Lec 02: Sets II

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Sets, so far...

- Sets are a collection of unique items, including other sets, that are designated using braces.
  - $\{1,2,3\}$, $\{red, blue\}$, $\{\}$, $\{5,\{4,3\},8\}$
- We described an element as a member of a set using $\in$ or not as a member using $\notin$.
  - $1 \in \{1,2,3\}$ and $7 \notin \{1,2,3\}$
- Two sets are equal $=$ if they contain the same elements, the order is irrelevant.
  - $\{1,2,3\} = \{2,1,3\}$ and $\{1,2,3\} \neq \{2,3,4\}$
- Set builder notation: $\{x \in S \mid P(x)\}$
  - $\{x \in \mathbb{Z} \quad | \quad x > 0 \text{ and } x \neq 15\}$
- Cardinality of a set is the number of elements contained in the set
  - $|\{1,2,3\}| = 3 \quad |\{1,\{2,3\}\} = 2 \quad \{x \in \mathbb{Z} \quad | \quad x > -5\} = \infty$
- If a the cardinality of a set is zero, we call that the empty set
  - $|\{\}| = 0$, $|\emptyset| = 0$
  - But the empty set can be member of a set, $|\{\{\}\}| = |\{\emptyset\}| = 1$

Today

- Subsets
- Union, Intersection, Complement and Difference
- Partitions
- Power Sets
- Order Pairs and Tuples
- Cartesian Products

Subsets

Definition

For two sets A and B, A is a subset of B (written $A \subseteq B$) if for all elements $x \in A$ then $x \in B$.

Example

For the set $A = \{9, 7, 22\}$ and $B = \{9, 22, 18, 42, 7\}$, $A \subseteq B$ (or $B \supseteq A$) because all the elements of A are also in B, that is $9 \in B$, $7 \in B$ and $22 \in B$.

Definition

A proper subset (written $A \subset B$) is a subset whereby $A \subseteq B$ but there exists at least on element $x \in B$ such that $x \notin A$

Exercises

For the set $A = \{42, 18, 3, 22\}$ and $B = \{22, 3, 42, 18\}$,
is $A \subset B$?
Is $A \subseteq B$?

For two sets, C and D, where $C = D$, is $C \subseteq D$?

For two sets E and F, if $E \subseteq F$ and $E \supseteq F$, then it must be the case that $E = F$. Why?

Is $\emptyset \subset \{5, 12, 2, 19\}$?

Is $\emptyset \subseteq \emptyset$?

Union and Intersection

Definition

The union of two sets A and B, written $A \cup B$, is the set that contains all elements in A *or* B.

Definition

The intersection of two sets A and B, written $A \cap B$, is the set that contains all elements in A *and* B.

Example

If $A = \{12, 7, 19, 5\}$ and $B = \{12, 3, 22, 42, 5\}$ then

$A \cup B = \{12, 7, 19, 5, 3, 22, 42\}$ and
$A \cap B = \{12, 5\}$

Complements or Difference

Definition

The difference between set B and A (or the relative complement of A in B), written $B \setminus A$ or $B - A$, is the set of elements in B that are not in A.

Example

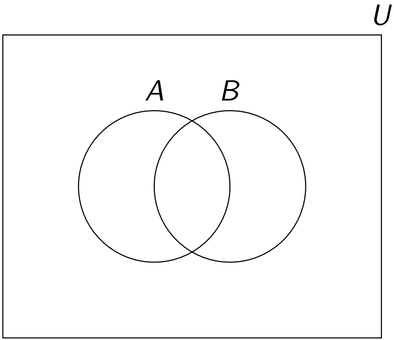
If $A = \{12, 7, 19, 5\}$ and $B = \{12, 3, 22, 42, 5\}$ then

$A \setminus B = A - B = \{7, 19, 5\}$ and
$B \setminus A = B - A = \{3, 22, 42\}$

Definition

The compliment of set A, written A^c is the set of all elements in the universe U from which sets are populated that are not in A.

Set Operators as Venn Diagrams



Exercises

For any set A, what is
$A \cup \emptyset$?
$A \cap \emptyset$?
$A \cup A$?
$A \cap A$?

Write the set builder notation equivalent for each operation, assuming a universal set U.
$A \cup B$
$A \cap B$
$B - A$
A^c

Operators on Indexed Collections of Sets

Consider an *indexed* collection of sets $A_1, A_2, A_3, \ldots, A_n$ where n is a positive integer, then we can define the following index operations over the sets:

- $\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n = \{x \,|\, x \in A_j \text{ for some } j = 1, 2 \ldots, n\}$
- $\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n = \{x \,|\, x \in A_j \text{ for all } j = 1, 2, \ldots, n\}$

We can even define these up to ∞:

- $\bigcup_{i=1}^{\infty} A_i = A_1 \cup A_2 \cup A_3 \cup \ldots = \{x \,|\, x \in A_j \text{ for some } j = 1, 2, \ldots\}$
- $\bigcap_{i=1}^{\infty} A_i = A_1 \cap A_2 \cap A_3 \cap \ldots = \{x \,|\, x \in A_j \text{ for all } j = 1, 2, \ldots\}$

Mutually Disjoint Sets

Definition

Sets $A_1, A_2, A_3, \ldots$ are **mutually disjoint** (or **pairwise disjoint** or **nonoverlapping**) if, and only if, no two sets A_i and A_j with distinct subscripts have any elements in common. More precisely, for all integers i and $j = 1, 2, 3, \ldots$

$$A_i \cap A_j = \emptyset \quad \text{whenever } i \neq j.$$

Definition

Collection of indexed sets $A_1, A_2, A_3, \ldots$ are called mutually disjoint (or pairwise disjoint) if and only if for every two sets A_i and A_j, $A_i \cap A_j = \emptyset$

