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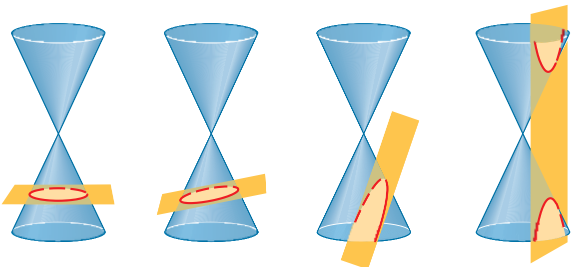
# 10 Conic Sections



**Conic sections** are the curves that are formed when a plane cuts a cone, as shown in the figure. For example, if a cone is cut horizontally, the cross section is a circle. So a circle is a conic section. Other ways of cutting a cone produce ellipses, parabolas, and hyperbolas.



Circle Ellipse Parabola Hyperbola

Our goal in this chapter is to find equations whose graphs are conic sections. We will find such equations by analyzing the geometric properties of conic sections. These properties make conic sections useful for many real-world applications. For instance, a reflecting surface with parabolic cross sections concentrates light at a single point. This property of a parabola is used in the construction of reflecting telescopes, as well as in describing the path of planets and comets. In the *Focus on Modeling* at the end of the chapter we explore how these curves are used in architecture.

# 10.1 Parabolas

■ Geometric Definition of a Parabola ■ Equations and Graphs of Parabolas ■ Applications

## ■ Geometric Definition of a Parabola

We saw in Section 3.1 that the graph of the equation

$$y = ax^2 + bx + c$$

is a U-shaped curve called a *parabola* that opens either upward or downward, depending on whether the number $a$ is positive or negative.

In this section we study parabolas from a geometric, rather than an algebraic, point of view. We begin with the geometric definition of a parabola and show how this leads to the algebraic formula that we are already familiar with.

### Geometric Definition of a Parabola

A **parabola** is the set of all points in the plane that are equidistant from a fixed point $F$ (called the **focus**) and a fixed line $l$ (called the **directrix**).

This definition is illustrated in Figure 1. The **vertex** $V$ of the parabola lies halfway between the focus and the directrix, and the **axis of symmetry** is the line that runs through the focus perpendicular to the directrix.

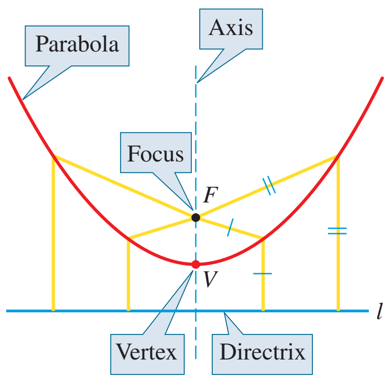



**Figure 1**

In this section we restrict our attention to parabolas that are situated with the vertex at the origin and that have a vertical or horizontal axis of symmetry. (Parabolas in more general positions will be considered in Section 10.4.) If the focus of such a parabola is the point $F(0, p)$, then the axis of symmetry must be vertical, and the directrix has the equation $y = -p$. Figure 2 illustrates the case $p > 0$.

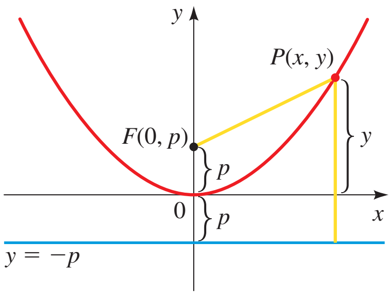



**Figure 2**

**Deriving the Equation of a Parabola** If $P(x, y)$ is any point on the parabola, then the distance from $P$ to the focus $F$ (using the Distance Formula) is

$$\sqrt{x^2 + (y - p)^2}$$

The distance from $P$ to the directrix is

$$|y - (-p)| = |y + p|$$

By the definition of a parabola these two distances must be equal.

$$\sqrt{x^2 + (y - p)^2} = |y + p|$$

$$x^2 + (y - p)^2 = |y + p|^2 = (y + p)^2 \qquad \text{Square both sides}$$

$$x^2 + y^2 - 2py + p^2 = y^2 + 2py + p^2 \qquad \text{Expand}$$

$$x^2 - 2py = 2py \qquad \text{Simplify}$$

$$x^2 = 4py$$

This is the **standard equation** of a parabola with vertical axis and vertex at the origin. If $p > 0$, then the parabola opens upward; if $p < 0$, it opens downward. When $x$ is replaced by $-x$, the equation remains unchanged, so the graph is symmetric about the $y$-axis.

## ■ Equations and Graphs of Parabolas

The following box summarizes what we have just proved about the equation and features of a parabola with a vertical axis.

### Parabola with Vertical Axis

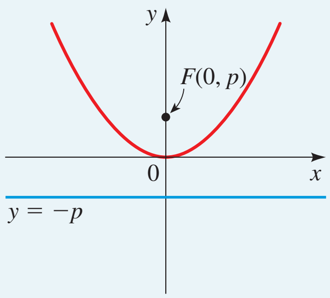
The graph of the equation

$$x^2 = 4py$$

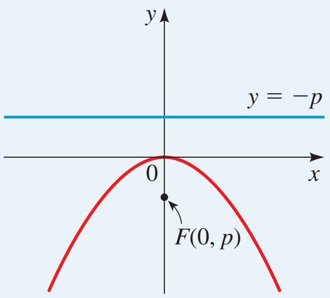
is a parabola with the following properties.

| | |
|---|---|
| **VERTEX** | $V(0, 0)$ |
| **FOCUS** | $F(0, p)$ |
| **DIRECTRIX** | $y = -p$ |

The parabola opens upward if $p > 0$ or downward if $p < 0$.




$x^2 = 4py$ with $p > 0$




$x^2 = 4py$ with $p < 0$

## Example 1 ■ Finding the Equation of a Parabola

Find the standard equation for the parabola with vertex $V(0, 0)$ and focus $F(0, 2)$, and sketch its graph.

**Solution** Since the focus is $F(0, 2)$, we conclude that $p = 2$ (so the directrix is $y = -2$). Thus the standard equation of the parabola is

$$x^2 = 4(2)y \qquad x^2 = 4py \text{ with } p = 2$$

$$x^2 = 8y$$

Since $p = 2 > 0$, the parabola opens upward. See Figure 3.

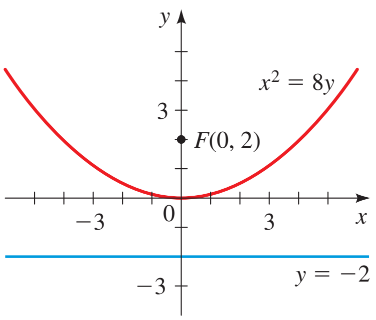
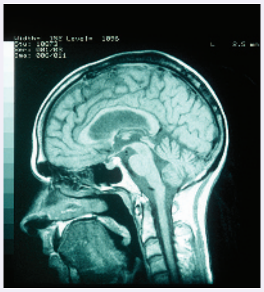



**Figure 3**

Now Try Exercises 31 and 49

**Mathematics in the Modern World**



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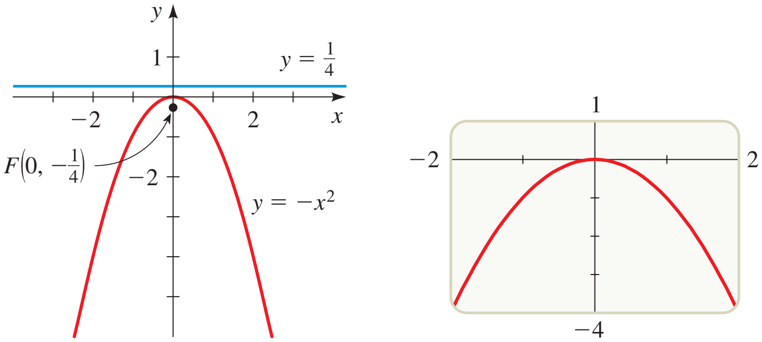
**Looking Inside Your Head**

Would you like to look inside your head? The idea isn't particularly appealing to most of us, but doctors often need to do just that. If they can look without invasive surgery, all the better. An X-ray doesn't really give a look inside, it simply gives a "graph" of the density of tissue the X-rays must pass through. So an X-ray is a "flattened" view in one direction. Suppose you get an X-ray view from many different directions. Can these "graphs" be used to reconstruct the three-dimensional inside view? This is a purely mathematical problem. The mathematician Johann Radon conceived of and solved this problem only because it was mathematically interesting to him. His solution, published in 1917, had no practical application at that time because reconstructing the inside view requires thousands of tedious computations. Today, high-speed computers make it possible to "look inside" by a process called computer-aided tomography (CAT scan). The first CAT scan machine was invented by Allan Cormack and Sir Godfrey Hounsfield in 1963 and was based on Radon's work. Mathematicians continue to search for better ways of using mathematics to reconstruct images. One of the latest techniques, called magnetic resonance imaging (MRI), combines molecular biology and mathematics for a clear "look inside."

## Example 2 ■ Finding the Focus and Directrix of a Parabola from Its Equation

Find the focus and directrix of the parabola $y = -x^2$, and sketch the graph.

**Solution** To find the focus and directrix, we put the given equation in the standard form $x^2 = -y$. Comparing this to the equation $x^2 = 4py$, we see that $4p = -1$, so $p = -\frac{1}{4}$. Thus the focus is $F\left(0, -\frac{1}{4}\right)$, and the directrix is $y = \frac{1}{4}$. The graph of the parabola, together with the focus and the directrix, is shown in Figure 4(a). We can also draw the graph using a graphing device, as shown in Figure 4(b).




(a) (b)

**Figure 4**

Now Try Exercise 11 ■

Reflecting the graph in Figure 2 about the diagonal line $y = x$ has the effect of interchanging the roles of $x$ and $y$. This results in a parabola with horizontal axis and vertex at the origin, with standard equation $y^2 = 4px$.

### Parabola with Horizontal Axis

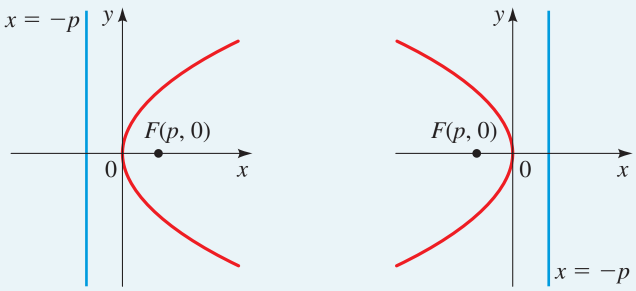
The graph of the equation

$$y^2 = 4px$$

is a parabola with the following properties.

| | |
|---|---|
| VERTEX | $V(0, 0)$ |
| FOCUS | $F(p, 0)$ |
| DIRECTRIX | $x = -p$ |

The parabola opens to the right if $p > 0$ or to the left if $p < 0$.




$y^2 = 4px$ with $p > 0$ $y^2 = 4px$ with $p < 0$

## Example 3 ■ A Parabola with Horizontal Axis

A parabola has the equation $6x + y^2 = 0$.

**(a)** Find the focus and directrix of the parabola, and sketch the graph.

**(b)** Use a graphing device to draw the graph.

### Solution

**(a)** To find the focus and directrix, we put the given equation in the standard form $y^2 = -6x$. Comparing this to the equation $y^2 = 4px$, we see that $4p = -6$, so $p = -\frac{3}{2}$. Thus the focus is $F\left(-\frac{3}{2}, 0\right)$, and the directrix is $x = \frac{3}{2}$. Since $p < 0$, the parabola opens to the left. The graph of the parabola, together with the focus and the directrix, is shown in Figure 5(a).

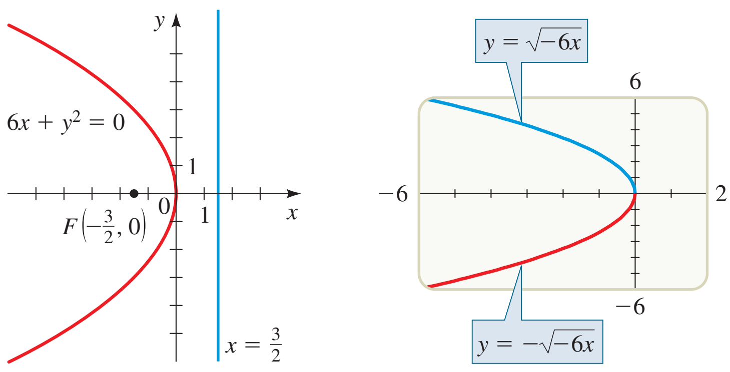
Equations that define functions are discussed in Section 2.2.

**(b)** The equation $6x + y^2 = 0$ does not define $y$ as a function of $x$, but most graphing devices can draw the graph of this equation, as shown in Figure 5(b).

**Using a Graphing Calculator** Many graphing calculators can only graph equations that define $y$ as a function of $x$. To use such a graphing calculator we first solve for $y$.

$$\begin{aligned} 6x + y^2 &= 0 \\ y^2 &= -6x && \text{Subtract } 6x \\ y &= \pm\sqrt{-6x} && \text{Take square roots} \end{aligned}$$

The graph of the parabola in Figure 5(b) is obtained by graphing both functions $y = \sqrt{-6x}$ and $y = -\sqrt{-6x}$.




(a) (b)

**Figure 5**

**Now Try Exercises 13 and 27** ■

We can use the coordinates of the focus to estimate the "width" of a parabola when sketching its graph. The line segment that runs through the focus perpendicular to the axis, with endpoints on the parabola, is called the **latus rectum**, and its length is the **focal diameter** of the parabola. From Figure 6 we can see that the distance from an endpoint $Q$ of the latus rectum to the directrix is $|2p|$. Thus the distance from $Q$ to the focus must be $|2p|$ as well (by the definition of a parabola), so the focal diameter is $|4p|$. In the next example we use the focal diameter to determine the "width" of a parabola when graphing it.

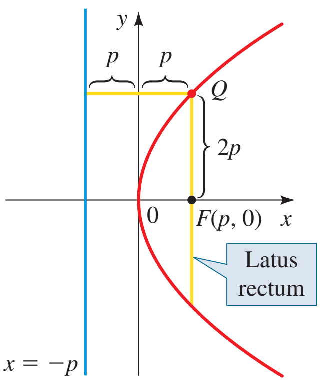



**Figure 6**

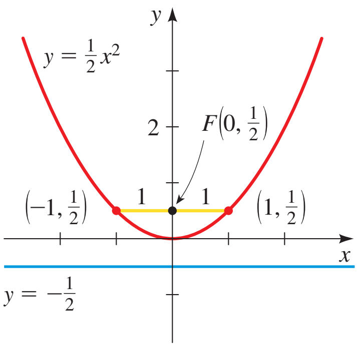



**Figure 7**

## Example 4 ■ The Focal Diameter of a Parabola

Find the focus, directrix, and focal diameter of the parabola $y = \frac{1}{2}x^2$, and sketch its graph.

**Solution** We first put the equation in the standard form $x^2 = 4py$.

$$y = \tfrac{1}{2}x^2$$

$$x^2 = 2y \qquad \text{Multiply by 2, switch sides}$$

From this equation we see that $4p = 2$, so the focal diameter is 2. Solving for $p$ gives $p = \frac{1}{2}$, so the focus is $\left(0, \frac{1}{2}\right)$, and the directrix is $y = -\frac{1}{2}$. Since the focal diameter is 2, the latus rectum extends 1 unit to the left and 1 unit to the right of the focus. The graph is sketched in Figure 7.

**Now Try Exercise 15** ■

In the next example we graph a family of parabolas to show how changing the distance between the focus and the vertex affects the "width" of a parabola.

## Example 5 ■ A Family of Parabolas

**(a)** Find equations for the parabolas with vertex at the origin and foci $F_1\left(0, \frac{1}{8}\right)$, $F_2\left(0, \frac{1}{2}\right)$, $F_3(0, 1)$, and $F_4(0, 4)$.

**(b)** Draw the graphs of the parabolas in part (a). What do you conclude?

**Solution**

**(a)** Since the foci are on the positive $y$-axis, the parabolas open upward and have equations of the form $x^2 = 4py$. This leads to the following equations.

| Focus | $p$ | Equation $x^2 = 4py$ | Form of the Equation for Graphing Calculator |
|---|---|---|---|
| $F_1\left(0, \frac{1}{8}\right)$ | $p = \frac{1}{8}$ | $x^2 = \frac{1}{2}y$ | $y = 2x^2$ |
| $F_2\left(0, \frac{1}{2}\right)$ | $p = \frac{1}{2}$ | $x^2 = 2y$ | $y = 0.5x^2$ |
| $F_3(0, 1)$ | $p = 1$ | $x^2 = 4y$ | $y = 0.25x^2$ |
| $F_4(0, 4)$ | $p = 4$ | $x^2 = 16y$ | $y = 0.0625x^2$ |

**(b)** The graphs are drawn in Figure 8. We see that the closer the focus is to the vertex, the narrower the parabola.

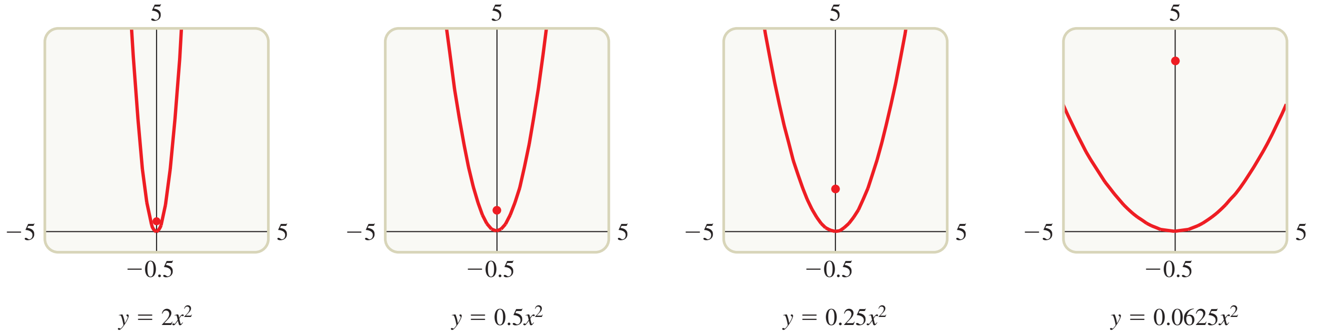



**Figure 8** | A family of parabolas

**Now Try Exercise 59** ■

## Applications

Parabolas have a property that makes them useful as reflectors for lamps and telescopes. Light from a source placed at the focus of a surface with parabolic cross section will be reflected in such a way that it travels parallel to the axis of the parabola (see Figure 9). Thus a parabolic mirror reflects the light into a beam of parallel rays. Conversely, light approaching the reflector in rays parallel to its axis of symmetry is concentrated to the focus. This *reflection property*, which can be proved by using calculus, is used in the construction of reflecting telescopes.

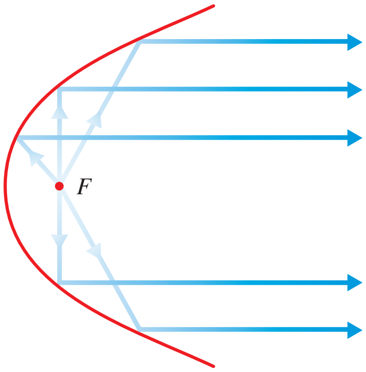



**Figure 9** | Parabolic reflector

### Example 6 ■ Finding the Focal Point of a Searchlight Reflector

A searchlight has a parabolic reflector that forms a "bowl," which is 12 in. wide from rim to rim and 8 in. deep, as shown in Figure 10. If the filament of the light bulb is located at the focus, how far from the vertex of the reflector is it?

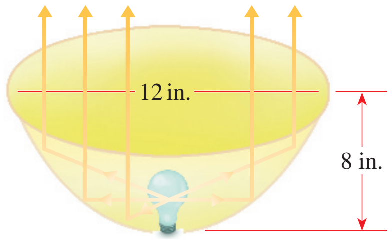



**Figure 10** | A parabolic reflector

**ARCHIMEDES** (287–212 B.C.) was the greatest mathematician of the ancient world. He was born in Syracuse, a Greek colony on Sicily, a generation after Euclid (see Section 7.1). One of his many discoveries is the Law of the Lever (see Exercise 1.7.77). He famously said, "Give me a place to stand and a fulcrum for my lever, and I can lift the earth."

Renowned as a mechanical genius for his many engineering inventions, he designed pulleys for lifting heavy ships and the spiral screw for transporting water to higher levels. He is said to have used parabolic mirrors to concentrate the rays of the sun to set fire to Roman ships attacking Syracuse.

King Hieron II of Syracuse once suspected a goldsmith of keeping part of the gold intended for the king's crown and replacing it with an equal amount of silver. The king asked Archimedes for advice. While in deep thought at a public bath, Archimedes discovered the solution to the king's problem when he noticed that his body's volume was the same as the volume of water it displaced from the tub. Using this insight, he was able to measure the volume of the crown and so determine what the weight of an all-gold crown should be (Exercise 9.1.74). As the story is told, he ran home, forgetting that he was naked, shouting, "Eureka, eureka!" ("I have found it, I have found it!") This incident attests to his enormous powers of concentration.

In spite of his engineering prowess, Archimedes was most proud of his mathematical discoveries. These include the formulas for the volume of a sphere, $\left(V = \frac{4}{3}\pi r^3\right)$ and the surface area of a sphere $\left(S = 4\pi r^2\right)$ and a careful analysis of the properties of parabolas and other conics.

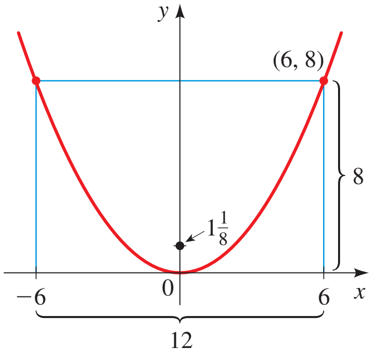



**Figure 11**

**Solution** We introduce a coordinate system and place a parabolic cross section of the reflector so that its vertex is at the origin and its axis is vertical (see Figure 11). Then the equation of this parabola has the form $x^2 = 4py$. From Figure 11 we see that the point $(6, 8)$ lies on the parabola. We use this to find $p$.

$$6^2 = 4p(8) \qquad \text{The point } (6, 8) \text{ satisfies the equation } x^2 = 4py$$

$$36 = 32p$$

$$p = \tfrac{9}{8}$$

The focus is $F\left(0, \frac{9}{8}\right)$, so the distance between the vertex and the focus is $\frac{9}{8} = 1\frac{1}{8}$ in. Because the filament is positioned at the focus, it is located $1\frac{1}{8}$ in. from the vertex of the reflector.

**Now Try Exercise 61** ■

## 10.1 Exercises

### Concepts

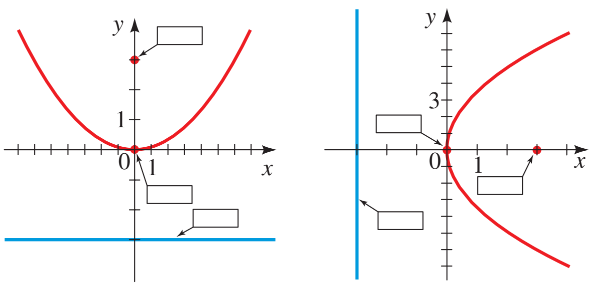
**1.** A parabola is the set of all points in the plane that are equidistant from a fixed point called the ________ and a fixed line called the ________ of the parabola.

**2.** The graph of the equation $x^2 = 4py$ is a parabola with focus $F($___, ___$)$, directrix $y =$ ________, and ________ (horizontal/vertical) axis. So the graph of $x^2 = 12y$ is a parabola with focus $F($___, ___$)$ and directrix $y =$ ________.

**3.** The graph of the equation $y^2 = 4px$ is a parabola with focus $F($___, ___$)$, directrix $x =$ ________, and ________ (horizontal/vertical) axis. So the graph of $y^2 = 12x$ is a parabola with focus $F($___, ___$)$ and directrix $x =$ ________.

**4.** Label the focus, directrix, and vertex on the graphs given for the parabolas in Exercises 2 and 3.

**(a)** $x^2 = 12y$ **(b)** $y^2 = 12x$




### Skills

**5–10 ■ Graphs of Parabolas** Match the equation with the graphs labeled I–VI. Give reasons for your answers.

**5.** $y^2 = 2x$

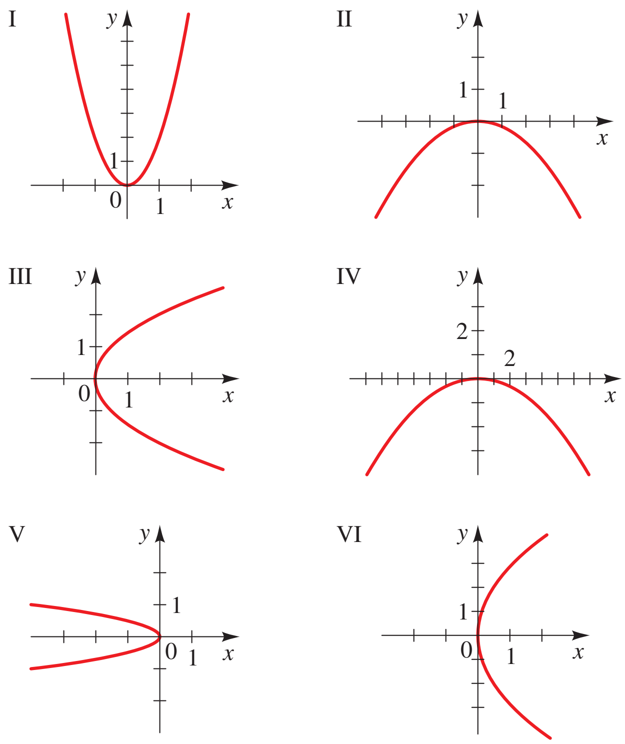
**6.** $y^2 = -\frac{1}{4}x$

**7.** $x^2 = -6y$

**8.** $2x^2 = y$

**9.** $y^2 - 8x = 0$

**10.** $12y + x^2 = 0$




**11–24 ■ Graphing Parabolas** An equation of a parabola is given. **(a)** Find the focus, directrix, and focal diameter of the parabola. **(b)** Sketch a graph of the parabola and its directrix.

**11.** $x^2 = 16y$

**12.** $x^2 = -8y$

**13.** $y^2 = -4x$

**14.** $y^2 = 24x$

**15.** $x = \frac{1}{16}y^2$

**16.** $y = \frac{1}{2}x^2$

**17.** $y = -2x^2$

**18.** $x = -\frac{1}{12}y^2$

**19.** $5y = x^2$

**20.** $9x = y^2$

**21.** $x^2 + 12y = 0$

**22.** $x + \frac{1}{5}y^2 = 0$

**23.** $5x + 3y^2 = 0$

**24.** $8x^2 + 12y = 0$

**25–30 ■ Graphing Parabolas** Use a graphing device to graph the parabola.

**25.** $x^2 = 20y$

**26.** $x^2 = -8y$

**27.** $y^2 = -\frac{1}{3}x$

**28.** $8y^2 = x$

**29.** $4x + y^2 = 0$

**30.** $x - 2y^2 = 0$

**31–48 ■ Finding the Equation of a Parabola** Find the standard equation for the parabola that has its vertex at the origin and satisfies the given condition(s).

