 3) position of this matrix to a 1 without changing the zeros in the (3, 1) and (3, 2) positions. But there is no way to accomplish this: No matter what multiple of rows 1 and/or 2 we add to row 3, we can't change the third zero in row 3 without changing the first or second zero as well. Thus we cannot change the left half to the identity matrix, so the original matrix doesn't have an inverse.

Now Try Exercise 21 ■

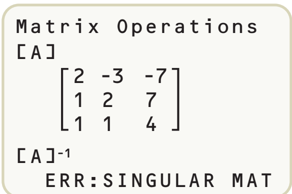


Figure 2

If we encounter a row of zeros on the left side when trying to find an inverse, as we did in Example 5, then the original matrix does not have an inverse. (A matrix that has no inverse is called *singular*.) If we try to calculate the inverse of the matrix from Example 5 on a graphing device, we get an error message like the one shown in Figure 2.

■ Matrix Equations

We saw in Example 9.4.6 that a system of linear equations can be written as a single matrix equation. For example, the system

$$\begin{cases} x - 2y - 4z = 7 \\ 2x - 3y - 6z = 5 \\ -3x + 6y + 15z = 0 \end{cases}$$

is equivalent to the matrix equation

$$\underset{A}{\begin{bmatrix} 1 & -2 & -4 \\ 2 & -3 & -6 \\ -3 & 6 & 15 \end{bmatrix}} \underset{X}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{B}{\begin{bmatrix} 7 \\ 5 \\ 0 \end{bmatrix}}$$

If we let

$$A = \begin{bmatrix} 1 & -2 & -4 \\ 2 & -3 & -6 \\ -3 & 6 & 15 \end{bmatrix} \qquad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad B = \begin{bmatrix} 7 \\ 5 \\ 0 \end{bmatrix}$$

then this matrix equation can be written as

$$AX = B$$

The matrix A is called the **coefficient matrix**. We can use matrix operations to solve for the matrix X and we get $X = A^{-1}B$. See the proof below.

In Example 4 we showed that

$$A^{-1} = \begin{bmatrix} -3 & 2 & 0 \\ -4 & 1 & -\frac{2}{3} \\ 1 & 0 & \frac{1}{3} \end{bmatrix}$$

So from $X = A^{-1}B$ we have

$$\underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} -3 & 2 & 0 \\ -4 & 1 & -\frac{2}{3} \\ 1 & 0 & \frac{1}{3} \end{bmatrix}}_{A^{-1}} \underbrace{\begin{bmatrix} 7 \\ 5 \\ 0 \end{bmatrix}}_{B} = \begin{bmatrix} -11 \\ -23 \\ 7 \end{bmatrix}$$

Thus $x = -11$, $y = -23$, $z = 7$ is the solution of the original system.

Solving a Matrix Equation

If A is a square $n \times n$ matrix that has an inverse A^{-1} and if X is a variable matrix and B is a known matrix, both with n rows, then the solution of the matrix equation

$$AX = B$$

is given by

$$X = A^{-1}B$$

Proof We solve the matrix equation by multiplying each side by the inverse of A.

$AX = B$	Matrix equation
$A^{-1}(AX) = A^{-1}B$	Multiply on left by A^{-1}
$(A^{-1}A)X = A^{-1}B$	Associative Property
$I_3X = A^{-1}B$	Property of inverses
$X = A^{-1}B$	Property of identity matrix ■

Solving the matrix equation $AX = B$ is similar to solving a real-number equation like

$$3x = 12$$

which we do by multiplying each side by the reciprocal (or inverse) of 3.

$$\tfrac{1}{3}(3x) = \tfrac{1}{3}(12)$$

$$x = 4$$

Example 6 ■ Solving a System Using a Matrix Inverse

A system of equations is given.

(a) Write the system of equations as a matrix equation.

(b) Solve the system by solving the matrix equation.

$$\begin{cases} 2x - 5y = 15 \\ 3x - 6y = 36 \end{cases}$$

Mathematics in the Modern World

Mathematical Ecology

In the 1970s humpback whales became a center of controversy. Environmentalists believed that whaling threatened the whales with imminent extinction; whalers saw their livelihood threatened by any attempt to stop whaling. Are whales really threatened to extinction by whaling? What level of whaling is safe to guarantee survival of the whales? These questions motivated mathematicians to study population patterns of whales and other species more closely.

As early as the 1920s Lotka and Volterra had founded the field of mathematical biology by creating predator-prey models. Their models, which draw on a branch of mathematics called differential equations, take into account the rates at which predator eats prey and the rates of growth of each population: As predator eats prey, the prey population decreases. This means less food supply for the predators, so their population begins to decrease. With fewer predators the prey population begins to increase, and so on. Normally, a state of equilibrium develops, and the two populations alternate between a minimum and a maximum value. However, if the predators eat the prey too fast, they will be left without food and will thus ensure their own extinction.

Since Lotka and Volterra's time, more detailed mathematical models of animal populations have been developed. For many species the population is divided into several stages: immature, juvenile, adult, and so on. The proportion of each stage that survives or reproduces in a given time period is entered into a matrix (called a transition matrix); matrix multiplication is then used to predict the population in succeeding time periods. (See *Discovery Project: Will the Species Survive?* at the book companion website: **www.stewartmath.com**.)

The power of mathematics to model and predict is an invaluable tool in the ongoing debate over the environment.

Solution

(a) We write the system as a matrix equation of the form $AX = B$.

$$\underbrace{\begin{bmatrix}2 & -5\\ 3 & -6\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}x\\ y\end{bmatrix}}_{X} \underset{=}{=} \underbrace{\begin{bmatrix}15\\ 36\end{bmatrix}}_{B}$$

(b) Using the rule for finding the inverse of a 2×2 matrix, we get

$$A^{-1} = \begin{bmatrix}2 & -5\\ 3 & -6\end{bmatrix}^{-1} = \frac{1}{2(-6) - (-5)3}\begin{bmatrix}-6 & -(-5)\\ -3 & 2\end{bmatrix} = \frac{1}{3}\begin{bmatrix}-6 & 5\\ -3 & 2\end{bmatrix}$$

Multiplying each side of the matrix equation by this inverse matrix, we get

$$\underbrace{\begin{bmatrix}x\\ y\end{bmatrix}}_{X} \underset{=}{=} \underbrace{\frac{1}{3}\begin{bmatrix}-6 & 5\\ -3 & 2\end{bmatrix}}_{A^{-1}}\underbrace{\begin{bmatrix}15\\ 36\end{bmatrix}}_{B} = \begin{bmatrix}30\\ 9\end{bmatrix}$$

So $x = 30$ and $y = 9$.

Now Try Exercise 39 ■

■ Modeling with Matrix Equations

Suppose we need to solve several systems of equations that have the same coefficient matrix. Then converting the systems to matrix equations provides an efficient way to obtain the solutions, because we need to find the inverse of the coefficient matrix only once. This procedure is particularly convenient if we use a graphing device to perform the matrix operations, as illustrated in the next example.

Example 7 ■ Modeling Nutritional Requirements Using Matrix Equations

A pet-store owner feeds hamsters and gerbils different mixtures of three types of rodent food: KayDee Food, Pet Pellets, and Rodent Chow. The animals should get the correct amount of each brand to satisfy their daily requirements for protein, fat, and carbohydrates. Suppose that hamsters require 340 mg of protein, 280 mg of fat, and 440 mg of carbohydrates, and gerbils need 480 mg of protein, 360 mg of fat, and 680 mg of carbohydrates each day. The amount of each nutrient (in mg) in 1 g of each brand is given in the table. How many grams of each food should the hamsters and gerbils be fed daily to satisfy their daily nutritional requirements?

	KayDee Food	Pet Pellets	Rodent Chow
Protein (mg)	10	0	20
Fat (mg)	10	20	10
Carbohydrates (mg)	5	10	30

Solution We let x_1, x_2, and x_3 be the respective amounts (in grams) of KayDee Food, Pet Pellets, and Rodent Chow that the hamsters should eat, and we let y_1, y_2, and y_3 be the corresponding amounts for the gerbils. Then we want to solve the matrix equations

$$\begin{bmatrix} 10 & 0 & 20 \\ 10 & 20 & 10 \\ 5 & 10 & 30 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 340 \\ 280 \\ 440 \end{bmatrix} \qquad \text{Hamster equation}$$

$$\begin{bmatrix} 10 & 0 & 20 \\ 10 & 20 & 10 \\ 5 & 10 & 30 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 480 \\ 360 \\ 680 \end{bmatrix} \qquad \text{Gerbil equation}$$

Let

$$A = \begin{bmatrix} 10 & 0 & 20 \\ 10 & 20 & 10 \\ 5 & 10 & 30 \end{bmatrix} \quad B = \begin{bmatrix} 340 \\ 280 \\ 440 \end{bmatrix} \quad C = \begin{bmatrix} 480 \\ 360 \\ 680 \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Then we can write these matrix equations as

$$AX = B \qquad \text{Hamster equation}$$

$$AY = C \qquad \text{Gerbil equation}$$

We want to solve for X and Y, so we multiply both sides of each equation by A^{-1}, the inverse of the coefficient matrix. We could find A^{-1} by hand, but it is more convenient to use a graphing device, as shown in Figure 3.

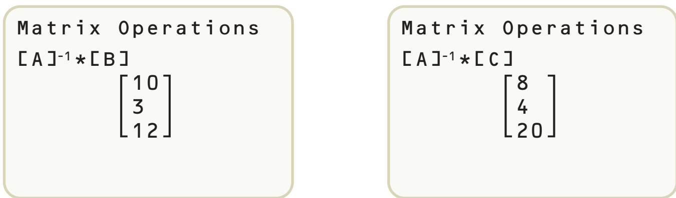


Figure 3 (a) (b)

So

$$X = A^{-1}B = \begin{bmatrix} 10 \\ 3 \\ 12 \end{bmatrix} \qquad Y = A^{-1}C = \begin{bmatrix} 8 \\ 4 \\ 20 \end{bmatrix}$$

Thus each hamster should be fed 10 g of KayDee Food, 3 g of Pet Pellets, and 12 g of Rodent Chow; and each gerbil should be fed 8 g of KayDee Food, 4 g of Pet Pellets, and 20 g of Rodent Chow daily.

Now Try Exercise 61 ■

9.5 Exercises

Concepts

1. (a) The matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is called an ________ matrix.

(b) If A is a 2×2 matrix, then $AI =$ ________ and $IA =$ ________.

(c) If A and B are 2×2 matrices with $AB = I$, then B is the ________ of A.

2. (a) Write the following system as a matrix equation $AX = B$.

System

$$5x + 3y = 4$$
$$3x + 2y = 3$$

Matrix equation

$$\begin{array}{ccccc} A & \cdot & X & = & B \end{array}$$
$$\begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix} \begin{bmatrix} \square \\ \square \end{bmatrix} = \begin{bmatrix} \square \\ \square \end{bmatrix}$$

(b) The inverse of A is $A^{-1} = \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix}$.

(c) The solution of the matrix equation is $X = A^{-1}B$.

$$\begin{array}{ccccccc} X & = & A^{-1} & & B & & \\ \begin{bmatrix} x \\ y \end{bmatrix} & = & \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix} & & \begin{bmatrix} \square \\ \square \end{bmatrix} & = & \begin{bmatrix} \square \\ \square \end{bmatrix} \end{array}$$

(d) The solution of the system is $x =$ ________, $y =$________.

Skills

3–6 ■ Verifying the Inverse of a Matrix Calculate the products AB and BA to verify that B is the inverse of A.

3. $A = \begin{bmatrix} 4 & 1 \\ 7 & 2 \end{bmatrix}$ $\quad B = \begin{bmatrix} 2 & -1 \\ -7 & 4 \end{bmatrix}$

4. $A = \begin{bmatrix} 2 & -3 \\ 4 & -7 \end{bmatrix}$ $\quad B = \begin{bmatrix} \frac{7}{2} & -\frac{3}{2} \\ 2 & -1 \end{bmatrix}$

5. $A = \begin{bmatrix} 1 & 3 & -1 \\ 1 & 4 & 0 \\ -1 & -3 & 2 \end{bmatrix}$ $\quad B = \begin{bmatrix} 8 & -3 & 4 \\ -2 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$

6. $A = \begin{bmatrix} 3 & 2 & 4 \\ 1 & 1 & -6 \\ 2 & 1 & 12 \end{bmatrix}$ $\quad B = \begin{bmatrix} 9 & -10 & -8 \\ -12 & 14 & 11 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

7–8 ■ The Inverse of a 2 × 2 Matrix Find the inverse of the matrix and verify that $A^{-1}A = AA^{-1} = I_2$ and $B^{-1}B = BB^{-1} = I_3$.

7. $A = \begin{bmatrix} 7 & 4 \\ 3 & 2 \end{bmatrix}$ $\quad$ **8.** $B = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 2 \\ -2 & -1 & 0 \end{bmatrix}$

9–10 ■ The Inverse of a 2 × 2 Matrix Use a graphing device to find the inverse of the matrix and to verify that $A^{-1}A = AA^{-1} = I_2$ and $B^{-1}B = BB^{-1} = I_3$. Use the appropriate command on your graphing device to obtain the answer in fractions.

9. $A = \begin{bmatrix} 1.2 & 0.3 \\ -1.2 & 0.2 \end{bmatrix}$ $\quad$ **10.** $B = \begin{bmatrix} 5 & -1 & 3 \\ 6 & -1 & 3 \\ 7 & 1 & -2 \end{bmatrix}$

11–26 ■ Finding the Inverse of a Matrix Find the inverse of the matrix if it exists.

11. $\begin{bmatrix} 3 & 2 \\ 13 & 9 \end{bmatrix}$ $\quad$ **12.** $\begin{bmatrix} 5 & 7 \\ 3 & 4 \end{bmatrix}$

13. $\begin{bmatrix} 2 & 5 \\ -5 & -13 \end{bmatrix}$ $\quad$ **14.** $\begin{bmatrix} -7 & 4 \\ 8 & -5 \end{bmatrix}$

15. $\begin{bmatrix} 6 & -3 \\ -8 & 4 \end{bmatrix}$ $\quad$ **16.** $\begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ 5 & 4 \end{bmatrix}$

17. $\begin{bmatrix} 0.4 & -1.2 \\ 0.3 & 0.6 \end{bmatrix}$ $\quad$ **18.** $\begin{bmatrix} 4 & 2 & 3 \\ 3 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}$

19. $\begin{bmatrix} 2 & 4 & 1 \\ -1 & 1 & -1 \\ 1 & 4 & 0 \end{bmatrix}$ $\quad$ **20.** $\begin{bmatrix} 5 & 7 & 4 \\ 3 & -1 & 3 \\ 6 & 7 & 5 \end{bmatrix}$

21. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & -1 \\ 1 & -1 & -10 \end{bmatrix}$ $\quad$ **22.** $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 4 \\ 2 & 1 & 2 \end{bmatrix}$

23. $\begin{bmatrix} 0 & -2 & 2 \\ 3 & 1 & 3 \\ 1 & -2 & 3 \end{bmatrix}$ $\quad$ **24.** $\begin{bmatrix} 3 & -2 & 0 \\ 5 & 1 & 1 \\ 2 & -2 & 0 \end{bmatrix}$

25. $\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & 0 & 2 \end{bmatrix}$ $\quad$ **26.** $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

27–34 ■ Finding the Inverse of a Matrix Use a graphing device to find the inverse of the matrix, if it exists. Use the appropriate command on your graphing device to obtain the answer in fractions.

27. $\begin{bmatrix} -3 & 2 & 3 \\ 0 & -1 & 3 \\ 1 & 0 & -2 \end{bmatrix}$ $\quad$ **28.** $\begin{bmatrix} -5 & 2 & 1 \\ 5 & 1 & 0 \\ 0 & -1 & -2 \end{bmatrix}$

29. $\begin{bmatrix} -1 & -4 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 4 & 1 & -2 \\ 2 & 2 & -2 & 0 \end{bmatrix}$ $\quad$ **30.** $\begin{bmatrix} -3 & 0 & -1 & 1 \\ 3 & -1 & 1 & -1 \\ 1 & 3 & 0 & 1 \\ -2 & -3 & 1 & 0 \end{bmatrix}$

31. $\begin{bmatrix} 1 & 7 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ $\quad$ **32.** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 5 & 0 & 0 \\ 4 & 2 & 3 & 0 \\ 5 & 1 & 2 & 1 \end{bmatrix}$

33. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}$ $\quad$ **34.** $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$

35–38 ■ Products Involving Matrices and Inverses The matrices A and B are defined as follows.

$$A = \begin{bmatrix} -1 & 0 & 2 \\ 0 & -2 & -1 \\ 4 & 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & -1 & -2 \\ 0 & 3 & 1 \\ -1 & 0 & 2 \end{bmatrix}$$

Use a graphing device to carry out the indicated algebraic operations, or explain why they cannot be performed. Use the appropriate command on your graphing device to obtain the answer in fractions.

35. $A^{-1}B$ $\quad$ **36.** AB^{-1} $\quad$ **37.** BAB^{-1} $\quad$ **38.** $B^{-1}AB$

39–46 ■ Solving a Linear System as a Matrix Equation Solve the system of equations by converting to a matrix equation and using the inverse of the coefficient matrix, as in Example 6. Use the inverses from Exercises 11–14, 19, 20, 23, and 25.

39. $\begin{cases} 3x + 2y = 1 \\ 13x + 9y = 3 \end{cases}$ $\quad$ **40.** $\begin{cases} 5x + 7y = -9 \\ 3x + 4y = -6 \end{cases}$

41. $\begin{cases} 2x + 5y = 2 \\ -5x - 13y = 20 \end{cases}$ $\quad$ **42.** $\begin{cases} -7x + 4y = 0 \\ 8x - 5y = 100 \end{cases}$

43. $\begin{cases} 2x + 4y + z = 7 \\ -x + y - z = 0 \\ x + 4y = -2 \end{cases}$ $\quad$ **44.** $\begin{cases} 5x + 7y + 4z = 1 \\ 3x - y + 3z = 1 \\ 6x + 7y + 5z = 1 \end{cases}$

45. $\begin{cases} -2y + 2z = 12 \\ 3x + y + 3z = -2 \\ x - 2y + 3z = 8 \end{cases}$

46. $\begin{cases} x + 2y + 3w = 0 \\ y + z + w = 1 \\ y + w = 2 \\ x + 2y + 2w = 3 \end{cases}$

47–52 ■ Solving a Linear System Solve the system of equations by converting to a matrix equation. Use a graphing device to perform the necessary matrix operations, as in Example 7.

47. $\begin{cases} x + y - 2z = 3 \\ 2x + 5z = 11 \\ 2x + 3y = 12 \end{cases}$

48. $\begin{cases} 3x + 4y - z = 2 \\ 2x - 3y + z = -5 \\ 5x - 2y + 2z = -3 \end{cases}$

49. $\begin{cases} 12x + \frac{1}{2}y - 7z = 21 \\ 11x - 2y + 3z = 43 \\ 13x + y - 4z = 29 \end{cases}$

50. $\begin{cases} x + \frac{1}{2}y - \frac{1}{3}z = 4 \\ x - \frac{1}{4}y + \frac{1}{6}z = 7 \\ x + y - z = -6 \end{cases}$

51. $\begin{cases} x + y - 3w = 0 \\ x - 2z = 8 \\ 2y - z + w = 5 \\ 2x + 3y - 2w = 13 \end{cases}$

52. $\begin{cases} x + y + z + w = 15 \\ x - y + z - w = 5 \\ x + 2y + 3z + 4w = 26 \\ x - 2y + 3z - 4w = 2 \end{cases}$

Skills Plus

53–54 ■ Solving a Matrix Equation Solve the matrix equation by multiplying each side by the appropriate inverse matrix.

53. $\begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} x & y & z \\ u & v & w \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \end{bmatrix}$

54. $\begin{bmatrix} 0 & -2 & 2 \\ 3 & 1 & 3 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x & u \\ y & v \\ z & w \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 12 \\ 0 & 0 \end{bmatrix}$

55–56 ■ Inverses of Special Matrices Find the inverse of the matrix.

55. $\begin{bmatrix} a & -a \\ a & a \end{bmatrix}$ $(a \neq 0)$

56. $\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{bmatrix}$ $(abcd \neq 0)$

57–60 ■ When Do Matrices Have Inverses? Find the inverse of the matrix. For what value(s) of x, if any, does the matrix have no inverse?

57. $\begin{bmatrix} 2 & x \\ x & x^2 \end{bmatrix}$

58. $\begin{bmatrix} e^x & -e^{2x} \\ e^{2x} & e^{3x} \end{bmatrix}$

59. $\begin{bmatrix} 1 & e^x & 0 \\ e^x & -e^{2x} & 0 \\ 0 & 0 & 2 \end{bmatrix}$

60. $\begin{bmatrix} x & 1 \\ -x & \dfrac{1}{x-1} \end{bmatrix}$

Applications

61. Nutrition A nutritionist is studying the effects of the nutrients folic acid, choline, and inositol. There are three types of food available, and each type contains the following amounts of these nutrients per ounce.

	Type A	Type B	Type C
Folic acid (mg)	3	1	3
Choline (mg)	4	2	4
Inositol (mg)	3	2	4

(a) Find the inverse of the matrix

$$\begin{bmatrix} 3 & 1 & 3 \\ 4 & 2 & 4 \\ 3 & 2 & 4 \end{bmatrix}$$

and use it to solve the remaining parts of this problem.

(b) How many ounces of each food should the laboratory rats be fed if their daily diet is to contain 10 mg of folic acid, 14 mg of choline, and 13 mg of inositol?

(c) How much of each food is needed to supply 9 mg of folic acid, 12 mg of choline, and 10 mg of inositol?

(d) Will any combination of these foods supply 2 mg of folic acid, 4 mg of choline, and 11 mg of inositol?

62. Nutrition Refer to Exercise 61. Suppose food type C has been improperly labeled, and it actually contains 4 mg of folic acid, 6 mg of choline, and 5 mg of inositol per ounce. Would it still be possible to use matrix inversion to solve parts (b), (c), and (d) of Exercise 61? Why or why not?

63. Sales Commissions A salesperson works at a kiosk that offers three different models of cell phones: standard with 64 GB capacity, deluxe with 128 GB capacity, and super-deluxe with 256 GB capacity. For each phone sold the salesperson earns a commission based on the cell phone model. One week 9 standard, 11 deluxe, and 8 super-deluxe are sold and the salesperson makes \$740 in commission. The next week 13 standard, 15 deluxe, and 16 super-deluxe are sold for a \$1204 commission. The third week 8 standard, 7 deluxe, and 14 super-deluxe are sold, earning the salesperson \$828 in commission.

(a) Let x, y, and z represent the commission the saleperson earns on standard, deluxe, and super-deluxe, respectively. Translate the given information into a system of equations in x, y, and z.

(b) Express the system of equations you found in part (a) as a matrix equation of the form $AX = B$.

(c) Find the inverse of the coefficient matrix A and use it to solve the matrix equation in part (b). How much commission does the salesperson earn on each model of cell phone?

Discuss ■ Discover ■ Prove ■ Write

64. Discuss: No Zero-Product Property for Matrices We have used the Zero-Product Property to solve algebraic equations. Matrices do *not* have this property. Let O represent the **2 × 2 zero matrix**

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Find 2×2 matrices $A \neq O$ and $B \neq O$ such that $AB = O$. Can you find a matrix $A \neq O$ such that $A^2 = O$?

65. Prove: The Inverse of a Product of Matrices Let A and B be $n \times n$ invertible matrices. Show that the product matrix AB is invertible, and its inverse is

$$(AB)^{-1} = B^{-1}A^{-1}$$

PS *Try to recognize something familiar.* Apply the definition of the inverse of a matrix to the matrix AB.

