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THE BEHAVIOR AS $\varepsilon \to 0^+$ OF SOLUTIONS TO $\varepsilon\nabla^2 w = (\partial/\partial y)w$ ON THE RECTANGLE $0 \leqq x \leqq l$, $|y| \leqq 1$*

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Abstract. The title problem is first examined in the limit of the semi-infinite strip $l = \infty$, for boundary data $w(x, -1) = f(x)$, $w(x, 1) = g(x)$, $w(0, y) = h(y)$. Here f, g, h are infinitely differentiable except at the corners where one-sided derivatives of all orders exist. Previous work on the infinite strip covers cases where $h = 0$ so that (by superposition) the present discussion may be narrowed to cases where $f = g = 0$; for these the solution is asymptotically zero for $x \geqq x_0 > 0$. Near $x = 0$ four regions are distinguished: the parabolic boundary layer $y \leqq y_1 < 1$, excluding $\varepsilon^{-1/2}x \leqq X_0$, $1 + y \leqq y_{-1}$, which is determined by the singular region $\varepsilon^{-1}(1 + y) \leqq y_{*\infty}$; and the two parts of the hyperbolic boundary layer $\varepsilon^{-1}(1 - y) \leqq Y_\infty$, namely $\varepsilon^{-1/2}x \geqq X_1 > 0$ and the transition zone $\varepsilon^{-1}x \leqq x_{*\infty}$, both of which are determined by the parabolic layer. By means of Fourier sine transforms the method of matched asymptotic expansions is proved valid to all orders in ε in each of the regions, which can be extended to overlap. Other assumptions about h are also considered. Finally the corresponding results for the rectangle are shown to follow from the superposition of two semi-infinite strip problems.

1. Introduction. We propose to examine the asymptotic properties as $\varepsilon \to 0^+$ of the solution to the equation

$$\varepsilon(\partial^2/\partial x^2 + \partial^2/\partial y^2)W - (\partial/\partial y)W = 0 \tag{1a}$$

on the semi-infinite strip $|y| \leqq 1, x \geqq 0$, under the boundary conditions

$$W(x, 1) = g(x), \qquad W(x, -1) = f(x), \qquad W(0, y) = h(y). \tag{1b}$$

Our goal is to prove that the method of matched asymptotic expansions does give the correct approximation to W to all orders in ε.

The method of attack is similar to and an extension of that used in our previous paper (Cook and Ludford [2]). In § 10 we shall show how to extend these results to cover the asymptotics of the equation (1a) in a rectangular region, the latter being of greater physical interest than either the infinite or semi-infinite strip.

Before outlining the method of proof, we simplify the problem in the following manner. The solution W, under the boundary conditions (1b), is the sum of the solution w satisfying the boundary conditions

$$w(x, \pm 1) = 0, \tag{1c}$$

$$w(0, y) = h(y), \tag{1d}$$

and the solution v satisfying the boundary conditions

$$v(x, 1) = g(x), \qquad v(x, -1) = f(x), \qquad v(0, y) = 0.$$

Now consider $u(x, y)$ which solves (1a) on the infinite strip $-\infty < x < \infty$, with the boundary conditions

$$u(x, 1) = G(x), \qquad u(x, -1) = F(x),$$

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where G, F are the odd extensions of g, f respectively. Clearly $u(x, y)$ is an odd function of x, so that $u(0, y) = 0$ and $v(x, y) = u(x, y)$ for $x \geqq 0$.

The proof that the method of matched asymptotic expansions is valid for u is found in Cook and Ludford [2]. The results depend on the differentiability properties of G and F. For example, if right and left derivatives of G, F exist at zero to order k_0, and if $G^{(k)}$, $F^{(k)}$ are integrable for $k \leqq k_0 + 1$, G and F being infinitely differentiable except at zero, then the results of the method of matched asymptotic expansions are valid to order ε^m where $m \leqq 2[(k_0 + 1)/4]$.

Since v is covered by the u of our previous paper, we may concentrate on w here. As in [2] the proof depends on having an explicit representation of the exact solution in terms of the Green's function, the latter consisting of the fundamental solution and its images in the (extended) boundaries. It can be seen immediately that, of the infinity of such terms, all but the first four can be ignored because they are a.e.s. (asymptotically exponentially small) throughout the strip. However, manipulating the remaining terms is difficult, and instead we consider their Fourier sine transforms. The latter are easily managed by expanding in Taylor series in ε. The basic difficulty is to prove that term-by-term inversion of the expansions in the transform plane does produce asymptotic expansions whose terms are those obtained by the method of matched asymptotic expansions.

2. The method of matched asymptotic expansions. As in our previous paper [2] we shall assume the reader is familiar with this method; see, for example,

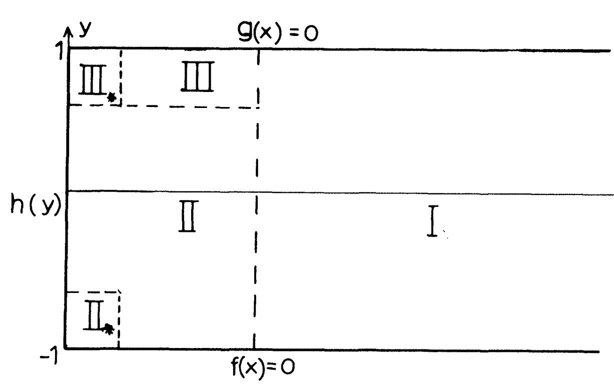

FIG. 1

Chapter 4 of Cole's book [1] where its application to problems such as ours is considered in some detail. In particular Cole discusses the locations of the boundary layers and their orders of magnitude, as well as the physical situations in which they occur (cf. also the Introduction of [2]). In this section we are solely concerned with collecting the results obtained by the method in a form that is suitable for our later proofs, without reproducing Cole's arguments for each step.

Figure 1 shows schematically the various regions of validity of the expansions referred to in the present section.

Assuming an expansion

$$w \sim \sum_{k=0}^{\infty} \varepsilon^k w_k^{\mathrm{I}}(x, y) \tag{2}$$

of the solution to the boundary value problem (1a), (1c), (1d), we obtain the recurrence relation

$$(\partial/\partial y)w_k^{\mathrm{I}} = (\partial^2/\partial x^2 + \partial^2/\partial y^2)w_{k-1}^{\mathrm{I}} \tag{3}$$

for the coefficient functions, by direct substitution. We must therefore take

$$w_k^{\mathrm{I}}(x, y) = 0$$

in order to satisfy the boundary conditions (1c). It is clear that the expansion cannot be uniformly valid since it does not satisfy the boundary condition at $x = 0$.

Therefore, we consider the substitution

$$X = \varepsilon^{-1/2}x \tag{4}$$

in order to make $\varepsilon\partial^2/\partial x^2 = \partial^2/\partial X^2$ comparable to $\partial/\partial y$. Then with an expansion

$$w \sim \sum_{k=0}^{\infty} \varepsilon^k w_{k-1}^{\mathrm{II}}(X, y), \tag{5}$$

we obtain the recurrence relation

$$(\partial^2/\partial X^2 - \partial/\partial y)w_k^{\mathrm{II}}(X, y) = -(\partial^2/\partial y^2)w_{k-1}^{\mathrm{II}}(X, y), \tag{6a}$$

for the coefficient functions. The appropriate boundary conditions for this inhomogeneous parabolic equation are

$$w_k^{\mathrm{II}}(X, -1) = 0, \qquad w_k^{\mathrm{II}}(0, y) = \begin{cases} h(y) & \text{for } k = 0, \\ 0 & \text{for } k > 0. \end{cases} \tag{6b}$$

Such an expansion cannot be valid near $X = 0$, $y = -1$, as is easily seen for the case $h = 1$: The functions

$$\begin{aligned} w_0^{\mathrm{II}} &= \operatorname{erfc}[X(y + 1)^{-1/2}/2], \\ w_1^{\mathrm{II}} &= -X^3/[8\pi^{1/2}(y + 1)^{5/2}]\exp[-X^2(y + 1)^{-1}/4] \end{aligned} \tag{7}$$

satisfy all conditions, and w_1^{II} becomes unbounded in a neighborhood of $X = 0$, $y = -1$. In fact, it is not even uniquely determined since, if singularities are admitted at $X = 0, y = -1$, certain solutions of the homogeneous diffusion equation may be added.

Such difficulties could have been anticipated since we are attempting to represent an elliptic singularity by means of solutions of parabolic equations. To consider the singular region we introduce the stretched coordinates

$$X_* = \varepsilon^{-1/2}X, \qquad y_* = \varepsilon^{-1}(y + 1) \tag{8}$$

in order to make all derivatives in (1a) of comparable order. With an expansion

$$w \sim \sum_{k=0}^{\infty} \varepsilon^k w_k^{\mathrm{II}*}(X_*, y_*), \tag{9}$$

we then see that the coefficient functions must satisfy the full elliptic equation

$$(\partial^2/\partial X_*^2 + \partial^2/\partial y_*^2 - \partial/\partial y_*)w_k^{\mathrm{II}*} = 0 \tag{10a}$$

and the boundary conditions

$$w_k^{\mathrm{II}*}(X_*, 0) = 0, \qquad w_k^{\mathrm{II}*}(0, y_*) = h^{(k)}(-1)y_*^k/k!. \tag{10b}$$

The $w_k^{\mathrm{II}*}$ are unique if in addition we require that they do not grow exponentially as $y_* \to \infty$, a condition which is necessary in order to match with (5). This matching then uniquely determines the coefficients of the expansion (5). More precisely, $X_*^2/y_*\,(=X^2/(y+1))$ is fixed as $\varepsilon \to 0$ (though $y_* \to \infty$) and we find that only full powers of ε are involved. Consequently, no half powers of ε are required in region II, as was anticipated in writing the expansion (5). In particular, we find that the homogeneous solution $3X/[4\pi^{1/2}(y+1)^{7/2}]\exp[-X^2(y+1)^{-1}/4]$ must be added to the w_1^{II} in (7).

Finally we consider the boundary layer at $y = 1$ which is needed to correct the II-expansion for the boundary condition at $y = 1$. With X and the stretched variable

$$Y = \varepsilon^{-1}(1 - y) \tag{11}$$

(so that $\varepsilon\partial^2/\partial y^2 = \varepsilon^{-1}\partial^2/\partial Y^2$ is comparable to $\partial/\partial y = \varepsilon^{-1}\partial/\partial Y$), we assume an expansion

$$w \sim \sum_{k=0}^{\infty} \varepsilon^k w_k^{\mathrm{III}}(X, Y). \tag{12}$$

The recurrence relation for the coefficient functions is then

$$(\partial^2/\partial Y^2 + \partial/\partial Y)w_k^{\mathrm{III}} = -(\partial^2/\partial X^2)w_{k-1}^{\mathrm{III}} \tag{13a}$$

and the boundary condition

$$w_k^{\mathrm{III}}(X, 0) = 0. \tag{13b}$$

At each stage an integration constant is obtained, and is uniquely determined by matching with the expansion (5). It is now clear that the boundary layer must occur at $y = 1$, and not at $y = -1$, in order to obtain exponentially decreasing functions for the matching.

