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Singularly Perturbed Linear Two-Point Boundary Value Problems*

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Dedicated to Bob Ackerberg

Abstract. Finding asymptotic solutions to two-point boundary value problems for linear singularly perturbed second-order ordinary differential equations is now quite well understood. Troublesome exceptions involving turning points, known as problems of boundary layer resonance, have been studied by many experts since the initial work of Ackerberg and O'Malley [*Stud. Appl. Math.*, 49 (1970), pp. 277–295]; this significant work is surveyed here. Closely related literature involves the canard phenomena. New, generally more geometric, approaches, including Gevrey asymptotics and "blowup," promise improved understanding of such sensitive asymptotics and their generalizations.

Key words. singular perturbations, boundary layer resonance, asymptotics

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1. Non-Turning-Point Problems.

How to asymptotically solve a two-point Dirichlet boundary value problem for the singularly perturbed linear ordinary differential equation (ODE)

$$\epsilon y'' + a(x)y' + b(x)y = 0 \tag{1}$$

on $0 \leq x \leq 1$, when $a(x)$ and $b(x)$ are infinitely smooth coefficients and ϵ is a small positive parameter, is a long-studied problem for prescribed values $y(0)$ and $y(1)$. The problem for a positive constant function $a(x)$ and a constant $b(x)$ was, indeed, considered in Prandtl's Goettingen lectures of 1931–1932 (cf. Schlichting and Gersten (2000)) as a simplified example of his fluid dynamical *boundary layer*. More generally, for any $a(x) > 0$, one can readily generate an asymptotic solution of the form

$$y(x, \epsilon) = Y(x, \epsilon) + \xi(x/\epsilon, \epsilon), \tag{2}$$

where the smooth *outer solution*

$$Y(x, \epsilon) \sim \sum_{j\geq 0} Y_j(x)\epsilon^j \tag{3}$$

can be developed termwise as an *asymptotic power series* in ϵ that formally satisfies (1) and the terminal condition, and where the terms of the power series for the *initial*

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(boundary) layer correction

$$\xi(X,\epsilon) \sim \sum_{j\geq 0} \xi_j(X)\epsilon^j \tag{4}$$

satisfy the appropriate complementary initial value and tend (exponentially fast) to zero as the stretched coordinate

$$X = x/\epsilon \tag{5}$$

tends to infinity (cf., e.g., O'Malley (1991)). This simplified *asymptotic matching* procedure shows that the solution of the given two-point problem converges nonuniformly from the prescribed $y(0)$ to the formally generated series $Y(0,\epsilon)$ in an $O(\epsilon)$-thick initial layer region abutting $x = 0$. Since the limiting solution for $x > 0$ is $Y_0(x)$, we naturally ask Y_0 to satisfy the *reduced problem*

$$a(x)Y_0' + b(x)Y_0 = 0, \quad Y_0(1) = y(1). \tag{6}$$

Its solution

$$Y_0(x) = e^{\int_x^1 \frac{b(s)}{a(s)}ds} y(1) \tag{7}$$

is defined throughout $0 \leq x \leq 1$. Higher-order coefficients $Y_j(x)$ in the *outer expansion* (3) will likewise follow uniquely as solutions of corresponding nonhomogeneous terminal value problems determined by equating successive coefficients termwise when the series (3) is substituted into the given differential equation. Breakdown of the asymptotic representation (2) will generally occur at any *turning point* x, where $a(x) = 0$, since the differential equation (6) for Y_0 has a singular point there.

Linearity implies that ξ must satisfy the *stretched* equation

$$\frac{d^2\xi}{dX^2} + a(\epsilon X)\frac{d\xi}{dX} + \epsilon b(\epsilon X)\xi = 0 \tag{8}$$

on $X \geq 0$, together with the initial condition

$$\xi(0,\epsilon) = y(0) - Y(0,\epsilon) \tag{9}$$

and the imposed *asymptotic stability* condition at $X = \infty$. Since $a(0) > 0$, the decaying solution of the limiting problem

$$\frac{d^2\xi_0}{dX^2} + a(0)\frac{d\xi_0}{dX} = 0,\ \xi_0(0) = y(0) - Y_0(0),$$

will be

$$\xi_0(X) = e^{-a(0)X}(y(0) - Y_0(0)). \tag{10}$$

Later terms ξ_j will then decay exponentially (like $X^j e^{-a(0)X}$) as $X \to \infty$. Because the x derivative of ξ blows up like $1/\epsilon$ at $x = 0$, we must expect fast dynamics in the initial layer. Thus, we obtain the *uniform* limiting solution

$$\begin{aligned} y(x,\epsilon) &= e^{\int_x^1 \frac{b(s)}{a(s)}ds} y(1) \\ &\quad + e^{-a(0)x/\epsilon}\left(y(0) - e^{\int_0^1 \frac{b(s)}{a(s)}ds} y(1)\right) + O(\epsilon) \end{aligned} \tag{11}$$

throughout $0 \leq x \leq 1$. Higher-order approximations for y and its derivatives follow without complication. If $a(x) < 0$ throughout the interval, we have a terminal boundary layer of nonuniform convergence near $x = 1$, as follows by introducing $r = 1 - x$ and seeking $y(r, \epsilon)$. For the nonhomogeneous differential equation corresponding to (1), the asymptotic solution for $a(x) > 0$ again has the form (2), where $Y_0(x)$ satisfies the nonhomogeneous version of (6). Even if y is a vector, this approach remains valid presuming the matrix $a(x)$ is positive definite. Complications naturally arise, however, when such an $n \times n$ matrix $a(x)$ is singular but positive semidefinite of fixed rank r, since the (homogeneous) reduced equation then has an $(n - r)$-dimensional manifold of solutions (cf. Vasil'eva, Butuzov, and Kalachev (1995)).

Using *two-timing* (cf., e.g., O'Malley (1974)) or the *WKB method* (cf., e.g., Olver (1974)), we instead obtain the approximate solution

$$y(x, \epsilon) = Y_0(x) + \eta_0(x, \kappa) + O(\epsilon) \tag{12}$$

with the *same* outer limit $Y_0(x)$, but instead of $\xi_0(X)$ the more elaborate *initial layer correction*

$$\eta_0(x, \kappa) = e^{-\kappa} \frac{a(0)}{a(x)} e^{\int_0^x \frac{b(s)}{a(s)} ds} \left(y(0) - e^{\int_0^1 \frac{b(s)}{a(s)} ds} y(1) \right) \tag{13}$$

now expressed in terms of x and the stretched variable

$$\kappa \equiv \frac{1}{\epsilon} \int_0^x a(s) ds, \tag{14}$$

which, like $\xi_0(X)$, decays exponentially to zero in an $O(\epsilon)$-neighborhood of $x = 0$ where κ (and X) $\to \infty$. Ou and Wong (2003) say that approximation (12) is correct. Both (11) and (12) are asymptotically correct, however, since the difference

$$\xi_0(X) - \eta_0(x, \kappa) = O(\epsilon) \tag{15}$$

uniformly throughout $0 \leq x \leq 1$. For any fixed ϵ, of course, the two-timing solution (12) can be expected to be numerically more accurate in the initial layer.

Examples. Consider the linear singularly perturbed equation

$$\epsilon y'' + (1 + x^2) y' - x^3 y = 0$$

on $0 \leq x \leq 1$ subject to the boundary conditions $y(0) = y(1) = 1$ (a boundary value problem which we can solve asymptotically, but not exactly). The corresponding limiting outer solution is uniquely found to be

$$Y_0(x) = \sqrt{\frac{2}{1 + x^2}} e^{(x^2 - 1)/2},$$

while the limiting initial layer correction is either

$$\xi_0(X) = e^{-X} \left(1 - \sqrt{\frac{2}{e}} \right)$$

for $X = x/\epsilon$ (via matching) or

$$\eta_0(x, \kappa) = e^{-\kappa} \frac{e^{-x^2}}{\sqrt{1 + x^2}} \left(1 - \sqrt{\frac{2}{e}} \right)$$

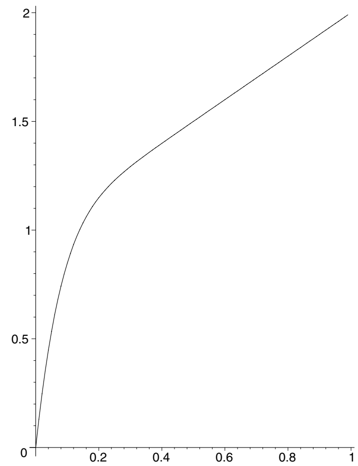

Fig. 1 *Solution of* $(*)$ *for* $\epsilon = 0.1$.