**31.** Focus: $F(0, 3)$

**32.** Focus: $F\left(0, -\frac{1}{8}\right)$

**33.** Focus: $F(-8, 0)$

**34.** Focus: $F(5, 0)$

**35.** Focus: $F\left(0, -\frac{3}{4}\right)$

**36.** Focus: $F\left(-\frac{1}{12}, 0\right)$

**37.** Directrix: $x = -2$

**38.** Directrix: $y = \frac{1}{4}$

**39.** Directrix: $y = \frac{1}{10}$

**40.** Directrix: $x = -\frac{1}{8}$

**41.** Directrix: $x = \frac{1}{20}$

**42.** Directrix: $y = -5$

**43.** Focus on the positive $x$-axis, 2 units away from the directrix

**44.** Focus on the negative $y$-axis, 6 units away from the directrix

**45.** Opens downward with focus 10 units away from the vertex

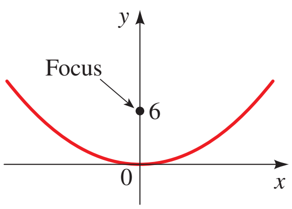
**46.** Opens upward with focus 5 units away from the vertex

**47.** Directrix has $y$-intercept 6

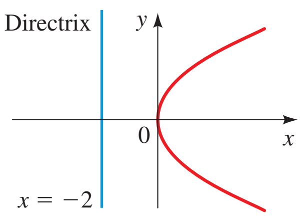
**48.** Focal diameter 8 and focus on the negative $y$-axis

**49–58 ■ Finding the Equation of a Parabola** Find the standard equation of the parabola whose graph is shown.

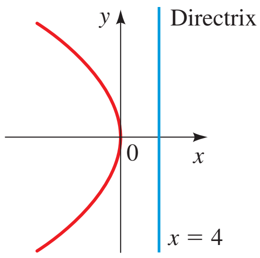
**49.**



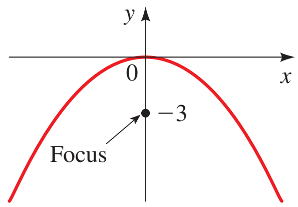

**50.**



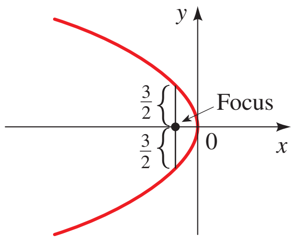

**51.**



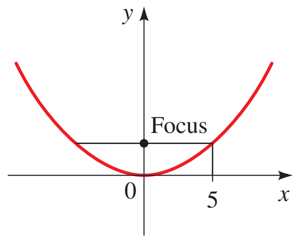

**52.**



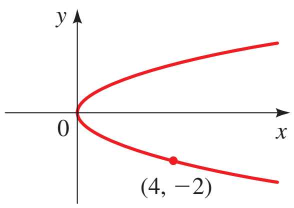

**53.**



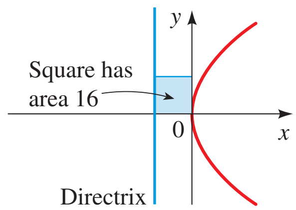

**54.**



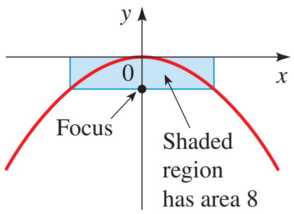

**55.**



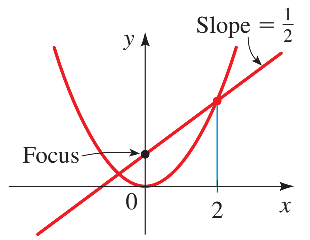

**56.**




**57.**




**58.**




**59–60 ■ Families of Parabolas** **(a)** Find equations for the family of parabolas with the given description. **(b)** Draw the graphs. What do you conclude?

**59.** The family of parabolas with vertex at the origin and with directrixes $y = \frac{1}{2}$, $y = 1$, $y = 4$, and $y = 8$

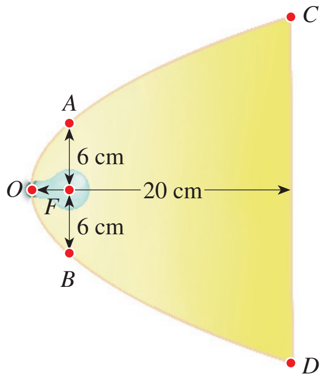
**60.** The family of parabolas with vertex at the origin, focus on the positive $y$-axis, and with focal diameters 1, 2, 4, and 8

## Applications

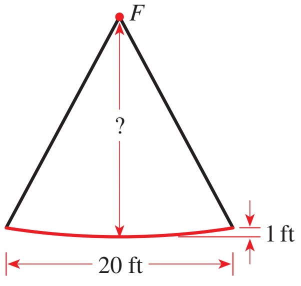
**61. Parabolic Reflector** A lamp with a parabolic reflector is shown in the figure. The bulb is placed at the focus, and the focal diameter is 12 cm.

**(a)** Find the standard equation of the parabola.

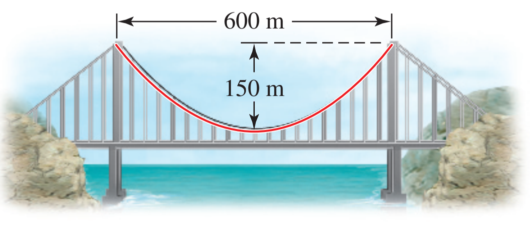
**(b)** Find the diameter $d(C, D)$ of the opening, 20 cm from the vertex.



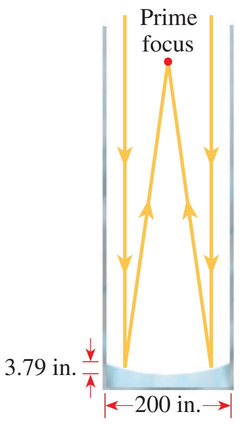

**62. Satellite Dish** A reflector for a satellite dish is parabolic in cross section, with the receiver at the focus $F$. The reflector is 1 ft deep and 20 ft wide from rim to rim (see the figure). How far is the receiver from the vertex of the parabolic reflector?




**63. Suspension Bridge** In a suspension bridge the shape of the suspension cables is parabolic. The bridge shown in the figure has towers that are 600 m apart, and the lowest point of the suspension cables is 150 m below the top of the towers. Find the equation of the parabolic part of the cables, placing the origin of the coordinate system at the vertex. [*Note*: This equation is used to find the length of cable needed in the construction of the bridge.]



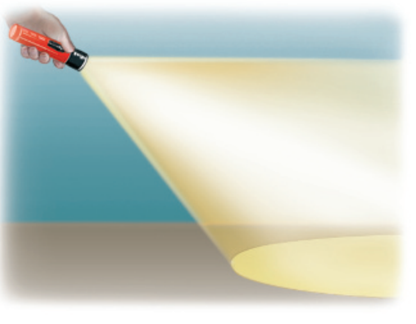

**64. Reflecting Telescope** The Hale telescope at the Mount Palomar Observatory has a 200-inch mirror, as shown in the figure. The mirror is constructed in a parabolic shape that collects light from the stars and focuses it at the *prime focus*, that is, the focus of the parabola. The mirror is 3.79 in. deep at its center. Find the **focal length** of this parabolic mirror, that is, the distance from the vertex to the focus.




## ■ Discuss ■ Discover ■ Prove ■ Write

**65. Discuss ■ Write: Parabolas in the Real World** Several examples of the uses of parabolas are given in the text. Find other situations in which parabolas occur.

**66. Discuss: Light Cone from a Flashlight** A flashlight is held to form a lighted area on the ground, as shown in the figure. Is it possible to angle the flashlight in such a way that the boundary of the lighted area is a parabola? Explain your answer.



# 10.2 Ellipses

■ Geometric Definition of an Ellipse ■ Equations and Graphs of Ellipses
■ Eccentricity of an Ellipse

## ■ Geometric Definition of an Ellipse

An ellipse is an oval curve that looks like an elongated circle. More precisely, we have the following definition.

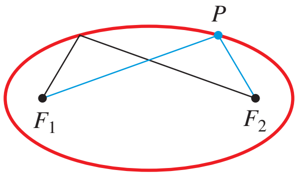



**Figure 1**

### Geometric Definition of an Ellipse

An **ellipse** is the set of all points in the plane the sum of whose distances from two fixed points $F_1$ and $F_2$ is a constant. (See Figure 1.) These two fixed points are the **foci** (plural of **focus**) of the ellipse.

The geometric definition suggests a simple method for drawing an ellipse. Place a sheet of paper on a drawing board, and insert thumbtacks at the two points that are to be the foci of the ellipse. Attach the ends of a string to the tacks, as shown in Figure 2(a). With the point of a pencil, hold the string taut. Then carefully move the pencil around the foci, keeping the string taut at all times. The pencil will trace out an ellipse, because the sum of the distances from the point of the pencil to the foci will always equal the length of the string, which is constant.

If the string is only slightly longer than the distance between the foci, then the ellipse that is traced out will be elongated in shape, as in Figure 2(a), but if the foci are close together relative to the length of the string, the ellipse will be almost circular, as shown in Figure 2(b).

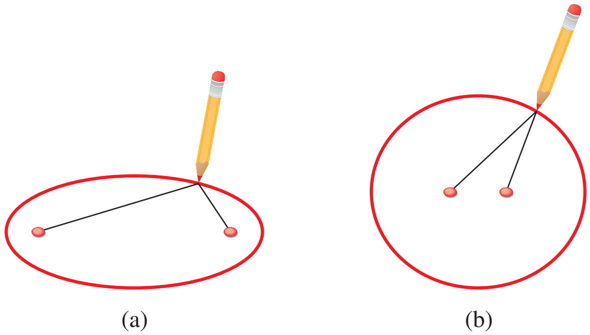



**Figure 2**

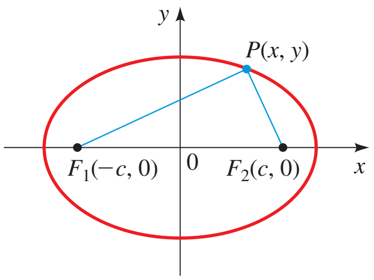



**Figure 3**

**Deriving the Equation of an Ellipse** To obtain the simplest equation for an ellipse, we place the foci on the $x$-axis at $F_1(-c, 0)$ and $F_2(c, 0)$ so that the origin is halfway between them (see Figure 3).

For later convenience we let the sum of the distances from a point on the ellipse to the foci be $2a$. Then if $P(x, y)$ is any point on the ellipse, we have

$$d(P, F_1) + d(P, F_2) = 2a$$

So from the Distance Formula we have

$$\sqrt{(x + c)^2 + y^2} + \sqrt{(x - c)^2 + y^2} = 2a$$

or

$$\sqrt{(x - c)^2 + y^2} = 2a - \sqrt{(x + c)^2 + y^2}$$

Squaring each side and expanding, we get

$$x^2 - 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + (x^2 + 2cx + c^2 + y^2)$$

which simplifies to

$$4a\sqrt{(x+c)^2 + y^2} = 4a^2 + 4cx$$

Dividing each side by 4 and squaring again, we get

$$a^2[(x+c)^2 + y^2] = (a^2 + cx)^2$$

$$a^2x^2 + 2a^2cx + a^2c^2 + a^2y^2 = a^4 + 2a^2cx + c^2x^2$$

$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2)$$

Since the sum of the distances from $P$ to the foci must be larger than the distance between the foci, we have that $2a > 2c$, or $a > c$. Thus $a^2 - c^2 > 0$; if we divide each side of the preceding equation by $a^2(a^2 - c^2)$, we get

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1$$

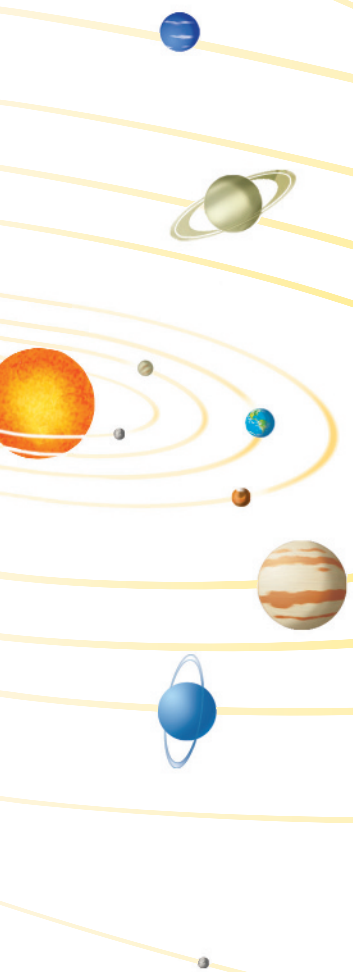
For convenience, let $b^2 = a^2 - c^2$ (with $b > 0$). Since $b^2 < a^2$, it follows that $b < a$. The preceding equation then becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (a > b)$$

This is the **standard equation** of an ellipse with foci on the $x$-axis and center at the origin. To graph it, we need to know the $x$- and $y$-intercepts. Setting $y = 0$, we get

$$\frac{x^2}{a^2} = 1$$

so $x^2 = a^2$, or $x = \pm a$. Thus the ellipse crosses the $x$-axis at $(a, 0)$ and $(-a, 0)$, as shown in Figure 4. These points are called the **vertices** of the ellipse, and the segment that joins them is called the **major axis**. Its length is $2a$.



The orbits of the planets are ellipses, with the sun at one focus.

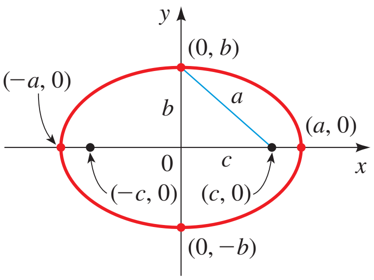



**Figure 4** | $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ with $a > b$

Similarly, if we set $x = 0$, we get $y = \pm b$, so the ellipse crosses the $y$-axis at $(0, b)$ and $(0, -b)$. The segment that joins these points is called the **minor axis**, and it has length $2b$. Note that $2a > 2b$, so the major axis is longer than the minor axis. The origin is the **center** of the ellipse.

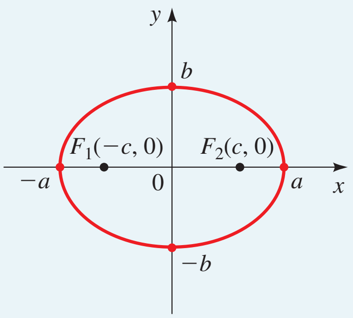
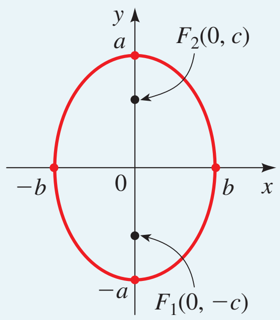
If the foci of the ellipse are placed on the $y$-axis at $(0, \pm c)$ rather than on the $x$-axis, then the roles of $x$ and $y$ are reversed in the preceding discussion, and we get a vertical ellipse.

## Equations and Graphs of Ellipses

The following box summarizes what we have just proved about ellipses centered at the origin.

### Ellipse with Center at the Origin

The graph of each of the following equations is an ellipse with center at the origin and having the given properties.

| | | |
|---|---|---|
| **EQUATION** | $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ <br> $a > b > 0$ | $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1$ <br> $a > b > 0$ |
| **VERTICES** | $(\pm a, 0)$ | $(0, \pm a)$ |
| **MAJOR AXIS** | Horizontal, length $2a$ | Vertical, length $2a$ |
| **MINOR AXIS** | Vertical, length $2b$ | Horizontal, length $2b$ |
| **FOCI** | $(\pm c, 0)$, $c^2 = a^2 - b^2$ | $(0, \pm c)$, $c^2 = a^2 - b^2$ |
| **GRAPH** |  |  |





In the standard equation for an ellipse, $a^2$ is the *larger* denominator, and $b^2$ is the *smaller*. To find $c^2$, we subtract: larger denominator minus smaller denominator.

## Example 1 ■ Sketching an Ellipse

An ellipse has the standard equation

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

**(a)** Find the foci, the vertices, and the lengths of the major and minor axes, and sketch the graph.

**(b)** Draw the graph using a graphing device.

### Solution

**(a)** Since the denominator of $x^2$ is larger, the ellipse has a horizontal major axis. This gives $a^2 = 9$ and $b^2 = 4$, so $c^2 = a^2 - b^2 = 9 - 4 = 5$. Thus $a = 3$, $b = 2$, and $c = \sqrt{5}$.

| | |
|---|---|
| **FOCI** | $(\pm\sqrt{5}, 0)$ |
| **VERTICES** | $(\pm 3, 0)$ |
| **LENGTH OF MAJOR AXIS** | 6 |
| **LENGTH OF MINOR AXIS** | 4 |

The graph is shown in Figure 5(a) on the next page.

**(b)** Most graphing devices can draw the graph of this equation, as shown in Figure 5(b).

**Using a Graphing Calculator** To graph the equation we first solve for $y$.

Note that the equation of an ellipse does not define $y$ as a function of $x$ (see Section 2.2).

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

$$y^2 = 4\left(1 - \frac{x^2}{9}\right) \qquad \text{Subtract } \frac{x^2}{9}\text{, multiply by 4}$$

$$y = \pm 2\sqrt{1 - \frac{x^2}{9}} \qquad \text{Take square roots}$$

The graph of the ellipse in Figure 5(b) is obtained by graphing both functions $y = 2\sqrt{1 - x^2/9}$ and $y = -2\sqrt{1 - x^2/9}$.

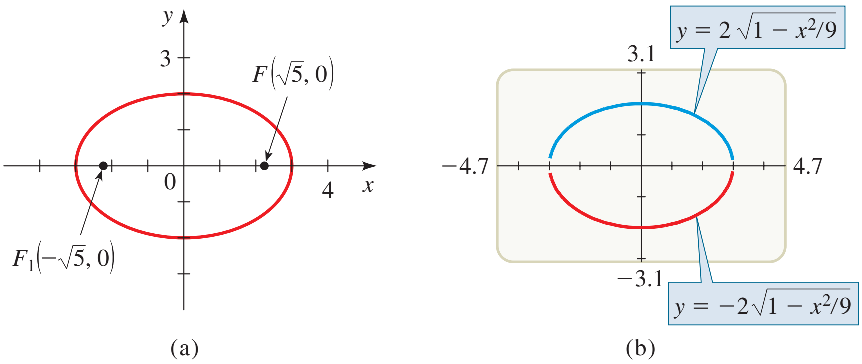



**Figure 5** | $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

**Now Try Exercises 9 and 35** ■

## Example 2 ■ Finding the Foci of an Ellipse

Find the foci of the ellipse $16x^2 + 9y^2 = 144$, and sketch its graph.

**Solution** First we put the equation in standard form. Dividing by 144, we get

$$\frac{x^2}{9} + \frac{y^2}{16} = 1$$

Since $16 > 9$, this is an ellipse with its foci on the $y$-axis and with $a = 4$ and $b = 3$:

$$c^2 = a^2 - b^2 = 16 - 9 = 7$$

$$c = \sqrt{7}$$

Thus the foci are $(0, \pm\sqrt{7})$. The graph is shown in Figure 6(a). We can also draw the graph using a graphing device as shown in Figure 6(b).

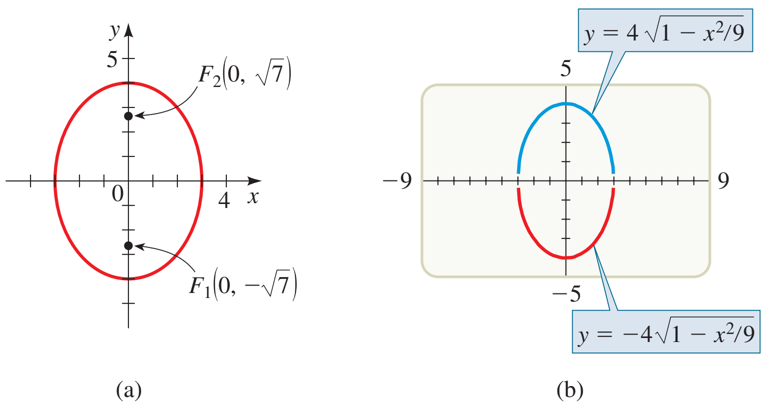



**Figure 6** | $16x^2 + 9y^2 = 144$

**Now Try Exercises 15 and 37** ■

## Example 3 ■ Finding the Equation of an Ellipse

The vertices of an ellipse are $(\pm 4, 0)$, and the foci are $(\pm 2, 0)$. Find the standard equation of the ellipse, and sketch the graph.

**Solution** Since the vertices are $(\pm 4, 0)$, we have $a = 4$, and the major axis is horizontal. The foci are $(\pm 2, 0)$, so $c = 2$. To write the equation, we need to find $b$. Since $c^2 = a^2 - b^2$, we have

$$2^2 = 4^2 - b^2$$

$$b^2 = 16 - 4 = 12$$

Thus the standard equation of the ellipse is

$$\frac{x^2}{16} + \frac{y^2}{12} = 1$$

The graph is shown in Figure 7.

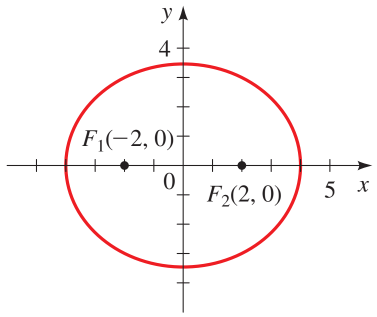



**Figure 7** | $\dfrac{x^2}{16} + \dfrac{y^2}{12} = 1$

Now Try Exercises 31 and 39 ■

## ■ Eccentricity of an Ellipse

We saw earlier in this section (Figure 2) that if $2a$ is only slightly greater than $2c$, the ellipse is long and thin, whereas if $2a$ is much greater than $2c$, the ellipse is almost circular. We measure the deviation of an ellipse from being circular by the ratio of $a$ and $c$.

### Definition of Eccentricity

For the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ or $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1$ (with $a > b > 0$), the **eccentricity *e*** is the number

$$e = \frac{c}{a}$$

where $c = \sqrt{a^2 - b^2}$. The eccentricity of every ellipse satisfies $0 < e < 1$.

If $a = b$ in the equation of an ellipse, then

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1$$

so $x^2 + y^2 = a^2$. In this case the "ellipse" is a circle with radius $a$ and eccentricity 0.

Thus if $e$ is close to 1, then $c$ is almost equal to $a$, and the ellipse is elongated in shape, but if $e$ is close to 0, then the ellipse is close to a circle in shape. The eccentricity is a measure of how "stretched" the ellipse is.

In Figure 8 we show a number of ellipses to demonstrate the effect of varying the eccentricity $e$.

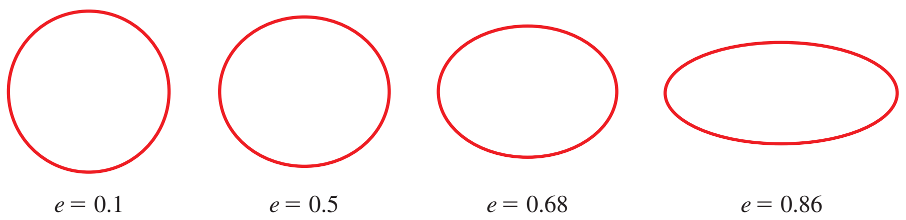



**Figure 8** | Ellipses with various eccentricities

## Example 4 ■ Finding the Equation of an Ellipse from Its Eccentricity and Foci

Find the standard equation of the ellipse with foci $(0, \pm 8)$ and eccentricity $e = \frac{4}{5}$, and sketch its graph.

**Solution** We are given $e = \frac{4}{5}$ and $c = 8$. Thus

$$\frac{4}{5} = \frac{8}{a} \qquad \text{Eccentricity } e = \frac{c}{a}$$

$$4a = 40 \qquad \text{Cross-multiply}$$

$$a = 10$$

To find $b$, we use the fact that $c^2 = a^2 - b^2$.

$$8^2 = 10^2 - b^2$$

$$b^2 = 10^2 - 8^2 = 36$$

$$b = 6$$

Thus the standard equation of the ellipse is

$$\frac{x^2}{36} + \frac{y^2}{100} = 1$$

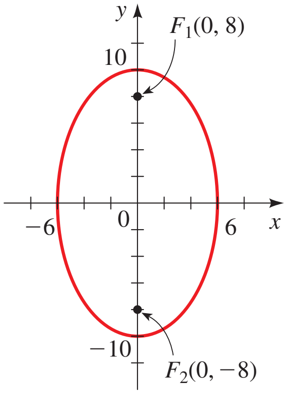



**Figure 9** | $\frac{x^2}{36} + \frac{y^2}{100} = 1$

Because the foci are on the $y$-axis, the ellipse is oriented vertically. To sketch the ellipse, we find the intercepts. The $x$-intercepts are $\pm 6$, and the $y$-intercepts are $\pm 10$. The graph is sketched in Figure 9.

**Now Try Exercise 53** ■

**Eccentricities of the Orbits of the Planets**

The orbits of the planets are ellipses with the sun at one focus. For most planets these ellipses have very small eccentricity, so they are nearly circular. However, Mercury and Pluto, the innermost and outermost known planets, respectively, have visibly elliptical orbits.

| Planet | Eccentricity |
|---|---|
| Mercury | 0.206 |
| Venus | 0.007 |
| Earth | 0.017 |
| Mars | 0.093 |
| Jupiter | 0.048 |
| Saturn | 0.056 |
| Uranus | 0.046 |
| Neptune | 0.010 |
| Pluto* | 0.248 |

*Pluto is a "dwarf planet."

Gravitational attraction causes the planets to move in elliptical orbits around the sun with the sun at one focus. This remarkable property was first observed by Johannes Kepler and was later deduced by Isaac Newton from his inverse square Law of Gravity, using calculus. The orbits of the planets have different eccentricities, but most are nearly circular (see the margin).

Ellipses, like parabolas, have an interesting *reflection property* that leads to a number of practical applications. If a light source is placed at one focus of a reflecting surface with elliptical cross sections, then all the light will be reflected off the surface to the other focus, as shown in Figure 10. This principle, which works for sound waves as well as for light, is used in *lithotripsy*, a treatment for kidney stones. The patient is placed in a tub of water with elliptical cross sections in such a way that the kidney stone is accurately located at one focus. High-intensity sound waves generated at the other focus are reflected to the stone and destroy it with minimal damage to surrounding tissue. The patient is spared the trauma of surgery and recovers within days instead of weeks.

The reflection property of ellipses is also used in the construction of *whispering galleries*. Sound coming from one focus bounces off the walls and ceiling of an elliptical room and passes through the other focus. In these rooms even quiet whispers spoken at one focus can be heard clearly at the other. Famous whispering galleries include the Mormon Tabernacle in Salt Lake City, Utah, and the National Statuary Hall of the US Capitol in Washington, D.C. (See the Focus on Modeling *Conics in Architecture* that follows this chapter.)

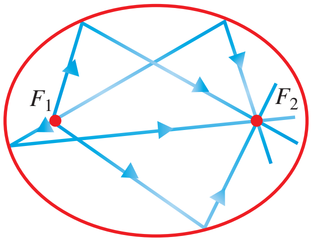



**Figure 10**

## 10.2 Exercises

### Concepts

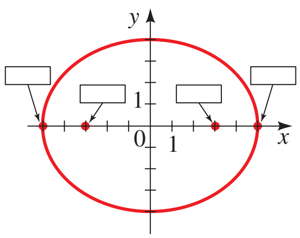
**1.** An ellipse is the set of all points in the plane for which the ________ of the distances from two fixed points $F_1$ and $F_2$ is constant. The points $F_1$ and $F_2$ are called the ________ of the ellipse.

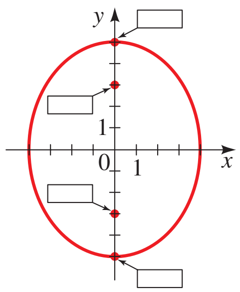
**2.** The graph of the equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ with $a > b > 0$ is an ellipse with ________ (horizontal/vertical) major axis, vertices (___, ___) and (___, ___) and foci $(\pm c, 0)$, where $c =$ ________. So the graph of $\dfrac{x^2}{5^2} + \dfrac{y^2}{4^2} = 1$ is an ellipse with vertices (___, ___) and (___, ___) and foci (___, ___) and (___, ___).

**3.** The graph of the equation $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1$ with $a > b > 0$ is an ellipse with ________ (horizontal/vertical) major axis, vertices (___, ___) and (___, ___) and foci $(0, \pm c)$, where $c =$ ________. So the graph of $\dfrac{x^2}{4^2} + \dfrac{y^2}{5^2} = 1$ is an ellipse with vertices (___, ___) and (___, ___) and foci (___, ___) and (___, ___).

**4.** Label the vertices and foci on the graphs given for the ellipses in Exercises 2 and 3.

**(a)** $\dfrac{x^2}{5^2} + \dfrac{y^2}{4^2} = 1$ **(b)** $\dfrac{x^2}{4^2} + \dfrac{y^2}{5^2} = 1$




### Skills

**5–8 ■ Graphs of Ellipses** Match the equation with the graphs labeled I–IV. Give reasons for your answers.

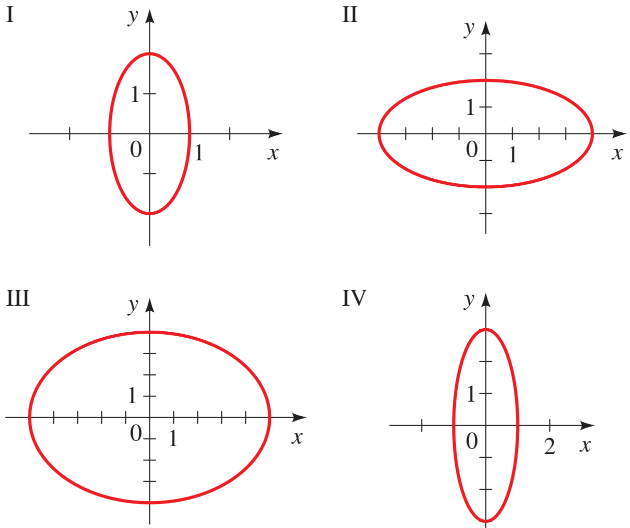
**5.** $\dfrac{x^2}{16} + \dfrac{y^2}{4} = 1$

**6.** $x^2 + \dfrac{y^2}{9} = 1$

**7.** $4x^2 + y^2 = 4$

**8.** $16x^2 + 25y^2 = 400$

*Graphs for Exercises 5–8:*




**9–28 ■ Graphing Ellipses** An equation of an ellipse is given. **(a)** Find the vertices, foci, and eccentricity of the ellipse. **(b)** Determine the lengths of the major and minor axes. **(c)** Sketch a graph of the ellipse.

