CSCI 1311: Problem Set 7

Early Deadline: 1 May 2023

Final Deadline: 3 May 2023

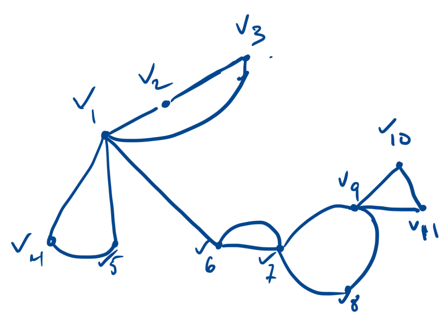
Instructions:

- You will submit your answers to via gradescope.
- Refer to gradescope to determine which questions or sub-questions are submitted individually.
- For each submission of a question or sub-question, you should submit an image of your answer, which can be a picture of your handwriting, a screenshot of a tablet, an image of something you typed, or a scan of a document. The most important point is that it's legible and clearly marked in gradescope.
- Submission by the *Early Deadline* will receive a 10 point bonus. Submission after the *Final Deadline* will not be accepted.
- You may ask to use your *one time amnesty* for this assignment to receive two additional days past the *Final Deadline*.
- Note that this problem set has 110 points, but it will be graded out of 100 points. So the max score you can get, with an early deadline bonus, is 120/100.

Question Weighting

Question:	1	2	3	4	5	Total
Points:	35	25	10	10	30	110

1. Consider the following underacted graph, G.



   (a) **[5 points]** What is the matrix representation of the graph?

(b) **[5 points]** Using the matrix representation, how many closed walks exist in the graph of length 3 that start/end at any given vertex?

(c) **[2 points]** Does the graph contain a Euler circuit? If so, why must it and provide such a circuit, if not, why not? For a negative answer, you should provide a proof; more than, it doesn't exist or you can't find one.

(d) **[3 points]** Does the graph contain a Euler trail? If so, why must it and provide such a circuit, if not, why not? For a negative answer, you should provide a proof; more than, it doesn't exist or you can't find one.

(e) **[5 points]** Does the graph contain a Hamiltonian Circuit? If so, provide it, otherwise, if not, why not? For a negative answer, you should provide a proof using an example from the graph; more than, it doesn't exist or you can't find one.

(f) **[5 points]** An *articulation point* (or *single point of failure*) of a connected graph is a node whose removal disconnects the graph. List the articulation points of G, if any.

(g) **[5 points]** Perform *two* Depth-First-Search traversal of the graph starting with vertex v_1 and v_7. Break ties by always choosing the edge that connect to a vertex of lower number first.

(h) **[5 points]** Perform *two* Breadth-First-Search traversals of the graph starting with vertex v_1 and v_7. Break ties by always choosing the edge that connects to a vertex of lower number first.

2. Consider the following simple graph $G = \{E, V\}$:

$$\begin{aligned} V =& \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\} \\ E =& \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_3, v_4\}, \{v_3, v_7\}, \{v_4, v_5\} \\ & \{v_5, v_7\}, \{v_6, v_7\}, \{v_7, v_8\}\} \end{aligned}$$

(a) **[10 points]** The *distance* function $d(u, v)$ between two vertices u and v is the length of the shortest path, as counted by the number of edges crossed in that path. Compute the distance matrix A based on G such that $d_{ij} = d(i, j)$. *Hint: consider BFS as a way to compute distances*

(b) **[2 points]** The *diameter* of a graph is the largest distance in that graph. What is the diameter of G?

(c) **[3 points]** The *radius* of the graph from a vertex v is the distance from v to the farthest node. For each vertex v in G, compute the radius of G from v.

(d) **[5 points]** The *center* of a graph is a vertex v where the radius of G from v is the smallest. If there are ties, the graph has multiple centers. Find the center(s) of G.

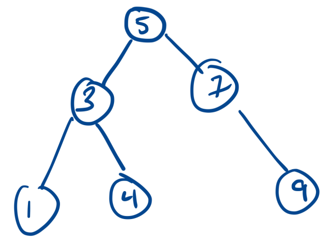
(e) **[5 points]** The *average radius* of the graph from a vertex u is the average distance from x to all the nodes of the graph.

For each vertex u in G, compute the average radius of G from u. Which vertex/vertices has/have the smallest average radius?

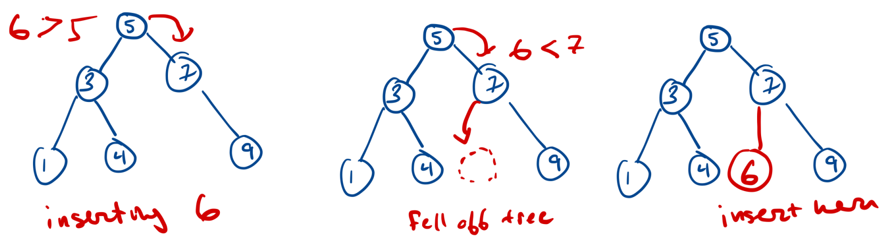
3. **[10 points]** Consider a graph where there exists a single center vertex v. The distance to that vertex is r. Prove that the diameter d of the graph is at most $d = 2r$. ***See the previous question for the definition of center and diameter of the***

4. **[10 points]** Let T_a and T_b be trees. Prove that if you create a single edge from any vertex in T_a to any vertex in T_b the resulting graph T_{ab} is also must be a tree.

5. A **binary search tree** (BST) is a rooted tree where (1) every vertex holds a numerical value (called a key), and (2) for any internal vertex in the tree (v), the keys of the vertices to the left of v is less-than-or-equal-to the key of v, the keys of the vertices to the right of v is greater-than-or-equal-to the key of v. For example, the following is a valid (non-full) BST:



If I were to insert a new vertex into the BST, say 6, I would start at root, and ask, is this vertex less than or equal to the given vertex? If it is greater than, I go right, if less than (or equal) I go left. Then I'd ask the same of the next vertex, and so on, until I fall off the tree; that's where the new vertex is inserted. Like in the visual below:



(a) **[10 points]** Prove (by induction on the number of vertices in the tree) that the largest key in a BST must always be associated with the right-most vertex. That is, the vertex that you reach by always taking the right branch from the root of the tree until you reach a leaf.

(b) **[20 points]** Prove (by induction on the height of the tree) that if the BST is a *perfect* or *full* binary tree with height $h \geq 0$, than the root of the tree must be median value of all the keys, that is it is middle value $n/2$ if the all the keys were aligned in a list of n values.