The mathematical daisy

The inner florets of a daisy—and many other plant parts—grow in spiral patterns that are fundamentally linked to the theory of numbers and the golden ratio known to the mathematicians of ancient Greece
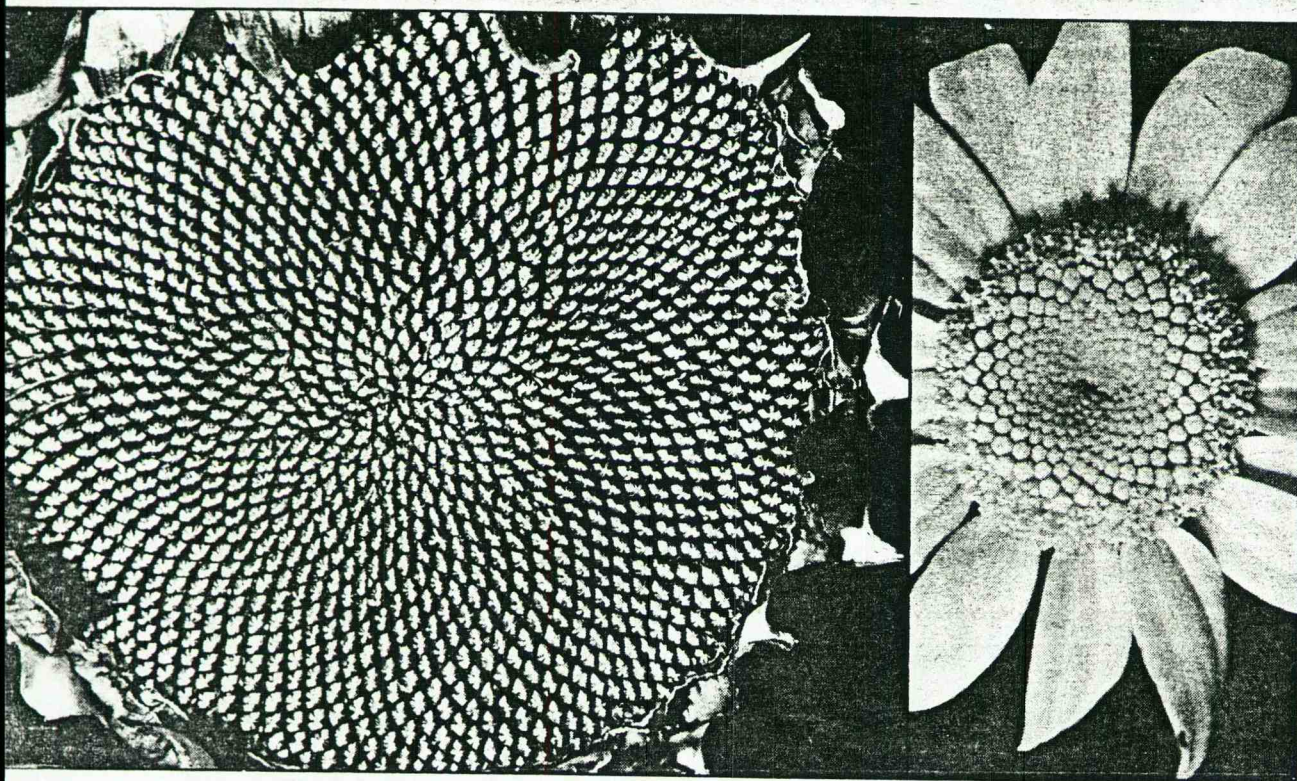

Figure 1 *A daisy-head (far left) and a sunflower show spiral patterns curling clockwise and anticlockwise from the centre. The florets, and ultimately the seeds, are generated at the centre, one after another in a single sequence. Each floret is pushed from the centre as new ones form, and as older florets grow in size. The single spiral of growth generates apparent "secondary" spirals. Count the number of spirals you see in either direction. The numbers you will find are Fibonacci numbers—members of the series 1,1,2,3,5,8,13,21, . . .*

Robert Dixon

Why do plants often appear to follow particular patterns as they grow? How do they know which pattern to follow? Can we explain the patterns using the theory of numbers, and if so, why? These questions had been intriguing me for some time when I realised that modern computer graphics provides the ideal way to draw out patterns of plant growth according to specified mathematical rules. Thus with the aid of the computer I have investigated the problem of spiral growth in a variety of plants.