**9.** $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$

**10.** $\dfrac{x^2}{16} + \dfrac{y^2}{25} = 1$

**11.** $\dfrac{x^2}{36} + \dfrac{y^2}{81} = 1$

**12.** $\dfrac{x^2}{4} + y^2 = 1$

**13.** $\dfrac{x^2}{49} + \dfrac{y^2}{25} = 1$

**14.** $\dfrac{x^2}{9} + \dfrac{y^2}{64} = 1$

**15.** $9x^2 + 4y^2 = 36$

**16.** $4x^2 + 25y^2 = 100$

**17.** $x^2 + 4y^2 = 16$

**18.** $4x^2 + y^2 = 16$

**19.** $16x^2 + 25y^2 = 1600$

**20.** $2x^2 + 49y^2 = 98$

**21.** $3x^2 + y^2 = 9$

**22.** $x^2 + 3y^2 = 9$

**23.** $2x^2 + y^2 = 4$

**24.** $3x^2 + 4y^2 = 12$

**25.** $x^2 + 4y^2 = 1$

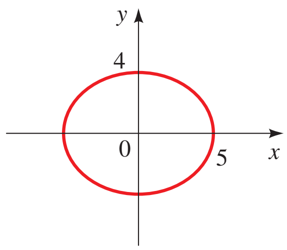
**26.** $9x^2 + 4y^2 = 1$

**27.** $x^2 = 4 - 2y^2$

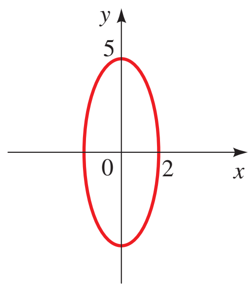
**28.** $y^2 = 1 - 2x^2$

**29–34 ■ Finding the Equation of an Ellipse** Find the standard equation for the ellipse whose graph is shown.

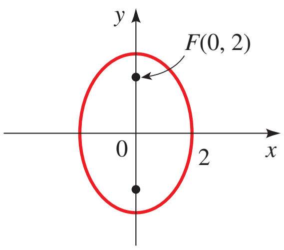
**29.**



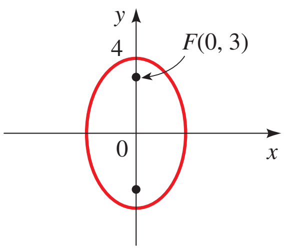

**30.**



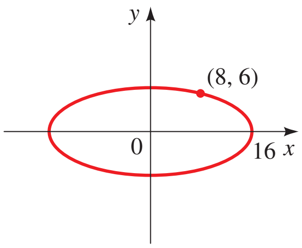
**31.**



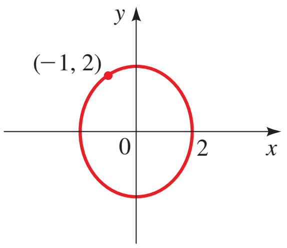

**32.**




**33.**




**34.**




**35–38 ■ Graphing Ellipses** Use a graphing device to graph the ellipse.

**35.** $\dfrac{x^2}{25} + \dfrac{y^2}{20} = 1$

**36.** $x^2 + \dfrac{y^2}{12} = 1$

**37.** $6x^2 + y^2 = 36$

**38.** $x^2 + 2y^2 = 8$

**39–56 ■ Finding the Equation of an Ellipse** Find the standard equation for the ellipse that satisfies the given conditions.

**39.** Foci: $(\pm 4, 0)$, vertices: $(\pm 5, 0)$

**40.** Foci: $(0, \pm 3)$, vertices: $(0, \pm 5)$

**41.** Foci: $F(\pm 1, 0)$, vertices: $(\pm 2, 0)$

**42.** Foci: $F(0, \pm 2)$, vertices: $(0, \pm 3)$

**43.** Foci: $F(0, \pm\sqrt{10})$, vertices: $(0, \pm 7)$

**44.** Foci: $F(\pm\sqrt{15}, 0)$, vertices: $(\pm 6, 0)$

**45.** Length of major axis: 4, length of minor axis: 2, foci on $y$-axis

**46.** Length of major axis: 6, length of minor axis: 4, foci on $x$-axis

**47.** Foci: $(0, \pm 2)$, length of minor axis: 6

**48.** Foci: $(\pm 5, 0)$, length of major axis: 12

**49.** Endpoints of major axis: $(\pm 10, 0)$, distance between foci: 6

**50.** Endpoints of minor axis: $(0, \pm 3)$, distance between foci: 8

**51.** Length of major axis: 10, foci on $x$-axis, ellipse passes through the point $(\sqrt{5}, 2)$

**52.** Length of minor axis: 10, foci on $y$-axis, ellipse passes through the point $(\sqrt{5}, \sqrt{40})$

**53.** Eccentricity: $\frac{1}{3}$, foci: $(0, \pm 2)$

**54.** Eccentricity: 0.75, foci: $(\pm 1.5, 0)$

**55.** Eccentricity: $\sqrt{3}/2$, foci on $y$-axis, length of major axis: 4

**56.** Eccentricity: $\sqrt{5}/3$, foci on $x$-axis, length of major axis: 12

## Skills Plus

**57–60 ■ Intersecting Ellipses** Find the intersection points of the pair of ellipses. Sketch the graphs of each pair of equations on the same coordinate axes, and label the points of intersection.

**57.** $\begin{cases} 4x^2 + y^2 = 4 \\ 4x^2 + 9y^2 = 36 \end{cases}$

**58.** $\begin{cases} \dfrac{x^2}{16} + \dfrac{y^2}{9} = 1 \\ \dfrac{x^2}{9} + \dfrac{y^2}{16} = 1 \end{cases}$

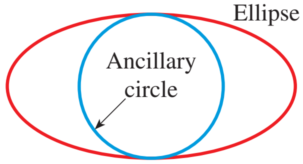
**59.** $\begin{cases} 100x^2 + 25y^2 = 100 \\ x^2 + \dfrac{y^2}{9} = 1 \end{cases}$

**60.** $\begin{cases} 25x^2 + 144y^2 = 3600 \\ 144x^2 + 25y^2 = 3600 \end{cases}$

**61. Ancillary Circle** The *ancillary circle* of an ellipse is the circle with radius equal to half the length of the minor axis and center the same as the ellipse (see the figure). The ancillary circle is thus the largest circle that can fit within an ellipse.

**(a)** Find an equation for the ancillary circle of the ellipse $x^2 + 4y^2 = 16$.

**(b)** For the ellipse and ancillary circle of part (a), show that if $(s, t)$ is a point on the ancillary circle, then $(2s, t)$ is a point on the ellipse.




**62. Family of Ellipses**

**(a)** Use a graphing device to sketch the top half (the portion in the first and second quadrants) of the family of ellipses $x^2 + ky^2 = 100$ for $k = 4$, 10, 25, and 50.

**(b)** What do the members of this family of ellipses have in common? How do they differ?

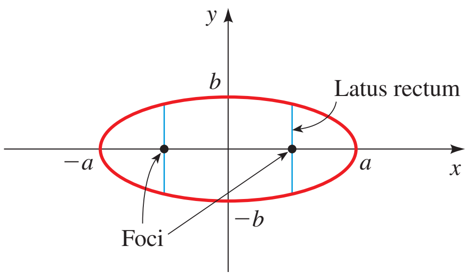
**63. Family of Ellipses** If $k > 0$, the following equation represents an ellipse:

$$\frac{x^2}{k} + \frac{y^2}{4 + k} = 1$$

Show that all the ellipses represented by this equation have the same foci, no matter what the value of $k$.

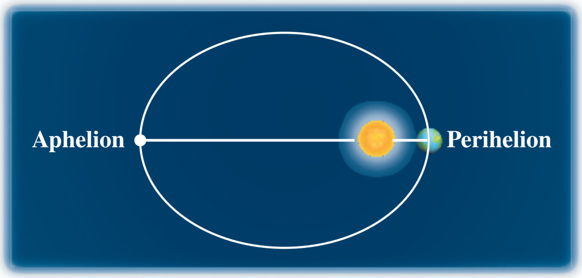
**64. How Wide Is an Ellipse at a Focus?** A *latus rectum* for an ellipse is a line segment perpendicular to the major axis at a focus, with endpoints on the ellipse, as shown in the figure. Show that the length of a latus rectum is $2b^2/a$ for the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (a > b)$$



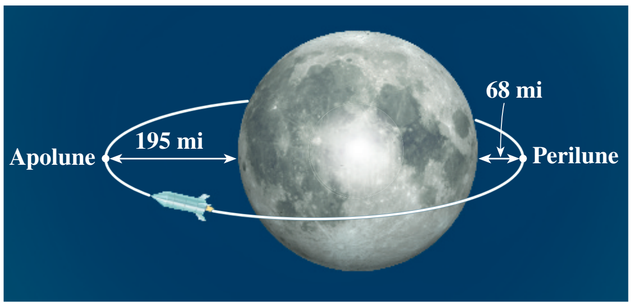
## Applications

**65. Perihelion and Aphelion** The planets move around the sun in elliptical orbits with the sun at one focus. The point in the orbit at which the planet is closest to the sun is called *perihelion*, and the point at which it is farthest is called *aphelion*. These points are the vertices of the orbit. The earth's distance from the sun is 147,000,000 km at perihelion and 153,000,000 km at aphelion. Find an equation for the earth's orbit. (Place the origin at the center of the orbit with the sun on the $x$-axis.)

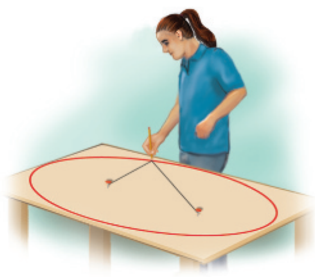



**66. The Orbit of Pluto** With an eccentricity of 0.25, Pluto's orbit is the most eccentric in the solar system. The length of the minor axis of its orbit is approximately 10,000,000,000 km. Find the distance between Pluto and the sun at perihelion and at aphelion. (See Exercise 65.)

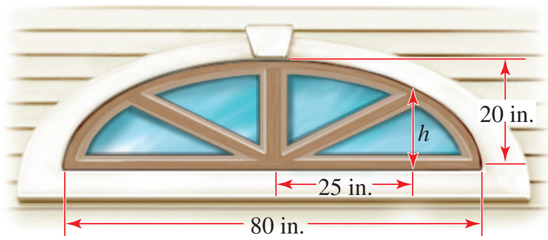
**67. Lunar Orbit** For an object in an elliptical orbit around the moon, the points in the orbit that are closest to and farthest from the center of the moon are called *perilune* and *apolune*, respectively. These are the vertices of the orbit. The center of the moon is at one focus of the orbit. The *Apollo 11* spacecraft was placed in a lunar orbit with perilune at 68 miles and apolune at 195 miles above the surface of the moon. Assuming that the moon is a sphere of radius 1075 miles, find an equation for the orbit of *Apollo 11*. (Place the coordinate axes so that the origin is at the center of the orbit and the foci are located on the $x$-axis.)




**68. Plywood Ellipse** A carpenter wishes to construct an elliptical table top from a 4 ft by 8 ft sheet of plywood, by tracing out the ellipse using the "thumbtack and string" method illustrated in Figures 2 and 3. What length of string should be used, and how far apart should the tacks be located, if the ellipse is to be the largest possible that can be cut out of the plywood sheet?



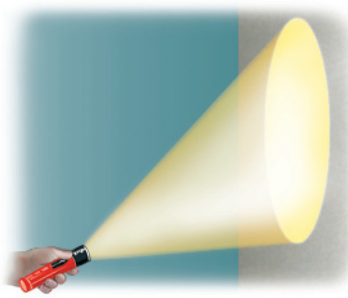
**69. Sunburst Window** A "sunburst" window above a doorway is constructed in the shape of the top half of an ellipse, as shown in the figure. The window is 20 inches tall at its highest point and 80 inches wide at the bottom. Find the height of the window 25 inches from the center of the base.




## Discuss ■ Discover ■ Prove ■ Write

**70. Discuss: Drawing an Ellipse on a Whiteboard** Try drawing an ellipse as accurately as possible on a whiteboard. How would a piece of string and two friends help this process?

**71. Discuss: Light Cone from a Flashlight** A flashlight shines on a wall, as shown in the figure. What is the shape of the boundary of the lighted area? Explain your answer.



**72. Discuss: Is It an Ellipse?** A piece of paper is wrapped around a cylindrical bottle, and then a compass is used to draw a circle on the paper, as shown in the figure. When the paper is laid flat, is the shape drawn on the paper an ellipse? (You don't need to prove your answer, but you might want to do the experiment and see what you get.)

# 10.3 Hyperbolas

■ Geometric Definition of a Hyperbola ■ Equations and Graphs of Hyperbolas

## ■ Geometric Definition of a Hyperbola

Although ellipses and hyperbolas have completely different shapes, their definitions and equations are similar. Instead of using the *sum* of distances from two fixed foci, as in the case of an ellipse, we use the *difference* to define a hyperbola.

### Geometric Definition of a Hyperbola

A **hyperbola** is the set of all points in the plane, the difference of whose distances from two fixed points $F_1$ and $F_2$ is a constant. (See Figure 1.) These two fixed points are the **foci** of the hyperbola.

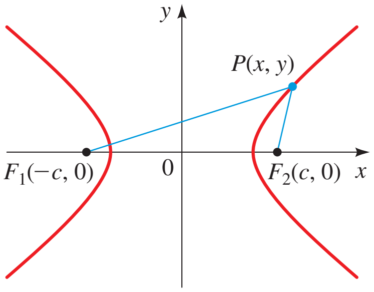



**Figure 1** | $P$ is on the hyperbola if $|d(P, F_1) - d(P, F_2)| = 2a$.

**Deriving the Equation of a Hyperbola** As in the case of the ellipse, we get the simplest equation for the hyperbola by placing the foci on the $x$-axis at $(\pm c, 0)$, as shown in Figure 1. By definition, if $P(x, y)$ lies on the hyperbola, then either $d(P, F_1) - d(P, F_2)$ or $d(P, F_2) - d(P, F_1)$ must equal some positive constant, which we call $2a$. Thus we have

$$d(P, F_1) - d(P, F_2) = \pm 2a$$

or

$$\sqrt{(x + c)^2 + y^2} - \sqrt{(x - c)^2 + y^2} = \pm 2a$$

Proceeding as we did in the case of the ellipse (Section 10.2), this simplifies to

$$(c^2 - a^2)x^2 - a^2y^2 = a^2(c^2 - a^2)$$

From triangle $PF_1F_2$ in Figure 1 we see that $|d(P, F_1) - d(P, F_2)| < 2c$. It follows that $2a < 2c$, or $a < c$. Thus $c^2 - a^2 > 0$, so we can set $b^2 = c^2 - a^2$. We then simplify the last displayed equation to get

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

This is the **standard equation** of a hyperbola with foci on the $x$-axis and center at the origin.

If we replace $x$ by $-x$ or $y$ by $-y$ in this equation, it remains unchanged, so the hyperbola is symmetric about both the $x$- and $y$-axes and about the origin. The $x$-intercepts are $\pm a$, and the points $(a, 0)$ and $(-a, 0)$ are the **vertices** of the hyperbola. There is no $y$-intercept, because setting $x = 0$ in the equation of the hyperbola leads to $-y^2 = b^2$, which has no real solution. Furthermore, the equation of the hyperbola implies that

$$\frac{x^2}{a^2} = \frac{y^2}{b^2} + 1 \geq 1$$

so $x^2/a^2 \geq 1$; thus $x^2 \geq a^2$, and hence $x \geq a$ or $x \leq -a$. This means that the hyperbola consists of two parts, called its **branches**. The segment joining the two vertices on the separate branches is the **transverse axis** of the hyperbola, and the origin is called its **center**.

If we place the foci of the hyperbola on the $y$-axis rather than on the $x$-axis, this has the effect of reversing the roles of $x$ and $y$ in the derivation of the equation of the hyperbola. This leads to a hyperbola with a vertical transverse axis.

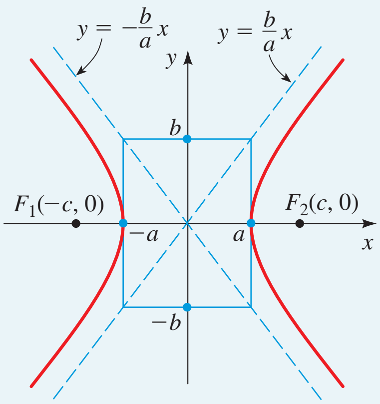
## Equations and Graphs of Hyperbolas

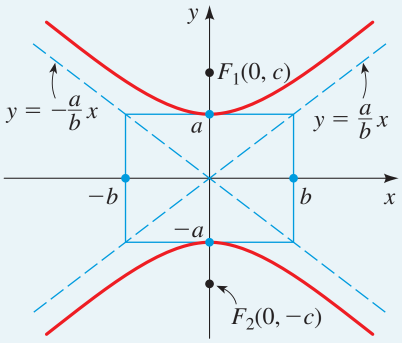
The main properties of hyperbolas are listed in the following box.

### Hyperbola with Center at the Origin

The graph of each equation is a hyperbola with center at the origin and having the given properties.

| | | |
|---|---|---|
| **EQUATION** | $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1 \quad (a > 0, b > 0)$ | $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1 \quad (a > 0, b > 0)$ |
| **VERTICES** | $(\pm a, 0)$ | $(0, \pm a)$ |
| **TRANSVERSE AXIS** | Horizontal, length $2a$ | Vertical, length $2a$ |
| **ASYMPTOTES** | $y = \pm \dfrac{b}{a} x$ | $y = \pm \dfrac{a}{b} x$ |
| **FOCI** | $(\pm c, 0), \quad c^2 = a^2 + b^2$ | $(0, \pm c), \quad c^2 = a^2 + b^2$ |
| **GRAPH** | | |




Asymptotes of rational functions are discussed in Section 3.6.

The *asymptotes* mentioned in this box are lines that the hyperbola approaches for large values of $x$ and $y$. To find the asymptotes in the first case in the box, we solve the equation for $y$ to get

$$\begin{aligned} y &= \pm \frac{b}{a}\sqrt{x^2 - a^2} \\ &= \pm \frac{b}{a} x \sqrt{1 - \frac{a^2}{x^2}} \end{aligned}$$

As $x$ gets large, $a^2/x^2$ gets closer to zero. In other words, as $x \to \infty$, we have $a^2/x^2 \to 0$. So for large $x$ the value of $y$ can be approximated as $y = \pm(b/a)x$. This shows that these lines are asymptotes of the hyperbola.

We use the following guidelines to sketch the graph of a hyperbola.

### How to Sketch a Hyperbola

1. **Sketch the Central Box.** This is the rectangle centered at the origin, with sides parallel to the axes, that crosses one axis at $\pm a$ and the other at $\pm b$.
2. **Sketch the Asymptotes.** These are the lines obtained by extending the diagonals of the central box.
3. **Plot the Vertices.** These are the two $x$-intercepts or the two $y$-intercepts.
4. **Sketch the Hyperbola.** Start at a vertex, and sketch a branch of the hyperbola, approaching the asymptotes. Sketch the other branch in the same way.

Asymptotes are an essential aid for graphing a hyperbola; they help us to determine its shape. A convenient way to find the asymptotes, for a hyperbola with horizontal transverse axis, is to first plot the points $(a, 0)$, $(-a, 0)$, $(0, b)$, and $(0, -b)$. Then sketch horizontal and vertical segments through these points to construct a rectangle, as shown in Figure 2(a). We call this rectangle the **central box** of the hyperbola. The slopes of the diagonals of the central box are $\pm b/a$, so by extending them, we obtain the asymptotes $y = \pm(b/a)x$, as sketched in Figure 2(b). Finally, we plot the vertices and use the asymptotes as a guide in sketching the hyperbola shown in Figure 2(c). (A similar procedure applies to graphing a hyperbola that has a vertical transverse axis.)

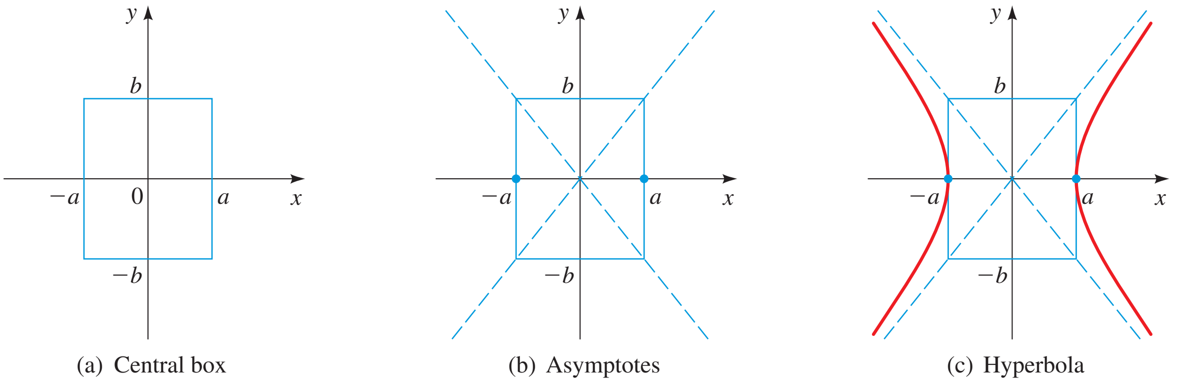



**Figure 2** | Steps in graphing the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

## Example 1 ■ A Hyperbola with Horizontal Transverse Axis

A hyperbola has the equation

$$9x^2 - 16y^2 = 144$$

**(a)** Find the vertices, foci, length of the transverse axis, and asymptotes, and sketch the graph.

**(b)** Draw the graph using a graphing device.

### Solution

**(a)** First we divide both sides of the equation by 144 to put it into standard form:

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

Because the $x^2$-term is positive, the hyperbola has a horizontal transverse axis; its vertices and foci are on the $x$-axis. Since $a^2 = 16$ and $b^2 = 9$, we get $a = 4$, $b = 3$, and $c = \sqrt{16 + 9} = 5$. Thus we have

| | |
|---|---|
| VERTICES | $(\pm 4, 0)$ |
| FOCI | $(\pm 5, 0)$ |
| ASYMPTOTES | $y = \pm\frac{3}{4}x$ |

The length of the transverse axis is $2a = 8$. After sketching the central box and asymptotes, we complete the sketch of the hyperbola as in Figure 3(a).

**(b)** Most graphing devices can draw the graph of this equation, as in Figure 3(b).

Note that the equation of a hyperbola does not define $y$ as a function of $x$ (see Section 2.2).

**Using a Graphing Calculator** To graph the equation we first solve for $y$.

$$9x^2 - 16y^2 = 144$$

$$-16y^2 = -9x^2 + 144 \qquad \text{Subtract } 9x^2$$

$$y^2 = 9\left(\frac{x^2}{16} - 1\right) \qquad \text{Divide by } -16 \text{ and factor } 9$$

$$y = \pm 3\sqrt{\frac{x^2}{16} - 1} \qquad \text{Take square roots}$$

The graph of the hyperbola in Figure 3(b) is obtained by graphing both functions $y = 3\sqrt{(x^2/16) - 1}$ and $y = -3\sqrt{(x^2/16) - 1}$.

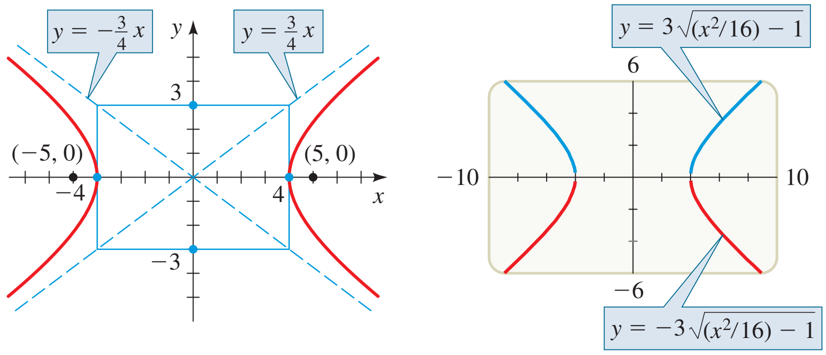



**Figure 3** | $9x^2 - 16y^2 = 144$ (a) (b)

**Now Try Exercises 9 and 33** ■

## Example 2 ■ A Hyperbola with Vertical Transverse Axis

Find the vertices, foci, length of the transverse axis, and asymptotes of the hyperbola, and sketch its graph.

$$x^2 - 9y^2 + 9 = 0$$

**Solution** We begin by writing the equation in the standard form for a hyperbola:

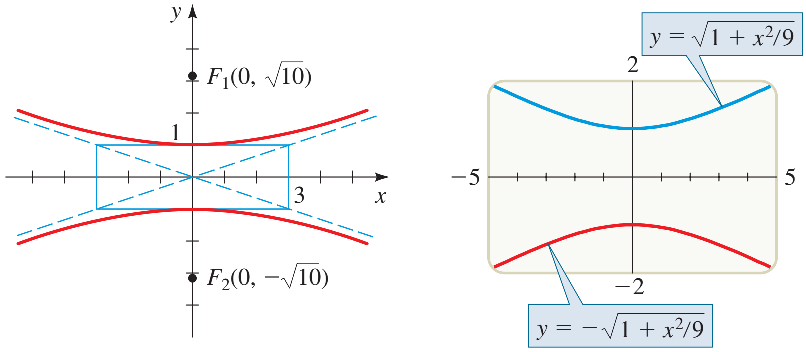
$$x^2 - 9y^2 = -9$$

$$y^2 - \frac{x^2}{9} = 1 \qquad \text{Divide by } -9$$

Because the $y^2$-term is positive, the hyperbola has a vertical transverse axis; its foci and vertices are on the $y$-axis. Since $a^2 = 1$ and $b^2 = 9$, we get $a = 1$, $b = 3$, and $c = \sqrt{1 + 9} = \sqrt{10}$. Thus we have

| | |
|---|---|
| VERTICES | $(0, \pm 1)$ |
| FOCI | $(0, \pm\sqrt{10})$ |
| ASYMPTOTES | $y = \pm\frac{1}{3}x$ |

The length of the transverse axis is $2a = 2$. We sketch the central box and asymptotes, then complete the graph, as shown in Figure 4(a) on the next page. We can also draw the graph using a graphing device, as shown in Figure 4(b).




(a) (b)

**Figure 4** | $x^2 - 9y^2 + 9 = 0$

**Now Try Exercises 21 and 35** ■

## Example 3 ■ Finding the Equation of a Hyperbola from Its Vertices and Foci

Find the standard equation of the hyperbola with vertices $(\pm 3, 0)$ and foci $(\pm 4, 0)$. Sketch the graph.

**Solution** Since the vertices are on the $x$-axis, the hyperbola has a horizontal transverse axis. Its equation is of the form

$$\frac{x^2}{3^2} - \frac{y^2}{b^2} = 1$$

We have $a = 3$ and $c = 4$. To find $b$, we use the relation $a^2 + b^2 = c^2$.

$$3^2 + b^2 = 4^2$$

$$b^2 = 4^2 - 3^2 = 7$$

$$b = \sqrt{7}$$

Thus the standard equation of the hyperbola is

$$\frac{x^2}{9} - \frac{y^2}{7} = 1$$

The graph is shown in Figure 5.

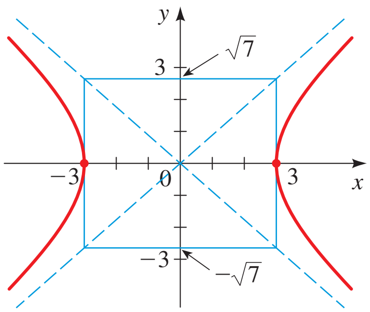



**Figure 5** | $\dfrac{x^2}{9} - \dfrac{y^2}{7} = 1$

**Now Try Exercises 27 and 37** ■

## Example 4 ■ Finding the Equation of a Hyperbola from Its Vertices and Asymptotes

Find the standard equation and the foci of the hyperbola with vertices $(0, \pm 2)$ and asymptotes $y = \pm 2x$. Sketch the graph.

**Solution** Since the vertices are on the $y$-axis, the hyperbola has a vertical transverse axis with $a = 2$. From the asymptote equation we see that $a/b = 2$. Since $a = 2$, we get $2/b = 2$, so $b = 1$. Thus the standard equation of the hyperbola is

$$\frac{y^2}{4} - x^2 = 1$$

To find the foci, we calculate $c^2 = a^2 + b^2 = 2^2 + 1^2 = 5$, so $c = \sqrt{5}$. Thus the foci are $(0, \pm\sqrt{5})$. The graph is shown in Figure 6.