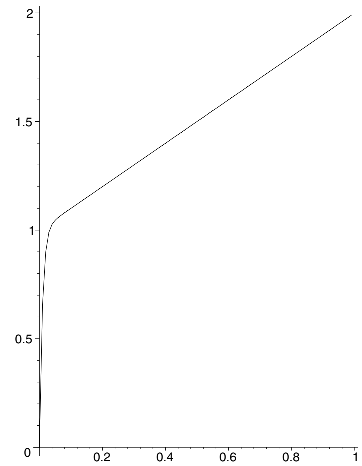

Fig. 2 *Solution of* $(*)$ *for* $\epsilon = 0.01$.

for $\kappa = \frac{1}{\epsilon}(x + \frac{x^3}{3})$ (via two-timing). Note that

$$\xi_0(X) - \eta_0(x,\kappa) = e^{-x/\epsilon}\left(1 - e^{-x^3/3\epsilon}\frac{e^{-x^2}}{\sqrt{1+x^2}}\right)\left(1 - \sqrt{\frac{2}{e}}\right) = O(\epsilon^2),$$

as can be shown numerically (cf. Bosley (1996)).

We note that Johnson (2005) analogously considers nonlinear examples like

$$(*) \qquad \epsilon y'' + (1+x)^2 y' - y^2 = 0, \quad y(0,\epsilon) = 0, \quad y(1,\epsilon) = 2.$$

He uses two-timing and obtains the outer limit

$$Y_0(x) = 1 + x$$

and the initial layer correction

$$\eta_0(x,\kappa) = (1+x)^{-4} e^{-\kappa}$$

for $\kappa = \frac{1}{\epsilon}\int_0^x (1+s)^2 ds = \frac{x}{3\epsilon}(3 + 3x + x^2)$. Figures 1 and 2 illustrate the thinness of the initial layer when ϵ is only moderately small. We point out that much work has been done to obtain asymptotic solutions to related nonlinear singularly perturbed boundary value problems. This is reported in Wollkind (1977), Chang and Howes (1984), Smith (1985), O'Malley (1991), Verhulst (2005), and Johnson (2005).

A direct way to determine the asymptotic behavior of the solution to the prescribed linear boundary value problem for (1) when $a(x) > 0$ is to first seek two linearly independent solutions of (1) throughout $0 \le x \le 1$ as

$$(16) \qquad Y_1(x,\epsilon) = e^{\int_x^1 \frac{b(s)}{a(s)} ds} C(x,\epsilon)$$

and

$$(17) \qquad Y_2(x,\epsilon) = e^{-\frac{1}{\epsilon}\int_0^x a(s) ds} \frac{a(0)}{a(x)} e^{\int_0^x \frac{b(s)}{a(s)} ds} D(x,\epsilon),$$

where the factors $C(x,\epsilon)$ and $D(x,\epsilon)$ are smooth and such that $C(1,\epsilon)=1=D(0,\epsilon)$. Setting

$$C(x,\epsilon)=1+\epsilon G(x,\epsilon)$$

and substituting directly into the ODE (1) requires that

$$a(x)G'=\left(\left(\frac{b}{a}\right)'-\left(\frac{b}{a}\right)^2\right)(1+\epsilon G)+2\epsilon\frac{b}{a}G'-\epsilon G'',$$

so G must satisfy the integral equation

$$\begin{aligned}G(x,\epsilon)=&-\int_x^1\frac{1}{a(s)}\left(\left(\frac{b(s)}{a(s)}\right)'-\left(\frac{b(s)}{a(s)}\right)^2\right)ds\\&-\epsilon\int_x^1\frac{1}{a(s)}\left(\left[\left(\frac{b(s)}{a(s)}\right)'-\left(\frac{b(s)}{a(s)}\right)^2\right]G(s,\epsilon)+2\frac{b(s)}{a(s)}G'(s,\epsilon)-G''(s,\epsilon)\right)ds.\end{aligned}$$

Iterating in the integral equation (using zero as the initial guess) then provides the asymptotic power series

$$C(x,\epsilon)\sim\sum_{j\geq 0}C_j(x)\epsilon^j, \tag{18}$$

where

$$\begin{aligned}&C_0(x)=1,\\&C_1(x)=-\int_x^1\frac{1}{a(s)}\left(\left(\frac{b(s)}{a(s)}\right)'-\left(\frac{b(s)}{a(s)}\right)^2\right)ds,\quad\text{etc.}\end{aligned}$$

The corresponding solution $C(x,\epsilon)$ is *smooth* throughout the interval and it corresponds to using a special initial value $C'(0,\epsilon)=\epsilon C_1'(0)+O(\epsilon^2)$. One can conveniently say that it lies on the *slow manifold* of its differential equation (cf. Strygin and Sobolev (1988) or Verhulst (2005)). Proceeding analogously, D must satisfy

$$\begin{aligned}a(x)D'(x,\epsilon)=\epsilon\Bigg[&\left(\left(\frac{b}{a}\right)^2+\left(\frac{a'}{a}\right)^2+\frac{2ba'}{a^2}+\left(\frac{b}{a}\right)'-\left(\frac{a'}{a}\right)'\right)D\\&+\left(2\left(\frac{b}{a}-\frac{a'}{a}\right)\right)D'+D''\Bigg].\end{aligned}$$

Upon integrating and iterating, the smooth solution $D(x,\epsilon)$ is found to have the asymptotic power series

$$D(x,\epsilon)\sim\sum_{j\geq 0}D_j(x)\epsilon^j, \tag{19}$$

with

$$\begin{aligned}&D_0(x)=1,\\&D_1(x)=\int_0^x\frac{1}{a^3(s)}\left[b^2(s)+b(s)a'(s)+b'(s)a(s)-a''(s)a(s)\right]ds,\quad\text{etc.}\end{aligned}$$

Existence of two such linearly independent solutions Y_1 and Y_2 to (1) throughout $0\leq x\leq 1$ for ϵ sufficiently small is proved using classical methods, e.g., in Chapter X of Wasow (1965).

Having obtained C and D and thereby Y_1 and Y_2 (at least asymptotically), we next naturally seek a solution of the given two-point problem in the form

$$(20)\qquad \begin{aligned} y(x,\epsilon) &= e^{\int_x^1 \frac{b(s)}{a(s)}ds} C(x,\epsilon)A(\epsilon) \\ &\quad + e^{-\frac{1}{\epsilon}\int_0^x a(s)ds}\frac{a(0)}{a(x)} e^{\int_0^x \frac{b(s)}{a(s)}ds} D(x,\epsilon)B(\epsilon). \end{aligned}$$

The prescribed boundary values then require that the unknowns A and B satisfy the linear equations

$$y(0) = e^{\int_0^1 \frac{b(s)}{a(s)}ds} C(0,\epsilon)A(\epsilon) + B(\epsilon)$$

and

$$y(1) = A(\epsilon) + e^{-\frac{1}{\epsilon}\int_0^1 a(s)ds}\frac{a(0)}{a(1)} e^{\int_0^1 \frac{b(s)}{a(s)}ds} D(1,\epsilon)B(\epsilon).$$

Since $e^{-\frac{1}{\epsilon}\int_0^1 a(s)ds}$ is asymptotically negligible, however,

$$A(\epsilon) \sim y(1),$$

while

$$B(\epsilon) \sim y(0) - e^{\int_0^1 \frac{b(s)}{a(s)}ds} C(0,\epsilon)y(1).$$

Thus, the asymptotic representation of the solution is uniquely obtained as the sum

$$(21)\qquad y(x,\epsilon) = Y(x,\epsilon) + \eta(x,\kappa,\epsilon)$$

throughout $[0,1]$ for the outer expansion

$$(22)\qquad Y(x,\epsilon) \sim e^{\int_x^1 \frac{b(s)}{a(s)}ds} C(x,\epsilon)y(1)$$

(providing the asymptotic solution to all orders ϵ^k for $x > 0$) and the initial layer correction

$$(23)\qquad \eta(x,\kappa,\epsilon) \sim e^{-\kappa}\frac{a(0)}{a(x)} e^{\int_0^x \frac{b(s)}{a(s)}ds} D(x,\epsilon)\left(y(0) - e^{\int_0^1 \frac{b(s)}{a(s)}ds} C(0,\epsilon)y(1)\right)$$

(which decays exponentially to zero in the initial layer as κ (cf. (14)) $\to \infty$). The asymptotic agreement of (2) and (21) holds to all orders of ϵ. We could also use the two linearly independent solutions (16) and (17) and variation of parameters to obtain a Green's function to solve various nonhomogeneous boundary value problems for (1). Its asymptotic structure is important for the numerical analysis of related linear and nonlinear boundary value problems (sfcf. Ascher, Mattheij, and Russell (1988) and Roos, Stynes, and Tobiska (1996)).