9.6 Determinants and Cramer's Rule

■ Determinant of a 2 × 2 Matrix ■ Determinant of an $n \times n$ Matrix ■ Row and Column Transformations ■ Cramer's Rule ■ Areas of Triangles Using Determinants

If a matrix is **square** (that is, it has the same number of rows as columns), then we can assign to it a number called its *determinant*. Determinants can be used to solve systems of linear equations, as we will see later in this section. They are also useful in determining whether a matrix has an inverse.

■ Determinant of a 2 × 2 Matrix

We denote the determinant of a square matrix A by the symbol $\det(A)$ or $|A|$. We first define $\det(A)$ for the simplest cases. If $A = [a]$ is a 1×1 matrix, then $\det(A) = a$. The following box gives the definition of a 2×2 determinant.

We will use both notations, $\det(A)$ and $|A|$, for the determinant of A. Although the symbol $|A|$ looks like the absolute value symbol, it will be clear from the context which meaning is intended.

Determinant of a 2 × 2 Matrix

The **determinant** of the 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

$$\det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Example 1 ■ Determinant of a 2 × 2 Matrix

Evaluate $|A|$ for $A = \begin{bmatrix} 6 & -3 \\ 2 & 3 \end{bmatrix}$.

Solution

$$\begin{vmatrix} 6 & -3 \\ 2 & 3 \end{vmatrix} = 6 \cdot 3 - (-3)2 = 18 - (-6) = 24$$

To evaluate a 2×2 determinant, we take the product of the diagonal from top left to bottom right and subtract the product from top right to bottom left, as indicated by the arrows.

Now Try Exercise 5 ■

■ Determinant of an $n \times n$ Matrix

To define the concept of determinant for an arbitrary $n \times n$ matrix, we need the following terminology.

Minors and Cofactors

Let A be an $n \times n$ matrix.

1. The **minor** M_{ij} of the element a_{ij} is the determinant of the matrix obtained by deleting the ith row and jth column of A.
2. The **cofactor** A_{ij} of the element a_{ij} is

$$A_{ij} = (-1)^{i+j}M_{ij}$$

Dr David Hilbert

DAVID HILBERT (1862–1943) was born in Königsberg, Germany, and became a professor at Göttingen University. He is considered by many to be the greatest mathematician of the 20th century. At the International Congress of Mathematicians held in Paris in 1900, Hilbert set the direction of mathematics for the 20th century by posing 23 problems that he believed to be of crucial importance. He said that "these are problems whose solutions we expect from the future." Most of Hilbert's problems have now been solved (see Julia Robinson, Section 9.4, and Alan Turing, Section 2.6), and their solutions have led to important new areas of mathematical research. Yet as of this writing, several of Hilbert's problems remain unsolved. In his work, Hilbert emphasized structure, logic, and the foundations of mathematics. Part of his genius lay in his ability to see the most general possible statement of a problem. For instance, Euler proved that every whole number is the sum of four squares; Hilbert proved a similar statement for all powers of positive integers. Hilbert firmly believed that every mathematical problem had a solution; he famously said "We must know, we will know."

For example, if A is the matrix

$$\begin{bmatrix} 2 & 3 & -1 \\ 0 & 2 & 4 \\ -2 & 5 & 6 \end{bmatrix}$$

then the minor M_{12} is the determinant of the matrix obtained by deleting the first row and second column from A. Thus

$$M_{12} = \begin{vmatrix} 2 & 3 & -1 \\ 0 & 2 & 4 \\ -2 & 5 & 6 \end{vmatrix} = \begin{vmatrix} 0 & 4 \\ -2 & 6 \end{vmatrix} = 0(6) - 4(-2) = 8$$

So the cofactor $A_{12} = (-1)^{1+2}M_{12} = -8$. Similarly,

$$M_{33} = \begin{vmatrix} 2 & 3 & -1 \\ 0 & 2 & 4 \\ -2 & 5 & 6 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 0 & 2 \end{vmatrix} = 2 \cdot 2 - 3 \cdot 0 = 4$$

So $A_{33} = (-1)^{3+3}M_{33} = 4$.

Note that the cofactor of a_{ij} is simply the minor of a_{ij} multiplied by either 1 or -1, depending on whether $i + j$ is even or odd. Thus in a 3×3 matrix we obtain the cofactor of any element by prefixing its minor with the sign obtained from the following checkerboard pattern.

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

We are now ready to define the determinant of any square matrix.

The Determinant of a Square Matrix

If A is an $n \times n$ matrix, then the **determinant** of A is obtained by multiplying each element of the first row by its cofactor and then adding the results. In symbols,

$$\det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}$$

Example 2 ■ Determinant of a 3 × 3 Matrix

Evaluate the determinant of the matrix

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 2 & 4 \\ -2 & 5 & 6 \end{bmatrix}$$

Solution

$$\det(A) = \begin{vmatrix} 2 & 3 & -1 \\ 0 & 2 & 4 \\ -2 & 5 & 6 \end{vmatrix} = 2\begin{vmatrix} 2 & 4 \\ 5 & 6 \end{vmatrix} - 3\begin{vmatrix} 0 & 4 \\ -2 & 6 \end{vmatrix} + (-1)\begin{vmatrix} 0 & 2 \\ -2 & 5 \end{vmatrix}$$

$$= 2(2 \cdot 6 - 4 \cdot 5) - 3[0 \cdot 6 - 4(-2)] - [0 \cdot 5 - 2(-2)]$$

$$= -16 - 24 - 4$$

$$= -44$$

Now Try Exercises 21 and 29 ■

In our definition of the determinant we used the cofactors of elements in the first row only. This is called **expanding the determinant by the first row**. In fact, *we can expand the determinant by any row or column in the same way and obtain the same result in each case* (although we won't prove this). The next example illustrates this principle.

Example 3 ■ Expanding a Determinant About a Row and a Column

Let A be the matrix of Example 2. Evaluate the determinant of A by expanding

(a) by the second row.

(b) by the third column.

Verify that each expansion gives the same value.

Solution

(a) Expanding by the second row, we get

$$\begin{aligned}\det(A) = \begin{vmatrix} 2 & 3 & -1 \\ 0 & 2 & 4 \\ -2 & 5 & 6 \end{vmatrix} &= -0\begin{vmatrix} 3 & -1 \\ 5 & 6 \end{vmatrix} + 2\begin{vmatrix} 2 & -1 \\ -2 & 6 \end{vmatrix} - 4\begin{vmatrix} 2 & 3 \\ -2 & 5 \end{vmatrix} \\ &= 0 + 2[2\cdot 6 - (-1)(-2)] - 4[2\cdot 5 - 3(-2)] \\ &= 0 + 20 - 64 = -44\end{aligned}$$

(b) Expanding by the third column gives

$$\begin{aligned}\det(A) &= \begin{vmatrix} 2 & 3 & -1 \\ 0 & 2 & 4 \\ -2 & 5 & 6 \end{vmatrix} \\ &= -1\begin{vmatrix} 0 & 2 \\ -2 & 5 \end{vmatrix} - 4\begin{vmatrix} 2 & 3 \\ -2 & 5 \end{vmatrix} + 6\begin{vmatrix} 2 & 3 \\ 0 & 2 \end{vmatrix} \\ &= -[0\cdot 5 - 2(-2)] - 4[2\cdot 5 - 3(-2)] + 6(2\cdot 2 - 3\cdot 0) \\ &= -4 - 64 + 24 = -44\end{aligned}$$

In both cases we obtain the same value for the determinant as when we expanded by the first row in Example 2.

We can also use a graphing device to compute determinants, as shown in Figure 1.

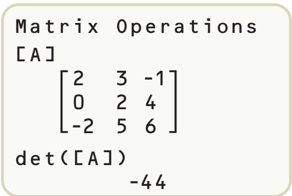


Figure 1

Now Try Exercise 39 ■

The following criterion allows us to determine whether a square matrix has an inverse without actually calculating the inverse. This is one of the most important uses of the determinant in matrix algebra, and it is the reason for the name *determinant*.

Invertibility Criterion

If A is a square matrix, then A has an inverse if and only if $\det(A) \neq 0$.

We will not prove this fact, but from the formula for the inverse of a 2×2 matrix (Section 9.5), you can see why it is true in the 2×2 case.

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EMMY NOETHER (1882–1935) was one of the foremost mathematicians of the early 20th century. Her groundbreaking work in abstract algebra provided much of the foundation for this field, and her work in invariant theory was essential in the development of Einstein's theory of general relativity. Although women weren't allowed to study at German universities in her time, she audited courses unofficially and went on to receive a doctorate at Erlangen *summa cum laude*, despite the opposition of the academic senate, which declared that women students would "overthrow all academic order." She subsequently taught mathematics at Göttingen, Moscow, and Frankfurt. In 1933 she left Germany to escape Nazi persecution, accepting a position at Bryn Mawr College in suburban Philadelphia. She lectured there and at the Institute for Advanced Study in Princeton, New Jersey, until her untimely death in 1935.

Example 4 ■ Using the Determinant to Show That a Matrix Is Not Invertible

Show that the matrix A has no inverse.

$$A = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 3 \\ 5 & 6 & 2 & 6 \\ 2 & 4 & 0 & 9 \end{bmatrix}$$

Solution We begin by calculating the determinant of A. Since all but one of the elements of the second row is zero, we expand the determinant by the second row. If we do this, we see from the following equation that only the cofactor A_{24} will have to be calculated.

$$\begin{aligned} \det(A) &= \begin{vmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 3 \\ 5 & 6 & 2 & 6 \\ 2 & 4 & 0 & 9 \end{vmatrix} \\ &= -0 \cdot A_{21} + 0 \cdot A_{22} - 0 \cdot A_{23} + 3 \cdot A_{24} = 3A_{24} \\ &= 3\begin{vmatrix} 1 & 2 & 0 \\ 5 & 6 & 2 \\ 2 & 4 & 0 \end{vmatrix} \qquad \text{Expand this by column 3} \\ &= 3(-2)\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} \\ &= 3(-2)(1 \cdot 4 - 2 \cdot 2) = 0 \end{aligned}$$

Since the determinant of A is zero, A cannot have an inverse, by the Invertibility Criterion.

Now Try Exercise 25 ■

■ Row and Column Transformations

The preceding example shows that if we expand a determinant about a row or column that contains many zeros, our work is reduced considerably because we don't have to evaluate the cofactors of the elements that are zero. The following principle often simplifies the process of finding a determinant by introducing zeros into the matrix without changing the value of the determinant.

Row and Column Transformations of a Determinant

If A is a square matrix and if the matrix B is obtained from A by adding a multiple of one row to another or a multiple of one column to another, then $\det(A) = \det(B)$.

Example 5 ■ Using Row and Column Transformations to Calculate a Determinant

Find the determinant of the matrix A. Does it have an inverse?

$$A = \begin{bmatrix} 8 & 2 & -1 & -4 \\ 3 & 5 & -3 & 11 \\ 24 & 6 & 1 & -12 \\ 2 & 2 & 7 & -1 \end{bmatrix}$$

Solution If we add -3 times row 1 to row 3, we change all but one element of row 3 to zeros.

$$\begin{bmatrix} 8 & 2 & -1 & -4 \\ 3 & 5 & -3 & 11 \\ 0 & 0 & 4 & 0 \\ 2 & 2 & 7 & -1 \end{bmatrix}$$

This new matrix has the same determinant as A, and if we expand its determinant by the third row, we get

$$\det(A) = 4\begin{vmatrix} 8 & 2 & -4 \\ 3 & 5 & 11 \\ 2 & 2 & -1 \end{vmatrix}$$

Now, adding 2 times column 3 to column 1 in this determinant gives us

$$\begin{aligned} \det(A) &= 4\begin{vmatrix} 0 & 2 & -4 \\ 25 & 5 & 11 \\ 0 & 2 & -1 \end{vmatrix} && \text{Expand this by column 1} \\ &= 4(-25)\begin{vmatrix} 2 & -4 \\ 2 & -1 \end{vmatrix} \\ &= 4(-25)[2(-1) - (-4)2] = -600 \end{aligned}$$

Since the determinant of A is not zero, A does have an inverse.

Now Try Exercise 35 ■

■ Cramer's Rule

The solutions of linear equations can sometimes be expressed by using determinants. To illustrate, let's solve the following pair of linear equations for the variable x.

$$\begin{cases} ax + by = r \\ cx + dy = s \end{cases}$$

To eliminate the variable y, we multiply the first equation by d and the second by b and subtract.

$$\begin{aligned} adx + bdy &= rd \\ \underline{bcx + bdy} &\underline{= bs} \\ adx - bcx &= rd - bs \end{aligned}$$

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Discovery Project ■ Computer Graphics II

Matrix algebra is the basic tool used in computer graphics. Properties of each pixel in an image are stored in a large matrix in the computer memory. In this project we discover how matrix multiplication can be used to "move" a point in the plane to a prescribed location. Combining such moves for each pixel in an image enables us to stretch, compress, translate, and otherwise transform an image on a computer screen by using matrix algebra. You can find the project at **www.stewartmath.com**.

Factoring the left-hand side, we get $(ad - bc)x = rd - bs$. Assuming that $ad - bc \neq 0$, we can now solve this equation for x:

$$x = \frac{rd - bs}{ad - bc}$$

Similarly, we find

$$y = \frac{as - cr}{ad - bc}$$

The numerator and denominator of the fractions for x and y are determinants of 2×2 matrices. So we can express the solution of the system using determinants as follows.

Cramer's Rule for Systems in Two Variables

The linear system

$$\begin{cases} ax + by = r \\ cx + dy = s \end{cases}$$

has the solution

$$x = \frac{\begin{vmatrix} r & b \\ s & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a & r \\ c & s \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

provided that $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$.

Using the notation

$$\underbrace{D = \begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{\text{Coefficient matrix}} \qquad \underbrace{D_x = \begin{bmatrix} r & b \\ s & d \end{bmatrix}}_{\text{Replace first column of } D \text{ by } r \text{ and } s} \qquad \underbrace{D_y = \begin{bmatrix} a & r \\ c & s \end{bmatrix}}_{\text{Replace second column of } D \text{ by } r \text{ and } s}$$

we can write the solution of the system as

$$\boxed{x = \frac{|D_x|}{|D|} \quad \text{and} \quad y = \frac{|D_y|}{|D|}}$$

Example 6 ■ Using Cramer's Rule to Solve a System with Two Variables

Use Cramer's Rule to solve the system.

$$\begin{cases} 2x + 6y = -1 \\ x + 8y = 2 \end{cases}$$

Solution For this system we have

$$|D| = \begin{vmatrix} 2 & 6 \\ 1 & 8 \end{vmatrix} = 2 \cdot 8 - 6 \cdot 1 = 10$$

$$|D_x| = \begin{vmatrix} -1 & 6 \\ 2 & 8 \end{vmatrix} = (-1)8 - 6 \cdot 2 = -20$$

$$|D_y| = \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} = 2 \cdot 2 - (-1)1 = 5$$

The solution is

$$x = \frac{|D_x|}{|D|} = \frac{-20}{10} = -2$$

$$y = \frac{|D_y|}{|D|} = \frac{5}{10} = \frac{1}{2}$$

Now Try Exercise 41 ■

Cramer's Rule can be extended to apply to any system of n linear equations in n variables in which the determinant of the coefficient matrix is not zero. As we saw in the preceding section, any such system can be written in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

By analogy with our derivation of Cramer's Rule in the case of two equations in two unknowns, we let D be the coefficient matrix in this system, and D_{x_i} be the matrix obtained by replacing the ith column of D by the numbers $b_1, b_2, \ldots, b_n$ that appear to the right of the equal sign. The solution of the system is then given by the following rule.

Cramer's Rule

If a system of n linear equations in the n variables $x_1, x_2, \ldots, x_n$ is equivalent to the matrix equation $DX = B$, and if $|D| \neq 0$, then its solutions are

$$x_1 = \frac{|D_{x_1}|}{|D|} \qquad x_2 = \frac{|D_{x_2}|}{|D|} \qquad \cdots \qquad x_n = \frac{|D_{x_n}|}{|D|}$$

where D_{x_i} is the matrix obtained by replacing the ith column of D by the $n \times 1$ matrix B.

Example 7 ■ Using Cramer's Rule to Solve a System with Three Variables

Use Cramer's Rule to solve the system.

$$\begin{cases} 2x - 3y + 4z = 1 \\ x + 6z = 0 \\ 3x - 2y = 5 \end{cases}$$

Solution First, we evaluate the determinants that appear in Cramer's Rule. Note that D is the coefficient matrix and that D_x, D_y, and D_z are obtained by replacing the first, second, and third columns of D by the constant terms.

$$|D| = \begin{vmatrix} 2 & -3 & 4 \\ 1 & 0 & 6 \\ 3 & -2 & 0 \end{vmatrix} = -38 \qquad |D_x| = \begin{vmatrix} 1 & -3 & 4 \\ 0 & 0 & 6 \\ 5 & -2 & 0 \end{vmatrix} = -78$$

$$|D_y| = \begin{vmatrix} 2 & 1 & 4 \\ 1 & 0 & 6 \\ 3 & 5 & 0 \end{vmatrix} = -22 \qquad |D_z| = \begin{vmatrix} 2 & -3 & 1 \\ 1 & 0 & 0 \\ 3 & -2 & 5 \end{vmatrix} = 13$$

Now we use Cramer's Rule to get the solution:

$$x = \frac{|D_x|}{|D|} = \frac{-78}{-38} = \frac{39}{19} \qquad y = \frac{|D_y|}{|D|} = \frac{-22}{-38} = \frac{11}{19}$$

$$z = \frac{|D_z|}{|D|} = \frac{13}{-38} = -\frac{13}{38}$$

Now Try Exercise 47 ■

Solving the system in Example 7 using Gaussian elimination would involve matrices whose elements are fractions with fairly large denominators. Thus in cases like Examples 6 and 7, Cramer's Rule gives us an efficient way to solve systems of linear equations. But in systems with more than three equations, evaluating the various determinants that are involved is usually a long and tedious task (unless you are using a graphing device). Moreover, the rule doesn't apply if $|D| = 0$ or if D is not a square matrix. So Cramer's Rule is a useful alternative to Gaussian elimination, but only in some situations.

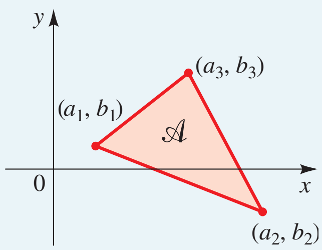
■ Areas of Triangles Using Determinants

Determinants provide a simple way to calculate the area of a triangle in the coordinate plane.

Area of a Triangle

If a triangle in the coordinate plane has vertices (a_1, b_1), (a_2, b_2), and (a_3, b_3), then its area is

$$\mathcal{A} = \pm\frac{1}{2}\begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$$



where the sign is chosen to make the area positive.

You are asked to prove this formula in Exercise 74.

Example 8 ■ Area of a Triangle

Find the area of the triangle shown in Figure 2.

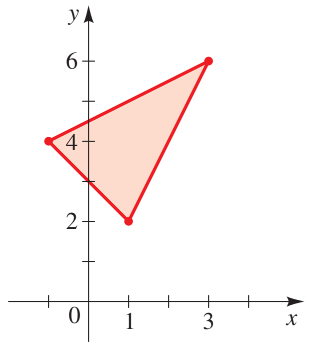


Figure 2

Solution The vertices are $(-1, 4)$, $(3, 6)$, and $(1, 2)$. Using the formula for the area of a triangle, we get

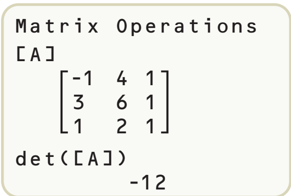
$$\mathscr{A} = \pm\tfrac{1}{2}\begin{vmatrix} -1 & 4 & 1 \\ 3 & 6 & 1 \\ 1 & 2 & 1 \end{vmatrix} = \pm\tfrac{1}{2}(-12)$$

To make the area positive, we choose the negative sign in the formula. Thus the area of the triangle is

$$\mathscr{A} = -\tfrac{1}{2}(-12) = 6$$

Now Try Exercise 57 ■

We can calculate the determinant in Example 8 by using the methods of this section or by using a graphing device.



9.6 Exercises

Concepts

1. *True or false*? $\det(A)$ is defined only for a square matrix A.

2. *True or false*? $\det(A)$ is a number, not a matrix.

3. *True or false*? If $\det(A) = 0$, then A is not invertible.

4. Fill in the blanks with appropriate numbers to calculate the determinant. Where there is "±", choose the appropriate sign (+ or −).

(a) $\begin{vmatrix} 2 & 1 \\ -3 & 4 \end{vmatrix} = \square\square - \square\square = \_\_\_\_\_$

(b) $\begin{vmatrix} 1 & 0 & 2 \\ 3 & 2 & 1 \\ 0 & -3 & 4 \end{vmatrix} = \pm\square(\square\square - \square\square) \pm \square(\square\square - \square\square)$

$\pm\square(\square\square - \square\square) = \_\_\_\_\_$

Skills

5–14 ■ Finding Determinants Find the determinant of the matrix, if it exists.

5. $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

6. $\begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}$

7. $\begin{bmatrix} \frac{3}{2} & 1 \\ -1 & -\frac{2}{3} \end{bmatrix}$

8. $\begin{bmatrix} 0.2 & 0.4 \\ -0.4 & -0.8 \end{bmatrix}$

9. $\begin{bmatrix} 4 & 5 \\ 0 & -1 \end{bmatrix}$

10. $\begin{bmatrix} -2 & 1 \\ 3 & -2 \end{bmatrix}$

11. $\begin{bmatrix} 2 & 5 \end{bmatrix}$

12. $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$

13. $\begin{bmatrix} \frac{1}{2} & \frac{1}{8} \\ 1 & \frac{1}{2} \end{bmatrix}$

14. $\begin{bmatrix} 2.2 & -1.4 \\ 0.5 & 1.0 \end{bmatrix}$

15–20 ■ Minors and Cofactors Evaluate the specified minor and cofactor using the matrix A.

$$A = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ -3 & 5 & 2 \\ 0 & 0 & 4 \end{bmatrix}$$

15. M_{11}, A_{11}

16. M_{33}, A_{33}

17. M_{12}, A_{12}

18. M_{13}, A_{13}

19. M_{23}, A_{23}

20. M_{32}, A_{32}

21–28 ■ Finding Determinants Find the determinant of the matrix. Determine whether the matrix has an inverse, but don't calculate the inverse.

21. $\begin{bmatrix} 2 & 1 & 0 \\ 0 & -2 & 4 \\ 0 & 1 & -3 \end{bmatrix}$

22. $\begin{bmatrix} 1 & 2 & 5 \\ -2 & -3 & 2 \\ 3 & 5 & 3 \end{bmatrix}$

23. $\begin{bmatrix} 30 & 0 & 20 \\ 0 & -10 & -20 \\ 40 & 0 & 10 \end{bmatrix}$

24. $\begin{bmatrix} -2 & -\frac{3}{2} & \frac{1}{2} \\ 2 & 4 & 0 \\ \frac{1}{2} & 2 & 1 \end{bmatrix}$

25. $\begin{bmatrix} 1 & 3 & 7 \\ 2 & 0 & 8 \\ 0 & 2 & 2 \end{bmatrix}$

26. $\begin{bmatrix} 0 & -1 & 0 \\ 2 & 6 & 4 \\ 1 & 0 & 3 \end{bmatrix}$

27. $\begin{bmatrix} 1 & 3 & 3 & 0 \\ 0 & 2 & 0 & 1 \\ -1 & 0 & 0 & 2 \\ 1 & 6 & 4 & 1 \end{bmatrix}$

28. $\begin{bmatrix} 1 & 2 & 0 & 2 \\ 3 & -4 & 0 & 4 \\ 0 & 1 & 6 & 0 \\ 1 & 0 & 2 & 0 \end{bmatrix}$

29–34 ■ Finding Determinants Use a graphing device to find the determinant of the matrix. Determine whether the matrix has an inverse, but don't calculate the inverse.