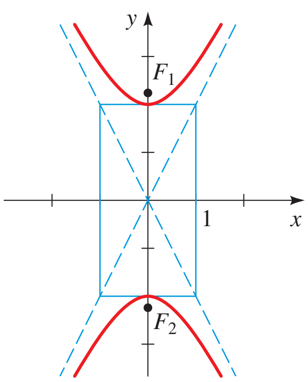



**Figure 6** | $\dfrac{y^2}{4} - x^2 = 1$

**Now Try Exercises 31 and 41** ■

Like parabolas and ellipses, hyperbolas have an interesting *reflection property*. Light aimed at one focus of a hyperbolic mirror is reflected toward the other focus, as shown in Figure 7. This property is used in the construction of Cassegrain-type telescopes. A hyperbolic mirror is placed in the telescope tube so that light reflected from the primary parabolic reflector is aimed at one focus of the hyperbolic mirror. The light is then refocused at a more accessible point below the primary reflector (Figure 8).

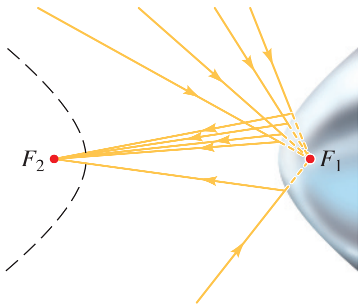



**Figure 7** | Reflection property of hyperbolas

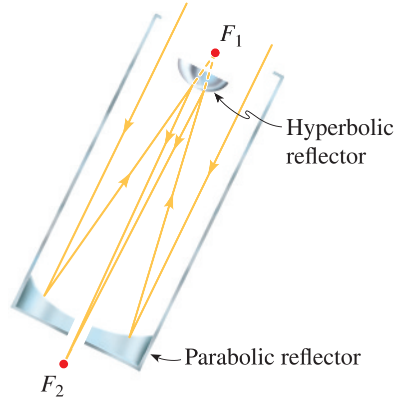



**Figure 8** | Cassegrain-type telescope

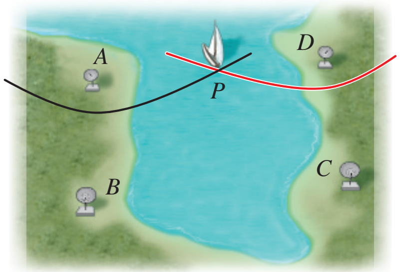



**Figure 9** | LORAN system for finding the location of a ship

The LORAN (LOng RAnge Navigation) system was used until the early 1990s; it has now been superseded by the GPS system (see Section 9.8). In the LORAN system, hyperbolas are used onboard a boat to determine its location. In Figure 9 radio stations at $A$ and $B$ transmit signals simultaneously for reception by the boat at $P$. The onboard computer converts the time difference in reception of these signals into a distance difference $d(P, A) - d(P, B)$. From the definition of a hyperbola this locates the boat on one branch of a hyperbola with foci at $A$ and $B$ (sketched in black in the figure). The same procedure is carried out with two other radio stations at $C$ and $D$, and this locates the boat on a second hyperbola (shown in red in the figure). (In practice, only three stations are needed because one station can be used as a focus for both hyperbolas.) The coordinates of the intersection point of these two hyperbolas, which can be calculated precisely by the computer, give the location of $P$.

## 10.3 Exercises

### Concepts

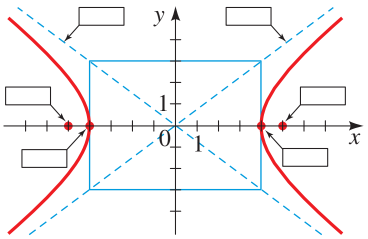
**1.** A hyperbola is the set of all points in the plane for which the ________ of the distances from two fixed points $F_1$ and $F_2$ is constant. The points $F_1$ and $F_2$ are called the ________ of the hyperbola.

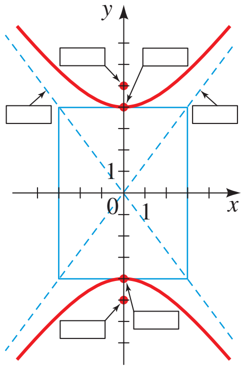
**2.** The graph of the equation $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ with $a > 0, b > 0$ is a hyperbola with ________ (horizontal/vertical) transverse axis, vertices (___, ___) and (___, ___) and foci $(\pm c, 0)$, where $c =$ ________. So the graph of $\dfrac{x^2}{4^2} - \dfrac{y^2}{3^2} = 1$ is a hyperbola with vertices (___, ___) and (___, ___) and foci (___, ___) and (___, ___).

**3.** The graph of the equation $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$ with $a > 0, b > 0$ is a hyperbola with ________ (horizontal/vertical) transverse axis, vertices (___, ___) and (___, ___) and foci $(0, \pm c)$, where $c =$ ________. So the graph of $\dfrac{y^2}{4^2} - \dfrac{x^2}{3^2} = 1$ is a hyperbola with vertices (___, ___) and (___, ___) and foci (___, ___) and (___, ___).

**4.** Label the vertices, foci, and asymptotes on the graphs given for the hyperbolas in Exercises 2 and 3.

**(a)** $\dfrac{x^2}{4^2} - \dfrac{y^2}{3^2} = 1$ **(b)** $\dfrac{y^2}{4^2} - \dfrac{x^2}{3^2} = 1$



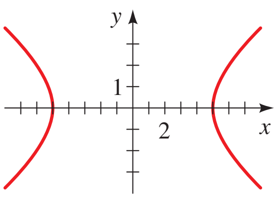

## Skills

**5–8 ■ Graphs of Hyperbolas** Match the equation with the graphs labeled I–IV. Give reasons for your answers.

**5.** $\dfrac{x^2}{4} - y^2 = 1$ **6.** $y^2 - \dfrac{x^2}{9} = 1$

**7.** $16y^2 - x^2 = 144$ **8.** $9x^2 - 25y^2 = 225$

I

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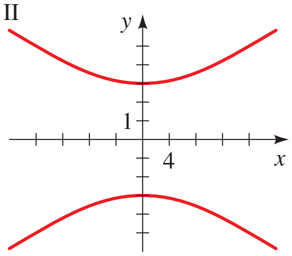
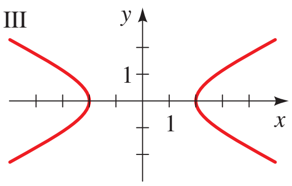
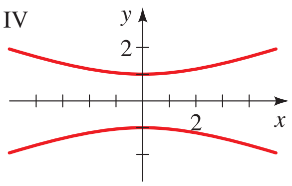



III




IV




**9–26 ■ Graphing Hyperbolas** An equation of a hyperbola is given. **(a)** Find the vertices, foci, and asymptotes of the hyperbola. **(b)** Determine the length of the transverse axis. **(c)** Sketch a graph of the hyperbola.

**9.** $\dfrac{x^2}{4} - \dfrac{y^2}{16} = 1$ **10.** $\dfrac{y^2}{9} - \dfrac{x^2}{16} = 1$

**11.** $\dfrac{y^2}{36} - \dfrac{x^2}{4} = 1$ **12.** $\dfrac{x^2}{9} - \dfrac{y^2}{64} = 1$

**13.** $y^2 - \dfrac{x^2}{25} = 1$ **14.** $\dfrac{x^2}{2} - y^2 = 1$

**15.** $x^2 - y^2 = 1$ **16.** $\dfrac{x^2}{16} - \dfrac{y^2}{12} = 1$

**17.** $9x^2 - 4y^2 = 36$ **18.** $25y^2 - 9x^2 = 225$

**19.** $4y^2 - 9x^2 = 144$ **20.** $y^2 - 25x^2 = 100$

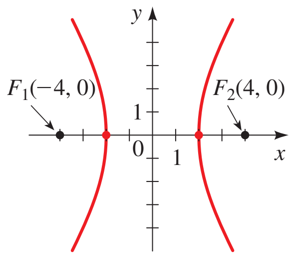
**21.** $x^2 - 4y^2 - 8 = 0$ **22.** $3y^2 - x^2 - 9 = 0$

**23.** $x^2 - y^2 + 4 = 0$ **24.** $x^2 - 3y^2 + 12 = 0$

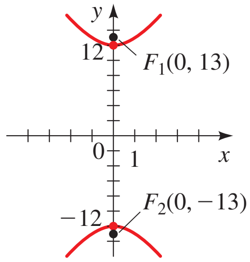
**25.** $4y^2 - x^2 = 1$ **26.** $9x^2 - 16y^2 = 1$

**27–32 ■ Finding an Equation of a Hyperbola** Find the standard equation for the hyperbola whose graph is shown.

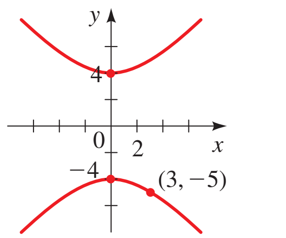
**27.**



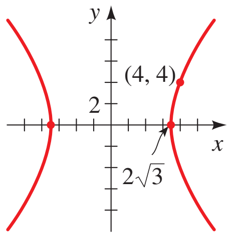

**28.**



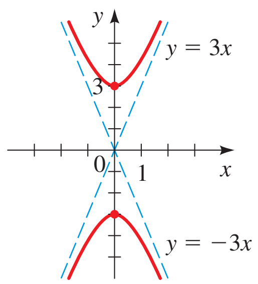

**29.**



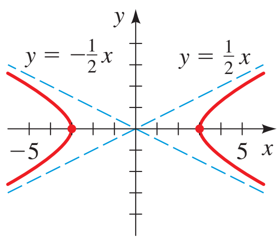

**30.**




**31.**




**32.**




**33–36 ■ Graphing Hyperbolas** Use a graphing device to graph the hyperbola.

**33.** $x^2 - 2y^2 = 8$ **34.** $3y^2 - 4x^2 = 24$

**35.** $\dfrac{y^2}{2} - \dfrac{x^2}{6} = 1$ **36.** $\dfrac{x^2}{100} - \dfrac{y^2}{64} = 1$

**37–50 ■ Finding the Equation of a Hyperbola** Find the standard equation for the hyperbola that satisfies the given conditions.

**37.** Foci: $(\pm 5, 0)$, vertices: $(\pm 3, 0)$

**38.** Foci: $(0, \pm 10)$, vertices: $(0, \pm 8)$

**39.** Foci: $(0, \pm 2)$, vertices: $(0, \pm 1)$

**40.** Foci: $(\pm 6, 0)$, vertices: $(\pm 2, 0)$

**41.** Vertices: $(\pm 1, 0)$, asymptotes: $y = \pm 5x$

**42.** Vertices: $(0, \pm 6)$, asymptotes: $y = \pm \frac{1}{3}x$

**43.** Vertices: $(0, \pm 6)$, hyperbola passes through $(-5, 9)$

**44.** Vertices: $(\pm 2, 0)$, hyperbola passes through $(3, \sqrt{30})$

**45.** Asymptotes: $y = \pm x$, hyperbola passes through $(5, 3)$

**46.** Asymptotes: $y = \pm x$, hyperbola passes through $(1, 2)$

**47.** Foci: $(0, \pm 3)$, hyperbola passes through $(1, 4)$

**48.** Foci: $(\pm\sqrt{10}, 0)$, hyperbola passes through $(4, \sqrt{18})$

**49.** Foci: $(\pm 5, 0)$, length of transverse axis: 6

**50.** Foci: $(0, \pm 1)$, length of transverse axis: 1

## Skills Plus

**51. Perpendicular Asymptotes**

**(a)** Show that the asymptotes of the hyperbola $x^2 - y^2 = 5$ are perpendicular to each other.

**(b)** Find the standard equation for the hyperbola with foci $(\pm c, 0)$ and with asymptotes perpendicular to each other.

**52. Conjugate Hyperbolas** The hyperbolas

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{and} \qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$$

are said to be *conjugate* to each other.

**(a)** Show that the hyperbolas

$$x^2 - 4y^2 + 16 = 0 \qquad \text{and} \qquad 4y^2 - x^2 + 16 = 0$$

are conjugate to each other, and sketch their graphs on the same coordinate axes.

**(b)** What do the hyperbolas of part (a) have in common?

**(c)** Show that any pair of conjugate hyperbolas have the relationship you discovered in part (b).

**53. Equation of a Hyperbola** In the derivation of the equation of the hyperbola at the beginning of this section we said that the equation

$$\sqrt{(x + c)^2 + y^2} - \sqrt{(x - c)^2 + y^2} = \pm 2a$$

simplifies to

$$(c^2 - a^2)x^2 - a^2y^2 = a^2(c^2 - a^2)$$

Supply the steps needed to show this.

**54. Verifying a Geometric Property of a Hyperbola**

**(a)** For the hyperbola

$$\frac{x^2}{9} - \frac{y^2}{16} = 1$$

determine the values of $a$, $b$, and $c$, and find the coordinates of the foci $F_1$ and $F_2$.

**(b)** Show that the point $P(5, \frac{16}{3})$ lies on this hyperbola.

**(c)** Find $d(P, F_1)$ and $d(P, F_2)$.

**(d)** Verify that the difference between $d(P, F_1)$ and $d(P, F_2)$ is $2a$.

**55. Confocal Hyperbolas** Hyperbolas are called *confocal* if they have the same foci.

**(a)** Show that the hyperbolas

$$\frac{y^2}{k} - \frac{x^2}{16 - k} = 1 \qquad (0 < k < 16)$$

are confocal.

**(b)** Use a graphing device to draw the top branches of the family of hyperbolas in part (a) for $k = 1, 4, 8,$ and 12. How does the shape of the graph change as $k$ increases?

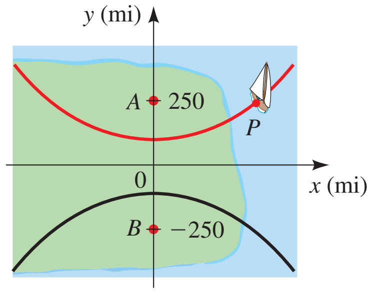
## Applications

**56. Navigation** In the figure, the LORAN stations at $A$ and $B$ are 500 mi apart, and the ship at $P$ receives station $A$'s signal 2640 microseconds (μs) before it receives the signal from station $B$.

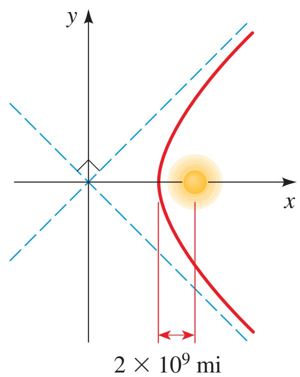
**(a)** Assuming that radio signals travel at 980 ft/μs, find $d(P, A) - d(P, B)$.

**(b)** Find an equation for the branch of the hyperbola indicated in red in the figure. (Use miles as the unit of distance.)

**(c)** If $A$ is due north of $B$ and if $P$ is due east of $A$, how far is $P$ from $A$?



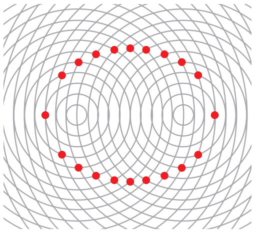

**57. Comet Trajectories** Some comets, such as Halley's comet, are a permanent part of the solar system, traveling in elliptical orbits around the sun. Other comets pass through the solar system only once, following a hyperbolic path with the sun at a focus. The figure shows the path of such a comet. Find an equation for the path, assuming that the closest the comet comes to the sun is $2 \times 10^9$ miles and that the path the comet was taking before it neared the solar system is at a right angle to the path it continues on after leaving the solar system.

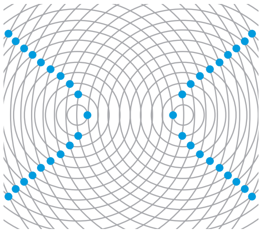


**58. Ripples in Pool** Two stones are dropped simultaneously into a calm pool of water. The crests of the resulting waves form equally spaced concentric circles, as shown in the figures. The waves interact with each other to create certain interference patterns.

**(a)** Explain why the red dots lie on an ellipse.

**(b)** Explain why the blue dots lie on a hyperbola.





■ Discuss ■ Discover ■ Prove ■ Write

**59. Discuss ■ Write: Hyperbolas in the Real World** Several examples of the uses of hyperbolas are given in the text. Find other situations in which hyperbolas occur.

**60. Discuss: Light from a Lamp** The light from a lamp forms a lighted area on a wall, as shown in the figure. Why is the boundary of this lighted area a hyperbola? How can one hold a flashlight so that its beam forms a hyperbola on the ground?

# 10.4 Shifted Conics

■ Shifting Graphs of Equations ■ Shifted Ellipses ■ Shifted Parabolas ■ Shifted Hyperbolas ■ The General Equation of a Shifted Conic

In the preceding sections we studied parabolas with vertices at the origin and ellipses and hyperbolas with centers at the origin. We restricted ourselves to these cases because these equations have the simplest form. In this section we consider conics whose vertices and centers are not necessarily at the origin, and we determine how this affects their equations.

## ■ Shifting Graphs of Equations

In Section 2.6 we studied transformations of functions that have the effect of shifting their graphs. In general, for any equation in $x$ and $y$, if we replace $x$ by $x - h$ or by $x + h$, the graph of the new equation is simply the old graph shifted horizontally; if $y$ is replaced by $y - k$ or by $y + k$, the graph is shifted vertically. The following box gives the details.

### Shifting Graphs of Equations

If $h$ and $k$ are positive real numbers, then replacing $x$ by $x - h$ or by $x + h$ and replacing $y$ by $y - k$ or by $y + k$ has the following effect(s) on the graph of any equation in $x$ and $y$.

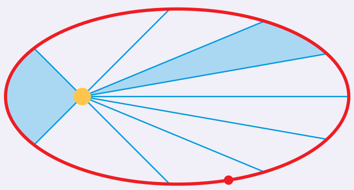
| Replacement | How the graph is shifted |
|---|---|
| **1.** $x$ replaced by $x - h$ | Right $h$ units |
| **2.** $x$ replaced by $x + h$ | Left $h$ units |
| **3.** $y$ replaced by $y - k$ | Upward $k$ units |
| **4.** $y$ replaced by $y + k$ | Downward $k$ units |

North Wind Picture Archives/Alamy Stock Photo

**JOHANNES KEPLER** (1571–1630) was the first to give a correct description of the motion of the planets. The cosmology of his time postulated complicated systems of circles moving on circles to describe these motions. Kepler sought a simpler and more harmonious description. As the official astronomer at the imperial court in Prague, he studied the astronomical observations of the Danish astronomer Tycho Brahe, whose data were the most accurate available at the time. After numerous attempts to find a theory, Kepler made the momentous discovery that the orbits of the planets are elliptical. His three great laws of planetary motion are

1. The orbit of each planet is an ellipse with the sun at one focus.
2. The line segment that joins the sun to a planet sweeps out equal areas in equal time (see the figure).
3. The square of the period of revolution of a planet is proportional to the cube of the length of the major axis of its orbit.

Kepler's formulation of these laws is perhaps the most impressive deduction from empirical data in the history of science.



## ■ Shifted Ellipses

Let's apply horizontal and vertical shifting to the ellipse with the standard equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{Standard equation of ellipse}$$

whose graph is shown in Figure 1. If we shift it so that its center is at the point $(h, k)$ instead of at the origin, then its standard equation becomes

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \qquad \text{Standard equation of shifted ellipse}$$

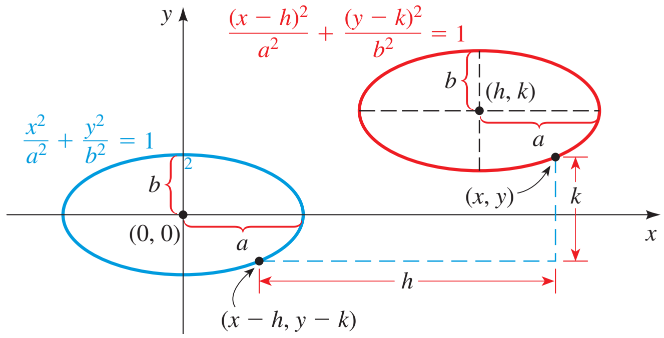



**Figure 1** | Shifted ellipse

## Example 1 ■ Sketching the Graph of a Shifted Ellipse

Sketch a graph of the ellipse

$$\frac{(x+1)^2}{4} + \frac{(y-2)^2}{9} = 1$$

and determine the coordinates of the foci.

**Solution** The ellipse

$$\frac{(x+1)^2}{4} + \frac{(y-2)^2}{9} = 1 \qquad \text{Shifted ellipse}$$

is shifted so that its center is at $(-1, 2)$. It is obtained from the ellipse

$$\frac{x^2}{4} + \frac{y^2}{9} = 1 \qquad \text{Ellipse with center at origin}$$

by shifting it left 1 unit and upward 2 units. The endpoints of the minor and major axes of the ellipse with center at the origin are $(2, 0)$, $(-2, 0)$, $(0, 3)$, $(0, -3)$. We apply the required shifts to these points to obtain the corresponding points on the shifted ellipse.

$$(2, 0) \rightarrow (2-1, 0+2) = (1, 2)$$
$$(-2, 0) \rightarrow (-2-1, 0+2) = (-3, 2)$$
$$(0, 3) \rightarrow (0-1, 3+2) = (-1, 5)$$
$$(0, -3) \rightarrow (0-1, -3+2) = (-1, -1)$$

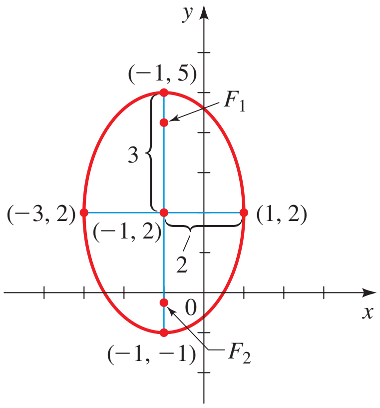



**Figure 2** | $\dfrac{(x+1)^2}{4} + \dfrac{(y-2)^2}{9} = 1$

This helps us sketch the graph in Figure 2.

To find the foci of the shifted ellipse, we first find the foci of the ellipse with center at the origin. Since $a^2 = 9$ and $b^2 = 4$, we have $c^2 = 9 - 4 = 5$, so $c = \sqrt{5}$. So the foci are $(0, \pm\sqrt{5})$. Shifting left 1 unit and upward 2 units, we get

$$(0, \sqrt{5}) \to (0 - 1, \sqrt{5} + 2) = (-1, 2 + \sqrt{5})$$

$$(0, -\sqrt{5}) \to (0 - 1, -\sqrt{5} + 2) = (-1, 2 - \sqrt{5})$$

Thus the foci of the shifted ellipse are

$$F_1(-1, 2 + \sqrt{5}) \qquad \text{and} \qquad F_2(-1, 2 - \sqrt{5})$$

**Now Try Exercise 7** ■

## Example 2 ■ Finding the Equation of a Shifted Ellipse

The vertices of an ellipse are $(-7, 3)$ and $(3, 3)$, and the foci are $(-6, 3)$ and $(2, 3)$. Find the standard equation for the ellipse, and sketch its graph.

**Solution** The center of the ellipse is the midpoint of the line segment between the vertices. By the Midpoint Formula the center is

The Midpoint Formula is given in Section 1.9.

$$\left(\frac{-7+3}{2}, \frac{3+3}{2}\right) = (-2, 3) \qquad \text{Center}$$

Since the vertices lie on a horizontal line, the major axis is horizontal. The length of the major axis is $3 - (-7) = 10$, so $a = 5$. The distance between the foci is $2 - (-6) = 8$, so $c = 4$. Since $c^2 = a^2 - b^2$, we have

$$4^2 = 5^2 - b^2 \qquad c = 4, a = 5$$

$$b^2 = 25 - 16 = 9 \qquad \text{Solve for } b^2$$

Thus the standard equation of the ellipse is

$$\frac{(x+2)^2}{25} + \frac{(y-3)^2}{9} = 1 \qquad \text{Standard equation of shifted ellipse}$$

The graph is shown in Figure 3.

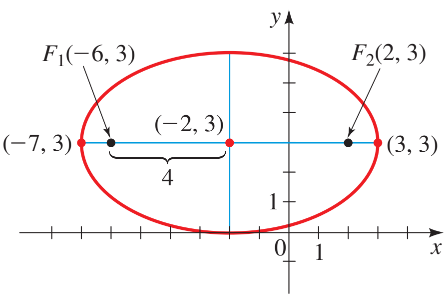
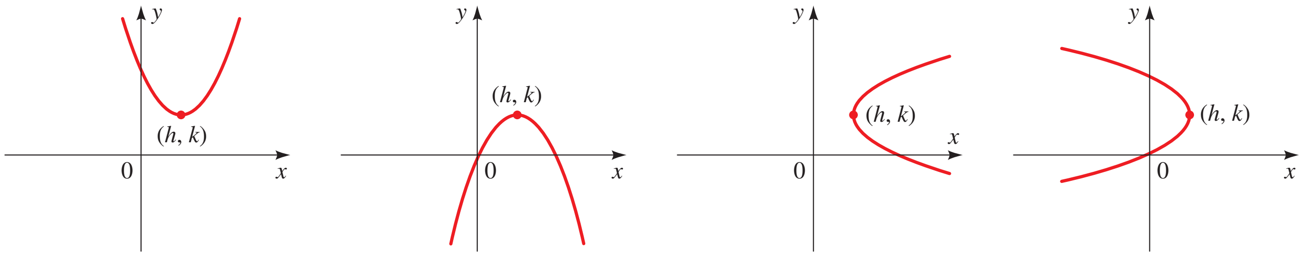



**Figure 3** | $\dfrac{(x+2)^2}{25} + \dfrac{(y-3)^2}{9} = 1$

**Now Try Exercise 35** ■

## Shifted Parabolas

Applying shifts to parabolas leads to the standard equations and graphs shown in Figure 4.




(a) $(x - h)^2 = 4p(y - k)$ $p > 0$ (b) $(x - h)^2 = 4p(y - k)$ $p < 0$ (c) $(y - k)^2 = 4p(x - h)$ $p > 0$ (d) $(y - k)^2 = 4p(x - h)$ $p < 0$

**Figure 4** | Shifted parabolas

### Example 3 ■ Graphing a Shifted Parabola

Determine the vertex, focus, and directrix, and sketch a graph of the parabola

$$x^2 - 4x = 8y - 28$$

**Solution** We complete the square in $x$ to put this equation into one of the standard forms given in Figure 4.

$$x^2 - 4x + 4 = 8y - 28 + 4 \qquad \text{Add 4 to complete the square}$$

$$(x - 2)^2 = 8y - 24 \qquad \text{Perfect square}$$

$$(x - 2)^2 = 8(y - 3) \qquad \text{Shifted parabola in standard form}$$

This parabola opens upward with vertex $(2, 3)$. It is obtained from the parabola

$$x^2 = 8y \qquad \text{Parabola with vertex at origin}$$

by shifting right 2 units and upward 3 units. Since $4p = 8$, we have $p = 2$, so the focus is 2 units above the vertex and the directrix is 2 units below the vertex. Thus the focus is $(2, 5)$ and the directrix is $y = 1$. The graph is shown in Figure 5.

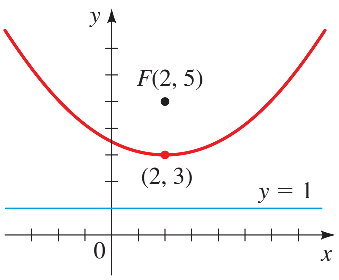
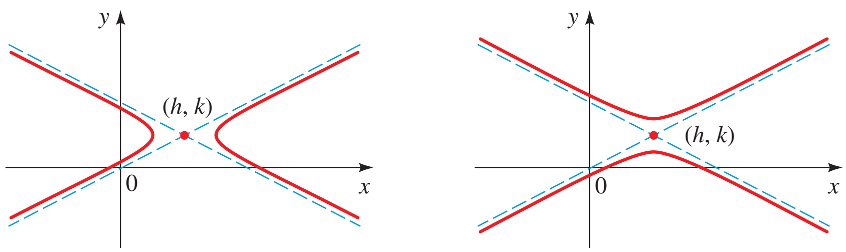



**Figure 5** | $x^2 - 4x = 8y - 28$

Now Try Exercises 13 and 19 ■

## Shifted Hyperbolas

Applying shifts to hyperbolas leads to the equations and graphs shown in Figure 6.




(a) $\dfrac{(x - h)^2}{a^2} - \dfrac{(y - k)^2}{b^2} = 1$ (b) $-\dfrac{(x - h)^2}{b^2} + \dfrac{(y - k)^2}{a^2} = 1$

**Figure 6** | Shifted hyperbolas

### Example 4 ■ Graphing a Shifted Hyperbola

A shifted conic has the equation $9x^2 - 72x - 16y^2 - 32y = 16$.