Once again this expansion cannot be uniformly valid in the boundary layer, as can be seen from $w_0^{\mathrm{III}} = C_0[1 - \exp(-Y)]$ where $C_0(X)$ is obtained by matching; in particular $C_0(0) = h(1)$. This violates the boundary condition on the y-axis for a distance ε down from $y = 1$.

Finally, to consider the top corner region we introduce

$$x_* = \varepsilon^{-1/2}X \tag{14}$$

so as to make $\varepsilon\partial^2/\partial X^2 = \partial^2/\partial x_*^2$ comparable to $\partial^2/\partial Y^2 + \partial/\partial Y$. Setting

$$w \sim \sum_{k=0}^{\infty} \varepsilon^{k/2} w_k^{\mathrm{III}*}(x_*, Y), \tag{15}$$

the recurrence relation for the coefficient functions is

$$(\partial^2/\partial x_*^2 + \partial^2/\partial Y^2 + \partial/\partial Y)w_k^{\mathrm{III}*} = 0, \tag{16a}$$

and the boundary conditions are

$$\text{(16b)} \qquad w_k^{\mathrm{III}*}(x_*, 0) = 0, \qquad w_k^{\mathrm{III}*}(0, Y) = \begin{cases} h^{(k)}(1)(-Y)^k/k! & \text{for } k \text{ even}, \\ 0 & \text{for } k \text{ odd}. \end{cases}$$

Such Dirichlet problems have unique solutions under the additional requirement of exponential decay as $(x_*^2 + Y^2)^{1/2} \to \infty$ with $x_* \neq 0$. Without such a requirement there are solutions with algebraic growth, but these are precisely what are needed to match the expansion in II. We may think of the solution in II running through the region III_*, which reacts to the violation of its boundary conditions with a correction that dies out exponentially away from $Y = 0$. Note that half powers of ε are induced in the III_*-expansion, since the II-expansion (5) will involve powers of $\varepsilon^{1/2}$ after the substitution of (11) and (14). Such terms did not appear in II_*.

These then are the results obtained by the method of matched asymptotic expansions. We shall now show them to be valid approximations to order m, where m depends on the differentiability of h. However, to begin with we assume that

$$\text{(17a)} \qquad h \text{ is infinitely differentiable on } (-1, 1),$$

$$\text{(17b)} \qquad h^{(k)}(-1 + 0) \quad \text{and} \quad h^{(k)}(+1 - 0) \quad \text{exist for all } k,$$

for which the method as given above can be carried on indefinitely. The regions of validity for the expansions are (see Fig. 2)

$$\begin{aligned}
\mathrm{I}&: \ x_0 \leqq x, \quad -1 \leqq y \leqq 1; \\
\mathrm{II}&: \ 0 \leqq X, \quad -1 \leqq y \leqq y_1 < 1 \quad \text{excluding} \quad 0 \leqq X \leqq X_0, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\; 0 \leqq y + 1 \leqq y_{-1}; \\
\mathrm{II}_*&: \ 0 \leqq X_*, \quad 0 \leqq y_* \leqq y_{*\infty}; \\
\mathrm{III}&: \ X_1 \leqq X, \quad 0 \leqq Y \leqq Y_\infty; \\
\mathrm{III}_*&: \ 0 \leqq x_* \leqq x_{*\infty}, \quad 0 \leqq Y \leqq Y_\infty.
\end{aligned}$$

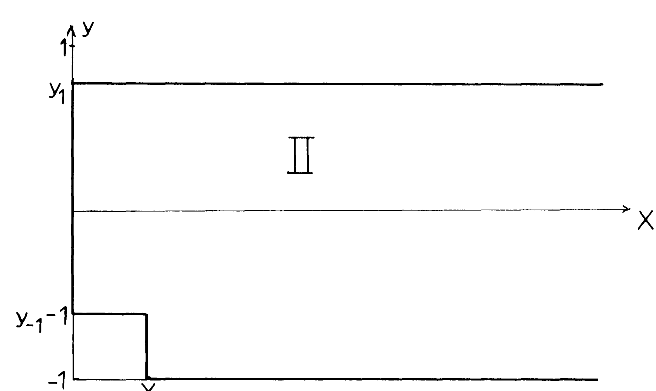

FIG. 2a

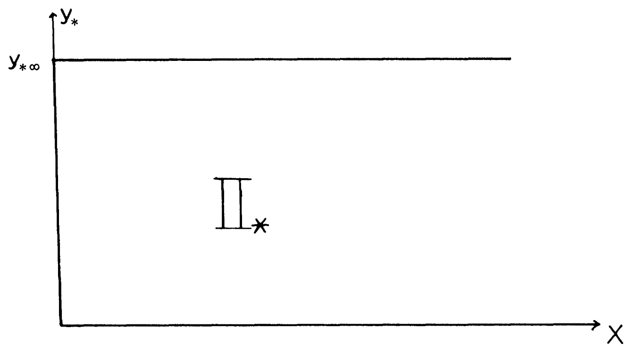

FIG. 2b

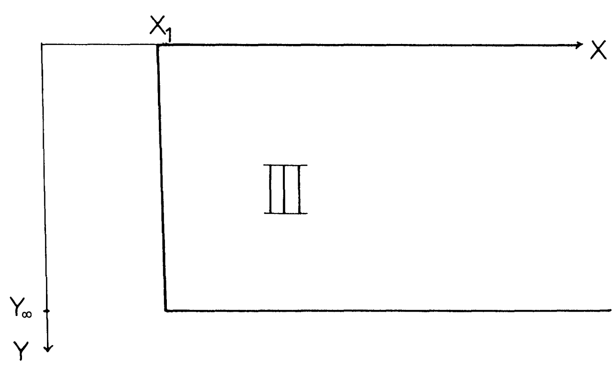

FIG. 2c

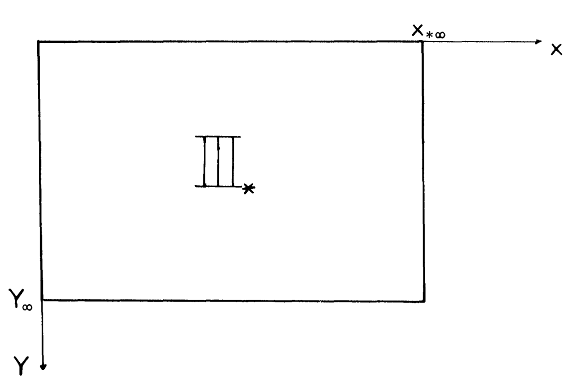

FIG. 2d

Here $x_0, y_1, X_0, y_{-1}, y_{*\infty}, X_1, Y_\infty, x_{*\infty}$ are first assumed to be fixed positive numbers, but it is later shown that the regions of validity can be extended to

$$x_0 = O(\varepsilon^{1/2-\delta}), \quad y_1 = O(\varepsilon^{1-\delta}), \quad X_0 = O(\varepsilon^{1/4-\delta}), \quad y_{-1} = O(\varepsilon^{1/2-\delta}),$$
$$y_{*\infty} = O(\varepsilon^{-1+\delta}), \quad Y_\infty = O(\varepsilon^{-1+\delta}), \quad X_1 = O(\varepsilon^{1/2-\delta}), \quad x_{*\infty} = O(\varepsilon^{-1/2+\delta}).$$

Here $\delta > 0$ is arbitrarily small.

The regions II_* and III_* at the bottom and top corners both arise from discontinuities in the boundary data, but otherwise they are quite different: the expansion in II is determined by that in II_* whereas the opposite is true of II and III_*. We may say that the parabolic layer on $x = 0$ is completely determined by its singular origin $x = 0$, $y = -1$, and in turn determines the top singularity $x = 0, y = 1$. That the structures of II_* and III_* are the same as in [2] can be seen from dividing the solution into three parts. The first is an infinitely smooth solution satisfying the data on $x = 0$, but not on $y = \pm 1$ where it is in general nonzero. The second part nullifies the first on $y = -1$ and is zero elsewhere on the boundary, while the third does the same for $y = +1$. The first part does not require the regions II_* and III_* (nor for that matter III). As was shown in the Introduction, the other two are covered by our previous paper [2], albeit extended to data depending on ε. They therefore involve regions II_* and III_*, respectively, of the type found there.

We have been unable to exhibit these three parts explicitly, but at least the asymptotic existence of the third is clear from our analysis. As indicated above, the III_*-expansion has two components: the II-expansion written in the x_*, Y-variables, corresponding to the sum of the first two parts; and a correction for the boundary conditions on $y = 1$, corresponding to the third part.

3. The exact solution. Taking the Fourier sine transform

$$\overline{(\cdot)} = \int_0^\infty (\cdot) \sin \xi x \, dx$$

of the differential equation (1a), we find

$$(-\varepsilon\xi^2 + \varepsilon\partial^2/\partial y^2 - \partial/\partial y)\bar{w} = -\varepsilon\xi h(y),$$

where the boundary condition (1d) has been incorporated. The boundary conditions (1c) then give

$$\bar{w}(\xi, y) = -\varepsilon\xi \int_{-1}^{+1} \bar{\mathscr{G}}(y, y'; \xi, \varepsilon) h(y') \, dy',$$

where the Green's function

$$\bar{\mathscr{G}} = 2\exp[(y - y')/2\varepsilon]/\{(1 + 4\varepsilon^2\xi^2)^{1/2} \sinh[(1 + 4\varepsilon^2\xi^2)^{1/2}/2\varepsilon]\}$$

$$\cdot\begin{cases} \sinh[(1 + 4\varepsilon^2\xi^2)^{1/2}(y + 1)/2\varepsilon] \sinh[(1 + 4\varepsilon^2\xi^2)^{1/2}(y' - 1)/2\varepsilon] & \text{for } y < y', \\ \sinh[(1 + 4\varepsilon^2\xi^2)^{1/2}(y' + 1)/2\varepsilon] \sinh[(1 + 4\varepsilon^2\xi^2)^{1/2}(y - 1)/2\varepsilon] & \text{for } y' < y \end{cases} \tag{18}$$

is actually the transform of that for the original problem. The exact solution of

(1a) under the boundary conditions (1c), (1d) is then

$$w(x, y) = \frac{2}{\pi} \int_0^\infty \bar{w}(\xi, y) \sin \xi x \, dx.$$

We are not concerned with a.e.s. contributions to the solution, so that terms which are uniformly a.e.s. for ξ on the real axis may be neglected. Thus, since $\exp[-(1 + 4\varepsilon^2\xi^2)^{1/2}/\varepsilon]$ is uniformly a.e.s., we may write

$$\begin{aligned} \bar{\mathscr{G}} \sim (1 + 4\varepsilon^2\xi^2)^{-1/2} \exp[(y - y')/2\varepsilon]\{&\exp[-(1 + 4\varepsilon^2\xi^2)^{1/2}(y + y' + 2)/2\varepsilon] \\ &- \exp[-(1 + 4\varepsilon^2\xi^2)^{1/2}|y - y'|/2\varepsilon] + \exp[(1 + 4\varepsilon^2\xi^2)^{1/2}(y + y' - 2)/2\varepsilon] \\ &- \exp[(1 + 4\varepsilon^2\xi^2)^{1/2}(|y - y'| - 4)/2\varepsilon]\}, \end{aligned} \tag{18'a}$$

as will be needed in regions III and III$_*$. But $\bar{\mathscr{G}}$ may be further simplified for y away from 1, so that

$$\begin{aligned} \bar{\mathscr{G}} \sim (1 + 4\varepsilon^2\xi^2)^{-1/2} \exp[(y - y')/2\varepsilon]\{&\exp[-(1 + 4\varepsilon^2\xi^2)^{1/2}(y + y' + 2)/2\varepsilon] \\ &- \exp[-(1 + 4\varepsilon^2\xi^2)^{1/2}|y - y'|/2\varepsilon]\} \end{aligned} \tag{18'b}$$

will be used in regions II and II$_*$.

