MATH-6620 PERTURBATION METHODS SPRING 2016

Homework-1

Assigned Tuesday January 26, 2016

Due Friday February 5, 2016

NOTES

1. Writing solutions in LaTeX is strongly recommended but not required.
2. Show all work. Illegible or undecipherable solutions will be **returned without grading**.
3. Figures, if any, should be neatly drawn (either by hand or by a drawing program), properly labelled and captioned.
4. Please make sure that the pages are stapled together.
5. The assignment can be submitted in the labelled box in Amos Eaton 301, at my office, or in class.

PROBLEMS

1. (a) As $\epsilon \to 0$, find a 3-term perturbation expansion for each root of $\epsilon x^3 + x - 1 = 0$.

   (b) For ϵ small, find the first three terms of the perturbation expansion of $x(\epsilon)$, the solution near zero, of

$$\sqrt{2}\sin\left(x + \frac{\pi}{4}\right) - 1 - x + \frac{x^2}{2} + \frac{1}{6}\epsilon = 0.$$

(a) We note immediately that the cubic degenerates into the linear equation $x - 1 = 0$ for $\epsilon = 0$. Therefore only the perturbation series for the root $x \approx 1$ can be found by the usual regular perturbation procedure. To find the other roots the problem will need to be rescaled. Let the first root be expanded as

$$x = 1 + \epsilon x_1 + \epsilon^2 x_2 + \cdots .$$

Substitution into the cubic yields

$$\epsilon[1 + \epsilon x_1 + \epsilon^2 x_2 + \cdots]^3 + 1 + \epsilon x_1 + \epsilon^2 x_2 + \cdots - 1 = 0,$$

or,

$$\epsilon[1 + 3\{\epsilon x_1 + \epsilon^2 x_2 + \cdots\} + 3\{\epsilon x_1 + \epsilon^2 x_2 + \cdots\}^2 + \{\epsilon x_1 + \epsilon^2 x_2 + \cdots\}^3] + \epsilon x_1 + \epsilon^2 x_2 + \cdots = 0,$$

or,

$$\epsilon[1 + 3\epsilon x_1 + 3\epsilon^2(x_2 + x_1^2) + \cdots] + \epsilon x_1 + \epsilon^2 x_2 + \cdots = 0.$$

On comparing coefficients of ϵ and ϵ^2 we get

$$\begin{aligned} 1 + x_1 &= 0, \\ 3x_1 + x_2 &= 0, \end{aligned}$$

whence $x_1 = -1$, $x_2 = 3$, and the following 3-term expansion emerges for the first root.

$$x^{(1)} = 1 - \epsilon + 3\epsilon^2 + \cdots .$$

To recover the other two roots we must reinstate the cubic term as $\epsilon \to 0$, which implies that we must let x become unbounded in the limit. Of the remaining terms in the cubic, x will be more important than 1, so that ϵx^3 and x must balance, yielding $x = O(1/\sqrt{\epsilon})$. This suggest a rescaling of the equation, and we let $x = X/\sqrt{\epsilon}$, to get the scaled cubic

$$X^3 + X - \sqrt{\epsilon} = 0.$$

We now expand the solution as

$$X = X_0 + \epsilon^{1/2} X_1 + \epsilon X_2 + \cdots .$$

Substitution into the cubic leads to

$$[X_0 + \epsilon^{1/2} X_1 + \epsilon X_2 + \cdots]^3 + [X_0 + \epsilon^{1/2} X_1 + \epsilon X_2 + \cdots] - \epsilon^{1/2} = 0,$$

or,

$$\begin{aligned}[X_0^3 + 3X_0^2\{\epsilon^{1/2}X_1 + \epsilon X_2 + \cdots\} &+ 3X_0\{\epsilon^{1/2}X_1 + \epsilon X_2 + \cdots\}^2 + \{\epsilon^{1/2}X_1 + \epsilon X_2 + \cdots\}^3] \\ &+ [X_0 + \epsilon^{1/2}X_1 + \epsilon X_2 + \cdots] - \epsilon^{1/2} = 0,\end{aligned}$$

or,

$$X_0^3 + X_0 + \epsilon^{1/2}\{3X_0^2 X_1 + X_1 - 1\} + \epsilon\{3X_0^2 X_2 + 3X_0 X_1^2 + X_2\} + \cdots = 0.$$

On comparing the coefficients of powers of ϵ we get

$$\begin{aligned} X_0^3 + X_0 &= 0, \\ 3X_0^2 X_1 + X_1 &= 1, \\ 3X_0^2 X_2 + 3X_0 X_1^2 + X_2 &= 0. \end{aligned}$$