**(a)** Complete the square in $x$ and $y$ to show that the equation represents a hyperbola.

**(b)** Find the center, vertices, foci, and asymptotes of the hyperbola, and sketch its graph.

**(c)** Draw the graph using a graphing device.

### Solution

**(a)** We complete the squares in both $x$ and $y$.

$$9(x^2 - 8x \qquad) - 16(y^2 + 2y \qquad) = 16 \qquad \text{Group terms and factor}$$

$$9(x^2 - 8x + 16) - 16(y^2 + 2y + 1) = 16 + 9 \cdot 16 - 16 \cdot 1 \qquad \text{Complete the squares}$$

$$9(x - 4)^2 - 16(y + 1)^2 = 144 \qquad \text{Divide this by 144}$$

$$\frac{(x - 4)^2}{16} - \frac{(y + 1)^2}{9} = 1 \qquad \text{Shifted hyperbola in standard form}$$

Comparing this to Figure 6(a), we see that this is the equation of a shifted hyperbola.

**(b)** The shifted hyperbola has center $(4, -1)$ and a horizontal transverse axis.

$$\text{CENTER} \qquad (4, -1)$$

Its graph will have the same shape as the unshifted hyperbola

$$\frac{x^2}{16} - \frac{y^2}{9} = 1 \qquad \text{Hyperbola with center at origin}$$

Since $a^2 = 16$ and $b^2 = 9$, we have $a = 4$, $b = 3$, and $c = \sqrt{a^2 + b^2} = \sqrt{16 + 9} = 5$. Thus the foci lie 5 units to the left and to the right of the center, and the vertices lie 4 units to either side of the center.

$$\text{FOCI} \qquad (-1, -1) \quad \text{and} \quad (9, -1)$$

$$\text{VERTICES} \qquad (0, -1) \quad \text{and} \quad (8, -1)$$

The asymptotes of the unshifted hyperbola are $y = \pm\frac{3}{4}x$, so the asymptotes of the shifted hyperbola are found as follows.

$$\text{ASYMPTOTES} \qquad y + 1 = \pm\tfrac{3}{4}(x - 4)$$

$$y + 1 = \pm\tfrac{3}{4}x \mp 3$$

$$y = \tfrac{3}{4}x - 4 \qquad \text{and} \qquad y = -\tfrac{3}{4}x + 2$$

To help us sketch the hyperbola, we draw the central box; it extends 4 units left and right from the center and 3 units upward and downward from the center. We

Lightspring/Shutterstock.com; Jopelka/Shutterstock.com; Serafima Antipova/Shutterstock.com

**Discovery Project** ■ **Symmetry**

We have learned about certain symmetry properties of graphs of equations and how to test an equation for symmetry about the coordinate axes or the origin. There are other types of symmetry. For example, we have the feeling that the starfish in the photo is somehow symmetric, even though it is not symmetric about any axis or about the origin. In this project we investigate the symmetries of figures in the plane, including the shifted and rotated conics. We also investigate symmetry in nature. You can find the project at **www.stewartmath.com**.

then draw the asymptotes and complete the graph of the shifted hyperbola as shown in Figure 7(a).

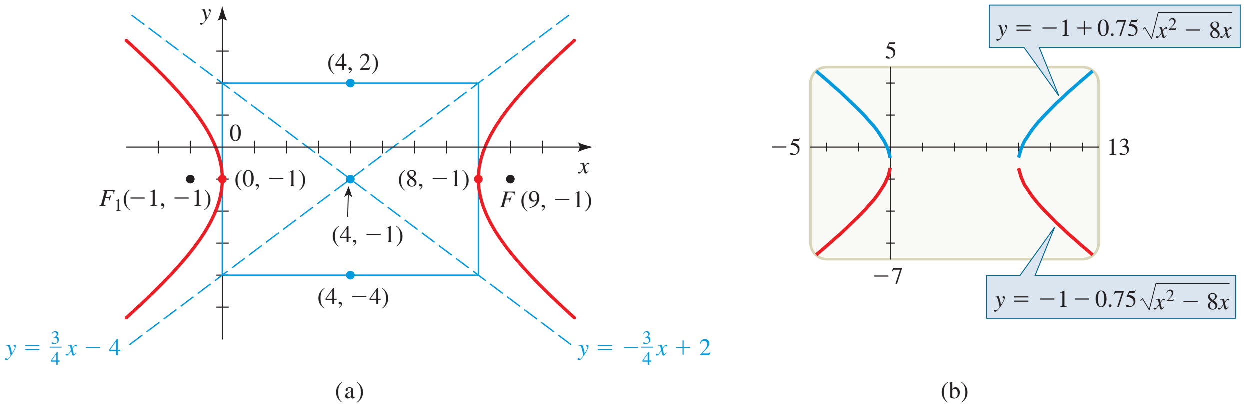



**Figure 7** | $9x^2 - 72x - 16y^2 - 32y = 16$

**(c)** Most graphing devices can draw the graph of this equation, as shown in Figure 7(b).

Note that the equation of a hyperbola does not define $y$ as a function of $x$ (see Section 2.2).

**Using a Graphing Calculator** To graph the equation we first solve for $y$. The given equation is a quadratic equation in $y$, so we use the Quadratic Formula to solve for $y$. Writing the equation in the form

$$\overbrace{16}^{a}y^2 + \overbrace{32}^{b}y + \overbrace{(-9x^2 + 72x + 16)}^{c} = 0$$

we get

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\begin{aligned} y &= \frac{-32 \pm \sqrt{32^2 - 4(16)(-9x^2 + 72x + 16)}}{2(16)} && \text{Quadratic Formula} \\ &= \frac{-32 \pm \sqrt{576x^2 - 4608x}}{32} && \text{Expand} \\ &= \frac{-32 \pm 24\sqrt{x^2 - 8x}}{32} && \text{Factor 576 from under the radical} \\ &= -1 \pm \tfrac{3}{4}\sqrt{x^2 - 8x} && \text{Simplify} \end{aligned}$$

The graph of the hyperbola in Figure 7(b) is obtained by graphing both functions

$$y = -1 + 0.75\sqrt{x^2 - 8x} \qquad \text{and} \qquad y = -1 - 0.75\sqrt{x^2 - 8x}$$

**Now Try Exercises 21, 27, and 61** ■

## ■ The General Equation of a Shifted Conic

If we expand and simplify the equations of any of the shifted conics illustrated in Figures 1, 4, and 6, then we will always obtain an equation of the form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

where $A$ and $C$ are not both 0. Conversely, if we begin with an equation of this form, then we can complete the square in $x$ and $y$ to see which type of conic section the equation represents. In some cases the graph of the equation turns out to be just a pair of lines or a single point, or there might be no graph at all. These cases are called **degenerate conics**. If the equation is not degenerate, then we can tell whether it represents a parabola, an ellipse, or a hyperbola by examining the signs of $A$ and $C$, as described in the following box.

### General Equation of a Shifted Conic

The graph of the equation

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

where $A$ and $C$ are not both 0, is a conic or a degenerate conic. In the non-degenerate cases the graph is

1. a parabola if $A$ or $C$ is 0,
2. an ellipse if $A$ and $C$ have the same sign (or a circle if $A = C$),
3. a hyperbola if $A$ and $C$ have opposite signs.

## Example 5 ■ An Equation That Leads to a Degenerate Conic

Sketch the graph of the equation

$$9x^2 - y^2 + 18x + 6y = 0$$

**Solution** Because the coefficients of $x^2$ and $y^2$ have opposite signs, this equation looks as if it should represent a hyperbola (like the equation of Example 4). To see whether this is in fact the case, we complete the squares.

$$
\begin{aligned}
9(x^2 + 2x \quad\;) - (y^2 - 6y \quad\;) &= 0 && \text{Group terms and factor 9} \\
9(x^2 + 2x + 1) - (y^2 - 6y + 9) &= 0 + 9 \cdot 1 - 9 && \text{Complete the squares} \\
9(x + 1)^2 - (y - 3)^2 &= 0 && \text{Factor} \\
(x + 1)^2 - \frac{(y - 3)^2}{9} &= 0 && \text{Divide by 9}
\end{aligned}
$$

For this to fit the form of the equation of a hyperbola, we would need a nonzero constant to the right of the equal sign. In fact, further analysis shows that this is the equation of a pair of intersecting lines.

$$
\begin{aligned}
(y - 3)^2 &= 9(x + 1)^2 \\
y - 3 &= \pm 3(x + 1) && \text{Take square roots}
\end{aligned}
$$

$$
\begin{aligned}
y &= 3(x + 1) + 3 &\quad \text{or} \quad&& y &= -3(x + 1) + 3 \\
y &= 3x + 6 &&& y &= -3x
\end{aligned}
$$

These lines are graphed in Figure 8.

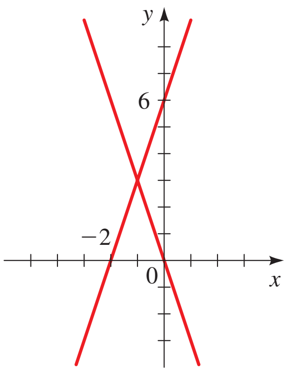


**Figure 8** | $9x^2 - y^2 + 18x + 6y = 0$

Now Try Exercise 55 ■

**Note** Because the equation in Example 5 looked at first glance like the equation of a hyperbola but, in fact, turned out to represent simply a pair of lines, we refer to its graph as a **degenerate hyperbola**. Degenerate ellipses and parabolas can also arise when we complete the square(s) in an equation that seems to represent a conic. For example, the equation

$$4x^2 + y^2 - 8x + 2y + 6 = 0$$

looks as if it should represent an ellipse, because the coefficients of $x^2$ and $y^2$ have the same sign. But completing the squares leads to

$$(x - 1)^2 + \frac{(y + 1)^2}{4} = -\frac{1}{4}$$

which has no solution at all (since the sum of two squares cannot be negative). This equation is therefore degenerate.

## 10.4 Exercises

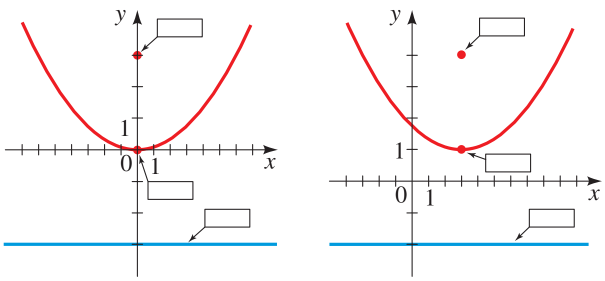
### Concepts

**1.** Suppose we want to graph an equation in $x$ and $y$.

**(a)** If we replace $x$ by $x - 3$, the graph of the equation is shifted to the ________ by 3 units. If we replace $x$ by $x + 3$, the graph of the equation is shifted to the ________ by 3 units.

**(b)** If we replace $y$ by $y - 1$, the graph of the equation is shifted ________ by 1 unit. If we replace $y$ by $y + 1$, the graph of the equation is shifted ________ by 1 unit.

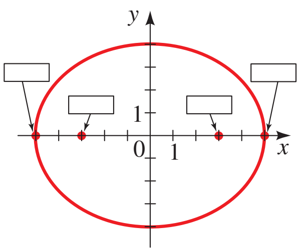
**2.** The graphs of $x^2 = 12y$ and $(x - 3)^2 = 12(y - 1)$ are given. Label the focus, directrix, and vertex on each parabola.

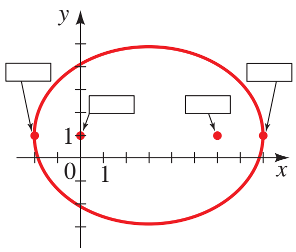



**3.** The graphs of the ellipses

$$\frac{x^2}{5^2} + \frac{y^2}{4^2} = 1 \qquad \text{and} \qquad \frac{(x-3)^2}{5^2} + \frac{(y-1)^2}{4^2} = 1$$

are given. Label the vertices and foci on each ellipse.

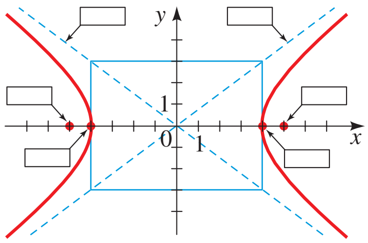


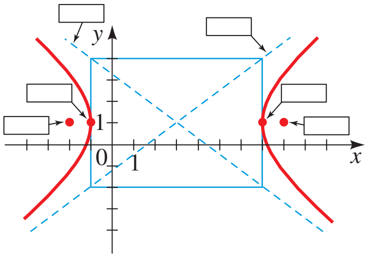




**4.** The graphs of the hyperbolas

$$\frac{x^2}{4^2} - \frac{y^2}{3^2} = 1 \qquad \text{and} \qquad \frac{(x-3)^2}{4^2} - \frac{(y-1)^2}{3^2} = 1$$

are given. Label the vertices, foci, and asymptotes on each hyperbola.




### Skills

**5–12 ■ Graphing Shifted Ellipses** An equation of an ellipse is given. **(a)** Find the center, vertices, and foci of the ellipse. **(b)** Determine the lengths of the major and minor axes. **(c)** Sketch a graph of the ellipse.

**5.** $\dfrac{(x-2)^2}{9} + \dfrac{(y-1)^2}{4} = 1$

**6.** $\dfrac{(x-3)^2}{16} + (y+3)^2 = 1$

**7.** $\dfrac{x^2}{9} + \dfrac{(y+5)^2}{25} = 1$

**8.** $x^2 + \dfrac{(y+2)^2}{4} = 1$

**9.** $\dfrac{(x+5)^2}{16} + \dfrac{(y-1)^2}{4} = 1$

**10.** $\dfrac{(x+1)^2}{36} + \dfrac{(y+1)^2}{64} = 1$

**11.** $4x^2 + 25y^2 - 50y = 75$

**12.** $9x^2 - 54x + y^2 + 2y + 46 = 0$

**13–20 ■ Graphing Shifted Parabolas** An equation of a parabola is given. **(a)** Find the vertex, focus, and directrix of the parabola. **(b)** Sketch a graph showing the parabola and its directrix.

**13.** $(x-3)^2 = 8(y+1)$

**14.** $(y+1)^2 = 16(x-3)$

**15.** $(y+5)^2 = -6x + 12$

**16.** $y^2 = 16x - 8$

**17.** $2(x-1)^2 = y$

**18.** $-4\left(x + \frac{1}{2}\right)^2 = y$

**19.** $y^2 - 6y - 12x + 33 = 0$

**20.** $x^2 + 2x - 20y + 41 = 0$

**21–28 ■ Graphing Shifted Hyperbolas** An equation of a hyperbola is given. **(a)** Find the center, vertices, foci, and asymptotes of the hyperbola. **(b)** Sketch a graph showing the hyperbola and its asymptotes.

**21.** $\dfrac{(x+1)^2}{9} - \dfrac{(y-3)^2}{16} = 1$

**22.** $(x-8)^2 - (y+6)^2 = 1$

**23.** $y^2 - \dfrac{(x+1)^2}{4} = 1$

**24.** $\dfrac{(y-1)^2}{25} - (x+3)^2 = 1$

**25.** $\dfrac{(x+1)^2}{9} - \dfrac{(y+1)^2}{4} = 1$

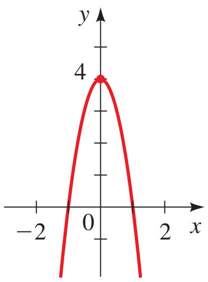
**26.** $\dfrac{(y+2)^2}{36} - \dfrac{x^2}{64} = 1$

**27.** $36x^2 + 72x - 4y^2 + 32y + 116 = 0$

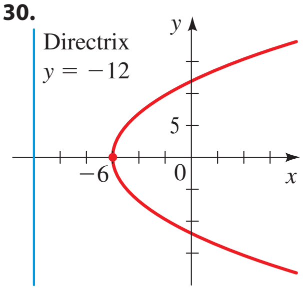
**28.** $25x^2 - 9y^2 - 54y = 306$

**29–34 ■ Finding the Equation of a Shifted Conic** Find an equation for the conic whose graph is shown.

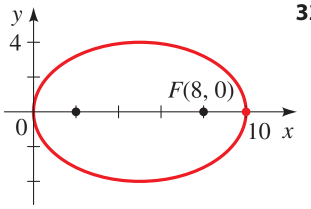
**29.**



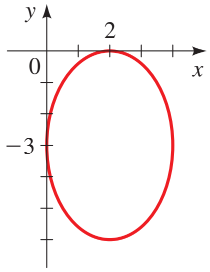

**30.**



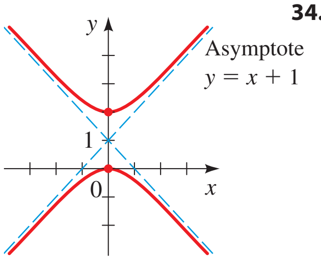

**31.**



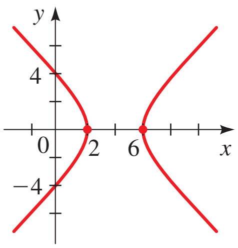

**32.**




**33.**




**34.**




**35–46 ■ Finding the Equation of a Shifted Conic** Find the standard equation for the conic section with the given properties.

**35.** The ellipse with center $C(2, -3)$, vertices $V_1(-8, -3)$ and $V_2(12, -3)$, and foci $F_1(-4, -3)$ and $F_2(8, -3)$

**36.** The ellipse with vertices $V_1(-1, -4)$ and $V_2(-1, 6)$ and foci $F_1(-1, -3)$ and $F_2(-1, 5)$

**37.** The hyperbola with center $C(-1, 4)$, vertices $V_1(-1, -3)$ and $V_2(-1, 11)$, and foci $F_1(-1, -5)$ and $F_2(-1, 13)$

**38.** The hyperbola with vertices $V_1(-1, -1)$ and $V_2(5, -1)$ and foci $F_1(-4, -1)$ and $F_2(8, -1)$

**39.** The parabola with vertex $V(-3, 5)$ and directrix $y = 2$

**40.** The parabola with focus $F(1, 3)$ and directrix $x = 3$

**41.** The hyperbola with foci $F_1(1, -5)$ and $F_2(1, 5)$ that passes through the point $(1, 4)$

**42.** The hyperbola with foci $F_1(-2, 2)$ and $F_2(4, 2)$ that passes through the point $(3, 2)$

**43.** The ellipse with foci $F_1(1, -4)$ and $F_2(5, -4)$ that passes through the point $(3, 1)$

**44.** The ellipse with foci $F_1(3, -4)$ and $F_2(3, 4)$, and $x$-intercepts 0 and 6

**45.** The parabola that passes through the point $(6, 1)$, with vertex $V(-1, 2)$ and horizontal axis of symmetry

**46.** The parabola that passes through the point $(6, -2)$, with vertex $V(4, -1)$ and vertical axis of symmetry

**47–58 ■ Graphing Shifted Conics** Complete the square to determine whether the graph of the equation is an ellipse, a parabola, a hyperbola, or a degenerate conic. If the graph is an ellipse, find the center, foci, vertices, and lengths of the major and minor axes. If it is a parabola, find the vertex, focus, and directrix. If it is a hyperbola, find the center, foci, vertices, and asymptotes. Then sketch the graph of the equation. If the equation has no graph, explain why.

**47.** $y^2 = 4(x + 2y)$

**48.** $9x^2 - 36x + 4y^2 = 0$

**49.** $x^2 - 5y^2 - 2x + 20y = 44$

**50.** $x^2 + 6x + 12y + 9 = 0$

**51.** $4x^2 + 25y^2 - 24x + 250y + 561 = 0$

**52.** $2x^2 + y^2 = 2y + 1$

**53.** $16x^2 - 9y^2 - 96x + 288 = 0$

**54.** $4x^2 - 4x - 8y + 9 = 0$

**55.** $x^2 + 16 = 4(y^2 + 2x)$

**56.** $x^2 - y^2 = 10(x - y) + 1$

**57.** $3x^2 + 4y^2 - 6x - 24y + 39 = 0$

**58.** $x^2 + 4y^2 + 20x - 40y + 300 = 0$

**59–62 ■ Graphing Shifted Conics** Use a graphing device to graph the conic.

**59.** $2x^2 - 4x + y + 5 = 0$

**60.** $4x^2 + 9y^2 - 36y = 0$

**61.** $9x^2 + 36 = y^2 + 36x + 6y$

**62.** $x^2 - 4y^2 + 4x + 8y = 0$

## Skills Plus

**63. Degenerate Conic** Determine what the value of $F$ must be if the graph of the equation

$$4x^2 + y^2 + 4(x - 2y) + F = 0$$

is **(a)** an ellipse, **(b)** a single point, or **(c)** the empty set.

**64. Common Focus and Vertex** Find an equation for the ellipse that shares a vertex and a focus with the parabola

$$x^2 + y = 100$$

and has its other focus at the origin.

**65. Confocal Parabolas** This exercise deals with *confocal* parabolas, that is, families of parabolas that have the same focus.

**(a)** Draw graphs of the family of parabolas

$$x^2 = 4p(y + p)$$

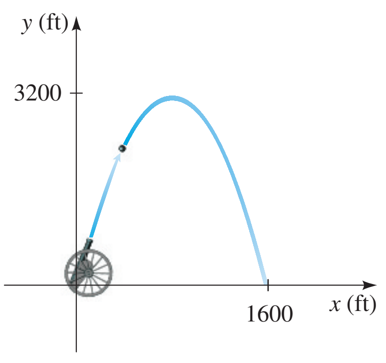
for $p = -2, -\frac{3}{2}, -1, -\frac{1}{2}, \frac{1}{2}, 1, \frac{3}{2}, 2$.

**(b)** Show that each parabola in this family has its focus at the origin.

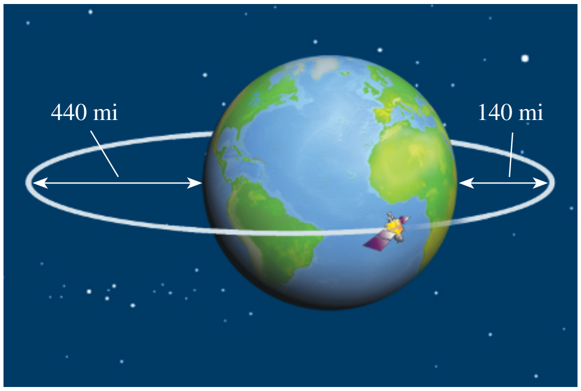
**(c)** Describe the effect on the graph of moving the vertex closer to the origin.

## Applications

**66. Path of a Cannonball** A cannon fires a cannonball as shown in the figure. The path of the cannonball is a parabola with vertex at the highest point of the path. If the cannonball lands 1600 ft from the cannon and the highest point it reaches is 3200 ft above the ground, find an equation for the path of the cannonball. Place the origin at the location of the cannon.




**67. Orbit of a Satellite** A satellite is in an elliptical orbit around the earth with the center of the earth at one focus, as shown in the figure. The height of the satellite above the earth varies between 140 mi and 440 mi. Assume that the earth is a sphere with radius 3960 mi. Find an equation for the path of the satellite with the origin at the center of the earth.




## Discuss ■ Discover ■ Prove ■ Write

**68. Discuss: A Family of Confocal Conics** Conics that share a focus are called **confocal**. Consider the family of conics that have a focus at $(0, 1)$ and a vertex at the origin, as shown in the figure.

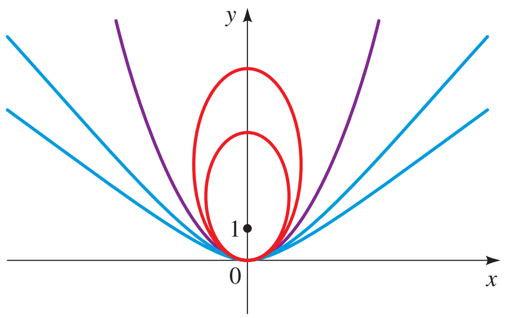
**(a)** Find equations of two different ellipses that have these properties.

**(b)** Find equations of two different hyperbolas that have these properties.

**(c)** Explain why only one parabola satisfies these properties. Find its equation.

**(d)** Sketch the conics you found in parts (a), (b), and (c) on the same coordinate axes (for the hyperbolas, sketch the top branches only).

**(e)** How are the ellipses and hyperbolas related to the parabola?




**69. Discuss ■ Discover: Different Forms of a Quadratic Function** We have used several different forms of quadratic functions.

| | |
|---|---|
| $y = ax^2 + bx + c$ | Quadratic function |
| $y = a(x - r_1)(x - r_2)$ | Factored form |
| $y = a(x - h)^2 + k$ | Vertex form |
| $(x - h)^2 = 4p(y - k)$ | Standard form |

Any quadratic function can be expressed in each of these equivalent forms. In each case the graph is the same parabola.

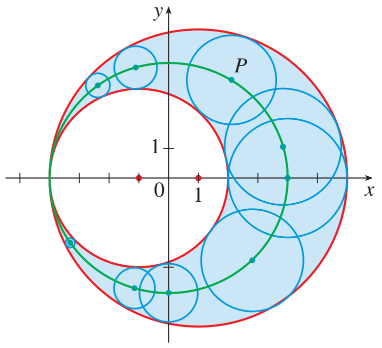
**(a)** Express the quadratic function

$$y = 2x^2 - 8x + 6$$

in each form.

**(b)** Which form would you use to do each of the following: find the minimum or maximum value of $y$, find the focus of the parabola, find the real zeros of the function, find the complex zeros of the function.

**70. Discuss ■ Prove: Tangent Circles** The graph shows two red circles with centers $(-1, 0)$ and $(1, 0)$ and radii 3 and 5, respectively. Consider the collection of all circles tangent to both of these circles. (Some of these circles are shown in blue.) Show that the centers of all such circles lie on an ellipse with foci $(\pm 1, 0)$. Find an equation of this ellipse.




PS *Look for something familiar.* Apply the geometric definition of an ellipse. Observe that for a blue circle with center $P$ and radius $r$, the distance from one focus to $P$ is $3 + r$ and from the other focus to $P$ is $5 - r$.

# 10.5 Rotation of Axes

■ Rotation of Axes ■ General Equation of a Conic ■ The Discriminant

In Section 10.4 we studied conics with equations of the form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

We saw that the graph is always an ellipse, parabola, or hyperbola with horizontal or vertical axes (except in the degenerate cases). In this section we study the most general second-degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

We will see that the graph of an equation of this form is also a conic. In fact, by rotating the coordinate axes through an appropriate angle, we can eliminate the term $Bxy$ and then use our knowledge of conic sections to analyze the graph.

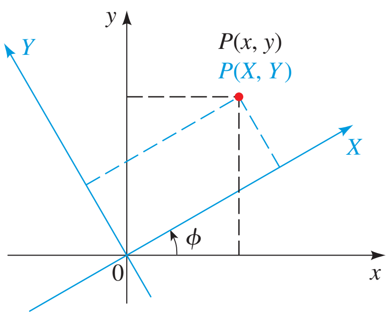



Figure 1

## ■ Rotation of Axes

In Figure 1 the $x$- and $y$-axes have been rotated through an acute angle $\phi$ about the origin to produce a new pair of axes, which we call the $X$- and $Y$-axes. A point $P$ that has coordinates $(x, y)$ in the old system has coordinates $(X, Y)$ in the new system. If we let $r$ denote the distance of $P$ from the origin and let $\theta$ be the angle that the segment $OP$ makes with the new $X$-axis, then we can see from Figure 2 (by considering the two right triangles in the figure) that

$$X = r\cos\theta \qquad Y = r\sin\theta$$
$$x = r\cos(\theta + \phi) \qquad y = r\sin(\theta + \phi)$$

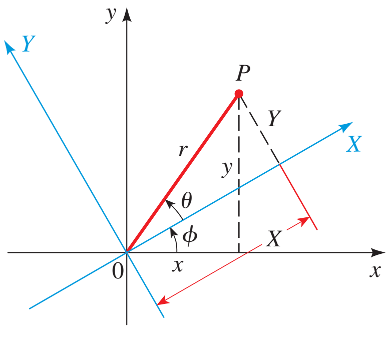



Figure 2

Using the Addition Formula for Cosine, we see that

$$\begin{aligned} x &= r\cos(\theta + \phi) \\ &= r(\cos\theta\cos\phi - \sin\theta\sin\phi) \\ &= (r\cos\theta)\cos\phi - (r\sin\theta)\sin\phi \\ &= X\cos\phi - Y\sin\phi \end{aligned}$$

Similarly, we can apply the Addition Formula for Sine to the expression for $y$ to obtain $y = X \sin \phi + Y \cos \phi$. By treating these equations for $x$ and $y$ as a system of linear equations in the variables $X$ and $Y$ (see Exercise 35), we obtain expressions for $X$ and $Y$ in terms of $x$ and $y$, as detailed in the following box.

### Rotation of Axes Formulas

Suppose the $x$- and $y$-axes in a coordinate plane are rotated through the acute angle $\phi$ to produce the $X$- and $Y$-axes, as shown in Figure 1. Then the coordinates $(x, y)$ and $(X, Y)$ of a point in the $xy$- and the $XY$-planes are related as follows.

$$x = X \cos \phi - Y \sin \phi \qquad X = x \cos \phi + y \sin \phi$$

$$y = X \sin \phi + Y \cos \phi \qquad Y = -x \sin \phi + y \cos \phi$$

## Example 1 ■ Rotation of Axes

If the coordinate axes are rotated through 30°, find the $XY$-coordinates of the point with $xy$-coordinates $(2, -4)$.