2. Boundary Layer Resonance.

(a) Introduction. One of the first-noticed situations in which classical asymptotic matching encountered difficulties was so-called *boundary layer resonance* (cf. Ackerberg and O'Malley (1970), whose authors were introduced in 1968 by their common acquaintance J. D. Murray, who knew that both were independently studying these curious problems at the Polytechnic Institute of Brooklyn and New York University, respectively). A prototypical example is provided by the singularly perturbed scalar equation

$$\epsilon y'' - xy' + \beta y = 0 \tag{24}$$

on $-1 \leq x \leq 1$ with $y(\pm 1)$ prescribed. Because $a(x) = -x > 0$ for $x < 0$, while $a(x) < 0$ for $x > 0$, one might expect to have $O(\epsilon)$-thick boundary layers of nonuniform convergence near both endpoints and a smooth limiting solution of the form $x^\beta C$, satisfying the reduced equation, within $(-1, 1)$ (or, at least, within $(-1, 0)$ and $(0, 1)$). A tipoff to upcoming complications is obtained by realizing that this outer limit is irregular at the turning point $x = 0$ if β is not a nonnegative integer. What boundary condition to apply to determine the respective constants C_L and C_R for C, say, remains unclear, however, unless one (successfully, but somewhat artificially) considers (24) to be an Euler–Lagrange equation for a corresponding functional (cf. Grasman and Matkowsky (1977) and much subsequent work, including Kath, Knessl, and Matkowsky (1987) and Srinivasan (1988)) or invokes a less-obvious symmetry argument (cf. Kevorkian and Cole (1996)). One can, indeed, anticipate that the limiting solution can be uniformly represented throughout $-1 \leq x \leq 1$ in the form

$$y_\epsilon(x) = x^\beta C + e^{-\frac{1}{\epsilon}(x+1)}(y(-1) - (-1)^\beta C) + e^{\frac{1}{\epsilon}(x-1)}(y(1) - C) + O(\epsilon)$$

(cf. Lagerstrom (1988)). (For related metastable parabolic equations, we also use such an ansatz with a slowly-varying C.) Matching fails to determine the constant C, although the appropriate introduction of some *exponential asymptotics* salvages the classical technique (cf., e.g., Cook and Eckhaus (1973), O'Malley and Ward (1996), and MacGillivray (1997)). To be more specific, one could use a local $O(\sqrt{\epsilon})$-stretching (and/or a WKB solution) in a neighborhood of $x = 0$ (cf., e.g., McHugh (1971) and Froman and Froman (2002)). It often provides an exponentially growing solution that is impossible to match with a nontrivial outer solution (cf. Pearson (1968)). That argument misses the provocative resonance phenomena, however. Further consideration of the Stokes phenomena is actually appropriate (cf. Olde Daalhuis et al. (1995) and Lakin (1972)). For trivial endvalues, one can profitably consider $-\beta$ as an ϵ-dependent eigenvalue corresponding to nontrivial eigenvectors (cf. de Groen (1977, 1980) and Lee and Ward (1995)) and employ Rayleigh quotient approximations for eigenfunctions. A partial differential equation (PDE) version of this boundary value problem arises as the stochastic exit problem (cf. Matkowsky (1980), Maier and Stein (1997) and Grasman and Herwaarden (1999)). De Groen (1974/1975) likewise considers related resonance problems for certain elliptic equations.

Our dilemma regarding (24), which we will call *resonance*, is clarified by introducing appropriate special functions. We first use the Liouville transformation

$$y = e^{x^2/4\epsilon} u$$

to reduce (24) to the Weber-like equation

$$\epsilon u'' + \left(-\frac{x^2}{4\epsilon} + \frac{1}{2} + \beta\right) u = 0.$$

Since the parabolic cylinder functions $D_n(\pm t)$ and $D_{-n-1}(\pm it)$ all satisfy $\frac{d^2 w}{dt^2} + (n + \frac{1}{2} - \frac{t^2}{4})w = 0$, we shall seek solutions y as the linear combination

$$y(x, \epsilon) = e^{\frac{x^2}{4\epsilon}} D_\beta \left(\frac{x}{\sqrt{\epsilon}}\right) A(\epsilon) + e^{\frac{x^2}{4\epsilon}} D_\beta \left(-\frac{x}{\sqrt{\epsilon}}\right) B(\epsilon) \tag{25}$$

for constants A and B (to find y as a function of the stretched variable $x/\sqrt{\epsilon}$, which varies from $-\infty$ to ∞ in an $O(\sqrt{\epsilon})$-neighborhood of the turning point $x = 0$). As an alternative, one could write the solution in terms of confluent hypergeometric functions or use combinations of $D_{-n-1}(it)$ and $D_n(-t)$ since they always remain linearly independent. The *connection formulas*

$$D_\beta(z) = e^{\mp\beta\pi i} D_\beta(-z) + \frac{\sqrt{2\pi}}{\Gamma(-\beta)} e^{\mp(\beta+1)\pi i/2} D_{-\beta-1}(\pm iz)$$

are needed in connecting asymptotic approximations across a Stokes line in the complex z plane (cf. Wasow (1985) and Kawai and Takei (2005)).

Using the asymptotic approximations

$$\begin{cases} D_\zeta(z) \sim z^\zeta e^{-z^2/4} & \text{as } z \to \infty \text{ for } |\arg z| < \frac{3\pi}{4} \\ \text{and} & \\ D_\zeta(z) \sim z^\zeta e^{-z^2/4} - \frac{\sqrt{2\pi}}{\Gamma(-\zeta)} \frac{e^{i\pi\zeta}}{z^{\zeta+1}} e^{z^2/4} & \text{as } z \to \infty \text{ for } \frac{\pi}{4} < \arg z < \frac{5\pi}{4} \end{cases} \tag{26}$$

(cf. Whittaker and Watson (1952)) allows us to approximate the ϵ-dependent constants A and B from the boundary conditions

$$y(\pm 1) = e^{\frac{1}{4\epsilon}} D_\beta\left(\pm\frac{1}{\sqrt{\epsilon}}\right) A + e^{\frac{1}{4\epsilon}} D_\beta\left(\mp\frac{1}{\sqrt{\epsilon}}\right) B.$$

The special functions used in (25) become linearly dependent when $\Gamma(-\beta) = 0$. For $\beta \neq 0, 1, 2, \ldots$, however, we will have

$$e^{x^2/4\epsilon} D_\beta(x/\sqrt{\epsilon}) \sim \left(\frac{x}{\sqrt{\epsilon}}\right)^\beta$$

and

$$e^{x^2/4\epsilon} D_\beta\left(-\frac{x}{\sqrt{\epsilon}}\right) \sim \frac{\sqrt{2\pi}}{\Gamma(-\beta)} \left(\frac{\sqrt{\epsilon}}{x}\right)^{\beta+1} e^{\frac{x^2}{2\epsilon}}$$

for $x > 0$. Solving the boundary conditions asymptotically for A and B, we find that

$$\begin{cases} y(x,\epsilon) \sim \frac{e^{-\frac{1}{2\epsilon}(1-x^2)}}{x^{\beta+1}} y(1) & \text{for } x > 0, \\ y(x,\epsilon) \sim \frac{e^{-\frac{1}{2\epsilon}(1-x^2)}}{(-x)^{\beta+1}} y(-1) & \text{for } x < 0, \\ \text{and} & \\ y(0,\epsilon) = O\left(\frac{e^{-\frac{1}{2\epsilon}}}{\epsilon^{(\beta+1)/2}}\right). & \end{cases} \tag{27}$$

Thus, the unique solution of (24) features an $O(\epsilon)$-thick boundary layer of nonuniform convergence near both endpoints and an asymptotically trivial limiting solution within $(-1, 1)$ (corresponding to selecting the constant $C = 0$ in the outer limit(s)). Nothing special happens at the turning point $x = 0$, where more generally we might have anticipated a singularity in x.

