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THE BEHAVIOR AS $\varepsilon \to +0$ OF SOLUTIONS TO $\varepsilon\nabla^2 w = \partial w/\partial y$ IN $|y| \leqq 1$ FOR DISCONTINUOUS BOUNDARY DATA*

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Abstract. The title problem is considered for boundary data $w(x, -1) = f(x)$, $w(x, 1) = g(x)$. Here f, g are infinitely differentiable except at $x = 0, a$ respectively, where they have right- and left-hand derivatives of all orders. With $g = 0$ five regions are distinguished: the core $0 < x_0 \leqq |x|$ and the free layer $\varepsilon^{-1/2}|x| \leqq X_\infty$, excluding $\varepsilon^{-1/2}|x| \leqq X_0$, $|y + 1| \leqq y_{-1}$, in $-1 \leqq y \leqq y_1 < 1$; their boundary layers in $\varepsilon^{-1}(1 - y) \leqq Y_\infty$; and the excluded region $\varepsilon^{-1}|x| \leqq X_{*\infty}$, $\varepsilon^{-1}(1 + y) \leqq y_{*\infty}$. The solution for $f = 0$ is asymptotically zero everywhere except in the boundary layer, where $0 < x_a \leqq |x - a|$ is distinguished from the transition zone $\varepsilon^{-1}|x - a| \leqq x_{*\infty}$. By means of Fourier transforms it is shown that the method of matched asymptotic expansions gives approximations to all orders in each of the regions, and that the latter can be extended to overlap. For the excluded region, which gives birth to the "parabolic" free layer, this contradicts what has previously been supposed. Of particular interest is the transition zone, which resolves a breakdown in the "hyperbolic" boundary layer. The expansion in the core is determined independently of the others, but not that in the free layer. As a consequence, the odd powers of $\varepsilon^{1/2}$ which appear in the free layer are absent in the core. Other assumptions concerning f and g are also considered.

1. Introduction. We propose to study the asymptotic properties, as $\varepsilon \to +0$, of the solution of the elliptic equation

$$\varepsilon\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) - \frac{\partial w}{\partial y} = 0 \quad \text{on the strip } |y| \leqq 1, \tag{1a}$$

which satisfies the boundary conditions

$$w(x, -1) = f(x), \qquad w(x, 1) = g(x). \tag{1b}$$

The equation arises in magnetohydrodynamics, where ε measures the importance of viscous force relative to the electromagnetic force, and in the theory of plate-membranes under tension in the y-direction, where ε measures the bending stiffness [6]. In either case the region is bounded, and treatment of the strip is intended to be a first step in understanding that more complicated situation. Certainly boundedness in the y-direction is the more important feature. For this reason we have not mentioned ordinary hydrodynamics, where the equation arises in Oseen's approximation: the region there is unbounded in the y-direction and the questions are of quite different character.

The present type of problem has been considered with varying degrees of generality by several authors. The classic paper on the subject is by Eckhaus and de Jager [3]. Certain aspects have been followed up by Mauss [7]–[10] and Grasman [5] as well as by Eckhaus himself [2]. But nobody has faced the question we shall treat: proving that the formal method of matched asymptotic expansions is correct to all orders, where we are especially interested in discontinuities in f, g, or their derivatives.

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The proof consists in deriving these asymptotic expansions directly from the exact solution, which is expressed in terms of the Green's function. Difficulty arises when this Green's function is written as the infinite sum of Bessel functions, corresponding to the fundamental solution and images in the two boundaries (see end of § 3). Although it is easily seen that all but the first few Bessel functions are asymptotically zero, it is difficult to see how the remaining ones should be manipulated to yield the various asymptotic expansions. However, when the Bessel functions are replaced by their Fourier integrals, the terms in the expansions are obtained simply from the Taylor series (in ε) of the corresponding transforms, and the validity of the expansions is established by estimating remainders. In short, we take Fourier transforms from the start and find, as is often the case, that it is relatively easy to deal with the transform of the Green's function.

Some care is still required to ensure that inverses exist and remainders are estimated correctly. Since there are enough of these questions to deal with, we shall ignore the more trivial ones such as whether an integral can be differentiated. In other words, a formal step will only receive attention if it is in fact not valid.

2. The method of matched asymptotic expansions.[1] The solution is assumed to have an asymptotic expansion[2]

$$w \sim \sum_{k=0}^{\infty} w_k^I(x, y)\varepsilon^k. \tag{2}$$

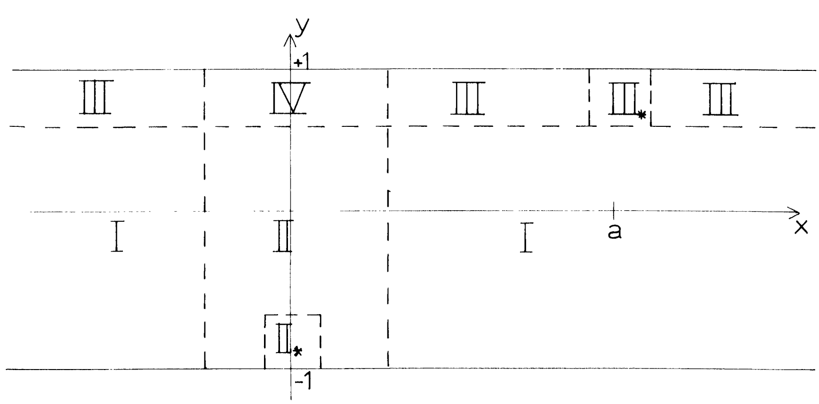

FIG. 1

Substitution in the boundary value problem (1) then yields the recurrence relation

$$\frac{\partial w_k^I}{\partial y} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) w_{k-1}^I \tag{3a}$$

for the coefficient functions (assuming that the derivatives are $O(1)$). At each stage a first order equation has to be solved, so that only one boundary condition can be

[1] See, for example, [1], [11].

[2] The superscripts I, II, etc. correspond to the regions in Fig. 1.

satisfied; for reasons that will be clear later, this must be

$$w_k^I(x, -1) = \begin{cases} f(x), & k = 0, \\ 0, & k \neq 0, \end{cases} \tag{3b}$$

and not the one at $y = 1$. There is no difficulty in calculating as many terms as desired, but the very first term

$$w_0^I(x, y) = f(x)$$

shows that the asymptotic expansion cannot be uniformly valid. The boundary condition at $y = 1$ is in general violated, and a discontinuity (implying that the x-derivative is not $O(1)$) occurs across any vertical line through a point of discontinuity of f. Note that these two types of breakdown are different: the first occurs however smooth f and g are at the value of x considered; the second is a direct consequence of a discontinuity in f.

For simplicity we shall assume that f has a single discontinuity at $x = 0$. Then, recognizing that the solution must have rapid changes across $x = 0$, we introduce the new coordinate

$$X = \varepsilon^{-1/2} x \tag{4}$$

so as to make $\varepsilon(\partial^2/\partial x^2) = \partial^2/\partial X^2$ explicitly comparable to $\partial/\partial y$ in the original equation (1a). The solution is now assumed to have the asymptotic expansion

$$w \sim \sum_{k=0}^{\infty} w_k^{II}(X, y)\varepsilon^{k/2}, \tag{5}$$

which leads to the recurrence relation

$$\left(\frac{\partial^2}{\partial X^2} - \frac{\partial}{\partial y}\right) w_k^{II} = -\frac{\partial^2}{\partial y^2} w_{k-2}^{II} \tag{6a}$$

and the boundary condition

$$w_k^{II}(X, -1) = f^{(k)}(\pm 0)\frac{X^k}{k!} \quad \text{when } X \gtrless 0 \tag{6b}$$

for the coefficient functions. Here $f^{(k)}(\pm 0)$ are right and left kth derivatives of f at $x = 0$. Once more only one boundary condition can be satisfied, and it must be the one at the lower boundary. However, because of the singularity at $X = 0$, $y = -1$, the w_k^{II} are not determined to within certain singular solutions of the homogeneous diffusion equation. Leaving aside this question for the moment, we see that in practice only a few terms can be calculated since at each stage an inhomogeneous diffusion equation must be solved; though, in principle, all terms can be determined. Note how half-integer powers are induced by the boundary values $f(x)$. No such terms arise for $g(x)$, which is associated with there being no equivalent to region II.

The expansion (5) cannot be valid near $X = 0$, $y = -1$, as is easily seen for the special case

$$f(\pm 0) = \pm 1, \quad f'(\pm 0) = 0, \quad f''(\pm 0) = 0.$$

The functions

$$w_0^{II} = \operatorname{erf}\left(\frac{X}{2(y+1)^{1/2}}\right), \quad w_1^{II} = 0, \quad w_2^{II} = \frac{X^3}{8\pi^{1/2}(y+1)^{5/2}} \exp\left(-\frac{X^2}{4(y+1)}\right)$$

satisfy all conditions, and w_2^{II} becomes unbounded as the discontinuity in f is approached along any path $X/(y+1)^{1/2} = \text{const.} \neq 0$. This is hardly surprising since we are attempting to approximate the solution of the elliptic equation (1a) near a singularity in its boundary data by means of the corresponding singular solutions of parabolic equations.

To take proper account of the rapid changes near the discontinuity we introduce the coordinates

$$(7) \qquad X_* = \varepsilon^{-1/2} X, \qquad y_* = \varepsilon^{-1}(1 + y)$$

so as to make the neglected term $\varepsilon(\partial^2/\partial y^2) = \varepsilon^{-1}(\partial^2/\partial y_*^2)$ explicitly comparable to $\partial^2/\partial X^2 - \partial/\partial y = \varepsilon^{-1}(\partial^2/\partial X^2 - \partial/\partial y_*)$. The asymptotic expansion

$$(8) \qquad w \sim \sum_{k=0}^{\infty} w_k^{II*}(X_*, y_*)\varepsilon^k$$

then leads to the full equation

$$(9a) \qquad \left(\frac{\partial^2}{\partial X_*^2} + \frac{\partial^2}{\partial y_*^2} - \frac{\partial}{\partial y_*}\right) w_k^{II*} = 0$$

for each of the coefficient functions, and the boundary conditions

$$(9b) \qquad w_k^{II*}(X_*, 0) = f^{(k)}(\pm 0)\frac{X_*^k}{k!} \quad \text{when } X_* \gtrless 0.$$

The solution at each stage is not completely determinate. However, there is only one solution which does not grow exponentially as $y_* \to \infty$, and this must be selected to ensure matching.

It is then through this matching that the indeterminacy in region II (noted above) is resolved. In particular, one finds that the homogeneous diffusion solution

$$\frac{3}{4\sqrt{\pi}} \frac{X}{(y+1)^{3/2}} \exp\left(-\frac{X^2}{4(y+1)}\right)$$

must be added to w_2^{II}. It is interesting to note that this choice satisfies the principle of minimum singularity: as $y \to -1$ the original w_2^{II} becomes a multiple of $\delta'(X)$ and the added term just cancels this behavior.