We could also write the exact solution in terms of the fundamental solution of equation (1a) and its images in $y = \pm 1$ together with their images in $x = 0$. The same result is obtained by expanding the denominator of $\bar{\mathscr{G}}$ in (18), to obtain the terms

$$\begin{aligned} (1 + 4\varepsilon^2\xi^2)^{-1/2} \exp[(y - y')/2\varepsilon]\{&\exp[(1 + 4\varepsilon^2\xi^2)^{1/2}(y + y' - 2 - 4r)/2\varepsilon] \\ &- \exp[-(1 + 4\varepsilon^2\xi^2)^{1/2}(|y - y'| + 4r)/2\varepsilon] \\ &+ \exp[-(1 + 4\varepsilon^2\xi^2)^{1/2}(y + y' + 2 - 4r)/2\varepsilon] \\ &- \exp[(1 + 4\varepsilon^2\xi^2)^{1/2}(|y - y'| - 4 - 4r)/2\varepsilon]\} \end{aligned}$$

with $r = 0, 1, 2, \cdots$. The inverse sine transforms of these are the Bessel functions obtained by the imaging process above. To uniformly a.e.s. terms then the solution could also be written

$$\begin{aligned} -\frac{1}{\pi} \int_{-1}^{1} & h(y') \exp[(y - y')/2\varepsilon](\partial/\partial x) \\ &\cdot \{K_0[(x^2 + (y - y')^2)^{1/2}/2\varepsilon] - K_0[(x^2 + (y + y' + 2)^2)^{1/2}/2\varepsilon] \\ &- K_0[(x^2 + (y + y' - 2)^2)^{1/2}/2\varepsilon] + K_0[(x^2 + (y' - y + 4)^2/2\varepsilon]\} \, dy'. \end{aligned}$$

4. The core region I. Consider this last representation of the exact solution in terms of Bessel functions. In the core region we have

$$w(x, y) = 0$$

to a.e.s. terms since $x \geqq x_0 > 0$.

Extension of the core region inward is limited by the behavior of

$$\exp\{[y - y' - (x^2 + (y - y')^2)^{1/2}]/2\varepsilon\}$$

which arises from the first Bessel function when its argument is large. With

$$x_0 = \varepsilon^\mu$$

the argument in the exponential is negative and at least $O(\varepsilon^{2\mu-1})$ for $|y - y'| \leqq 2$ and $x \geqq x_0$. Hence $w(x, y)$ remains a.e.s. when

$$\mu < 1/2. \tag{19}$$

The core-region expansion is valid for $-1 \leqq y \leqq 1$, i.e., even into the boundary layer region for x restricted as above. This is to be expected since the zero expansion does in fact agree with the given boundary condition at $y = 1$. In other words, data at $x = 0$ has no asymptotic influence away from $x = 0$.

5. The side layer II. As suggested by the limitation (19) we introduce the stretched variable (4) in order to describe the solution near $x = 0$. In terms of the appropriate transform variable $\eta = \varepsilon^{1/2}\xi$,

$$w(X, y; \varepsilon) = \frac{2}{\pi}\int_0^\infty \sin \eta X \tilde{w}(\eta, y; \varepsilon)\, \partial\eta, \tag{20}$$

where

$$\begin{aligned}\tilde{w} \sim \eta(1 + 4\varepsilon\eta^2)^{-1/2} &\int_{-1}^{1} h(y') \exp[(y - y')/2\varepsilon] \\ &\cdot \{\exp[-(1 + 4\varepsilon\eta^2)^{1/2}(y + y' + 2)/2\varepsilon] \\ &\quad - \exp[-(1 + 4\varepsilon\eta^2)^{1/2}|y - y'|/2\varepsilon]\}\, dy'.\end{aligned} \tag{20$'$}$$

On expanding in a Taylor series in ε we obtain

$$\tilde{w}(\eta, y; \varepsilon) \sim \sum_{k=0}^{m-1} \varepsilon^k \tilde{w}_k^{\mathrm{II}}(\eta, y) + \varepsilon^m \tilde{R}_m(\eta, y; \varepsilon); \tag{21}$$

it is not necessary to write down $\tilde{w}_k^{\mathrm{II}}$ and $\tilde{R}_m$ explicitly. We now show that under inversion: (i) the coefficient functions $w_k^{\mathrm{II}}(X, y)$ satisfy the recurrence relation (6a) together with the boundary conditions (6b); and (ii) $R_m(X, y; \varepsilon)$ is bounded independently of ε in region II. Proof of matching with the expansion in II_* will however be postponed until the next section. From now on, we shall also use $\tilde{w}(\eta, y; \varepsilon)$ to denote its asymptotic approximation (20′).

(i) It can be checked that $(-\eta^2 + \varepsilon\partial^2/\partial y^2 - \partial/\partial y)\tilde{w}(\eta, y; \varepsilon) = -\eta h(y)$. Substituting the expansion (21) and equating coefficients of corresponding powers of ε yields

$$(\eta^2 + \partial/\partial y)\tilde{w}_k^{\mathrm{II}} = \begin{cases} \eta h(y) & \text{for } k = 0, \\ \partial^2/\partial y^2 \tilde{w}_{k-1}^{\mathrm{II}} & \text{for } k > 0. \end{cases}$$

Since $\tilde{w}(\eta, -1) = 0$, we also have

$$\tilde{w}_k^{\mathrm{II}}(\eta, -1) = 0 \quad \text{for all } k.$$

Hence, provided the $\tilde{w}_k^{\mathrm{II}}(\eta, y)$ are invertible, a fact that will be proved when $\tilde{R}_m$ is discussed, the inverses $w_k^{\mathrm{II}}(X, y)$ do satisfy the recurrence relation and boundary conditions as desired.

(ii) Except for a constant factor, $\tilde{R}_m(\eta, y; \varepsilon)$ is the mth derivative of $\tilde{w}(\eta, y; \varepsilon)$ with respect to ε, evaluated at εt, $0 < t < 1$. It is not immediately clear that the inverse of such a derivative exists from the form (20′), since expansion of $(1 + 4\varepsilon\eta^2)^{-1/2}$ alone generates powers of η^2 multiplying terms apparently bounded as $\eta \to \infty$. In order to see that the inverse does exist we integrate (20′) by parts m times. Noticing that terms of the form $(2\varepsilon)^k[1 + (1 + 4\varepsilon\eta^2)^{1/2}]^{-k} \exp\{[(y + 1) \cdot (1 - (1 + 4\varepsilon\eta^2)^{1/2}) - 2(1 + (1 + 4\varepsilon\eta^2)^{1/2})]/2\varepsilon\}$ can be ignored since they are uniformly a.e.s. and those of the form

$$(2\varepsilon)^k(1 + (1 + 4\varepsilon\eta^2)^{1/2})^{-k} \exp[(y - 1)(1 + (1 + 4\varepsilon\eta^2)^{1/2})/2\varepsilon] \tag{22}$$

can be ignored since they are uniformly a.e.s. for y away from 1, we obtain

$$\begin{aligned}\tilde{w} \sim{}& \eta(1 + 4\varepsilon\eta^2)^{-1/2}\\ &\cdot\Bigg[\sum_{k=0}^{m-1} \{(2\varepsilon)^{k+1}(1 + (1 + 4\varepsilon\eta^2)^{1/2})^{-k-1} - (-1 - (1 + 4\varepsilon\eta^2)^{1/2})^{k+1}(2\eta^2)^{-k-1}\}\\ &\quad\cdot\{h^{(k)}(-1)\exp[-2\eta^2(y + 1)/(1 + (1 + 4\varepsilon\eta^2)^{1/2})] - h^{(k)}(y)\}\\ &- (-1 - (1 + 4\varepsilon\eta^2)^{1/2})^m(2\eta^2)^{-m}\int_{-1}^{y} h^{(m)}(y')\\ &\quad\cdot\exp[-2\eta^2(y - y')/(1 + (1 + 4\varepsilon\eta^2)^{1/2})]\,dy' - (2\varepsilon)^m(1 + (1 + 4\varepsilon\eta^2)^{1/2})^{-m}\\ &\quad\cdot\Bigg\{\int_y^1 h^{(m)}(y')\exp[(y - y')(1 + (1 + 4\varepsilon\eta^2)^{1/2})/2\varepsilon]\,dy' - \int_{-1}^{1} h^{(m)}(y')\\ &\quad\cdot\exp[-2\eta^2(y + 1)/(1 + (1 + 4\varepsilon\eta^2)^{1/2}) - (y' + 1)(1 + (1 + 4\varepsilon\eta^2)^{1/2})/2\varepsilon]\,dy'\Bigg\}\Bigg].\end{aligned}$$

Note immediately that the last two terms are invertible and $O(\varepsilon^m)$ in the X, y-plane, and hence can be dropped. This can be seen by integrating by parts once more to obtain

$$\eta(1 + 4\varepsilon\eta^2)^{-1/2}(2\varepsilon)^{m+1}[1 + (1 + 4\varepsilon\eta^2)^{1/2}]^{-m-1}$$

times terms which are bounded independently of η, y and ε. The result then follows on setting $\eta_* = \varepsilon^{1/2}\eta$ in the inversion integral.