When $\beta = N$, a nonnegative integer,

$$D_N(z) = He_N(z) e^{-z^2/4} = (-1)^N e^{z^2/4} \frac{d^N}{dz^N}(e^{-z^2/2})$$

for the Hermite polynomial He_N of degree N; cf. Hille (1976). (Note, indeed, that Lebedev (1965) defines the Hermite function $H_\varphi(z)$ for an arbitrary complex degree φ.) This implies the linear independence mentioned above and forces us to now seek the general solution of (24) in the form

$$y(x,\epsilon) = He_N\left(\frac{x}{\sqrt{\epsilon}}\right)A(\epsilon) + e^{x^2/4\epsilon}D_{-N-1}\left(\frac{ix}{\sqrt{\epsilon}}\right)B(\epsilon), \tag{28}$$

instead of (25), where A and B must now be determined from the linear system

$$y(\pm 1) = (\pm 1)^N He_N\left(\frac{1}{\sqrt{\epsilon}}\right)A(\epsilon) + e^{\frac{1}{4\epsilon}}D_{-N-1}\left(\pm\frac{i}{\sqrt{\epsilon}}\right)B(\epsilon).$$

Using the asymptotic expansion for $D_{-N-1}(\frac{ix}{\sqrt{\epsilon}})$ determines

$$A(\epsilon) \sim \frac{1}{2He_N(\frac{1}{\sqrt{\epsilon}})}(y(1) + (-1)^N y(-1))$$

and

$$B(\epsilon) \sim \frac{1}{2}\left(\frac{i}{\sqrt{\epsilon}}\right)^{N+1} e^{-\frac{1}{2\epsilon}}(y(1) - (-1)^N y(-1)).$$

Since $\frac{He_N(x/\sqrt{\epsilon})}{He_N(1/\sqrt{\epsilon})} \sim x^N + O(\epsilon)$ for $x \neq 0$ because (as a referee observed) these polynomials are either even or odd, we get

$$\begin{cases} y(x,\epsilon) \sim \frac{1}{2}x^N(y(1) + (-1)^N y(-1)) \\ \qquad + \frac{1}{2}\frac{e^{-(1-x^2)/2\epsilon}}{x^{N+1}}(y(1) - (-1)^N y(-1)) \quad \text{for } x \neq 0 \\ \text{and} \\ y(0,\epsilon) = O(\epsilon^{N/2}). \end{cases} \tag{29}$$

Thus, we generally obtain a nontrivial limiting solution within $(-1,1)$ when $\beta = N$, corresponding to the unique constant

$$C = \frac{1}{2}(y(1) + (-1)^N y(-1)).$$

(The outer limit satisfies the reduced equation and an averaged boundary value.) We then have boundary layer behavior near both endpoints unless $C = (\pm 1)^N y(\pm 1)$, when the outer limit satisfies both boundary conditions. The first term in (28) clearly provides the asymptotic solution to all orders ϵ^j within $(-1,1)$, while the second term provides the necessary boundary layer corrections near both $x = \pm 1$. The condition

$$\beta = N,$$

which allows a nontrivial limiting outer solution for $C \neq 0$, is naturally interpreted as a *resonance* condition for (24). Readers should challenge themselves by considering what happens when $\beta(\epsilon) - N$ is asymptotically negligible (cf. Skinner (1987)). Such supersensitive turning-point problems, studied in Hemker (1977), continue to provide *numerical nightmares*. Note that the solution of the two-point problem is asymptotically unique, whether there is resonance or not.

(b) Related Problems. The exceptional nature of the polynomial limiting solution to (24) was described geometrically in Callot (1981). If one utilizes the u-x phase plane for

$$u = y'/y, \text{ so } y(x) = e^{\int_{-1}^{x} u(s)ds} y(-1),$$

one gets the slow-fast system

$$\begin{cases} x' = 1, \\ \epsilon u' = xu - \beta - \epsilon u^2. \end{cases} \tag{30}$$

(We observe, however, that the transformation $w = \epsilon y'/y$, used by Fruchard and Schaefke (2003), seems preferable because it keeps $\epsilon y'$ bounded.)

The important related idea of the *canard* or *delayed bifurcation* is most simply illustrated by considering solutions of the related initial value problem for

$$\epsilon w' = xw$$

on $x \geq -1$. The trivial solution $w = 0$ of the reduced equation is *attractive* (or stable to perturbations) for $x < 0$, but *repulsive* for $x > 0$. Nonetheless, the solution

$$w(x, \epsilon) = e^{(x^2-1)/2\epsilon} w(-1)$$

is asymptotically negligible within $(-1, 1)$, after the initial layer.

The surprise is the solution's *blowup* about $x = 1$ (from exponentially small to exponentially large in ϵ), well beyond the turning point at $x = 0$. Such an elementary example is important because the most popular example of a *canard* is for the much more complicated forced van der Pol equation (cf. Diener and Diener (1995), Benoit (1991), and Eckhaus (1983)). Note that by contrast the solution of the slightly perturbed equation

$$\epsilon w' = xw + \epsilon$$

blows up at the turning point, while the bifurcation remains delayed beyond $x = 0$ when the forcing is appropriately asymptotically negligible, such as $e^{-C/\epsilon}$ for a sufficiently large $C > 0$ (see Figures 3 and 4). Fruchard and Schaefke (2003, 2004) have indeed shown that boundary layer resonance is equivalent to the presence of canard solutions to the associated perturbed Riccati equation (30), as anticipated by Callot. Related problems in chemical kinetics are discussed in Butuzov, Nefedov, and Schneider (2004), while Gavin et al. (2006) gives simple examples of false canards, feeble canards, and two-faced canards! (Sobolev suggested that feeble canards might also be called *lame ducks*.) Also see Sobolev, Pokrovskii, and Shchepakina (2008). Curiously, the Butuzov paper considers the Riccati equation

$$\epsilon \dot{u} = u(t - u)$$

on $t \geq -1$ for $u(-1) > -1$ and shows that a delayed bifurcation from $u = 0$ to $u = t$ occurs near $t = 1$. The transformation $u = t - w$ then shows the corresponding delayed bifurcation for

$$\epsilon \dot{w} = w(w - t) + \epsilon$$

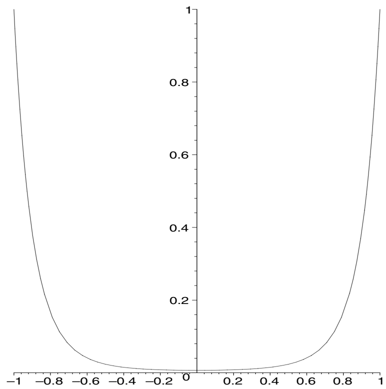

Fig. 3 *The solution of $\epsilon w' = xw$ for $w(-1) = 1$ and $\epsilon = 0.1$.*

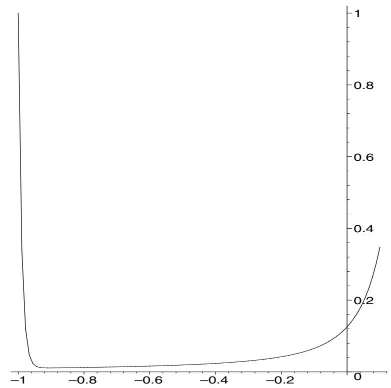

Fig. 4 *The solution of $\epsilon w' = xw + \epsilon$ for $w(-1) = 1$ and $\epsilon = 0.1$.*

on $t \geq -1$ from $w = t$ to 0 provided $w(-1) < 0$. For the slightly perturbed Riccati equation

$$\epsilon \dot{v} = v(v - t),$$

however, the corresponding solution transfers from $v = 0$ to $v = t$ at $t = 0$, precisely where the two roots of the limiting equation cross. (Recalling that O'Malley (1991) called the solution w "Dahlquist's knee" (in reference to an example from Dahlquist et al. (1982)), we might now say that the perturbation causes the late Germund's knee to break at the delayed time $t = 1$.)

Among many other related problems, we mention the *spurious solutions* of Carrier and Pearson (1968). For them, matched asymptotic expansions provide formal "solutions" that do not correspond to any actual solutions. These nonlinear problems typically have multiple solutions that are amenable to techniques like those used for the resonance problem (cf. O'Malley (1976), Lange (1983), Kath, Knessl, and Matkowsky (1987), and MacGillivray (1997)), no doubt because the linearized problem may have turning points and asymptotically negligible eigenvalues. Another related complication with asymptotic matching is the possibility of *transcendental switchback*, where unanticipated terms, typically scaled like $\epsilon|\log \epsilon|$, need to be introduced to make matching work (cf. Lagerstrom (1988)). This requires additional study (cf. Popović (2005), however). One anticipates that further asymptotic analysis on such problems would allow us to study more general resonance problems.