The first equation yields $X_0 = 0$ and $X_0 = \pm i$. The first of these solutions corresponds to the root already found. For the other two the second and third equations above yield

$$X_1 = \frac{1}{3X_0^2 + 1}, \quad X_2 = -\frac{3X_0 X_1^2}{3X_0^2 + 1}.$$

Then, the required expansions are

$$\begin{aligned} X^{(2)} &= i - \epsilon^{1/2}\frac{1}{2} + \epsilon\frac{3i}{8} + \cdots , \\ X^{(3)} &= -i - \epsilon^{1/2}\frac{1}{2} - \epsilon\frac{3i}{8} + \cdots . \end{aligned}$$

Correspondingly,

$$\begin{aligned} x^{(2)} &= \epsilon^{-1/2} i - \frac{1}{2} + \epsilon^{1/2}\frac{3i}{8} + \cdots , \\ x^{(3)} &= -\epsilon^{-1/2} i - \frac{1}{2} - \epsilon^{1/2}\frac{3i}{8} + \cdots . \end{aligned}$$

(b) We are given that to leading order the solution is zero. Before seeking the expansion in x we expand the sign in a power series around $x = 0$, so that

$$\sqrt{2}\left(\sin\frac{\pi}{4} + x\cos\frac{\pi}{4} - \frac{x^2}{2}\sin\frac{\pi}{4} - \frac{x^3}{6}\cos\frac{\pi}{4} + \frac{x^4}{24}\sin\frac{\pi}{4} + \frac{x^5}{120}\cos\frac{\pi}{4} + \cdots\right) - 1 - x + \frac{x^2}{2} + \frac{1}{6}\epsilon = 0,$$

or,

$$\left(1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots\right) - 1 - x + \frac{x^2}{2} + \frac{1}{6}\epsilon = 0,$$

which simplifies to

$$-x^3 + \frac{x^4}{4} + \frac{x^5}{20} + \cdots + \epsilon = 0.$$

As $\epsilon \to 0$ the dominant balance is between x^3 and ϵ, suggesting that $x = O(\epsilon^{1/3})$ to leading order. Thus we seek the expansion

$$x = \epsilon^{1/3} x_1 + \epsilon^{2/3} x_2 + \epsilon x_3 + \cdots .$$

Then the equation for x yields

$$-[\epsilon^{1/3}x_1+\epsilon^{2/3}x_2+\epsilon x_3+\cdots]^3+\frac{1}{4}[\epsilon^{1/3}x_1+\epsilon^{2/3}x_2+\epsilon x_3+\cdots]^4+\frac{1}{20}[\epsilon^{1/3}x_1+\epsilon^{2/3}x_2+\epsilon x_3+\cdots]^5+\cdots+\epsilon = 0,$$

or,

$$\begin{aligned} -\left[\epsilon x_1^3 + 3\epsilon^{4/3}x_1^2x_2 + 3\epsilon^{5/3}(x_1^2x_3 + x_1x_2^2) + \cdots\right] &+ \frac{1}{4}\left[\epsilon^{4/3}x_1^4 + 4\epsilon^{5/3}x_1^3x_2 + \cdots\right] \\ &+ \frac{1}{20}\left[\epsilon^{5/3}x_1^5 + \cdots\right] + \cdots + \epsilon = 0. \end{aligned}$$

A gathering of terms leads to

$$\epsilon\left[-x_1^3+1\right] + \epsilon^{4/3}\left[-3x_1^2x_2 + \frac{1}{4}x_1^4\right] + \epsilon^{5/3}\left[-3x_1^2x_3 - 3x_1x_2^2 + x_1^3x_2 + \frac{1}{20}x_1^5\right] + \cdots = 0.$$

On comparing coefficients of powers of ϵ we get

$$\begin{aligned} x_1^3 &= 1, \\ -3x_1^2x_2 + \frac{1}{4}x_1^4 &= 0, \quad \text{or} \quad x_2 = \frac{x_1^2}{12}, \\ -3x_1^2x_3 - 3x_1x_2^2 + x_1^3x_2 + \frac{1}{20}x_1^5 &= 0, \quad \text{or} \quad x_3 = -\frac{x_2^2}{x_1} + \frac{x_1x_2}{3} + \frac{x_1^3}{60}. \end{aligned}$$

The real solution is

$$x_1 = 1,\ x_2 = \frac{1}{12},\ \text{and}\ x_3 = \frac{3}{80}.$$

Then,

$$x = \epsilon^{1/3} + \frac{1}{12}\epsilon^{2/3} + \frac{3}{80}\epsilon + \cdots .$$

2. Find a 2-term expansion, for ϵ small, of the solution of the initial-value problem

$$y' = 2x + \epsilon y^2, \quad y = 0 \text{ at } x = 0.$$

Check whether your expansion is uniformly valid in the interval (i) $0 \le x \le 1$, and (ii) $x \ge 0$.

