Perturbation Methods
Test 1 Solutions, Spring '16

PROBLEMS

1. (a) Noting that $\tan x \to \infty$ as $x \to \pi/2-$, obtain three terms of a perturbative solution to

$$\tan x = 1/\epsilon.$$

(b) Find a 3-term asymptotic expansion of the function $x(\epsilon)$ defined by the equation

$$\epsilon e^x = 1 + \frac{\epsilon}{1+x}.$$

(a) Let $x = \pi/2 - y$, where y is expected to approach zero as $\epsilon \to 0$. Then $\tan(\pi/2 - y) = 1/\epsilon$, or, $\tan y = \epsilon$. On using the Taylor expansion of $\tan y$ we have

$$y + \frac{y^3}{3} + \cdots = \epsilon.$$

Therefore,

$$y = \epsilon - \frac{y^3}{3} + \cdots = \epsilon - \frac{\epsilon^3}{3} + \cdots .$$

Finally,

$$x = \frac{\pi}{2} - y = \frac{\pi}{2} - \epsilon + \frac{\epsilon^3}{3} + \cdots .$$

Thus we have the required 3-term expansion.

(b) To begin let us assume that x is bounded as $\epsilon \to 0$. Then we can write

$$1 + x + \epsilon - \epsilon(1+x)e^x = 0.$$

Let

$$x \sim -1 + \epsilon x_1 + \epsilon^2 x_2.$$

Then

$$\epsilon x_1 + \epsilon^2 x_2 + \epsilon - \epsilon(\epsilon x_1 + \cdots)e^{-1+\epsilon x_1 + \cdots} = 0,$$

or,

$$\epsilon x_1 + \epsilon^2 x_2 + \epsilon - \epsilon(\epsilon x_1 + \cdots)e^{-1}(1 + \epsilon x_1 + \cdots) = 0,$$

or,

$$\epsilon x_1 + \epsilon^2 x_2 + \epsilon - \epsilon e^{-1}(\epsilon x_1 + \cdots) = 0.$$

On comparing coefficients of ϵ we get

$$\begin{aligned} x_1 + 1 &= 0, \\ x_2 - e^{-1}x_1 &= 0, \end{aligned}$$

so that

$$x_1 = -1, \;\; x_2 = -e^{-1}.$$

Thus we are led to the 3-term expansion

$$x \sim -1 - \epsilon - \epsilon^2 e^{-1}.$$

Now let us assume that x is unbounded as $\epsilon \to 0$. Then we can write

$$e^x = \frac{1}{\epsilon}\left(1 + \frac{\epsilon}{1+x}\right),$$

or,

$$\begin{aligned} x &= \ln(1/\epsilon) + \ln\left(1 + \frac{\epsilon}{1+x}\right) \\ &= \ln(1/\epsilon) + \frac{\epsilon}{1+x} - \frac{1}{2}\left(\frac{\epsilon}{1+x}\right)^2 + \cdots. \end{aligned}$$

We note that $x = \ln(1/\epsilon) + y$, where $y = o(\epsilon)$. Therefore we have

$$\begin{aligned} y &= \frac{\epsilon}{\ln(1/\epsilon)+1+y} - \frac{1}{2}\left(\frac{\epsilon}{\ln(1/\epsilon)+1+y}\right)^2 + \cdots \\ &= \frac{\epsilon}{\ln(1/\epsilon)+1}\left(1 + \frac{y}{\ln(1/\epsilon)+1}\right)^{-1} - \frac{1}{2}\left(\frac{\epsilon}{\ln(1/\epsilon)+1}\right)^2\left(1 + \frac{y}{\ln(1/\epsilon)+1}\right)^{-2} + \cdots \\ &= \frac{\epsilon}{\ln(1/\epsilon)+1}\left(1 - \frac{y}{\ln(1/\epsilon)+1} + \cdots\right) - \frac{1}{2}\left(\frac{\epsilon}{\ln(1/\epsilon)+1}\right)^2\left(1 - 2\frac{y}{\ln(1/\epsilon)+1} + \cdots\right) + \cdots. \end{aligned}$$

We see that to leading order,

$$y \sim \frac{\epsilon}{\ln(1/\epsilon)+1} = o(\epsilon).$$

Therefore

$$\frac{y}{\ln(1/\epsilon)+1} = o\left(\frac{\epsilon}{\ln(1/\epsilon)+1}\right),$$

which allows us to order the above expansion for y as

$$y \sim \frac{\epsilon}{\ln(1/\epsilon)+1} - \frac{1}{2}\left(\frac{\epsilon}{\ln(1/\epsilon)+1}\right)^2.$$

This leads us to the result

$$x \sim \ln(1/\epsilon) + \frac{\epsilon}{\ln(1/\epsilon)+1} - \frac{1}{2}\left(\frac{\epsilon}{\ln(1/\epsilon)+1}\right)^2.$$

NOTE: A different choice of the gauge sequence would lead to the expansion

$$x \sim \ln(1/\epsilon) + \frac{\epsilon}{\ln(1/\epsilon)} - \frac{\epsilon}{[\ln(1/\epsilon)]^2}.$$

This is a perfectly valid expansion but not as accurate as the one found above.