(c) Other Resonance Problems. The limiting solution to our prototype equation (24) differs when the interval $[-1, 1]$ is shortened to $[-\alpha, 1]$, $0 < \alpha < 1$. First suppose $\beta \neq N$. We naturally again seek a solution of the form (25), where A and B must now satisfy

$$y(-\alpha) \sim \frac{\sqrt{2\pi}}{\Gamma(-\beta)} \left(\frac{\sqrt{\epsilon}}{\alpha}\right)^{\beta+1} e^{\frac{\alpha^2}{2\epsilon}} A + \left(\frac{\alpha}{\sqrt{\epsilon}}\right)^{\beta} B$$

and

$$y(1) \sim \left(\frac{1}{\sqrt{\epsilon}}\right)^{\beta} A + \frac{\sqrt{2\pi}}{\Gamma(-\beta)} \left(\sqrt{\epsilon}\right)^{\beta+1} e^{\frac{1}{2\epsilon}} B,$$

yielding

$$A \sim \frac{\Gamma(-\beta)}{\sqrt{2\pi}} \left(\frac{\alpha}{\sqrt{\epsilon}}\right)^{\beta+1} e^{-\frac{\alpha^2}{2\epsilon}} y(-\alpha) \quad \text{and} \quad B \sim \frac{\Gamma(-\beta)}{\sqrt{2\pi}} \frac{1}{(\sqrt{\epsilon})^{\beta+1}} e^{-\frac{1}{2\epsilon}} y(1).$$

Thus, we again obtain an asymptotically negligible outer solution, i.e.,

$$y(x,\epsilon) \sim \begin{cases} \frac{e^{\frac{x^2-1}{2\epsilon}}}{x^{\beta+1}} y(1) & \text{for } 0 < x \le 1, \\ O\left(\frac{e^{-\alpha^2/2\epsilon}}{(\sqrt{\epsilon})^{\beta+1}}\right) & \text{for } x = 0, \\ (\frac{\alpha}{x})^{\beta+1} e^{\frac{x^2-\alpha^2}{2\epsilon}} y(-\alpha) & \text{for } -\alpha \le x < 0, \end{cases} \tag{31}$$

with an $O(\epsilon)$-thick boundary layer at each endpoint and the trivial outer solution to all orders in ϵ. When $\beta = N$, however, we use the representation (28), but the boundary conditions then imply that

$$y(-\alpha) \sim \left(-\frac{\alpha}{\sqrt{\epsilon}}\right)^N A + e^{\frac{\alpha^2}{4\epsilon}} \left(\frac{-i\alpha}{\sqrt{\epsilon}}\right)^{-N-1} B$$

and

$$y(1) \sim \left(\frac{1}{\sqrt{\epsilon}}\right)^N A + e^{\frac{1}{4\epsilon}} \left(\frac{i}{\sqrt{\epsilon}}\right)^{-N-1} B,$$

so $A \sim (-\frac{\sqrt{\epsilon}}{\alpha})^N y(-\alpha)$, $B \sim (\frac{i}{\sqrt{\epsilon}})^{N+1} e^{-\frac{1}{2\epsilon}}[y(1) - \frac{y(-\alpha)}{(-\alpha)^N}]$, and

$$y(x,\epsilon) \sim \begin{cases} (-\frac{x}{\alpha})^N y(-\alpha) + \frac{1}{x^{N+1}} e^{\frac{x^2-1}{2\epsilon}} \left(y(1) - \frac{y(-\alpha)}{(-\alpha)^N}\right) & \text{for } 0 < x \le 1, \\ O(\epsilon^{N/2}) & \text{for } x = 0, \\ (-\frac{x}{\alpha})^N y(-\alpha) & \text{for } -\alpha \le x < 0. \end{cases} \tag{32}$$

Note, in particular, that the limiting solution $(-\frac{x}{\alpha})^N y(-\alpha)$ for $x < 1$ now satisfies the reduced problem $-xY_0' + NY_0 = 0$, $Y_0(-\alpha) = y(-\alpha)$, while there is an $O(\epsilon)$-thick terminal layer at the more distant endpoint $x = 1$. The boundary value problem on $[-\alpha, 1]$ for $\alpha > 1$ would instead feature a trivial outer limit except in resonance and then a limit satisfying $-xY_0' + NY_0 = 0$, $Y_0(1) = y(1)$, for $x > -\alpha$. One needs both initial and terminal boundary layers at resonance when $\alpha = 1$, but again one would also expect some transition when $\alpha - 1$ is asymptotically negligible.

If we now consider (24) on the interval $0 \le x \le 1$, $\alpha = 0$ and the turning point coincides with the initial point. We might expect a terminal layer because of the sign of the y' coefficient for $x > 0$. When we again look for a solution of the form (25) for $\beta \neq N$, the boundary conditions imply that $y(0) = D_\beta(0)(A + B)$, while

$$y(1) = e^{\frac{1}{4\epsilon}} D_\beta\left(\frac{1}{\sqrt{\epsilon}}\right) A + e^{\frac{1}{4\epsilon}} D_\beta\left(-\frac{1}{\sqrt{\epsilon}}\right) B \sim \left(\frac{1}{\sqrt{\epsilon}}\right)^\beta A + \frac{\sqrt{2\pi}}{\Gamma(-\beta)} (\sqrt{\epsilon})^{\beta+1} e^{\frac{1}{2\epsilon}} B.$$

For $x > 0$, then, we obtain the asymptotic approximation

$$y(x,\epsilon) \sim \left(\frac{x}{\sqrt{\epsilon}}\right)^\beta y(0) + \frac{1}{x^{\beta+1}} e^{\frac{x^2-1}{2\epsilon}} y(1). \tag{33}$$

For $\beta < 0$, then, the limiting solution on $x \gg O(\sqrt{\epsilon})$ is bounded. For $\beta > 0$, however, it is bounded only if $y(0) = 0$. In the special case $\beta = 0$, 1 and $\int_0^x e^{s^2/2\epsilon} ds$ are linearly independent solutions of the differential equation, so the unique solution of the two-point problem is

$$y(x) = y(0) + \left(\frac{\int_0^x e^{s^2/2\epsilon} ds}{\int_0^1 e^{s^2/2\epsilon} ds} \right) (y(1) - y(0)).$$

Writing $\int^x e^{s^2/2\epsilon} ds = \int^x \frac{\epsilon}{s} \frac{d}{ds} (e^{s^2/2\epsilon}) ds \sim \frac{\epsilon}{x} e^{x^2/2\epsilon}$ for $x > 0$ provides the asymptotic limit

$$y(x) \sim y(0) + \frac{1}{x} e^{\frac{x^2-1}{2\epsilon}} (y(1) - y(0)).$$

Thus, the outer limit is the constant $y(0)$ and a terminal layer of $O(\epsilon)$-thickness occurs at $x = 1$. Solutions for $\beta = 1$ can be obtained by integrating solutions for $\beta = 0$ (since differentiating the ODE for y provides the same ODE for y' with β shifted by one). Thus, the general solution of $\epsilon y'' - xy' + y = 0$ will be a linear combination of x and $\int^x \int_0^r e^{s^2/2\epsilon} ds dr$, i.e., $y(x, \epsilon) = kx + c(x \int_0^x e^{s^2/2\epsilon} ds - \epsilon e^{x^2/2\epsilon})$. Applying the boundary conditions,

$$\begin{aligned} y(x, \epsilon) &= \left[y(1) + y(0) \left(\frac{1}{\epsilon} \int_0^1 e^{s^2/2\epsilon} ds - e^{\frac{1}{2\epsilon}} \right) \right] x + y(0) \left(e^{\frac{x^2}{2\epsilon}} - \frac{x}{\epsilon} \int_0^x e^{\frac{s^2}{2\epsilon}} ds \right) \\ &= y(1)x + y(0) \left(\frac{x}{\epsilon} \int_x^1 e^{s^2/2\epsilon} ds + e^{\frac{x^2}{2\epsilon}} - x e^{\frac{1}{2\epsilon}} \right) = y(1)x + y(0)x \int_x^1 \frac{e^{\frac{s^2}{2\epsilon}}}{s^2} ds. \end{aligned}$$

Expanding the integral asymptotically, however, shows the anticipated exponential blowup for $x > 0$.