Thus we need only consider the expansion of the remaining terms which, since $2\varepsilon[1 + (1 + 4\varepsilon\eta^2)^{1/2}]^{-1} = [-1 + (1 + 4\varepsilon\eta^2)^{1/2}](2\eta^2)^{-1}$ can be written as

$$\begin{aligned}&\eta^{-1}\sum_{k=0}^{m-1}(-2\eta^2)^{-k}\sum_{t=0}^{[k/2]}\binom{k+1}{2t+1}(1 + 4\varepsilon\eta^2)^t\\ &\qquad\cdot\{h^{(k)}(-1)\exp[-2\eta^2(y + 1)/(1 + (1 + 4\varepsilon\eta^2)^{1/2})] - h^{(k)}(y)\}\\ &\quad - \eta(1 + 4\varepsilon\eta^2)^{-1/2}[-1 - (1 + 4\varepsilon\eta^2)^{1/2}]^m(2\eta^2)^{-m}\int_{-1}^{y} h^{(m)}(y')\\ &\qquad\cdot\exp[-2\eta^2(y - y')/(1 + (1 + 4\varepsilon\eta^2)^{1/2})]\,dy'.\end{aligned} \tag{23}$$

Differentiating m times with respect to ε gives terms of the form

$$\begin{aligned}&\eta^{-1}(\eta^2)^{-j+m+l}(1 + 4\varepsilon\eta^2)^{t-m/2-\alpha_1/2}[1 + (1 + 4\varepsilon\eta^2)^{1/2}]^{-2l-\alpha_2}(y + 1)^l h^{(j)}(-1)\\ &\quad\cdot\exp[-2\eta^2(y + 1)/(1 + (1 + 4\varepsilon\eta^2)^{1/2})]\end{aligned} \tag{24a}$$

and

$$\eta(\eta^2)^l(1 + 4\varepsilon\eta^2)^{(1+m+\alpha_1)/2}(1 + (1 + 4\varepsilon\eta^2)^{1/2})^{\alpha_1 - l}\int_{-1}^{y} h^{(m)}(y')(y - y')^l$$
$$\cdot \exp[-2\eta^2(y - y')/(1 + (1 + 4\varepsilon\eta^2)^{1/2})]\,dy',$$

where

$$0 \leqq l + \alpha_1 + \alpha_2 \leqq m, \qquad 0 \leqq l, \alpha_1, \alpha_2 \leqq m, \qquad 0 \leqq j \leqq m - 1.$$

Integrating by parts $l + 1$ times on the last term in order to absorb the powers η^2 we can replace it with terms

$$\text{(24b)}\quad \eta(\eta^2)^{l-\beta_1}(1 + 4\varepsilon\eta^2)^{-(1+m+\alpha_1)/2}[1 + (1 + 4\varepsilon\eta^2)^{1/2}]^{-l+\alpha_1+\beta_1}$$
$$\cdot (y + 1)^{l-\beta_1+\beta_2+1}h^{(m+\beta_2)}(-1)\exp[-2\eta^2(y + 1)/(1 + (1 + 4\varepsilon\eta^2)^{1/2})],$$

$$\text{(24c)}\quad \eta^{-1}(1 + 4\varepsilon\eta^2)^{-(1+m+\alpha_1)/2}[1 + (1 + 4\varepsilon\eta^2)^{1/2}]^{1+\alpha_1}$$
$$\cdot \{h^{(m)}(y) - h^{(m)}(-1)\exp[-2\eta^2(y + 1)/(1 + (1 + 4\varepsilon\eta^2)^{1/2})]\},$$

$$\text{(24d)}\quad \eta^{-1}(1 + 4\varepsilon\eta^2)^{-(1+m+\alpha_1)/2}[1 + (1 + 4\varepsilon\eta^2)^{1/2}]^{1+\alpha_1}\int_{-1}^{y} h^{(m+\beta_3+1)}(y')(y - y')^{\beta_3}$$
$$\cdot \exp[-2\eta^2(y - y')/(1 + (1 + 4\varepsilon\eta^2)^{1/2})]\,dy',$$

where $0 \leqq \beta_2 \leqq \beta_1 - 1, 0 \leqq \beta_1 \leqq l + 1, 0 \leqq \beta_3 \leqq l$. The bounding of R_m is now reduced to the bounding of the inverses of (24).

The integrand in the inversion integral of (24d) can be rounded by $c/(1 + |\eta|^3)$, where c is independent of η, y and ε. (The bound for η large is obtained by integrating by parts once more and bounding the resulting terms.) Thus, (24d) is invertible and the result is bounded in ε. The terms in (24c) are invertible as they stand and the inverses are bounded as $X \to 0$. (The limit must be used to define the inverse functions at $X = 0$ since sine inversions automatically give zero there —this point is discussed further when we come to the coefficient functions.)

In dealing with (24a), (24b) we note that the same difficulty occurs as in the case of the infinite strip [2], leading to the exclusion of a region near $X = 0$, $y = -1$. The inversion integrals are convergent and $O(1)$ in ε for y away from -1, since we then obtain help from the exponentials. Although each integral is divergent for $y = -1$, we can obtain convergence for X bounded away from zero as follows. Rewrite $\sin \eta X$ as $(e^{i\eta X} - e^{-i\eta X})/2i$ and, in the resulting two integrals, bend the ends of the integration line upwards into the complex η-plane in the first and downwards in the second. They are then convergent for all y.

Thus we see that $R_m(X, y; \varepsilon)$ is $O(1)$ in ε in region II.

The terms $w_k^{\mathrm{II}}(X, y)$ may be treated similarly. Using the expansion (23) with $m = k + 1$ and noting that $\tilde{w}_k^{\mathrm{II}}(\eta, y)$ is the kth derivative with respect to ε evaluated at $\varepsilon = 0$, we find that $\tilde{w}_k^{\mathrm{II}}(\eta, y)$ is composed of terms

$$\text{(25a)}\quad \eta^{-1}(\eta^2)^{-j+k+l}(y + 1)^l \exp[-\eta^2(y + 1)]h^{(j)}(-1),$$

$$\text{(25b)}\quad \eta^{-1}(\eta^2)^{l-\beta_1}(y + 1)^{l-\beta_1+\beta_2+1}\exp[-\eta^2(y + 1)]h^{(k+1+\beta_2)}(-1),$$

$$\text{(25c)}\quad \eta^{-1}\int_{-1}^{y}(y - y')^{\beta_3}\exp[-\eta^2(y - y')]h^{(k+\beta_3+1)}(y')\,dy',$$

where $0 \leqq l \leqq k$, $0 \leqq \beta_1 \leqq l$, $0 \leqq \beta_2 \leqq \beta_1 - 1$, $0 \leqq \beta_3 \leqq l$, and for $k = 0$, the additional term

$$\eta^{-1}h(y). \tag{25d}$$

The inversion of these terms follows easily (when we deform the contour again for the first two) except for $\eta^{-1}h(y)$. Then the inversion integral is not uniformly convergent, so we must first take $X \neq 0$ and let $X \to 0$. Thus the sine transform forces a zero value at $X = 0$, whereas we want the limiting value as $X \to 0$, which is in general not zero.

Extension of region II upwards is limited to $y_1 = \varepsilon^{1-\delta}$ because we have omitted terms of the form (22) from the expansion and ignored the last two terms in (18′a) as being uniformly a.e.s. Extension outwards is unlimited since no such limitation was required for the above bounding.

Extension of region II into the corner is limited by the inversion of (24a), (24b) which for points near $X_0 = \varepsilon^{\kappa}$, $y_{-1} = \varepsilon^{\lambda}$ with $\kappa, \lambda > 0$ involves the exponential of $-2\eta^2\varepsilon^{\lambda}/[1 + (1 + 4\varepsilon\eta^2)^{1/2}] \pm i\eta\varepsilon^{\kappa}$ in the integrands. After the deformation of the integration line both terms have negative real parts, one of which may be prevented from vanishing in the limit $\varepsilon \to 0$ by the transformation $\eta = \varepsilon^{-\kappa}\tau$ when $\lambda \geqq 2\kappa$ or $\eta = \varepsilon^{-\lambda/2}\tau$ when $\lambda \leqq 2\kappa$. The terms (24a), (24b) are then of order $\varepsilon^{-4\kappa m}$ or $\varepsilon^{-2\lambda m}$ at worst. Thus λ can be arbitrarily large so long as

$$\kappa < 1/4, \tag{26a}$$

and κ can be arbitrarily large so long as

$$\lambda < 1/2. \tag{26b}$$

6. The Singular Region II$_*$. The limitations (26) are misleading: we must in fact introduce the stretched variables (8) in order to describe the solution near $X = 0$, $y = -1$. Using a hat to denote the appropriate Fourier transform (with variable $\eta_* = \varepsilon^{1/2}\eta$) we obtain

$$\begin{aligned} \hat{w} \sim \eta_*(1 + 4\eta_*^2)^{-1/2} \int_0^{2/\varepsilon} & h(\varepsilon y_*' - 1) \\ \cdot \{\exp[(1 - (1 + 4\eta_*^2)^{1/2})y_*/2 & - (1 + (1 + 4\varepsilon\eta^2)^{1/2})y_*'/2] \\ & - \exp[(y_* - y_*')/2 - |y_* - y_*'|(1 + 4\varepsilon\eta_*^2)^{1/2}/2]\}\, dy_*'. \end{aligned} \tag{27}$$

Expanding h in a Taylor series in ε we find

$$\hat{w}(\eta_*, y_*; \varepsilon) \sim \sum_{k=0}^{m-1} \varepsilon^k \hat{w}_k^{\mathrm{II}*}(\eta_*, y_*) + \varepsilon^m \hat{R}_m^*(\eta_*, y_*; \varepsilon), \tag{28}$$

where

$$\begin{aligned} \hat{w}_k^{\mathrm{II}*} = (\eta_*(1 + 4\eta_*^2)^{-1/2} h^{(k)}(-1)/k!) \int_0^{\infty} & y_*'^k \\ \cdot \{\exp[(1 - (1 + 4\eta_*^2)^{1/2})y_*/2 & - (1 + (1 + 4\eta_*^2)^{1/2})y_*'/2] \\ & - \exp[(y_* - y_*')/2 - |y_* - y_*'|(1 + 4\eta_*^2)^{1/2}/2]\}\, dy_*', \end{aligned} \tag{28′a}$$

$$\begin{aligned}\hat{R}_m^* = (\eta_*(1 + 4\eta_*^2)^{-1/2}/m!)\int_0^{2/\varepsilon} y'^m_* h^{(m)}(\varepsilon t y'_* - 1) \\ \cdot\{\exp[(1 - (1 + 4\eta_*^2)^{1/2})y_*/2 - (1 + (1 + 4\eta_*^2)^{1/2})y'_*/2] \\ - \exp[(y_* - y'_*)/2 - |y_* - y'_*|(1 + 4\eta_*^2)^{1/2}/2]\}\, dy'_*.\end{aligned} \tag{28'b}$$

The integrals in $\hat{w}_k^{\mathrm{II}*}$ have been extended to infinity, the added pieces being uniformly a.e.s. for y_* bounded. This renders the coefficient functions independent of ε.