The patterns into which plants grow represent optimum solutions to the geometric problems they face: of how to pack all the constituent parts into a structure that can grow continuously, or how to occupy a volume of air and receive sunlight with maximum economy. Botanists and mathematicians use the term "phyllotaxis", whose original meaning is "leaf-arrangement", when classifying and analysing the arrangements of any repetitive parts of a plant, including florets, seeds, petals, branches and so on. One elementary and widespread pattern of such parts in plant life is the formation of a particular type of spiral, named after the medieval Italian mathematician, Leonardo Fibonacci.

I first looked at this pattern as it occurs in composite flower heads, such as daisies. The way the florets are packed together on a daisy head looks somewhat similar to the structure of a honeycomb from a beehive, but the differences are significant. Instead of a symmetrical array of identical units, as in the honeycomb, the daisy is a collection of florets all at different stages of growth. What you see if you look carefully at a daisy

are spirals, running both clockwise and anticlockwise (Figure 1). If you count the number of spirals in either direction you will find that the total is not any number, but a number that is a member of the so-called Fibonacci series: 1,1,2,3,5,8,13, . . . , where the three dots indicate that the series continues indefinitely.

Fibonacci published his influential *Liber Abaci* in 1202. It is a book on the abacus in which he set down the case for adopting the Arabic system of notation for numbers instead of the Roman—29 instead of XXIX, and so on. He also posed a problem about the breeding pattern of rabbits whose periods of maturation and gestation are both one month. If each pregnancy yields one new pair, and if you start with a single newborn pair on 1 January, how many pairs will there be on the first day of subsequent months? Answer: a series of numbers which has since been named after Fibonacci.

But why do the spirals in daisies occur only in the quantities given by the Fibonacci series? Of course this is really two questions rolled into one: why spirals, and why Fibonacci numbers? The simple numerical pattern in the Fibonacci series does not suggest an obvious explanation for "Fibonacci spirals" in plants. In fact, despite the almost perennial appearance of fresh studies dating back to the 1830s, when a rigorously mathematical approach to the puzzle was first introduced, it has proved difficult to provide a completely satisfactory explanation.

The most successful line of analysis begins with the observation that the different spirals that you see on the daisy head are formed rather by a single spiral sequence of florets. And in this sequence the florets follow each other at an angle of $137 \cdot 50776$ . . .°. The single spiral is the "primary" spiral; the apparent spirals that correspond to the Fibonacci numbers are "secondary" spirals.

I shall come back to the significance of the precise angle of 137·50776 . . . °; suffice it to say that this special angle not only produces the apparent secondary spirals, but also provides a uniquely flexible design for the harmonious and efficient spacing of plant parts repeated around a central stem, even as they grow. The mathematical difficulties lie in turning these intuitively spatial ideas into a sound theory.

While the analysis and proofs are difficult, the basic geometric ideas of a spiral sequence and a regular angular interval are relatively simple. It is possible to consider the pattern visually, and in particular computerised drawing allows one to construct the mathematically-idealised pattern with great speed and accuracy. But more of this shortly.

Besides the daisy head, other common examples of Fibonacci phyllotaxis—plant growth on Fibonacci spirals —include the arrangement of the sunflower's seeds, the pine cone, the petal sequence in a rose or a lotus, the sequence of leaves on a thistle, the fruit partitions of a pineapple, and the succession of twigs branching from the stem of a pear tree. The form of a daisy head is a flat disc with a central point, while other examples take the forms of a cone or a cylinder. This variety of forms is unified, however, in the continuous spatial transformation from one to another: a cylinder becomes a cone by reducing the radius of one end to a point; a cone becomes a flat disc by reducing its height to zero. And when a flat disc is extended to form a cylinder, the pattern of spirals on the disc becomes a pattern of helices.