29. $\begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

30. $\begin{bmatrix} 10 & -20 & 31 \\ 10 & -11 & 45 \\ -20 & 40 & -50 \end{bmatrix}$

31. $\begin{bmatrix} 1 & 10 & 2 & 7 \\ 2 & 18 & 18 & 13 \\ -3 & -30 & -4 & -24 \\ 1 & 10 & 2 & 10 \end{bmatrix}$

32. $\begin{bmatrix} 1 & 3 & -2 & 5 \\ -3 & -9 & 11 & 5 \\ 2 & 6 & 0 & 31 \\ 5 & 15 & -10 & 39 \end{bmatrix}$

33. $\begin{bmatrix} 4 & 3 & -2 & 10 \\ -8 & -6 & 24 & -1 \\ 20 & 15 & 3 & 27 \\ 12 & 9 & -6 & -1 \end{bmatrix}$

34. $\begin{bmatrix} 2 & 3 & -5 & 10 \\ -2 & -2 & 26 & 3 \\ 6 & 9 & -16 & 45 \\ -8 & -12 & 20 & -36 \end{bmatrix}$

35–38 ■ Determinants Using Row and Column Operations Evaluate the determinant, using row or column operations whenever possible to simplify your work.

35. $\begin{vmatrix} 0 & 0 & 4 & 6 \\ 2 & 1 & 1 & 3 \\ 2 & 1 & 2 & 3 \\ 3 & 0 & 1 & 7 \end{vmatrix}$

36. $\begin{vmatrix} -2 & 3 & -1 & 7 \\ 4 & 6 & -2 & 3 \\ 7 & 7 & 0 & 5 \\ 3 & -12 & 4 & 0 \end{vmatrix}$

37. $\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 4 & 6 & 8 \\ 0 & 0 & 3 & 6 & 9 \\ 0 & 0 & 0 & 4 & 8 \\ 0 & 0 & 0 & 0 & 5 \end{vmatrix}$

38. $\begin{vmatrix} 2 & -1 & 6 & 4 \\ 7 & 2 & -2 & 5 \\ 4 & -2 & 10 & 8 \\ 6 & 1 & 1 & 4 \end{vmatrix}$

39. Calculating a Determinant in Different Ways Consider the matrix

$$B = \begin{bmatrix} 4 & 1 & 0 \\ -2 & -1 & 1 \\ 4 & 0 & 3 \end{bmatrix}$$

(a) Evaluate $\det(B)$ by expanding by the second row.

(b) Evaluate $\det(B)$ by expanding by the third column.

(c) Do your results in parts (a) and (b) agree?

40. Determinant of a Special Matrix Find the determinant of a 10×10 matrix which has a 2 in each main diagonal entry and zeros everywhere else.

41–56 ■ Cramer's Rule Use Cramer's Rule to solve the system.

41. $\begin{cases} 2x - y = -9 \\ x + 2y = 8 \end{cases}$

42. $\begin{cases} 6x + 12y = 33 \\ 4x + 7y = 20 \end{cases}$

43. $\begin{cases} x - 6y = 3 \\ 3x + 2y = 1 \end{cases}$

44. $\begin{cases} \frac{1}{2}x + \frac{1}{3}y = 1 \\ \frac{1}{4}x - \frac{1}{6}y = -\frac{3}{2} \end{cases}$

45. $\begin{cases} 0.4x + 1.2y = 0.4 \\ 1.2x + 1.6y = 3.2 \end{cases}$

46. $\begin{cases} 10x - 17y = 21 \\ 20x - 31y = 39 \end{cases}$

47. $\begin{cases} x - y + 2z = 0 \\ 3x + z = 11 \\ -x + 2y = 0 \end{cases}$

48. $\begin{cases} 5x - 3y + z = 6 \\ 4y - 6z = 22 \\ 7x + 10y = -13 \end{cases}$

49. $\begin{cases} 2x_1 + 3x_2 - 5x_3 = 1 \\ x_1 + x_2 - x_3 = 2 \\ 2x_2 + x_3 = 8 \end{cases}$

50. $\begin{cases} -2a + c = 2 \\ a + 2b - c = 9 \\ 3a + 5b + 2c = 22 \end{cases}$

51. $\begin{cases} \frac{1}{3}x - \frac{1}{5}y + \frac{1}{2}z = \frac{7}{10} \\ -\frac{2}{3}x + \frac{2}{5}y + \frac{3}{2}z = \frac{11}{10} \\ x - \frac{4}{5}y + z = \frac{9}{5} \end{cases}$

52. $\begin{cases} 2x - y = 5 \\ 5x + 3z = 19 \\ 4y + 7z = 17 \end{cases}$

53. $\begin{cases} 3y + 5z = 4 \\ 2x - z = 10 \\ 4x + 7y = 0 \end{cases}$

54. $\begin{cases} 2x - 5y = 4 \\ x + y - z = 8 \\ 3x + 5z = 0 \end{cases}$

55. $\begin{cases} x + y + z + w = 0 \\ 2x + w = 0 \\ y - z = 0 \\ x + 2z = 1 \end{cases}$

56. $\begin{cases} x + y = 1 \\ y + z = 2 \\ z + w = 3 \\ w - x = 4 \end{cases}$

57–60 ■ Area of a Triangle Sketch the triangle with the given vertices, and use a determinant to find its area.

57. $(0, 0), (6, 2), (3, 8)$

58. $(1, 0), (3, 5), (-2, 2)$

59. $(-1, 3), (2, 9), (5, -6)$

60. $(-2, 5), (7, 2), (3, -4)$

Skills Plus

61–62 ■ Determinants of Special Matrices Evaluate the determinants.

61. $\begin{vmatrix} a & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & d & 0 \\ 0 & 0 & 0 & 0 & e \end{vmatrix}$ **62.** $\begin{vmatrix} a & a & a & a & a \\ 0 & a & a & a & a \\ 0 & 0 & a & a & a \\ 0 & 0 & 0 & a & a \\ 0 & 0 & 0 & 0 & a \end{vmatrix}$

63–66 ■ Determinant Equations Solve for x.

63. $\begin{vmatrix} x & 12 & 13 \\ 0 & x-1 & 23 \\ 0 & 0 & x-2 \end{vmatrix} = 0$ **64.** $\begin{vmatrix} x & 1 & 1 \\ 1 & 1 & x \\ x & 1 & x \end{vmatrix} = 0$

65. $\begin{vmatrix} 1 & 0 & x \\ x^2 & 1 & 0 \\ x & 0 & 1 \end{vmatrix} = 0$ **66.** $\begin{vmatrix} a & b & x-a \\ x & x+b & x \\ 0 & 1 & 1 \end{vmatrix} = 0$

67. Using Determinants Show that

$$\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (x-y)(y-z)(z-x)$$

68. Number of Solutions of a Linear System Consider the system

$$\begin{cases} x + 2y + 6z = 5 \\ -3x - 6y + 5z = 8 \\ 2x + 6y + 9z = 7 \end{cases}$$

(a) Verify that $x = -1$, $y = 0$, $z = 1$ is a solution of the system.

(b) Find the determinant of the coefficient matrix.

(c) Without solving the system, determine whether there are any other solutions.

(d) Can Cramer's Rule be used to solve this system? Why or why not?

69. Collinear Points and Determinants

(a) If three points lie on a line, what is the area of the "triangle" that they determine? Use the answer to this question, together with the determinant formula for the area of a triangle, to explain why the points (a_1, b_1), (a_2, b_2), and (a_3, b_3) are collinear if and only if

$$\begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix} = 0$$

(b) Use a determinant to check whether each set of points is collinear. Graph them to verify your answer.

(i) $(-6, 4)$, $(2, 10)$, $(6, 13)$

(ii) $(-5, 10)$, $(2, 6)$, $(15, -2)$

70. Determinant Form for the Equation of a Line

(a) Use the result of Exercise 69(a) to show that the equation of the line containing the points (x_1, y_1) and (x_2, y_2) is

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

(b) Use the result of part (a) to find an equation for the line containing the points $(20, 50)$ and $(-10, 25)$.

Applications

71. Buying Fruit A roadside fruit stand sells apples at 75¢ a pound, peaches at 90¢ a pound, and pears at 60¢ a pound. A customer buys a total of 18 lb of these fruits at a total cost of \$13.80. The peaches and pears together cost \$1.80 more than the apples.

(a) Set up a linear system for the number of pounds of apples, peaches, and pears the customer purchased.

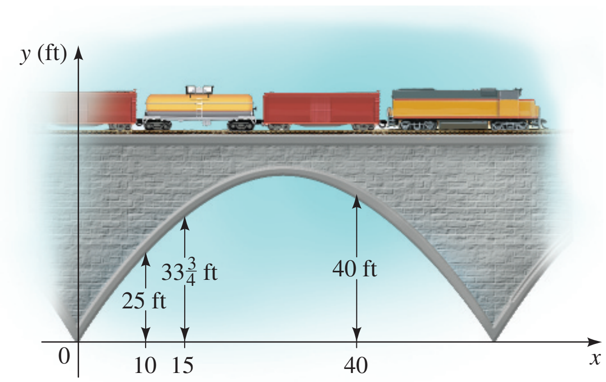
(b) Solve the system using Cramer's Rule.

72. The Arch of a Bridge The opening of a railway bridge over a roadway is in the shape of a parabola. A surveyor measures the heights of three points on the bridge, as shown in the figure. The shape of the arch can be modeled by an equation of the form

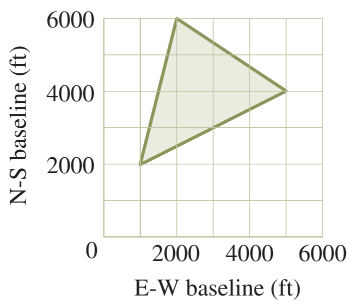
$$y = ax^2 + bx + c$$

(a) Use the surveyed points to set up a system of linear equations for the unknown coefficients a, b, and c.

(b) Solve the system using Cramer's Rule.



73. A Triangular Plot of Land An outdoors club is purchasing land to set up a conservation area. The last remaining piece they need to buy is the triangular plot shown in the figure. Use the determinant formula for the area of a triangle to find the area of the plot.



■ Discuss ■ Discover ■ Prove ■ Write

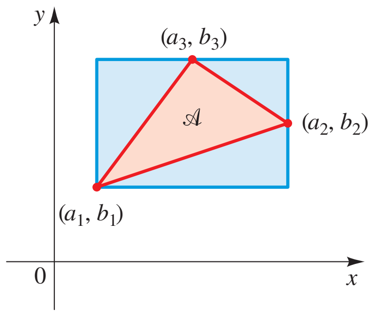
74. Discover ■ Prove: Determinant Formula for the Area of a Triangle The figure shows a triangle in the plane with vertices (a_1, b_1), (a_2, b_2), and (a_3, b_3).

(a) Find the coordinates of the vertices of the surrounding rectangle, and find its area.

(b) Find the area of the red triangle by subtracting the areas of the three blue triangles from the area of the rectangle.

(c) Use your answer to part (b) to show that the area $\mathscr{A}$ of the red triangle is given by

$$\mathscr{A} = \pm\frac{1}{2}\begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$$



75. Discuss: Matrices with Determinant Zero Use the definition of determinant and the elementary row and column operations to explain why matrices of the following types have determinant 0.

(a) A matrix with a row or column consisting entirely of zeros

(b) A matrix with two rows that are the same or two columns that are the same

(c) A matrix in which one row is a multiple of another row, or one column is a multiple of another column

76. Discuss ■ Write: Solving Linear Systems Suppose you have to solve a linear system with five equations and five variables without the assistance of a calculator or computer. Which method would you prefer: Cramer's Rule or Gaussian elimination? Write a short paragraph explaining the reasons for your answer.

77. Prove: The Determinant of a Product of Matrices Let A and B be $n \times n$ matrices. It is known that

$$\det(AB) = \det(A)\det(B)$$

Use this fact to find a formula for $\det(A^{-1})$ in terms of $\det(A)$.

PS *Try to recognize something familiar.* Use the fact that $AA^{-1} = I_n$ and $\det I_n = 1$.

9.7 Partial Fractions

■ Distinct Linear Factors ■ Repeated Linear Factors ■ Irreducible Quadratic Factors ■ Repeated Irreducible Quadratic Factors

Common denominator

$$\frac{1}{x-1} + \frac{1}{2x+1} = \frac{3x}{2x^2 - x - 1}$$

Partial fractions

To write a sum or difference of fractional expressions as a single fraction, we bring them to a common denominator. For example,

$$\frac{1}{x-1} + \frac{1}{2x+1} = \frac{(2x+1)+(x-1)}{(x-1)(2x+1)} = \frac{3x}{2x^2 - x - 1}$$

But for some applications of algebra to calculus we must reverse this process—that is, we must express a fraction such as $3x/(2x^2 - x - 1)$ as the sum of the simpler fractions $1/(x-1)$ and $1/(2x+1)$. These simpler fractions are called *partial fractions*; we learn how to find them in this section.

Let r be the rational function

$$r(x) = \frac{P(x)}{Q(x)}$$

where the degree of P is less than the degree of Q. By the Linear and Quadratic Factors Theorem in Section 3.5, every polynomial with real coefficients can be factored completely into linear and irreducible quadratic factors, that is, factors of the form $ax + b$ and $ax^2 + bx + c$, where a, b, and c are real numbers. For instance,

$$x^4 - 1 = (x^2 - 1)(x^2 + 1) = (x - 1)(x + 1)(x^2 + 1)$$

After we have completely factored the denominator Q of r, we can express $r(x)$ as a sum of **partial fractions** of the form

$$\frac{A}{(ax+b)^i} \quad \text{and} \quad \frac{Ax+B}{(ax^2+bx+c)^j}$$

This sum is called the **partial fraction decomposition** of r. Let's examine the details of the four possible cases.

Distinct Linear Factors

We first consider the case in which the denominator factors into distinct linear factors.

Case 1: The Denominator Is a Product of Distinct Linear Factors

Suppose that we can factor $Q(x)$ as

$$Q(x) = (a_1x + b_1)(a_2x + b_2)\cdots(a_nx + b_n)$$

with no factor repeated. In this case the partial fraction decomposition of $P(x)/Q(x)$ takes the form

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_n}{a_nx + b_n}$$

The constants $A_1, A_2, \ldots, A_n$ are determined as in the following example.

Example 1 ■ Distinct Linear Factors

Find the partial fraction decomposition of $\dfrac{5x+7}{x^3+2x^2-x-2}$.

Solution The denominator factors as follows.

$$\begin{aligned} x^3 + 2x^2 - x - 2 &= x^2(x+2) - (x+2) = (x^2-1)(x+2) \\ &= (x-1)(x+1)(x+2) \end{aligned}$$

This gives us the partial fraction decomposition

$$\frac{5x+7}{x^3+2x^2-x-2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+2}$$

Multiplying each side by the common denominator, $(x-1)(x+1)(x+2)$, we get

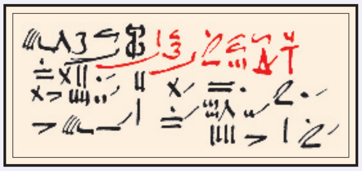
$$\begin{aligned} 5x + 7 &= A(x+1)(x+2) + B(x-1)(x+2) + C(x-1)(x+1) \\ &= A(x^2+3x+2) + B(x^2+x-2) + C(x^2-1) && \text{Expand} \\ &= (A+B+C)x^2 + (3A+B)x + (2A-2B-C) && \text{Combine like terms} \end{aligned}$$

If two polynomials are equal, then their coefficients are equal. Thus because $5x+7$ has no x^2-term, we have $A+B+C=0$. Similarly, by comparing the coefficients of x, we see that $3A+B=5$, and by comparing constant terms, we get $2A-2B-C=7$. This leads to the following system of linear equations for A, B, and C.

$$\begin{cases} A + B + C = 0 & \text{Equation 1: Coefficients of } x^2 \\ 3A + B = 5 & \text{Equation 2: Coefficients of } x \\ 2A - 2B - C = 7 & \text{Equation 3: Constant coefficients} \end{cases}$$

THE RHIND PAPYRUS is the oldest known mathematical document. It is an Egyptian scroll written in 1650 B.C. by the scribe Ahmes, who explains that it is an exact copy of a scroll written 200 years earlier. Ahmes claims that his papyrus contains "a thorough study of all things, insight into all that exists, knowledge of all obscure secrets." In fact, the document contains rules for doing arithmetic, including multiplication and division of fractions and several exercises with solutions. The exercise shown below reads: "A heap and its seventh make nineteen; how large is the heap?" In solving problems of this sort, the Egyptians used partial fractions because their number system required all fractions to be written as sums of reciprocals of whole numbers. For example, $\frac{7}{12}$ would be written as $\frac{1}{3} + \frac{1}{4}$.

The papyrus gives a correct formula for the volume of a truncated pyramid, which the ancient Egyptians probably used when building the pyramids at Giza. It also gives the formula $A = \left(\frac{8}{9}d\right)^2$ for the area of a circle with diameter d. How close is this to the actual area?



We use Gaussian elimination to solve this system.

$$\begin{cases} A + B + C = 0 \\ \quad -2B - 3C = 5 & \text{Equation 2} + (-3) \times \text{Equation 1} \\ \quad -4B - 3C = 7 & \text{Equation 3} + (-2) \times \text{Equation 1} \end{cases}$$

$$\begin{cases} A + B + C = 0 \\ \quad -2B - 3C = 5 \\ \qquad\quad 3C = -3 & \text{Equation 3} + (-2) \times \text{Equation 2} \end{cases}$$

From the third equation we get $C = -1$. Back-substituting, we find that $B = -1$ and $A = 2$. So the partial fraction decomposition is

$$\frac{5x + 7}{x^3 + 2x^2 - x - 2} = \frac{2}{x - 1} + \frac{-1}{x + 1} + \frac{-1}{x + 2}$$

Now Try Exercises 3 and 13 ■

The same approach works in the remaining cases: Set up the partial fraction decomposition with the unknown constants $A, B, C, \ldots$. Then multiply each side of the resulting equation by the common denominator, combine like terms on the right-hand side of the equation, and equate coefficients. This gives a set of linear equations that will always have a unique solution (provided that the partial fraction decomposition has been set up correctly).

■ Repeated Linear Factors

We now consider the case in which the denominator factors into linear factors, some of which are repeated.

Case 2: The Denominator Is a Product of Linear Factors, Some of Which Are Repeated

Suppose the complete factorization of $Q(x)$ contains the linear factor $ax + b$ repeated k times; that is, $(ax + b)^k$ is a factor of $Q(x)$. Then, corresponding to each such factor, the partial fraction decomposition for $P(x)/Q(x)$ contains

$$\frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \cdots + \frac{A_k}{(ax + b)^k}$$

Example 2 ■ Repeated Linear Factors

Find the partial fraction decomposition of $\dfrac{x^2 + 1}{x(x - 1)^3}$.

Solution Because the factor $x - 1$ is repeated three times in the denominator, the partial fraction decomposition has the form

$$\frac{x^2 + 1}{x(x - 1)^3} = \frac{A}{x} + \frac{B}{x - 1} + \frac{C}{(x - 1)^2} + \frac{D}{(x - 1)^3}$$

Multiplying each side by the common denominator, $x(x - 1)^3$, gives

$$\begin{aligned} x^2 + 1 &= A(x - 1)^3 + Bx(x - 1)^2 + Cx(x - 1) + Dx \\ &= A(x^3 - 3x^2 + 3x - 1) + B(x^3 - 2x^2 + x) + C(x^2 - x) + Dx && \text{Expand} \\ &= (A + B)x^3 + (-3A - 2B + C)x^2 + (3A + B - C + D)x - A && \text{Combine like terms} \end{aligned}$$

Equating coefficients, we get the following equations.

$$\begin{cases} A + B = 0 & \text{Coefficients of } x^3 \\ -3A - 2B + C = 1 & \text{Coefficients of } x^2 \\ 3A + B - C + D = 0 & \text{Coefficients of } x \\ -A = 1 & \text{Constant coefficients} \end{cases}$$

If we rearrange these equations by putting the last one in the first position, we can see (using substitution) that the solution to the system is $A = -1$, $B = 1$, $C = 0$, $D = 2$, so the partial fraction decomposition is

$$\frac{x^2 + 1}{x(x-1)^3} = \frac{-1}{x} + \frac{1}{x-1} + \frac{2}{(x-1)^3}$$

Now Try Exercises 5 and 29 ■

■ Irreducible Quadratic Factors

We now consider the case in which the denominator has distinct irreducible quadratic factors.

Case 3: The Denominator Has Irreducible Quadratic Factors, None of Which Is Repeated

Suppose the complete factorization of $Q(x)$ contains the quadratic factor $ax^2 + bx + c$ (which can't be factored further). Then, corresponding to this, the partial fraction decomposition of $P(x)/Q(x)$ will have a term of the form

$$\frac{Ax + B}{ax^2 + bx + c}$$

Example 3 ■ Distinct Quadratic Factors

Find the partial fraction decomposition of $\dfrac{2x^2 - x + 4}{x^3 + 4x}$.

Solution Since $x^3 + 4x = x(x^2 + 4)$, which can't be factored further, we write

$$\frac{2x^2 - x + 4}{x^3 + 4x} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}$$

Multiplying by $x(x^2 + 4)$, we get

$$\begin{aligned} 2x^2 - x + 4 &= A(x^2 + 4) + (Bx + C)x \\ &= (A + B)x^2 + Cx + 4A \end{aligned}$$

Equating coefficients gives us the equations

$$\begin{cases} A + B = 2 & \text{Coefficients of } x^2 \\ C = -1 & \text{Coefficients of } x \\ 4A = 4 & \text{Constant coefficients} \end{cases}$$

so $A = 1$, $B = 1$, and $C = -1$. The required partial fraction decomposition is

$$\frac{2x^2 - x + 4}{x^3 + 4x} = \frac{1}{x} + \frac{x - 1}{x^2 + 4}$$

Now Try Exercises 7 and 37 ■

Repeated Irreducible Quadratic Factors

We now consider the case in which the denominator has irreducible quadratic factors, some of which are repeated.

Case 4: The Denominator Has a Repeated Irreducible Quadratic Factor

Suppose the complete factorization of $Q(x)$ contains the factor $(ax^2 + bx + c)^k$, where $ax^2 + bx + c$ can't be factored further. Then the partial fraction decomposition of $P(x)/Q(x)$ will have the terms

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_kx + B_k}{(ax^2 + bx + c)^k}$$

Example 4 ■ Repeated Quadratic Factors

Write the form of the partial fraction decomposition of

$$\frac{x^5 - 3x^2 + 12x - 1}{x^3(x^2 + x + 1)(x^2 + 2)^3}$$

Solution The irreducible quadratic factor $x^2 + 2$ is repeated three times.

$$\frac{x^5 - 3x^2 + 12x - 1}{x^3(x^2 + x + 1)(x^2 + 2)^3}$$
$$= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Dx + E}{x^2 + x + 1} + \frac{Fx + G}{x^2 + 2} + \frac{Hx + I}{(x^2 + 2)^2} + \frac{Jx + K}{(x^2 + 2)^3}$$

Now Try Exercises 11 and 41 ■

To find the values of $A, B, C, D, E, F, G, H, I, J$, and K in Example 4, we would have to solve a system of 11 linear equations. Although possible, this would certainly involve a great deal of work!

The techniques that we have described in this section apply only to rational functions $P(x)/Q(x)$ in which the degree of P is less than the degree of Q. If this isn't the case, we must first use long division to divide Q into P.