In contrast to (24), solutions to

$$\epsilon y'' + xy' + \gamma y = 0 \quad \text{on} \quad -1 \le x \le 1 \tag{34}$$

with endvalues $y(\pm 1)$ prescribed can, due to the opposite sign of the coefficient of y', be expected to feature an interior shock layer at the turning point. (McKelvey and Bohac (1976) call this the *interior flaring* phenomenon, while others relate it to a *false canard*. A more complicated but related *boundary resonance* is considered in Wazwaz and Hanson (1986) and de Groen (1988).) The Liouville transformation $y = e^{-x^2/4\epsilon} u$ converts the ODE into a Weber equation and shows that any solution has the form

$$y(x, \epsilon) = e^{-\frac{x^2}{4\epsilon}} D_{\gamma - 1} \left(\frac{x}{\sqrt{\epsilon}} \right) A(\epsilon) + e^{-\frac{x^2}{4\epsilon}} D_{\gamma - 1} \left(-\frac{x}{\sqrt{\epsilon}} \right) B(\epsilon)$$

as long as these parabolic cylinder functions are linearly independent (i.e., $\gamma \neq 1, 2, \ldots$). Because

$$y(1) \sim \left(\frac{1}{\sqrt{\epsilon}} \right)^{\gamma - 1} e^{-\frac{1}{2\epsilon}} A + \frac{\sqrt{2\pi}}{\Gamma(1 - \gamma)} (\sqrt{\epsilon})^{\gamma} B$$

and

$$y(-1) \sim \frac{\sqrt{2\pi}}{\Gamma(1 - \gamma)} (\sqrt{\epsilon})^{\gamma} A + \left(\frac{1}{\sqrt{\epsilon}} \right)^{\gamma - 1} e^{-\frac{1}{2\epsilon}} B,$$

we can solve asymptotically for A and B to find that

$$y(x,\epsilon) \sim \begin{cases} \frac{y(-1)}{(-x)^\gamma} + (-\frac{x}{\sqrt{\epsilon}})^{\gamma-1} e^{-\frac{x^2}{2\epsilon}} \frac{\Gamma(1-\gamma)}{\sqrt{2\pi\epsilon}} y(1) & \text{for } x < 0, \\ \frac{\Gamma(1-\gamma)}{\sqrt{2\pi}(\sqrt{\epsilon})^\gamma} D_{\gamma-1}(0)(y(-1)+y(1)) & \text{for } x = 0, \\ \frac{y(1)}{x^\gamma} + \frac{\Gamma(1-\gamma)}{\sqrt{2\pi\epsilon}} (\frac{x}{\epsilon})^{\gamma-1} e^{-\frac{x^2}{2\epsilon}} y(-1) & \text{for } x > 0. \end{cases} \tag{35}$$

For $x < 0$, then, the limiting solution satisfies the reduced problem

$$xy' + \gamma y = 0, \quad y(-1) \text{ given,}$$

so $C_L = (-1)^\gamma y(-1)$, while for $x > 0$, it satisfies

$$xy' + \gamma y = 0, \quad y(1) \text{ given.}$$

There is a transition between these natural outer limits near $x = 0$, which becomes algebraically unbounded as $\epsilon \to 0$ if $\gamma > 0$. It might then, indeed, be instructive to model the impulsive behavior about the turning point by introducing an appropriate delta function. For the exceptional values $\gamma = 1, 2, \ldots,$ one finds a new resonance and the asymptotic solution blows up exponentially within $(-1, 1)$. When γ is negative, derivatives of the solution may jump at the turning point where a *corner layer* occurs.

The simplest example with an interior transition layer may be

$$\epsilon y'' + xy' = 0,$$

i.e., (33) for $\gamma = 0$. Its limiting solution is constant for $x \neq 0$ and the monotonic transition there between the limits $y(-1)$ and $y(1)$ can be expressed in terms of error functions as

$$y(x,\epsilon) = y(-1) + \left(\frac{\int_{-1/\sqrt{2\epsilon}}^{x/\sqrt{2\epsilon}} e^{-r^2} dr}{\int_{-1/\sqrt{2\epsilon}}^{1/\sqrt{2\epsilon}} e^{-r^2} dr} \right) (y(1) - y(-1)).$$

Note how the ratio of integrals switches asymptotically from 0 to 1 in an $O(\sqrt{\epsilon})$-neighborhood of $x = 0$.

The asymptotic behavior described in this section is certainly not limited to the examples described. Note first the modified version of (24),

$$\epsilon y'' - xy' + (N + x + \delta\epsilon)y = 0.$$

Setting $y = e^{x^2/4\epsilon} u$ provides the transformed Weber equation

$$\epsilon u'' + \left(- \left(\frac{(x-2\epsilon)^2}{4\epsilon} \right) + \frac{1}{2} + N + (\delta+1)\epsilon \right) u = 0,$$

so our preceding analysis shows directly that resonance occurs when $N + (\delta + 1)\epsilon$ is a nonnegative integer, i.e., if N is nonnegative and $\delta = -1$. Thus, as Watts (1971) first showed, the equation $\epsilon y'' - xy' + (N + x)y = 0$ is nonresonant, while $\epsilon y'' - xy' + (N + x - \epsilon)y = 0$ is resonant. "Higher-order" conditions for resonance are needed. Ackerberg and O'Malley (1970) showed that a first necessary condition for

$$\epsilon y'' - xy' + b(x,\epsilon)y = 0$$

to be resonant is that $b(0,0)$ be a nonnegative integer. Thus, the example $\epsilon y'' - xy' + \frac{1}{2}(1+x^2)y = 0$, considered by Wong and Yang (2002a,b), is nonresonant. Its limiting solution can be obtained by matching (cf. Bender and Orszag (1978)) or by a WKB technique. To retain resonance, the de Groen (1980) analysis of the related eigenvalue problem implies that one cannot perturb the coefficient b by more than an asymptotically negligible amount. Matkowsky (1975) boldly suggested that resonance follows when (24) with a power series $\beta(\epsilon)$ (or its generalization (38), to be considered below) has a smooth nontrivial formal power series solution. Indeed, this also suggests perturbations that violate this so-called Matkowsky condition for resonance (cf. Matkowsky (1975), Olver (1978a), Kopell (1980), Sibuya (1981), Lin (1983), and Sibuya (1990)) by violating the analyticity of β in ϵ. One faces great uncertainty, especially if one is accustomed to using only a few terms in an asymptotic approximation. Even if one shows that the first sixteen necessary conditions for resonance hold, the seventeenth could make the limiting solution trivial.

Special cases of the preceding include

$$\epsilon y'' + a(x)y' = 0 \tag{36}$$

for $a(x) = \pm x$. With a more general coefficient $a(x)$, we obtain solutions

$$y(x,\epsilon) = y(-1) + \frac{\Psi(x,\epsilon)}{\Psi(1,\epsilon)}(y(1) - y(-1)) \tag{37}$$

for $\Psi(x,\epsilon) = \int_{-1}^{x} e^{-\frac{1}{\epsilon}\int_{-1}^{r} a(s)ds} dr$. The asymptotic behavior of these solutions can be determined by using Laplace's method (cf. Olver (1974)). Limiting solutions are constant except for transitions where $\int_{-1}^{x} a(s)ds$ has an absolute minimum (cf. de Groen (1980)). Many examples have been discussed in the literature, including those where $a(x)$ has a number of zeros (cf. Wazwaz (1990)) and those where $a(x)$ has a higher-order zero at $x = 0$ (cf. Matkowsky (1975)). Solving such specific examples is important because they can serve as *comparison equations* when uniform reduction methods are applied (cf. Lynn and Keller (1970)). We note that Wasow's 1942 NYU thesis showed that asymptotic solutions to the nonhomogeneous equation

$$\epsilon y'' + xc(x)y' = f(x)$$

for $c(x) \neq 0$ are not simple because the reduced problem is singular at the turning point when $f(0) \neq 0$ (cf. Wasow (1970) and Jung and Temam (2007)).