The pattern of leaves growing from a stem reflects an elementary predicament that plants face—that of how to occupy space, collect sunlight and breathe, in the most economic way. To begin with, a plant grows along an axis, extending its occupation of space along one line to gather more and more sunlight. Then periodically it sprouts leaves which branch out from the stem to occupy the surrounding space. But in which direction do they sprout? At every point on the stem the plant has 360° around the stem to choose from. In response to this choice, plants have evolved several systematic branching patterns, each species following one or other. These patterns represent the relatively few optimum solutions to the geometric problem. One such pattern is the spiral/helical succession of branches at every 137·50776 . . . °, or Fibonacci phyllotaxis (Figure 2).
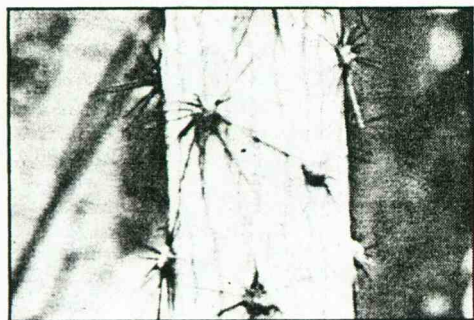
To see why this is an ideal angle, or to see why any interval of spiral succession is better than any other, consider the Sun's-eye view of the plant after several leaves have sprouted. Each leaf needs its own equal place in the Sun. So each new leaf seeks a direction from the stem not already occupied by a lower leaf. And one by one the available gaps are further divided, as the circle becomes crowded with more and more leaves. The significance of 137·50776 . . . ° is that it is the angle which most evenly and gradually divides the circle.

You might wonder what would happen if other angles were used as the repeated interval, and so query the uniqueness of 137·50776 . . . °. First, any exact fraction of a whole circle has the drawback of repeating itself after a finite sequence of directions. This leaves the full circle of directions with gaps which can never be closed.

But what happens if each new member of the sequence falls between the direction of two previous members? If it falls exactly mid-way between its two neighbours then the sequential interval is again an exact fraction with its consequent drawback. On the other hand, it is no good if the new member is too near either of its neighbours.

With an interval of 137·50776 . . . °, each new member of the sequence divides the gap between its two preceding nearest neighbours in the so-called "golden mean". Describing this angle of 137·50776 . . . ° in degrees disguises its identity somewhat. It is an expression of Euclid's "golden section", but in a less traditional form—the golden section of the circle. Three geometric figures that traditionally express the "golden ratio" are the golden section of a straight line segment, the golden rectangle, and the regular pentagon. Figure 3 shows all four examples together.

Figure 3 *The golden section, golden rectangle, regular pentagon and circle provide four ways of expressing the golden ratio, τ:1, where τ=1·618034 . . .*

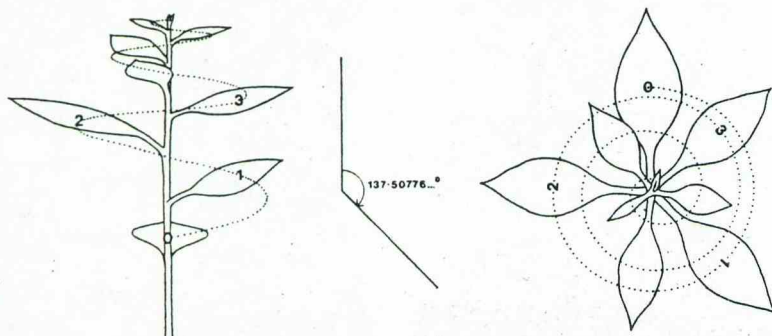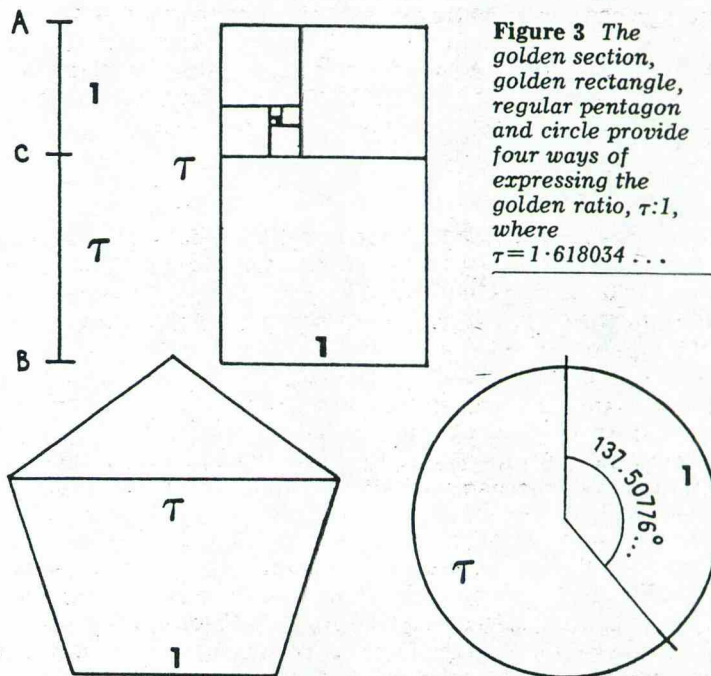
Figure 2 *The ideal spiral arrangement for leaves and for branches—every 137·50776 . . .°. Many plants show a tendency to approximate to this pattern—as does the* Pereskia Grandiflora, *left*

If you divide a straight line *AB* at *C* so that the larger part (*BC*) to the smaller (*AC*) bears the same ratio of lengths as the whole segment (*AB*) to the larger part, then you have made a golden section. Contemporary mathematicians denote the ratio *BC*:*AC* by τ:1, using the Greek letter tau after *tomos* which means cut. The definition puts the precise value of τ = 1·618034 . . . on the golden ratio.