We now consider the violated boundary condition. It is convenient to treat the part of the solution due to g separately and to consider $g = 0$ first. The anticipated rapid change in the solution as $y \to 1$ suggests the new coordinate

$$(10) \qquad Y = \varepsilon^{-1}(1 - y),$$

so that $\varepsilon(\partial^2/\partial y^2) = \varepsilon^{-1}(\partial^2/\partial Y^2)$ is explicitly comparable to $\partial/\partial y = -\varepsilon^{-1}(\partial/\partial Y)$ in (1a). The asymptotic expansion

$$(11) \qquad w \sim \sum_{k=0}^{\infty} w_k^{III}(x, Y)\varepsilon^k$$

then leads to the recurrence relation

$$\left(\frac{\partial^2}{\partial Y^2} + \frac{\partial}{\partial Y}\right) w_k^{III} = -\frac{\partial^2}{\partial x^2} w_{k-2}^{III} \tag{12a}$$

and the boundary condition

$$w_k^{III}(x, 0) = 0 \tag{12b}$$

for the coefficient functions. This time the functions are not uniquely determined since at each stage an integration constant is introduced. The N constants in the truncation of the series (11) after N terms are obtained by the matching principle. Y is replaced by $\varepsilon^{-1}(1 - y)$ in the truncated series, which is then expanded to $O(\varepsilon^M)$. Similarly y is replaced by $1 - \varepsilon Y$ in the M-term truncation of the series (2), which is then expanded to $O(\varepsilon^N)$. The results must be identical under the transformation (10).

It is now clear why the boundary condition (3b) had to be satisfied: the matching procedure would fail at $y = -1$. With $Y = \varepsilon^{-1}(1 + y)$, the operator in the recurrence relation (12a) is replaced by $\partial^2/\partial Y^2 - \partial/\partial Y$, so that the function e^Y appears giving terms which cannot be matched since they are of exponential order in ε for fixed y.

Near $x = 0$ we must also introduce the coordinate X and write

$$w \sim \sum_{k=0}^{\infty} w_k^{IV}(X, Y)\varepsilon^{k/2}. \tag{13}$$

The recurrence relation is now

$$\left(\frac{\partial^2}{\partial Y^2} + \frac{\partial}{\partial Y}\right) w_k^{IV} = -\frac{\partial^2}{\partial X^2} w_{k-2}^{IV} \tag{14a}$$

with the boundary condition

$$w_k^{IV}(X, 0) = 0. \tag{14b}$$

Once again there is an integration constant at each stage which is determined by matching with the expansion (5).

Finally we consider the part of the solution due to g. Except near $y = 1$ it will be asymptotically zero, as is seen by setting $f \equiv 0$ above. Near $y = 1$ we again use the coordinate (10), the expansion (11), and the recurrence relation (12a); but instead of the boundary condition (12b), we take

$$w_k^{III}(x, 0) = \begin{cases} g(x) & \text{for } k = 0, \\ 0 & \text{for } k \neq 0. \end{cases} \tag{12c}$$

The integration constant at each stage is determined by matching with zero. Note that $w_k^{III}(x, Y) = 0$ for k odd; in other words, the odd powers of ε are induced by the solution away from $y = 1$. The first coefficient function is clearly

$$w_0^{III}(x, Y) = g(x)\, e^{-Y}$$

so that a discontinuity occurs across any vertical line extending down a distance $O(\varepsilon)$ from a point of discontinuity of g.

For simplicity we shall assume that g has a single discontinuity at $x = a$. Then, introducing

$$x_* = \varepsilon^{-1}(x - a), \tag{15}$$

so as to make $\varepsilon(\partial^2/\partial x^2) = \varepsilon^{-1}(\partial^2/\partial x_*^2)$ explicitly comparable to $\varepsilon^{-1}(\partial^2/\partial Y^2 - \partial/\partial Y)$, we set

$$w \sim \sum_{k=0}^{\infty} w_k^{III*}(x_*, Y)\varepsilon^k. \tag{16}$$

The corresponding recurrence relation is

$$\left(\frac{\partial^2}{\partial x_*^2} + \frac{\partial^2}{\partial Y^2} + \frac{\partial}{\partial Y}\right) w_k^{III*} = 0 \tag{17a}$$

with the boundary condition

$$w_k^{III*}(x_*, 0) = g^{(k)}(a \pm 0)(x_*^k/k!) \quad \text{for } x_* \gtrless 0, \tag{17b}$$

where $g^{(k)}(a \pm 0)$ are right and left kth derivatives of g at $x = a$. At each state we must select the (unique) solution which vanishes as $Y \to \infty$, in order to match with zero.

These then are the results obtained by the method of matched asymptotic expansions, and the object of the present paper is to show that they are uniformly valid representations of the exact solution to all orders in ε. To this end it is necessary to place certain conditions on f, g and to make more precise the regions of validity.

The conditions:

(18a) f is infinitely differentiable for $x \neq 0$ and $f^{(k)}(\pm 0)$ exist for all k;

(18b) g is infinitely differentiable for $x \neq a$ and $g^{(k)}(a \pm 0)$ exist for all k;

are implicitly assumed in using the method. For example, $w_k^I(w_k^{III})$ involves any given derivative of $f(g)$ for k sufficiently large. The further conditions

$$\int_{-\infty}^{\infty} |f^{(k)}(x)|\, dx, \int_{-\infty}^{\infty} |g^{(k)}(x)|\, dx < \infty \quad \text{for all } k \tag{18c}$$

are then a technicality: the derivatives must now die out sufficiently rapidly at $\pm\infty$, but the data there has no asymptotic influence at any finite point.

The regions of validity for the f-expansions are:

(a) I: $x_0 \leqq |x|, \quad -1 \leqq y \leqq y_1 < 1$;

(b) II: $|X| \leqq X_\infty, \quad -1 \leqq y \leqq y_1 < 1$ excluding $|X| \leqq X_0$, $y + 1 \leqq y_{-1}$;

(c) II_*: $|X_*| \leqq X_{*\infty}, \quad 0 \leqq y_* \leqq y_{*\infty}$;

(d) III: $x_0 \leqq |x|, \quad 0 \leqq Y \leqq Y_\infty$;

(e) IV: $|X| \leqq X_\infty, \quad 0 \leqq Y \leqq Y_\infty$.

