MATH-6620 PERTURBATION METHODS SPRING 2016

Homework-1

Assigned Tuesday January 26, 2016

Due Friday February 5, 2016

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PROBLEMS

1. (a) As $\epsilon \to 0$, find a 3-term perturbation expansion for each root of $\epsilon x^3 + x - 1 = 0$.

   (b) For ϵ small, find the first three terms of the perturbation expansion of $x(\epsilon)$, the solution near zero, of

$$\sqrt{2}\sin\left(x + \frac{\pi}{4}\right) - 1 - x + \frac{x^2}{2} + \frac{1}{6}\epsilon = 0.$$

Solution:

(a) We begin by expanding x:

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + ...$$

Thus the equation becomes

$$\epsilon(x_0 + \epsilon x_1 + \epsilon^2 x_2 + ...)^3 + x_0 + \epsilon x_1 + \epsilon^2 x_2 + ... - 1 = 0.$$

Keeping only the ϵ^2 terms, the above simplifies by expanding the cube and reordering terms to the equation

$$x_0 - 1 + \epsilon(x_0^3 + x_1) + \epsilon^2(3x_0^2 x_1 + x_2) + ... = 0.$$

However, we can only solve for one root. We get the equations
$\epsilon^0 :\ x_0 = 1$
$\epsilon^1 :\ x_0^3 + x_1 = 0 \implies x_1 = -x_0^3 = -1$
$\epsilon^2 :\ 3x_0^2 x_1 + x_2 = 0 \implies x_2 = -3x_0^2 x_1 = 3.$
Thus one root is $x^{(1)} = 1 - \epsilon + 3\epsilon^2 + ...$

We realize the procedure has broken down because our expansion of x fails to balance the equation. As $\epsilon \to 0$, $\epsilon x^3 + x$ must balance. Thus

$$\epsilon x^3 \approx x \implies \epsilon x^2 \approx 1 \implies x^2 \approx \frac{1}{\epsilon} \implies x \approx \frac{1}{\sqrt{\epsilon}}.$$

We then scale the equation with $y = \frac{x}{\sqrt{\epsilon}}$ which turns the original equation into

$$\epsilon\frac{y^3}{\epsilon^{3/2}} + \frac{y}{\sqrt{\epsilon}} - 1 = 0.$$

$$\implies y^3 + y - \sqrt{\epsilon} = 0.$$

Thus we let $y = y_0 + \sqrt{\epsilon}y_1 + \epsilon y_2 + \epsilon^{3/2}y_3 + ...$ which gives

$$(y_0 + \sqrt{\epsilon}y_1 + \epsilon y_2 + \epsilon^{3/2}y_3 + ...)^3 + y_0 + \sqrt{\epsilon}y_1 + \epsilon y_2 + \epsilon^{3/2}y_3 + ... - \sqrt{\epsilon} = 0.$$

Expanding, reordering, and keeping all terms up to $\epsilon^{3/2}$ results in

$$y_0^3+y_0+\sqrt{\epsilon}(3y_0^2y_1+y_1-1)+\epsilon(3y_0y_1^2+3y_0^2y_2+y_2)+\epsilon^{3/2}(3y_0^2y_3+4y_0y_1y_3+y_1^3+2y_0y_1^2+y_3)+...=0$$

Thus we can solve for all three roots. Beginning with the ϵ^0 terms, and continuing in increasing powers of ϵ we solve:
$\epsilon^0:\ y_0^3+y_0=0 \implies y_0(y_0^2+1)=0 \implies y_0=0,i,-i$
$\epsilon^{1/2}:\ 3y_0^2y_1+y_1-1=0 \implies y_1=\frac{1}{3y_0^2+1}$
$\epsilon^1:\ 3y_0y_1^2+3y_0^2y_2+y_2=0 \implies y_2=\frac{-3y_0y_1^2}{3y_0^2+1}$
$\epsilon^{3/2}:\ 3y_0^2y_3+4y_0y_1y_3+y_1^3+2y_0y_1^2+y_3=0 \implies y_3=\frac{-y_1^3-2y_0y_1^2}{3y_0^2+4y_0y_1+1}$
Thus we have three solutions for y:

$$y^{(1)}=\sqrt{\epsilon}-\epsilon^{3/2}+...$$

$$y^{(2)}=i-\frac{1}{2}\epsilon^{1/2}+\frac{3}{8}i\epsilon+\frac{4i-1}{16(i+1)}\epsilon^{3/2}$$

$$y^{(3)}=-i-\frac{1}{2}\epsilon^{1/2}-\frac{3}{8}i\epsilon+\frac{4i+1}{16(i-1)}\epsilon^{3/2}.$$

Finally, we then use the identity $y=\frac{x}{\sqrt{\epsilon}}$ to get the solutions to the original equation:

$$x^{(1)}=1-\epsilon+3\epsilon^2+...\text{ (from the unscaled solution attempt)}$$

$$x^{(2)}=i\epsilon^{1/2}-\frac{1}{2}\epsilon+\frac{3}{8}i\epsilon^{3/2}+\frac{4i-1}{16(i+1)}\epsilon^2+...$$

$$x^{(3)}=-i\epsilon^{1/2}-\frac{1}{2}\epsilon-\frac{3}{8}i\epsilon^{3/2}+\frac{4i+1}{16(i-1)}\epsilon^2+...$$

(b) We begin by rewriting the equation using the trigonometric identity $\sin(a+b)=\sin(a)\cos(b)+\cos(a)\sin(b)$:

$$\sqrt{2}\sin(x+\frac{\pi}{4})-1-x+\frac{x^2}{2}+\frac{1}{6}\epsilon=0$$

becomes

$$\sin(x)+\cos(x)-1-x+\frac{x^2}{2}+\frac{1}{6}\epsilon=0.$$

We also expand the $\sin(x)$ and $\cos(x)$ terms to get the equation

$$x-\frac{x^3}{3!}+...+1-\frac{x^2}{2!}+...-1-x+\frac{x^2}{2}+\frac{1}{6}\epsilon=0.$$

This simplifies to $x=\epsilon^{1/3}$. Thus we expand x in powers of $\epsilon^{1/3}$, $x=x_0+x_1\epsilon^{1/3}+x_2\epsilon^{2/3}+...$ However, since we are looking for the solution near $x=0$ we can let $x_0=0$. Thus we expand $x=x_1\epsilon^{1/3}+x_2\epsilon^{2/3}+x_3\epsilon+...$ and begin with

$$x-\frac{x^3}{3!}+\frac{x^5}{5!}+...+1-\frac{x^2}{2!}+\frac{x^4}{4!}+...-1-x+\frac{x^2}{2}+\frac{1}{6}\epsilon=0.$$

After cancellation the equation becomes

$$-\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\frac{1}{6}\epsilon+...=0$$

Then substituting the expanded x gives

$$-\frac{(x_1\epsilon^{1/3}+x_2\epsilon^{2/3}+x_3\epsilon+...)^3}{6}+\frac{(x_1\epsilon^{1/3}+x_2\epsilon^{2/3}+x_3\epsilon+...)^4}{24}+\frac{(x_1\epsilon^{1/3}+x_2\epsilon^{2/3}+x_3\epsilon+...)^5}{5!}+\frac{1}{6}\epsilon=0.$$

After some simplification and rearrangement we have

$$-\frac{1}{6}x_1\epsilon+\frac{1}{6}\epsilon+(\frac{1}{24}x_1^4-\frac{1}{2}x_1^2x_2\epsilon^{4/3})\epsilon^{4/3}+(\frac{1}{120}x_1^5+\frac{1}{6}x_1^3x_2-\frac{1}{2}x_1x_2^2-\frac{1}{2}x_1^2x_3)\epsilon^{5/3}+...=0.$$