A rectangle whose proportions are in the golden ratio, that is, whose long side is τ times the length of the shorter, is a golden rectangle. The diagram shows its inherent pattern of recursive squares. Remove a square

from a golden rectangle—the square which fits the shorter side—and you are left with another, smaller golden rectangle. And so on indefinitely. The sequence of squares spiral forever inwards towards a limiting point—following a logarithmic spiral. Conversely if you add the square to a golden rectangle you get a larger rectangle, which is itself a golden rectangle. The square is an addition which leaves the shape of the golden rectangle unchanged, while changing its size and orientation. The addition increases the size by the same proportion every time, giving a pattern of uniform growth.

In a regular pentagon a diagonal, the line joining any two non-adjacent corners, is τ times as long as a side. Not for nothing was the pentagram, the continuous star formed by all five diagonals of a regular pentagon, the symbol of membership to Pythagorean brotherhood.

The golden section of a circle divides the whole circumference into two arcs, the longer being τ times the shorter. Again, the ratio of the longer to the shorter part is the same as that of the whole to the longer. But unlike the straight-line segment which has two ends and needs only one cut, the circle has no end and needs two cuts. The golden section of a circle subtends a "golden angle" at the centre—which is none other than 137·50776 . . .°.

So there is an ideal angle that guarantees that the parts of a plant—be they seeds, leaves, florets, twigs or petals—which branch out in spiral succession will fall into place in an ideal way, and this angle is the golden section of a circle. But why do the Fibonacci numbers turn up in daisies and other plants? The answer to this question lies in the number theory of the golden ratio.

Certain geometric quantities cannot be exactly represented in whole-number ratios. This discovery troubled the Greeks, and continues to this day to pose a tricky hurdle in our mathematical education. Classic examples are the relative sizes of the circumference and diameter in a circle, given by $\pi = 3 \cdot 1426 \ldots$, and the relative sizes of diagonal and side in a square, given by $\sqrt{2} = 1 \cdot 414 \ldots$—post-Greek mathematicians describe such quantities as *irrational* numbers (no insanity implied).

So, the circumference of a circle is approximately three times the diameter, but not quite—22/7 is a better approximation; 31 416/10 000 is much closer, but still not exact. The decimal notation, $3 \cdot 1416 \ldots$, expresses this last approximating ratio of whole numbers, and also indicates in the three dots that better and better approximations can be obtained indefinitely by finding the value of more and more decimal places. Another such geometric quantity which cannot be exactly expressed in whole numbers is the golden ratio, τ. Like $\sqrt{2}$, but unlike π, τ is the solution to an algebraic equation, namely $x^2 = x + 1$.

In decimal notation, and for all practical purposes, $\tau = 1 \cdot 618034 \ldots$ But if we express τ as a decimal we miss the fundamental truth—that τ can be derived, or at least approximated, by coupling Fibonacci numbers (2, 3, 5, 8, 13, . . .). Thus a rough approximation to the value of τ is given by 3/2, the first two numbers in the Fibonacci series greater than 1. But 5/3 approximates more closely; 8/5 even more closely, and so on. The fractions in this series, 3/2, 5/3, 8/5, 13/8, 21/13, . . . , approach successively closer to the limiting value of τ, while alternating above and below its ultimate value.