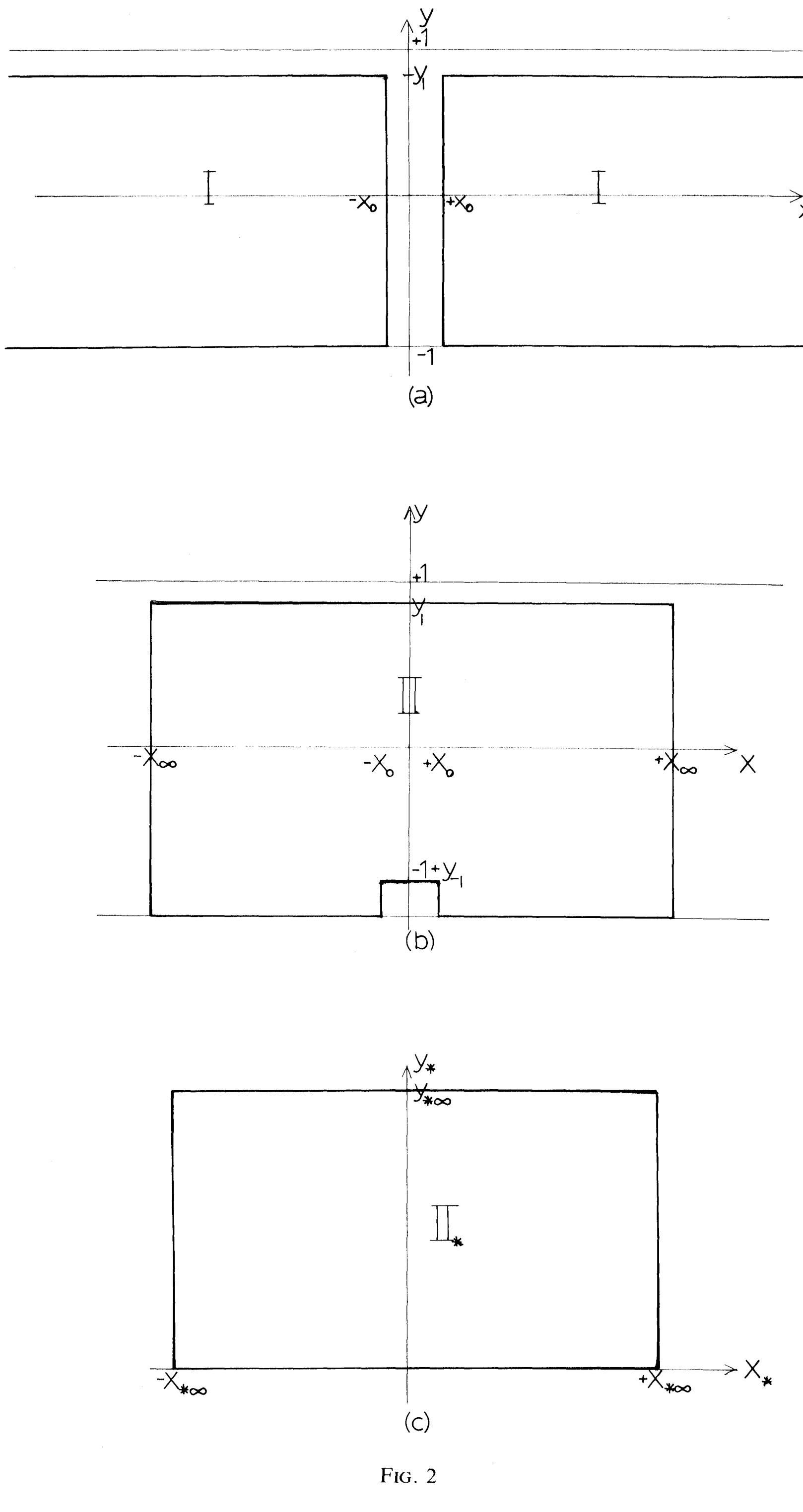

FIG. 2

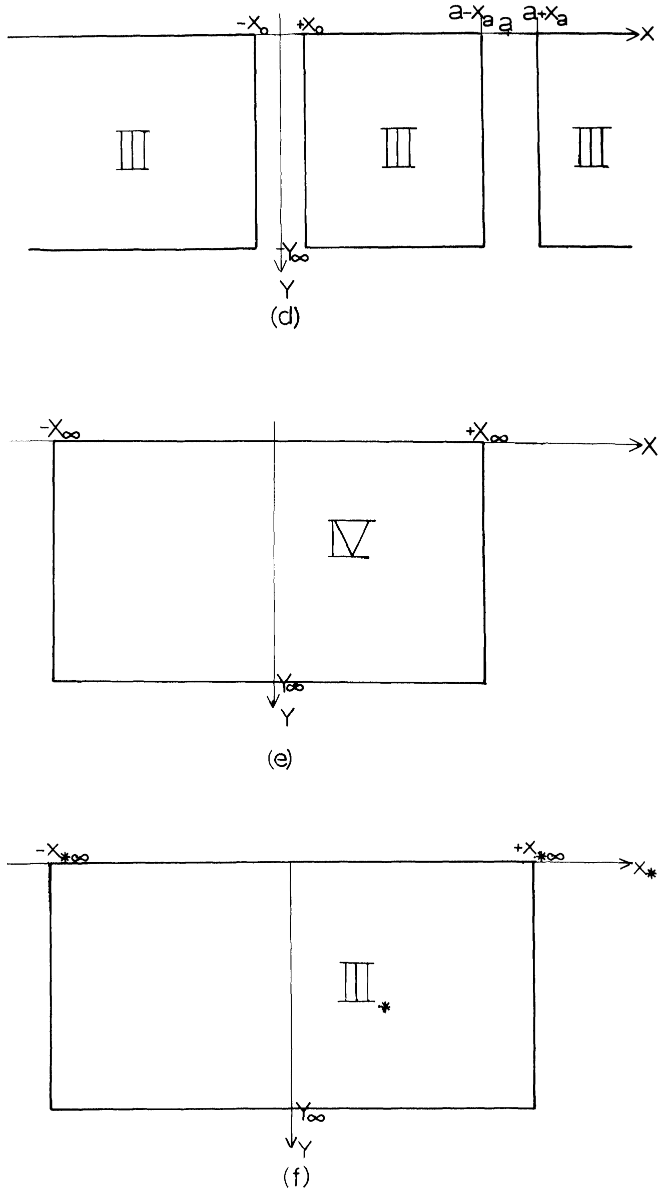

FIG. 2 (*continued*)

For the g-expansions we have

$$\text{(d)} \qquad III: \quad x_a \leqq |x - a|, \qquad 0 \leqq Y \leqq Y_\infty;$$

$$\text{(f)} \qquad III_*: \quad |x_*| \leqq x_{*\infty}, \qquad 0 \leqq Y \leqq Y_\infty.$$

The lettering corresponds to the parts of Fig. 2.

Here x_0, y_1, X_∞, X_0, y_{-1}, $X_{*\infty}$, $y_{*\infty}$, Y_∞, x_a, $x_{*\infty}$ are in the first instance fixed positive numbers; but we shall show that the regions can be extended to

$$x_0 = \varepsilon^{1/2-\delta}, \quad 1 - y_1 = \varepsilon^{1-\delta}, \quad X_\infty = \varepsilon^{-1/2+\delta}, \quad X_0 = \varepsilon^{1/4-\delta}, \qquad y_{-1} = \varepsilon^{1/2-\delta},$$
$$X_{*\infty} = \varepsilon^{-1+\delta}, \quad y_{*\infty} = (2-\delta)\varepsilon^{-1}, \quad Y_\infty = \varepsilon^{-1+\delta}, \quad x_a = \varepsilon^{1-\delta}, \quad x_{*\infty} = \varepsilon^{-1+\delta},$$

if weaker asymptotic approximations are allowed. Here $\delta > 0$ is arbitrarily small; and, in the case of g, the Y_∞ can in fact be arbitrarily large. Adjacent regions now overlap.

Note that X_0, y_{-1} do not reach the scale of the inner region II_* while $X_{*\infty}$, $y_{*\infty}$ go beyond the scale of region II. Extension is therefore not necessarily a guide to the new scale.

Because the solution is governed by the inhomogeneous diffusion equation (6a) in the region II, Eckhaus and de Jager [3] have called the latter a parabolic layer. Note that there is no difficulty in applying the method of matched asymptotic expansions to the origin of this layer, namely the region II_* (cf. Grasman [5]). For similar reasons we may call the region III a hyperbolic layer and IV a hyperbolic intersection region. The transition zone III_*, and in particular its essential difference from the singular region II_*, has been overlooked in the literature.

3. The exact solution. Taking the Fourier transform

$$\overline{(\cdot)} = \int_{-\infty}^{\infty} e^{-i\xi x}(\cdot)\,dx$$

of the differential equation (1a) and using the boundary condition (1b) we find

$$\bar{w}(\xi, y) = e^{(y+1)/(2\varepsilon)}\left[\frac{e^{-r(y+1)} - e^{r(y-3)}}{1 - e^{-4r}}\right]\bar{f}(\xi) + e^{(y-1)/(2\varepsilon)}\left[\frac{e^{r(y-1)} - e^{-r(y+3)}}{1 - e^{-4r}}\right]\bar{g}(\xi), \tag{19}$$

where

$$r = \sqrt{1 + 4\varepsilon^2\xi^2}/(2\varepsilon).$$

The exact solution of our boundary value problem is then the inverse

$$w(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x}\bar{w}(\xi, y)\,d\xi.$$

We are concerned with the asymptotic properties of this solution and will use the abbreviation a.e.s. for "asymptotically exponentially small." Any function which is uniformly a.e.s. in a region can be omitted. Thus e^{-4r} is a.e.s. uniformly in ξ and so will lead to functions which can be ignored throughout the strip. We may therefore write

$$\bar{w} = e^{(y+1)/(2\varepsilon)}[e^{-r(y+1)} - e^{r(y-3)}]\bar{f} + e^{(y-1)/(2\varepsilon)}[e^{r(y-1)} - e^{-r(y+3)}]\bar{g}(\xi).$$

The solution may be further simplified by separating regions near $y = 1$ from the others. Thus in regions I, II, II_*,

$$\bar{w} \sim \exp\left[\left(\frac{1}{2\varepsilon} - r\right)(y + 1)\right]\bar{f} \tag{19$'$a}$$

since the remaining terms are uniformly a.e.s., provided y_1 approaches 1 more slowly than ε tends to zero. In regions III, III_*, IV,

$$\bar{w} \sim \left\{\exp\left[\left(\frac{1}{2\varepsilon} - r\right)(2 - \varepsilon Y)\right] - \exp\left[\frac{2 - \varepsilon Y}{2\varepsilon} - r(2 + \varepsilon Y)\right]\right\}\bar{f} + \exp\left[-\left(\frac{1}{2\varepsilon} + r\right)\varepsilon Y\right]\bar{g}, \tag{19$'$b}$$

the remaining terms being uniformly a.e.s., provided Y_∞ tends to infinity more slowly than ε^{-1}.

The brackets multiplying $\bar{f}$ and $\bar{g}$ in the Fourier transform (19) of the exact solution come from the Green's function, and in order to see the separate effects of the Bessel functions mentioned in the Introduction, their common denominator should be expanded as a series of exponentials. The first term in the $\bar{f}$-bracket then gives rise to

$$\exp\left(\frac{y + 1}{2\varepsilon}\right)\exp\left[-r(y + 4k + 1)\right], \qquad k = 0, 1, 2, 3, \cdots,$$

where the second factor is the transform of the normal derivative at the lower boundary of a Bessel function representing the (equal) effects of the fundamental solution and the images at $y + 4k$ and $-(y + 4k + 2)$. Similarly the second term yields

$$-\exp\left(\frac{y + 1}{2\varepsilon}\right)\exp\left[-r(2 - y + 4k + 1)\right], \qquad k = 0, 1, 2, 3, \cdots,$$

representing the (equal) effects of images at $2 - y + 4k$ and $y - 4k - 4$. Together these account for the fundamental solution and all images in $y = \pm 1$, at $2l + (-1)^l y$, $l = 0, \pm 1, \pm 2, \cdots$. Similarly the bracket multiplying $\bar{g}$ leads to

$$\exp\left(\frac{y - 1}{2\varepsilon}\right)\exp\left[-r(2 - y + 4k - 1)\right] \quad \text{and}$$

$$-\exp\left(\frac{y - 1}{2\varepsilon}\right)\exp\left[-r(y + 4k + 4 - 1)\right],$$

where the second factors are derived from the same Bessel functions, this time the normal derivative at the upper boundary being taken. The pairings $2 - y + 4k$, $y - 4k$ and $y + 4k + 4$, $-(y + 4k + 2)$ are different because the boundary is, but the same points are involved.

Only the fundamental solution and its image in the lower boundary contribute to the simplified form (19$'$a). In addition to these, only the image of the fundamental solutions in the upper boundary plus its further image in the lower boundary contribute to the $\bar{f}$-term in (19$'$b). (Note that y and $2 - y$ approach each other as

$y \to 1$.) The $\bar{g}$-term has only the fundamental solution and its image in the upper boundary.

4. The core region I. Here we have the representation (19′a), which may be written

(20) $$\bar{w} \sim \bar{K}\bar{f}, \quad \text{where} \quad \bar{K}(\xi, y; \varepsilon) = \exp\left[(1/(2\varepsilon) - r)(y + 1)\right].$$

$\bar{K}$ is in fact the transform of the Green's function of the original equation (1a) for the half-plane $y \geqq -1$, so that it satisfies

(21) $$\frac{\partial}{\partial y}\bar{K} = \varepsilon\left(\frac{\partial^2}{\partial y^2} - \xi^2\right)\bar{K}, \qquad \bar{K}(\xi, -1; \varepsilon) = 1$$

as can easily be seen directly.

Consider now the Taylor expansion,

(22) $$\bar{K}(\xi, y; \varepsilon) = \sum_{k=0}^{m-1} \bar{K}^{(k)}(\xi, y; 0)\frac{\varepsilon^k}{k!} + \bar{R}_m(\xi, y; \varepsilon),$$

where

$$\bar{R}_m(\xi, y; \varepsilon) = \bar{K}^{(m)}(\xi, y; t\varepsilon)\frac{\varepsilon^m}{m!}, \qquad 0 < t < 1,$$

to any order m. The series apparently leads to the first m terms of the expansion (2) with coefficient functions which are the inverses of $\bar{f}\bar{K}^{(k)}/k!$. However these inverses do not exist (in the ordinary sense) since $\bar{K}^{(k)}$ is $O(\xi^{2k})$ for large ξ and, in order to obtain ones which do, we replace f with a function F such that

(23) $$F \equiv f \quad \text{for } x_0/2 \leqq |x|,$$
$$F \in C^\infty \quad \text{and} \quad \int_{-\infty}^{\infty} |F^{(k)}(x)|\, dx < \infty \quad \text{for all } k.$$

The construction and properties of F are given in the Appendix. Since it has integrable derivatives of all orders, its transform is $o(\xi^{-N})$ for every N and the inverses mentioned above exist for every k.