Example 5 ■ Using Long Division to Prepare for Partial Fractions

Find the partial fraction decomposition of

$$\frac{2x^4 + 4x^3 - 2x^2 + x + 7}{x^3 + 2x^2 - x - 2}$$

Solution Since the degree of the numerator is larger than the degree of the denominator, we use long division to obtain

$$\begin{array}{r} 2x \\ x^3 + 2x^2 - x - 2 \,\big)\, \overline{2x^4 + 4x^3 - 2x^2 + x + 7} \\ \underline{2x^4 + 4x^3 - 2x^2 - 4x} \\ 5x + 7 \end{array}$$

$$\frac{2x^4 + 4x^3 - 2x^2 + x + 7}{x^3 + 2x^2 - x - 2} = 2x + \frac{5x + 7}{x^3 + 2x^2 - x - 2}$$

The remainder term now satisfies the requirement that the degree of the numerator is less than the degree of the denominator. At this point we proceed as in Example 1 to obtain the decomposition

$$\frac{2x^4 + 4x^3 - 2x^2 + x + 7}{x^3 + 2x^2 - x - 2} = 2x + \frac{2}{x - 1} + \frac{-1}{x + 1} + \frac{-1}{x + 2}$$

Now Try Exercise 43 ■

9.7 Exercises

Concepts

1–2 ■ For each rational function r, choose from (i)–(iv) the appropriate form for its partial fraction decomposition.

1. $r(x) = \dfrac{4}{x(x-2)^2}$

(i) $\dfrac{A}{x} + \dfrac{B}{x-2}$ (ii) $\dfrac{A}{x} + \dfrac{B}{(x-2)^2}$

(iii) $\dfrac{A}{x} + \dfrac{B}{x-2} + \dfrac{C}{(x-2)^2}$ (iv) $\dfrac{A}{x} + \dfrac{B}{x-2} + \dfrac{Cx+D}{(x-2)^2}$

2. $r(x) = \dfrac{2x+8}{(x-1)(x^2+4)}$

(i) $\dfrac{A}{x-1} + \dfrac{B}{x^2+4}$

(ii) $\dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+4}$

(iii) $\dfrac{A}{x-1} + \dfrac{B}{x+2} + \dfrac{C}{x^2+4}$

(iv) $\dfrac{Ax+B}{x-1} + \dfrac{Cx+D}{x^2+4}$

Skills

3–12 ■ **Form of the Partial Fraction Decomposition** Write the form of the partial fraction decomposition of the function (as in Example 4). Do not determine the numerical values of the coefficients.

3. $\dfrac{1}{(x-1)(x+2)}$ **4.** $\dfrac{x}{x^2+3x-4}$

5. $\dfrac{x^2-3x+5}{(x-2)^2(x+4)}$ **6.** $\dfrac{1}{x^4-x^3}$

7. $\dfrac{x^2}{(x-3)(x^2+4)}$ **8.** $\dfrac{1}{x^4-1}$

9. $\dfrac{x^3-4x^2+2}{(x^2+1)(x^2+2)}$ **10.** $\dfrac{x^4+x^2+1}{x^2(x^2+4)^2}$

11. $\dfrac{x^3+x+1}{x(2x-5)^3(x^2+2x+5)^2}$ **12.** $\dfrac{1}{(x^3-1)(x^2-1)}$

13–44 ■ **Partial Fraction Decomposition** Find the partial fraction decomposition of the rational function.

13. $\dfrac{2}{(x-1)(x+1)}$ **14.** $\dfrac{2x}{(x-1)(x+1)}$

15. $\dfrac{5}{(x-1)(x+4)}$ **16.** $\dfrac{x+6}{x(x+3)}$

17. $\dfrac{12}{x^2-9}$ **18.** $\dfrac{x-12}{x^2-4x}$

19. $\dfrac{4}{x^2-4}$ **20.** $\dfrac{2x+1}{x^2+x-2}$

21. $\dfrac{x+14}{x^2-2x-8}$ **22.** $\dfrac{8x-3}{2x^2-x}$

23. $\dfrac{x}{8x^2-10x+3}$ **24.** $\dfrac{7x-3}{x^3+2x^2-3x}$

25. $\dfrac{9x^2-9x+6}{2x^3-x^2-8x+4}$ **26.** $\dfrac{-3x^2-3x+27}{(x+2)(2x^2+3x-9)}$

27. $\dfrac{x^2+1}{x^3+x^2}$ **28.** $\dfrac{3x^2+5x-13}{(3x+2)(x^2-4x+4)}$

29. $\dfrac{2x}{4x^2+12x+9}$ **30.** $\dfrac{x-4}{(2x-5)^2}$

31. $\dfrac{4x^2-x-2}{x^4+2x^3}$ **32.** $\dfrac{x^3-2x^2-4x+3}{x^4}$

33. $\dfrac{-10x^2+27x-14}{(x-1)^3(x+2)}$ **34.** $\dfrac{-2x^2+5x-1}{x^4-2x^3+2x-1}$

35. $\dfrac{3x^3+22x^2+53x+41}{(x+2)^2(x+3)^2}$ **36.** $\dfrac{3x^2+12x-20}{x^4-8x^2+16}$

37. $\dfrac{x-3}{x^3+3x}$ **38.** $\dfrac{3x^2-2x+8}{x^3-x^2+2x-2}$

39. $\dfrac{2x^3+7x+5}{(x^2+x+2)(x^2+1)}$ **40.** $\dfrac{x^2+x+1}{2x^4+3x^2+1}$

41. $\dfrac{x^4+x^3+x^2-x+1}{x(x^2+1)^2}$ **42.** $\dfrac{2x^2-x+8}{(x^2+4)^2}$

43. $\dfrac{x^5-2x^4+x^3+x+5}{x^3-2x^2+x-2}$

44. $\dfrac{x^5-3x^4+3x^3-4x^2+4x+12}{(x-2)^2(x^2+2)}$

Skills Plus

45. Partial Fractions Determine A and B in terms of a and b.

$$\frac{ax+b}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}$$

46. Partial Fractions Determine A, B, C, and D in terms of a and b.

$$\frac{ax^3+bx^2}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$$

Discuss ■ Discover ■ Prove ■ Write

47. Discuss: Recognizing Partial Fraction Decompositions For each expression, determine whether it is already a partial fraction decomposition or whether it can be decomposed further.

(a) $\dfrac{x}{x^2+1} + \dfrac{1}{x+1}$ **(b)** $\dfrac{x}{(x+1)^2}$

(c) $\dfrac{1}{x+1} + \dfrac{2}{(x+1)^2}$ **(d)** $\dfrac{x+2}{(x^2+1)^2}$

48. Discuss: Assembling and Disassembling Partial Fractions The following expression is a partial fraction decomposition.

$$\frac{2}{x-1} + \frac{1}{(x-1)^2} + \frac{1}{x+1}$$

Use a common denominator to combine the terms into one fraction. Then use the techniques of this section to find its partial fraction decomposition. Did you get back the original expression?

9.8 Systems of Nonlinear Equations

■ Substitution and Elimination Methods ■ Graphical Method

In this section we solve systems of equations in which the equations are not all linear. The methods we learned in Section 9.1 can also be used to solve nonlinear systems.

■ Substitution and Elimination Methods

To solve a system of nonlinear equations, we can use the substitution or elimination method, as illustrated in the next examples.

Example 1 ■ Substitution Method

Find all solutions of the system.

$$\begin{cases} x^2 + y^2 = 100 & \text{Equation 1} \\ 3x - y = 10 & \text{Equation 2} \end{cases}$$

Solution **Solve for one variable.** We start by solving for y in the second equation.

$$y = 3x - 10 \qquad \text{Solve for } y \text{ in Equation 2}$$

Substitute. Next we substitute for y in the first equation and solve for x.

$$\begin{aligned} x^2 + (3x - 10)^2 &= 100 && \text{Substitute } y = 3x - 10 \text{ into Equation 1} \\ x^2 + (9x^2 - 60x + 100) &= 100 && \text{Expand} \\ 10x^2 - 60x &= 0 && \text{Simplify} \\ 10x(x - 6) &= 0 && \text{Factor} \\ x = 0 \quad \text{or} \quad x &= 6 && \text{Solve for } x \end{aligned}$$

Back-substitute. Now we back-substitute these values of x into the equation $y = 3x - 10$.

$$\begin{aligned} &\text{For } x = 0: && y = 3(0) - 10 = -10 && \text{Back-substitute} \\ &\text{For } x = 6: && y = 3(6) - 10 = 8 && \text{Back-substitute} \end{aligned}$$

So we have two solutions: $(0, -10)$ and $(6, 8)$.

The graph of the first equation is a circle, and the graph of the second equation is a line. Figure 1 shows that the graphs intersect at the two points $(0, -10)$ and $(6, 8)$.

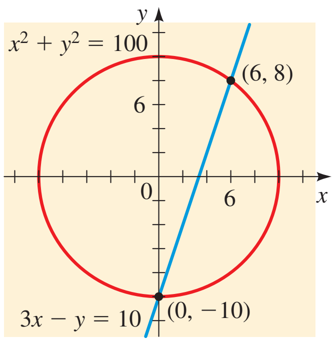


Figure 1

Check Your Answers

$x = 0, y = -10$:

$$\begin{cases} (0)^2 + (-10)^2 = 100 \\ 3(0) - (-10) = 10 \end{cases}$$

$x = 6, y = 8$:

$$\begin{cases} (6)^2 + (8)^2 = 36 + 64 = 100 \\ 3(6) - (8) = 18 - 8 = 10 \end{cases}$$

Now Try Exercise 5 ■

Example 2 ■ Elimination Method

Find all solutions of the system.

$$\begin{cases} 3x^2 + 2y = 26 & \text{Equation 1} \\ 5x^2 + 7y = 3 & \text{Equation 2} \end{cases}$$

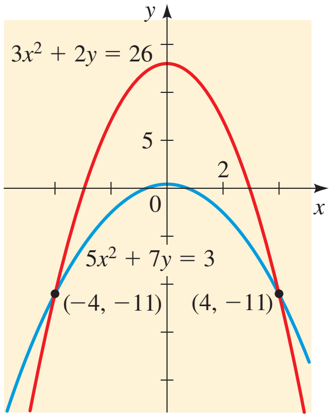


Figure 2

Check Your Answers

$x = -4, y = -11$:

$$\begin{cases} 3(-4)^2 + 2(-11) = 26 \\ 5(-4)^2 + 7(-11) = 3 \end{cases}$$ ✓

$x = 4, y = -11$:

$$\begin{cases} 3(4)^2 + 2(-11) = 26 \\ 5(4)^2 + 7(-11) = 3 \end{cases}$$ ✓

Solution **Adjust coefficints.** We choose to eliminate the terms containing x, so we multiply the first equation by 5 and the second equation by -3.

Eliminate a variable. We add the two equations and solve for y.

$$\begin{cases} 15x^2 + 10y = 130 & 5 \times \text{Equation 1} \\ -15x^2 - 21y = -9 & (-3) \times \text{Equation 2} \end{cases}$$

$$\begin{aligned} -11y &= 121 && \text{Add} \\ y &= -11 && \text{Solve for } y \end{aligned}$$

Back-substitute. Now we back-substitute $y = -11$ into one of the original equations, say $3x^2 + 2y = 26$, and solve for x.

$$\begin{aligned} 3x^2 + 2(-11) &= 26 && \text{Back-substitute } y = -11 \text{ into Equation 1} \\ 3x^2 &= 48 && \text{Add 22} \\ x^2 &= 16 && \text{Divide by 3} \\ x = -4 \quad \text{or} \quad x &= 4 && \text{Solve for } x \end{aligned}$$

So we have two solutions: $(-4, -11)$ and $(4, -11)$.

The graphs of both equations are parabolas (see Section 3.1). Figure 2 shows that the graphs intersect at the two points $(-4, -11)$ and $(4, -11)$.

Now Try Exercise 11 ■

■ Graphical Method

The graphical method is particularly useful in solving systems of nonlinear equations.

Example 3 ■ Graphical Method

Find all solutions of the system

$$\begin{cases} x^2 - y = 2 \\ 2x - y = -1 \end{cases}$$

Solution **Graph each equation.** To graph, we solve for y in each equation.

$$\begin{cases} y = x^2 - 2 \\ y = 2x + 1 \end{cases}$$

Find intersection points. Figure 3 shows that the graphs of these equations intersect at two points. From the graph we see that the solutions are

$$(-1, -1) \quad \text{and} \quad (3, 7)$$

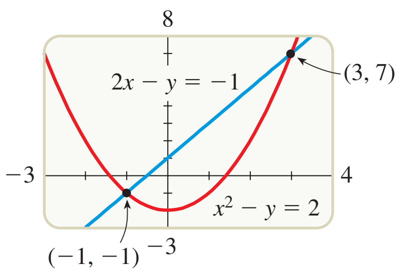


Figure 3

Check Your Answers

$x = -1, y = -1$:

$$\begin{cases} (-1)^2 - (-1) = 2 \\ 2(-1) - (-1) = -1 \end{cases}$$ ✓

$x = 3, y = 7$:

$$\begin{cases} 3^2 - 7 = 2 \\ 2(3) - 7 = -1 \end{cases}$$ ✓

Now Try Exercise 33 ■

Example 4 ■ Solving a System of Equations Graphically

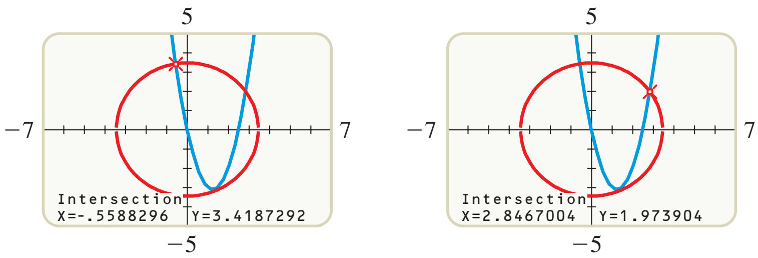
Find all solutions of the system, rounded to one decimal place.

$$\begin{cases} x^2 + y^2 = 12 & \text{Equation 1} \\ y = 2x^2 - 5x & \text{Equation 2} \end{cases}$$

Solution **Graph each equation.** The graph of the first equation is a circle, and the graph of the second is a parabola. Using a graphing device, we graph the circle and the parabola on the same screen, as shown in Figure 4.

Find intersection points. The graphs intersect in Quadrants I and II. Zooming in, we see that the intersection points are $(-0.559, 3.419)$ and $(2.847, 1.974)$. There also appears to be an intersection point in Quadrant IV. However, when we zoom in, we see that the curves come close to each other but don't intersect (see Figure 5). Thus the system has two solutions; rounded to the nearest tenth, they are

$$(-0.6, 3.4) \quad \text{and} \quad (2.8, 2.0)$$



(a) (b)

Figure 4 | $x^2 + y^2 = 12,\ y = 2x^2 - 5x$

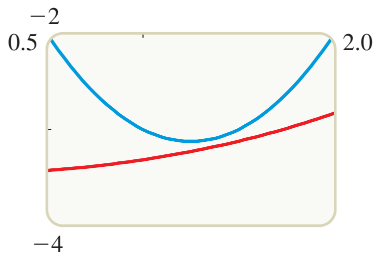


Figure 5 | Zooming in

Now Try Exercise 37 ■

Note Some graphing devices only graph functions, so to graph the circle in Example 4 we must first solve for y in terms of x.

$$\begin{aligned} x^2 + y^2 &= 12 \\ y^2 &= 12 - x^2 && \text{Isolate } y^2 \text{ on LHS} \\ y &= \pm\sqrt{12 - x^2} && \text{Take square roots} \end{aligned}$$

To graph the complete circle, we must graph both functions.

$$y = \sqrt{12 - x^2} \qquad \text{and} \qquad y = -\sqrt{12 - x^2}$$

Mathematics in the Modern World

Courtesy of NASA

Global Positioning System (GPS)

On a cold, foggy day in 1707 a British naval fleet was sailing home at a fast clip. The fleet's navigators didn't know it, but the fleet was only a few yards from the rocky shores of England. In the ensuing disaster the fleet was totally destroyed. This tragedy could have been avoided had the navigators known their positions. In those days latitude was determined by the position of the North Star (and this could be done only at night in good weather), and longitude was determined by the position of the sun relative to where it would be in England *at that same time*. So navigation required an accurate method of telling time on ships. (The invention of the spring-loaded clock brought about the eventual solution.)

Since then, several different methods have been developed to determine position, and all rely heavily on mathematics (see LORAN, Section 10.3). The latest method, called the Global Positioning System (GPS), uses triangulation. In this system, 24 satellites are strategically located above the surface of the earth. A GPS device measures distance from a satellite, using the travel time of radio signals emitted from the satellite. Knowing the distances to three different satellites tells us that we are at the point of intersection of three different spheres. This uniquely determines our position (see Exercise 51).

9.8 Exercises

Concepts

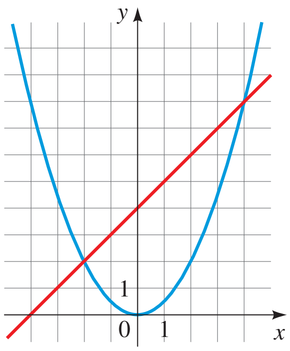
1–2 ■ The system of equations

$$\begin{cases} 2y - x^2 = 0 \\ y - x = 4 \end{cases}$$

is graphed below.

1. Use the graph to find the solutions of the system.

2. Check that the solutions you found in Exercise 1 satisfy the system.



Skills

3–8 ■ **Substitution Method** Use the substitution method to find all solutions of the system of equations.

3. $\begin{cases} y = x^2 \\ y = x + 12 \end{cases}$

4. $\begin{cases} x^2 + y^2 = 25 \\ y = 2x \end{cases}$

5. $\begin{cases} x^2 + y^2 = 8 \\ x + y = 0 \end{cases}$

6. $\begin{cases} x^2 + y = 9 \\ x - y + 3 = 0 \end{cases}$

7. $\begin{cases} x + y^2 = 0 \\ 2x + 5y^2 = 75 \end{cases}$

8. $\begin{cases} x^2 - y = 1 \\ 2x^2 + 3y = 17 \end{cases}$

9–14 ■ **Elimination Method** Use the elimination method to find all solutions of the system of equations.

9. $\begin{cases} x^2 - 2y = 1 \\ x^2 + 5y = 29 \end{cases}$

10. $\begin{cases} 3x^2 + 4y = 17 \\ 2x^2 + 5y = 2 \end{cases}$

11. $\begin{cases} 3x^2 - y^2 = 11 \\ x^2 + 4y^2 = 8 \end{cases}$

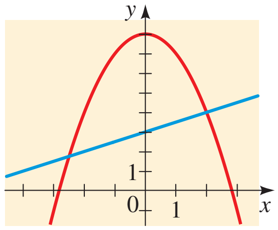
12. $\begin{cases} 2x^2 + 4y = 13 \\ x^2 - y^2 = \frac{7}{2} \end{cases}$

13. $\begin{cases} x - y^2 + 3 = 0 \\ 2x^2 + y^2 - 4 = 0 \end{cases}$

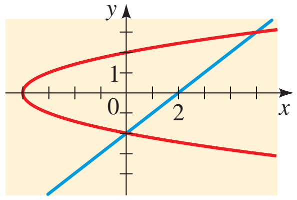
14. $\begin{cases} x^2 - y^2 = 1 \\ 2x^2 - y^2 = x + 3 \end{cases}$

15–18 ■ **Finding Intersection Points Graphically** Two equations and their graphs are given. Estimate the intersection point from the graph and check that the point is a solution to the system.

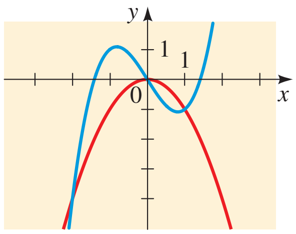
15. $\begin{cases} x^2 + y = 8 \\ x - 2y = -6 \end{cases}$



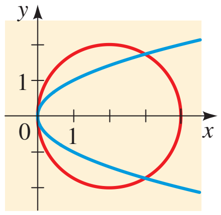
16. $\begin{cases} x - y^2 = -4 \\ x - y = 2 \end{cases}$



17. $\begin{cases} x^2 + y = 0 \\ x^3 - 2x - y = 0 \end{cases}$



18. $\begin{cases} x^2 + y^2 = 4x \\ x = y^2 \end{cases}$



19–32 ■ **Solving Nonlinear Systems** Find all solutions of the system of equations.

19. $\begin{cases} y + x^2 = 4x \\ y + 4x = 16 \end{cases}$

20. $\begin{cases} x - y^2 = 0 \\ y - x^2 = 0 \end{cases}$

21. $\begin{cases} x - 2y = 2 \\ y^2 - x^2 = 2x + 4 \end{cases}$

22. $\begin{cases} y = 4 - x^2 \\ y = x^2 - 4 \end{cases}$

23. $\begin{cases} x - y = 4 \\ xy = 12 \end{cases}$

24. $\begin{cases} xy = 24 \\ 2x^2 - y^2 + 4 = 0 \end{cases}$

25. $\begin{cases} x^2y = 16 \\ x^2 + 4y + 16 = 0 \end{cases}$

26. $\begin{cases} x + \sqrt{y} = 0 \\ y^2 - 4x^2 = 12 \end{cases}$

27. $\begin{cases} x^2 + y^2 = 9 \\ x^2 - y^2 = 1 \end{cases}$

28. $\begin{cases} x^2 + 2y^2 = 2 \\ 2x^2 - 3y = 15 \end{cases}$

29. $\begin{cases} 2x^2 - 8y^3 = 19 \\ 4x^2 + 16y^3 = 34 \end{cases}$

30. $\begin{cases} x^4 + y^3 = 17 \\ 3x^4 + 5y^3 = 53 \end{cases}$

31. $\begin{cases} \dfrac{2}{x} - \dfrac{3}{y} = 1 \\ -\dfrac{4}{x} + \dfrac{7}{y} = 1 \end{cases}$

32. $\begin{cases} \dfrac{4}{x^2} + \dfrac{6}{y^4} = \dfrac{7}{2} \\ \dfrac{1}{x^2} - \dfrac{2}{y^4} = 0 \end{cases}$

33–40 ■ Graphical Method Use the graphical method to find all solutions of the system of equations, rounded to two decimal places.

33. $\begin{cases} y = x^2 + 8x \\ y = 2x + 16 \end{cases}$

34. $\begin{cases} y = x^2 - 4x \\ 2x - y = 2 \end{cases}$

35. $\begin{cases} x^2 + y^2 = 25 \\ x + 3y = 2 \end{cases}$

36. $\begin{cases} x^2 + y^2 = 17 \\ x^2 - 2x + y^2 = 13 \end{cases}$

37. $\begin{cases} \dfrac{x^2}{9} + \dfrac{y^2}{18} = 1 \\ y = -x^2 + 6x - 2 \end{cases}$

38. $\begin{cases} x^2 - y^2 = 3 \\ y = x^2 - 2x - 8 \end{cases}$

39. $\begin{cases} x^4 + 16y^4 = 32 \\ x^2 + 2x + y = 0 \end{cases}$

40. $\begin{cases} y = e^x + e^{-x} \\ y = 5 - x^2 \end{cases}$

Skills Plus

41–44 ■ Some Trickier Systems Follow the hints and solve the systems.

41. $\begin{cases} \log x + \log y = \frac{3}{2} \\ 2 \log x - \log y = 0 \end{cases}$ [*Hint:* Add the equations.]

42. $\begin{cases} 2^x + 2^y = 10 \\ 4^x + 4^y = 68 \end{cases}$ [*Hint:* Note that $4^x = 2^{2x} = (2^x)^2$.]

43. $\begin{cases} x - y = 3 \\ x^3 - y^3 = 387 \end{cases}$ [*Hint:* Factor the left-hand side of the second equation.]

44. $\begin{cases} x^2 + xy = 1 \\ xy + y^2 = 3 \end{cases}$ [*Hint:* Add the equations, and factor the result.]