In this sequence of fractions which converge on the value of τ, the numerators (above the lines) and the denominators (below them) are from the Fibonacci series. The irrational value of τ therefore finds a completely ordered sequence of approximations in the whole numbers. Moreover, these whole numbers are the num-
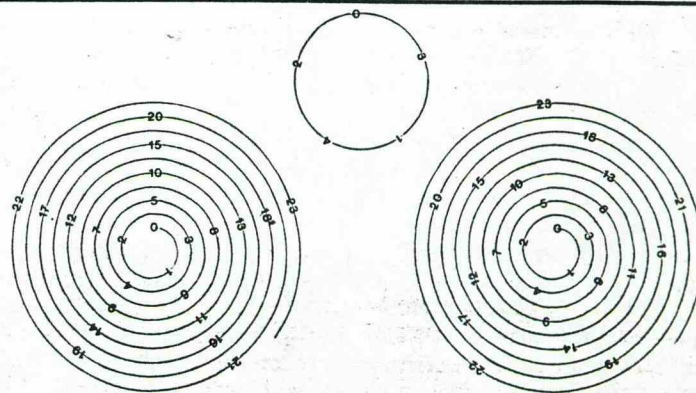

Figure 4 *Repeated steps of 2/5 of a turn on a circle give five positions (top); on a spiral, five rays emerge (left). But a step of slightly less than 2/5, creates five secondary spirals (right)*

bers which turn up in daisies and so on. But how do numerators and denominators become spirals?

In Figure 4 I try to explain the answer to this last question. The first diagram shows the sequence of positions reached by a moving point circling clockwise through regular discrete steps of 2/5 of a whole turn. After five steps—the number given by the denominator—you come back to where you started, after which you repeat the same five positions over and over. The next diagram shows what happens if at the same time as rotating clockwise you steadily increase the radial distance from the centre, forming a spiral sequence. The moving point now reaches a new position every time, but they are restricted to five directions from the centre. The positions together form a pattern of five straight rays. If the fraction of rotation had been, say 3/8 of a turn instead, the pattern would have been eight straight rays. The denominator gives the number of rays.
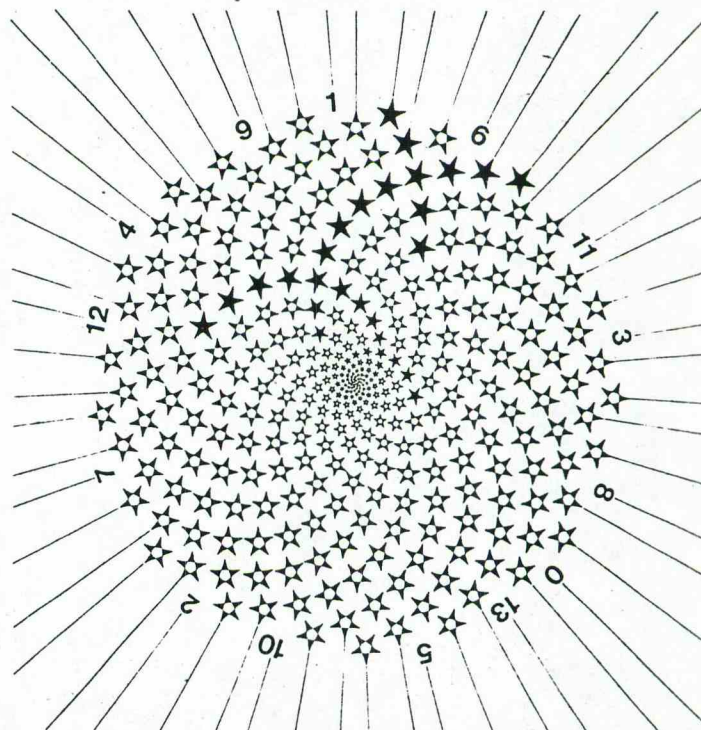

Keith Laws

Figure 5 *An "electronic" daisy drawn by computer graphics. The single spiral sequence starts at the centre, each floret following every 137·50776 . . .°. The spiral winds more than 100 times before it reaches the periphery. The outer 14 florets—which would be the first 14 formed in a plant—have been replaced by numbers 0-13. These florets lie on the primary spiral which connects them in order. The florets grow before reaching a maximum size; as they grow the Fibonacci numbers of secondary spirals emerge*