It will now be shown that: (i) the $w_k^{\mathrm{II}*}$ do satisfy the proper recurrence relation and boundary conditions for region II_*; and (ii) R_m^* is bounded independently of ε there. We shall in addition complete the treatment of region II by showing that its expansion matches the one here.

(i) Direct substitution shows that

$$(\eta_*^2 + \partial/\partial y_* - \partial^2/\partial y_*^2)\hat{w} = \eta_* h(\varepsilon y_* - 1),$$

where $\hat{w}$ denotes its own asymptotic expansion (27). Expanding about $\varepsilon = 0$ for $h(\varepsilon y_* - 1)$ and equating coefficients of corresponding powers of ε yields the relations

$$(\eta_*^2 + \partial/\partial y_* - \partial^2/\partial y_*^2)\hat{w}_k^{\mathrm{II}*} = \eta_* h^{(k)}(-1)y_*^k/k!.$$

Also it is clear that $\hat{w}(\eta_*, 0; \varepsilon) = 0$. So in the original plane we have

$$(\partial^2/\partial X_*^2 + \partial^2/\partial y_*^2 - \partial/\partial y_*)w_k^{\mathrm{II}*} = 0,$$

$$w_k^{\mathrm{II}*}(X_*, 0) = 0, \qquad w_k^{\mathrm{II}*}(0, y_*) = h^{(k)}(-1)y_*^k/k!,$$

as desired, provided $\hat{w}_k^{\mathrm{II}*}$ is invertible. The latter is covered by our treatment of R_m^* below.

To investigate the matching of the II- and II_*-expansions we note that for $y \neq -1$ the integral (20′) can be written in the form

$$\begin{aligned}&I(\varepsilon/(y + 1), X/(y + 1)^{1/2}, h(-1 + \varepsilon y'_*)) \\ &= \sum_{k=0}^{m-1} (\varepsilon/(y + 1))^k I_k(X/(y + 1)^{1/2}, h(-1 + \varepsilon y'_*)) + O(\varepsilon^m),\end{aligned}$$

where I_k is a linear operator on functions of y'_* which depends only on $X/(y + 1)^{1/2}$ and the order symbol refers to fixed X, y. This result can be obtained by setting $\eta = \tau/(y + 1)^{1/2}$, $y' = -1 + \varepsilon y'_*$, letting the y'_*-integration range to infinity instead of $2/\varepsilon$ (thereby introducing a.e.s. error), and expanding on $\varepsilon/(y + 1)$ as in § 5 (i.e., by integration by parts). Beyond $2/\varepsilon$ the function $h(-1 + \varepsilon y'_*)$ is defined as the polynomial $\sum_{p=1}^N h^{(p)}(1)(y'_* - 2/\varepsilon)^p/p!$, where N is sufficiently large to ensure whatever continuity of derivatives at $y'_* = 2/\varepsilon$ is required in the following. It is easily checked that, in operating on functions $O(1)$ in ε, I_k produces $O(1)$ functions. Consequently Taylor-series remainders can be ignored in writing the m-term II-expansion

$$\sum_{k=0}^{m-1} (\varepsilon/(y + 1))^k I_k(X/(y + 1)^{1/2}, \sum_{j=0}^{m-1-k} \varepsilon^j h^{(j)}(-1)y'^j_*/j!)$$

so that its n-term II_*-expansion is

$$\sum_{k=0}^{m-1} y_*^{-k} \sum_{j=0}^{\min(n-1,m-k-1)} \varepsilon^j h^{(j)}(-1) I_k(X_*/y_*^{1/2}, y_*'^j)/j!. \tag{29}$$

On the other hand, when η_* is replaced by $\tau/y_*^{1/2}$ the n-term II_*-expansion given by (28) is seen to be

$$I\left(1/y_*, X_*/y_*^{1/2}, \sum_{j=0}^{n-1} \varepsilon^j h^{(j)}(-1) y_*'^j/j!\right)$$

and its m-term II-expansion is

$$\sum_{k=0}^{m-1} (\varepsilon/(y+1))^k \sum_{j=0}^{\min(n-1,m-k-1)} \varepsilon^j h^{(j)}(-1) I_k(X/(y+1)^{1/2}, y_*'^j)/j!. \tag{30}$$

Clearly the expressions (29) and (30) are identical under the transformation $X = \varepsilon^{1/2} X_*$, $y + 1 = \varepsilon y_*$ so that matching is established.

(ii) Integrating by parts the terms in $\hat{R}_m^*$ which have $y_*'(1 + (1 + 4\eta_*^2)^{1/2})$ in the exponential, and noticing that $(2/\varepsilon)^j \exp[-(1 + (1 + 4\eta_*^2)^{1/2})/\varepsilon]$ is uniformly a.e.s., we obtain

$$\begin{aligned} m!\hat{R}_m^* \sim\ & -2\eta_*(1 + 4\eta_*^2)^{-1/2}(1 + (1 + 4\eta_*^2)^{1/2})^{-1} \\ & \cdot \Bigg\{ y_*^m h^{(m)}(\varepsilon t y_* - 1) + \int_0^{2/\varepsilon} [m y_*'^{m-1} h^{(m)}(\varepsilon t y_*' - 1) + y_*'^m \varepsilon t h^{(m+1)}(\varepsilon t y_*' - 1)] \\ & \qquad \cdot \exp[y_*(1 - (1 + 4\eta_*^2)^{1/2})/2 - y_*'(1 + (1 + 4\eta_*^2)^{1/2})/2]\, dy_*' \\ & \qquad + \int_{y_*}^{2/\varepsilon} [m y_*'^{m-1} h^{(m)}(\varepsilon t y_*' - 1) + y_*'^m \varepsilon t h^{(m+1)}(\varepsilon t y_*' - 1)] \\ & \qquad\qquad \cdot \exp[(y_* - y_*')(1 + (1 + 4\eta_*^2)^{1/2})/2]\, dy_*' \Bigg\} \\ & + \eta_*(1 + 4\eta_*^2)^{-1/2} \int_0^{y_*} y_*'^m h^{(m)}(\varepsilon t y_*' - 1) \\ & \qquad \cdot \exp[(y_* - y_*')(1 - (1 + 4\eta_*^2)^{1/2})/2]\, dy_*'. \end{aligned}$$

All but the first and last terms are clearly invertible and their inverses are $O(1)$ in ε since, after integrating by parts once more, they behave like η_*^{-2} for η_* large, independently of ε. The first term can be rewritten as $h^{(m)}(\varepsilon t y_* - 1) y_*^n$ times $(1 - (1 + 4\eta_*^2)^{1/2})/(2\eta_*(1 + 4\eta_*^2)^{1/2}) = (2\eta_*(1 + 4\eta_*^2)^{1/2})^{-1} - (2\eta_*)^{-1}$ and hence is invertible with inverse $O(1)$. Note that the value for $X_* = 0$ must again be interpreted as the limit for $X_* \to 0$. The last term inverts to

$$\begin{aligned} X_* \int_0^{y_*} & y_*'^m h^{(m)}(\varepsilon t y_*' - 1) \exp[(y_* - y_*')/2] K_1[(X_*^2 + (y_* - y_*')^2)^{1/2}/2] \\ & \cdot (X_*^2 + (y_* - y_*')^2)^{-1/2}\, dy_*', \end{aligned}$$

and thus is $O(1)$ in ε.

The invertibility of $\hat{w}_k^{\mathrm{II}*}$ is also covered by the above analysis, since the same terms, with m replaced by k and ε set zero, are involved.

Extension of region II_* outward is limited only by

$$y_{*\infty} = \varepsilon^{-1+\delta},$$

which ensures that the original asymptotic expression (27) is still valid and that the inverse of (28′b) is small compared to ε^{-m}. Nowhere was it necessary to bound X_*.

7. The boundary layer III. The expansion found in region II is not asymptotic to w near $y = 1$ since it does not include boundary layer terms. Such terms arise from the parts of the exact solution which were omitted for being uniformly a.e.s. away from $y = 1$ (see (18′) and (22)). Written in the stretched variable $Y = (1 - y)/\varepsilon$, these parts are given by the inverse of

$$\begin{aligned}\tilde{\tau} = {} & \eta(1 + 4\varepsilon\eta^2)^{-1/2} \exp[-Y(1 + 4\varepsilon\eta^2)^{1/2}/2] \\ & \cdot\left\{\int_{-1}^{1} h(y') \exp[(1 - y')(1 - (1 + 4\varepsilon\eta^2)^{1/2})/2\varepsilon]\, dy'\right. \\ & - \int_{-1}^{1} h(y') \exp[\{(1 - y')(1 + (1 + 4\varepsilon\eta^2)^{1/2} - 4(1 + 4\varepsilon\eta^2)^{1/2}\}/2\varepsilon]\, dy' \\ & \left. + \sum_{k=0}^{m} [2\varepsilon(1 + (1 + 4\varepsilon\eta^2)^{1/2})^{-k-1} h^{(k)}(1)\right\}.\end{aligned} \tag{31}$$

Thus to obtain the expansion in the boundary layer we must consider the contribution from the boundary layer correction τ as well as that from the asymptotic form used previously, now applied in the boundary layer.

Expansion of these two in a Taylor series in ε gives

$$\tilde{w}(\eta, Y; \varepsilon) \sim \sum_{k=0}^{m-1} \varepsilon^k \tilde{w}_k^{\mathrm{III}}(\eta, Y) + \varepsilon^m \tilde{\mathscr{R}}_m(\eta, Y; \varepsilon),$$

where it is not necessary to write out $\tilde{w}_k^{\mathrm{III}}$ and $\tilde{\mathscr{R}}_m$ explicitly. The task is now to demonstrate that (i) the $w_k^{\mathrm{III}}(X, Y)$ satisfy the recurrence relation (13a) together with the boundary conditions and matching mentioned there; and (ii) $\mathscr{R}_m(X, Y; \varepsilon)$ is $O(1)$ in ε in region III.

(i) By direct substitution we obtain

$$(\partial^2/\partial Y^2 + \partial/\partial Y - \varepsilon\eta^2)\tilde{w} = -\eta h(1 - \varepsilon Y), \tag{32}$$

where again $\tilde{w}$ stands for its asymptotic form; furthermore $\tilde{w}(\eta, 0; \varepsilon) = 0$. It follows that

$$(\partial^2/\partial Y^2 + \partial/\partial Y)\tilde{w}_k^{\mathrm{III}} = -\eta^2 \tilde{w}_{k-1}^{\mathrm{III}} - (-Y)^k \eta h^{(k-1)}(1)/k!,$$

$$\tilde{w}_k^{\mathrm{III}}(\eta, 0; \varepsilon) = 0,$$

so that the transforms of the recurrence relation and boundary conditions are satisfied. If the $\tilde{w}_k^{\mathrm{III}}$ are invertible, as will be proved along with $\tilde{\mathscr{R}}_m$, only the matching remains.

