PERTURBATION METHODS
SPRING 2016

LESSON 4: The WKBJ Method

The WKBJ method is examined via examples. This lesson supplements Chapter 4 of the Text.

1 Introduction

We consider differential equations of the form

$$w''(x) + f(x;\lambda)\, w(x) = 0. \tag{1.1}$$

Here λ is a large parameter, and the asymptotic expansion of w in the limit $\lambda \to \infty$ is of interest. The above equation for $f(x;\lambda) = \lambda^2 f_0(x)$ was first examined by Liouville (1835). The more general case $f(x;\lambda) = \lambda^2 F_0(x) + \lambda F_1(x) + F_2(x)$ was studied later (1920's) by similar methods by Wentzel, Kramers, Brillouin and Jeffreys. Equations of type (1.1) arise in a number of contexts, including wave propagation, diffusion, quantum mechanics, transmission-line theory, and elsewhere. We consider two physical examples.

Example 1.1. Let us consider the 1-D wave equation

$$v_{tt} = \alpha^2 v_{xx}, \tag{1.2}$$

and examine solutions periodic in t, *i.e.*,

$$v(x,t) = w(x)e^{i\omega t}.$$

Then w satisfies

$$w''(x) + \frac{\omega^2}{\alpha^2} w(x) = 0.$$

This equation is of the same form as (1.1) if $\alpha = \alpha(x)$, and one is concerned with high-frequency waves ($\omega \to \infty$). Note that for α constant the solution is

$$w(x) = ae^{i\omega x/\alpha} + be^{-i\omega x/\alpha}.$$

We are interested in an asymptotic solution when $\alpha = \alpha(x)$.

Example 1.2. Consider shear waves in a stratified medium. Assume that v, the displacement in the y-direction, is independent of y, so that $v = v(x,z,t)$. The governing equation is

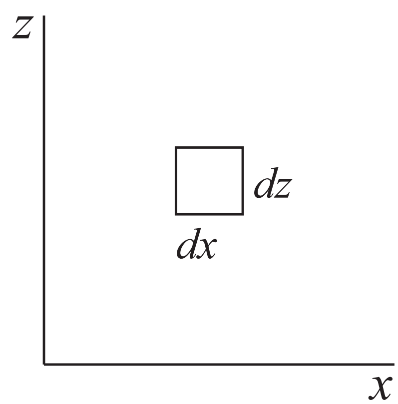

Figure 1: Configuration for shear waves

$$\rho v_{tt} = (\mu v_x)_x + (\mu v_z)_z, \tag{1.3}$$

where ρ is the density, assumed constant, and $\mu = \mu(z)$ is the shear modulus for the medium stratified in the z- direction. We assume $\mu'(z) > 0$, *i.e.*, the medium gets stiffer with increasing z.

We assume a plane wave,

$$v(x,z,t) = V(z)e^{ik(ct-x)},$$

where k and c are given. Then, substitution into (1.3) yields the equation

$$\mu V'' + \mu' V' + k^2 \rho (c^2 - \alpha^2) V = 0. \tag{1.4}$$

Here,

$$\alpha^2 = \frac{\mu}{\rho},$$

α being the local wave speed. We let

$$V(z) = w(z) \exp\left(-\frac{1}{2}\int \frac{\mu'}{\mu}\, dz\right) = \mu^{-1/2} w(z).$$

Then w satisfies

$$w''(z) + \left[k^2\left(\frac{c^2}{\alpha^2} - 1\right) + \frac{\mu'^2}{4\mu} - \frac{\mu''}{2\mu}\right] w = 0.$$

For high frequency (or short wave length, $k \gg 1$), this equation is the same as

$$w''(x) = [\lambda^2 F_0(x) + \lambda F_1(x) + F_2] w(x), \quad \lambda \gg 1, \tag{1.5}$$

if we make the following identifications:

$$\lambda = k,\ x = z,\ F_0(x) = 1 - c^2/\alpha^2,\ F_1(x) = 0,\ F_2(x) = \mu''/2\mu - \mu'^2/(4\mu).$$

Two items are of interest.