**Solution** Using the Rotation of Axes Formulas with $x = 2$, $y = -4$, and $\phi = 30°$, we arrive at

$$X = 2 \cos 30° + (-4) \sin 30° = 2\left(\frac{\sqrt{3}}{2}\right) - 4\left(\frac{1}{2}\right) = \sqrt{3} - 2$$

$$Y = -2 \sin 30° + (-4) \cos 30° = -2\left(\frac{1}{2}\right) - 4\left(\frac{\sqrt{3}}{2}\right) = -1 - 2\sqrt{3}$$

The $XY$-coordinates are $(-2 + \sqrt{3}, -1 - 2\sqrt{3})$.

Now Try Exercise 3 ■

## Example 2 ■ Rotating a Hyperbola

Rotate the coordinate axes through 45° to show that the graph of the equation $xy = 2$ is a hyperbola.

**Solution** We use the Rotation of Axes Formulas with $\phi = 45°$ to obtain

$$x = X \cos 45° - Y \sin 45° = \frac{X}{\sqrt{2}} - \frac{Y}{\sqrt{2}}$$

$$y = X \sin 45° + Y \cos 45° = \frac{X}{\sqrt{2}} + \frac{Y}{\sqrt{2}}$$

Substituting these expressions into the original equation gives

$$\left(\frac{X}{\sqrt{2}} - \frac{Y}{\sqrt{2}}\right)\left(\frac{X}{\sqrt{2}} + \frac{Y}{\sqrt{2}}\right) = 2$$

$$\frac{X^2}{2} - \frac{Y^2}{2} = 2$$

$$\frac{X^2}{4} - \frac{Y^2}{4} = 1$$

We recognize this as a hyperbola with vertices $(\pm 2, 0)$ in the $XY$-coordinate system. Its asymptotes are $Y = \pm X$, which correspond to the coordinate axes in the $xy$-system (see Figure 3).

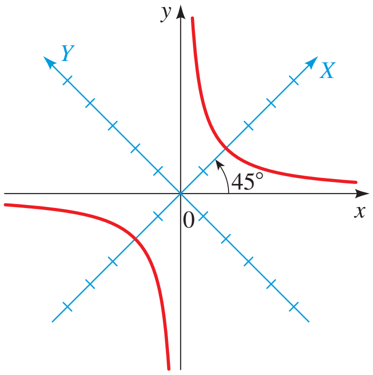



**Figure 3** | $xy = 2$

Now Try Exercise 11

## General Equation of a Conic

The method of Example 2 can be used to transform any equation of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

into an equation in $X$ and $Y$ that doesn't contain an $XY$-term by choosing an appropriate angle of rotation. To find the angle that works, we rotate the axes through an angle $\phi$ and substitute for $x$ and $y$ using the Rotation of Axes Formulas.

$$\begin{aligned} A(X\cos\phi - Y\sin\phi)^2 &+ B(X\cos\phi - Y\sin\phi)(X\sin\phi + Y\cos\phi) \\ &+ C(X\sin\phi + Y\cos\phi)^2 + D(X\cos\phi - Y\sin\phi) \\ &+ E(X\sin\phi + Y\cos\phi) + F = 0 \end{aligned}$$

If we expand this and collect like terms, we obtain an equation of the form

$$A'X^2 + B'XY + C'Y^2 + D'X + E'Y + F' = 0$$

where

$$\begin{aligned} A' &= A\cos^2\phi + B\sin\phi\cos\phi + C\sin^2\phi \\ B' &= 2(C - A)\sin\phi\cos\phi + B(\cos^2\phi - \sin^2\phi) \\ C' &= A\sin^2\phi - B\sin\phi\cos\phi + C\cos^2\phi \\ D' &= D\cos\phi + E\sin\phi \\ E' &= -D\sin\phi + E\cos\phi \\ F' &= F \end{aligned}$$

To eliminate the $XY$-term, we would like to choose $\phi$ so that $B' = 0$, that is,

**Double-Angle Formulas**

$\sin 2\phi = 2\sin\phi\cos\phi$

$\cos 2\phi = \cos^2\phi - \sin^2\phi$

$$\begin{aligned} 2(C - A)\sin\phi\cos\phi + B(\cos^2\phi - \sin^2\phi) &= 0 \\ (C - A)\sin 2\phi + B\cos 2\phi &= 0 && \text{Double-Angle Formulas for Sine and Cosine} \\ B\cos 2\phi &= (A - C)\sin 2\phi \\ \cot 2\phi &= \frac{A - C}{B} && \text{Divide by } B\sin 2\phi \end{aligned}$$

The preceding calculation proves the following theorem.

### Simplifying the General Conic Equation

To eliminate the $xy$-term in the general conic equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

rotate the axes through the acute angle $\phi$ that satisfies

$$\cot 2\phi = \frac{A - C}{B}$$

## Example 3 ■ Eliminating the *xy*-Term

Use a rotation of axes to eliminate the $xy$-term in the equation

$$6\sqrt{3}x^2 + 6xy + 4\sqrt{3}y^2 = 21\sqrt{3}$$

Identify and sketch the curve.

**Solution** To eliminate the $xy$-term, we rotate the axes through an angle $\phi$ that satisfies

$$\cot 2\phi = \frac{A - C}{B} = \frac{6\sqrt{3} - 4\sqrt{3}}{6} = \frac{\sqrt{3}}{3}$$

Thus $2\phi = 60°$, and hence $\phi = 30°$. With this value of $\phi$ we get

$$x = X\left(\frac{\sqrt{3}}{2}\right) - Y\left(\frac{1}{2}\right) \qquad \text{Rotation of Axes Formulas}$$

$$y = X\left(\frac{1}{2}\right) + Y\left(\frac{\sqrt{3}}{2}\right) \qquad \cos\phi = \frac{\sqrt{3}}{2},\ \sin\phi = \frac{1}{2}$$

Substituting these values for $x$ and $y$ into the given equation leads to

$$6\sqrt{3}\left(\frac{X\sqrt{3}}{2} - \frac{Y}{2}\right)^2 + 6\left(\frac{X\sqrt{3}}{2} - \frac{Y}{2}\right)\left(\frac{X}{2} + \frac{Y\sqrt{3}}{2}\right) + 4\sqrt{3}\left(\frac{X}{2} + \frac{Y\sqrt{3}}{2}\right)^2 = 21\sqrt{3}$$

Expanding and collecting like terms, we get

$$7\sqrt{3}X^2 + 3\sqrt{3}Y^2 = 21\sqrt{3}$$

$$\frac{X^2}{3} + \frac{Y^2}{7} = 1 \qquad \text{Divide by } 21\sqrt{3}$$

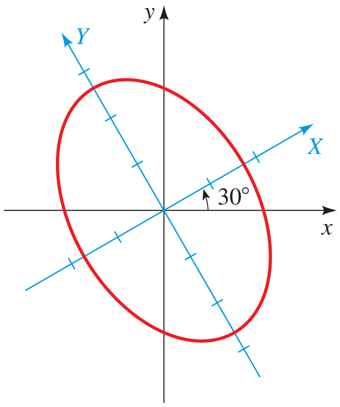



**Figure 4** | $6\sqrt{3}x^2 + 6xy + 4\sqrt{3}y^2 = 21\sqrt{3}$

This is the equation of an ellipse in the $XY$-coordinate system. The foci lie on the $Y$-axis. Because $a^2 = 7$ and $b^2 = 3$, the length of the major axis is $2\sqrt{7}$, and the length of the minor axis is $2\sqrt{3}$. The ellipse is sketched in Figure 4.

Now Try Exercise 17 ■

**Note** In the preceding example we were able to determine $\phi$ without difficulty because we remembered that $\cot 60° = \sqrt{3}/3$. In general, finding $\phi$ is not quite so easy. The next example illustrates how the following Half-Angle Formulas, which are valid for $0 < \phi < \pi/2$, are useful in determining $\phi$. (See Section 7.3.)

$$\cos\phi = \sqrt{\frac{1 + \cos 2\phi}{2}} \qquad \sin\phi = \sqrt{\frac{1 - \cos 2\phi}{2}}$$

## Example 4 ■ Graphing a Rotated Conic

A conic has the equation

$$64x^2 + 96xy + 36y^2 - 15x + 20y - 25 = 0$$

**(a)** Use a rotation of axes to eliminate the $xy$-term.

**(b)** Identify and sketch the graph.

**(c)** Draw the graph using a graphing device.

### Solution

**(a)** To eliminate the $xy$-term, we rotate the axes through an angle $\phi$ that satisfies

$$\cot 2\phi = \frac{A - C}{B} = \frac{64 - 36}{96} = \frac{7}{24}$$

In Figure 5 we sketch a triangle with $\cot 2\phi = \frac{7}{24}$. We see that

$$\cos 2\phi = \tfrac{7}{25}$$

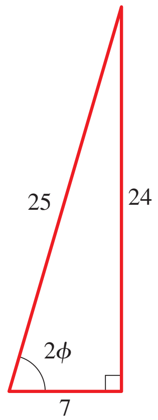



**Figure 5**

so, using the Half-Angle Formulas, we get

$$\cos \phi = \sqrt{\frac{1 + \frac{7}{25}}{2}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

$$\sin \phi = \sqrt{\frac{1 - \frac{7}{25}}{2}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$$

The Rotation of Axes Formulas then give

$$x = \tfrac{4}{5}X - \tfrac{3}{5}Y \qquad \text{and} \qquad y = \tfrac{3}{5}X + \tfrac{4}{5}Y$$

Substituting into the given equation, we have

$$64\left(\tfrac{4}{5}X - \tfrac{3}{5}Y\right)^2 + 96\left(\tfrac{4}{5}X - \tfrac{3}{5}Y\right)\left(\tfrac{3}{5}X + \tfrac{4}{5}Y\right)$$
$$+ 36\left(\tfrac{3}{5}X + \tfrac{4}{5}Y\right)^2 - 15\left(\tfrac{4}{5}X - \tfrac{3}{5}Y\right) + 20\left(\tfrac{3}{5}X + \tfrac{4}{5}Y\right) - 25 = 0$$

Expanding and collecting like terms, we get

$$100X^2 + 25Y - 25 = 0$$

$$4X^2 = -Y + 1 \qquad \text{Simplify}$$

$$X^2 = -\tfrac{1}{4}(Y - 1) \qquad \text{Divide by 4}$$

**(b)** We recognize this as the equation of a parabola that opens along the negative $Y$-axis and has vertex $(0, 1)$ in $XY$-coordinates. Since $4p = -\frac{1}{4}$, we have $p = -\frac{1}{16}$, so the focus is $\left(0, \frac{15}{16}\right)$ and the directrix is $Y = \frac{17}{16}$. Using

$$\phi = \cos^{-1}\left(\tfrac{4}{5}\right) \approx 37°$$


**Discovery Project ■ Computer Graphics III**

An image on a computer screen is stored in the computer memory as a large matrix. In the *Discovery Projects* Computer Graphics I and II, we used matrix operations to transform an image—adjust contrast, stretch, shrink, reflect, or shear. But rotating an image requires knowledge of the rotation formulas in this section. In this project we express the rotation formulas in matrix form (see Exercise 10.5.37) and experiment with using rotation matrices to rotate an image. You can find the project at **www.stewartmath.com**.

we sketch the graph in Figure 6(a).

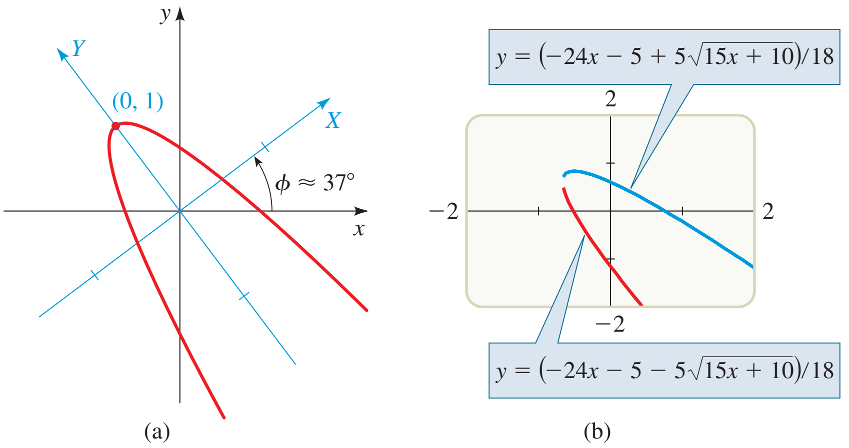



**Figure 6** | $64x^2 + 96xy + 36y^2 - 15x + 20y - 25 = 0$

**(c)** Most graphing devices can draw the graph of this equation, as shown in Figure 6(b).

**Using a Graphing Calculator** To graph the equation we first solve for $y$. Writing the equation in the form

$$36y^2 + (96x + 20)y + (64x^2 - 15x - 25) = 0$$

we see that this equation is a quadratic equation in $y$, so we can use the Quadratic Formula (as in Example 10.4.4) to solve for $y$. You can check that we get

$$y = (-24x - 5 + 5\sqrt{15x + 10})/18 \qquad \text{and} \qquad y = (-24x - 5 - 5\sqrt{15x + 10})/18$$

The graph of the parabola in Figure 6(b) is obtained by graphing both functions.

Now Try Exercise 23

## The Discriminant

In Examples 3 and 4 we were able to identify the type of conic by rotating the axes. The next theorem gives rules for identifying the type of conic directly from the equation, without rotating axes.

### Identifying Conics by the Discriminant

The graph of the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

is either a conic or a degenerate conic. In the nondegenerate cases the graph is

1. a parabola if $B^2 - 4AC = 0$,
2. an ellipse if $B^2 - 4AC < 0$,
3. a hyperbola if $B^2 - 4AC > 0$.

The quantity $B^2 - 4AC$ is the **discriminant** of the equation.

**Proof** If we rotate the axes through an angle $\phi$, we get an equation of the form

$$A'X^2 + B'XY + C'Y^2 + D'X + E'Y + F' = 0$$

where $A'$, $B'$, $C'$, . . . are given by the formulas in this section. A straightforward calculation shows that

$$(B')^2 - 4A'C' = B^2 - 4AC$$

Thus the expression $B^2 - 4AC$ remains unchanged for any rotation. In particular, if we choose a rotation that eliminates the $xy$-term $(B' = 0)$, we get

$$A'X^2 + C'Y^2 + D'X + E'Y + F' = 0$$

In this case $B^2 - 4AC = -4A'C'$. So $B^2 - 4AC = 0$ if either $A'$ or $C'$ is zero; $B^2 - 4AC < 0$ if $A'$ and $C'$ have the same sign; and $B^2 - 4AC > 0$ if $A'$ and $C'$ have opposite signs. As we observed in Section 10.4, these cases correspond to the graph of the last displayed equation being a parabola, an ellipse, or a hyperbola, respectively. ■

In the proof we indicated that the discriminant is unchanged by any rotation; for this reason the discriminant is said to be **invariant** under rotation.

### Example 5 ■ Identifying a Conic by the Discriminant

A conic has the equation

$$3x^2 + 5xy - 2y^2 + x - y + 4 = 0$$

**(a)** Use the discriminant to identify the conic.

**(b)** Confirm your answer to part (a) by graphing the conic with a graphing device.

**Solution**

**(a)** Since $A = 3$, $B = 5$, and $C = -2$, the discriminant is

$$B^2 - 4AC = 5^2 - 4(3)(-2) = 49 > 0$$

So the conic is a hyperbola.

**(b)** Using a graphing device, we obtain the graph shown in Figure 7. The graph confirms that the equation represents a hyperbola.

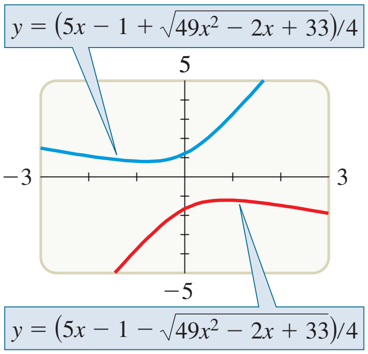



**Figure 7**

Now Try Exercise 29 ■

## 10.5 Exercises

### Concepts

**1.** Suppose the $x$- and $y$-axes are rotated through an acute angle $\phi$ to produce the new $X$- and $Y$-axes. A point $P$ in the plane can be described by its $xy$-coordinates $(x, y)$ or its $XY$-coordinates $(X, Y)$. These coordinates are related by the following formulas.

$x =$ ________ $X =$ ________

$y =$ ________ $Y =$ ________

**2.** Consider the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

**(a)** In general, the graph of this equation is a ________.

**(b)** To eliminate the $xy$-term from this equation, we rotate the axes through an angle $\phi$ that satisfies

$\cot 2\phi =$ ________.

**(c)** The discriminant of this equation is ________.

If the discriminant is 0, the graph is ________;

if it is negative, the graph is ________; and

if it is positive, the graph is ________.

### Skills

**3–8 ■ Rotation of Axes** Determine the $XY$-coordinates of the given point if the coordinate axes are rotated through the indicated angle.

**3.** $(1, 1)$, $\phi = 45°$

**4.** $(-2, 1)$, $\phi = 30°$

**5.** $(3, -\sqrt{3})$, $\phi = 60°$

**6.** $(2, 0)$, $\phi = 15°$

**7.** $(0, 2)$, $\phi = 55°$

**8.** $(\sqrt{2}, 4\sqrt{2})$, $\phi = 45°$

**9–14 ■ Finding the Equation for a Rotated Conic** Determine the equation of the given conic in $XY$-coordinates when the coordinate axes are rotated through the indicated angle.

**9.** $x^2 - 3y^2 = 4, \quad \phi = 60°$

**10.** $y = (x - 1)^2, \quad \phi = 45°$

**11.** $x^2 - y^2 = 2y, \quad \phi = \cos^{-1}\left(\frac{3}{5}\right)$

**12.** $x^2 + 2y^2 = 16, \quad \phi = \sin^{-1}\left(\frac{3}{5}\right)$

**13.** $x^2 + 2\sqrt{3}xy - y^2 = 4, \quad \phi = 30°$

**14.** $xy = x + y, \quad \phi = \pi/4$

**15–28 ■ Graphing a Rotated Conic** **(a)** Use the discriminant to determine whether the graph of the equation is a parabola, an ellipse, or a hyperbola. **(b)** Use a rotation of axes to eliminate the $xy$-term. **(c)** Sketch the graph.

**15.** $xy = 8$

**16.** $xy + 4 = 0$

**17.** $x^2 + 2\sqrt{3}xy - y^2 + 2 = 0$

**18.** $13x^2 + 6\sqrt{3}xy + 7y^2 = 16$

**19.** $11x^2 - 24xy + 4y^2 + 20 = 0$

**20.** $21x^2 + 10\sqrt{3}xy + 31y^2 = 144$

**21.** $\sqrt{3}x^2 + 3xy = 3$

**22.** $153x^2 + 192xy + 97y^2 = 225$

**23.** $x^2 + 2xy + y^2 + x - y = 0$

**24.** $25x^2 - 120xy + 144y^2 - 156x - 65y = 0$

**25.** $2\sqrt{3}x^2 - 6xy + \sqrt{3}x + 3y = 0$

**26.** $9x^2 - 24xy + 16y^2 = 100(x - y - 1)$

**27.** $52x^2 + 72xy + 73y^2 = 40x - 30y + 75$

**28.** $(7x + 24y)^2 = 600x - 175y + 25$

**29–32 ■ Identifying a Conic from Its Discriminant** **(a)** Use the discriminant to identify the conic. **(b)** Confirm your answer by graphing the conic using a graphing device.

**29.** $2x^2 - 4xy + 2y^2 - 5x - 5 = 0$

**30.** $x^2 - 2xy + 3y^2 = 8$

**31.** $6x^2 + 10xy + 3y^2 - 6y = 36$

**32.** $9x^2 - 6xy + y^2 + 6x - 2y = 0$

## Skills Plus

**33. Identifying a Hyperbola Using Rotation of Axes**

**(a)** Use rotation of axes to show that the following equation represents a hyperbola.

$$7x^2 + 48xy - 7y^2 - 200x - 150y + 600 = 0$$

**(b)** Find the $XY$- and $xy$-coordinates of the center, vertices, and foci.

**(c)** Find the equations of the asymptotes in $XY$- and $xy$-coordinates.

**34. Identifying a Parabola Using Rotation of Axes**

**(a)** Use rotation of axes to show that the following equation represents a parabola.

$$2\sqrt{2}(x + y)^2 = 7x + 9y$$

**(b)** Find the $XY$- and $xy$-coordinates of the vertex and focus.

**(c)** Find the equation of the directrix in $XY$- and $xy$-coordinates.

**35. Rotation of Axes Formulas** Solve the equations

$$x = X\cos\phi - Y\sin\phi$$
$$y = X\sin\phi + Y\cos\phi$$

for $X$ and $Y$ in terms of $x$ and $y$. [*Hint:* To begin, multiply the first equation by $\cos\phi$ and the second by $\sin\phi$, and then add the two equations to solve for $X$.]

**36. Graphing an Equation Using Rotation of Axes** Show that the graph of the equation

$$\sqrt{x} + \sqrt{y} = 1$$

is part of a parabola by rotating the axes through an angle of 45°. [*Hint:* First convert the equation to one that does not involve radicals.]

## Discuss ■ Discover ■ Prove ■ Write

**37. Prove: Matrix Form of Rotation of Axes Formulas** Let $Z$, $Z'$, and $R$ be the matrices

$$Z = \begin{bmatrix} x \\ y \end{bmatrix} \qquad Z' = \begin{bmatrix} X \\ Y \end{bmatrix}$$

$$R = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

**(a)** Show that the Rotation of Axes Formulas can be written as

$$Z = RZ' \qquad \text{and} \qquad Z' = R^{-1}Z$$

**(b)** Let $R_1$ and $R_2$ be matrices that represent rotations through the angles $\phi_1$ and $\phi_2$, respectively. Show that the product matrix $R_1R_2$ represents a rotation through an angle $\phi_1 + \phi_2$. [*Hint:* Use the Addition Formulas for Sine and Cosine to simplify the entries of the matrix $R_1R_2$.]

**38. Prove: Algebraic Invariants** A quantity is invariant under rotation if it does not change when the axes are rotated. It was stated in the text that for the general equation of a conic the quantity $B^2 - 4AC$ is invariant under rotation.

**(a)** Use the formulas for $A'$, $B'$, and $C'$ in this section to prove that the quantity $B^2 - 4AC$ is invariant under rotation; that is, show that

$$B^2 - 4AC = B'^2 - 4A'C'$$

**(b)** Prove that $A + C$ is invariant under rotation.

**(c)** Is the quantity $F$ invariant under rotation?

**39. Discover ■ Prove: Geometric Invariants** Do you expect that the distance between two points is invariant under rotation? Prove your answer by comparing the distance $d(P, Q)$ and $d(P', Q')$ where $P'$ and $Q'$ are the images of $P$ and $Q$ under a rotation of axes.

# 10.6 Polar Equations of Conics

■ A Unified Geometric Description of Conics ■ Polar Equations of Conics

## ■ A Unified Geometric Description of Conics

Earlier in this chapter, we defined a parabola in terms of a focus and directrix, but we defined the ellipse and hyperbola in terms of two foci. In this section we give a more unified treatment of all three types of conics in terms of a focus and directrix. If we place one focus at the origin, then a conic section has a simple polar equation. Moreover, in polar form, rotation of conics becomes a simple matter. Polar equations of ellipses are crucial in the derivation of Kepler's Laws (see Section 10.4).

### Equivalent Description of Conics

Let $F$ be a fixed point (the **focus**), $\ell$ a fixed line (the **directrix**), and let $e$ be a fixed positive number (the **eccentricity**). The set of all points $P$ such that the ratio of the distance from $P$ to $F$ to the distance from $P$ to $\ell$ is the constant $e$ is a conic. That is, the set of all points $P$ such that

$$\frac{d(P, F)}{d(P, \ell)} = e$$

is a conic. The conic is a parabola if $e = 1$, an ellipse if $e < 1$, or a hyperbola if $e > 1$.

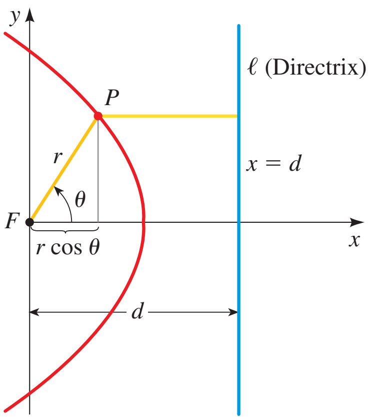



**Figure 1**

**Proof** If $e = 1$, then $d(P, F) = d(P, \ell)$, and so the given condition becomes the definition of a parabola as given in Section 10.1.

Now, suppose $e \neq 1$. Let's place the focus $F$ at the origin and the directrix parallel to the $y$-axis and $d$ units to the right. In this case the directrix has equation $x = d$ and is perpendicular to the polar axis. If the point $P$ has polar coordinates $(r, \theta)$, we see from Figure 1 that $d(P, F) = r$ and $d(P, \ell) = d - r\cos\theta$. Thus the condition $d(P, F)/d(P, \ell) = e$, or $d(P, F) = e \cdot d(P, \ell)$, becomes

$$r = e(d - r\cos\theta)$$

If we square both sides of this polar equation and convert to rectangular coordinates, we get

$$x^2 + y^2 = e^2(d - x)^2$$

$$(1 - e^2)x^2 + 2de^2x + y^2 = e^2d^2 \qquad \text{Expand and simplify}$$

$$\left(x + \frac{e^2d}{1 - e^2}\right)^2 + \frac{y^2}{1 - e^2} = \frac{e^2d^2}{(1 - e^2)^2} \qquad \text{Divide by } 1 - e^2 \text{ and complete the square}$$

If $e < 1$, then dividing both sides of this equation by $e^2d^2/(1 - e^2)^2$ gives an equation of the form

$$\frac{(x - h)^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $\qquad h = \dfrac{-e^2d}{1 - e^2} \qquad a^2 = \dfrac{e^2d^2}{(1 - e^2)^2} \qquad b^2 = \dfrac{e^2d^2}{1 - e^2}$

This is the equation of an ellipse with center $(h, 0)$. In Section 10.2 we found that the foci of an ellipse are a distance $c$ from the center, where $c^2 = a^2 - b^2$. In our case

$$c^2 = a^2 - b^2 = \frac{e^4 d^2}{(1 - e^2)^2}$$

Thus $c = e^2 d/(1 - e^2) = -h$, which confirms that the focus defined in the theorem (namely the origin) is the same as the focus defined in Section 10.2. It also follows that

$$\boxed{e = \frac{c}{a}}$$

If $e > 1$, a similar proof shows that the conic is a hyperbola with $e = c/a$, where $c^2 = a^2 + b^2$. ■

## Polar Equations of Conics

In the proof we saw that the polar equation of the conic in Figure 1 is $r = e(d - r\cos\theta)$. Solving for $r$, we get

$$r = \frac{ed}{1 + e\cos\theta}$$

If the directrix is chosen to be to the *left* of the focus $(x = -d)$, then we get the equation $r = ed/(1 - e\cos\theta)$. If the directrix is *parallel* to the polar axis $(y = d$ or $y = -d)$, then we get $\sin\theta$ instead of $\cos\theta$ in the equation. These observations are summarized in the following box and in Figure 2.

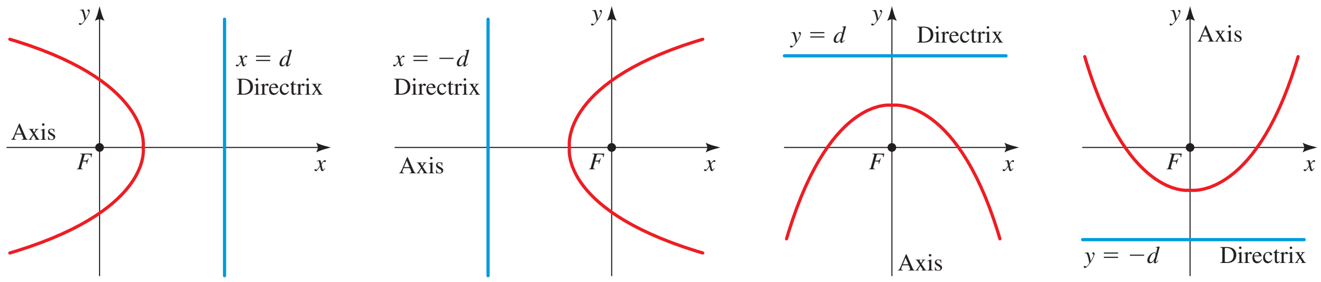
### Polar Equations of Conics

A polar equation of the form

$$r = \frac{ed}{1 \pm e\cos\theta} \qquad \text{or} \qquad r = \frac{ed}{1 \pm e\sin\theta}$$

represents a conic with one focus at the origin and with eccentricity $e$. The conic is

1. a parabola if $e = 1$,
2. an ellipse if $0 < e < 1$,
3. a hyperbola if $e > 1$.