Notice that $\tilde{\tau}(\eta, Y; \varepsilon)$ is a solution of the homogeneous differential equation (32) with $\tilde{\tau}(\eta, 0; \varepsilon)$ the negative value of (20′) at $y = 1$; it is therefore purely the correction for the boundary condition at $y = 1$. Since $\tilde{\tau}(\eta, (1 - y)/\varepsilon; \varepsilon)$ is a.e.s. for fixed $y \neq 1$, we need only prove that $\tilde{w} - \tilde{\tau}$ matches with the $\tilde{w}$ of region II, that is, we need only show that the $\tilde{w}$ used in region II satisfies the matching principle in the variables y, Y. But this follows directly from Fraenkel's Theorem 1 [3]: the $\mathscr{R}_m(\eta, y)$ in (21) remains $O(1)$ for all y in $-1 < y \leqq 1$ and the $\tilde{w}_k^{\mathrm{II}}(\eta, y)$, since they have the forms (25), clearly satisfy assumption 2 of Fraenkel's theorem. The expansion in region III is now seen to match that in region II since inversion preserves matching.

(ii) To consider the existence of the coefficient functions $w_k^{\mathrm{III}}(X, Y)$ and the bounding of $\mathscr{R}_m(X, Y; \varepsilon)$, recall that the corresponding expansion is the superposition of that for $\tau(X, Y; \varepsilon)$ and the expansion in region II rewritten in terms of the boundary layer variable Y.

In treating the terms from region II, i.e. omitting the boundary layer terms, the estimate of the remainder remains valid for y near 1. In other words the expansion (21) holds uniformly up to the top boundary. Thus for its contribution to the expansion in region III we need only substitute the boundary layer variable Y and expand to order ε^m. Since the $\tilde{w}_s^{\mathrm{II}}(\eta, y)$ have the forms (25), the remainder after Taylor series expansion in ε has terms of the form

$$\eta^{-1}(\eta^2)^{-j+s+l+p} Y^{m-s}(2 - \varepsilon t Y)^{l-m+p} \exp(-2\eta^2 + \varepsilon t \eta^2 Y) h^{(j)}(-1), \tag{33a}$$

$$\eta(\eta^2)^{l-s+p} Y^{m-s}(2 - \varepsilon t Y)^{l-\beta_1+1+\beta_2-m+p} \exp(-2\eta^2 + \varepsilon t \eta^2 Y) h^{(s+1+\beta_2)}(-1), \tag{33b}$$

$$\eta^{-1} Y^m h^{(m)}(1 - \varepsilon t Y), \tag{33c}$$

$$\eta^{-1}\eta^{2l_1} Y^{m-s}(2 - \varepsilon Y)^{\beta_3-\gamma_2}\int_0^1 (1 - y')^{\beta_3+l_1} y'^{\gamma_1} \cdot \exp[-\eta^2(1 - y')(2-\varepsilon t Y)] h^{(s+\beta_3+\gamma_1)}(y'(2 - \varepsilon t Y) - 1)\, dy',$$

where $0 \leqq p \leqq m - s$, $l_1 + \gamma_1 - 1 + \gamma_2 = m - s$ and the other parameters satisfy the previous conditions still. The last term was obtained by replacing y' by $(y + 1)y' - 1$ as the integration variable in (25c).

Inversion in the first two terms is valid and $O(1)$ in ε for Y bounded because of the exponential convergence. The third term is also invertible and its inverse, defined for $X = 0$ again by the limit, is $O(1)$ for Y bounded. In dealing with the last term we must integrate by parts l_1 times, to obtain terms of the form

$$\eta^{-1}(\eta^2)^{\alpha_1}(2 - \varepsilon Y)^{\beta_3-\gamma_2-l+\alpha_1} Y^{m-s} \exp[-\eta^2(2 - \varepsilon t Y)] h^{(s+\beta_3+\gamma_1+\alpha_2-1)}(-1), \tag{33d}$$

$$\eta^{-1}(2 - \varepsilon Y)^{\beta_3-\gamma_2-l_1} Y^{m-s}\int_0^1 (1 - y')^{\beta_3+r_2+r_3} y'^{\gamma_1-r_2} \cdot \exp[-\eta^2(2 - \varepsilon t Y)(1 - y')] h^{(s+l_1+\gamma_1+r_3)}(y'(2 - \varepsilon t Y) - 1)\, dy', \tag{33e}$$

where $\alpha_1 \leqq l_1$ and $0 \leqq r_2 + r_3 \leqq l_1$. Again the terms are invertible and $O(1)$ in ε for Y bounded. We conclude that the contribution to the remainder is uniformly $O(\varepsilon^m)$ in region III.

Treating now the contribution to the expansion from $\tau(\eta, Y; \varepsilon)$ we first integrate (31) by parts m times:

$$
\begin{aligned}
\tilde{\tau} = {} & \exp[-Y(1+(1+4\varepsilon\eta^2)^{1/2})/2]\left\{\sum_{k=0}^{m-1} \eta^{-1}(-2\eta^2)^{-k} \sum_{t=0}^{[k/2]} (1+4\varepsilon\eta^2)^t \binom{k+1}{2t+1}\right. \\
& \cdot [h^{(k)}(1) - h^{(k)}(-1)\exp[-4\eta^2/(1+(1+4\varepsilon\eta^2)^{1/2})]] \\
& + \eta(1+4\varepsilon\eta^2)^{-1/2}\int_{-1}^{1} [(-1)^{m-1}(1+(1+4\varepsilon\eta^2)^{1/2})^m(2\eta^2)^{-m}h^{(m)}(y') \\
& \cdot \exp[-2\eta^2(1-y')/(1+(1+4\varepsilon\eta^2)^{1/2})] + (2\varepsilon)^m(1+(1+4\varepsilon\eta^2)^{1/2})^{-m}h^{(m)}(y') \\
& \left.\cdot \exp[\{(1-y')(1+(1+4\varepsilon\eta^2)^{1/2}) - 4(1+4\varepsilon\eta^2)^{1/2}\}/2\varepsilon]]\, dy'\right\}.
\end{aligned} \tag{34}
$$

On integrating by parts once more and letting $\eta_* = \varepsilon^{1/2}\eta$ in the inversion integral, it is clear that the last term is invertible and $O(\varepsilon^m)$. Consequently we need only consider the expansion of the first three groups of terms. By expansion in Taylor series, their contribution to $\tilde{\mathscr{R}}_m$ is their mth derivative with respect to ε evaluated at $t\varepsilon$, which produces terms of the form

$$\eta^{2(m-k)}\exp[-Y(1+(1+4\varepsilon\eta^2)^{1/2})/2]h^{(k)}(1)/[\eta(1+4\varepsilon\eta^2)^{m-t-\alpha_1/2}], \tag{35a}$$

$$
\begin{aligned}
& \eta^{2(m-k+l)}\exp[-Y(1+(1+4\varepsilon\eta^2)^{1/2})/2 - 4\eta^2/(1+(1+4\varepsilon\eta^2)^{1/2})] \\
& \quad \cdot h^{(k)}(-1)/[\eta(1+4\varepsilon\eta^2)^{m-t-\alpha_1/2}(1+(1+4\varepsilon\eta^2)^{1/2})^{2l+\alpha_2}],
\end{aligned} \tag{35b}
$$

$$
\begin{aligned}
& \eta^{2(l-\beta_1)}\eta\exp[-Y(1+(1+4\varepsilon\eta^2)^{1/2})/2 - 4\eta^2/(1+(1+4\varepsilon\eta^2)^{1/2}) \\
& \quad \cdot h^{(m+\beta_2)}(-1)/[(1+4\varepsilon\eta^2)^{(1+m+\alpha_1)/2}(1+(1+4\varepsilon\eta^2)^{1/2})^{l-\alpha_1-\beta_1}],
\end{aligned} \tag{35c}
$$

$$
\begin{aligned}
& \{\exp[-Y(1+(1+4\varepsilon\eta^2)^{1/2})/2]h^{(m)}(1) \\
& \quad - \exp[-Y(1+(1+4\varepsilon\eta^2)^{1/2})/2 - 4\eta^2/(1+(1+4\varepsilon\eta^2)^{1/2})]h^{(m)}(-1)\} \\
& \quad \cdot (1+(1+4\varepsilon\eta^2)^{1/2})^{\alpha_1+1}/[\eta(1+4\varepsilon\eta^2)^{(1+m+\alpha_1)/2}],
\end{aligned} \tag{35d}
$$

$$
\begin{aligned}
& \exp[-Y(1+(1+4\varepsilon\eta^2)^{1/2}/2]\int_{-1}^{1} h^{(m+\beta_3+1)}(y')(1-y')^{\beta_3} \\
& \quad \cdot \exp[-2\eta^2(1-y')/(1-(1+4\varepsilon\eta^2)^{1/2})]\, dy' \\
& \quad \cdot (1+(1+4\varepsilon\eta^2)^{1/2})^{\alpha_1+1}/[\eta(1+4\varepsilon\eta^2)^{(1+m+\alpha_1)/2}],
\end{aligned} \tag{35e}
$$

where

$$0 \leqq \alpha_1 \leqq m, \quad 0 \leqq \alpha_1 + l \leqq m, \quad 0 \leqq \beta_1 \leqq l-1, \quad 0 \leqq \beta_3 \leqq l, \quad 0 \leqq \beta_2 \leqq \beta_1 - 1.$$

Notice that (as in region II) it was necessary to integrate by parts, after having differentiated, to remove the powers of η^2 which emerged.

The terms (35d), (35e) can be bounded under inversion in precisely the same manner as (24c), (24d) in region II. The extra factor $\exp[-Y(1+(1+4\varepsilon\eta^2)^{1/2})/2]$ only improves convergence of the inversion integral. The terms (35b), (35c) are invertible and their inverses are bounded in ε because they provide exponential convergence.

The first term (35a) is clearly invertible for $\varepsilon \neq 0$, $Y \neq 0$ but we require its inverse for all ε, Y. To this end bound X away from zero, so that the inversion line can be bent upwards/downwards respectively for the $\exp(\pm i\eta X)$, of which $\sin \eta X$

is composed, to provide convergent integrals $O(1)$ in ε uniformly for $Y \geqq 0$. (There is no question of reaching $X = 0$ by bounding Y away from zero, as there was in region II.)

Thus we are assured that $\mathscr{R}_m$ is $O(1)$ in region III, as desired.