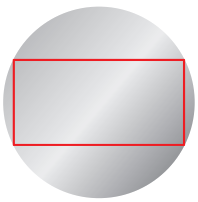
Applications

45. Dimensions of a Rectangle A rectangle has an area of 180 cm^2 and a perimeter of 54 cm. What are its dimensions?

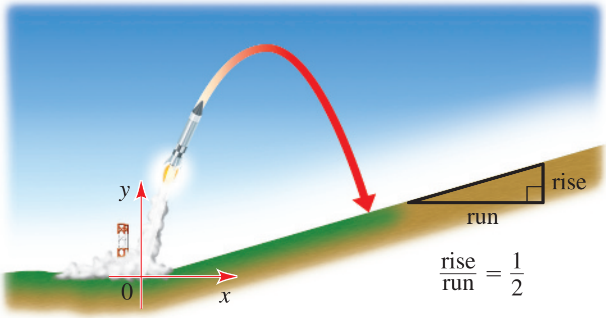
46. Legs of a Right Triangle A right triangle has an area of 84 ft^2 and a hypotenuse 25 ft long. What are the lengths of its other two sides?

47. Dimensions of a Rectangle The perimeter of a rectangle is 70, and its diagonal is 25. Find its length and width.

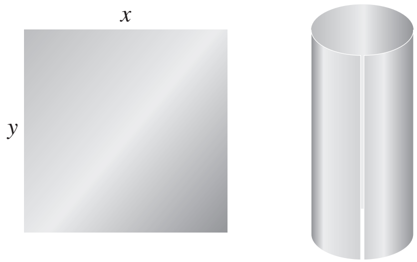
48. Dimensions of a Rectangle A circular piece of sheet metal has a diameter of 20 in. The edges are to be cut off to form a rectangle of area 160 in^2. (See the figure.) What are the dimensions of the rectangle?



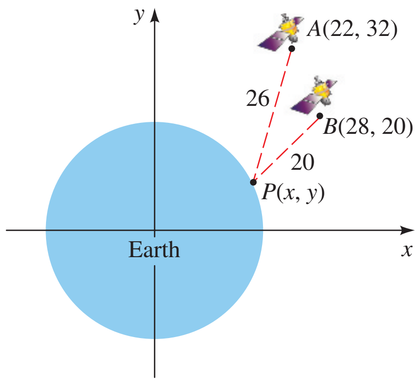
49. Flight of a Rocket A hill is inclined so that its "slope" is $\frac{1}{2}$, as shown in the figure. We introduce a coordinate system with the origin at the base of the hill and with the scales on the axes measured in meters. A rocket is fired from the base of the hill in such a way that its trajectory is the parabola $y = -x^2 + 401x$. At what point does the rocket strike the hillside? How far is this point from the base of the hill (to the nearest centimeter)?



50. Making a Stovepipe A rectangular piece of sheet metal with an area of 1200 in^2 is to be bent into a cylindrical length of stovepipe having a volume of 600 in^3. What are the dimensions of the sheet metal?



51. Global Positioning System (GPS) The Global Positioning System determines the location of an object from its distances to satellites in orbit around the earth. From the simplified, two-dimensional situation shown in the figure, determine the coordinates of P, using the fact that P is 26 units from satellite A and 20 units from satellite B.



■ Discuss ■ Discover ■ Prove ■ Write

52. Discover ■ Prove: Intersection of a Parabola and a Line On a sheet of graph paper or using a graphing device, draw the parabola $y = x^2$. Then draw the graphs of the linear equation $y = x + k$ on the same coordinate plane for various values of k. Try to choose values of k so that the line and the parabola intersect at two points for some of your k's and not for others. For what value of k is there exactly one intersection point? Use the results of your experiment to make a conjecture about the values of k for which the following system has two solutions, one solution, and no solution. Prove your conjecture.

$$\begin{cases} y = x^2 \\ y = x + k \end{cases}$$

9.9 Systems of Inequalities

■ Graphing an Inequality ■ Systems of Inequalities ■ Systems of Linear Inequalities ■ Application: Feasible Regions

In this section we study systems of inequalities in two variables from a graphical point of view.

■ Graphing an Inequality

We begin by considering the graph of a single inequality. We already know that the graph of $y = x^2$, for example, is the *parabola* in Figure 1.

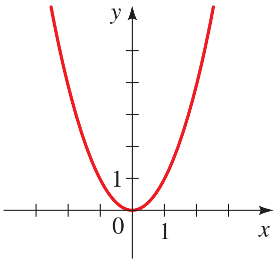
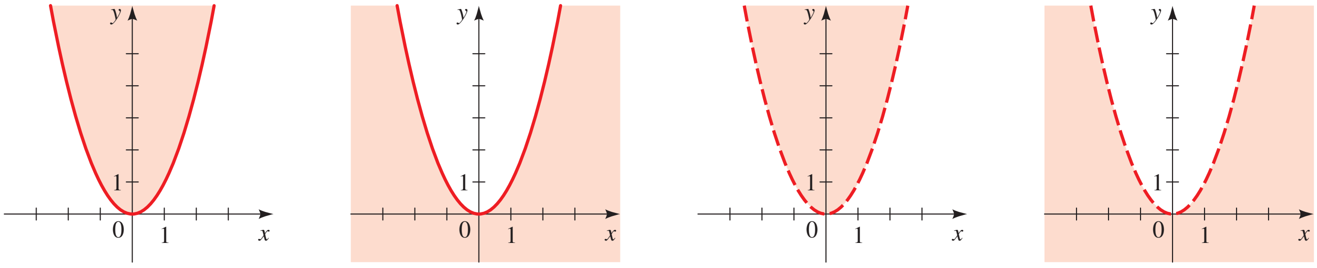


Figure 1 | $y = x^2$

If we replace the equal sign by the symbol $\geq$, we obtain the *inequality* $y \geq x^2$. Its graph consists of not just the parabola in Figure 1, but also every point whose y-coordinate is *larger* than x^2. We indicate the solution in Figure 2(a) by shading the points *above* the parabola.

Similarly, the graph of $y \leq x^2$ in Figure 2(b) consists of all points on and *below* the parabola. However, the graphs of $y > x^2$ and $y < x^2$ do not include the points on the parabola itself, as indicated by the dashed curves in Figures 2(c) and 2(d).



(a) $y \geq x^2$ (b) $y \leq x^2$ (c) $y > x^2$ (d) $y < x^2$

Figure 2

The graph of an inequality, in general, consists of a region in the plane whose boundary is the graph of the equation obtained by replacing the inequality sign ($\geq$, $\leq$, $>$, or $<$) with an equal sign. To determine which side of the graph gives the solution set of the inequality, we need only check **test points**.

Graphing an Inequality

To graph an inequality, we carry out the following steps.

1. **Graph the Equation.** Graph the equation that corresponds to the inequality. Use a dashed curve for $>$ or $<$ and a solid curve for $\leq$ or $\geq$.
2. **Graph the Inequality.** The graph of the inequality consists of all the points on one side of the curve that we graphed in Step 1. We use **test points** on either side of the curve to determine whether the points on each side satisfy the inequality. If the point satisfies the inequality, then all the points on that side of the curve satisfy the inequality. In that case, **shade that side of the curve** to indicate that it is part of the graph. If the test point does not satisfy the inequality, then the region isn't part of the graph.

Example 1 ■ Graphs of Inequalities

Graph each inequality.

(a) $x^2 + y^2 < 25$ **(b)** $x + 2y \geq 5$

Solution We follow the guidelines stated in the preceding box.

(a) **Graph the equation.** The graph of the equation $x^2 + y^2 = 25$ is a circle of radius 5 centered at the origin. The points on the circle itself do not satisfy the inequality because it is of the form $<$, so we graph the circle with a dashed curve, as shown in Figure 3.

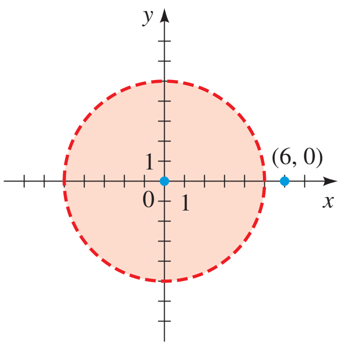


Figure 3 | Graph of $x^2 + y^2 < 25$

Graph the inequality. To determine whether the inside or the outside of the circle satisfies the inequality, we use the test points $(0, 0)$ on the inside and $(6, 0)$ on the outside. To do this, we substitute the coordinates of each point into the inequality and check whether the result satisfies the inequality.

Note that *any* point inside or outside the circle can serve as a test point. We have chosen these points for simplicity.

Test Point	Inequality $x^2 + y^2 < 25$	Conclusion
$(0, 0)$	$0^2 + 0^2 \overset{?}{<} 25$ ✓	Part of graph
$(6, 0)$	$6^2 + 0^2 \overset{?}{<} 25$ ✗	Not part of graph

Our check shows that the points *inside* the circle satisfy the inequality. A graph of the inequality is shown in Figure 3.

(b) **Graph the equation.** We first graph the equation $x + 2y = 5$. The graph is the line shown in Figure 4, on the next page.

Graph the inequality. Let's use the test points $(0, 0)$ and $(5, 5)$ on either side of the line.

Test Point	Inequality $x + 2y \geq 5$	Conclusion
$(0, 0)$	$0 + 2(0) \overset{?}{\geq} 5$ ✗	Not part of graph
$(5, 5)$	$5 + 2(5) \overset{?}{\geq} 5$ ✓	Part of graph

We can write the inequality in Example 1(b) as

$$y \geq -\tfrac{1}{2}x + \tfrac{5}{2}$$

From this form of the inequality we see that the solution consists of the points with y-values *on* or *above* the line $y = -\frac{1}{2}x + \frac{5}{2}$. So the graph of the inequality is the region *above* the line.

Our check shows that the points *above* the line satisfy the inequality. A graph of the inequality is shown in Figure 4.

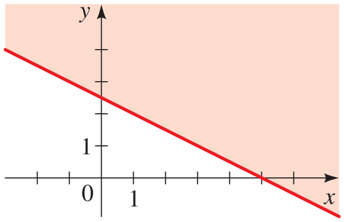


Figure 4 | Graph of $x + 2y \geq 5$

Now Try Exercises 15 and 21 ■

Systems of Inequalities

We now consider *systems* of inequalities. The **solution set of a system of inequalities** in two variables is the set of all points in the coordinate plane that satisfy every inequality in the system. The **graph of a system of inequalities** is the graph of the solution set.

To find the solution of a system of inequalities, we first graph each inequality in the system. The solution of the system consists of those points in the coordinate plane that belong to the solution of each inequality in the system. In other words, the solution of the system is the intersection of the solutions of the individual inequalities in the system. So to solve a system of inequalities, we use the following guidelines.

The Solution of a System of Inequalities

To graph the solution of a system of inequalities, we carry out the following steps.

1. **Graph Each Inequality.** Graph each inequality in the system on the same graph.
2. **Graph the Solution of the System.** Shade the region where the graphs of all the inequalities intersect. All the points in this region satisfy each inequality, so they belong to the solution of the system.
3. **Find the Vertices.** Label the vertices of the region that you shaded in Step 2.

Example 2 ■ A System of Two Inequalities

Graph the solution of the system of inequalities, and label its vertices.

$$\begin{cases} x^2 + y^2 < 25 \\ x + 2y \geq 5 \end{cases}$$

Solution These are the two inequalities of Example 1. Here we want to graph only those points that simultaneously satisfy both inequalities.

Graph each inequality. In Figure 5 we graph the solutions of the two inequalities on the same axes (in different colors).

Graph the solution of the system. The solution of the system of inequalities is the intersection of the two graphs. This is the region where the two regions overlap, which is the purple region graphed in Figure 6.

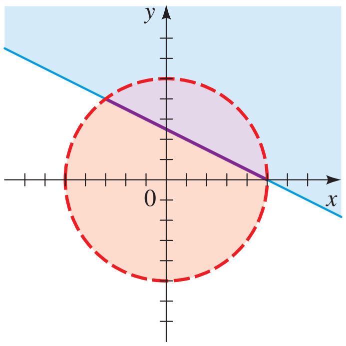


Figure 5

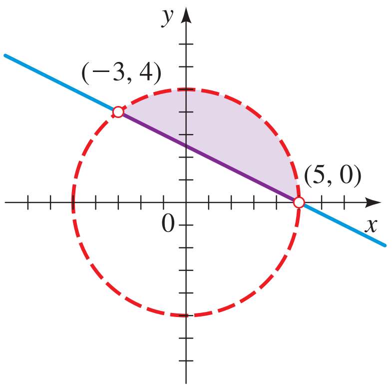


Figure 6 | $\begin{cases} x^2 + y^2 < 25 \\ x + 2y \geq 5 \end{cases}$

Find the Vertices. The points $(-3, 4)$ and $(5, 0)$ in Figure 6 are the **vertices** of the solution set. They are obtained by solving the system of *equations*

$$\begin{cases} x^2 + y^2 = 25 \\ x + 2y = 5 \end{cases}$$

We solve this system of equations by substitution. Solving for x in the second equation gives $x = 5 - 2y$, and substituting this into the first equation gives

$$(5 - 2y)^2 + y^2 = 25 \qquad \text{Substitute } x = 5 - 2y$$

$$(25 - 20y + 4y^2) + y^2 = 25 \qquad \text{Expand}$$

$$-20y + 5y^2 = 0 \qquad \text{Simplify}$$

$$-5y(4 - y) = 0 \qquad \text{Factor}$$

Thus $y = 0$ or $y = 4$. When $y = 0$, we have $x = 5 - 2(0) = 5$, and when $y = 4$, we have $x = 5 - 2(4) = -3$. So the points of intersection of these curves are $(5, 0)$ and $(-3, 4)$, as shown in Figure 6.

Note that in this case the vertices are not part of the solution set because they don't satisfy the inequality $x^2 + y^2 < 25$ (so they are graphed as open circles in the figure). They simply show where the "corners" of the solution set lie.

Now Try Exercise 43 ■

■ Systems of Linear Inequalities

An inequality is **linear** if it can be put into one of the following forms:

$$ax + by \geq c \qquad ax + by \leq c \qquad ax + by > c \qquad ax + by < c$$

In the next example we graph the solution set of a system of linear inequalities.

Example 3 ■ A System of Four Linear Inequalities

Graph the solution set of the system, and label its vertices.

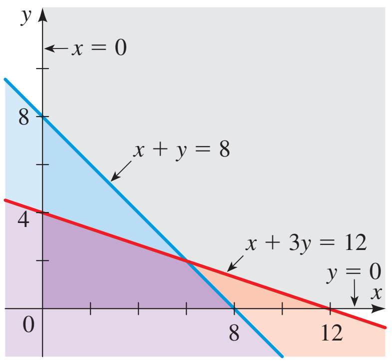
$$\begin{cases} x + 3y \leq 12 \\ x + y \leq 8 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

Solution **Graph each inequality.** In Figure 6 we first graph the lines given by the equations that correspond to each inequality. To determine the graphs of the first two inequalities, we need to check only one test point. For simplicity let's use the point $(0, 0)$.

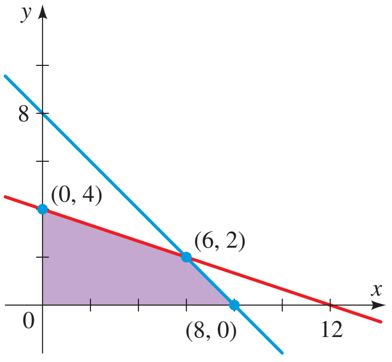
Inequality	Test Point $(0, 0)$	Conclusion
$x + 3y \leq 12$	$0 + 3(0) \stackrel{?}{\leq} 12$ ✓	Satisfies inequality
$x + y \leq 8$	$0 + 0 \stackrel{?}{\leq} 8$ ✓	Satisfies inequality

Since $(0, 0)$ is below the line $x + 3y = 12$, our check shows that the region on or below the line must satisfy the inequality. Likewise, since $(0, 0)$ is below the line $x + y = 8$, our check shows that the region on or below this line must satisfy the inequality. The inequalities $x \geq 0$ and $y \geq 0$ say that x and y are nonnegative. These regions are sketched in Figure 7(a).

Graph the solution of the system. The solution of the system of inequalities is the intersection of the graphs. This is the purple region graphed in Figure 7(b).



(a)



(b)

Figure 7

Find the Vertices. The coordinates of each vertex are obtained by simultaneously solving the equations of the lines that intersect at that vertex. From the system

$$\begin{cases} x + 3y = 12 \\ x + y = 8 \end{cases}$$

we get the vertex (6, 2). The origin (0, 0) is also clearly a vertex. The other two vertices are at the x- and y-intercepts of the corresponding lines: (8, 0) and (0, 4). In this case all the vertices *are* part of the solution set.

Now Try Exercise 51 ■

Example 4 ■ A System of Linear Inequalities

Graph the solution set of the system of inequalities, and label the vertices.

(a) $\begin{cases} 10x + 20y \geq 60 \\ 30x + 20y \geq 100 \\ 10x + 40y \geq 80 \\ x \geq 0, \quad y \geq 0 \end{cases}$ **(b)** $\begin{cases} 10x + 20y \leq 60 \\ 30x + 20y \geq 100 \\ 10x + 40y \geq 80 \\ x \geq 0, \quad y \geq 0 \end{cases}$

Solution

(a) **Graph each inequality.** We must graph the lines that correspond to these inequalities and then shade the appropriate regions. The graph of $10x + 20y \geq 60$ is the region above the line $y = 3 - \frac{1}{2}x$. The graph of $30x + 20y \geq 100$ is the region above the line $y = 5 - \frac{3}{2}x$, and the graph of $10x + 40y \geq 80$ is the region above the line $y = 2 - \frac{1}{4}x$.

Graph the solution of the system. The inequalities $x \geq 0$ and $y \geq 0$ indicate that the region is in the first quadrant. With this information we graph the system of inequalities in Figure 8.

Find the vertices. We determine the vertices of the region by finding the points of intersection of the appropriate lines. You can check that the vertices of the region are the ones indicated in Figure 8.

(b) The graph of the first inequality $10x + 20y \leq 60$ is the region below the line $y = 3 - \frac{1}{2}x$, and all the other inequalities are the same as those in part (a), so the solution to the system is the region shown in purple in Figure 9.

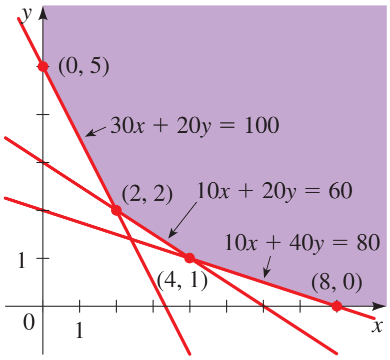


Figure 8

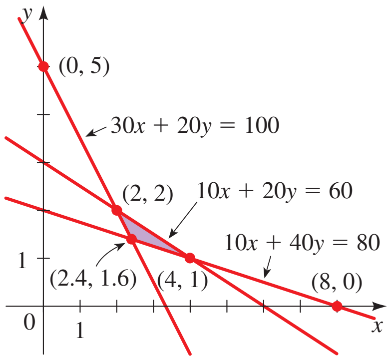


Figure 9

Now Try Exercises 59 and 63 ■

In the next example we use a graphing device to graph the inequalities and solve the system.

Example 5 ■ A System of Linear Inequalities

Graph the solution set of the system of inequalities, and label the vertices.

$$\begin{cases} x + 2y \geq 8 \\ -x + 2y \leq 4 \\ 3x - 2y \leq 8 \end{cases}$$

Solution **Graph each inequality.** We use a graphing device to obtain the graph in Figure 10(a). The device shades each region in a different pattern (or a different color).

Graph the solution of the system. The solution set is the triangular region that is shaded in all three patterns (or all three colors). The solution set is graphed in Figure 10(b).

Find the vertices. We use the graphing device to find the vertices of the region. The vertices are labeled in Figure 10(b).

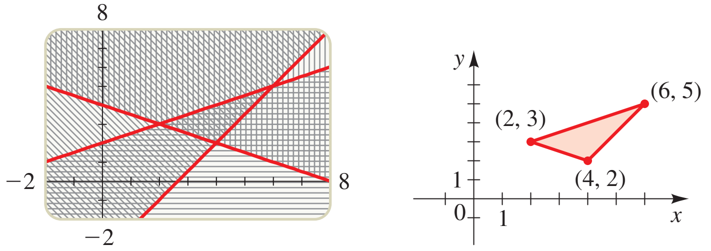


Figure 10 (a) Graphing device output (b) Graph of solution set

Now Try Exercise 65 ■

A region in the plane is called **bounded** if it can be enclosed in a (sufficiently large) circle. A region that is not bounded is called **unbounded**. For example, the regions graphed in Figures 3, 6, 7(b), 9, and 10 are bounded because they can be enclosed in a circle, as illustrated in Figure 11(a). But the regions graphed in Figures 2, 4, and 8 are unbounded because we cannot enclose them in a circle, as illustrated in Figure 11(b).

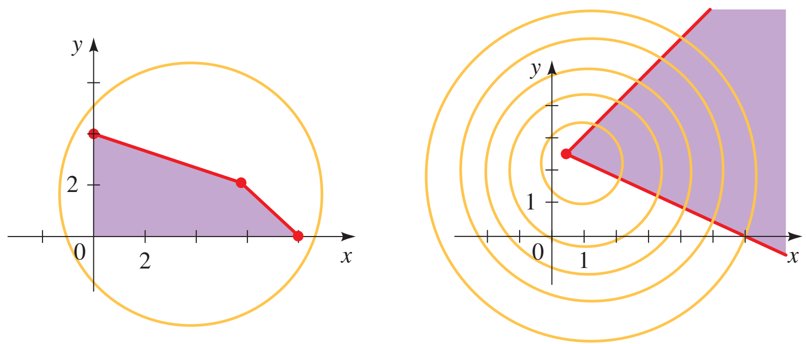


Figure 11 (a) A bounded region can be enclosed in a circle. (b) An unbounded region cannot be enclosed in a circle.

■ Application: Feasible Regions

Many applied problems involve **constraints** on the variables. For instance, a factory manager has only a certain number of workers who can be assigned to perform jobs on the factory floor. A farmer deciding what crops to cultivate has only a certain amount of land that can be seeded. Such constraints or limitations can usually be expressed as systems of inequalities. When dealing with applied inequalities, we usually refer to the solution set of a system as a **feasible region**, because the points in the solution set represent feasible (or possible) values for the quantities being studied.

Example 6 ■ Restricting Pollutant Outputs

A factory produces two agricultural pesticides, A and B. For every barrel of pesticide A, the factory emits 0.25 kg of carbon monoxide (CO) and 0.60 kg of sulfur dioxide (SO_2); and for every barrel of pesticide B, it emits 0.50 kg of CO and 0.20 kg of SO_2. Pollution laws restrict the factory's output of CO to a maximum of 75 kg per day and its output of SO_2 to a maximum of 90 kg per day.

(a) Find a system of inequalities that describes the number of barrels of each pesticide the factory can produce per day and still satisfy the pollution laws. Graph the feasible region.

(b) Would it be legal for the factory to produce 100 barrels of pesticide A and 80 barrels of pesticide B per day?

(c) Would it be legal for the factory to produce 60 barrels of pesticide A and 160 barrels of pesticide B per day?

Solution

(a) We state the constraints as a system of inequalities and then graph the solution of the system.

Set up the inequalities. We first identify and name the variables, and we then express each statement in the problem in terms of the variables. We let the variable x represent the number of barrels of A produced per day and let y be the number of barrels of B produced per day. We can organize the information in the problem as follows.