We may therefore write

$$w \sim \sum_{k=0}^{m-1} w_k^I \varepsilon^k + R_m * F + K * (f - F),$$

where

(24a) $$w_k^I = \frac{1}{2\pi k!}\int_{-\infty}^{\infty} \bar{K}^{(k)}(\xi, y; 0)\bar{F}(\xi)\, e^{i\xi x}\, d\xi,$$

(24b) $$R_m * F = \frac{\varepsilon^m}{2\pi m!}\int_{-\infty}^{\infty} \bar{K}^{(m)}(\xi, y; t\varepsilon)\bar{F}(\xi)\, e^{i\xi x}\, d\xi,$$

(24c) $$K * (f - F) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{K}(\xi, y; \varepsilon)[\bar{f}(\xi) - \bar{F}(\xi)]\, e^{i\xi x}\, d\xi.$$

We shall now show that: (i) the w_k^I satisfy the recurrence relation (3a) and the boundary conditions (3b); and that, uniformly in the region I (with x_0, y_1 fixed),

(ii) $R_m * F = O(\varepsilon^m)$ for every m and (iii) $K * (f - F)$ is a.e.s. The validity of the expansion (2) will thereby be established.

(i) By substituting the expansion (22) into (21) we find

$$\frac{\partial}{\partial y}\bar{K}^{(k)}(\xi, y; 0) = k\left(\frac{\partial^2}{\partial y^2} - \xi^2\right)\bar{K}^{(k-1)}(\xi, y; 0), \qquad \bar{K}^{(k)}(\xi, -1, 0) = \begin{cases} 1, & k = 0, \\ 0, & k \neq 0. \end{cases}$$

From these it is easily seen that the integrals (24a) satisfy the recurrence relation and boundary conditions.

(ii) Since $\bar{K}^{(m)}(\xi, y; \varepsilon)$ is the sum of terms of the form

$$\varepsilon^{\alpha_1}(1 + 4\varepsilon^2\xi^2)^{-\alpha_2/2}(1 + (1 + 4\varepsilon^2\xi^2)^{1/2})^{-\alpha_3}(y + 1)^{\beta}\xi^{2\gamma}\exp\left[\frac{-2\varepsilon\xi^2(y + 1)}{1 + \sqrt{1 + 4\varepsilon^2\xi^2}}\right],$$

where $\alpha_1, \alpha_2, \alpha_3, \beta, \gamma$ are nonnegative integers with $\gamma \leqq m$, there exists a constant C_m such that

$$|\bar{K}^{(m)}(\xi, y; t\varepsilon)| \leqq C_m(1 + \xi^2)^m$$

when ε is bounded. It follows that

$$(25) \qquad |R_m * F| < \frac{C_m\varepsilon^m}{2\pi m!}\int_{-\infty}^{\infty}(1 + \xi^2)^m|\bar{F}(\xi)|\,d\xi = O(\varepsilon^m)$$

since $\bar{F}$ is $o(\xi^{-N})$ for every N.

(iii) From the convolution theorem and the definition (24c) we may write

$$(26) \qquad K * (f - F) = \frac{y + 1}{2\pi\varepsilon}e^{(y+1)/(2\varepsilon)}\int_{-x_0/2}^{x_0/2}\frac{K_1(\sqrt{(x - x')^2 + (y + 1)^2}/(2\varepsilon))}{\sqrt{(x - x')^2 + (y + 1)^2}} \cdot [f(x') - F(x')]\,dx',$$

where K_1 is the modified Bessel function. Since the integration variable x' is always bounded away from x when the latter lies in region I, we deduce from the exponential behavior of K_1 for large values of its argument that $K * (f - F)$ is a.e.s. uniformly in the region I.

There remains the question of extending the region of validity by accepting a weaker asymptotic approximation. The limitation

$$(27) \qquad (1 - y_1) = \varepsilon^{1-\delta}$$

is accepted in using the simplified form (19′a) and no further condition is imposed by the analysis of the present section. Hence we may concentrate on x_0, and the problem is to determine the asymptotic behavior of $R_m * F$ and $K * (f - F)$ when

$$x_0 = \varepsilon^{\kappa} \quad \text{with} \quad \kappa > 0.$$

$R_m * F$ requires a more careful estimate of $\bar{F}$, which can be obtained from

$$\bar{F}(\xi) = (i\xi)^{-(2m+1)}\int_{-\infty}^{\infty}F^{(2m+1)}(x)\,e^{-i\xi x}\,dx.$$

Since

$$F^{(2m+1)}(x_0 x) = x_0^{-(2m+1)} F_m(x; x_0),$$

where F_m is bounded as $x_0 \to 0$ (see Appendix), it follows that

$$\bar{F}\left(\frac{\xi}{x_0}\right) = x_0 (i\xi)^{-(2m+1)} \int_{-\infty}^{\infty} F_m(x; x_0)\, e^{-i\xi x}\, dx$$

is bounded by $x_0 A_m \xi^{-(2m+2)}$ as $|\xi| \to \infty$. Here A_m depends on x_0 but, being bounded as $x_0 \to 0$, it may be replaced by a constant. By changing ξ^2 into $(1 + \xi^2)$ and the constant A_m appropriately, we then have a bound for all ξ, which can be used in the estimate (25) to give

$$|R_m * F| < \frac{AC_m}{2\pi m!} \varepsilon^m x_0^{-2m} \int_{-\infty}^{\infty} \frac{(x_0^2 + \xi^2)^m}{(1 + \xi^2)^{m+1}}\, d\xi = O(\varepsilon^{m(1-2\kappa)}).$$

Hence asymptotic approximation on a weaker scale is obtained provided

$$\kappa < \tfrac{1}{2}.$$

$K * (f - F)$ remains a.e.s. for such an x_0.

5. The free layer *II*. The last limitation suggests introducing the coordinate (4) to describe the solution near $x = 0$. Correspondingly the transform variable is changed to

$$\eta = \varepsilon^{1/2} \xi$$

so that (with tildes denoting the new transforms)

$$\tilde{w} \sim \varepsilon^{-1/2} \tilde{L} \bar{f}(\varepsilon^{-1/2}\eta),$$

where

$$\tilde{L}(\eta, y; \varepsilon) = \exp\left[\left(\frac{1}{2\varepsilon} - s\right)(y + 1)\right] \quad \text{and} \quad s = \frac{\sqrt{1 + 4\varepsilon\eta^2}}{2\varepsilon}. \tag{28}$$

The kernel $\tilde{L}$ satisfies

$$\left(\frac{\partial}{\partial y} + \eta^2\right)\tilde{L} = \varepsilon \frac{\partial^2}{\partial y^2} \tilde{L}, \qquad \tilde{L}(\eta, -1; \varepsilon) = 1. \tag{29}$$

As before, the Taylor expansion

$$\tilde{L}(\eta, y; \varepsilon) = \sum_{k=0}^{m-1} \tilde{L}^{(k)}(\eta, y; 0) \frac{\varepsilon^k}{k!} + \tilde{S}_m(\eta, y; \varepsilon), \tag{30}$$

where

$$\tilde{S}_m(\eta, y; \varepsilon) = \tilde{L}^{(m)}(\eta, y; t\varepsilon) \frac{\varepsilon^m}{m!},$$

will be needed. Simultaneous expansion of $\bar{f}$ and inversion of the coefficients of successive powers of ε then apparently lead to the first m terms of the series (5).

However this involves divergent integrals, which may be avoided by using the convolution theorem to invert before expanding f. With

$$f(\varepsilon^{1/2}x) = \sum_{k=0}^{m-1} f^{(k)}(\pm 0)X^k \frac{\varepsilon^{k/2}}{k!} + f^{(m)}(t\varepsilon^{1/2}X)X^m \frac{\varepsilon^{m/2}}{m!} \quad \text{for } X \gtrless 0,$$

we find

$$w \sim \sum_{k=0}^{m-1} w_k^{II}\varepsilon^{k/2} + S_m * f + T_m,$$

where

$$\text{(31a)} \quad \omega_k^{II} = \sum_{j=0}^{[k/2]} \frac{1}{j!(k-2j)!} \int_{-\infty}^{\infty} L^{(j)}(X - X', y; 0) f_{k-2j}(X') X'^{(k-2j)}\, dX',$$

$$\text{(31b)} \quad S_m * f = \frac{\varepsilon^n}{n!} \int_{-\infty}^{\infty} L^{(n)}(X - X', y; t\varepsilon) f(\varepsilon^{1/2}X')\, dX',$$

$$\text{(31c)} \quad T_m = \varepsilon^{m/2} \sum_{j=0}^{n-1} \frac{1}{j!(m-2j)!} \int_{-\infty}^{\infty} L^{(j)}(X - X', y; 0) f^{(m-2j)}(t\varepsilon^{1/2}X') X'^{(m-2j)}\, dX',$$

with

$$\text{(31')} \quad n = \left[\frac{m+1}{2}\right] = \begin{cases} (m+1)/2 & \text{for } m \text{ odd}, \\ m/2 & \text{for } m \text{ even}, \end{cases}$$

$$f_{k-2j}(X) = f^{(k-2j)}(\pm 0) \quad \text{for } X \gtrless 0,$$

and t may vary from function to function. We shall now show that: (i) the w_k^{II} satisfy the recurrence relation (6a) and the boundary conditions (6b); and that (ii) $S_m * f$ and T_m are $O(\varepsilon^{m/2})$, for every m, uniformly in the region II (with X_0, X_∞, y_{-1} and y_1 fixed). Proof of matching with the II_*-expansion will however be postponed to § 6. Expansion (5) will then have been validated.

(i) Substitution of the expansion (30) into the equations (29) and inversion show that

$$\left(\frac{\partial^2}{\partial X^2} - \frac{\partial}{\partial y}\right) L^{(k)}(X, y; 0) = -k \frac{\partial^2}{\partial y^2} L^{(k-1)}(X, y; 0),$$

$$L^{(k)}(X, -1; 0) = \begin{cases} \partial(X), & k = 0, \\ 0, & k \neq 0. \end{cases}$$

The sum of integrals (31a) is now seen to satisfy the recurrence relation and boundary conditions.

(ii) We need only prove that each of the integrals (31b), (31c) is bounded in the region II whenever ε is bounded. But some care is needed, as is easily seen for $S_m * f$ from the terms

$$\text{(32)} \quad (1 + 4\varepsilon\eta^2)^{-\alpha/2}[1 + (1 + 4\varepsilon\eta^2)^{1/2}]^{-\beta}(y+1)^\gamma \eta^{2\delta} \exp[(\tfrac{1}{2\varepsilon} - s)(y+1)]$$

which form $\tilde{L}^{(j)}(\eta, y; \varepsilon)$ for $j \neq 0$, where α, β, γ, δ are positive integers with $\alpha + \beta \geqq 3$ and $4 \leqq \delta \leqq 2j$. At $y = -1$ no help is obtained from the exponential, and the corresponding inversion integral is divergent for all x.