The sth coefficient function involves the same terms (33), (35) with m replaced by s and ε set equal to zero. With the exception of terms corresponding to (35a), which are now positive powers of η, the existence of the coefficient function is therefore covered by our discussion above. The exceptional terms are also covered if the inversion line is deformed before expanding in Taylor series.

Extension of region III downwards to $Y_\infty = \varepsilon^{-\lambda}$ is restricted to

$$\lambda < 1$$

because we have implicitly assumed $2 - \varepsilon t Y$ is positive. Extension outwards in X is unrestricted since it was nowhere necessary to bound it (e.g. terms of the form $\eta^{2p+1} \exp(-2\eta^2)$ invert into $X^p \exp[-X^2/2\sqrt{2}]$ which are bounded as $X \to \infty$).

Extension into the corner is limited by the integrals resulting from deformation of the inversion line. Letting $\eta X = t$ shows that the terms behave at worst like X^{-2m}, so that for $X = \varepsilon^\kappa$ we must have

$$\kappa < 1/2.$$

8. The transition zone III$_*$. Motivated by the last restriction, we introduce the stretched variable $x_* = \varepsilon^{-1/2} X$ in order to describe the boundary layer near $X = 0$. The structure of the III$_*$-expansion is similar to that of the III-expansion in that it is composed of two parts, namely the II-expansion, expanded in the III$_*$-variables, and the boundary layer correction. We shall show that

$$w(x_*, Y; \varepsilon) \sim \sum_{k=0}^{m-1} \varepsilon^{k/2} w_k^{\mathrm{III}*}(x_*, Y) + \varepsilon^{m/2} S_m(x_*, Y; \varepsilon), \tag{36}$$

where: (i) the $w_k^{\mathrm{III}*}$ satisfy the recurrence relation (16a), the boundary conditions (16b), and the appropriate matching conditions, and (ii) S_m is $O(1)$ in ε uniformly in region III$_*$.

(i) Once it is known that the asymptotic expansion (36) holds in region III$_*$, the fact that w (as formed from (18′a)) satisfies (1a) and the boundary conditions (1c), (1d) to within a.e.s. terms is sufficient to ensure that the $w_k^{\mathrm{III}*}$ satisfy (16a), (16b). The validity of (36) is established by the boundedness of S_m and the existence of the $w_k^{\mathrm{III}*}$ as proved in (ii) below.

As noted above, the III$_*$-expansion is a superposition of the II-expansion, expanded in the III$_*$-variables, and the boundary layer correction terms. The latter terms, as the name suggests, are a.e.s. out of the boundary layer; and so the matching of the III$_*$-expansion with the II-expansion is assured if the II-expansion matches with itself expanded in the III$_*$-variables. In the next section it is proved that the w_k^{II} have asymptotic expansions in the III$_*$-variables which are polynomials in $\varepsilon^{1/2} x_*$ and εY. Therefore we are assured of matching by Fraenkel's theorem.

(ii) To order $[(m + 1)/2]$ the III-expansion of the II-expansion is

$$\omega = \sum_{k=0}^{[(m-1)/2]} \varepsilon^k \frac{2}{\pi} \int_0^\infty \tilde{w}_k^{\mathrm{II}}(\eta, Y) \sin \eta X \, d\eta + \varepsilon^{[(m+1)/2]} \mathscr{S}_{[(m+1)/2]}, \tag{37}$$

where we have already shown that $\mathscr{S}_{[(m+1)/2]}$ is bounded in ε not only in III but also in III_*. To obtain the contribution of (37) to the expansion (36) we shall deform the inversion line near $\eta = 0$ in the complex η-plane so as to avoid later convergence difficulties at that point. Now, in the boundary layer $\tilde{\omega}_k^{\mathrm{II}}$ is composed of terms of the form (33), with $\varepsilon = 0$ and m replaced by k, and hence is an odd function of η. Thus the integrals in (37), with $X = \varepsilon^{-1/2}x_*$, can be written

$$\begin{aligned}(38)\qquad &\int_0^\infty \tilde{\omega}_k^{\mathrm{II}}(\eta, Y)\sin(\eta\varepsilon^{1/2}x_*)\,d\eta \\ &\qquad = (2i)^{-1}\oint_{-\infty}^{\infty} \tilde{\omega}_k^{\mathrm{II}}(\eta, Y)\exp(i\eta\varepsilon^{1/2}x_*)\,d\eta - (\pi/2)r_k(Y),\end{aligned}$$

where $r_k(Y)$ is the residue of $\tilde{\omega}_k^{\mathrm{II}}$ at its simple pole $\eta = 0$ and hence is a polynomial in Y. Instead of changing to the corresponding transform variable $\eta_* = \varepsilon^{1/2}\eta$, which would lead us to troublesome terms $\exp(-2\eta_*^2/\varepsilon)$, we would like to expand $\exp(i\eta\varepsilon^{1/2}x_*)$. But this contributes powers of η to the integrand which apparently destroy the convergence as $\eta \to \infty$. Therefore we first rewrite the integral terms of $\tilde{\omega}_k^{\mathrm{II}}$ by integrating by parts $[(m - 2k)/2] + 1$ times with respect to y', to obtain

(39a) $\eta^{-1}\eta^{-2\gamma_2}Y^{\gamma_1}h^{(\gamma_3)}(1)$,

(39b) $\eta^{-1}\eta^{2\gamma_4}Y^{\gamma_5}h^{(\gamma_6)}(1)\exp(-2\eta^2)$,

(39c) $\eta^{-1}\eta^{-2\{[(m-2k)/2]+1\}}Y^{\gamma_7}\int_0^1 (1 - y')^{\gamma_8}y'^{\gamma_9}h^{(\gamma_{10})}(2y' - 1)\exp[-2\eta^2(1 - y')]\,dy'$,

where

$$0 \leqq \gamma_2 \leqq \left[\frac{m - 2k}{2}\right] + 1, \qquad 0 \leqq \gamma_4.$$

The set of terms (39a), (39b) can be integrated explicitly in (38) to give powers of εx_*^2 and Y; so that they, as well as the residue terms, provide $O(1)$ contributions to the coefficient functions and remainder in III_*. Having taken care of these terms, we can expand the exponential in the integrand (38) for the remaining terms without losing convergence. If $\tilde{\omega}_k$ denotes their contribution to $\tilde{\omega}_k^{\mathrm{II}}$, we have

$$\begin{aligned}(2i)^{-1}\oint_{-\infty}^{\infty} \tilde{\omega}_k \exp(i\varepsilon^{1/2}\eta x_*)\,d\eta &= \sum_{j=0}^{m-1-2k} (i)^{j-1}\varepsilon^{j/2}x_*^j(2j!)^{-1}\oint_{-\infty}^{\infty} \tilde{\omega}_k\eta^j\,d\eta \\ &\quad + (i)^{m-2k-1}(\varepsilon^{1/2}x_*)^{m-2k}(2(m - 2k)!)^{-1}\oint_{-\infty}^{\infty} \tilde{\omega}_k\eta^{m/2-k}\exp(i\eta\varepsilon^{1/2}x_*)\,d\eta,\end{aligned}$$

where the Taylor series has been taken to a remainder providing $O(\varepsilon^{m/2})$ in (38). Two things must be proved about this last expansion, namely that the integrals in the sum exist and that the remainder integral is $O(1)$ uniformly in III_*. But these facts are clear since for the former the integrands behave exponentially as $n \to \infty$, while for the latter they converge at least as well as η^{-2}. It was to obtain this last convergence property that integration by parts was performed on the integral terms of $\tilde{\omega}_k^{\mathrm{II}}$.

The boundary layer correction terms (34) can be written as

$$\exp[-Y(1+(1+4\varepsilon\eta^2)^{1/2})/2] \tag{40}$$
$$\cdot\left\{\tilde{\varpi}(\eta,\varepsilon)+\eta(1+4\varepsilon\eta^2)^{-1/2}\sum_{k=0}^{m-1}[2\varepsilon/(1+(1+4\varepsilon\eta^2)^{1/2})]^{k+1}h^{(k)}(1)\right\},$$

where

$$\tilde{\varpi}=\eta(1+4\varepsilon\eta^2)^{-1/2}\int_{-1}^{1}h(y')\{\exp[(1-y')(1-(1+4\varepsilon\eta^2)^{1/2})/2\varepsilon]$$
$$-\exp[\{(1-y')(1+(1+4\varepsilon\eta^2)^{1/2})-4(1+4\varepsilon\eta^2)^{1/2}\}/2\varepsilon]\}\,dy$$

is the approximation (20′) from which the II-expansion was obtained, evaluated at $y=1$. So, as proved in the preceding paragraph,

$$\varpi(x_*,\varepsilon)=\sum_{k=0}^{m-1}\varepsilon^{k/2}\varpi_k^{\mathrm{II}}(x_*)+\varepsilon^{m/2}\mathscr{S}_{m/2}(x_*,\varepsilon)$$

holds in III_*. But we must now incorporate the exponential factor in (40), and the fact that ϖ is an odd function of η enables us to do this by convolution. Changing to the variables x_* and $\eta_*=\varepsilon^{1/2}\eta$, we find

$$\frac{2}{\pi}\int_0^\infty\exp[-Y(1+(1+4\eta_*^2)^{1/2})/2]\varepsilon^{-1/2}\tilde{\varpi}(\varepsilon^{-1/2}\eta_*,\varepsilon)\sin\eta_*x_*\,d\eta_*$$
$$=\frac{1}{\pi i}\int_{-\infty}^{\infty}\exp[-Y(1+(1+4\eta^2)^{1/2})/2]\varepsilon^{-1/2}\tilde{\varpi}(\varepsilon^{-1/2}\eta_*,\varepsilon)\exp(i\eta_*x_*)\,d\eta_*$$
$$=\frac{1}{2\pi}\exp(-Y/2)\int_{-\infty}^{\infty}\varpi(x'_*,\varepsilon)YK_1[((x_*-x'_*)^2+Y^2)^{1/2}/2]$$
$$\cdot((x_*-x'_*)^2+Y^2)^{-1/2}\,dx'_*$$
$$=\sum_{k=0}^{m-1}\varepsilon^{k/2}Y\exp(-Y/2)\int_{-\infty}^{\infty}\varpi_k^{\mathrm{II}}(x'_*)K_1[((x_*-x'_*)^2+Y^2)^{1/2}/2]$$
$$\cdot[2\pi((x_*-x'_*)^2+Y^2)^{1/2}]^{-1}\,dx'_*$$
$$+\varepsilon^{m/2}Y\exp(-Y/2)\int_{-\infty}^{\infty}\mathscr{S}_{m/2}(x'_*,\varepsilon)K_1[((x_*-x'_*)^2+Y^2)^{1/2}/2]$$
$$\cdot[2\pi((x_*-x'_*)^2+Y^2)^{1/2}]^{-1}\,dx'_*.$$

Since ϖ_k, S_m behave as polynomials in x_* and

$$\int_0^\infty x^{2\mu+1}K_1[(x^2+Y^2)^{1/2}/2](x^2+Y^2)^{-1/2}\,dx=2^{2\mu+1}\Gamma(\mu+1)Y^\mu K_{-\mu}(Y/2),$$

these integrals are bounded independently of ε throughout III_*.