Let

$$y(x;\epsilon) \sim y_0(x) + \epsilon y_1(x).$$

Then the ODE and the initial condition become

$$y_0' + \epsilon y_1' + \cdots = 2x + \epsilon[y_0^2 + \epsilon 2y_0y_1 + \cdots], \quad y_0(0) + \epsilon y_1(0) + \cdots = 0.$$

The following hierarchy of problems emerges.

At $O(1)$ the reduced problem is

$$y_0' = 2x, \quad y_0(0) = 0, \quad \text{yielding} \quad y_0 = x^2.$$

At $O(\epsilon)$, we have

$$y_1' = y_0^2 = x^4, \quad y_1(0) = 0, \quad \text{with solution} \quad y_1 = \frac{x^5}{5}.$$

Thus we obtain the asymptotic expansion

$$y \sim x^2 + \epsilon \frac{x^5}{5}.$$

In the interval $0 \le x \le 1$ the coefficients of the expansion are bounded. Therefore the second term is always $O(\epsilon)$ relative to the first, so that the expansion is uniform.

In the interval $x \ge 0$ the coefficients are unbounded. The ratio of the second term to the first is of order ϵx^3, and for $\epsilon \to 0$ the second term ceases to be of higher order than the first when $x = O(\epsilon^{-1/3})$. Thus the expansion loses validity for x large enough.

3. Expand each of the functions below in a power series in ϵ, upto and including the $O(\epsilon^3)$ term. The result of each part will be useful in the subsequent parts.

(a)
$$\frac{\epsilon}{\sqrt{4-\epsilon^2}},$$

(b)
$$\sin\left(\frac{\epsilon}{\sqrt{4-\epsilon^2}}\right),$$

(c)
$$\ln\left[2 + \sin\left(\frac{\epsilon}{\sqrt{4-\epsilon^2}}\right)\right].$$

(a) We use the binomial expansion to write

$$\begin{aligned} f(\epsilon) \equiv \frac{\epsilon}{\sqrt{4-\epsilon^2}} &= \frac{1}{2}\epsilon\left(1 - \frac{\epsilon^2}{4}\right)^{-1/2} \\ &= \frac{1}{2}\epsilon\left(1 + \frac{1}{8}\epsilon^2 + \cdots\right) \\ &= \frac{1}{2}\epsilon + \frac{1}{16}\epsilon^3 + \cdots. \end{aligned}$$

(b) We note that $f(\epsilon)$, the argument of the sine, $\to 0$ as $\epsilon \to 0$. Therefore we write down a Taylor expansion of the sine about $f = 0$ to get

$$\sin f = f - \frac{f^3}{3!} + \cdots.$$

Now we substitute for f from part (a), leading to

$$\begin{aligned} g(\epsilon) \equiv \sin\left(\frac{\epsilon}{\sqrt{4-\epsilon^2}}\right) + \cdots &= \left(\frac{1}{2}\epsilon + \frac{1}{16}\epsilon^3 + \cdots\right) - \frac{1}{6}\left(\frac{1}{2}\epsilon + \frac{1}{16}\epsilon^3 + \cdots\right)^3 + \cdots \\ &= \left(\frac{1}{2}\epsilon + \frac{1}{16}\epsilon^3 + \cdots\right) - \frac{1}{6}\left(\frac{1}{8}\epsilon^3 + \cdots\right) + \cdots \\ &= \frac{1}{2}\epsilon + \frac{1}{24}\epsilon^3 + \cdots. \end{aligned}$$

(c) Now we have $h(\epsilon) \equiv \ln[2 + g(\epsilon)]$. According to the part (b), $g(\epsilon) \to 0$ as $\epsilon \to 0$. Therefore we expand $h(\epsilon)$ by using the following known Taylor expansion of the logarithm,

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots.$$

We have

$$\begin{aligned} h(\epsilon) &= \ln[2+g(\epsilon)] \\ &= \ln 2 + \ln\left(1+\frac{1}{2}g\right) \\ &= \ln 2 + \frac{g}{2} - \frac{g^2}{8} + \frac{g^3}{24} + \cdots \end{aligned}$$