We then break the equation down into powers of ϵ:
ϵ^1 : $\frac{1}{6}x_1=\frac{1}{6} \implies x_1=1$
$\epsilon^{4/3}$ $-\frac{1}{2}x_1^2x_2+\frac{1}{12}x+1^2=0 \implies x_2=\frac{1}{12}$
$\epsilon^{5/3}$ $-\frac{1}{2}x_1^2x_3-\frac{1}{2}x_1x_2^2+\frac{1}{6}x_1^3x_2+\frac{1}{120}x_1^5=0 \implies x_3=\frac{197}{5445}$
Therefore we have the perturbation expansion of the solution near zero

$$x(\epsilon)=\epsilon^{1/3}+\frac{1}{12}\epsilon^{2/3}+\frac{197}{5445}\epsilon+...$$

2. Find a 2-term expansion, for ϵ small, of the solution of the initial-value problem

$$y'=2x+\epsilon y^2, \quad y=0 \text{ at } x=0.$$

Check whether your expansion is uniformly valid in the interval (i) $0\le x\le 1$, and (ii) $x\ge 0$.

Solution:

We begin by letting $y=y_0+\epsilon y_1+...$ Thus the differential equation becomes

$$y_0'+\epsilon y_1'+...=2x+\epsilon(y_0+\epsilon y_1+...)^2$$

with initial conditions

$$y_0(0)+\epsilon y_1(0)+...=0.$$

By expanding and reordering the equation we get

$$y_0'-2x+\epsilon y_1'+...=\epsilon y_0^2+...$$

We then split the equation up into equations of powers of ϵ:
ϵ^0 : $y_0'=2x \implies y=x^2+C,\ \ y_0(0)=0 \implies y_0=x^2$
ϵ^1 : $y_1'=y_0^2=x^4 \implies y_1=\frac{1}{5}x^5+D,\ \ y_1(0)=0 \implies y_1=\frac{1}{5}x^5$.
Thus a 2-term expansion for ϵ small is $y=x^2+\epsilon\frac{x^5}{5}+...$

We now check the uniformity of the expansion.

(a) $0\le x\le 1$
For $x=O(1)$, clearly, $y=O(1)$. However, for x near 0 some non-uniformity is introduced. It is easy to see that if $x=O(\epsilon)$, then $y=O(\epsilon^2)$.

(b) $x\ge 0$
For x near 0, and $x=O(1)$ the results above hold. However, if x gets large, say $x=\frac{\xi}{\epsilon}$ then we see that $y=o(\epsilon^{-4})$.

3. Expand each of the functions below in a power series in ϵ, upto and including the $O(\epsilon^3)$ term. The result of each part will be useful in the subsequent parts.

(a)
$$\frac{\epsilon}{\sqrt{4-\epsilon^2}},$$

(b)
$$\sin\left(\frac{\epsilon}{\sqrt{4-\epsilon^2}}\right),$$

(c)

$$\ln\left[2+\sin\left(\frac{\epsilon}{\sqrt{4-\epsilon^2}}\right)\right].$$

Solution:

(a) $\frac{\epsilon}{\sqrt{4-\epsilon^2}}$

$$f(\epsilon) = \frac{\epsilon}{\sqrt{4-\epsilon^2}} \implies f(0) = 0$$
$$f'(\epsilon) = \frac{4}{(4-\epsilon^2)^{3/2}} \implies f'(0) = \frac{1}{2}$$
$$f''(\epsilon) = \frac{12\epsilon}{(4-\epsilon^2)^{5/2}} \implies f''(0) = 0$$
$$f'''(\epsilon) = \frac{48(\epsilon^2+1)}{(4-\epsilon^2)^{7/2}} \implies f'''(0) = \frac{3}{8}$$

Thus we get an expansion

$$\frac{\epsilon}{\sqrt{4-\epsilon^2}} = \frac{1}{2}\epsilon + \frac{1}{16}\epsilon^3 + ...$$