If we consider the boundary value problem for

$$\epsilon y'' + \left(x^2 - \frac{1}{4}\right) y' = 0 \tag{$*$}$$

on $-1 \leq x \leq 1$, we will have turning points at $x = \pm 1/2$. The two-point problem with $y(\pm 1)$ prescribed can be considered as the concatenation of the corresponding two-point problems on the adjacent intervals $-1 \leq x \leq 0$ and $0 \leq x \leq 1$. These problems would, respectively, feature an $O(\epsilon)$-thick initial layer at $x = -1$ (presuming $y(-1) \neq y(0)$) and an $O(\sqrt{\epsilon})$-thick shock layer about $x = 1/2$, with constant limits elsewhere. Since $x = 0$ is an ordinary point, we take $y(0) \sim y(-1)$ in order to make the solution smooth there, and thereby obtain the limiting solution

$$y(x,\epsilon) \sim \begin{cases} y(-1) & \text{for } -1 < x < \frac{1}{2}, \\ y(1) & \text{for } \frac{1}{2} < x \leq 1. \end{cases}$$

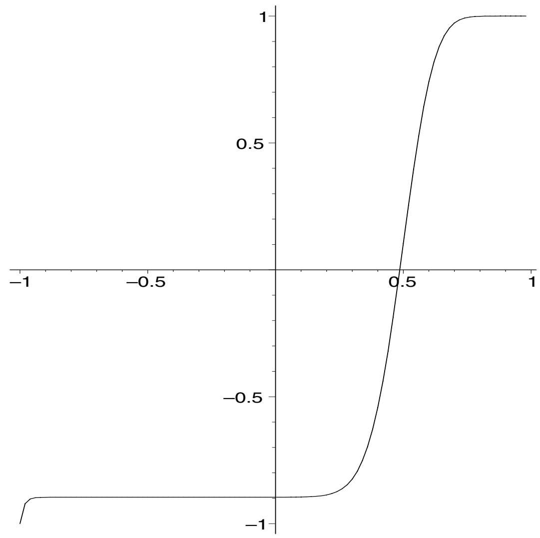

Fig. 5 *Solution of* $(*)$ *for* $y(-1) = -1$, $y(1) = 1$, *and* $\epsilon = 0.01$.

More specifically, the solution has the form (37) for $a(x) = x^2 - 1/4$. Because the integrands of Ψ are asymptotically negligible away from $s = -1$ and $s = 1/2$, Laplace's method shows that the contribution from $x = -1^+$ is asymptotically $\frac{4}{3}\epsilon$, while those from $x = \frac{1^\pm}{2}$ are both asymptotically $\sqrt{2\pi\epsilon}$. This implies an $O(\sqrt{\epsilon})$ initial layer jump for y followed by a shock layer jump about $x = 1/2$ from $y(-1) + O(\sqrt{\epsilon})$ to $y(1)$. See Figure 5 for an illustration of a solution.

3. Uniform Reduction and Alternatives. Consider the linear singularly perturbed equation

$$\epsilon y'' + f(x,\epsilon)y' + g(x,\epsilon)y = 0 \tag{38}$$

on the interval $-1 \le x \le 1$, where f and g are real and analytic and where

$$f(x,0) = -xh(x,0) \text{ for some } h(x,0) > 0. \tag{39}$$

We will say that (38), generalizing (24), then has a simple turning point at $x = 0$. Such an equation is satisfied by the angular velocity for viscous flow between counterrotating disks (cf. Matkowsky and Siegmann (1976)), which provided motivation for early studies. Other possible resonance problems are suggested in McKelvey and Bohac (1976). We shall seek higher-order conditions for resonance, generalizing those found for (24). We can first nearly transform the equation into Weber's equation by introducing

$$\eta^2(x) = 2\int_0^x sh(s,0)ds \tag{40}$$

and selecting $\eta > 0$ for $x > 0$. Since η has a simple zero at $x = 0$, differentiation implies that $\eta'(x) = \frac{xh(x,0)}{\eta(x)} > 0$. Moreover,

$$\epsilon\frac{d^2y}{d\eta^2} + F(\eta,\epsilon)\frac{dy}{d\eta} + G(\eta,\epsilon)y = 0 \tag{41}$$

for $F(\eta,\epsilon)=\frac{f(x,\epsilon)}{\eta'(x)}+\epsilon\frac{\eta''(x)}{(\eta'(x))^2}\equiv-\eta+\epsilon F_1(x,\epsilon)$, $G(\eta,\epsilon)=\frac{g(x,\epsilon)}{(\eta'(x))^2}$, and

$$\eta_-\equiv\eta(-1)\le\eta\le\eta(1)\equiv\eta_+.$$

Further, the Liouville transformation

$$y=e^{-\frac{1}{2\epsilon}\int_0^\eta F(s,\epsilon)ds}z(\eta,\epsilon) \tag{42}$$

shows that z will satisfy the perturbed Weber equation

$$\epsilon\frac{d^2z}{d\eta^2}-\left(\frac{\eta^2}{4\epsilon}+R(\eta,\epsilon)\right)z=0 \tag{43}$$

for $R(\eta,\epsilon)=\frac{1}{4\epsilon}(\eta^2-F^2)+G-\frac{1}{2}F_\eta$.

Classical turning-point theory (cf. Wasow (1965) or Wasow (1985)) implies that we can find power series M, N, and σ in ϵ so that z can be represented as the linear combination

$$z=M(\eta,\epsilon)w+\epsilon N(\eta,\epsilon)w_\eta, \tag{44}$$

where w satisfies the *comparison* equation

$$\epsilon\frac{d^2w}{d\eta^2}-\left(\frac{\eta^2}{4\epsilon}+\sigma(\epsilon)\right)w=0 \tag{45}$$

(with solutions $D_{\sigma-\frac{1}{2}}(\pm\frac{\eta}{\sqrt{\epsilon}})$) and where $M(0,\epsilon)=1$. (In place of (44), note that Rubenfeld and Willner (1977) use the simpler ansatz

$$z=(w_\eta(\eta,\epsilon))^{-1/2}Q(w(\eta,\epsilon),\epsilon)$$

for a power series Q in ϵ.) Differentiation then implies that

$$\epsilon\frac{dz}{d\eta}=\left[\epsilon M_\eta+\left(\frac{\eta^2}{4}+\epsilon\sigma\right)N\right]w+(M+\epsilon N_\eta)\epsilon w_\eta,$$

while

$$\epsilon^2\frac{d^2z}{d\eta^2}=\left[\epsilon\left(\epsilon M_\eta+\left(\frac{\eta^2}{4}+\epsilon\sigma\right)N\right)_\eta+(M+\epsilon N_\eta)\left(\frac{\eta^2}{4}+\epsilon\sigma\right)\right]w$$
$$+\left[\epsilon M_\eta+\left(\frac{\eta^2}{4}+\epsilon\sigma\right)N+\epsilon(M+\epsilon N_\eta)_\eta\right]\epsilon w_\eta.$$

Since $\epsilon^2\frac{d^2z}{d\eta^2}=(\frac{\eta^2}{4}+\epsilon R)z$, separating coefficients of w and w_η finally implies that M and N must satisfy a perturbed first-order linear system for M and ηN,

$$\begin{cases}\frac{\eta}{2}(\eta N)_\eta+(\sigma-R)M=-\epsilon(M_{\eta\eta}+2\sigma N_\eta)\\ \text{and}\\ M_\eta+\frac{1}{2}(\sigma-R)N=-\frac{\epsilon}{2}N_{\eta\eta}\end{cases} \tag{46}$$

(with a singular point at $\eta=0$). We will seek smooth series solutions for M, N, and σ termwise. When $\epsilon=0$, we get

$$2M_{0\eta}+\frac{1}{\eta}(\sigma_0-R(\eta,0))\eta N_0=0\quad\text{and}\quad(\eta N_0)_\eta+\frac{2}{\eta}(\sigma_0-R(\eta,0))M_0=0.$$

To obtain smooth coefficients at $\eta = 0$, we must select

$$\sigma_0 = R(0,0) = -\frac{g(0,0)}{h(0)} - \frac{1}{2}, \tag{47}$$

leaving the system

$$(2M_0)_\eta - S(\eta)\eta N_0 = 0 \quad \text{and} \quad (\eta N_0)_\eta - 2S(\eta)M_0 = 0$$

for $S(\eta) \equiv \frac{1}{\eta}(R(\eta, 0) - R(0,0))$. Since $M_0(0) = 1$, $(2M_0)^2 - (\eta N_0)^2 = 4$ and

$$2M_0(\eta) = \cosh\left(\int_0^\eta S(r)dr\right) \quad \text{and} \quad \eta N_0(\eta) = 2\sinh\left(\int_0^\eta S(r)dr\right). \tag{48}$$

From the $O(\epsilon)$-coefficients, we likewise get

$$2M_{1\eta} - S(\eta)\eta N_1 = -N_{0\eta\eta}$$

and

$$(\eta N_1)_\eta - 2S(\eta)M_1 = -\frac{2}{\eta}[M_{0\eta\eta} + 2\sigma_0 N_{0\eta} + 2\sigma_1 M_0].$$

For the right-hand side to be bounded at $\eta = 0$, we must select

$$\sigma_1 = -\sigma_0 N_{0\eta}(0) - \frac{1}{2}M_{0\eta\eta}(0). \tag{49}$$

Because

$$P(\eta) \equiv \begin{pmatrix} \cosh(\int_0^\eta S(t)dt) & \sinh(\int_0^\eta S(t)dt) \\ \sinh(\int_0^\eta S(t)dt) & \cosh(\int_0^\eta S(t)dt)) \end{pmatrix}$$

is a fundamental matrix for the linear system, variation of parameters implies that

$$\begin{pmatrix} 2M_1 \\ \eta N_1 \end{pmatrix} = -P(\eta)\int_0^\eta P^{-1}(r)\begin{pmatrix} N_{0\eta\eta} \\ \frac{2}{\eta}(M_{0\eta\eta} + 2\sigma_0 N_{0\eta} + 2\sigma_1 M_0) \end{pmatrix} dr. \tag{50}$$

The process can clearly be continued termwise.