2. Find a 1-term composite expansion to the solution of the initial-value problem

$$\epsilon\frac{d^2u}{dt^2} + 2t\frac{du}{dt} = t^3, \quad u(0) = 0, \quad u'(0) = \frac{1}{\epsilon}.$$

Hint: Construct a provisional outer solution, then the inner solution, and based upon the behavior of the inner solution, a revised outer solution.

Outer solution. We assume the expansion $u(t;\epsilon) \sim u_0(t)$. Then, u_0 satisfies the reduced equation

$$2t\frac{du_0}{dt} = t^3,$$

with solution

$$u_0(t) = c_0 + \frac{t^3}{6}.$$

We apply neither initial condition at this stage, but note that not both will be satisfied in general as there is only one constant to be determined, c_0.

Inner solution. Dominant balance suggests an inner region of thickness $\sqrt{\epsilon}$ at $t = 0$. We set $t = \sqrt{\epsilon}\sigma$ and $u(t;\epsilon) = v(\sigma;\epsilon)$, and note that

$$\frac{du}{dt} = \frac{1}{\sqrt{\epsilon}}\frac{dv}{d\sigma}.$$

Therefore the inner problem becomes

$$\frac{d^2v}{d\sigma^2} + 2\sigma\frac{dv}{d\sigma} = \epsilon^{3/2}\sigma^3, \quad v(0) = 0,\ v'(0) = \sqrt{\epsilon}u'(0) = \frac{1}{\sqrt{\epsilon}}.$$

The second initial condition suggests the expansion

$$v \sim \frac{1}{\sqrt{\epsilon}}v_0.$$

The leading-order term v_0 satisfies the reduced problem

$$\frac{d^2v_0}{d\sigma^2} + 2\sigma\frac{dv_0}{d\sigma} = 0, \quad v_0(0) = 0,\ v_0'(0) = 1.$$

The solution is easily seen to be

$$v_0(\sigma) = \frac{\sqrt{\pi}}{2}\operatorname{erf}\sigma.$$

Then

$$v \sim \frac{1}{\sqrt{\epsilon}}\frac{\sqrt{\pi}}{2}\operatorname{erf}\sigma \to \frac{1}{\sqrt{\epsilon}}\frac{\sqrt{\pi}}{2} \text{ as } \sigma \to \infty.$$

This behavior dictates that for matching to be successful, the outer solution must be $O(1/\sqrt{\epsilon})$ to leading order. Therefore we discard the outer solution found above and instead, seek the expansion

$$u(t;\epsilon) \sim \frac{1}{\sqrt{\epsilon}}u_0(t).$$

Then u_0 satisfies

$$\frac{du_0}{dt} = 0,$$

with solution $u_0(t) = b_0$, a constant. Therefore

$$u \sim \frac{1}{\sqrt{\epsilon}}b_0.$$

Matching determines

$$b_0 = \frac{\sqrt{\pi}}{2}.$$

The leading-order composite solution is the inner solution itself, *i.e.*,

$$u_c \sim \frac{1}{\sqrt{\epsilon}}\frac{\sqrt{\pi}}{2}\operatorname{erf}\sigma.$$

3. Consider the problem

$$(1+\epsilon)x^2y' = \epsilon[(1-\epsilon)xy^2 - (1+\epsilon)x + y^3 + 2\epsilon y^2], \quad y(1) = 1, \quad 0 \le x \le 1.$$

(a) Compute three terms of an outer expansion.

(b) Deduce the location and width of the inner region.

(c) Compute two terms of an inner expansion. (You may find it convenient to compute one term and match before computing the second term.)

(d) Form a composite expansion.

Outer Expansion. Let the outer expansion be

$$y \sim y_0 + \epsilon y_1 + \epsilon^2 y_2.$$

Then the following problems emerge at each stage.

At order unity,

$$x^2 y_0' = 0, \quad y_0(1) = 1.$$

The solution is $y_0 = 1$.

At $O(\epsilon)$,

$$x^2 y_1' = -x^2 y_0' + x y_0^2 - x + y_0^3 = 1, \quad y_1(1) = 0.$$

The solution is

$$y_1 = 1 - \frac{1}{x}.$$

At $O(\epsilon^2)$,

$$x^2 y_2' = -x^2 y_1' + 2y_0 y_1 x - x y_0^2 - x + 3y_0^2 y_1 + 2y_0^2 = 2 - \frac{3}{x}, \quad y_2(0) = 0.$$

The solution is

$$y_2 = \frac{3}{2x^2} - \frac{2}{x} + \frac{1}{2}.$$

Thus the outer expansion, satisfying $y(1) = 1$, turns out to be

$$y \sim 1 + \epsilon\left(1 - \frac{1}{x}\right) + \epsilon^2\left(\frac{3}{2x^2} - \frac{2}{x} + \frac{1}{2}\right).$$

This expansion is disordered when $x = O(\epsilon)$. Then, each of the three terms above becomes $O(1)$. This is a symptom of the singular nature of the problem.