In Words	In Algebra
Barrels of A produced	x
Barrels of B produced	y
Total CO produced	$0.25x + 0.50y$
Total SO_2 produced	$0.60x + 0.20y$

From the information in the problem and the fact that x and y can't be negative we obtain the following inequalities.

$$\begin{cases} 0.25x + 0.50y \le 75 & \text{At most 75 kg of CO can be produced} \\ 0.60x + 0.20y \le 90 & \text{At most 90 kg of } SO_2 \text{ can be produced} \\ x \ge 0, \quad y \ge 0 \end{cases}$$

Multiplying the first inequality by 4 and the second by 5 simplifies the system to the following:

$$\begin{cases} x + 2y \le 300 \\ 3x + y \le 450 \\ x \ge 0, \quad y \ge 0 \end{cases}$$

Graph the solution set. We first graph the equations

$$x + 2y = 300$$
$$3x + y = 450$$

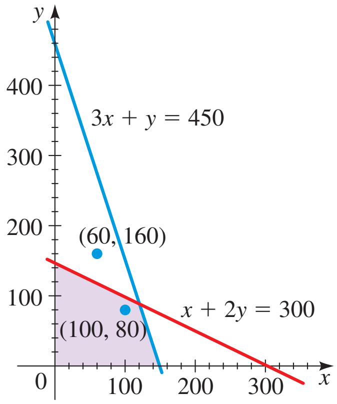


Figure 12

The graphs are the two lines shown in Figure 12. Using the test point $(0, 0)$, we see that the solution set of each of these inequalities is the region below the corresponding line. So the solution to the system is the intersection of these sets, as shown in Figure 12.

(b) Since the point $(100, 80)$ lies inside the feasible region, this production plan is legal (see Figure 12).

(c) Since the point $(60, 160)$ lies outside the feasible region, this production plan is not legal. It violates the CO restriction, although it does not violate the SO_2 restriction (see Figure 12).

Now Try Exercise 69 ■

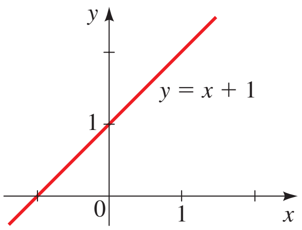
9.9 Exercises

Concepts

1. If the point $(2, 3)$ is a solution of an inequality in x and y, then the inequality is satisfied when we replace x by ________ and y by ________. Is the point $(2, 3)$ a solution of the inequality $4x - 2y \geq 1$?

2. To graph an inequality, we first graph the corresponding ________. So to graph the inequality $y \leq x + 1$, we first graph the equation ________. To decide which side of the graph of the equation is the graph of the inequality, we use ________ points. Complete the table, and sketch a graph of the inequality by shading the appropriate region of the graph shown below.

Test Point	Inequality $y \leq x + 1$	Conclusion
$(0, 0)$		
$(0, 2)$		

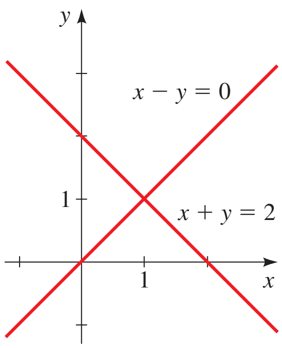


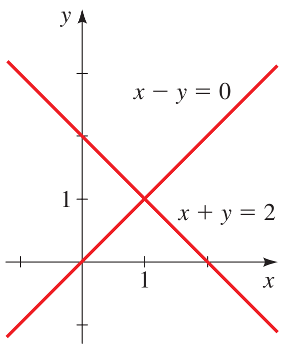
3. If the point $(2, 3)$ is a solution of a *system* of inequalities in x and y, then *each* inequality is satisfied when we replace x by ________ and y by ________. Is the point $(2, 3)$ a solution of the following system?

$$\begin{cases} 2x + 4y \leq 17 \\ 6x + 5y \leq 29 \end{cases}$$

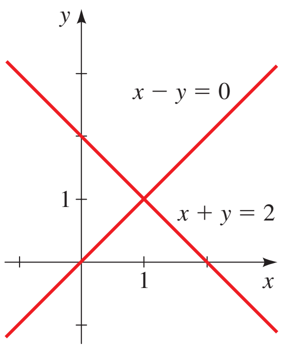
4. Shade the solution of each system of inequalities on the given graph.

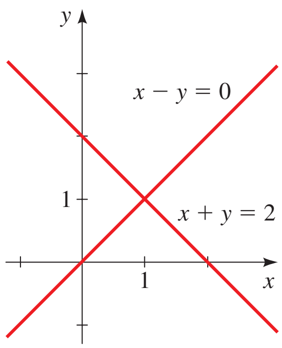
(a) $\begin{cases} x - y \geq 0 \\ x + y \geq 2 \end{cases}$ **(b)** $\begin{cases} x - y \leq 0 \\ x + y \leq 2 \end{cases}$





(c) $\begin{cases} x - y \geq 0 \\ x + y \leq 2 \end{cases}$ **(d)** $\begin{cases} x - y \leq 0 \\ x + y \geq 2 \end{cases}$





Skills

5–6 ■ Solutions of Inequalities An inequality and several points are given. Determine which points are solutions of the inequality.

5. $x - 5y > 3$; $(-1, -2), (1, -2), (1, 2), (8, 1)$

6. $3x + 2y \le 2$; $(-2, 1), (1, 3), (1, -3), (0, 1)$

7–8 ■ Solutions of Systems of Inequalities A system of inequalities and several points are given. Determine which points are solutions of the system.

7. $\begin{cases} 3x - 2y \le 5 \\ 2x + y \ge 3 \end{cases}$; $(0, 0), (1, 2), (1, 1), (3, 1)$

8. $\begin{cases} x + 2y \ge 4 \\ 4x + 3y \ge 11 \end{cases}$; $(0, 0), (1, 3), (3, 0), (1, 2)$

9–22 ■ Graphing Inequalities Graph the inequality.

9. $y < -2x$ **10.** $y \ge 3x$

11. $y \ge 2$ **12.** $x \le -1$

13. $x < 2$ **14.** $y > 1$

15. $y > x - 3$ **16.** $y \le 1 - x$

17. $2x - y \ge -4$ **18.** $3x - y - 9 < 0$

19. $-x^2 + y \ge 5$ **20.** $y > x^2 + 1$

21. $x^2 + y^2 > 9$ **22.** $x^2 + (y - 2)^2 \le 4$

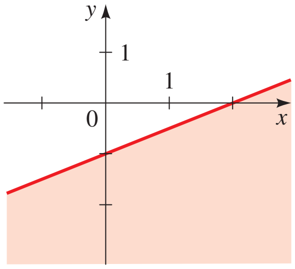
23–26 ■ Graphing Inequalities Use a graphing device to graph the linear inequality.

23. $3x - 2y \ge 18$ **24.** $4x + 3y \le 9$

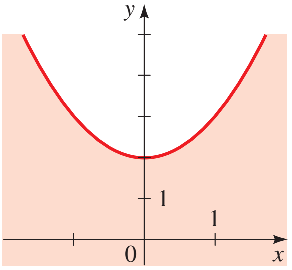
25. $5x + 2y > 8$ **26.** $5x - 3y \ge 15$

27–30 ■ Finding Inequalities from a Graph An equation and its graph are given. Find an inequality whose solution is the shaded region.

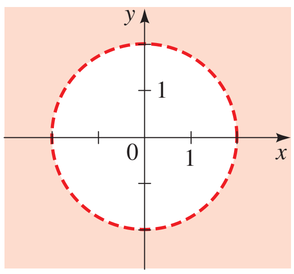
27. $y = \frac{1}{2}x - 1$



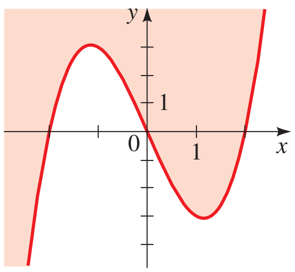
28. $y = x^2 + 2$



29. $x^2 + y^2 = 4$



30. $y = x^3 - 4x$



31–58 ■ Systems of Inequalities Graph the solution set of the system of inequalities. Find the coordinates of all vertices, and determine whether the solution set is bounded.

31. $\begin{cases} x + y \le 4 \\ y \ge x \end{cases}$

32. $\begin{cases} 2x + 3y > 12 \\ 3x - y < 21 \end{cases}$

33. $\begin{cases} y < \frac{1}{4}x + 2 \\ y \ge 2x - 5 \end{cases}$

34. $\begin{cases} x - y > 0 \\ 4 + y \le 2x \end{cases}$

35. $\begin{cases} y \le -2x + 8 \\ y \le -\frac{1}{2}x + 5 \\ x \ge 0, \quad y \ge 0 \end{cases}$

36. $\begin{cases} 4x + 3y \le 18 \\ 2x + y \le 8 \\ x \ge 0, \quad y \ge 0 \end{cases}$

37. $\begin{cases} x \ge 0 \\ y \ge 0 \\ 3x + 5y \le 15 \\ 3x + 2y \le 9 \end{cases}$

38. $\begin{cases} x > 2 \\ y < 12 \\ 2x - 4y > 8 \end{cases}$

39. $\begin{cases} y \le 9 - x^2 \\ x \ge 0, \quad y \ge 0 \end{cases}$

40. $\begin{cases} y \ge x^2 \\ y \le 4 \\ x \ge 0 \end{cases}$

41. $\begin{cases} y < 9 - x^2 \\ y \ge x + 3 \end{cases}$

42. $\begin{cases} y \ge x^2 \\ x + y \ge 6 \end{cases}$

43. $\begin{cases} x^2 + y^2 \le 4 \\ x - y > 0 \end{cases}$

44. $\begin{cases} x > 0 \\ y > 0 \\ x + y < 10 \\ x^2 + y^2 > 9 \end{cases}$

45. $\begin{cases} x^2 - y \le 0 \\ 2x^2 + y \le 12 \end{cases}$

46. $\begin{cases} 2x^2 + y > 4 \\ x^2 - y \le 8 \end{cases}$

47. $\begin{cases} x^2 + y^2 \le 9 \\ 2x + y^2 \le 1 \end{cases}$

48. $\begin{cases} x^2 + y^2 \le 4 \\ x^2 - 2y > 1 \end{cases}$

49. $\begin{cases} x + 2y \le 14 \\ 3x - y \ge 0 \\ x - y \ge 2 \end{cases}$

50. $\begin{cases} y < x + 6 \\ 3x + 2y \ge 12 \\ x - 2y \le 2 \end{cases}$

51. $\begin{cases} x \ge 0 \\ y \ge 0 \\ x \le 5 \\ x + y \le 7 \end{cases}$

52. $\begin{cases} x \ge 0 \\ y \ge 0 \\ y \le 4 \\ 2x + y \le 8 \end{cases}$

53. $\begin{cases} y > x + 1 \\ x + 2y \le 12 \\ x + 1 > 0 \end{cases}$

54. $\begin{cases} x + y > 12 \\ y < \frac{1}{2}x - 6 \\ 3x + y < 6 \end{cases}$

55. $\begin{cases} x^2 + y^2 \le 8 \\ x \ge 2 \\ y \ge 0 \end{cases}$

56. $\begin{cases} x^2 - y \ge 0 \\ x + y < 6 \\ x - y < 6 \end{cases}$

57. $\begin{cases} x^2 + y^2 < 9 \\ x + y > 0 \\ x \le 0 \end{cases}$

58. $\begin{cases} y \ge x^3 \\ y \le 2x + 4 \\ x + y \ge 0 \end{cases}$

59–64 ■ Systems of Inequalities Graph the system of inequalities, label the vertices, and determine whether the region is bounded or unbounded.

59. $\begin{cases} x + 2y \le 14 \\ 3x - y \ge 0 \\ x - y \le 2 \end{cases}$

60. $\begin{cases} x + 2y \le 14 \\ 3x - y \ge 0 \\ x - y \ge 2 \end{cases}$

61. $\begin{cases} x + y \le 12 \\ y \le \frac{1}{2}x - 6 \\ y \le 2x + 6 \end{cases}$

62. $\begin{cases} y \ge x + 1 \\ x + 2y \le 12 \\ x + 1 \ge 0 \end{cases}$

63. $\begin{cases} 30x + 10y \ge 50 \\ 10x + 20y \ge 50 \\ 10x + 60y \ge 90 \\ x \ge 0, \quad y \ge 0 \end{cases}$

64. $\begin{cases} x + y \ge 6 \\ 4x + 7y \le 39 \\ x + 5y \ge 13 \\ x \ge 0, \quad y \ge 0 \end{cases}$

65–68 ■ Graphing Systems of Inequalities Use a graphing device to graph the solution of the system of inequalities. Find the coordinates of all vertices, rounded to one decimal place.

65. $\begin{cases} y \ge x - 3 \\ y \ge -2x + 6 \\ y \le 8 \end{cases}$

66. $\begin{cases} x + y \ge 12 \\ 2x + y \le 24 \\ x - y \ge -6 \end{cases}$

67. $\begin{cases} y \le 6x - x^2 \\ x + y \ge 4 \end{cases}$

68. $\begin{cases} y \ge x^3 \\ 2x + y \ge 0 \\ y \le 2x + 6 \end{cases}$

Applications

69. Planting Crops A farmer has 500 acres of arable land on which to plant potatoes and corn. The farmer has \$40,000 available for planting and \$30,000 for fertilizer. Planting one acre of potatoes costs \$90, and planting one acre of corn costs \$50. Fertilizer costs \$30 for one acre of potatoes and \$80 for one acre of corn.

(a) Find a system of inequalities that describes the number of acres of each crop that the farmer can plant with the available resources. Graph the feasible region.

(b) Can the farmer plant 300 acres of potatoes and 180 acres of corn?

(c) Can the farmer plant 150 acres of potatoes and 325 acres of corn?

70. Planting Crops A farmer has 300 acres of arable land for planting cauliflower and cabbage. The farmer has \$17,500 available for planting and \$12,000 for fertilizer. Planting one acre of cauliflower costs \$70, and planting one acre of cabbage costs \$35. Fertilizer costs \$25 for one acre of cauliflower and \$55 for one acre of cabbage.

(a) Find a system of inequalities that describes the number of acres of each crop that the farmer can plant with the available resources. Graph the feasible region.

(b) Can the farmer plant 155 acres of cauliflower and 115 acres of cabbage?

(c) Can the farmer plant 115 acres of cauliflower and 175 acres of cabbage?

71. Publishing Books A publishing company publishes a total of no more than 100 books every year. At least 20 of these are nonfiction, but the company always publishes at least as much fiction as nonfiction. Find a system of inequalities that describes the possible numbers of fiction and nonfiction books that the company can produce each year consistent with these policies. Graph the solution set.

72. Furniture Manufacturing A furniture maker manufactures unfinished tables and chairs. Each table requires 3 hours of sawing and 1 hour of assembly. Each chair requires 2 hours of sawing and 2 hours of assembly. The furniture maker can put in up to 12 hours of sawing and 8 hours of assembly work each day. Find a system of inequalities that describes all possible combinations of tables and chairs that the furniture maker can make daily. Graph the solution set.

73. Coffee Blends A coffee merchant sells two different coffee blends. The Standard blend uses 4 oz of arabica and 12 oz of robusta beans per package; the Deluxe blend uses 10 oz of arabica and 6 oz of robusta beans per package. The merchant has 80 lb of arabica and 90 lb of robusta beans available. Find a system of inequalities that describes the possible number of Standard and Deluxe packages the merchant can make. Graph the solution set.

74. Nutrition A cat-food manufacturer uses fish and beef by-products. The fish contains 12 g of protein and 3 g of fat per ounce. The beef contains 6 g of protein and 9 g of fat per ounce. Each can of cat food must contain at least 60 g of protein and 45 g of fat. Find a system of inequalities that describes the possible number of ounces of fish and beef by-products that can be used in each can to satisfy these minimum requirements. Graph the solution set.

Discuss ■ Discover ■ Prove ■ Write

75. Discuss: Shading Unwanted Regions To graph the solution of a system of inequalities, we have shaded the solution of each inequality in a different color; the solution of the system is the region where all the shaded parts overlap. Here is a different method: For each inequality, shade the region that does *not* satisfy the inequality. Explain why the part of the plane that is left unshaded is the solution of the system. Solve the following system by both methods. Which do you prefer? Why?

$$\begin{cases} x + 2y > 4 \\ -x + y < 1 \\ x + 3y < 9 \\ x < 3 \end{cases}$$

Chapter 9 Review

Properties and Formulas

Systems of Equations | Section 9.1

A **system of equations** is a set of equations that involve the same variables. A **system of linear equations** is a system of equations in which each equation is linear. Systems of linear equations in two variables (x and y) and three variables (x, y, and z) have the following forms:

Linear system 2 variables

$$\begin{aligned} a_{11}x + a_{12}y &= b_1 \\ a_{21}x + a_{22}y &= b_2 \end{aligned}$$

Linear system 3 variables

$$\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= b_1 \\ a_{21}x + a_{22}y + a_{23}z &= b_2 \\ a_{31}x + a_{32}y + a_{33}z &= b_3 \end{aligned}$$

A **solution** of a system of equations is an assignment of values for the variables that makes *each* equation in the system true. To **solve** a system means to find all solutions of the system.

Substitution Method | Section 9.1

To solve a pair of equations in two variables by substitution:

1. **Solve for one variable** in terms of the other variable in one equation.
2. **Substitute** into the other equation to get an equation in one variable, and solve for this variable.
3. **Back-substitute** the value(s) of the variable you have found into either original equation, and solve for the remaining variable.

Elimination Method | Section 9.1

To solve a pair of equations in two variables by elimination:

1. **Adjust the coefficients** by multiplying the equations by appropriate constants so that the term(s) involving one of the variables are of opposite signs in the equations.
2. **Add the equations** to eliminate that one variable; this gives an equation in the other variable. Solve for this variable.
3. **Back-substitute** the value(s) of the variable that you have found into either original equation, and solve for the remaining variable.

Graphical Method | Section 9.1

To solve a pair of equations in two variables graphically, first put each equation in function form, $y = f(x)$.

1. **Graph the equations** on a common screen.
2. **Find the points of intersection** of the graphs. The solutions are the x- and y-coordinates of the points of intersection.

Gaussian Elimination | Section 9.2

To use **Gaussian elimination** to solve a system of linear equations, use the following operations to change the system to an **equivalent** simpler system:

1. Add a nonzero multiple of one equation to another.
2. Multiply an equation by a nonzero constant.
3. Interchange the position of two equations in the system.

Number of Solutions of a Linear System | Section 9.2

A system of linear equations can have:

1. A unique solution for each variable.
2. No solution, in which case the system is **inconsistent**.
3. Infinitely many solutions, in which case the system is **dependent**.

How to Determine the Number of Solutions of a Linear System | Section 9.2

When **Gaussian elimination** is used to solve a system of linear equations, then we can tell that the system has:

1. **No solution** (is *inconsistent*) if we arrive at a false equation of the form $0 = c$, where c is nonzero.
2. **Infinitely many solutions** (is *dependent*) if the system is consistent but we end up with fewer equations than variables (after discarding redundant equations of the form $0 = 0$).

Matrices | Section 9.3

A **matrix** A of **dimension** $m \times n$ is a rectangular array of numbers with m **rows** and n **columns**:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Augmented Matrix of a System | Section 9.3

The **augmented matrix** of a system of linear equations is the matrix consisting of the coefficients and the constant terms. For example, for the two-variable system

$$\begin{aligned} a_{11}x + a_{12}x &= b_1 \\ a_{21}x + a_{22}x &= b_2 \end{aligned}$$

the augmented matrix is

$$\begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{bmatrix}$$

Elementary Row Operations | Section 9.3

To solve a system of linear equations using the augmented matrix of the system, the following operations can be used to transform the rows of the matrix:

1. Add a nonzero multiple of one row to another.
2. Multiply a row by a nonzero constant.
3. Interchange two rows.

Row-Echelon Form of a Matrix | Section 9.3

A matrix is in **row-echelon form** if its entries satisfy the following conditions:

1. The first nonzero entry in each row (the **leading entry**) is the number 1.

2. The leading entry of each row is to the right of the leading entry in the row above it.
3. All rows consisting entirely of zeros are at the bottom of the matrix.

If the matrix also satisfies the following condition, it is in **reduced row-echelon** form:

4. If a column contains a leading entry, then every other entry in that column is a 0.

Number of Solutions of a Linear System | Section 9.3

If the augmented matrix of a system of linear equations has been reduced to row-echelon form using elementary row operations, then the system has:

1. **No solution** if the row-echelon form contains a row that represents the equation $0 = 1$. In this case the system is **inconsistent**.
2. **One solution** if each variable in the row-echelon form is a leading variable.
3. **Infinitely many solutions** if the system is not inconsistent but not every variable is a leading variable. In this case the system is **dependent**.

Operations on Matrices | Section 9.4

If A and B are $m \times n$ matrices and c is a scalar (real number), then:

1. The **sum** $A + B$ is the $m \times n$ matrix that is obtained by adding corresponding entries of A and B.
2. The **difference** $A - B$ is the $m \times n$ matrix that is obtained by subtracting corresponding entries of A and B.
3. The **scalar product** cA is the $m \times n$ matrix that is obtained by multiplying each entry of A by c.

Multiplication of Matrices | Section 9.4

If A is an $m \times n$ matrix and B is an $n \times k$ matrix (so the number of columns of matrix A is the same as the number of rows of matrix B), then the **matrix product** AB is the $m \times k$ matrix whose ij-entry is the inner product of the ith row of A and the jth column of B.

Properties of Matrix Operations | Section 9.4

If A, B, and C are matrices of compatible dimensions, then the following properties hold:

1. **Commutativity of addition:**

$$A + B = B + A$$

2. **Associativity:**

$$(A + B) + C = A + (B + C)$$

$$(AB)C = A(BC)$$

3. **Distributivity:**

$$A(B + C) = AB + AC$$

$$(B + C)A = BA + CA$$

(Note that matrix *multiplication* is *not* commutative.)

Identity Matrix | Section 9.5

The **identity matrix** I_n is the $n \times n$ matrix whose main diagonal entries are all 1 and whose other entries are all 0:

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

If A is an $m \times n$ matrix, then

$$AI_n = A \qquad \text{and} \qquad I_m A = A$$

Inverse of a Matrix | Section 9.5

If A is an $n \times n$ matrix, then the inverse of A is the $n \times n$ matrix A^{-1} with the following properties:

$$A^{-1}A = I_n \qquad \text{and} \qquad AA^{-1} = I_n$$

To find the inverse of a matrix, we use a procedure involving elementary row operations. (Note that *some* square matrices do not have an inverse.)

Inverse of a 2 × 2 Matrix | Section 9.5

For 2×2 matrices the following special rule provides a shortcut for finding the inverse:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \Rightarrow \quad A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Writing a Linear System as a Matrix Equation | Section 9.5

A system of n linear equations in n variables can be written as a single matrix equation

$$AX = B$$

where A is the $n \times n$ matrix of coefficients, X is the $n \times 1$ matrix of the variables, and B is the $n \times 1$ matrix of the constants. For example, the linear system of two equations in two variables

$$a_{11}x + a_{12}x = b_1$$

$$a_{21}x + a_{22}x = b_2$$

can be expressed as

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Solving Matrix Equations | Section 9.5

If A is an invertible $n \times n$ matrix, X is an $n \times 1$ variable matrix, and B is an $n \times 1$ constant matrix, then the matrix equation

$$AX = B$$

has the unique solution

$$X = A^{-1}B$$

Determinant of a 2 × 2 Matrix | Section 9.6

The **determinant** of the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is the *number*

$$\det(A) = |A| = ad - bc$$

Minors and Cofactors | Section 9.6

If $A = |a_{ij}|$ is an $n \times n$ matrix, then the **minor** M_{ij} of the entry a_{ij} is the determinant of the matrix obtained by deleting the ith row and the jth column of A.

The **cofactor** A_{ij} of the entry a_{ij} is

$$A_{ij} = (-1)^{i+j} M_{ij}$$

(Thus, the minor and the cofactor of each entry either are the same or are negatives of each other.)