Nevertheless, the integral has a limit as $y \to -1$ if $X \neq 0$, so we shall manipulate it until this limit is exhibited when y is set equal to -1. The first step is to write the corresponding term in $L^{(j)}(X, y; \varepsilon)$ in the form

$$\frac{(-1)^\delta}{2\pi}(y+1)^\gamma \frac{\partial^{2\delta}}{\partial X^{2\delta}} \int_{-\infty}^{\infty} (1 + 4\varepsilon\eta^2)^{-\alpha/2} \cdot [1 + (1 + 4\varepsilon\eta^2)^{1/2}]^{-\beta} \exp\left[\left(\tfrac{1}{2\varepsilon} - s\right)(y+1)\right] e^{i\eta X}\, d\eta$$

and integrate on X' by parts 2δ times. Because of the discontinuities in f and its derivatives at $X = 0$, the integrated terms will be multiples of

$$\varepsilon^{\delta-(\mu+1)/2}(y+1)^\gamma \int_{-\infty}^{\infty} (1 + 4\varepsilon\eta^2)^{-\alpha/2}[1 + (1 + 4\varepsilon\eta^2)^{1/2}]^{-\beta} \cdot \exp\left[\left(\tfrac{1}{2\varepsilon} - s'\right)(y+1)\right](i\eta)^\mu e^{i\eta X}\, d\eta, \qquad \text{where} \quad 0 \leqq \mu \leqq 2\delta - 1; \tag{33a}$$

the remaining integral is a multiple of

$$\varepsilon^\delta (y+1)^\gamma \int_{-\infty}^{\infty} f^{(2\delta)}(\varepsilon^{1/2}X')\, dX' \int_{-\infty}^{\infty} (1 + 4\varepsilon\eta^2)^{-\alpha/2} \cdot [1 + (1 + 4\varepsilon\eta^2)^{1/2}]^{-\beta} \exp\left[\left(\tfrac{1}{2\varepsilon} - s'\right)(y+1)\right] e^{i\eta(X-X')}\, d\eta. \tag{33b}$$

Here M_2' denotes M_2 with ε replaced by $t\varepsilon$. The integral in (33a) is bounded for all X, ε when y is bounded away from -1; but it is still divergent for $y = -1$. Convergence at $y = -1$ for X positive can be ensured by bending the ends of the integration path upwards in the complex η-plane, so that they asymptote at an angle to the real axis. The integral is then seen to be convergent for $y = -1$ and bounded for all y, when X is bounded away from zero. Deform downwards for X negative. In short, the terms (33a) are uniformly bounded in the region *II*. In bounding the integral in (33b) we note that the η-integral is bounded by a multiple of $\varepsilon^{-1/2}$, as can be seen by using $t^{1/2}\varepsilon^{1/2}\eta$ for integration variable and remembering $\alpha + \beta > 1$. The absolute integrability of $f^{(2\delta)}$ then ensures the contribution (33b) to be bounded in region *II* (in fact $O(\varepsilon^{\delta-1})$) since $\delta \geqq 1$.

The treatment of (31c) is similar. Each integral (including $j = 0$ now) involves the sum of terms of the form (32) with $\varepsilon = 0$, on each of which integration by parts is performed 2δ times. In place of the expression (33a) we now have

$$\varepsilon^{\nu/2}(y+1)^\gamma \int_{-\infty}^{\infty} \exp[-\eta^2(y+1)](i\eta)^\mu e^{i\eta X}\, d\eta = \sqrt{\pi}\,\varepsilon^{\nu/2}(y+1)^{\gamma-1/2}\frac{\partial^\mu}{\partial X^\mu}\exp\left[\frac{-X^2}{y+1}\right], \tag{34a}$$

where ν is a nonnegative integer, which is clearly uniformly bounded in the region

II. The expression (33b) is likewise replaced by

$$(y+1)^{\gamma}\int_{-\infty}^{\infty}\mathscr{F}(X')\,dX'\int_{-\infty}^{\infty}\exp\left[-\eta^2(y+1)\right]e^{i\eta(X-X')}\,d\eta \tag{34b}$$

$$=(y+1)^{\gamma}\int_{-\infty}^{\infty}\mathscr{F}(X')\left(\frac{\pi}{y+1}\right)^{1/2}\exp\left[\frac{-(X-X')^2}{4(y+1)}\right]dX',$$

where

$$\mathscr{F}(X')=\frac{\partial^{2\delta}}{\partial X'^{2\delta}}[f^{(m-2j)}(t\varepsilon^{1/2}X')X'^{(m-2j)}]$$

is bounded by a power of X'. It follows that the integral is uniformly bounded in the region *II*.

Extension of the region upwards to

$$1-y_1=\varepsilon^{1-\delta}$$

is valid, as in § 4. Sideways, the limitation

$$X_\infty=\varepsilon^{-1/2+\delta}$$

arises from the integrals (34b). For $X=\varepsilon^{-\kappa}$ they behave like $\varepsilon^{\delta-\kappa(m-2j)}$, of which the worst is $\varepsilon^{-\kappa m}$. From (31c) we then see that κ must be less than 1/2.

Extension towards the point $X=0$, $y=-1$ is limited by the exponent in the integrals (33a), which for points near $X_0=\varepsilon^{\kappa}$, $y_{-1}=\varepsilon^{\lambda}$ with $\kappa,\lambda>0$ becomes

$$\frac{-2\eta^2}{1+(1+4\varepsilon\eta^2)^{1/2}}\varepsilon^{\lambda}\pm i\eta\varepsilon^{\kappa}.$$

Both terms have negative real parts (after deformation), one of which may be prevented from vanishing in the limit $\varepsilon\to 0$ by the transformation $\eta=\varepsilon^{-\kappa}\tau$ when $\lambda\geqq 2\kappa$ or $\eta=\varepsilon^{-\lambda/2}\tau$ when $\lambda\leqq 2\kappa$. The terms (33a) are then of order $\varepsilon^{\delta-(\mu+1)/2}$ times $\varepsilon^{-(\mu+1)\kappa}$ or $\varepsilon^{-(\mu+1)\lambda/2}$ so that the worst is of order $\varepsilon^{-4\kappa n}$ or $\varepsilon^{-2\lambda n}$. From (31b) we then see that λ can be arbitrarily large provided

$$\kappa<\tfrac{1}{4},$$

while κ can be arbitrarily large provided

$$\lambda<\tfrac{1}{2}.$$

It is noteworthy that the region *II* can only be extended down to half the scale of II_*: extension is not a reliable guide to the new scale.

6. The singular region II_*. The limitations on the extension of region *II* suggest that the coordinates (7) are needed to describe the solution near $X=0$, $y=-1$. The transform variable is likewise changed to

$$\eta_*=\varepsilon^{1/2}\eta$$

so that (with hats denoting the new transforms)

$$\hat{w}\sim\varepsilon^{-1}\hat{L}_*\bar{f}(\varepsilon^{-1}\eta_*),$$

where

$$\hat{L}_*(\eta_*, y_*) = \exp\left[(\tfrac{1}{2} - s_*)y_*\right] \quad \text{and} \quad s_* = \sqrt{1 + 4\eta_*^2}/2.$$

The kernel $\hat{L}_*$, which is now independent of ε, satisfies

$$\left(\frac{\partial^2}{\partial y_*^2} - \frac{\partial}{\partial y_*} - \eta_*^2\right)\hat{L}_* = 0, \qquad \hat{L}_*(\eta_*, 0) = 1; \tag{35}$$

and, in place of a second boundary condition, it does not grow exponentially as $y_* \to \infty$.

No expansion of the kernel is involved this time, but it is again necessary to invert by convolution before expanding f to avoid divergent integrals. We find

$$w \sim \sum_{k=0}^{m-1} w_k^{II*}(X_*, y_*)\varepsilon^k + L_* * f^{(m)},$$

where

$$w_k^{II*} = \frac{1}{k!}\int_{-\infty}^{\infty} L_*(X_* - X'_*, y_*) f_k(X'_*) X'^k_* \, dX'_*, \tag{36a}$$

$$L_* * f^{(m)} = \frac{\varepsilon^m}{m!}\int_{-\infty}^{\infty} L_*(X_* - X'_*, y_*) f^{(m)}(t\varepsilon X'_*) X'^m_* \, dX'_*. \tag{36b}$$

There is no difficulty in showing that: (i) the w_k^{II*} satisfy the equation (9a), the boundary conditions (9b), and the matching conditions noted after them; and that (ii) $L_* * f^{(m)} = O(\varepsilon^m)$ uniformly in the region II_* (with $X_{*\infty}$, $y_{*\infty}$ fixed). The validity of the expansion (8) is thereby established.

(i) Substitute the integrals (36a) directly into the equation and boundary conditions to show that they satisfy them by virtue of the equations (35). The series formed from them matches that formed from the integrals (31a) by virtue of the matching of $\hat{L}_*$ and $\tilde{L}$.

(ii) The integral in $L_* * f^{(m)}$ is actually

$$\frac{y_* e^{y_*/2}}{2\pi}\int_{-\infty}^{\infty} \frac{K_1(\frac{1}{2}\sqrt{(X_* - X'_*)^2 + y_*^2})}{\sqrt{(X_* - X'_*)^2 + y_*^2}} X'^m_* f^{(m)}(t\varepsilon X'_*)\, dX'_*$$

which is seen to be bounded in II_* when ε is bounded.

Extending the region of validity to

$$X_{*\infty} = \varepsilon^{-\kappa}, \qquad y_{*\infty} = \varepsilon^{-\lambda} \quad \text{with } \kappa, \lambda > 0$$

requires an estimation of the last integral for such values of X_*, y_*. On interchanging $X_* - X'_*$ and X'_* and noting that $f^{(m)}$ is bounded, we see that there remains

$$\begin{aligned} &y_* e^{y_*/2}\int_{-\infty}^{\infty} \frac{K_1(\frac{1}{2}\sqrt{X'^2_* + y_*^2})}{\sqrt{X'^2_* + y_*^2}} \sum_{s=0}^{m} \binom{m}{s} |X_*|^{m-s} |X'_*|^s \, dX'_* \\ &\qquad = \sum_{s=0}^{m} c_s |X_*|^{m-s} y_*^{(s+1)/2} e^{y_*/2} K_{(1-s)/2}\left(\frac{y_*}{2}\right), \end{aligned} \tag{37}$$

where

$$c_s = 2^{s+1}\Gamma\left(\frac{s+1}{2}\right)\binom{m}{s}.$$

The terms in the series are clearly of order $\varepsilon^{-(m-s)\kappa}\varepsilon^{-s\lambda/2} = \varepsilon^{-m\kappa+s(\kappa-\lambda/2)}$, there being no singularities in the y_*-functions at $y_* = 0$. The worst term is of order

$$\varepsilon^{-m\kappa} \quad \text{for } \kappa \geqq \lambda/2 \quad \text{and} \quad \varepsilon^{-m\lambda/2} \quad \text{for } \kappa \leqq \lambda/2;$$

in either case we must have

$$\kappa < 1, \qquad \lambda < 2$$

if the remainder (36b) is to be asymptotically small (on a weaker scale).

Thus the expansion gives an approximation even in the extended free layer itself. In fact, since its accuracy in the region II is the same as that of expansion (5), it must then be identical and this is easily checked. Note that values of λ greater than 1 are not of interest here since we have already accepted the limitation (27).