Example

The following sets are mutually disjoint

- $N_0 = \{n \mid n = 3k \text{ for some } k \in \mathbb{N}\} = \{0, 3, 6, \ldots\}$
- $N_1 = \{n \mid n = 3k + 1 \text{ for some } k \in \mathbb{N}\} = \{1, 4, 7, \ldots\}$
- $N_2 = \{n \mid n = 3k + 2 \text{ for some } k \in \mathbb{N}\} = \{2, 5, 8, \ldots\}$

Mutually Disjoint Sets

a. Let $A_1 = \{3, 5\}$, $A_2 = \{1, 4, 6\}$, and $A_3 = \{2\}$. Are A_1, A_2, and A_3 mutually disjoint?

b. Let $B_1 = \{2, 4, 6\}$, $B_2 = \{3, 7\}$, and $B_3 = \{4, 5\}$. Are B_1, B_2, and B_3 mutually disjoint?

Solution

a. Yes. A_1 and A_2 have no elements in common, A_1 and A_3 have no elements in common, and A_2 and A_3 have no elements in common.

b. No. B_1 and B_3 both contain 4.

Partition

Definition

A finite or infinite collection of nonempty sets $\{A_1, A_2, A_3, \dots\}$ is a **partition** of a set A if, and only if,

1. A is the union of all the A_i;
2. the sets $A_1, A_2, A_3, \dots$ are mutually disjoint.

Definition

Let $A_1, A_2, A_3, \dots$ be a partition of the set A if $A_1, A_2, A_3, \dots$ are mutually disjoint and $\bigcup_i A_i = A$

Example

The following sets are a partition of $\mathbb{N}$ (natural numbers)

- $N_0 = \{n \mid n = 3k \text{ for some } k \in \mathbb{N}\} = \{0, 3, 6, \dots\}$
- $N_1 = \{n \mid n = 3k + 1 \text{ for some } k \in \mathbb{N}\} = \{1, 4, 7, \dots\}$
- $N_2 = \{n \mid n = 3k + 2 \text{ for some } k \in \mathbb{N}\} = \{2, 5, 8, \dots\}$

Because N_0, N_1, N_2 are mutually disjoint and $N_0 \cup N_1 \cup N_2 = \mathbb{N}$

Exercises

Write down three sets that are mutually disjoint

Define a partition for $\mathbb{Z}$

Is the following a valid partition of the set $M = \{1, 2, 3, \ldots, 9\}$:

- $\{x \in M \mid x \notin \{1, 2, 3\}\}$
- $\{x \in M \mid x \notin \{4, 5, 6\}\}$
- $\{x \in M \mid x \notin \{7, 8, 9\}\}$

Power Sets

Definition

The power set of A, written $\mathcal{P}(A)$, is the set of all subsets of A.

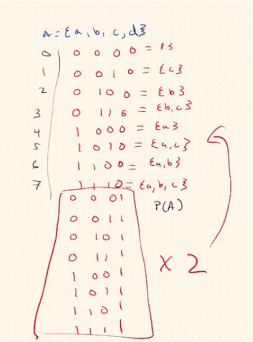
Example

The power set of $A = \{1, 2, 3\}$ is

$$\begin{aligned}\mathcal{P}(A) = &\{\emptyset,\\ &\{1\}, \{1,2\}, \{1,3\}, \{1,2,3\},\\ &\{2\}, \{2,3\},\\ &\{3\}\}\end{aligned}$$

The cadinality of a power set, $|\mathcal{P}(A)|$ is a power of 2, namely, $|\mathcal{P}(A)| = 2^{|A|}$

Why?



The empty set $\emptyset$ is a member of any power set, why?

Because the size of the power set is always a power of 2, we also denote the power set of a set A as 2^A

Example

$$\mathcal{P}(\{a,b\}) = 2^{\{a,b\}} = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$$

Ordered Pairs

Definition

An ordered pair of two elements a and b, written (a, b) is a grouping of a and b where a is the *first element* and b is the *second element*.

Definition

Two ordered pairs are equal, written $(a, b) = (c, d)$, if and only if $a = b$ and $c = d$.

Example

An ordered pair is used to define the Cartesian plane, where the first element is the x component and the second element is the y component.

Two points in the plain are only equal when they have the same x and y

Generalizing ordered pairs into tuples

We can define ordered collections of more than two elements by combining ordered pairs. For example, an ordered collection of three elements and four elements can be written:

$$x = (a, (b, c)) \quad y = (a, (b, (c, d)))$$

As this is cumbersome, we can combine these elements together to form a tuple, denoted like so

$$x = (a, b, c) \quad y = (a, b, c, d)$$

A tuple has the same properties as an ordered pair, where each element of the two tuples must be equal for the tuples to be equal. For example,

$$(10, 12, 3, 5) \neq (10, 2, 3, 5)$$

Cartesian Products (or Cross Products)

Definition

The Cartesian product (or cross product) of two sets A and B, written $A \times B$ and read "A cross B", is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$.

Example

We can denote the cross product of two sets using set builder notation:

$$A \times B = \{(a, b) \,|\, a \in A \text{ and } b \in B\}$$

We can further generalize the result for multiple products,

$$A_1 \times A_2 \times A_3 \times \ldots = \{(a_1, a_2, a_3, \ldots) \,|\, a_1 \in A_1, a_2 \in A_2, a_3 \in A_3, \ldots\}$$

Exercises

Let $A = \{1, 2, 3\}$ and $B = \{u, v\}$, what is:
$A \times B$
$B \times A$
$A \times A$
$B \times B$

Let $D = \{p, q\}$, what is $2^D \times B$?