- First, an asymptotic approximation of w when k is large and $1 - c^2/\alpha^2$ retains its sign. When $\alpha < c$ the solution is expected to be oscillatory and and for $\alpha > c$ it is expected to be exponential.
- Second, passage of $1 - c^2/\alpha^2$ through zero changes the type of the solution. Physically, only the exponentially decaying solution is allowed for $\alpha > c$. This should join smoothly with a certain linear combination of the two oscillatory solutions for $\alpha < c$. Otherwise stated, for each incident wave there is a unique reflected wave for which the transmitted signal is exponentially decaying (total internal reflection). The point in the strata where $\alpha = c$ is called the *turning point*.

2 The WKBJ procedure

We now focus on equation (1.5) as we go through the details of the expansion procedure. We assume that the turning point is at $x = 0$, so that $F_0(0) = 0$. Away from $x = 0$ let

$$w(x) = e^{\lambda\omega(x)} \phi(x) f(x;\lambda), \tag{2.1}$$

where

$$f(x;\lambda) = 1 + \frac{1}{\lambda} f_1(x) + \frac{1}{\lambda^2} f_2(x) + \cdots. \tag{2.2}$$

Then,

$$\begin{aligned} w' &= e^{\lambda\omega}[\lambda\omega'\phi f + (\phi f)'], \\ w'' &= e^{\lambda\omega}[\lambda^2\omega'^2\phi f + 2\lambda\omega'(\phi f)' + \lambda\omega''\phi f + (\phi f)''] \end{aligned} \tag{2.3}$$

In view of (2.3), (1.5) becomes

$$\lambda^2(\omega'^2 - F_0)\phi f + \lambda[2\omega'(\phi f)' + (\omega'' - F_1)\phi f] + (\phi f)'' - F_2 \phi f = 0.$$

It is convenient to rewrite the above equation as

$$\lambda^2(\omega'^2 - F_0)\phi f + \lambda[\{2\omega'\phi' + (\omega'' - F_1)\phi\}f + 2\omega'\phi f'] + (\phi'' - F_2\phi)f + 2\phi' f' + \phi f'' = 0. \tag{2.4}$$

We now compare coefficients of powers of λ in (2.4), keeping in mind that f has its own expansion, given by (2.2).

$O(\lambda^2)$:

$$\phi f(\omega'^2 - F_0) = 0, \quad \text{leading to} \quad \omega'^2 - F_0 = 0.$$

This nonlinear equation, called the *eikonal equation*, governs the fast variations of the system. It yields

$$\omega' = \pm F_0^{1/2},$$

and upon integration,

$$\omega = \pm \int F_0^{1/2}\, dx. \tag{2.5}$$

$O(\lambda)$:

$$(2\omega'\phi' + (\omega'' - F_1)\phi)f = 0, \quad \text{leading to} \quad 2\omega'\phi' + (\omega'' - F_1)\phi = 0.$$

This linear equation, called the *transport equation*, governs the leading-order slow variations. It can be written as

$$\frac{\phi'}{\phi} = -\frac{\omega''}{2\omega'} + \frac{F_1}{2\omega'},$$

and then integrated to yield

$$\phi = F_0^{-1/4} \exp\left(\pm \int \frac{F_1}{2F_0^{1/2}}\, dx \right). \tag{2.6}$$

Remarks.

- Up to this order, F_2 does not play any role.
- In equation (2.6) is manifested the previously anticipated difficulty at the turning point $x = 0$ ($F_0(0) = 0$).

$O(1)$:

$$2\omega'\phi f_1' + (\phi'' - \phi F_2) = 0.$$

Continuing in this vein,

$O(\lambda^{-n})$:

$$2\omega'\phi f_{n+1}' + (\phi'' - \phi F_2)f_n + 2\phi' f_n' + \phi f_n'' = 0, \;\; n = 1, 2, \cdots.$$

The results (2.5) and (2.6), when substituted into (2.1), provide the approximation,

$$w \sim F_0^{-1/4} \exp\left[\pm \int \left(\lambda F_0^{1/2} + \frac{F_1}{2F_0^{1/2}} \right) dx \right] \left[1 + O\left(\frac{1}{\lambda} \right) \right]. \tag{2.7}$$

The full solution will be a linear combination of the two linearly independent solutions corresponding to the $\pm$ signs above.

Before discussing the behavior near a turning point, we consider some examples.