7. The boundary layer III: $g \equiv 0$. We must now use the representation (19′b) with $\bar{g} \equiv 0$, which will be written

$$\bar{w} \sim \bar{\mathscr{K}}\bar{f},$$

where

$$\bar{\mathscr{K}}(\xi, Y; \varepsilon) = \exp\left[\left(\frac{1}{2\varepsilon} - r\right)(2 - \varepsilon Y)\right] - \exp\left[\frac{(2 - \varepsilon Y)}{2\varepsilon} - r(2 + \varepsilon Y)\right].$$

Clearly,

$$\left(\frac{\partial^2}{\partial Y^2} + \frac{\partial}{\partial Y}\right)\bar{\mathscr{K}} = \varepsilon^2\xi^2\bar{\mathscr{K}}, \qquad \bar{\mathscr{K}}(\xi, 0; \varepsilon) = 0, \tag{38}$$

but there is no second boundary condition. Instead we shall show that $\bar{\mathscr{K}}$ matches the $\bar{K}$ of definition (20) to all orders.

First note that there is no contribution from $\exp[(2 - \varepsilon Y)/(2\varepsilon) - r(2 + \varepsilon Y)]$: with $Y = (1 - y)/\varepsilon$ and $-1 \leqq y \leqq y_1$, it is a.e.s. uniformly for ξ real. Accordingly we must show that $\bar{K} = \exp[(1/(2\varepsilon) - r)(y + 1)] = \exp[(1/(2\varepsilon) - r)(2 - \varepsilon Y)]$ satisfies the matching principle. While it would be difficult to believe otherwise, a formal proof is as follows. $\bar{K}$ has the expansion (22) for $|y| \leqq 1$ for every m. Moreover, since $\bar{K}^{(k)}(\xi, y; 0)$ is a polynomial in $(y + 1)$ of degree k (with coefficients depending on ξ^2—see § 4(ii)), it certainly has an inner expansion (in Y). Hence, according to Fraenkel's Theorem 1 [4], the matching principle holds to all orders.

We now introduce the Taylor expansion

$$\bar{\mathscr{K}}(\xi, Y; \varepsilon) = \sum_{k=0}^{m-1} \bar{\mathscr{K}}^{(k)}(\xi, Y; 0)\frac{\varepsilon^k}{k!} + \bar{\mathscr{R}}_m(\xi, Y; \varepsilon), \tag{39}$$

where

$$\bar{\mathscr{R}}_m(\xi, Y; \varepsilon) = \bar{\mathscr{K}}^{(m)}(\xi, Y; t\varepsilon)\frac{\varepsilon^m}{m!}.$$

The inverse of $\bar{\mathscr{K}}^{(k)}$ does not exist, as may be expected from the similar difficulty in the core region since there is matching. As in § 4 term-by-term inversion must be done after F, the smoothed version (23) of f, has been introduced; so that

$$w \sim \sum_{k=0}^{m-1} w_k^{III} \varepsilon^k + \mathscr{R}_m * F + \mathscr{K} * (f - F),$$

where

$$w_k^{III} = \frac{1}{2\pi k!} \int_{-\infty}^{\infty} \bar{\mathscr{K}}^{(k)}(\xi, Y; 0) \bar{F}(\xi)\, e^{i\xi x}\, d\xi, \tag{40a}$$

$$\mathscr{R}_m * F = \frac{\varepsilon^m}{2\pi m!} \int_{-\infty}^{\infty} \bar{\mathscr{K}}^{(m)}(\xi, Y; t\varepsilon) \bar{F}(\xi)\, e^{i\xi x}\, d\xi, \tag{40b}$$

$$\mathscr{K} * (f - F) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{\mathscr{K}}(\xi, Y; \varepsilon)[\bar{f}(\xi) - \bar{F}(\xi)]\, e^{i\xi x}\, d\xi. \tag{40c}$$

It remains to be shown that (i) the w_k^{III} satisfy the recurrence relation (12a), the boundary conditions (12b), and the matching conditions mentioned after them; and that (ii) $\mathscr{R}_m * F = O(\varepsilon^m)$ for every m and (iii) $\mathscr{K} * (f - F)$ is a.e.s., both uniformly in the region III (with x_0, Y_∞ fixed). In other words, the validity of the expansion (11) will be established. The proof is similar to that in region I (§ 4).

(i) By substituting the expansion (39) into the equations (38), we find

$$\left(\frac{\partial^2}{\partial Y^2} + \frac{\partial}{\partial Y}\right)\bar{\mathscr{K}}^{(k)}(\xi, Y; 0) = k(k-1)\xi^2 \bar{\mathscr{K}}^{(k-2)}(\xi, Y; 0), \qquad \bar{\mathscr{K}}^{(k)}(\xi, 0; 0) = 0$$

for all k. Hence the integrals (40a) satisfy the recurrence relation and boundary conditions. The fact that the series formed from them matches the series formed from the integrals (24a) follows from the matching of $\bar{\mathscr{K}}$ and $\bar{K}$ as proved above.

(ii) $\bar{\mathscr{K}}^{(m)}(\xi, Y; \varepsilon)$ is the sum of terms of the form

$$(\varepsilon Y)^{\alpha_1}(1 + 4\varepsilon^2\xi^2)^{-\alpha_2/2}(1 + \sqrt{1 + 4\varepsilon^2\xi^2})^{-\alpha_3} Y^\beta \xi^{2\gamma} \tag{41}$$

times

$$\exp\left[\frac{-2\varepsilon\xi^2(2 - \varepsilon Y)}{1 + \sqrt{1 + 4\varepsilon^2\xi^2}}\right] \quad \text{or} \quad \exp\left[\frac{-4\varepsilon\xi^2}{1 + \sqrt{1 + 4\varepsilon^2\xi^2}} - \frac{(1 + \sqrt{1 + 4\varepsilon^2\xi^2})Y}{2}\right],$$

where $\alpha_1, \alpha_2, \alpha_3, \beta, \gamma$ are nonnegative integers with $\beta \leqq m - \gamma$ and $\gamma \leqq m$. Hence $\bar{\mathscr{K}}^{(m)}$ can be bounded as in § 4(ii), so that the smoothness of F ensures $\mathscr{R}_m * F$ is $O(\varepsilon^m)$.

(iii) The inverse of $\bar{\mathscr{K}}$ is

$$\frac{\exp[(2 - \varepsilon Y)/(2\varepsilon)]}{2\pi\varepsilon}\left\{\frac{2 - \varepsilon Y}{\sqrt{x^2 + (2 - \varepsilon Y)^2}} K_1\left(\frac{\sqrt{x^2 + (2 - \varepsilon Y)^2}}{2\varepsilon}\right)\right.$$

$$\left. - \frac{2 + \varepsilon Y}{\sqrt{x^2 + (2 + \varepsilon Y)^2}} K_1\left(\frac{\sqrt{x^2 + (2 + \varepsilon Y)^2}}{2\varepsilon}\right)\right\}$$

so that the convolution argument in § 3(iii) shows $\mathscr{K} * (f - F)$ to be a.e.s. uniformly in region III.

Extending the region of validity by setting

$$x_0 = \varepsilon^k, \qquad Y_\infty = \varepsilon^{-\lambda} \qquad \text{with } \kappa, \lambda > 0$$

follows the same lines as for region I (see end of § 4). Because of the occurrence of Y in the terms (41) making up the derivatives of $\bar{\mathscr{K}}$, each of the terms in $\mathscr{R}_m * F$ must be bounded separately using different estimates of $\bar{F}$. Anticipating that λ is not greater than 1 (so that εY is bounded), we see that the worst terms are those containing $Y^{m-\gamma}\xi^{2\gamma}$; and if the estimate of $\bar{F}$ obtained from $F^{(2\gamma+1)}$ is used (cf. end of § 3), their contribution to $\mathscr{R}_m * F$ is seen to be at most $O(\varepsilon^m x_0^{-2\gamma} Y_\infty^{m-\gamma})$. Letting γ range from 0 to m now shows that we must have

$$\kappa < \tfrac{1}{2} \quad \text{and} \quad \lambda < 1.$$

It is easily checked that $\mathscr{K} * (f - F)$ remains a.e.s. for such x_0, Y_∞.

8. The intersection IV of the layers: $g = 0$. To describe the solution near $x = 0$ we once more introduce the coordinate (4) and the transform variable of § 5. Then

$$\tilde{w} \sim \varepsilon^{-1/2}\tilde{\mathscr{L}}\bar{f}(\varepsilon^{-1/2}\eta),$$

where

$$\tilde{\mathscr{L}}(\eta, Y; \varepsilon) = \exp\left[\left(\frac{1}{2\varepsilon} - s\right)(2 - \varepsilon Y)\right] - \exp\left[\frac{2 - \varepsilon Y}{2\varepsilon} - s(2 + \varepsilon Y)\right].$$

(s is given by the formula (28).) We have

$$\left(\frac{\partial^2}{\partial Y^2} + \frac{\partial}{\partial Y}\right)\tilde{\mathscr{L}} = \varepsilon\eta^2\tilde{\mathscr{L}}, \qquad \tilde{\mathscr{L}}(\eta, 0; \varepsilon) = 0; \tag{42}$$

and in place of a second boundary condition, the matching of $\tilde{\mathscr{L}}$ with the $\tilde{L}$ of definition (28) to all orders (as will now be shown).

The argument is similar to that in § 6. The term $\exp[(2 - \varepsilon Y)/(2\varepsilon) - s(2 + \varepsilon Y)]$ can be neglected: with $Y = (1 - y)/\varepsilon$ and $-1 \leqq y \leqq y_1$, it is a.e.s. uniformly for η real. We need only show that $\tilde{L} = \exp[(1/(2\varepsilon) - s)(y + 1)] = \exp[(1/(2\varepsilon) - s) \cdot (2 - \varepsilon Y)]$ satisfies the matching principle. But $\tilde{L}$ has the expansion (30) for $|y| \leqq 1$ and every m, while $\tilde{L}^{(k)}(\eta, y; 0)$ has an inner (Y) expansion to all orders. Fraenkel's Theorem 1 therefore ensures that the matching principle holds to all orders.