Upon substituting the expansion of g from part (b) we obtain

$$\begin{aligned} h(\epsilon) &= \ln 2 + \frac{1}{2}\left(\frac{1}{2}\epsilon + \frac{1}{24}\epsilon^3 + \cdots\right) - \frac{1}{8}\left(\frac{1}{2}\epsilon + \frac{1}{24}\epsilon^3 + \cdots\right)^2 + \frac{1}{24}\left(\frac{1}{2}\epsilon + \frac{1}{24}\epsilon^3 + \cdots\right)^3 + \cdots \\ &= \ln 2 + \frac{1}{2}\left(\frac{1}{2}\epsilon + \frac{1}{24}\epsilon^3 + \cdots\right) - \frac{1}{8}\left(\frac{1}{4}\epsilon^2 + \frac{1}{24}\epsilon^4 + \cdots\right) + \frac{1}{24}\left(\frac{1}{8}\epsilon^3 + \cdots\right) + \cdots \\ &= \ln 2 + \frac{1}{4}\epsilon - \frac{1}{32}\epsilon^2 + \frac{5}{192}\epsilon^3 + \cdots . \end{aligned}$$

4. Consider the following sequence.

$$\phi_1(\epsilon) = \ln(1+2\epsilon^2),\ \phi_2(\epsilon) = \arcsin(\epsilon),\ \phi_3(\epsilon) = \frac{\sqrt{1+\epsilon}}{\sin\epsilon},\ \phi_4(\epsilon) = \epsilon\ln[\sinh(1/\epsilon)],\ \phi_5(\epsilon) = \frac{1}{1-\cos\epsilon}.$$

Arrange the terms of the sequence so that each term is of higher order than (*i.e.*, is little 'oh' compared to) the one preceding it, as $\epsilon \to 0+$. One strategy is to first find the order of each term in powers of ϵ.

The order of a term can be established relatively easily by computing the leading term in its Taylor series, should such a series exist. Thus we have

$$\begin{aligned} \phi_1(\epsilon) = \ln(1+2\epsilon^2) &= 2\epsilon^2 + \cdots, \\ \phi_2(\epsilon) = \arcsin(\epsilon) &= \epsilon + \cdots, \\ \phi_3(\epsilon) = \frac{\sqrt{1+\epsilon}}{\sin\epsilon} &= \frac{1}{\epsilon} + \cdots, \\ 1-\cos\epsilon &= 1 - \left(1 - \frac{1}{2}\epsilon^2 + \cdots\right) = \frac{1}{2}\epsilon^2 + \cdots, \quad \text{whereby} \\ \phi_5(\epsilon) = \frac{1}{1-\cos\epsilon} &= \frac{2}{\epsilon^2} + \cdots . \end{aligned}$$

We can now use the definition of Big 'Oh' to find the orders. This entails computing the relevant limits as $\epsilon \to 0+$. We have

$$\begin{aligned} \lim \frac{\phi_1}{\epsilon^2} &= \lim \frac{\ln(1+2\epsilon^2)}{\epsilon^2} = 2, \\ \lim \frac{\phi_2}{\epsilon} &= \lim \frac{\arcsin\epsilon}{\epsilon} = 1, \\ \lim \frac{\phi_3}{1/\epsilon} &= \lim \frac{\sqrt{1+\epsilon}/\sin\epsilon}{1/\epsilon} = 1, \\ \lim \frac{\phi_5}{1/\epsilon^2} &= \lim \frac{1/(1-\cos\epsilon)}{1/\epsilon^2} = 2. \end{aligned}$$

Therefore $\phi_1 = O(\epsilon^2)$, $\phi_2 = O(\epsilon)$, $\phi_3 = O(1/\epsilon)$ and $\phi_5 = O(1/\epsilon^2)$. It remains to consider

$$\begin{aligned} \phi_4(\epsilon) &= \epsilon\ln[\sinh(1/\epsilon)] = \epsilon\ln\frac{e^{1/\epsilon} - e^{-1/\epsilon}}{2} = \epsilon\ln(1/2) + \epsilon\ln[e^{1/\epsilon}[1-e^{-2/\epsilon}] \\ &= 1 + \ln[1-e^{-2/\epsilon}] + \epsilon\ln(1/2) \to 1 \quad \text{as} \quad \epsilon \to 0. \end{aligned}$$

Therefore $\phi_4 = O(1)$. Thus the correct arrangement is

$$\phi_5,\ \phi_3,\ \phi_4,\ \phi_2,\ \phi_1.$$