Example 2.1. Consider the ODE

$$w''(x) + \lambda^2(1+x^2)^2 w(x) = 0. \tag{2.8}$$

Here, $F_0 = -(1+x^2)^2$, while $F_1 = F_2 = 0$. Substitution into the solution (2.7) yields the result

$$\begin{aligned} w(x) &= \frac{1}{\sqrt{1+x^2}} \exp\left[\pm \int \lambda i(1+x^2)\, dx\right]\left[1 + O\left(\frac{1}{\lambda}\right)\right] \\ &= \frac{1}{\sqrt{1+x^2}} \exp\left[\pm \lambda i \left(x + \frac{x^3}{3}\right)\right]\left[1 + O\left(\frac{1}{\lambda}\right)\right]. \end{aligned}$$

Expressing the exponential in terms of trigonometric functions, the result is

$$w(x) = \frac{A}{\sqrt{1+x^2}} \cos\left\{\lambda\left(x + \frac{x^3}{3}\right)\right\}\left[1 + O\left(\frac{1}{\lambda}\right)\right] + \frac{B}{\sqrt{1+x^2}} \sin\left\{\lambda\left(x + \frac{x^3}{3}\right)\right\}\left[1 + O\left(\frac{1}{\lambda}\right)\right].$$

For an initial-value problem the constants A and B will be determined by initial conditions.
If the boundary conditions are homogeneous, then we are faced with an eigenvalue problem, and nontrivial solutions will exist only for special values of λ. For example, application of Dirichlet boundary conditions $w(0) = w(1) = 0$ leads to $A = 0$ and

$$\sin(4\lambda/3) \approx 0$$

for λ large. Thus $4\lambda/3 \approx n\pi$, or $\lambda_n \approx 3n\pi/4$, where the integer n is large.

Example 2.2. Consider the ODE

$$w''(x) = \left(\lambda^2 x^4 + \lambda x^3 + \frac{x^2}{4}\right) w(x). \tag{2.9}$$

Here, $F_0 = x^4$, $F_1 = x^3$ and $F_2 = x^2/4$. We shall consider the WKBJ approximation to $O(1/\lambda)$ away from the turning point $x = 0$. Let

$$w = e^{\lambda\omega(x)}\phi(x) f = e^{\lambda\omega(x)}\phi(x)\left[1 + \frac{f_1(x)}{\lambda} + O\left(\frac{1}{\lambda^2}\right)\right]. \tag{2.10}$$

The the eikonal equation is

$$\omega' = \pm F_0^{1/2} = \pm x^2,$$

so that

$$\omega = \pm \frac{x^3}{3}.$$

The transport equation is

$$2\omega'\phi' + (\omega'' - F_1)\phi = 0,$$

which becomes

$$\frac{\phi'}{\phi} = -\frac{\omega''}{2\omega'} + \frac{F_1}{2\omega'}$$

and integrates to

$$\begin{aligned} \ln|\phi| &= -\frac{1}{2}\ln|\omega'| \pm \int \frac{x^3}{2x^2} \\ &= -\frac{1}{2}\ln x^2 \pm \frac{x^2}{4}, \end{aligned}$$

so that

$$\phi = \frac{1}{x} e^{\pm x^2/4}.$$

At the next order, the governing equation is

$$2\omega'\phi f_1' + (\phi'' - \phi F_2) = 0.$$

Upon substituting for ϕ, ω and F_2, this equation reduces to

$$f_1' = \mp\frac{1}{x^4} + \frac{1}{4x^4}.$$

An integration yields

$$f_1 = \pm\frac{1}{3x^3} - \frac{1}{4x}.$$

Upon substituting the above expressions for ω, ϕ and f_1 into (2.10), we get

$$w = \frac{1}{x}e^{\pm\lambda x^3/3 \pm x^2/4}\left[1 + \frac{1}{\lambda}\left(\pm\frac{1}{3x^3} - \frac{1}{4x}\right) + O\left(\frac{1}{\lambda^2}\right)\right].$$

The two linearly independent solutions found above can be combines into the general solution

$$w = \frac{A}{x}e^{\lambda x^3/3 + x^2/4}\left[1 + \frac{1}{\lambda}\left(\frac{1}{3x^3} - \frac{1}{4x}\right) + O\left(\frac{1}{\lambda^2}\right)\right] + \frac{B}{x}e^{-\lambda x^3/3 - x^2/4}\left[1 - \frac{1}{\lambda}\left(\frac{1}{3x^3} + \frac{1}{4x}\right) + O\left(\frac{1}{\lambda^2}\right)\right].$$

We reiterate that this solution is valid away from $x = 0$.