Once again a Taylor expansion

$$\tilde{\mathscr{L}}(\eta, Y; \varepsilon) = \sum_{k=0}^{m-1} \tilde{\mathscr{L}}^{(k)}(\eta, y; 0)\frac{\varepsilon^k}{k!} + \tilde{\mathscr{S}}_m(\eta, y; \varepsilon), \tag{43}$$

where

$$\tilde{\mathscr{S}}_m(\eta, y; \varepsilon) = \tilde{\mathscr{L}}^{(m)}(\eta, Y; t\varepsilon)\varepsilon^m/m!,$$

will be needed. Then, if as before (§ 5) divergent integrals are avoided by inverting through the convolution theorem before expanding f, we obtain

$$w \sim \sum_{k=0}^{m-1} w_k^{IV}(X, Y)\varepsilon^{k/2} + \mathscr{S}_m * f + \mathscr{T}_m,$$

where

$$w_k^{IV} = \sum_{j=0}^{[k/2]} \frac{1}{j!(k-2j)!} \int_{-\infty}^{\infty} \mathscr{L}^{(j)}(X - X', Y; 0) f_{k-2j}(X') X'^{(k-2j)}\, dX', \tag{44a}$$

$$\mathscr{S}_m * f = \frac{\varepsilon^n}{n!} \int_{-\infty}^{\infty} \mathscr{L}^{(n)}(X - X', Y; t\varepsilon) f(\varepsilon^{1/2} X')\, dX', \tag{44b}$$

$$\mathscr{T}_m = \varepsilon^{m/2} \sum_{j=0}^{n-1} \frac{1}{j!(m-2j)!} \int_{-\infty}^{\infty} \mathscr{L}^{(j)}(X - X', Y; 0) f^{(m-2j)}(t\varepsilon^{1/2} X') X'^{(m-2j)}\, dX', \tag{44c}$$

and the definitions (31′) still hold. It remains to be shown that (i) the w_k^{IV} satisfy the recurrence relation (14a), the boundary conditions (14b), and the matching conditions mentioned after them; and that (ii) $\mathscr{S}_m * f$ and $\mathscr{T}_m$ are $O(\varepsilon^{m/2})$, for every m, uniformly in the region IV (with X_∞, Y_∞ fixed). In this case the validity of the expansion (13) is established. The proof is similar to that in region II (§ 5).

(i) On substituting the expansion (43) into (42) and inverting, we find

$$\left(\frac{\partial^2}{\partial Y^2} + \frac{\partial}{\partial Y}\right) \mathscr{L}^{(k)}(X, Y; 0) = -k \frac{\partial^2 \mathscr{L}^{(k-1)}}{\partial X^2}(X, Y; 0), \qquad \mathscr{L}^{(k)}(X, 0; 0) = 0$$

for all k. Hence the sums of integrals (44a) satisfy the recurrence relation and boundary conditions. Moreover, the series formed from them matches the series formed from the integrals (31a) by virtue of the matching of $\mathscr{L}$ and L, which is ensured by the matching of $\tilde{\mathscr{L}}$ and $\tilde{L}$ proved above.

(ii) We have to show that each of the integrals (44b), (44c) is bounded in the region IV for ε sufficiently small; this turns out to be more straightforward than in § 5. As there, we must look at the individual terms in $\tilde{\mathscr{L}}^{(j)}(\eta, Y; \varepsilon)$. They are

$$(\varepsilon Y)^{\alpha_1}(1 + 4\varepsilon\eta^2)^{-\alpha_2/2}(1 + \sqrt{1 + 4\varepsilon\eta^2})^{-\alpha_3} Y^{\beta} \eta^{2\gamma} \tag{45}$$

times

$$\exp\left[\frac{-2\eta^2(2 - \varepsilon Y)}{1 + \sqrt{1 + 4\varepsilon\eta^2}}\right] \quad \text{or} \quad \exp\left[\frac{-4\eta^2}{1 + \sqrt{1 + 4\varepsilon\eta^2}} - (1 + \sqrt{1 + 4\varepsilon\eta^2})\frac{Y}{2}\right],$$

where α_1, α_2, α_3, β, γ are nonnegative integers with

$$\beta \leqq \gamma \quad \text{and} \quad \beta + \gamma \leqq 2j. \tag{45′}$$

The exponential factors ensure that the inverse of any such term is bounded for all X provided Y is bounded and, for the first factor, ε is sufficiently small to give a negative exponent. An immediate estimate of the integral in $\mathscr{S}_m * f$ is therefore $O(\varepsilon^{-1/2})$, but this can be improved by expanding $\mathscr{L}^{(n)}$ once more to give

$$\int_{-\infty}^{\infty} \mathscr{L}^{(n)}(X - X', Y; 0) f(\varepsilon^{1/2} X')\, dX'$$

plus a remainder which is now $O(\varepsilon^{1/2})$. So we are left with this last integral, which can be treated along with the integrals (44c).

For $\varepsilon = 0$ the nonzero terms (45) have the inverses

$$(-1)^\gamma Y^\beta\, \partial^{2\gamma}[\sqrt{\pi/2}\exp(-X^2/8)]/\partial X^{2\gamma}$$

times 1 or e^{-Y}, so that we are concerned with integrals of the form

$$\int_{-\infty}^{\infty} Y^\beta \exp[-X'^2/8]\mathscr{F}(X - X')\,dX', \tag{46}$$

where

$$\mathscr{F}(X) = \frac{\partial^{2\gamma}}{\partial X^{2\gamma}}[f^{(m-2j)}(t\varepsilon^{1/2}X')X'^{(m-2j)}], \qquad j = 0, 1, \cdots, n-1 \text{ or } m/2.$$

Note that X' and $X - X'$ have been interchanged, and that the limitations for $j = m/2$ are $\beta \leqq \gamma$, $\beta + \gamma \leqq 2n$ (and not m). Clearly these integrals are bounded if X, Y and ε are.

Extension of the region to

$$X_\infty = \varepsilon^{-\kappa}, \quad Y_\infty = \varepsilon^{-\lambda} \quad \text{with } \kappa, \lambda > 0$$

affects these results in two ways. Anticipating that λ is not greater than 1, so that εY is bounded, we see that the critical factor in the terms (45) is Y^β, which at worst changes the bound on their inverses to $O(\varepsilon^{-\lambda j})$. The contribution to $\mathscr{S}_m * f$ (corresponding to the remainder above) is then $O[\varepsilon^{n+1/2-\lambda(n+1)}]$ so that for

$$\lambda < 1$$

a weaker asymptotic approximation is attained as soon as n is larger than $(\lambda - 1/2)/(1 + \lambda)$.

The same change occurs in each of the integrals (46), but in addition the powers of X which arise from expanding the powers of $(X - X')$ will provide at worst $O(\varepsilon^{-\kappa'(m-2j-2\gamma)})$, where $\kappa' = 1/2$ or κ accordingly as $m - 2j \gtrless 2\gamma$, if we anticipate $\kappa < 1/2$. Thus the integrals (46) for a given $j \neq m/2$ are at worst $O(\varepsilon^{-q})$, where $q = \max[\lambda\beta + \kappa'(m - 2j - 2\gamma)]$ on the triangle (45′). But for $\varepsilon = 0$ the nonzero terms (45) have $\gamma \geqq j$, so that for fixed j the maximum value of the bracket is $\kappa' m + (\lambda - 4\kappa')j$, attained for $\beta = \gamma = j$. Hence $q = \lambda m/4$ or κm according as $\kappa \lessgtr \lambda/4$, and we must have

$$\kappa < \tfrac{1}{2}.$$

(No further restriction arises for $j = m/2$.)

9. The boundary layer III: $f \equiv 0$. Outside the boundary layer the part of the solution due to g is uniformly a.e.s., as has already been noted in writing down the representation (19′a). Inside the boundary layer the representation (19′b) gives

$$\bar{w} \sim \overline{\mathscr{K}}_0 \bar{g}, \quad \text{where} \quad \overline{\mathscr{K}}_0(\xi, Y; \varepsilon^2) = \exp\left[-\left(\frac{1}{2\varepsilon} + r\right)(\varepsilon Y)\right].$$

Clearly,

$$\left(\frac{\partial^2}{\partial Y^2} + \frac{\partial}{\partial Y}\right)\overline{\mathscr{K}}_0 = \varepsilon^2\xi^2\overline{\mathscr{K}}_0, \qquad \overline{\mathscr{K}}_0(\xi, 0; \varepsilon^2) = 1. \tag{47}$$

Once again there is no second boundary condition, but instead $\bar{\mathscr{K}}_0$ is seen to match the zero function in y.

The Taylor expansion of the kernel in ε^2 is

$$\bar{\mathscr{K}}_0(\xi, Y; \varepsilon^2) = \sum_{k=0}^{m-1} \bar{\mathscr{K}}_0^{(k)}(\xi, Y; 0)\frac{\varepsilon^{2k}}{k!} + \bar{\mathscr{R}}_{om}(\xi, Y; \varepsilon^2), \tag{48}$$

where

$$\bar{\mathscr{R}}_{0m}(\xi, Y; \varepsilon^2) = \bar{\mathscr{K}}_0^{(m)}(\xi, Y; t\varepsilon^2)\varepsilon^{2m}/m!$$

and derivatives are taken with respect to ε^2. Inversion term-by-term can only be carried out after a smoothed version of g has been introduced (cf. §§ 4, 7). With

$$G \equiv g \quad \text{for} \quad x_0/2 \leqq |x - a|,$$

$$G \in C^\infty \quad \text{and} \quad \int_{-\infty}^{\infty} |G^{(k)}(X)|\, dX < \infty \quad \text{for all } k$$

(constructed in the Appendix), we may write

$$w \sim \sum_{k=0}^{m-1} w_{2k}^{III}\varepsilon^{2k} + \mathscr{R}_{0m} * G + \mathscr{K}_0 * (g - G),$$

where

$$w_{2k}^{III} = \frac{1}{2\pi k!}\int_{-\infty}^{\infty} \bar{\mathscr{K}}_0^{(k)}(\xi, Y; 0)\bar{G}(\xi)\, e^{i\xi x}\, d\xi, \tag{49a}$$

$$\mathscr{R}_{0m} * G = \frac{\varepsilon^{2m}}{2\pi m!}\int_{-\infty}^{\infty} \bar{\mathscr{K}}_0^{(m)}(\xi, Y; t\varepsilon^2)\bar{G}(\xi)\, e^{i\xi x}\, d\xi, \tag{49b}$$

$$\mathscr{K} * (g - G) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{\mathscr{K}}_0(\xi, Y; \varepsilon^2)[\bar{g}(\xi) - \bar{G}(\xi)]\, e^{i\xi x}\, d\xi. \tag{49c}$$

We shall show that (i) the w_{2k}^{III} satisfy the recurrence relation (12a), the boundary conditions (12c), and the matching conditions mentioned after them; and that (ii) $\mathscr{R}_{0m} * G = O(\varepsilon^{2m})$ for every m and (iii) $\mathscr{K} * (g - G)$ is a.e.s., both uniformly in the region III (with x_0, Y_∞ fixed). The expansion (11), containing only even powers of ε, will then have been proved valid. The steps are similar to but simpler than those for f in § 7.

(i) According to equations (47), the coefficient functions in the expansion (48) satisfy

$$\left(\frac{\partial^2}{\partial Y^2} + \frac{\partial}{\partial Y}\right)\bar{\mathscr{K}}^{(k)}(\xi, Y; 0) = k\xi^2\bar{\mathscr{K}}_0(\xi, Y; 0),$$

$$\bar{\mathscr{K}}_0^{(k)}(\xi, 0; 0) = \begin{cases} 1 & \text{for } k = 0, \\ 0 & \text{for } k \neq 0. \end{cases}$$

It follows that the integrals (49a) satisfy the recurrence relation and boundary conditions. Matching with the zero function outside the boundary layer is ensured by the matching of $\bar{\mathscr{K}}_0$ with it.