Determinant of an $n \times n$ Matrix | Section 9.6

To find the **determinant** of the $n \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

we choose a row or column to **expand**, and then we calculate the number that is obtained by multiplying each element of that row or column by its cofactor and then adding the resulting products. For example, if we choose to expand about the first row, we get

$$\det(A) = |A| = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}$$

Invertibility Criterion | Section 9.6

A square matrix has an inverse if and only if its determinant is not 0.

Row and Column Transformations | Section 9.6

If we add a nonzero multiple of one row to another row in a square matrix or a nonzero multiple of one column to another column, then the determinant of the matrix is unchanged.

Cramer's Rule | Section 9.6

If a system of n linear equations in the n variables $x_1, x_2, \ldots, x_n$ is equivalent to the matrix equation $DX = B$ and if $|D| \neq 0$, then the solutions of the system are

$$x_1 = \frac{|D_{x_1}|}{|D|} \qquad x_2 = \frac{|D_{x_2}|}{|D|} \qquad \cdots \qquad x_n = \frac{|D_{x_n}|}{|D|}$$

where D_{x_i} is the matrix that is obtained from D by replacing its ith column by the constant matrix B.

Area of a Triangle Using Determinants | Section 9.6

If a triangle in the coordinate plane has vertices (a_1, b_1), (a_2, b_2), and (a_3, b_3), then the area of the triangle is given by

$$\mathscr{A} = \pm\frac{1}{2}\begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$$

where the sign is chosen to make the area positive.

Partial Fractions | Section 9.7

The *partial fraction decomposition* of a rational function

$$r(x) = \frac{P(x)}{Q(x)}$$

(where the degree of P is less than the degree of Q) is a sum of simpler fractional expressions that equal $r(x)$ when brought to a common denominator. The denominator of each simpler fraction is either a linear or quadratic factor of $Q(x)$ or a power of such a linear or quadratic factor. To find the terms of the partial fraction decomposition, we first factor $Q(x)$ into linear and irreducible quadratic factors. The terms then have the following forms, depending on the factors of $Q(x)$.

1. For every **distinct linear factor** $ax + b$ there is a term of the form
$$\frac{A}{ax + b}$$
2. For every **repeated linear factor** $(ax + b)^m$ there are terms of the form
$$\frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \cdots + \frac{A_m}{(ax + b)^m}$$
3. For every **distinct quadratic factor** $ax^2 + bx + c$ there is a term of the form
$$\frac{Ax + B}{ax^2 + bx + c}$$
4. For every **repeated quadratic factor** $(ax^2 + bx + c)^m$ there are terms of the form
$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_mx + B_m}{(ax^2 + bx + c)^m}$$

Graphing Inequalities | Section 9.9

To graph an inequality:

1. Graph the equation that corresponds to the inequality. This "boundary curve" divides the coordinate plane into separate regions.
2. Use **test points** to determine which region(s) satisfy the inequality.
3. Shade the region(s) that satisfy the inequality, and use a solid line for the boundary curve if it satisfies the inequality ($\leq$ or $\geq$) and a dashed line if it does not ($<$ or $>$).

Graphing Systems of Inequalities | Section 9.9

To graph the solution of a system of inequalities (or **feasible region** determined by the inequalities):

1. Graph all the inequalities on the same coordinate plane.
2. The solution is the intersection of the solutions of all the inequalities, so shade the region that satisfies all the inequalities.
3. Determine the coordinates of the intersection points of all the boundary curves that touch the solution set of the system. These points are the **vertices** of the solution.

Concept Check

1. (a) What are the three methods we use to solve a system of equations?

(b) Solve the system by the elimination method and by the graphical method.

$$\begin{cases} x + y = 3 \\ 3x - y = 1 \end{cases}$$

2. For a system of two linear equations in two variables:

(a) How many solutions are possible?

(b) What is meant by an inconsistent system? a dependent system?

3. What operations can be performed on a linear system so as to arrive at an equivalent system?

4. (a) Explain how Gaussian elimination works.

(b) Use Gaussian elimination to put the following system in triangular form, and then solve the system.

System	**Triangular form**
$\begin{cases} x + y - 2z = 3 \\ x + 2y + z = 5 \\ 3x - y + 5z = 1 \end{cases}$	

5. What does it mean to say that A is a matrix with dimension $m \times n$?

6. What is the row-echelon form of a matrix? What is a leading entry?

7. (a) What is the augmented matrix of a system? What are leading variables?

(b) What are the elementary row operations on an augmented matrix?

(c) How do we solve a system using the augmented matrix?

(d) Write the augmented matrix of the following system of linear equations.

System	**Augmented Matrix**
$\begin{cases} x + y - 2z = 3 \\ x + 2y + z = 5 \\ 3x - y + 5z = 1 \end{cases}$	

(e) Solve the system in part (d).

8. Suppose you have used Gaussian elimination to transform the augmented matrix of a linear system into row-echelon form. How can you tell whether the system has exactly one solution? no solution? infinitely many solutions?

9. What is the reduced row-echelon form of a matrix?

10. (a) How do Gaussian elimination and Gauss-Jordan elimination differ?

(b) Use Gauss-Jordan elimination to solve the linear system in 7(d).

11. If A and B are matrices with the same dimension and k is a real number, how do you find $A + B$ and kA?

12. (a) What must be true of the dimensions of A and B for the product AB to be defined?

(b) If A has dimension 2×3 and if B has dimension 3×2, is the product AB defined? If so, what is the dimension of AB?

(c) Find the matrix product.

$$\begin{bmatrix} 2 & 1 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 3 & 4 & 1 \\ 5 & 1 & 2 \end{bmatrix}$$

13. (a) What is an identity matrix I_n? If A is an $n \times n$ matrix, what are the products AI_n and I_nA?

(b) If A is an $n \times n$ matrix, what is its inverse matrix?

(c) Complete the formula for the inverse of a 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

(d) Find the inverse of A.

$$A = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$$

14. (a) Express the system in 1(b) as a matrix equation $AX = B$.

(b) If a linear system is expressed as a matrix equation $AX = B$, how do we solve the system? Solve the system in part (a).

15. (a) Is it true that the determinant $\det(A)$ of a matrix A is defined only if A is a square matrix?

(b) Find the determinant of the matrix A in 13(d).

(c) Use Cramer's Rule to solve the system in 1(b).

16. (a) How do we express a rational function r as a partial fraction decomposition?

(b) Give the form of each partial fraction decomposition.

(i) $\dfrac{2x}{(x-5)(x-2)^2}$ (ii) $\dfrac{2x}{(x-5)(x^2+1)}$

17. (a) How do we graph an inequality in two variables?

(b) Graph the solution set of the inequality $x + y \geq 3$.

(c) Graph the solution set of the system of inequalities: $x + y \geq 3$, $3x - y \geq 1$.

Answers to the Concept Check can be found at the book companion website stewartmath.com.

Exercises

1–6 ■ Systems of Linear Equations in Two Variables Solve the system of equations, and graph the lines.

1. $\begin{cases} 3x - y = 5 \\ 2x + y = 5 \end{cases}$

2. $\begin{cases} y = 2x + 6 \\ y = -x + 3 \end{cases}$

3. $\begin{cases} 2x - 7y = 28 \\ y = \frac{2}{7}x - 4 \end{cases}$

4. $\begin{cases} 6x - 8y = 15 \\ -\frac{3}{2}x + 2y = -4 \end{cases}$

5. $\begin{cases} 2x - y = 1 \\ x + 3y = 10 \\ 3x + 4y = 15 \end{cases}$

6. $\begin{cases} 2x + 5y = 9 \\ -x + 3y = 1 \\ 7x - 2y = 14 \end{cases}$

7–10 ■ Systems of Nonlinear Equations Solve the system of equations.

7. $\begin{cases} y = x^2 + 2x \\ y = 6 + x \end{cases}$

8. $\begin{cases} x^2 + y^2 = 8 \\ y = x + 2 \end{cases}$

9. $\begin{cases} 3x + \dfrac{4}{y} = 6 \\ x - \dfrac{8}{y} = 4 \end{cases}$

10. $\begin{cases} x^2 + y^2 = 10 \\ x^2 + 2y^2 - 7y = 0 \end{cases}$

11–14 ■ Systems of Nonlinear Equations Use a graphing device to solve the system. Round answers to two decimal places.

11. $\begin{cases} 0.32x + 0.43y = 0 \\ 7x - 12y = 341 \end{cases}$

12. $\begin{cases} \sqrt{12}x - 3\sqrt{2}y = 660 \\ 7137x + 3931y = 20{,}000 \end{cases}$

13. $\begin{cases} x - y^2 = 10 \\ x = \frac{1}{22}y + 12 \end{cases}$

14. $\begin{cases} y = 5^x + x \\ y = x^5 + 5 \end{cases}$

15–20 ■ Matrices A matrix is given.

(a) State the dimension of the matrix.
(b) Is the matrix in row-echelon form?
(c) Is the matrix in reduced row-echelon form?
(d) Write the system of equations for which the given matrix is the augmented matrix.

15. $\begin{bmatrix} 1 & 2 & -5 \\ 0 & 1 & 3 \end{bmatrix}$

16. $\begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & 0 \end{bmatrix}$

17. $\begin{bmatrix} 1 & 0 & 8 & 0 \\ 0 & 1 & 5 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

18. $\begin{bmatrix} 1 & 3 & 6 & 2 \\ 2 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

19. $\begin{bmatrix} 0 & 1 & -3 & 4 \\ 1 & 1 & 0 & 7 \\ 1 & 2 & 1 & 2 \end{bmatrix}$

20. $\begin{bmatrix} 1 & 8 & 6 & -4 \\ 0 & 1 & -3 & 5 \\ 0 & 0 & 2 & -7 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

21–42 ■ Systems of Linear Equations in Several Variables Find the complete solution of the system, or show that the system has no solution.

21. $\begin{cases} x + y + 2z = 6 \\ 2x + 5z = 12 \\ x + 2y + 3z = 9 \end{cases}$

22. $\begin{cases} x - 2y + 3z = 1 \\ x - 3y - z = 0 \\ 2x - 6z = 6 \end{cases}$

23. $\begin{cases} x - 2y + 3z = 1 \\ 2x - y + z = 3 \\ 2x - 7y + 11z = 2 \end{cases}$

24. $\begin{cases} x + y + z + w = 2 \\ 2x - 3z = 5 \\ x - 2y + 4w = 9 \\ x + y + 2z + 3w = 5 \end{cases}$

25. $\begin{cases} x + 2y + 2z = 6 \\ x - y = -1 \\ 2x + y + 3z = 7 \end{cases}$

26. $\begin{cases} x - y + z = 2 \\ x + y + 3z = 6 \\ 2y + 3z = 5 \end{cases}$

27. $\begin{cases} x - 2y + 3z = -2 \\ 2x - y + z = 2 \\ 2x - 7y + 11z = -9 \end{cases}$

28. $\begin{cases} x - y + z = 2 \\ x + y + 3z = 6 \\ 3x - y + 5z = 10 \end{cases}$

29. $\begin{cases} x + y + z + w = 0 \\ x - y - 4z - w = -1 \\ x - 2y + 4w = -7 \\ 2x + 2y + 3z + 4w = -3 \end{cases}$

30. $\begin{cases} x + 3z = -1 \\ y - 4w = 5 \\ 2y + z + w = 0 \\ 2x + y + 5z - 4w = 4 \end{cases}$

31. $\begin{cases} x - 3y + z = 4 \\ 4x - y + 15z = 5 \end{cases}$

32. $\begin{cases} 2x - 3y + 4z = 3 \\ 4x - 5y + 9z = 13 \\ 2x + 7z = 0 \end{cases}$

33. $\begin{cases} -x + 4y + z = 8 \\ 2x - 6y + z = -9 \\ x - 6y - 4z = -15 \end{cases}$

34. $\begin{cases} x - z + w = 2 \\ 2x + y - 2w = 12 \\ 3y + z + w = 4 \\ x + y - z = 10 \end{cases}$

35. $\begin{cases} x - y + 3z = 2 \\ 2x + y + z = 2 \\ 3x + 4z = 4 \end{cases}$

36. $\begin{cases} x - y = 1 \\ x + y + 2z = 3 \\ x - 3y - 2z = -1 \end{cases}$

37. $\begin{cases} x - y + z - w = 0 \\ 3x - y - z - w = 2 \end{cases}$

38. $\begin{cases} x - y = 3 \\ 2x + y = 6 \\ x - 2y = 9 \end{cases}$

39. $\begin{cases} x - y + z = 0 \\ 3x + 2y - z = 6 \\ x + 4y - 3z = 3 \end{cases}$

40. $\begin{cases} x + 2y + 3z = 2 \\ 2x - y - 5z = 1 \\ 4x + 3y + z = 6 \end{cases}$

41. $\begin{cases} x + y - z - w = 2 \\ x - y + z - w = 0 \\ 2x + 2w = 2 \\ 2x + 4y - 4z - 2w = 6 \end{cases}$

42. $\begin{cases} x - y - 2z + 3w = 0 \\ y - z + w = 1 \\ 3x - 2y - 7z + 10w = 2 \end{cases}$

43. Investments An investor has savings in two accounts, one paying 6% interest per year and the other paying 7%. Twice as much is invested in the 7% account as in the 6% account, and the annual interest income is $600. How much is invested in each account?

44. Number of Coins A piggy bank contains 50 coins, all of them nickels, dimes, or quarters. The total value of the coins is \$5.60, and the value of the dimes is five times the value of the nickels. How many coins of each type are there?

45. Investments An amount of \$60,000 is invested in money-market accounts at three different banks. Bank A pays 2% interest per year, bank B pays 2.5%, and bank C pays 3%. Twice as much is invested in bank B as in the other two banks combined. After one year, the investment has earned \$1575 in interest. How much is invested in each bank?

46. Number of Fish Caught A commercial fishing boat fishes for haddock, sea bass, and red snapper. The haddock sells for \$3.75/lb, sea bass for \$2.25/lb, and red snapper for \$6.00/lb. Yesterday the fishing boat caught 560 lb of fish worth \$1725. The haddock and red snapper together are worth \$960. How many pounds of each fish did the fishing boat catch?

47–58 ■ Matrix Operations Let

$$A = \begin{bmatrix} 2 & 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 4 \\ -2 & 1 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} \frac{1}{2} & 3 \\ 2 & \frac{3}{2} \\ -2 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 4 \\ 0 & -1 \\ 2 & 0 \end{bmatrix}$$

$$E = \begin{bmatrix} 2 & -1 \\ -\frac{1}{2} & 1 \end{bmatrix} \qquad F = \begin{bmatrix} 4 & 0 & 2 \\ -1 & 1 & 0 \\ 7 & 5 & 0 \end{bmatrix} \qquad G = \begin{bmatrix} 5 \end{bmatrix}$$

Carry out the indicated operation, or explain why it cannot be performed.

47. $A + B$ **48.** $C - D$

49. $2C + 3D$ **50.** $5B - 2C$

51. GA **52.** AG

53. BC **54.** CB

55. BF **56.** FC

57. $(C + D)E$ **58.** $F(2C - D)$

59–60 ■ Inverse Matrices Verify that matrices A and B are inverses of each other by calculating the products AB and BA.

59. $A = \begin{bmatrix} 2 & -5 \\ -2 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 3 & \frac{5}{2} \\ 1 & 1 \end{bmatrix}$

60. $A = \begin{bmatrix} 2 & -1 & 3 \\ 2 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -\frac{3}{2} & 2 & \frac{5}{2} \\ -1 & 1 & 2 \\ 1 & -1 & -1 \end{bmatrix}$

61–66 ■ Matrix Equations Solve the matrix equation for the unknown matrix X, or show that no solution exists, where

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 1 & 3 \\ -2 & 4 & 0 \end{bmatrix}$$

61. $A + 3X = B$ **62.** $\frac{1}{2}(X - 2B) = A$

63. $2(X - A) = 3B$ **64.** $2X + C = 5A$

65. $AX = C$ **66.** $AX = B$

67–74 ■ Determinants and Inverse Matrices Find the determinant and, if possible, the inverse of the matrix.

67. $\begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix}$ **68.** $\begin{bmatrix} 2 & 2 \\ 1 & -3 \end{bmatrix}$

69. $\begin{bmatrix} 4 & -12 \\ -2 & 6 \end{bmatrix}$ **70.** $\begin{bmatrix} 2 & 4 & 0 \\ -1 & 1 & 2 \\ 0 & 3 & 2 \end{bmatrix}$

71. $\begin{bmatrix} 3 & 0 & 1 \\ 2 & -3 & 0 \\ 4 & -2 & 1 \end{bmatrix}$ **72.** $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 2 & 5 & 6 \end{bmatrix}$

73. $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ **74.** $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 2 & 1 & 2 \end{bmatrix}$

75–78 ■ Using Inverse Matrices to Solve a System Express the system of linear equations as a matrix equation. Then solve the matrix equation by multiplying each side by the inverse of the coefficient matrix.

75. $\begin{cases} 12x - 5y = 10 \\ 5x - 2y = 17 \end{cases}$ **76.** $\begin{cases} 6x - 5y = 1 \\ 8x - 7y = -1 \end{cases}$

77. $\begin{cases} 2x + y + 5z = \frac{1}{3} \\ x + 2y + 2z = \frac{1}{4} \\ x + 3z = \frac{1}{6} \end{cases}$ **78.** $\begin{cases} 2x + 3z = 5 \\ x + y + 6z = 0 \\ 3x - y + z = 5 \end{cases}$

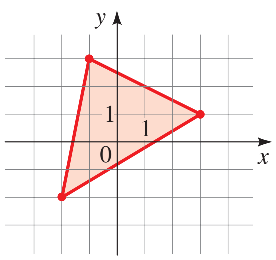
79–82 ■ Using Cramer's Rule to Solve a System Solve the system using Cramer's Rule.

79. $\begin{cases} 2x + 7y = 13 \\ 6x + 16y = 30 \end{cases}$ **80.** $\begin{cases} 12x - 11y = 140 \\ 7x + 9y = 20 \end{cases}$

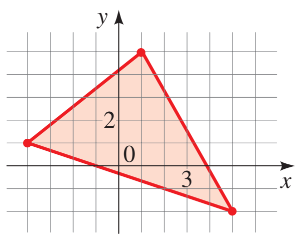
81. $\begin{cases} 2x - y + 5z = 0 \\ -x + 7y = 9 \\ 5x + 4y + 3z = -9 \end{cases}$ **82.** $\begin{cases} 3x + 4y - z = 10 \\ x - 4z = 20 \\ 2x + y + 5z = 30 \end{cases}$

83–84 ■ Area of a Triangle Use the determinant formula for the area of a triangle to find the area of the triangle in the figure.

83.



84.



85–90 ■ Partial Fraction Decomposition Find the partial fraction decomposition of the rational expression.

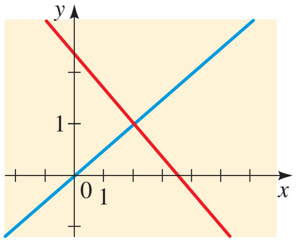
85. $\dfrac{3x + 1}{x^2 - 2x - 15}$ **86.** $\dfrac{8}{x^3 - 4x}$

87. $\dfrac{2x - 4}{x(x - 1)^2}$ **88.** $\dfrac{x + 6}{x^3 - 2x^2 + 4x - 8}$

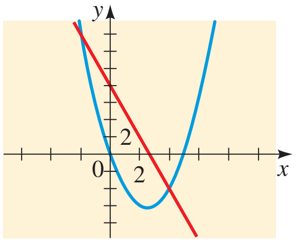
89. $\dfrac{2x - 1}{x^3 + x}$ **90.** $\dfrac{5x^2 - 3x + 10}{x^4 + x^2 - 2}$

91–94 ■ Intersection Points Two equations and their graphs are given. Estimate the intersection point from the graph and check that the point is a solution to the system.

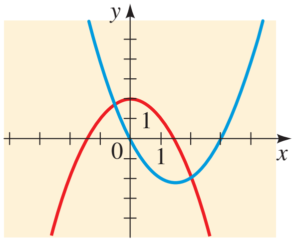
91. $\begin{cases} 2x + 3y = 7 \\ \ \ x - 2y = 0 \end{cases}$



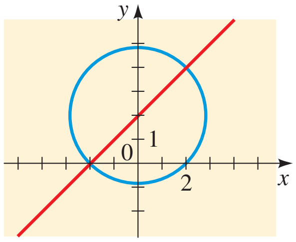
92. $\begin{cases} 3x + y = 8 \\ y = x^2 - 5x \end{cases}$



93. $\begin{cases} x^2 + y = 2 \\ x^2 - 3x - y = 0 \end{cases}$

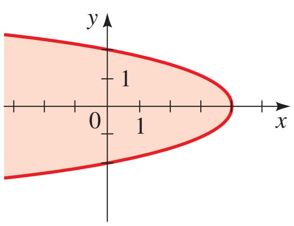


94. $\begin{cases} x - y = -2 \\ x^2 + y^2 - 4y = 4 \end{cases}$

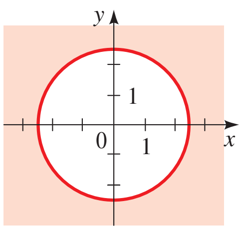


95–96 ■ Finding an Inequality from a Graph An equation and its graph are given. Find an inequality whose solution is the shaded region.

95. $x + y^2 = 4$



96. $x^2 + y^2 = 8$



97–100 ■ Graphing Inequalities Graph the inequality.

97. $3x + y \leq 6$

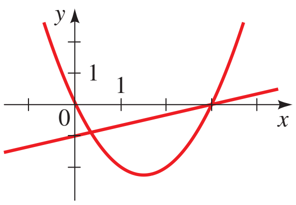
98. $y \geq x^2 - 3$

99. $x^2 + y^2 \geq 9$

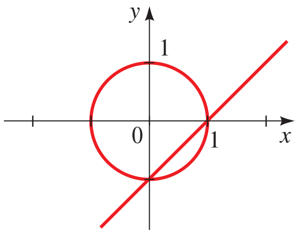
100. $x - y^2 < 4$

101–104 ■ Solution Set of a System of Inequalities The figure shows the graphs of the equations corresponding to the given inequalities. Shade the solution set of the system of inequalities.

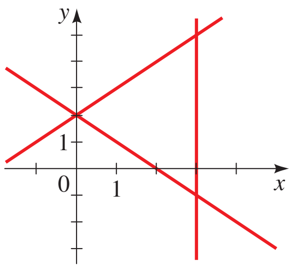
101. $\begin{cases} y \geq x^2 - 3x \\ y \leq \frac{1}{3}x - 1 \end{cases}$



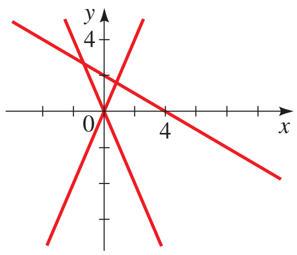
102. $\begin{cases} y \geq x - 1 \\ x^2 + y^2 \leq 1 \end{cases}$



103. $\begin{cases} x + y \geq 2 \\ y - x \leq 2 \\ x \leq 3 \end{cases}$



104. $\begin{cases} y \geq -2x \\ y \leq 2x \\ y \leq -\frac{1}{2}x + 2 \end{cases}$



105–108 ■ Systems of Inequalities Graph the solution set of the system of inequalities. Find the coordinates of all vertices, and determine whether the solution set is bounded or unbounded.

105. $\begin{cases} x^2 + y^2 < 9 \\ x + y < 0 \end{cases}$

106. $\begin{cases} y - x^2 \geq 4 \\ y < 20 \end{cases}$

107. $\begin{cases} x \geq 0, \quad y \geq 0 \\ x + 2y \leq 12 \\ y \leq x + 4 \end{cases}$

108. $\begin{cases} x \geq 4 \\ x + y \geq 24 \\ x \leq 2y + 12 \end{cases}$

109–110 ■ General Systems of Equations Solve for x, y, and z in terms of a, b, and c.