Inner expansion. The breakdown of the outer solution at $x = 0$ calls for a layer of thickness ϵ, in which we set $x = \epsilon\xi$, $y(x) = Y(\xi)$. The inner equation is

$$(1+\epsilon)\xi^2 Y' = (\epsilon - \epsilon^2)\xi Y^2 - (\epsilon + \epsilon^2)\xi + Y^3 + 2\epsilon Y^2.$$

We seek the inner expansion as

$$Y \sim Y_0 + \epsilon Y_1.$$

Then Y_0 satisfies

$$\xi^2 Y_0' = Y_0^3,$$

yielding

$$Y_0 = \left(\frac{\xi}{2 + c_0\xi}\right)^{1/2}.$$

Leading-order match with the outer solution yields $c_0 = 1$ whence

$$Y_0 = \left(\frac{\xi}{2 + \xi}\right)^{1/2}.$$

At the next order the reduced ODE is

$$\xi^2 Y_1' = -\xi^2 Y_0' + \xi Y_0^2 - \xi + 3Y_0^2 Y_1 + 2Y_0^2,$$

or,

$$
\begin{aligned}
\xi^2 Y_1' - 3Y_0^2 Y_1 &= -\xi^2 Y_0' + \xi Y_0^2 - \xi + 2Y_0^2 \\
&= -Y_0^3 + \xi Y_0^2 - \xi + 2Y_0^2.
\end{aligned}
$$

Upon substitution for Y_0, we get

$$
\xi^2 Y_1' - 3\left(\frac{\xi}{2+\xi}\right) Y_1 = -\left(\frac{\xi}{2+\xi}\right)^{3/2},
$$

or,

$$
Y_1' - \frac{3}{\xi(2+\xi)} Y_1 = -\frac{1}{\xi^2}\left(\frac{\xi}{2+\xi}\right)^{3/2}.
$$

This first-order ODE can be written as

$$
\frac{d}{d\xi}\left[\left(\frac{2+\xi}{\xi}\right)^{3/2} Y_1\right] = -\frac{1}{\xi^2},
$$

and integrates to

$$
Y_1 = \frac{\xi^{1/2}}{(2+\xi)^{3/2}} + C_1\left(\frac{\xi}{2+\xi}\right)^{3/2}.
$$

Thus the 2-term inner expansion is

$$
Y \sim \left(\frac{\xi}{2+\xi}\right)^{1/2} + \epsilon\left[\frac{\xi^{1/2}}{(2+\xi)^{3/2}} + C_1\left(\frac{\xi}{2+\xi}\right)^{3/2}\right].
$$

Matching.

Inner expansion written in the outer variable,

$$
Y \sim \left(\frac{x}{2\epsilon+x}\right)^{1/2} + \epsilon\left[\frac{\epsilon x^{1/2}}{(2\epsilon+x)^{3/2}} + C_1\left(\frac{x}{2\epsilon+x}\right)^{3/2}\right].
$$

Expanded to ϵ^2 for x fixed and $\epsilon \to 0$,

$$
Y \sim 1 - \frac{\epsilon}{x} + \frac{3\epsilon^2}{2x^2} + \frac{\epsilon^2}{x} + \epsilon C_1 - \frac{3\epsilon^2 C_1}{x}.
$$

Outer expansion written in the inner variable,

$$
y \sim 1 + \epsilon\left(1 - \frac{1}{\epsilon\xi}\right) + \epsilon^2\left(\frac{3}{2\epsilon^2\xi^2} - \frac{2}{\epsilon\xi} + \frac{1}{2}\right).
$$

Expanded to ϵ for ξ fixed and $\epsilon \to 0$,

$$
y \sim 1 + \epsilon - \frac{1}{\xi} + \frac{3}{2\xi^2} - \frac{2\epsilon}{\xi}.
$$

Re-expressed in the outer variable,

$$
y \sim 1 + \epsilon - \frac{\epsilon}{x} + \frac{3\epsilon^2}{2x^2} - \frac{2\epsilon^2}{x}.
$$

Matching determines that $C_1 = 1$.

Composite expansion.

The composite expansion is found by adding the inner and outer expansions and subtracting out the matched part. The result is

$$
\begin{aligned}
y_c \sim\; & 1+\epsilon\left(1-\frac{1}{x}\right)+\epsilon^2\left(\frac{3}{2x^2}-\frac{2}{x}+\frac{1}{2}\right)+\left(\frac{x}{2\epsilon+x}\right)^{1/2}+\epsilon\left[\frac{\epsilon x^{1/2}}{(2\epsilon+x)^{3/2}}+\left(\frac{x}{2\epsilon+x}\right)^{3/2}\right] \\
& -\left(1+\epsilon-\frac{\epsilon}{x}+\frac{3\epsilon^2}{2x^2}-\frac{2\epsilon^2}{x}\right) \\
=\; & \left(\frac{x}{2\epsilon+x}\right)^{1/2}+\epsilon\left[\frac{\epsilon x^{1/2}}{(2\epsilon+x)^{3/2}}+\left(\frac{x}{2\epsilon+x}\right)^{3/2}\right]+\frac{1}{2}\epsilon^2.
\end{aligned}
$$