2.1 Turning point - inner approximation

We return to the ODE (1.5) and the asymptotic approximation (2.7) to its solution. Both of these are reproduced below, but we shall continue referring to them by their original equation numbers.

$$w''(x) = [\lambda^2 F_0(x) + \lambda F_1(x) + F_2]w(x), \quad \lambda \gg 1, \tag{1.5}$$

$$w \sim F_0^{-1/4}\exp\left[\pm\int\left(\lambda F_0^{1/2} + \frac{F_1}{2F_0^{1/2}}\right)dx\right]\left[1 + O\left(\frac{1}{\lambda}\right)\right]. \tag{2.7}$$

Let $F_0(0) = 0$, $F_0'(0) > 0$. Then $x = 0$ is called a first-order turning point. The solution (2.7) clearly does not hold at the turning point, where $F_0(x) \sim xF_0'(0)$ and $\lambda^2 F_0(x)$ is no longer of order λ^2.

We introduce an inner coordinate ξ via the stretching $x = \lambda^{-p}\xi$, where $p > 0$ is to be determined, and let $w(x) = y(\xi)$. Then the governing equation (1.5) transforms into

$$\begin{aligned}\lambda^{2p}y''(\xi) &= [\lambda^2 F_0(\lambda^{-p}\xi) + \lambda F_1(\lambda^{-p}\xi) + F_2(\lambda^{-p}\xi)]y \\ &= [\lambda^{2-p}F_0'(0)\xi + O(\lambda, \lambda^{2-2p})]y.\end{aligned}$$

Dominant balance is achieved when $2p = 2 - p$, or $p = 2/3$. Then the above equation becomes

$$y''(\xi) = F_0'(0)\xi\, y(\xi) + O(\lambda^{-1/3}).$$

We seek the inner expansion

$$y \sim \mu_0(\lambda)y_0(\xi), \tag{2.11}$$

where the gauge function $\mu_0(\lambda)$ will be determined by matching. Then y_0 satisfies

$$y_0''(\xi) = F_0'(0)\xi\, y_0(\xi).$$

The transformations

$$q = \{F_0'(0)\}^{1/3}, \tag{2.12}$$

and

$$q\xi = \eta, \; y_0(\xi) = z(\eta),$$

reduce the above equation to

$$z''(\eta) = \eta\, z(\eta). \tag{2.13}$$

This is the *Airy equation*, and its solutions are the Airy functions $\mathrm{Ai}(\eta)$ and $\mathrm{Bi}(\eta)$. (The Airy equation is a special case of the Bessel equation, and the Airy functions can be expressed in terms of Bessel functions of order $1/3$.) For $\eta > 0$ the Airy functions are like exponentials, with Ai exponentially decaying and Bi exponentially growing. For $\eta < 0$ they are both like damped trigonometric functions. Their asymptotic behavior is as follows.

$$\mathrm{Ai}(\eta) \sim \frac{1}{2\sqrt{\pi}}\eta^{-1/4}e^{-(2/3)\eta^{3/2}}, \quad \eta \to \infty, \tag{2.14}$$

$$\mathrm{Bi}(\eta) \sim \frac{1}{\sqrt{\pi}}\eta^{-1/4}e^{(2/3)\eta^{3/2}}, \quad \eta \to \infty, \tag{2.15}$$

$$\mathrm{Ai}(\eta) \sim \frac{1}{\sqrt{\pi}}\eta^{-1/4}\sin\left(\frac{2}{3}|\eta|^{3/2} + \frac{\pi}{4}\right), \quad \eta \to -\infty, \tag{2.16}$$

$$\mathrm{Bi}(\eta) \sim \frac{1}{\sqrt{\pi}}\eta^{-1/4}\cos\left(\frac{2}{3}|\eta|^{3/2} + \frac{\pi}{4}\right), \quad \eta \to -\infty. \tag{2.17}$$

Plots of $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ are displayed in Figure 2.