109. $\begin{cases} -x + y + z = a \\ \ \ x - y + z = b \\ \ \ x + y - z = c \end{cases}$

110. $\begin{cases} ax + by + cz = a - b + c \\ bx + by + cz = c \\ cx + cy + cz = c \end{cases} \qquad (a \neq b, b \neq c, c \neq 0)$

111. General Systems of Equations For what values of k do the following three lines have a common point of intersection?

$$x + y = 12$$
$$kx - y = 0$$
$$y - x = 2k$$

112. General Systems of Equations For what value of k does the following system have infinitely many solutions?

$$\begin{cases} kx + \ \ y + \ \ z = 0 \\ \ \ x + 2y + kz = 0 \\ -x \qquad\ \ + 3z = 0 \end{cases}$$

Chapter 9 | Test

1–2 ■ A system of equations is given. **(a)** Determine whether the system is linear or nonlinear. **(b)** Find all solutions of the system.

1. $\begin{cases} x + 3y = 7 \\ 5x + 2y = -4 \end{cases}$

2. $\begin{cases} 6x + y^2 = 10 \\ 3x - y = 5 \end{cases}$

3. Use a graphing device to find all solutions of the system, rounded to two decimal places.

$$\begin{cases} x - 2y = 1 \\ y = x^3 - 2x^2 \end{cases}$$

4. In $2\frac{1}{2}$ hours an airplane travels 600 km against the wind. It takes 50 minutes to travel 300 km with the wind. Find the speed of the wind and the speed of the airplane in still air.

5. Determine whether each matrix is in reduced row-echelon form, row-echelon form, or neither.

(a) $\begin{bmatrix} 1 & 2 & 4 & -6 \\ 0 & 1 & -3 & 0 \end{bmatrix}$ **(b)** $\begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ **(c)** $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix}$

6. Use Gaussian elimination to find the complete solution of the system, or show that no solution exists.

(a) $\begin{cases} x - y + 2z = 0 \\ 2x - 4y + 5z = -5 \\ 2y - 3z = 5 \end{cases}$ **(b)** $\begin{cases} 2x - 3y + z = 3 \\ x + 2y + 2z = -1 \\ 4x + y + 5z = 4 \end{cases}$

7. Use Gauss-Jordan elimination to find the complete solution of the system.

$$\begin{cases} x + 3y - z = 0 \\ 3x + 4y - 2z = -1 \\ -x + 2y = 1 \end{cases}$$

8. Three friends enter a coffee shop. The first orders two coffees, one juice, and two doughnuts and pays \$6.25. The second orders one coffee and three doughnuts and pays \$3.75. The third orders three coffees, one juice, and four doughnuts and pays \$9.25. Find the price of coffee, juice, and doughnuts at this coffee shop.

9. Let

$$A = \begin{bmatrix} 2 & 3 \\ 2 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 4 \\ -1 & 1 \\ 3 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 4 \\ -1 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix}$$

Carry out the indicated operation, or explain why it cannot be performed.

(a) $A + B$ **(b)** AB **(c)** $BA - 3B$ **(d)** CBA

(e) A^{-1} **(f)** B^{-1} **(g)** $\det(B)$ **(h)** $\det(C)$

10. (a) Write a matrix equation equivalent to the following system.

$$\begin{cases} 4x - 3y = 10 \\ 3x - 2y = 30 \end{cases}$$

(b) Find the inverse of the coefficient matrix, and use it to solve the system.

11. Only one of the following matrices has an inverse. Find the determinant of each matrix, and use the determinants to identify the one that has an inverse. Then find the inverse.

$$A = \begin{bmatrix} 1 & 4 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 2 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

12. Solve using Cramer's Rule:

$$\begin{cases} 2x \qquad\quad - \ z = 14 \\ 3x - \ y + 5z = 0 \\ 4x + 2y + 3z = -2 \end{cases}$$

13. Find the partial fraction decomposition of each rational expression.

(a) $\dfrac{4x - 1}{(x - 1)^2(x + 2)}$ **(b)** $\dfrac{2x - 3}{x^3 + 3x}$

14. Graph the solution set of the system of inequalities. Label the vertices with their coordinates.

(a) $\begin{cases} 2x + \ y \leq 8 \\ x - \ y \geq -2 \\ x + 2y \geq 4 \end{cases}$ **(b)** $\begin{cases} x^2 + y \leq 5 \\ y \geq 2x + 5 \end{cases}$

Linear programming is a modeling technique that is used to determine the optimal allocation of resources in business, the military, and other areas of human endeavor. For example, a manufacturer who makes several different products from the same raw materials can use linear programming to determine how much of each product should be produced to maximize the profit. This modeling technique is probably the most important practical application of systems of linear inequalities. In 1975 Leonid Kantorovich and T. C. Koopmans won the Nobel Prize in economics for their work in the development of this technique.

Linear programming can be applied to complex problems with hundreds or even thousands of variables. Here we consider problems involving two variables (x and y) to which the graphical methods of Section 9.9 can be applied. Each linear programming problem includes restrictions, called **constraints**, that lead to a system of linear inequalities whose solution is called the **feasible region**. The objective of the problem is to maximize or minimize a linear function in the variables x and y, called the **objective function**. This function always attains its largest and smallest values at **vertices** of the feasible region. The following guidelines show the steps used to set up and solve a linear programming problem.

Guidelines for Linear Programming

1. **Choose the Variables.** Decide what variable quantities in the problem should be named x and y.
2. **Find the Objective Function.** Write an expression for the function we want to maximize or minimize.
3. **Graph the Feasible Region.** Express the constraints as a system of inequalities, and graph the solution of this system (the feasible region).
4. **Find the Maximum or Minimum.** Evaluate the objective function at the vertices of the feasible region to determine its maximum or minimum value.

Example 1 ■ Manufacturing for Maximum Profit

A small shoe manufacturer makes two styles of shoes: oxfords and loafers. Two machines are used in the process: a cutting machine and a sewing machine. Each type of shoe requires 15 minutes per pair on the cutting machine. Oxfords require 10 minutes of sewing per pair, and loafers require 20 minutes of sewing per pair. Because the manufacturer can hire only one operator for each machine, each process is available for just 8 hours per day. If the profit is \$15 on each pair of oxfords and \$20 on each pair of loafers, how many pairs of each type should be produced per day for maximum profit?

Because loafers produce more profit, it would seem best to manufacture only loafers. Surprisingly, this does not turn out to be the most profitable solution.

Solution First we organize the given information into a table. To be consistent, let's convert all times to hours.

	Oxfords	Loafers	Time Available
Time on cutting machine (h)	$\frac{1}{4}$	$\frac{1}{4}$	8
Time on sewing machine (h)	$\frac{1}{6}$	$\frac{1}{3}$	8
Profit	\$15	\$20	

We describe the model and solve the problem in four steps.

Linear Programming helps the telephone industry to determine the most efficient way to route telephone calls. The computerized routing decisions must be made very rapidly so that callers are not kept waiting for connections. Since the database of customers and routes is huge, an extremely fast method for solving linear programming problems is essential. In 1984 the 28-year-old mathematician **Narendra Karmarkar**, working at Bell Labs in Murray Hill, New Jersey, discovered just such a method. His idea is so ingenious and his method so fast that the discovery caused a sensation in the mathematical world. Although mathematical discoveries rarely make the news, this one was reported in *Time*, on December 3, 1984. Today airlines routinely use Karmarkar's technique to minimize costs in scheduling passengers, flight personnel, fuel, baggage, and maintenance workers.

Choose the variables. To make a mathematical model, we first give names to the variable quantities. For this problem we let

$$x = \text{number of pairs of oxfords made daily}$$
$$y = \text{number of pairs of loafers made daily}$$

Find the objective function. Our goal is to determine which values for x and y give maximum profit. Since each pair of oxfords provides \$15 profit and each pair of loafers provides \$20, the total profit is given by

$$P = 15x + 20y$$

This function is the *objective function*.

Graph the feasible region. As x and y increase, so does the profit. But we cannot choose arbitrarily large values for these variables because of the restrictions, or *constraints*, in the problem. Each restriction is an inequality in the variables.

In this problem the total number of cutting hours needed is $\frac{1}{4}x + \frac{1}{4}y$. Since only 8 hours are available on the cutting machine, we have

$$\tfrac{1}{4}x + \tfrac{1}{4}y \le 8$$

Similarly, by considering the amount of time needed and available on the sewing machine, we get

$$\tfrac{1}{6}x + \tfrac{1}{3}y \le 8$$

We cannot produce a negative number of shoes, so we also have

$$x \ge 0 \qquad \text{and} \qquad y \ge 0$$

Thus x and y must satisfy the constraints

$$\begin{cases} \frac{1}{4}x + \frac{1}{4}y \le 8 \\ \frac{1}{6}x + \frac{1}{3}y \le 8 \\ x \ge 0, \quad y \ge 0 \end{cases}$$

If we multiply the first inequality by 4 and the second by 6, we obtain the simplified system

$$\begin{cases} x + y \le 32 \\ x + 2y \le 48 \\ x \ge 0, \quad y \ge 0 \end{cases}$$

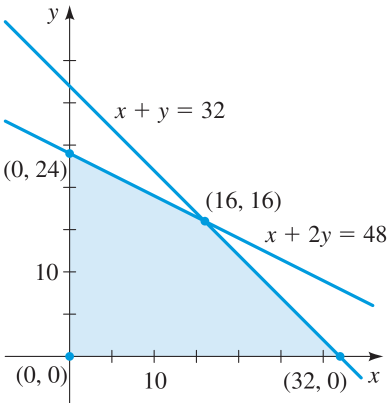


Figure 1

The solution of this system (with vertices labeled) is sketched in Figure 1. The only values that satisfy the restrictions of the problem are the ones that correspond to points of the shaded region in Figure 1. This is the *feasible region* for the problem.

Find the maximum profit. As x or y increases, profit increases as well. Thus it seems reasonable that the maximum profit will occur at a point on one of the outside edges of the feasible region, where it is impossible to increase x or y without going outside the region. In fact, it can be shown that the maximum value occurs at a vertex (see Problem 16). This means that we need to check the profit only at the vertices. From the table we see that the largest value of P occurs at the point (16, 16), where $P =$ \$560. Thus the manufacturer should make 16 pairs of oxfords and 16 pairs of loafers, for a maximum daily profit of \$560.

Vertex	$P = 15x + 20y$
(0, 0)	0
(0, 24)	15(0) + 20(24) = \$480
(16, 16)	15(16) + 20(16) = \$560
(32, 0)	15(32) + 20(0) = \$480

Maximum profit

Example 2 ■ A Shipping Problem

A car dealer has warehouses in Millville and Trenton and dealerships in Camden and Atlantic City. Every car that is sold at the dealerships must be delivered from one of the warehouses. On a certain day the Camden dealers sell 10 cars, and the Atlantic City dealers sell 12. The Millville warehouse has 15 cars available, and the Trenton warehouse has 10. The cost of shipping one car is \$50 from Millville to Camden, \$40 from Millville to Atlantic City, \$60 from Trenton to Camden, and \$55 from Trenton to Atlantic City. How many cars should be moved from each warehouse to each dealership to fill the orders at minimum cost?

Solution Our first step is to organize the given information. Rather than construct a table, we draw a diagram to show the flow of cars from the warehouses to the dealerships (see Figure 2 below). The diagram shows the number of cars available at each warehouse or required at each dealership and the cost of shipping between these locations.

Choose the variables. The arrows in Figure 2 indicate four possible routes, so the problem seems to involve four variables. We let

x = number of cars to be shipped from Millville to Camden

y = number of cars to be shipped from Millville to Atlantic City

To fill the orders, we must have

$10 - x$ = number of cars shipped from Trenton to Camden

$12 - y$ = number of cars shipped from Trenton to Atlantic City

So the only variables in the problem are x and y.

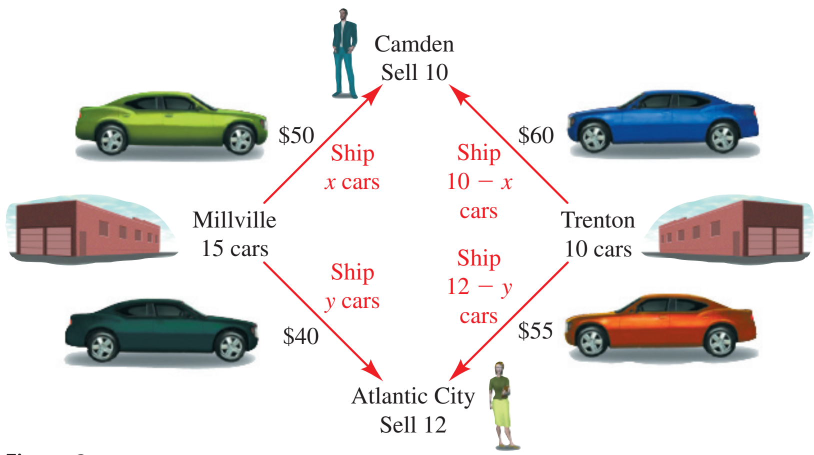


Figure 2

Find the objective function. The objective of this problem is to minimize cost. From Figure 2 we see that the total cost C of shipping the cars is

$$\begin{aligned} C &= 50x + 40y + 60(10 - x) + 55(12 - y) \\ &= 50x + 40y + 600 - 60x + 660 - 55y \\ &= 1260 - 10x - 15y \end{aligned}$$

This is the objective function.

Graph the feasible region. Now we derive the constraint inequalities that define the feasible region. First, the number of cars shipped on each route can't be negative, so we have

$$\begin{aligned} x &\geq 0 & y &\geq 0 \\ 10 - x &\geq 0 & 12 - y &\geq 0 \end{aligned}$$

Second, the total number of cars shipped from each warehouse can't exceed the number of cars available there, so

$$\begin{aligned} x + y &\le 15 \\ (10 - x) + (12 - y) &\le 10 \end{aligned}$$

Simplifying the latter inequality, we get

$$\begin{aligned} 22 - x - y &\le 10 \\ -x - y &\le -12 \\ x + y &\ge 12 \end{aligned}$$

The inequalities $10 - x \ge 0$ and $12 - y \ge 0$ can be rewritten as $x \le 10$ and $y \le 12$. Thus the feasible region is described by the constraints

$$\begin{cases} x + y \le 15 \\ x + y \ge 12 \\ 0 \le x \le 10 \\ 0 \le y \le 12 \end{cases}$$

The feasible region is graphed in Figure 3.

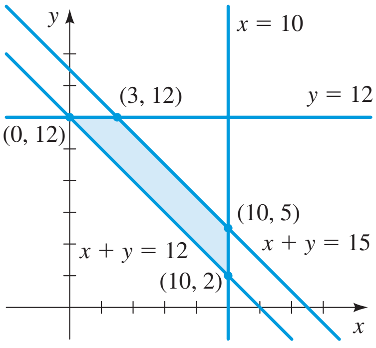


Figure 3

Find the minimum cost. We check the value of the objective function at each vertex of the feasible region.

Vertex	$C = 1260 - 10x - 15y$
(0, 12)	$1260 - 10(0) - 15(12) = \$1080$
(3, 12)	$1260 - 10(3) - 15(12) = \$1050$ ← Minimum cost
(10, 5)	$1260 - 10(10) - 15(5) = \$1085$
(10, 2)	$1260 - 10(10) - 15(2) = \$1130$

The lowest cost is incurred at the point (3, 12). Thus the dealer should ship

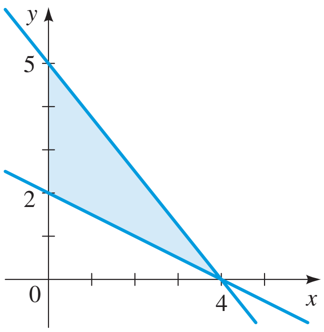
3 cars from Millville to Camden
12 cars from Millville to Atlantic City
7 cars from Trenton to Camden
0 cars from Trenton to Atlantic City ■

In the 1940s mathematicians developed matrix methods for solving linear programming problems that involve more than two variables. These methods were first used by the Allies in World War II to solve supply problems similar to (but, of course, much more complicated than) Example 2. Improving such matrix methods is an active and exciting area of current mathematical research.

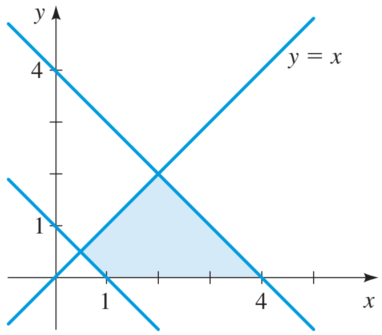
Problems

1–4 ■ Find the maximum and minimum values of the given objective function on the indicated feasible region.

1. $M = 200 - x - y$



2. $N = \frac{1}{2}x + \frac{1}{4}y + 40$



3. $P = 140 - x + 3y$

$$\begin{cases} x \geq 0, y \geq 0 \\ 2x + y \leq 10 \\ 2x + 4y \leq 28 \end{cases}$$

4. $Q = 70x + 82y$

$$\begin{cases} x \geq 0, y \geq 0 \\ x \leq 10, y \leq 20 \\ x + y \geq 5 \\ x + 2y \leq 18 \end{cases}$$

5. Making Furniture A furniture manufacturer makes wooden tables and chairs. The production process involves two basic types of labor: carpentry and finishing. A table requires 2 hours of carpentry and 1 hour of finishing, and a chair requires 3 hours of carpentry and $\frac{1}{2}$ hour of finishing. The profit is \$35 per table and \$20 per chair. The manufacturer's employees can supply a maximum of 108 hours of carpentry work and 20 hours of finishing work per day. How many tables and chairs should be made each day to maximize profit?

6. A Housing Development A housing contractor has subdivided a farm into 100 building lots. There are two types of homes for these lots: colonial and ranch style. A colonial requires \$30,000 of capital and produces a profit of \$4000 when sold. A ranch-style house requires \$40,000 of capital and provides an \$8000 profit. If the contractor has \$3.6 million of capital on hand, how many houses of each type should be built for maximum profit? Will any of the lots be left vacant?

7. Hauling Fruit A trucking company transports citrus fruit from Florida to Montreal. Each crate of oranges is 4 ft^3 in volume and weighs 80 lb. Each crate of grapefruit has a volume of 6 ft^3 and weighs 100 lb. Each company truck has a maximum capacity of 300 ft^3 and can carry no more than 5600 lb. Moreover, each truck is not permitted to carry more crates of grapefruit than crates of oranges. If the profit is \$2.50 on each crate of oranges and \$4 on each crate of grapefruit, how many crates of each fruit should a truck carry for maximum profit?

8. Manufacturing Calculators A manufacturer of calculators produces two models: standard and scientific. The long-term demand for the two models mandates that the company manufacture at least 100 standard and 80 scientific calculators each day. However, because of limitations on production capacity, no more than 200 standard and 170 scientific calculators can be made daily. To satisfy a shipping contract, a total of at least 200 calculators must be shipped every day.

(a) If the production cost is \$5 for a standard calculator and \$7 for a scientific one, how many of each model should be produced daily to minimize this cost?

(b) If each standard calculator results in a \$2 loss but each scientific one produces a \$5 profit, how many of each model should be made daily to maximize profit?

9. Shipping Televisions An electronics discount chain has a sale on a certain brand of 60-inch high-definition television set. The chain has stores in Santa Monica and El Toro and warehouses in Long Beach and Pasadena. To satisfy rush orders, 15 sets must be shipped from the warehouses to the Santa Monica store, and 19 must be shipped to the El Toro store. The cost of shipping a set is \$5 from Long Beach to Santa Monica, \$6 from Long Beach to El Toro, \$4 from Pasadena to Santa Monica, and \$5.50 from Pasadena to El Toro. If the Long Beach warehouse has 24 sets and the Pasadena warehouse has 18 sets in stock, how many sets should be shipped from each warehouse to each store to fill the orders at a minimum shipping cost?

10. Delivering Plywood A building supply company has two warehouses, one on the east side and one on the west side of a city. Two customers order some $\frac{1}{2}$-inch plywood. Customer A needs 50 sheets, and customer B needs 70 sheets. The east-side warehouse has 80 sheets, and the west-side warehouse has 45 sheets of this plywood in stock. The east-side warehouse's delivery costs per sheet are \$0.50 to customer A and \$0.60 to customer B. The west-side warehouse's delivery costs per sheet are \$0.40 to customer A and \$0.55 to customer B. How many sheets should be shipped from each warehouse to each customer to minimize delivery costs?

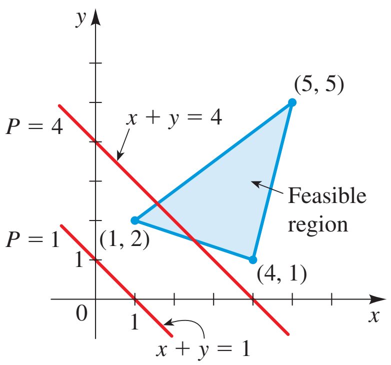
11. Packaging Nuts A confectioner sells two types of nut mixtures. The standard-mixture package contains 100 g of cashews and 200 g of peanuts and sells for \$1.95. The deluxe-mixture package contains 150 g of cashews and 50 g of peanuts and sells for \$2.25. The confectioner has 15 kg of cashews and 20 kg of peanuts available. On the basis of past sales, the confectioner needs to have at least as many standard as deluxe packages available. How many bags of each mixture should be packaged to maximize revenue?

12. Feeding Lab Rabbits A biologist wishes to feed laboratory rabbits a mixture of two types of foods. Type I contains 8 g of fat, 12 g of carbohydrate, and 2 g of protein per ounce. Type II contains 12 g of fat, 12 g of carbohydrate, and 1 g of protein per ounce. Type I costs \$0.20 per ounce and type II costs \$0.30 per ounce. Each rabbit receives a daily minimum of 24 g of fat, 36 g of carbohydrate, and 4 g of protein, but get no more than 5 oz of food per day. How many ounces of each food type should be fed to each rabbit daily to satisfy the dietary requirements at minimum cost?

13. Investing in Bonds A financial advisor needs to invest \$12,000 in three types of bonds: municipal bonds paying 7% interest per year, bank certificates paying 8%, and high-risk bonds paying 12%. For tax reasons the amount invested in municipal bonds should be at least three times the amount invested in bank certificates. To keep the level of risk manageable, no more than \$2000 should be invested in high-risk bonds. How much should be invested in each type of bond to maximize the annual interest yield? [*Hint:* Let x = amount in municipal bonds and y = amount in bank certificates. Then the amount in high-risk bonds will be $12{,}000 - x - y$.]

14. Annual Interest Yield Refer to Problem 13. Suppose the investor decides to increase the maximum invested in high-risk bonds to \$3000 but leaves the other conditions unchanged. By how much will the maximum possible interest yield increase?

15. Business Strategy A small software company publishes computer games, educational software, and utility software. Their business strategy is to market a total of 36 new programs each year, at least four of these being games. The number of utility programs published is never more than twice the number of educational programs. On average, the company makes an annual profit of \$5000 on each computer game, \$8000 on each educational program, and \$6000 on each utility program. How many of each type of software should the company publish annually for maximum profit?



16. Extreme Values and Vertices This exercise illustrates why the minimum and maximum values of the objective function occur at vertices of the feasible region. The feasible region for the following linear programming problem is graphed in the figure.

$$\begin{cases} 4y - 3x \le 5 \\ 4x - y \le 15 \\ x + 3y \ge 7 \end{cases}$$

$$P = x + y$$

For each value of P the graph of the objective function is a line; the lines for $P = 1$ and $P = 4$ are shown in the figure. On the given graph, sketch the lines corresponding to increasing values of P, starting at $P = 1$. What are the minimum and maximum values of P for which the line $P = x + y$ intersects the feasible region? Explain why these are the minimum and maximum values of P on the feasible region. At what points of the feasible region do these extreme values occur?