(a) $r = \dfrac{ed}{1 + e\cos\theta}$ (b) $r = \dfrac{ed}{1 - e\cos\theta}$ (c) $r = \dfrac{ed}{1 + e\sin\theta}$ (d) $r = \dfrac{ed}{1 - e\sin\theta}$

**Figure 2** | The form of the polar equation of a conic indicates the location of the directrix.

To graph the polar equation of a conic, we first determine the location of the directrix from the form of the equation. The four cases that arise are shown in Figure 2. (The figure shows only the parts of the graphs that are close to the focus at the origin. The shape of the rest of the graph depends on whether the equation represents a parabola, an ellipse, or a hyperbola.) The axis of a conic is perpendicular to the directrix—specifically we have the following:

1. For a parabola the axis of symmetry is perpendicular to the directrix.
2. For an ellipse the major axis is perpendicular to the directrix.
3. For a hyperbola the transverse axis is perpendicular to the directrix.

## Example 1 ■ Finding a Polar Equation for a Conic

Find a polar equation for the parabola that has its focus at the origin and whose directrix is the line $y = -6$.

**Solution** Using $e = 1$ and $d = 6$ and using part (d) of Figure 2, we see that the polar equation of the parabola is

$$r = \frac{6}{1 - \sin\theta}$$

Now Try Exercise 3 ■

**Note** To graph a polar conic, it is helpful to plot the points for which $\theta = 0, \pi/2, \pi$, and $3\pi/2$. Using these points and a knowledge of the type of conic (which we obtain from the eccentricity), we get a rough idea of the shape and location of the graph.

## Example 2 ■ Identifying and Sketching a Conic

A conic is given by the polar equation

$$r = \frac{10}{3 - 2\cos\theta}$$

**(a)** Show that the conic is an ellipse, and sketch its graph.

**(b)** Find the center of the ellipse and the lengths of the major and minor axes.

**Solution**

**(a)** To put the polar equation into one of the forms shown in Figure 2, we divide the numerator and denominator by 3:

$$r = \frac{ed}{1 - e\cos\theta}$$

$$r = \frac{\frac{10}{3}}{1 - \frac{2}{3}\cos\theta}$$

Since $e = \frac{2}{3} < 1$, the equation represents an ellipse. For a rough graph we plot the points for which $\theta = 0, \pi/2, \pi, 3\pi/2$. (See Figure 3.)

| $\theta$ | $r$ |
|---|---|
| $0$ | $10$ |
| $\frac{\pi}{2}$ | $\frac{10}{3}$ |
| $\pi$ | $2$ |
| $\frac{3\pi}{2}$ | $\frac{10}{3}$ |

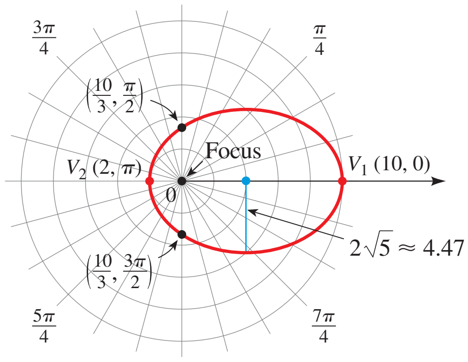
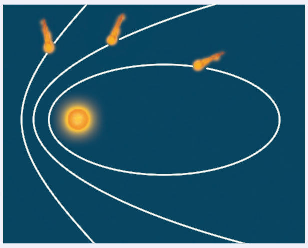



**Figure 3** | $r = \dfrac{10}{3 - 2\cos\theta}$



**Paths of Comets**
The path of a comet is an ellipse, a parabola, or a hyperbola with the sun at a focus. This fact can be proved by using calculus and Newton's Laws of Motion.* If the path is a parabola or a hyperbola, the comet will never return. If the path is an ellipse, it can be determined precisely when and where the comet can be seen again. Halley's comet has an elliptical path and returns every 75 years; it was last seen in 1987. The brightest comet of the 20th century was comet Hale-Bopp, seen in 1997. Its orbit is a very eccentric ellipse; it is expected to return to the inner solar system around the year 4385.

*James Stewart, Daniel Clegg, and Saleem Watson, *Calculus: Early Transcendentals*, 9th ed. (Boston, MA: Cengage, 2021), pages 921 and 925.

**(b)** Comparing the equation to the four equations given in Figure 2, we see that the major axis is horizontal. Thus the endpoints of the major axis are $V_1(10, 0)$ and $V_2(2, \pi)$. So the center of the ellipse is at $C(4, 0)$, the midpoint of $V_1V_2$.

The distance between the vertices $V_1$ and $V_2$ is 12; thus the length of the major axis is $2a = 12$, so $a = 6$. To determine the length of the minor axis, we need to find $b$. From the definition of eccentricity in this section, we have $c = ae = 6\left(\frac{2}{3}\right) = 4$, so

$$b^2 = a^2 - c^2 = 6^2 - 4^2 = 20$$

Thus $b = \sqrt{20} = 2\sqrt{5} \approx 4.47$, and the length of the minor axis is $2b = 4\sqrt{5} \approx 8.94$.

Now Try Exercises 17 and 21 ■

## Example 3 ■ Identifying and Sketching a Conic

A conic is given by the polar equation

$$r = \frac{12}{2 + 4\sin\theta}$$

**(a)** Show that the conic is a hyperbola, and sketch its graph.

**(b)** Find the center of the hyperbola, and sketch the asymptotes.

### Solution

**(a)** Dividing the numerator and denominator by 2, we have

$$r = \frac{6}{1 + 2\sin\theta}$$

Since $e = 2 > 1$, the equation represents a hyperbola. For a rough graph we plot the points for which $\theta = 0, \pi/2, \pi, 3\pi/2$. (See Figure 4.)

**(b)** Comparing the equation to the four equations given in Figure 2, we see that the transverse axis is vertical. Thus the endpoints of the transverse axis (the vertices of the hyperbola) are $V_1(2, \pi/2)$ and $V_2(-6, 3\pi/2) = V_2(6, \pi/2)$. So the center of the hyperbola is $C(4, \pi/2)$, the midpoint of $V_1V_2$.

To sketch the asymptotes, we need to find $a$ and $b$. The distance between $V_1$ and $V_2$ is 4; thus the length of the transverse axis is $2a = 4$, so $a = 2$. To find $b$, we first find $c$. From the definition of eccentricity in this section, we have $c = ae = 2 \cdot 2 = 4$, so

$$b^2 = c^2 - a^2 = 4^2 - 2^2 = 12$$

Thus $b = \sqrt{12} = 2\sqrt{3} \approx 3.46$. Knowing $a$ and $b$ allows us to sketch the central box, from which we obtain the asymptotes shown in Figure 4.

| $\theta$ | $r$ |
|---|---|
| $0$ | $6$ |
| $\frac{\pi}{2}$ | $2$ |
| $\pi$ | $6$ |
| $\frac{3\pi}{2}$ | $-6$ |

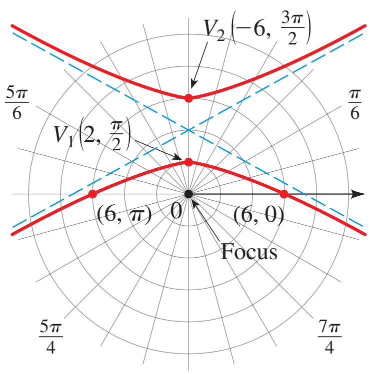



**Figure 4** | $r = \dfrac{12}{2 + 4\sin\theta}$

Now Try Exercise 25 ■

To rotate conic sections, it is more convenient to use polar equations than Cartesian equations. We observe that the graph of $r = f(\theta - \alpha)$ is the graph of $r = f(\theta)$ rotated counterclockwise about the origin through an angle $\alpha$ (see Exercise 8.2.65).

### Example 4 ■ Rotating an Ellipse

Suppose the ellipse of Example 2 is rotated through an angle $\pi/4$ about the origin. Find a polar equation for the resulting ellipse, and draw its graph.

**Solution** We get the equation of the rotated ellipse by replacing $\theta$ with $\theta - \pi/4$ in the equation given in Example 2. So the new equation is

$$r = \frac{10}{3 - 2\cos(\theta - \pi/4)}$$

We use this equation to graph the rotated ellipse in Figure 5. Notice that the ellipse has been rotated about the focus at the origin.

Now Try Exercise 37 ■

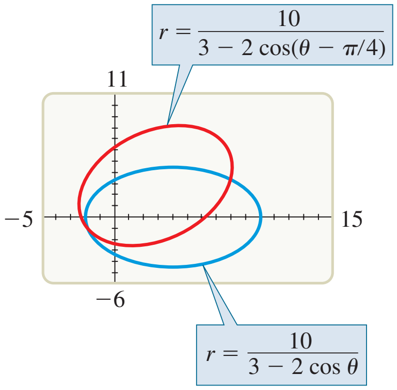



**Figure 5**

In Figure 6 we use a computer to sketch a number of conics to demonstrate the effect of varying the eccentricity $e$. Notice that when $e$ is close to 0, the ellipse is nearly circular, and it becomes more elongated as $e$ increases. When $e = 1$, of course, the conic is a parabola. As $e$ increases beyond 1, the conic is an ever steeper hyperbola.


**Figure 6**

## 10.6 Exercises

### Concepts

**1.** All conics can be described geometrically by using a fixed point $F$ called the ______ and a fixed line $\ell$ called the ______. For a fixed positive number $e$ the set of all points $P$ satisfying

$$\frac{\square}{\square} = e$$

is a ______. If $e = 1$, the conic is a(n) ______; if $e < 1$, the conic is a(n) ______; and if $e > 1$, the conic is a(n) ______. The number $e$ is called the ______ of the conic.

**2.** The polar equation of a conic with eccentricity $e$ has one of the following forms:

$r =$ ______ or $r =$ ______

### Skills

**3–10 ■ Finding a Polar Equation for a Conic** Write a polar equation of a conic that has its focus at the origin and satisfies the given conditions.

**3.** Ellipse, eccentricity $\frac{2}{3}$, directrix $x = 3$

**4.** Hyperbola, eccentricity $\frac{4}{3}$, directrix $x = -3$

**5.** Parabola, directrix $y = 2$

**6.** Ellipse, eccentricity $\frac{1}{2}$, directrix $y = -4$

**7.** Hyperbola, eccentricity 4, directrix $r = 5 \sec\theta$

**8.** Ellipse, eccentricity 0.6, directrix $r = 2 \csc\theta$

**9.** Parabola, vertex at $(5, \pi/2)$

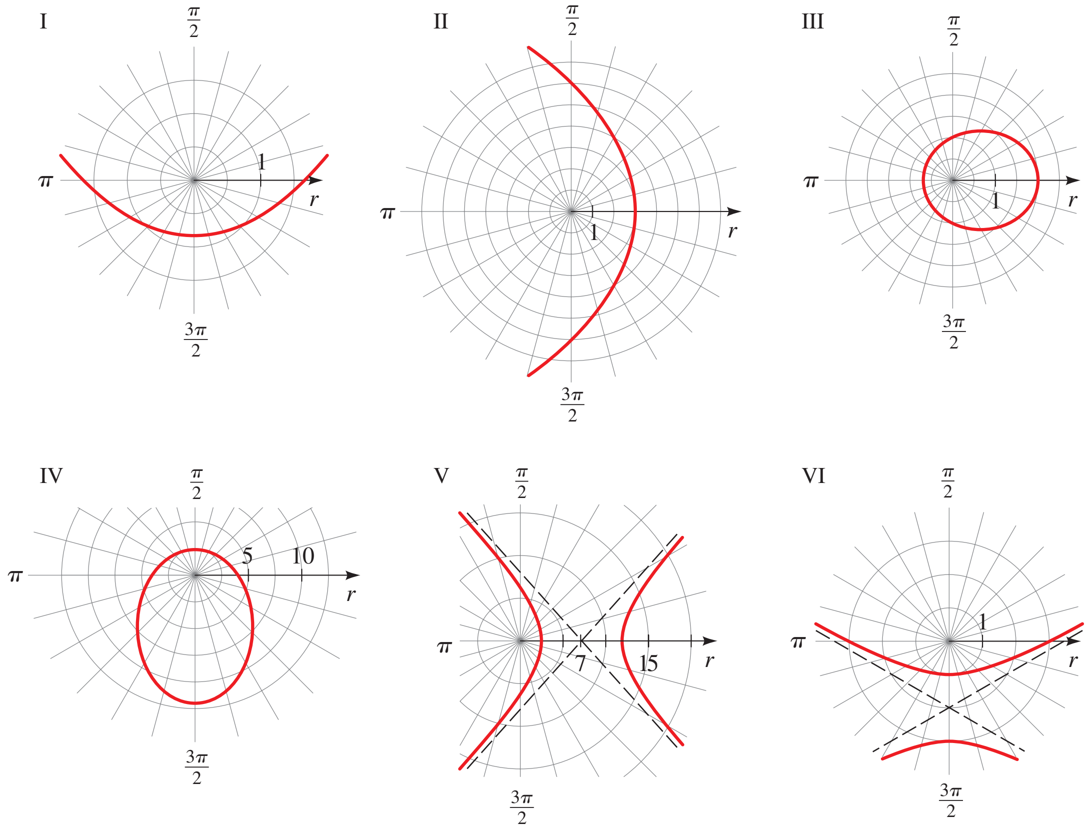
**10.** Ellipse, eccentricity 0.4, vertex at $(2, 0)$

**11–16 ■ Graphs of Polar Equations of Conics** Match the polar equations with the graphs labeled I–VI. Give reasons for your answers.

**11.** $r = \dfrac{6}{1 + \cos\theta}$ **12.** $r = \dfrac{2}{2 - \cos\theta}$

**13.** $r = \dfrac{3}{1 - 2\sin\theta}$ **14.** $r = \dfrac{5}{3 - 3\sin\theta}$

**15.** $r = \dfrac{12}{3 + 2\sin\theta}$ **16.** $r = \dfrac{12}{2 + 3\cos\theta}$




**17–20 ■ Polar Equation for a Parabola** A polar equation of a conic is given. **(a)** Show that the conic is a parabola, and sketch its graph. **(b)** Find the vertex and directrix, and indicate them on the graph.

**17.** $r = \dfrac{4}{1 - \sin\theta}$ **18.** $r = \dfrac{3}{2 + 2\sin\theta}$

**19.** $r = \dfrac{5}{3 + 3\cos\theta}$ **20.** $r = \dfrac{2}{5 - 5\cos\theta}$

**21–24 ■ Polar Equation for an Ellipse** A polar equation of a conic is given. **(a)** Show that the conic is an ellipse, and sketch its graph. **(b)** Find the vertices and directrix, and indicate them on the graph. **(c)** Find the center of the ellipse and the lengths of the major and minor axes.

**21.** $r = \dfrac{4}{2 - \cos\theta}$ **22.** $r = \dfrac{6}{3 - 2\sin\theta}$

**23.** $r = \dfrac{12}{4 + 3\sin\theta}$ **24.** $r = \dfrac{18}{4 + 3\cos\theta}$

**25–28 ■ Polar Equation for a Hyperbola** A polar equation of a conic is given. **(a)** Show that the conic is a hyperbola, and sketch its graph. **(b)** Find the vertices and directrix, and indicate them on the graph. **(c)** Find the center of the hyperbola, and sketch the asymptotes.

**25.** $r = \dfrac{8}{1 + 2\cos\theta}$ **26.** $r = \dfrac{10}{1 - 4\sin\theta}$

**27.** $r = \dfrac{20}{2 - 3\sin\theta}$ **28.** $r = \dfrac{6}{2 + 7\cos\theta}$

**29–36 ■ Identifying and Graphing a Conic** **(a)** Find the eccentricity, and identify the conic. **(b)** Sketch the conic, and label the vertices.

**29.** $r = \dfrac{4}{1 + 3\cos\theta}$ **30.** $r = \dfrac{8}{3 + 3\cos\theta}$

**31.** $r = \dfrac{2}{1 - \cos\theta}$ **32.** $r = \dfrac{10}{3 - 2\sin\theta}$

**33.** $r = \dfrac{6}{2 + \sin\theta}$ **34.** $r = \dfrac{5}{2 - 3\sin\theta}$

**35.** $r = \dfrac{7}{2 - 5\sin\theta}$ **36.** $r = \dfrac{8}{3 + \cos\theta}$

**37–40 ■ Rotating a Conic** A polar equation of a conic is given. **(a)** Find the eccentricity and the directrix of the conic. **(b)** If this conic is rotated about the origin through the given angle $\theta$, write the resulting equation. **(c)** Draw graphs of the original conic and the rotated conic in the same viewing rectangle.

**37.** $r = \dfrac{1}{4 - 3\cos\theta}$; $\theta = \dfrac{\pi}{3}$

**38.** $r = \dfrac{2}{5 - 3\sin\theta}$; $\theta = \dfrac{2\pi}{3}$

**39.** $r = \dfrac{2}{1 + \sin\theta}$; $\theta = -\dfrac{\pi}{4}$

**40.** $r = \dfrac{9}{2 + 2\cos\theta}$; $\theta = -\dfrac{5\pi}{6}$

## Skills Plus

**41. Families of Conics** Graph the conics $r = e/(1 - e\cos\theta)$ with $e = 0.4, 0.6, 0.8$, and $1.0$ in the same viewing rectangle. How does the value of $e$ affect the shape of the curve?

**42. Families of Conics**

**(a)** Graph the conics

$$r = \frac{ed}{(1 + e\sin\theta)}$$

for $e = 1$ and various values of $d$. How does the value of $d$ affect the shape of the conic?

**(b)** Graph these conics for $d = 1$ and various values of $e$. How does the value of $e$ affect the shape of the conic?

## Applications

**43. Orbit of the Earth** The polar equation of an ellipse can be expressed in terms of its eccentricity $e$ and the length $a$ of its major axis.

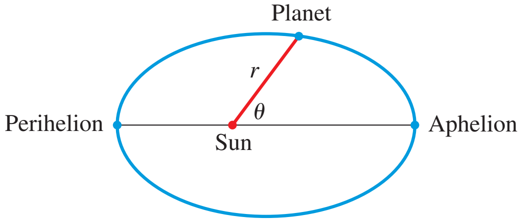
**(a)** Show that the polar equation of an ellipse with directrix $x = -d$ can be written in the form

$$r = \frac{a(1 - e^2)}{1 - e\cos\theta}$$

[*Hint:* Use the relation $a^2 = e^2d^2/(1 - e^2)^2$ given in the proof at the beginning of this section.]

**(b)** Find an approximate polar equation for the elliptical orbit of the earth around the sun (at one focus) given that the eccentricity is about 0.017 and the length of the major axis is about $2.99 \times 10^8$ km.

**44. Perihelion and Aphelion** The planets move around the sun in elliptical orbits with the sun at one focus. The positions of a planet that are closest to, and farthest from, the sun are called its **perihelion** and **aphelion**, respectively.




**(a)** Use Exercise 43(a) to show that the perihelion distance from a planet to the sun is $a(1 - e)$ and the aphelion distance is $a(1 + e)$.

**(b)** Use the data of Exercise 43(b) to find the distances from the earth to the sun at perihelion and at aphelion.

**45. Orbit of Pluto** The distance from Pluto to the sun is $4.43 \times 10^9$ km at perihelion and $7.37 \times 10^9$ km at aphelion. Use Exercise 44 to find the eccentricity of Pluto's orbit.

## Discuss ■ Discover ■ Prove ■ Write

**46. Discuss: Distance to a Focus** When we found polar equations for the conics, we placed one focus at the pole. It's easy to find the distance from that focus to any point on the conic. Explain how the polar equation gives us this distance.

**47. Discuss: Polar Equations of Orbits** When a satellite orbits the earth, its path is an ellipse with one focus at the center of the earth. Why do scientists use polar (rather than rectangular) coordinates to track the position of satellites? [*Hint:* Your answer to Exercise 46 is relevant here.]

# Chapter 10 Review

## Properties and Formulas

### Geometric Definition of a Parabola | Section 10.1

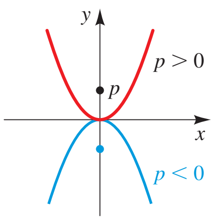
A **parabola** is the set of points in the plane that are equidistant from a fixed point $F$ (the **focus**) and a fixed line $l$ (the **directrix**).

### Graphs of Parabolas with Vertex at the Origin | Section 10.1

A parabola with vertex at the origin has one of the following standard equations.

**Vertical Axis**

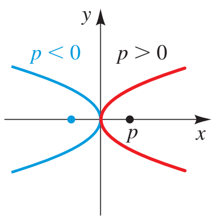
$x^2 = 4py$




Focus $(0, p)$, directrix $y = -p$

**Horizontal Axis**

$y^2 = 4px$




Focus $(p, 0)$, directrix $x = -p$

### Geometric Definition of an Ellipse | Section 10.2

An **ellipse** is the set of all points in the plane for which the sum of the distances to each of two given points $F_1$ and $F_2$ (the **foci**) is a fixed constant.

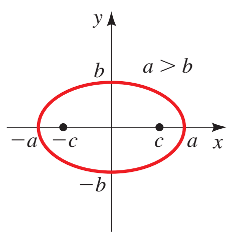
### Graphs of Ellipses with Center at the Origin | Section 10.2

An ellipse with center at the origin has one of the following standard equations.

**Horizontal Axis**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$(a > b > 0)$



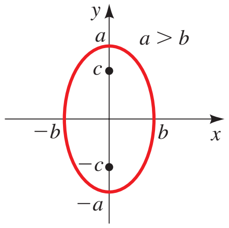

Foci $(\pm c, 0)$, $c^2 = a^2 - b^2$

**Vertical Axis**

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

$$(a > b > 0)$$

Foci $(0, \pm c)$, $c^2 = a^2 - b^2$




## Eccentricity of an Ellipse | Section 10.2

The **eccentricity** of an ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ or $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ (where $a > b > 0$) is the number

$$e = \frac{c}{a}$$

where $c = \sqrt{a^2 - b^2}$. The eccentricity $e$ of any ellipse is a number between 0 and 1. If $e$ is close to 0, then the ellipse is nearly circular; the closer $e$ gets to 1, the more elongated the ellipse becomes.

## Geometric Definition of a Hyperbola | Section 10.3

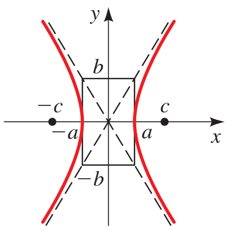
A **hyperbola** is the set of all points in the plane for which the absolute value of the difference of the distances to each of two given points $F_1$ and $F_2$ (the **foci**) is a fixed constant.

## Graphs of Hyperbolas with Center at the Origin | Section 10.3

A **hyperbola** with center at the origin has one of the following standard equations.

**Horizontal Axis**

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

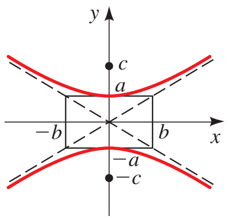



Foci $(\pm c, 0)$, $c^2 = a^2 + b^2$

Asymptotes: $y = \pm \frac{b}{a} x$

**Vertical Axis**

$$-\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$




Foci $(0, \pm c)$, $c^2 = a^2 + b^2$

Asymptotes: $y = \pm \frac{a}{b} x$

## Shifted Conics | Section 10.4

If the vertex of a parabola or the center of an ellipse or a hyperbola does not lie at the origin but rather at the point $(h, k)$, then we refer to the curve as a **shifted conic**. To find the equation of the shifted conic, we use the "unshifted" form for the appropriate curve and replace $x$ by $x - h$ and $y$ by $y - k$.

## General Equation of a Shifted Conic | Section 10.4

The graph of the equation

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

(where $A$ and $C$ are not both 0) is either a conic or a degenerate conic. In the nondegenerate cases the graph is

1. a **parabola** if $A = 0$ or $C = 0$,
2. an **ellipse** if $A$ and $C$ have the same sign (or a circle if $A = C$),
3. a **hyperbola** if $A$ and $C$ have opposite signs.

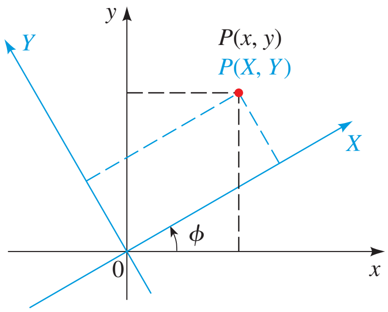
To graph a conic whose equation is given in general form, complete the squares in $x$ and $y$ to put the equation in standard form for a parabola, an ellipse, or a hyperbola.

## Rotation of Axes | Section 10.5

Suppose the $x$- and $y$-axes in a coordinate plane are rotated through the acute angle $\phi$ to produce the $X$- and $Y$-axes, as shown in the figure. Then the coordinates of a point in the $xy$- and the $XY$-planes are related as follows:

$$x = X\cos\phi - Y\sin\phi \qquad X = x\cos\phi + y\sin\phi$$

$$y = X\sin\phi + Y\cos\phi \qquad Y = -x\sin\phi + y\cos\phi$$




## The General Conic Equation | Section 10.5

The general equation of a conic is of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

The quantity $B^2 - 4AC$ is called the **discriminant** of the equation. The graph is

1. a parabola if $B^2 - 4AC = 0$,
2. an ellipse if $B^2 - 4AC < 0$,
3. a hyperbola if $B^2 - 4AC > 0$.

To eliminate the $xy$-term in the general equation of a conic, rotate the axes through an angle $\phi$ that satisfies

$$\cot 2\phi = \frac{A - C}{B}$$

## Polar Equations of Conics | Section 10.6

A polar equation of the form

$$r = \frac{ed}{1 \pm e\cos\theta} \qquad \text{or} \qquad r = \frac{ed}{1 \pm e\sin\theta}$$

represents a conic with one focus at the origin and with eccentricity $e$. The conic is

1. a parabola if $e = 1$,
2. an ellipse if $0 < e < 1$,
3. a hyperbola if $e > 1$.

## Concept Check

**1. (a)** Give the geometric definition of a parabola.

**(b)** Give the standard equation of a parabola with vertex at the origin and with vertical axis. Where is the focus? What is the directrix?

**(c)** Graph the equation $x^2 = 8y$. Indicate the focus on the graph.

**2. (a)** Give the geometric definition of an ellipse.

**(b)** Give the standard equation of an ellipse with center at the origin and with major axis along the $x$-axis. How long is the major axis? How long is the minor axis? Where are the foci? What is the eccentricity of the ellipse?

**(c)** Graph the equation $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$. What are the lengths of the major and minor axes? Where are the foci?

**3. (a)** Give the geometric definition of a hyperbola.

**(b)** Give the standard equation of a hyperbola with center at the origin and with transverse axis along the $x$-axis. How long is the transverse axis? Where are the vertices? What are the asymptotes? Where are the foci?

**(c)** What is a good first step in graphing the hyperbola that is described in part (b)?

**(d)** Graph the equation $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$. What are the asymptotes? Where are the vertices? Where are the foci? What is the length of the transverse axis?

**4. (a)** Suppose we are given an equation in $x$ and $y$. Let $h$ and $k$ be positive numbers. What is the effect on the graph of the equation if $x$ is replaced by $x - h$ or $x + h$ and if $y$ is replaced by $y - k$ or $y + k$?

**(b)** Sketch a graph of $\dfrac{(x+2)^2}{16} + \dfrac{(y-4)^2}{9} = 1$

**5. (a)** How can you tell whether the following nondegenerate conic is a parabola, an ellipse, or a hyperbola?

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

**(b)** What conic does $3x^2 - 5y^2 + 4x + 5y - 8 = 0$ represent?

**6. (a)** Suppose that the $x$- and $y$-axes are rotated through an acute angle $\phi$ to produce the $X$- and $Y$-axes. What are the equations that relate the coordinates $(x, y)$ and $(X, Y)$ of a point in the $xy$-plane and $XY$-plane, respectively?

**(b)** In the equation below, how do you eliminate the $xy$-term?

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

**(c)** Use a rotation of axes to eliminate the $xy$-term in the equation

$$25x^2 - 14xy + 25y^2 = 288$$

Graph the equation.

**7. (a)** What is the discriminant of the equation in Exercise 6(b)? How can you use the discriminant to determine the type of conic that the equation represents?

**(b)** Use the discriminant to identify the equation in Exercise 6(c).

**8. (a)** Write polar equations that represent a conic with eccentricity $e$. For what values of $e$ is the conic an ellipse? a hyperbola? a parabola?

**(b)** What conic does the polar equation $r = 2/(1 - \cos\theta)$ represent? Graph the conic.

**Answers to the Concept Check can be found at the book companion website stewartmath.com.**

## Exercises

**1–12 ■ Graphing Parabolas** An equation of a parabola is given. **(a)** Find the vertex, focus, and directrix of the parabola. **(b)** Sketch a graph of the parabola and its directrix.