(ii) The derivative $\bar{\mathscr{K}}^{(m)}(\xi, Y; \varepsilon^2)$ is the sum of terms

$$(1 + 4\varepsilon^2\xi^2)^{-\alpha/2} Y^{\beta}\xi^{2m} \exp[-(1 + \sqrt{1 + 4\varepsilon^2\xi^2})Y/2], \tag{50}$$

where α and $\beta \leqq m$ are nonnegative integers. Hence it can be bounded as in § 4 (ii), so that the smoothness of G ensures $\mathscr{R}_{0m} * G$ is $O(\varepsilon^m)$.

(iii) In convolution form

$$\mathscr{K} * (g - G) = \frac{Y}{2\pi} e^{-Y/2} \int_{a-(x_0/2)}^{a+(x_0/2)} \frac{K_1(\sqrt{(x - x')^2 + \varepsilon^2 Y^2}/(2\varepsilon))}{\sqrt{(x - x')^2 + \varepsilon^2 Y^2}} [g(x') - G(x')]\, dx'$$

which is seen to be a.e.s. uniformly in the region III.

Extension of the region to

$$x_a = \varepsilon^{\kappa}, \quad Y_{\infty} = \varepsilon^{-\lambda} \quad \text{with} \quad \kappa, \lambda > 0$$

uses a simple version of the argument at the end of § 7. Because of the exponential, the terms (50) are worst when Y is finite, and then they contribute at most $O(\varepsilon^{2m}x_a^{-2m})$ to $\mathscr{R}_{0m} * G$. Hence

$$\kappa < 1$$

but λ is arbitrary. In fact the two parts of the extended region may be joined across $x = x_a$ for any value of Y which tends to infinity algebraically in $1/\varepsilon$. No further restriction comes from $\mathscr{K} * (g - G)$, which remains a.e.s.

Note that the excluded region does not shrink down to a point, but only to the line $x = x_a$ in the boundary layer. Even though the regions II_* and III_* have the same asymptotic dimensions, the character of the solution in them is quite different.

10. The transition zone III_*. The solution near the vertical line $x = a$ in the boundary layer through the discontinuity in g is described by means of the coordinate (15) and the corresponding transform variable

$$\xi_* = \varepsilon\xi.$$

The different notation X_*, y_* and x_*, Y is designed to emphasize the different nature of the regions II_* and III_*: the former resolves a breakdown in a parabolic layer where two coordinates are involved; the latter resolves a breakdown in a hyperbolic layer where only one coordinate is involved.

We must now consider (using hats again for the transform)

$$\hat{w} \sim \varepsilon^{-1}\hat{\mathscr{K}}_*\bar{g}(\varepsilon^{-1}\xi_*)\, e^{ia\xi_*/\varepsilon},$$

where

$$\hat{\mathscr{K}}_*(\xi_*, Y) = \exp[(-\tfrac{1}{2} + r_*)Y] \quad \text{and} \quad r_* = \sqrt{(1 + 4\xi_*^2)}/2.$$

Clearly,

$$\left(\frac{\partial^2}{\partial Y^2} + \frac{\partial}{\partial Y} - \xi_*^2\right)\hat{\mathscr{K}}_* = 0, \qquad \hat{\mathscr{K}}_*(\xi_*, 0) = 1, \tag{51}$$

and, in place of a second boundary condition, $\hat{\mathscr{K}}_*$ is seen to match with the zero function in y.

As in region II_* no expansion is involved, but inversion by convolution is necessary before expanding g. We find

$$w \sim \sum_{k=0}^{m-1} w_k^{III*}(x_*, Y)\varepsilon^k + \mathscr{K}_* * g^{(m)},$$

where

$$\text{(52a)} \qquad w_k^{III*} = \frac{1}{k!}\int_{-\infty}^{\infty} \mathscr{K}_*(x_* - x'_*, Y)g_k(x'_*)x'^k_* \, dx'_*,$$

$$\text{(52b)} \qquad \mathscr{K}_* * g^{(m)} = \frac{\varepsilon^m}{m!}\int_{-\infty}^{\infty} \mathscr{K}_*(x_* - x'_*, Y)g^{(m)}(a + t\varepsilon x'_*)x'^m_* \, dx'_*,$$

with

$$\text{(52')} \qquad g_{(k)}(x_*) = g^{(k)}(a \pm 0) \quad \text{for } x_* \gtrless 0.$$

The expansion (16) will therefore be established if we show that (i) the w_k^{III*} satisfy (17a), the boundary conditions (17b) and the matching conditions noted after them; and that (ii) $\mathscr{K}_* * g^{(m)} = O(\varepsilon^m)$ uniformly in the region III_* (with $x_{*\infty}$, Y_∞ fixed).

(i) Substitute the integrals (52a) directly into the equation and boundary conditions to show that they satisfy them by virtue of (51). The series formed from them matches the zero function in y because $\hat{\mathscr{K}}_*$ does.

(ii) The integral in $\mathscr{K}_* * g^{(m)}$ is actually

$$\frac{Ye^{-Y/2}}{2\pi}\int_{-\infty}^{\infty} \frac{K_1(\sqrt{(x_* - x'_*)^2 + Y^2}/2)}{\sqrt{(x_* - x'_*)^2 + Y^2}} g^{(m)}(a + t\varepsilon x'_*)x'^m_* \, dx'_*$$

which is bounded in III_* so long as ε is bounded.

That the region can be extended to

$$x_{*\infty} = \varepsilon^{-\kappa}, \quad Y_\infty = \varepsilon^{-\lambda}, \quad \text{with} \quad \kappa, \lambda > 0,$$

is seen from the corresponding treatment of II_* (§ 6). The expression (37) is replaced by

$$e^{-Y}\left[\sum_{s=0}^{m} c_s|x_*|^{m-s}Y^{(s+1)/2}\, e^{Y/2}K_{(1-s)/2}(Y/2)\right].$$

The extra exponential factor results from the change $e^{y_*/2}$ to $e^{-Y/2}$, which in turn is traceable to the kernel having M_{1*} in place of M_{2*}. It suppresses the powers of Y so that Y_∞ plays no role. Thus

$$\kappa < 1 \quad \text{and any } \lambda$$

will do.

In fact, for any Y which tends to infinity algebraically in $1/\varepsilon$ every remainder is a.e.s. and the expansion asymptotes zero. In particular, this holds in the core, where εY is constant.

11. Concluding remarks. There are two variations of the basic conditions (18) on f and g which are of considerable importance. First the condition (18a) may be strengthened to:

$$f^{(k)} \text{ is continuous at } x = 0 \text{ for } k \leqq k_0. \tag{53}$$

The question then is at what stage the free layer must be introduced. Similarly the condition (18b) may be strengthened. On the other hand, the condition (18a) may be weakened to:

$$f^{(k)}(\pm 0) \text{ exist for } k \leqq k_0,$$

while still insisting that

$$\int_{-\infty}^{\infty} |f^{(k)}(x)|\, dx < \infty \quad \text{for } k \leqq k_0 + 1$$

remains from condition (18c), though nothing is said about later derivatives. The question then is the order to which the various asymptotic expansions are valid. Clearly a similar question arises for g. We shall consider these points in turn.

There is no need to introduce the free shear layer until the expansion in region I fails to be valid near $x = 0$, i.e., so long as F can be avoided. We must therefore determine the largest integer m for which the integral in the remainder (24b) can still be bounded when F is replaced by f. But by analogy with the bound (25), $\bar{f}$ must be small compared to $\xi^{-(2m+1)}$ as $\xi \to \infty$, while the condition (53) ensures that $\bar{f}$ is at worst of order $\xi^{-(k_0+1)}$. Hence

$$\max m = [(k_0 - 1)/2].$$

Obviously the expansion in region III is also valid to this order near $x = 0$. Similar arguments apply to g.

When only a finite number of left- and right-derivatives of f exist at $x = 0$, the expansions fail first in the free layer and its intersection with the boundary layer. In fact, they never fail outside if we still require

$$\left(\int_{-\infty}^{-x_0} + \int_{x_0}^{\infty} \right) |f^{(k)}(x)|\, dx < \infty \quad \text{for all } k;$$

see, for example, the bounding (25).

In region II all derivatives up to $f^{(2\delta-1)}(\pm 0)$ are used in estimating the remainder as well as $\int_{-\infty}^{\infty} |f^{(2\delta)}(x')|\, dx' < \infty$; see § 5(ii). Since $\delta \leqq 2j$ and $j \leqq n$ we must have $4n \leqq k_0 + 1$ so that

$$\max m = 2[(k_0 + 1)/4].$$

This applies also in region IV, and no further restriction arises in region II_*. Similar arguments apply to g.

Note that there is no difficulty in calculating the coefficient functions (31a) much further, namely up to $k = k_0$ corresponding to

$$\max m = k_0 + 1.$$

However we can only prove the approximation is $O(\varepsilon^{[(k_0+1)/4]})$. For example, if $k_0 = 3$ the four coefficient functions w_0^{II}, w_1^{II}, w_2^{II}, w_3^{II} can be calculated, but the resulting approximation is only known to be $O(\varepsilon)$; w_2^{II} and w_3^{II} are useless.

Appendix. The construction of $F(x)$, the smoothed version of f introduced in § 4, will be based on the C^∞-function

$$\sigma(x) = \begin{cases} e^{-1/x} & \text{for } x > 0, \\ 0 & \text{for } x < 0. \end{cases}$$

Consider

$$\tau(x) = \int_{-\infty}^{x} \sigma(x' - \tfrac{1}{2})\sigma(1 - x')\,dx' \Big/ \int_{-\infty}^{\infty} \sigma(x' - \tfrac{1}{2})\sigma(1 - x')\,dx',$$

where the common integrand vanishes for $x' < \frac{1}{2}$ and $x' > 1$. Clearly τ is C^∞ and takes the values

$$\tau(x) = \begin{cases} 0 & \text{for } x < \frac{1}{2}, \\ 1 & \text{for } x > 1. \end{cases}$$

Now set

$$F(x) = [\tau(2x/x_0) + \tau(-2x/x_0)]f(x);$$

then F has the properties (23), the last by virtue of $\int_{-\infty}^{\infty} |f^{(k)}(x)|\,dx < \infty$ for all k. (In fact, it is zero for $|x| \leqq x_0/4$.)

As $x_0 \to 0$ the most singular contributions to $F^{(2m+1)}$ come from letting all $2m + 1$ derivatives fall on the functions $\tau(\pm 2x/x_0)$. Thus the worst terms in $F^{(2m+1)}(x_0 x)$ are $(\pm 2/x_0)^{2m+1}\tau^{(2m+1)}(\pm 2x)f(x)$, so that it can be written $x_0^{-(2m+1)}F_m(x; x_0)$, where F_m is bounded as $x_0 \to 0$. This property is used at the end of § 4.

Similarly the smoothed version of g used in § 9 is

$$G(x) = [\tau(2(x - a)/x_a) + \tau(2(a - x)/x_a)]g(x).$$

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