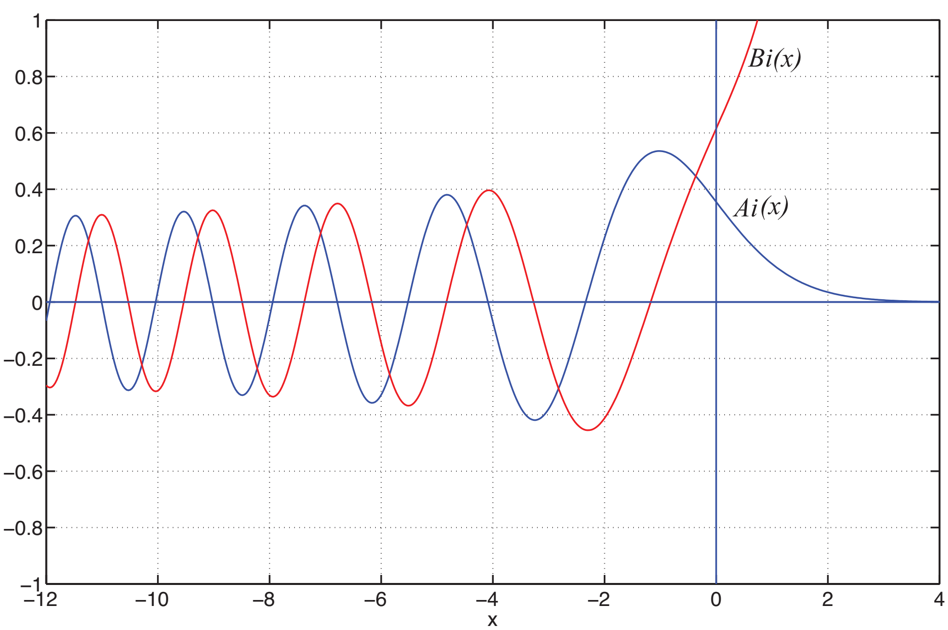

Figure 2: Plots of $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$.

To be specific, let us take $F_1 = 0$, and assume that the outer solution (2.7) is exponentially decaying. Then we must select the negative sign for the exponent, so that the outer solution, *at leading order*, is

$$w_0(x) = \{F_0(x)\}^{-1/4}\exp\left(-\lambda\int_0^x \{F_0(s)\}^{1/2}ds\right). \tag{2.18}$$

We may now match the outer solution with a certain linear combination of $\mathrm{Ai}(q\xi)$ and $\mathrm{Bi}(q\xi)$. First, some preliminaries. We start by noting that

$$\{F_0(x)\}^{-1/4} \sim \{F_0'(0)x\}^{-1/4} = q^{-3/4}x^{-1/4} \;\text{ as }\; x \to 0.$$

Next, examine the integral in (2.18) and note that for x small,

$$\begin{aligned}\int_0^x \{F_0(s)\}^{1/2} ds &\sim \int_0^x \{F_0'(0)s\}^{1/2}\, ds \\ &= \frac{2}{3}\{F_0'(0)\}^{1/2}x^{3/2} = \frac{2}{3}q^{3/2}x^{3/2}.\end{aligned}$$

These results will be used in what follows. Now the matching can proceed.

The outer solution expanded to one term in the inner variable:

$$\begin{aligned}w_0(\lambda^{-2/3}\xi) &= \{F_0(\lambda^{-2/3}\xi)\}^{-1/4} \exp\left(-\lambda \int_0^{\lambda^{-2/3}\xi} \{F_0(s)\}^{1/2} ds\right) \\ &= \lambda^{1/6} q^{-3/4} \xi^{-1/4} \exp\left(-\frac{2}{3}q^{3/2}\xi^{3/2}\right). \qquad (2.19)\end{aligned}$$

The exponentially decaying behavior of the outer solution as $\xi \to \infty$ suggests that for matching, we must select the relevant solution of the inner equation (2.13) to be the Airy function Ai.