**1.** $y^2 = 4x$ **2.** $x = \frac{1}{12}y^2$

**3.** $\frac{1}{8}x^2 = y$ **4.** $x^2 = -8y$

**5.** $x^2 + 8y = 0$ **6.** $2x - y^2 = 0$

**7.** $(y-2)^2 = 4(x+2)$ **8.** $(x+3)^2 = -20(y+2)$

**9.** $\frac{1}{2}(y-3)^2 + x = 0$ **10.** $2(x+1)^2 = y$

**11.** $\frac{1}{2}x^2 + 2x = 2y + 4$ **12.** $x^2 = 3(x+y)$

**13–24 ■ Graphing Ellipses** An equation of an ellipse is given. **(a)** Find the center, vertices, and foci of the ellipse. **(b)** Determine the lengths of the major and minor axes. **(c)** Sketch a graph of the ellipse.

**13.** $\dfrac{x^2}{9} + \dfrac{y^2}{25} = 1$ **14.** $\dfrac{x^2}{49} + \dfrac{y^2}{9} = 1$

**15.** $\dfrac{x^2}{49} + \dfrac{y^2}{4} = 1$ **16.** $\dfrac{x^2}{4} + \dfrac{y^2}{36} = 1$

**17.** $x^2 + 4y^2 = 16$ **18.** $9x^2 + 4y^2 = 1$

**19.** $\dfrac{(x-3)^2}{9} + \dfrac{y^2}{16} = 1$ **20.** $\dfrac{(x-2)^2}{25} + \dfrac{(y+3)^2}{16} = 1$

**21.** $\dfrac{(x-2)^2}{9} + \dfrac{(y+3)^2}{36} = 1$ **22.** $\dfrac{x^2}{3} + \dfrac{(y+5)^2}{25} = 1$

**23.** $4x^2 + 9y^2 = 36y$ **24.** $2x^2 + y^2 = 2 + 4(x - y)$

**25–36 ■ Graphing Hyperbolas** An equation of a hyperbola is given. **(a)** Find the center, vertices, foci, and asymptotes of the hyperbola. **(b)** Sketch a graph of the hyperbola.

**25.** $-\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$ **26.** $\dfrac{x^2}{49} - \dfrac{y^2}{32} = 1$

**27.** $\dfrac{x^2}{4} - \dfrac{y^2}{49} = 1$ **28.** $\dfrac{y^2}{25} - \dfrac{x^2}{4} = 1$

**29.** $x^2 - 2y^2 = 16$

**30.** $x^2 - 4y^2 + 16 = 0$

**31.** $\dfrac{(x+4)^2}{16} - \dfrac{y^2}{16} = 1$

**32.** $\dfrac{(x-2)^2}{8} - \dfrac{(y+2)^2}{8} = 1$

**33.** $\dfrac{(y-3)^2}{4} - \dfrac{(x+1)^2}{36} = 1$

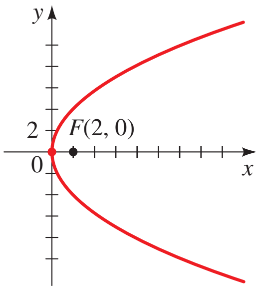
**34.** $\dfrac{(y-3)^2}{3} - \dfrac{x^2}{16} = 1$

**35.** $9y^2 + 18y = x^2 + 6x + 18$

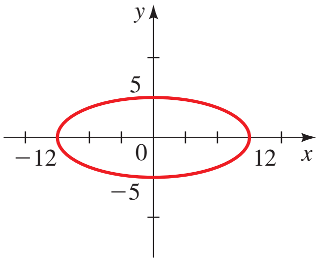
**36.** $y^2 = x^2 + 6y$

**37–42 ■ Finding the Equation of a Conic** Find the standard equation for the conic whose graph is shown.

**37.**




**38.**

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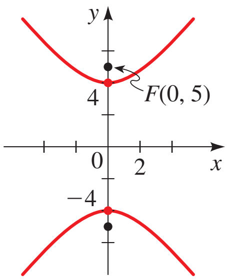


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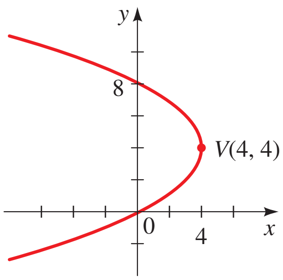
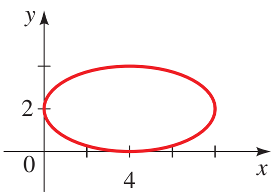
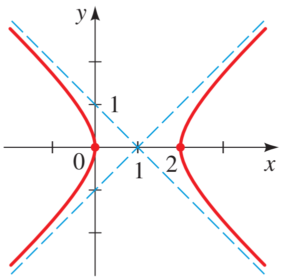



**41.**




**42.**




**43–54 ■ Identifying and Graphing a Conic** Determine whether the equation represents an ellipse, a parabola, a hyperbola, or a degenerate conic. If the graph is an ellipse, find the center, foci, and vertices. If it is a parabola, find the vertex, focus, and directrix. If it is a hyperbola, find the center, foci, vertices, and asymptotes. Then sketch the graph of the equation. If the equation has no graph, explain why.

**43.** $\dfrac{x^2}{12} + y = 1$

**44.** $\dfrac{x^2}{12} + \dfrac{y^2}{144} = \dfrac{y}{12}$

**45.** $x^2 - y^2 + 144 = 0$

**46.** $x^2 + 6x = 9y^2$

**47.** $4x^2 + y^2 = 8(x + y)$

**48.** $3x^2 - 6(x + y) = 10$

**49.** $x = y^2 - 16y$

**50.** $2x^2 + 4 = 4x + y^2$

**51.** $2x^2 - 12x + y^2 + 6y + 26 = 0$

**52.** $36x^2 - 4y^2 - 36x - 8y = 31$

**53.** $9x^2 + 8y^2 - 15x + 8y + 27 = 0$

**54.** $x^2 + 4y^2 = 4x + 8$

**55–64 ■ Finding the Equation of a Conic** Find an equation for the conic section with the given properties.

**55.** The parabola with focus $F(0, 1)$ and directrix $y = -1$

**56.** The parabola with vertex at the origin and focus $F(5, 0)$

**57.** The ellipse with center at the origin and with $x$-intercepts $\pm 2$ and $y$-intercepts $\pm 5$

**58.** The hyperbola with vertices $V(0, \pm 2)$ and asymptotes $y = \pm\frac{1}{2}x$

**59.** The ellipse with center $C(0, 4)$, foci $F_1(0, 0)$ and $F_2(0, 8)$, and major axis of length 10

**60.** The hyperbola with center $C(2, 4)$, foci $F_1(2, 1)$ and $F_2(2, 7)$, and vertices $V_1(2, 6)$ and $V_2(2, 2)$

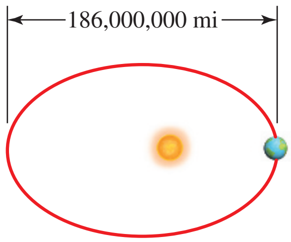
**61.** The ellipse with foci $F_1(1, 1)$ and $F_2(1, 3)$ and with one vertex on the $x$-axis

**62.** The parabola with vertex $V(5, 5)$ and directrix the $y$-axis

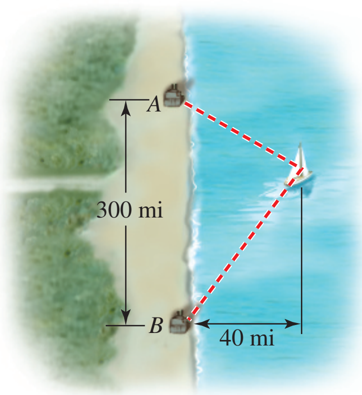
**63.** The ellipse with vertices $V_1(7, 12)$ and $V_2(7, -8)$ and passing through the point $P(1, 8)$

**64.** The parabola with vertex $V(-1, 0)$ and horizontal axis of symmetry and crossing the $y$-axis at $y = 2$

**65. Path of the Earth** The path of the earth around the sun is an ellipse with the sun at one focus. The ellipse has major axis of length 186,000,000 mi and eccentricity 0.017. Find the distance between the earth and the sun when the earth is **(a)** closest to the sun and **(b)** farthest from the sun.




**66. LORAN** A boat is located 40 miles from a straight shoreline. LORAN stations are located at points $A$ and $B$ on the shoreline, 300 miles apart. From the LORAN signals, the captain determines that the boat is 80 miles closer to $A$ than to $B$. Find the location of the boat. (Place $A$ and $B$ on the $y$-axis with the $x$-axis halfway between them. Find the $x$- and $y$-coordinates of the boat.)



**67. Families of Ellipses**

**(a)** Draw graphs of the following family of ellipses for $k = 1, 2, 4$, and 8.

$$\frac{x^2}{16 + k^2} + \frac{y^2}{k^2} = 1$$

**(b)** Prove that all the ellipses in part (a) have the same foci.

**68. Families of Parabolas**

**(a)** Draw graphs of the following family of parabolas for $k = \frac{1}{2}, 1, 2$, and 4.

$$y = kx^2$$

**(b)** Find the foci of the parabolas in part (a).

**(c)** How does the location of the focus change as $k$ increases?

**69–72 ■ Identifying a Conic** An equation of a conic is given. **(a)** Use the discriminant to determine whether the graph of the equation is a parabola, an ellipse, or a hyperbola. **(b)** Use a rotation of axes to eliminate the $xy$-term. **(c)** Sketch the graph.

**69.** $x^2 + 4xy + y^2 = 1$

**70.** $5x^2 - 6xy + 5y^2 - 8\sqrt{2}x + 8\sqrt{2}y - 4 = 0$

**71.** $7x^2 - 6\sqrt{3}xy + 13y^2 - 4\sqrt{3}x - 4y = 0$

**72.** $9x^2 + 24xy + 16y^2 = 25$

**73–76 ■ Identify a Conic from Its Graph** Use a graphing device to graph the conic. Identify the type of conic from the graph.

**73.** $5x^2 + 3y^2 = 60$

**74.** $9x^2 - 12y^2 + 36 = 0$

**75.** $6x + y^2 - 12y = 30$

**76.** $52x^2 - 72xy + 73y^2 = 100$

**77–80 ■ Polar Equations of Conics** A polar equation of a conic is given. **(a)** Find the eccentricity, and identify the conic. **(b)** Sketch the conic, and label the vertices.

**77.** $r = \dfrac{1}{1 - \cos\theta}$

**78.** $r = \dfrac{2}{3 + 2\sin\theta}$

**79.** $r = \dfrac{4}{1 + 2\sin\theta}$

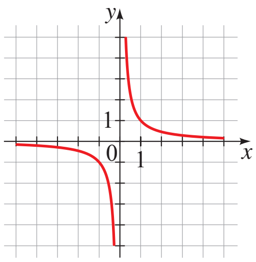
**80.** $r = \dfrac{12}{1 - 4\cos\theta}$

## Matching

**81. Equations and Their Graphs** Match the equation with its graph. Give reasons for your answers. (Don't use a graphing device.)

**(a)** $2x^2 + y^2 = 4$ **(b)** $3x^2 + 4y = 12$ **(c)** $x^2 + y^2 = 4$ **(d)** $x^2 + 2xy + y^2 + x - y = 0$

**(e)** $xy = 1$ **(f)** $4y^2 - x^2 + 8y = 0$ **(g)** $2y^2 - 5x - 4y = 8$ **(h)** $153x^2 + 192xy + 97y^2 = 225$

I
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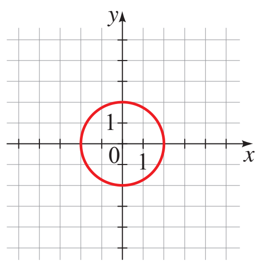


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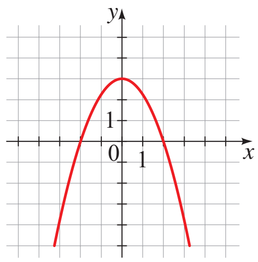


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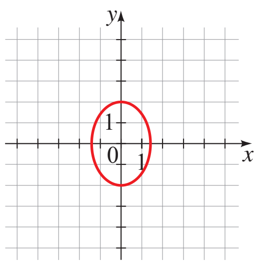


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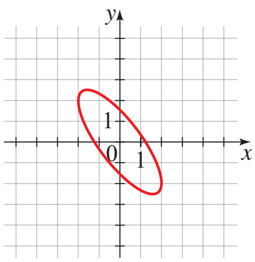


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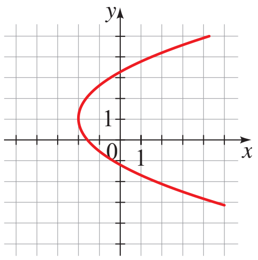


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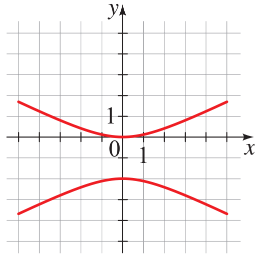


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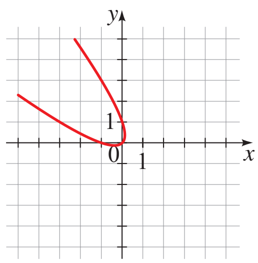


## Chapter 10 | Test

**1.** Find the focus and directrix of the parabola $x^2 = -12y$, and sketch its graph.

**2.** Find the vertices, foci, and the lengths of the major and minor axes for the ellipse $\frac{x^2}{16} + \frac{y^2}{4} = 1$. Then sketch its graph.

**3.** Find the vertices, foci, and asymptotes of the hyperbola $\frac{y^2}{9} - \frac{x^2}{16} = 1$. Then sketch its graph.

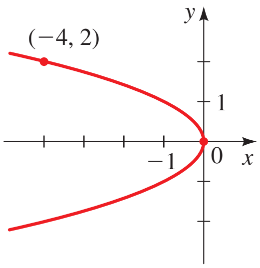
**4.** Find an equation for the parabola with vertex $(0, 0)$ and focus $(4, 0)$.

**5.** Find an equation for the ellipse with foci $(\pm 3, 0)$ and vertices $(\pm 4, 0)$.

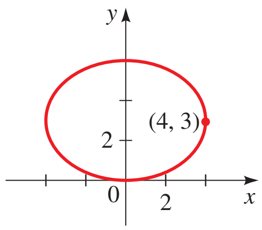
**6.** Find an equation for the hyperbola with foci $(0, \pm 5)$ and with asymptotes $y = \pm\frac{3}{4}x$.

**7–9** ■ Find the standard equation for the conic whose graph is shown.

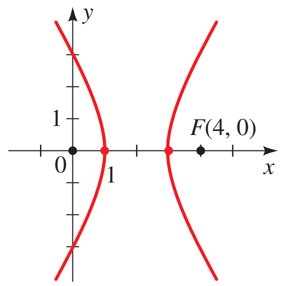
**7.**




**8.**




**9.**




**10–12** ■ Determine whether the equation represents an ellipse, a parabola, or a hyperbola. If the graph is an ellipse, find the center, foci, and vertices. If it is a parabola, find the vertex, focus, and directrix. If it is a hyperbola, find the center, foci, vertices, and asymptotes. Then sketch the graph of the equation.

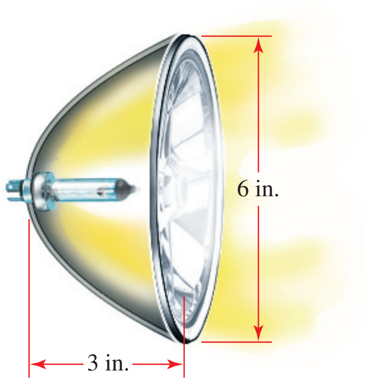
**10.** $16x^2 + 36y^2 - 96x + 36y + 9 = 0$

**11.** $9x^2 - 8y^2 + 36x + 64y = 164$

**12.** $2x + y^2 + 8y + 8 = 0$

**13.** Find an equation for the ellipse with center $(2, 0)$, foci $(2, \pm 3)$, and major axis of length 8.

**14.** Find an equation for the parabola with focus $(2, 4)$ and directrix the $x$-axis.




**15.** A parabolic reflector for a car headlight forms a bowl shape that is 6 inches wide at its opening and 3 inches deep, as shown in the figure at the left. How far from the vertex should the filament of the bulb be placed if it is to be located at the focus?

**16.** **(a)** Use the discriminant to determine whether the graph of the following equation is a parabola, an ellipse, or a hyperbola:

$$5x^2 + 4xy + 2y^2 = 18$$

**(b)** Use rotation of axes to eliminate the $xy$-term in the equation.

**(c)** Sketch a graph of the equation.

**(d)** Find the coordinates of the vertices of this conic (in the $xy$-coordinate system).

**17.** **(a)** Find the polar equation of the conic that has a focus at the origin, eccentricity $e = \frac{1}{2}$, and directrix $x = 2$. Sketch a graph of the conic.

**(b)** What type of conic is represented by the following equation? Sketch its graph.

$$r = \frac{3}{2 - \sin\theta}$$

**A Cumulative Review Test for Chapters 8, 9, and 10 can be found at the book companion website www.stewartmath.com.**

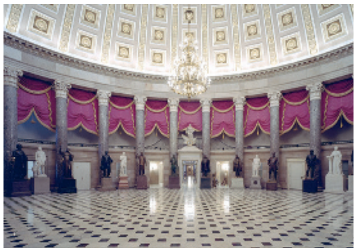
# Focus on Modeling | Conics in Architecture

Many buildings employ conic sections in their design. Architects have various reasons for using these curves, ranging from structural stability to simple beauty. But how can a huge parabola, ellipse, or hyperbola be accurately constructed in concrete and steel? In this *Focus on Modeling,* we will see how the geometric properties of the conics can be used to construct these shapes.

## Conics in Buildings

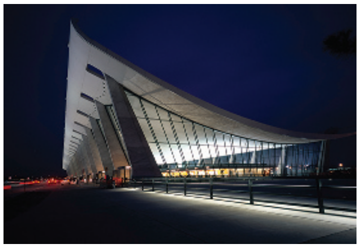
In ancient times architecture was part of mathematics, so architects had to be mathematicians. Many of the structures they built—pyramids, temples, amphitheaters, and irrigation projects—still stand. In modern times architects apply even more sophisticated mathematical principles. The photographs below show some structures that employ conic sections in their design.

Roman Amphitheater in Alexandria, Egypt (circle)
Nik Wheeler/Getty Images



Ceiling of Statuary Hall in the US Capitol (ellipse)
Architect of the Capitol

Roof of the Skydome in Toronto, Canada (parabola)
Walter Schmid/The Image Bank/Getty Images



Roof of Washington Dulles Airport (hyperbola and parabola)
Andrew Holt/Getty Images

McDonnell Planetarium, St. Louis, MO (hyperbola)
VisionsofAmerica/Joe Sohm/Getty Images

Attic in La Pedrera, Barcelona, Spain (parabola)
O. Alamany & E. Vicens/Getty Images

Architects have different reasons for using conics in their designs. For example, the Spanish architect Antoni Gaudí used parabolas in the attic of La Pedrera (see photo above). He reasoned that since a rope suspended between two points with an equally distributed load (as in a suspension bridge) has the shape of a parabola, an inverted parabola would provide the best support for a flat roof.

## Constructing Conics

The equations of the conics are helpful in manufacturing small objects, because a computer-controlled cutting tool can accurately trace a curve given by an equation. But in a building project, how can we construct a portion of a parabola, ellipse, or hyperbola that spans the ceiling or walls of a building? The geometric properties of the conics provide practical ways of constructing them. For example, if you were building a circular tower, you would choose a center point, then make sure that the walls of the tower were a fixed distance from that point. Elliptical walls can be constructed by using a string anchored at two points, as shown in Figure 1.

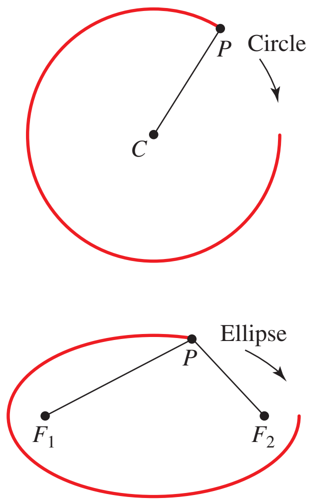



**Figure 1** | Constructing a circle and an ellipse

To construct a parabola, we can use the apparatus shown in Figure 2. A piece of string of length $a$ is anchored at $F$ and $A$. The T-square, also of length $a$, slides along the straight bar $L$. A pencil at $P$ holds the string taut against the T-square. As the T-square slides to the right, the pencil traces out a curve.

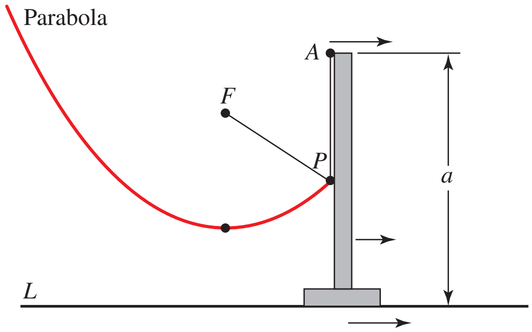



**Figure 2** | Constructing a parabola

From the figure we see that

$$d(F, P) + d(P, A) = a \qquad \text{The string is of length } a$$

$$d(L, P) + d(P, A) = a \qquad \text{The T-square is of length } a$$

It follows that $d(F, P) + d(P, A) = d(L, P) + d(P, A)$. Subtracting $d(P, A)$ from each side, we get

$$d(F, P) = d(L, P)$$

The last equation says that the distance from $F$ to $P$ is equal to the distance from $P$ to the line $L$. Thus the curve is a parabola with focus $F$ and directrix $L$.

In building projects, it is easier to construct a straight line than a curve. So in some buildings, such as in the Kobe Tower (see Problem 4), a curved surface is produced by using many straight lines. We can also produce a curve using straight lines, such as the parabola shown in Figure 3.

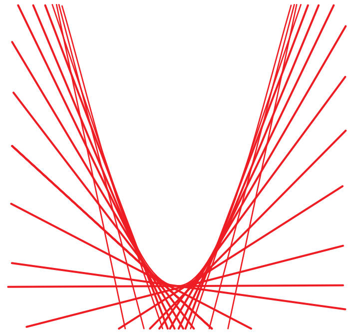


**Figure 3** | Tangent lines to a parabola

Each line is **tangent** to the parabola; that is, the line meets the parabola at exactly one point and does not cross the parabola. The line tangent to the parabola $y = x^2$ at the point $(a, a^2)$ is

$$y = 2ax - a^2$$

You are asked to show this in Problem 5. The parabola is called the **envelope** of all such lines.

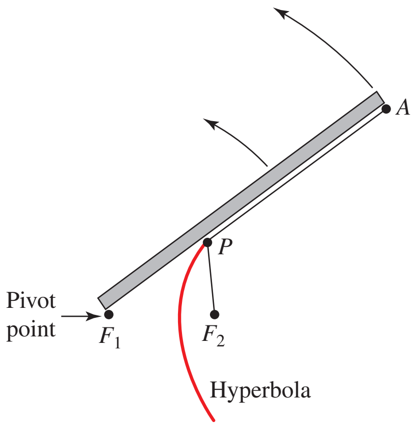
## Problems

1. **Conics in Architecture** The photographs at the beginning of this section show six examples of buildings that contain conic sections. Search the Internet to find other examples of structures that employ parabolas, ellipses, or hyperbolas in their design. Find at least one example for each type of conic.

2. **Constructing a Hyperbola** In this problem we construct a hyperbola. The wooden bar in the figure can pivot at $F_1$. A string that is shorter than the bar is anchored at $F_2$ and at $A$, the other end of the bar. A pencil at $P$ holds the string taut against the bar as it rotates counterclockwise around $F_1$.

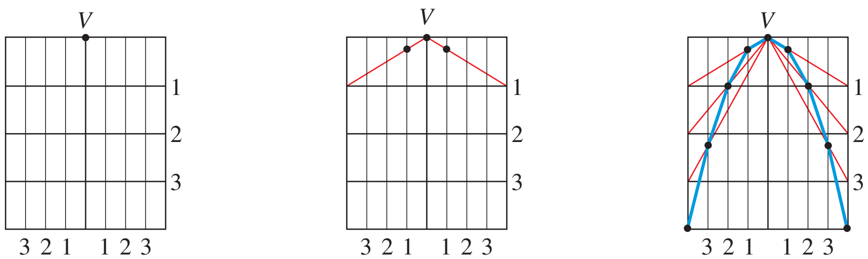
   **(a)** Show that the curve traced out by the pencil is one branch of a hyperbola with foci at $F_1$ and $F_2$.

   **(b)** How should the apparatus be reconfigured to draw the other branch of the hyperbola?




3. **A Parabola in a Rectangle** The following method can be used to construct a parabola that fits in a given rectangle. The parabola will be approximated by many short line segments.

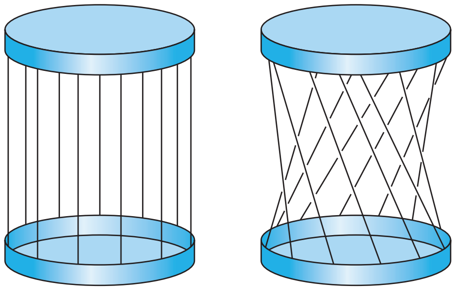
   First, draw a rectangle. Divide the rectangle in half by a vertical line segment, and label the top endpoint $V$. Next, divide the length and width of each half rectangle into an equal number of parts to form grid lines, as shown in the figure. Draw lines from $V$ to the endpoints of horizontal grid line 1, and mark the points where these lines cross the vertical grid lines labeled 1. Next, draw lines from $V$ to the endpoints of horizontal grid line 2, and mark the points where these lines cross the vertical grid lines labeled 2. Continue in this way until you have used all the horizontal grid lines. Now use line segments to connect the points you have marked to obtain an approximation to the desired parabola. Apply this procedure to draw a parabola that fits into a 6 ft by 10 ft rectangle on a lawn.

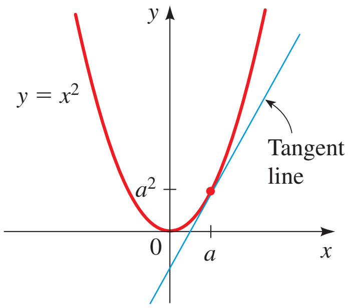



4. **Hyperbolas from Straight Lines** In this problem we construct hyperbolic shapes using straight lines. Punch equally spaced holes into the edges of two large plastic lids. Connect corresponding holes with strings of equal lengths as shown in the figure on the next page. Holding the strings taut, twist one lid against the other. An imaginary surface passing through the strings has hyperbolic cross sections. (An architectural example of this is the


Martin Mette/Shutterstock.com

Kobe Tower in Japan, shown in the photograph.) What happens to the vertices of the hyperbolic cross sections as the lids are twisted more?






**5. Tangent Lines to a Parabola** In this problem we show that the line tangent to the parabola $y = x^2$ at the point $(a, a^2)$ has the equation $y = 2ax - a^2$.

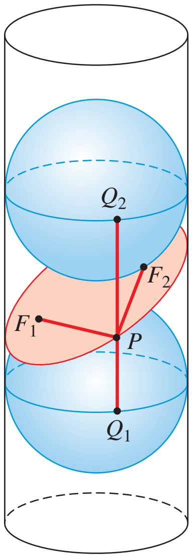
**(a)** Let $m$ be the slope of the tangent line at $(a, a^2)$. Show that the equation of the tangent line is $y - a^2 = m(x - a)$.

**(b)** Use the fact that the tangent line intersects the parabola at only one point to show that $(a, a^2)$ is the only solution of the system.

$$\begin{cases} y - a^2 = m(x - a) \\ y = x^2 \end{cases}$$

**(c)** Eliminate $y$ from the system in part (b) to get a quadratic equation in $x$. Show that the discriminant of this quadratic is $(m - 2a)^2$. Since the system in part (b) has exactly one solution, the discriminant must equal 0. Find $m$.

**(d)** Substitute the value for $m$ you found in part (c) into the equation in part (a), and simplify to get the equation of the tangent line.




**6. A Cut Cylinder** In this problem we prove that when a cylinder is cut by a plane, an ellipse is formed. An architectural example of this is the Tycho Brahe Planetarium in Copenhagen (see the photograph). In the figure, a cylinder is cut by a plane, resulting in the red curve. Two spheres with the same radius as the cylinder slide inside the cylinder so that they just touch the plane at $F_1$ and $F_2$. Choose an arbitrary point $P$ on the curve, and let $Q_1$ and $Q_2$ be the two points on the cylinder where a vertical line through $P$ touches the "equator" of each sphere.

**(a)** Show that $PF_1 = PQ_1$ and $PF_2 = PQ_2$. [*Hint:* Use the fact that all tangents to a sphere from a given point outside the sphere are of the same length.]

**(b)** Explain why $PQ_1 + PQ_2$ is the same for all points $P$ on the curve.

**(c)** Show that $PF_1 + PF_2$ is the same for all points $P$ on the curve.

**(d)** Conclude that the curve is an ellipse with foci $F_1$ and $F_2$.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