The uniform simplification process has thereby reduced representation of the solution of (38) to solving the simple comparison equation (45) for w. Moreover, it explains the previously mentioned Matkowsky condition: resonance occurs when a smooth nontrivial solution of the differential equation (38) can be found (cf. Sibuya (1981)). Using the infinite number of terms one generates for the asymptotic series for $\sigma(\epsilon)$ in (45), we determine that awkward countable infinity of necessary conditions for resonance. Moreover, resonance for the given equation (38) is equivalent to resonance for the comparison equation (45). An alternative to uniform reduction might involve careful matching at the endpoints and the turning point (cf. Il'in (1992) and Fedoryuk (1993)). Note that Cheng (2007) considers a wide variety of turning-point problems from both the WKB and the special function perspectives.

Somewhat analogously, Lewis (1982) uses a sequence of changes of dependent and independent variables to obtain comparison equations

$$\epsilon z_m'' + (-x + \epsilon^{m+1} f_m(x,\epsilon))z_m' + \left(\sum_{j=0}^{m-1} \beta_j \epsilon^j + \epsilon^m g_m(x,\epsilon)\right) z_m = 0.$$

Wong and Yang (2002a,b) likewise convert an equation $\epsilon y'' + a(x)y' + b(x)y = 0$ to the form

$$\epsilon\ddot{U} + \alpha t\dot{U} + \beta U = \epsilon[f(t)\dot{U} + g(t)U].$$

By using a Green's function, they find the asymptotic solution by successive approximations. Wong and Yang (2003) suggest that $\alpha < 0$, $\beta/\alpha = 1, 2, \ldots$, is a necessary and sufficient condition for resonance. This, however, contradicts the counterexample of Watts (1971) and the clear need for higher-order conditions.

As another alternative to uniform reduction, Kreiss and Parter (1974) (see also Kreiss (1981) and Berger, Han, and Kellogg (1984)) used a *maximum principle* argument based on the fact that the derivative $\frac{\partial^j y(x,\epsilon)}{\partial x^j}$ will satisfy

$$\epsilon v_j'' + f(x,\epsilon)v_j' + (g(x,\epsilon) + jf'(x,\epsilon))v_j = \sum_{s=0}^{j-1} A_{js}(x)v_s,$$

where $A_{js}(x)$ is a specific linear combination of $f^{(j+1-s)}(x)$ and $g^{(j-s)}(x)$ and where the coefficient of v_j will ultimately become negative.

Kreiss and Parter also introduced the examples

$$\begin{cases} \epsilon y'' - xy' + \frac{x}{1+x}y = 0 \\ \text{and} \\ \epsilon y'' - x(1+x)y' + xy = 0, \end{cases}$$

which both have the exact solution $y(x,\epsilon) = 1 + x$. These suggest that explicit solutions exhibiting resonance could be found for equations somewhat different from the Hermite equation (24). Note, too, that the last equation has a turning point at $x = 0$ and another at $x = -1$. Such problems are briefly discussed in de Groen (1980a). Additional computational examples are considered in Miranker and Morreeuw (1974) and Miranker (1981).

Olver (1978a) likewise solves such boundary value problems by using four asymptotic solutions (and their error bounds), with two being uniformly valid on $0 \le x \le 1$ and the other two on $-1 \le x \le 0$. Olver (1978b) likewise finds asymptotic approximations of solutions to equations with much more general transition points.

The phenomenon of boundary layer resonance requires the consideration of terms that are exponentially small in ϵ. Traditional power series techniques do not capture such behavior, so it is more natural to introduce Gevrey, rather than Poincaré, asymptotics (cf. Ramis (1991)). One says that

$$f(z) \sim \sum_{n=0}^{\infty} a_n z^n$$

is of Gevrey order 1 in a sector when

$$\left| f(z) - \sum_{n=0}^{N-1} a_n z^n \right| \le CN!(A|z|)^N$$

there for all integers $N \ge 1$ and for positive constants C and A. The factorial bound imposed implies that the typically divergent series arising from singularly perturbed differential equations feature "least term truncation" and an asymptotically negligible error for that approximation. Moreover, the Borel transform of such series converge. This approach has been recently taken by a number of authors, including Hsieh and Sibuya (1999), Canalis-Durand et al. (2000), Balser (2000), and Sibuya (2000). They consider slow-fast systems of the form

$$\begin{cases} \dot{y} = f(x,y,z), \\ \epsilon\dot{z} = g(x,y,z), \end{cases}$$

where the dot represents a derivative with respect to x, under various appropriate hypotheses. A *blowup* technique (cf. Dumortier and Roussarie (1996) and van Gils, Krupa, and Szmolyan (2005)) is also of much value. The *charts* they employ are analogous to Zwaan's method of avoiding the turning point (cf. McHugh (1971)). In particular, this approach may ultimately provide a justification of asymptotic matching that is more complete than Fraenkel (1969), Eckhaus (1979), and Il'in (1992). We further observe that Fraenkel's effective use of Hardy's *logarithmic-exponential* functions corresponds to the use of *transseries* by Costin (2005) and van der Hoeven (2006).

The most recent paper on boundary layer resonance, De Maesschalck (2007b), uses blowup and Gevrey asymptotics to study equations with a single higher-order turning point. (We note that Bender and Wang (2001) consider the associated eigenvalue problem and that Laforgue (1997) had considered certain problems earlier, relating the asymptotically negligible eigenvalues in resonance to metastability for the corresponding time-dependent PDE.) De Maesschalck considers (38) when

$$\frac{\partial^j f(0,0)}{\partial x^j} = 0 \text{ for } j = 0,1,\ldots,p-1 \quad \text{and} \quad \frac{\partial^p f(0,0)}{\partial x^p} < 0 \tag{51}$$

for some odd integer p and raises the issue of whether resonance doesn't occur when $g(x,0)$ is not zero up to order $p-1$ or when

$$N = -p\frac{\frac{\partial^{p-1} g(0,0)}{\partial x^{p-1}}}{\frac{\partial^p f(0,0)}{\partial x^p}} \tag{52}$$

is not a nonnegative integer. He finds resonance, like Kopell (1980), when the invariant manifolds (i.e., outer solutions) defined for $x < 0$ and for $x > 0$ coincide appropriately asymptotically. Indeed, he finds resonance by perturbing the coefficient g by

$$\delta(\epsilon^{1/p+1})x^{p-1}$$

when $\delta(0) = 0$ and $N = 0$ or $1 (\text{mod}\,(p+1))$. Surprisingly, he explicitly shows that the resonant solution $\varphi(x,\epsilon)$ may not be smooth, with $\varphi(0,\epsilon)$ and $\varphi'(0,\epsilon)$ of the form

$$\epsilon^{\frac{N}{p+1}+\xi(\epsilon^{\frac{1}{p+1}})}r(\epsilon^{\frac{1}{p+1}})$$

for C^∞ functions ξ and r. When $\xi = 0$, he shows that the Matkowsky condition suffices for resonance. Moreover, he also shows that

$$\epsilon y'' - x^3(1+x)y' + x^2(\delta + x^4)y = 0$$

is resonant for

$$\delta = -3\epsilon + O(\epsilon^{5/4})$$

and for

$$\xi = 1 + \epsilon + O(\epsilon^{5/4}).$$

Results from De Maesschalck and Dumortier (2006) are employed. Corresponding results for canards are continued in De Maesschalck (2007a, 2008), Forget (2006), Liu (2006), and ongoing work.

Conclusion. The problem of boundary layer resonance remains of interest after thirty-five years of study. The original question concerning the sufficiency of asymptotic matching remains. New techniques have substantial potential for explaining the phenomenon. Whether related resonance phenomena hold for other special functions or for nonlinear differential equations remains largely unexplored. Attempts to do so should, however, benefit our current understanding and growing arsenal of asymptotic methods.

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