The inner solution expanded to one term in the outer variable:

$$\begin{aligned}\mu_0(\lambda) y_0(\xi) &= \mu_0(\lambda)\mathrm{Ai}(q\xi) \\ &\sim \mu_0(\lambda)\, \frac{1}{2\sqrt{\pi}} q^{-1/4}\xi^{-1/4} \exp\left(-\frac{2}{3}q^{3/2}\xi^{3/2}\right). \qquad (2.20)\end{aligned}$$

A comparison of (2.19) and (2.20) reveals that

$$\lambda^{1/6} q^{-3/4}\xi^{-1/4} = \mu_0(\lambda)\, \frac{1}{2\sqrt{\pi}} q^{-1/4},$$

which determines μ_0 to be

$$\mu_0(\lambda) = \lambda^{1/6}\, 2\sqrt{\pi/q}.$$

Therefore the inner solution is

$$y \sim \lambda^{1/6}\, 2\sqrt{\pi/q}\ \mathrm{Ai}(q\xi). \qquad (2.21)$$

As $\xi \to -\infty$, the above solution has the behavior

$$\begin{aligned}y &\sim \lambda^{1/6}\, 2\sqrt{\pi} q^{-3/4} \frac{1}{\sqrt{\pi}} (-\xi)^{-1/4} \sin\left(\frac{2}{3}(-q\xi)^{3/2} + \frac{\pi}{4}\right), \\ &\sim \lambda^{1/6}\, 2q^{-3/4}(-\xi)^{-1/4} \sin\left(\frac{2}{3}(-q\xi)^{3/2} + \frac{\pi}{4}\right), \qquad (2.22)\end{aligned}$$

where the asymptotic expression (2.16) has been used.

For $x < 0$ the outer solution (2.7), for $F_1 = 0$, may be written as

$$(-F_0)^{-1/4} \exp\left(\pm i\lambda \int_0^x (-F_0)^{1/2}\, dx\right).$$

A linear combination leading to the outer solution expressed in a real form is

$$w \sim \nu_0(\lambda)(-F_0)^{-1/4} \sin\left(\lambda \int_0^x (-F_0)^{1/2}\, dx + \theta\right). \qquad (2.23)$$

The quantities $\nu_0(\lambda)$ and θ will be determined by matching with the inner solution (2.21) in the limit $x \to 0-$, or more precisely, $x = \lambda^{-2/3}\xi < 0$, ξ fixed, $\lambda \to \infty$. Now,

$$-F_0(x) \sim -\lambda^{-2/3}q^3\xi,$$

so that

$$(-F_0)^{-1/4} \sim \lambda^{1/6} q^{-3/4} (-\xi)^{-1/4},$$

and hence,

$$\begin{aligned}
\int_0^x (-F_0)^{1/2}\, dx &= \int_0^{\lambda^{-2/3}\xi} (-q^3 x)^{1/2}\, dx \\
&= \frac{2}{3} q^{3/2} (-x)^{3/2}\Big|_0^{\lambda^{-2/3}\xi} \\
&= \frac{2}{3} q^{3/2} \lambda^{-1} (-\xi)^{3/2}.
\end{aligned}$$

Then, (2.23) can be expressed as

$$w \sim \nu_0(\lambda) \lambda^{1/6} q^{-3/4} (-\xi)^{-1/4} \sin\left(\frac{2}{3} q^{3/2} (-\xi)^{3/2} + \theta\right). \tag{2.24}$$

This is the inner expansion of the outer solution as $x \to 0-$. A comparison with (2.22) determines

$$\nu_0(\lambda) = 2, \quad \theta = \pi/4.$$

Summary. We have now determined the outer solutions away from the turning point in regions $x > 0$ and $x < 0$, and the inner solution in the region $x \approx 0$. The outer and inner solutions have been matched. The results are collected below.

$$\begin{aligned}
x > 0: \quad w &\sim \{F_0(x)\}^{-1/4} \exp\left(-\lambda \int_0^x \{F_0(s)\}^{1/2} ds\right), \\
x < 0: \quad w &\sim 2(-F_0)^{-1/4} \sin\left(\lambda \int_0^x (-F_0)^{1/2}\, dx + \theta\right), \\
x \approx 0: \quad w &\sim \lambda^{1/6}\, 2\sqrt{\pi/q}\, \mathrm{Ai}(q\xi).
\end{aligned}$$