Finally we must deal with the contribution to the expansion (36) from the series in (40). But, under change of transform variable to η_*, these terms become

$$\exp[-Y(1+(1+4\varepsilon\eta^2)^{1/2})/2]\eta_*(1+4\eta_*^2)^{-1/2}$$
$$\cdot\sum_{k=0}^{m-1}2(2\varepsilon)^kh^{(k)}(1)/(1+(1+4\varepsilon\eta^2)^{1/2})^{k+1},$$

each of which is clearly invertible. Each therefore contributes to one of the coefficient functions of (36) or the remainder.

This completes the proof that there is indeed a valid asymptotic expansion throughout region III_*.

Extension of region III_* to

$$Y_\infty = \varepsilon^{-\lambda}, \qquad x_{*\infty} = \varepsilon^{-\mu}$$

is limited to

$$\lambda < 1, \qquad \mu < 1/2.$$

The restriction on λ is a carry-over from the bounding of the remainder in region III. The limitation on μ arises from the terms

$$\varepsilon^k(\varepsilon x_*^2)^{\gamma_4}$$

which come from (39b). We obtain the worst behavior for $k = 0$, when they behave like $(\varepsilon^{1-2\mu})^{\gamma_4}$—hence the restriction on μ.

9. Other assumptions about $h(y)$**.** It is of interest to see the effect of varying the conditions (17) on $h(y)$. First we strengthen them with

$$h^{(k)}(-1+0) = 0 \quad \text{or} \quad h^{(k)}(1-0) = 0 \quad \text{for } k \leqq k_0,$$

and ask at what stage in the approximation of w it is necessary to introduce the corner region II_* or III_*. Secondly we weaken them to

$$h \text{ is infinitely differentiable on } (-1, 1) \text{ except at } y = a,$$

$$h^{(k)}(a \pm 0),\ h^{(k)}(-1+0),\ h^{(k)}(1-0) \text{ exist for all } k,$$

when it is necessary to introduce a new region about $y = a$.

If $h^{(k)}(-1+0) = 0$ for $k \leqq k_0$, it is not necessary to introduce region II_* until $m = [k_0/2] + 1$. Thus, the exclusion of the lower corner from II arises in bounding the terms (24a), (24b) and these now vanish for $m \leqq [k_0/2]$. All remaining bounds extend into the corner. Similarly if $h^{(k)}(1-0) = 0$ for $k \leqq k_0$, it is not necessary to introduce the region III_* until $m = k_0 + 2$, since the terms (35a) in III are zero for $m \leqq k_0 + 1$ and again all bounds extend into the corner.

To facilitate the discussion when h and its derivatives are allowed a discontinuity at $y = a$, we define a new function

$$h_a(y) = \begin{cases} h(y) & \text{for } y < a, \\ \displaystyle\sum_{k=0}^{M} (y-a)^k h^{(k)}(a-0)/k! & \text{for } y \geqq a, \end{cases}$$

where M depends on m. The original boundary value problem (1a), (1c), (1d) is now written as the superposition of two semi-infinite strip problems with the respective boundary data

$$w_1(0, y) = h_a(y), \qquad w_1(x, \pm 1) = 0,$$

$$w_2(0, y) = h(y) - h_a(y), \qquad w_2(x, \pm 1) = 0.$$

Since the function h_a has continuous derivatives to order M, the proof that the method of matched asymptotic expansions is valid for w_1 to order m is contained in the preceding portion of this paper, when M is taken sufficiently large for all derivatives involved to be continuous.

For w_2 the results for region II follow as before except that we must also deal with terms

$$\text{(41a)}\quad \begin{aligned} &-\eta(1 + 4\varepsilon\eta^2)^{-1/2} \sum_{k=0}^{m-1} (-1 - (1 + 4\varepsilon\eta^2)^{1/2})^{k+1}(2\eta^2)^{-k-1} \\ &\quad \cdot \{h^{(k)}(a + 0) - h^{(k)}(a - 0)\} \exp[-2\eta^2(y - a)/(1 + (1 + 4\varepsilon\eta^2)^{1/2})], \end{aligned}$$

for $y > a$, and

$$\text{(41b)}\quad \begin{aligned} &-\eta(1 + 4\varepsilon\eta^2)^{-1/2} \sum_{k=0}^{m-1} (2\varepsilon)^{k+1}(1 + (1 + 4\varepsilon\eta^2)^{1/2})^{-k-1} \\ &\quad \cdot \{h^{(k)}(a + 0) - h^{(k)}(a - 0)\} \exp[(y - a)(1 + (1 + 4\varepsilon\eta^2)^{1/2})/2\varepsilon], \end{aligned}$$

for $y \leqq a$, which arise on integrating (20′) by parts due to the discontinuity at $y = a$.

The terms (41a) present no problem for $y \geqq a + \delta$ ($\delta > 0$) since even after m differentiations with respect to ε the inversion integral retains exponential convergence. As for region II_* (see § 5), if we rewrite $\sin \eta X$ as exponentials and deform the integration contour, we obtain convergence for all $y \geqq a$ when $X \geqq X_0 > 0$.

The terms (41b) are uniformly a.e.s. if $y \leqq a - \delta$, and analysis as in the previous paragraph shows they have similar properties for $y \leqq a$, $X \geqq X_0 > 0$.

It is therefore necessary to introduce a "corner" region near $X = 0$, $y = a$, similar to the II_*-region. In terms of the stretched variables $y_a = \varepsilon^{-1}(y - a)$, $x_* = \varepsilon^{-1/2}X$ and the corresponding transform variable $\eta_* = \varepsilon^{1/2}\eta$, we have

$$\begin{aligned} \hat{w} \sim \eta_*(1 + 4\eta_*^2)^{-1/2} &\int_0^{(1-a)/\varepsilon} \{h(\varepsilon y_a' + a) - h_a(\varepsilon y_a' + a)\} \\ &\cdot \exp[\{(y_a - y_a') - |y_a - y_a'|(1 + 4\eta_*^2)^{1/2}\}/2]\, dy_a', \end{aligned}$$

to uniformly a.e.s. terms.

Now expand $h - h_a$ in its Taylor series about a to m terms and note that, as in region II_*, the upper limit of integration can be extended to ∞ for these terms (but not the remainder), thereby introducing only uniformly a.e.s. terms and making the integral independent of ε. Bounding follows precisely as in region II_*.

The regions III and III_* present no problem since the additional terms due to the discontinuity at $y = a$ are invertible and the corresponding remainders are $O(\varepsilon^m)$. For instance in the formula (34) we must add

$$\begin{aligned} &\eta(1 + 4\varepsilon\eta^2)^{-1/2} \exp[-Y(1 + (1 + 4\varepsilon\eta^2)^{1/2})/2 + (a - 1)2\eta^2/(1 + (1 + 4\varepsilon\eta^2)^{1/2})] \\ &\quad \cdot [(-1 - (1 + 4\varepsilon\eta^2)^{1/2})/2\eta^2]^{k+1}\{h^{(k)}(a + 0) - h^{(k)}(a - 0)\}, \end{aligned}$$

which always provides exponentially convergent inversion integrals.

10. The rectangle. We are now in a position to show that the method of matched asymptotic expansions is valid for (1a) on the rectangle $0 \leqq x \leqq l$, $|y| \leqq 1$ with the boundary data

$$\begin{aligned} w(x,-1) &= f(x), & w(0,y) &= h(y), \\ w(x,1) &= g(x), & w(l,y) &= \mathscr{s}(y). \end{aligned} \tag{42}$$

The proof consists in writing w as the superposition of three functions, the first two of which are solutions on semi-infinite strips and the third a solution on the rectangle with data which is a.e.s.

In order to formulate these problems we define new functions f_1, f_2, g_1, g_2, where

$$f_1 = \begin{cases} fH & \text{for } 0 \leqq x \leqq l, \\ 0 & \text{for } x > l, \end{cases} \qquad f_2 = \begin{cases} f - f_1 & \text{for } 0 \leqq x \leqq l, \\ 0 & \text{for } x < 0, \end{cases}$$

and g_1, g_2 are similar. H is an infinitely differentiable function of x such that

$$H = \begin{cases} 1 & \text{for } -\infty < x \leqq l/4, \\ 0 & \text{for } 3l/4 \leqq x < \infty, \end{cases}$$

and is introduced purely as an artifice. That is, if we now let w_1, w_2 be the solutions satisfying the respective boundary conditions

$$\begin{aligned} &w_1(0,y) = h(y), \\ &w_1(x,-1) = f_1(x), \qquad w_1(x,1) = g_1(x) \quad \text{for } 0 \leqq x < \infty, \\ &w_2(l,y) = \mathscr{s}(y), \\ &w_2(x,-1) = f_2(x), \qquad w_2(x,1) = g_2(x) \quad \text{for } -\infty < x \leqq l, \end{aligned}$$

then the boundary data for w_1 is zero in a neighborhood of $x = l$ and that for w_2 is zero in a neighborhood of $x = 0$. It follows that $w_1(l, y)$ and $w_2(0, y)$ are a.e.s., a result on which the proof hinges. (The infinite differentiability of H ensures that no spurious layers are introduced by w_1 and w_2.)

On the rectangle we may write $w(x, y) = w_1(x, y) + w_2(x, y) - w_3(x, y)$, where w_3 is the solution with the boundary values

$$\begin{aligned} w_3(x,-1) &= 0, & w_3(0,y) &= w_2(0,y), \\ w_3(x,1) &= 0, & w_3(l,y) &= w_1(l,y). \end{aligned} \tag{43}$$

Note that these boundary values depend on, and are a.e.s. in, ε; and that w_3 is the correction needed to annihilate the boundary values of w_1 and w_2 at the sides $x = l$ and $x = 0$, respectively (which were omitted from their definitions as solutions of semi-infinite strip problems). Note also that the data (43) is continuous on the boundary, including the corners. It now follows from the maximum principle that $|w_3|$ is bounded by its maximum on the boundary and hence is a.e.s. throughout the rectangle. Its contribution to w may therefore be ignored.

The proof that the method of matched asymptotic expansions is valid for w_1 and w_2 separately is the preceding portion of this paper. Note that the entire problem is linear so that the approximation to w obtained by the method of matched asymptotic expansions in any region is the sum of the approximations obtained there for w_1 and w_2. This shows that the method is valid for w.

No mention has been made in this section about the differentiability properties of the boundary data. The order to which the approximations can be carried out will depend on these in the ways described in § 9 and our earlier paper [2].

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