MATH-6620 PERTURBATION METHODS SPRING 2016

Homework-2

Assigned Tuesday February 9, 2016

Due Friday February 19, 2016

NOTES

1. Writing solutions in LaTeX is strongly recommended but not required.
2. Show all work. Illegible or undecipherable solutions will be **returned without grading**.
3. Figures, if any, should be neatly drawn (either by hand or by a drawing program), properly labelled and captioned.
4. Please make sure that the pages are stapled together.
5. The assignment can be submitted in the labelled box in Amos Eaton 301, at my office, or in class.

PROBLEMS

1. Consider the function
$$f(x;\epsilon) = \frac{1+\epsilon x+\sqrt{x+\epsilon}}{1+\sqrt{x+\epsilon e^{-x/\epsilon}}}, \quad x \in [0,1].$$
   (a) Explain, analytically, why this function has a layer of thickness ϵ at $x=0$.
   (b) Compute $F_0(x)+\cdots+\epsilon F_1(x)$, the outer expansion of f to order ϵ. This expansion corresponds to the outer limit process $\epsilon \to 0$, x fixed.
   (c) Let $\xi = x/\epsilon$. Compute $G_0(\xi)+\cdots+\epsilon G_1(\xi)$, the inner expansion of f to order ϵ. This expansion corresponds to the inner limit process $\epsilon \to 0$, ξ fixed.
   (d) Let $x = \mu\eta$ define an intermediate variable, where $\mu(\epsilon)$ is to be determined. Find the restrictions on μ so that F_0 and G_0 match to order unity (in the intermediate limit), *i.e.,*
$$\lim_{\epsilon\to 0}[F_0(\mu\eta) - G_0(\mu\eta/\epsilon] = 0.$$
   (e) Find the restrictions on μ so that $F_0+\cdots+\epsilon F_1$ and $G_0+\cdots+\epsilon G_1$ match to order ϵ (in the intermediate limit), *i.e.,*
$$\lim_{\epsilon\to 0}\frac{[\{F_0(\mu\eta)+\cdots+\epsilon F_1(\mu\eta)\} - \{G_0(\mu\eta/\epsilon)+\cdots+\epsilon G_1(\mu\eta/\epsilon)\}]}{\epsilon} = 0.$$
   If the above limit does not exist, then consider relaxing the order of ϵ to which the two expansions match, and explain the consequences and meaning of such a relaxation.

In each of the following problems, anticipate (if possible) the location of the inner region(s). You may use the Van Dyke Principle for matching.

2. For the BVP $\epsilon y'' - y' + \epsilon x^2 y = 2x$, $0<x<1$, $y(0;\epsilon)=2$, $y(1;\epsilon) = 2+\epsilon$, find the first two terms in the outer and the inner solutions, and the composite approximation.
3. For the BVP $\epsilon y'' + x^{1/3}y' + y^2 = 0$, $-1<x<1$, $y(-1;\epsilon) = 2/9$, $y(1) = 1/3$, find the leading-order outer and inner solutions, and the composite approximation.
4. For the BVP $\epsilon y'' + e^x(xy'-y) = x^2$, $-1<x<1$, $y(-1)=1$, $y(1)=-1$, find the leading-order outer and inner solutions, and the composite approximation. How does the situation alter if the ODE is changed to $\epsilon y'' - e^x(xy'-y) = -x^2$ while the boundary conditions remain the same?
5. Find the leading-order outer and inner solutions and the composite approximation to the solution of the boundary-value problem
$$\epsilon y'' - (1+3x^2)y - x = 0, \;\; 0<x<1, \;\; y(0;\epsilon) = y(1;\epsilon) = 1.$$