The third diagram of figure 4 modifies the pattern a stage further. Instead of an exact 2/5 of a turn I have used an angular interval which is nearly, but slightly less than 2/5. The resultant pattern approximates to the previous diagram, but the five straight rays become five *spirals*, displaying a slightly out-of-phase effect. This is how secondary spirals emerge from the single primary generating spiral sequence. The figure also shows that the number of secondary spirals is the denominator of the fraction to which the angular interval approximates —2/5. Similarly you generate Fibonacci spirals when the angular interval is $1/\tau$ of a turn, the golden angle.
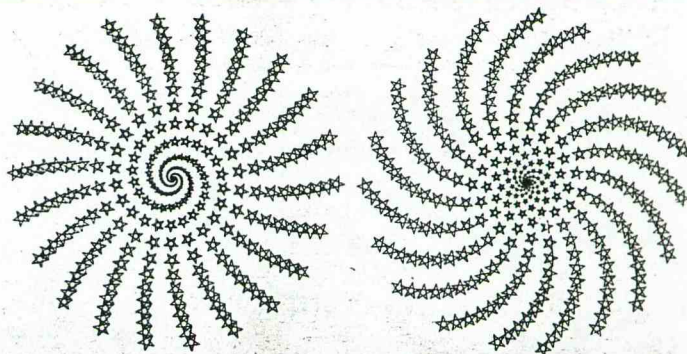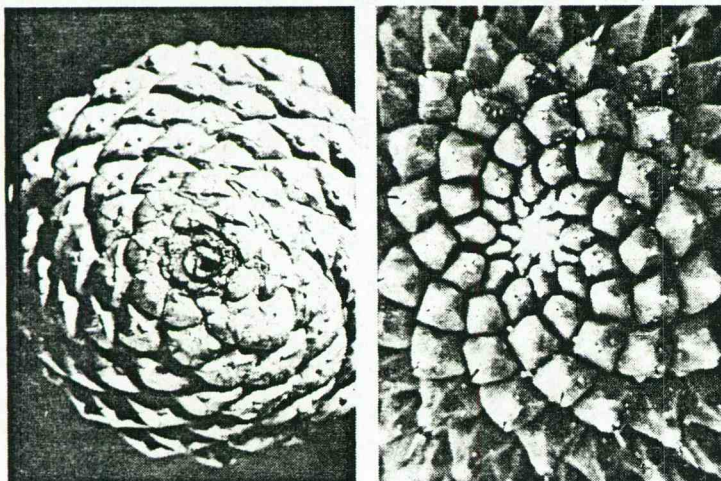
Spiral growth—on a cone (left) and a succulent

Computer graphics provides a powerful means of exploring patterns like Fibonacci phyllotaxis. The computer turns theory directly into a drawing—Figure 5 shows such an "electronic daisy". The tiny pentagrams represent the florets, in a sequence that begins at the centre and spirals steadily outwards at each turn of the golden angle. I allowed the florets to grow non-uniformly (increasing exponentially at first, but slowing down to an upper limiting size) and then formulated the primary spiral to enclose an area which is always in step with the accumulating area of florets. The results reveal secondary Fibonacci spirals and a quasi-regular space-filling array.

Also, the spirals of contacting florets change from one pair of Fibonacci numbers to another. This corresponds to the increasing number of florets needed to pack around the growing circumference as the growth rate of the florets slows down. The 8 by 13 system at the centre gives way to a 13 by 21 system, followed by a 21 by 34 system at the periphery. While the growth rate is exponential (increasing by the same proportion from one term to the next) the Fibonacci numbers of contacting florets stays the same. But the smooth way in which the contacting systems switch from one set to another without the pattern departing too far from a regular packing is perhaps the crucial and unique property of Fibonacci phyllotaxis. The comparable phenomenon using non-golden angles is always less regular.

Having seen the computer demonstrating how a plant can achieve such an efficient arrangement by using the golden angle, I was eager to see what happens when you use other angles. Figure 6 shows what happened when I repeated the drawing of the electronic daisy but with different irrational angles, namely, $1/\pi$ and $1/e$ ($e = 2\cdot718 \ldots$). The units I am using here are whole turns.

Again, one spiral sequence makes a pattern of secondary spirals. They occur in numbers which are not Fibonacci numbers, but the denominators of the relevant

Figure 6 *"Electronic" daisies based on non-golden angles—$1/\pi$, left, and $1/e$. In each of these the approximating whole number fractions are indicated by the numbers of secondary spirals*

approximating whole number fractions. For example, the first example in Figure 6 uses the familiar irrational number π, in the form of the fraction of a turn $1/\pi$. The first whole number fraction to approximate to this is 1/3, followed by 7/22; thus three spirals dominate the centre of the pattern, slowly switching to 22 spirals.

The mathematics of the electronic daisy describe the path followed by the computerised pen-point across the drawing paper plane. This is not to be confused with the path along which a flower grows. Each drawing represents a sort of snap-shot in time. To represent the whole growth of the flower requires a sequence of drawings with good continuity from one to another. They could be shown in rapid succession on a single screen, making an animated film. A continuous sequence of Fibonacci patterns of this kind represents an ideal pathway of quasi-regular packing, along which a flower can flow—a pathway of least resistance. □