(b) $\sin\left(\frac{\epsilon}{\sqrt{4-\epsilon^2}}\right)$

Here we use the fact that $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + ...$

$$\sin\left(\frac{\epsilon}{\sqrt{4-\epsilon^2}}\right) = (\frac{1}{2}\epsilon + \frac{1}{16}\epsilon^3 + ...) - \frac{\frac{1}{8}\epsilon^3 + ...}{6} + ... = \frac{1}{2}\epsilon + \frac{1}{24}\epsilon^3 + ...$$

(c) $\ln\left[2+\sin\left(\frac{\epsilon}{\sqrt{4-\epsilon^2}}\right)\right]$

We begin by noting that $\ln(2+x) = \ln(2(1+\frac{x}{2}) = \ln(2) + \ln(1+\frac{x}{2})$. We then expand $\ln(1+\frac{x}{2})$ to get

$$\ln(1+\frac{x}{2}) = \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{24} + ...$$

Thus combining these results and setting $x = \frac{1}{2}\epsilon + \frac{1}{24}\epsilon^3 + ...$ we get the solution

$$\ln\left[2+\sin\left(\frac{\epsilon}{\sqrt{4-\epsilon^2}}\right)\right] = \ln(2) = \frac{1}{4}\epsilon - \frac{1}{32}\epsilon^2 + \frac{5}{192}\epsilon^3 + ...$$

4. Consider the following sequence.

$$\phi_1(\epsilon) = \ln(1+2\epsilon^2),\ \phi_2(\epsilon) = \arcsin(\epsilon),\ \phi_3(\epsilon) = \frac{\sqrt{1+\epsilon}}{\sin\epsilon},\ \phi_4(\epsilon) = \epsilon\ln[\sinh(1/\epsilon)],\ \phi_5(\epsilon) = \frac{1}{1-\cos\epsilon}.$$

Arrange the terms of the sequence so that each term is of higher order than (*i.e.*, is little 'oh' compared to) the one preceding it, as $\epsilon \to 0+$. One strategy is to first find the order of each term in powers of ϵ.

Solution:

The solution is $O(\phi_5) < O(\phi_3) < O(\phi_4) < O(\phi_2) < O(\phi_1)$. We begin by finding $\lim_{\epsilon\to 0}\phi_i$.

$$\lim_{\epsilon\to 0}\phi_1 = 0 \quad \lim_{\epsilon\to 0}\phi_2 = 0 \quad \lim_{\epsilon\to 0}\phi_3 = \infty$$
$$\lim_{\epsilon\to 0}\phi_4 = 1 \quad \lim_{\epsilon\to 0}\phi_5 = \infty$$

Thus immediately we see that

$$\phi_4 = o(\phi_3) \text{ and } \phi_4 = o(\phi_5)$$
$$\phi_1 = o(\phi_4) \text{ and } \phi_2 = o(\phi_4).$$

Then we already have the ordering

$$O(\phi_5), O(\phi_3) < O(\phi_4) < O(\phi_2), O(\phi_1),$$

and require only two calculations. The first:

$$\lim_{\epsilon \to 0} \frac{\phi_1}{\phi_2} = \lim_{\epsilon \to 0} \frac{\ln(1+2\epsilon^2)}{\arcsin(\epsilon)} = \lim_{\epsilon \to 0} \frac{4\epsilon}{1+2\epsilon^2} \sqrt{1-\epsilon^2} = 0$$

Therefore $\phi_1 = o(\phi_2)$.
Now we check the second relationship:

$$\lim_{\epsilon \to 0} \frac{\phi_3}{\phi_5} = \lim_{\epsilon \to 0} \frac{\frac{\sqrt{1+\epsilon}}{\sin(\epsilon)}}{1-\cos(\epsilon)} = 0$$

Hence $\phi_3 = o(\phi_5)$ and the ordering listed above holds.