MATH-6620 PERTURBATION METHODS SPRING 2016

Homework-5 Solutions

Assigned Tuesday April 5, 2016

Due Monday April 18, 2016.

PROBLEMS

1. Use the multiple-scales technique, with t as the fast time and $\tau = \epsilon t$ the slow time, to compute the leading-order approximation, valid for long time, to the problem

$$\frac{d^2u}{dt^2} + u + \epsilon\left(u - a\frac{du}{dt}\right)^3 = 0, \quad t > 0; \quad u(0) = 1, \ \frac{du}{dt}(0) = 0.$$

Here the constant $a > 0$ is independent of ϵ. Explain what happens to your solution when $a < 0$.

As the time scales are specified, we set $u(t;\epsilon) = v(t,\tau;\epsilon)$, and note that

$$\begin{aligned} \frac{du}{dt} &= v_t + \epsilon v_\tau, \\ \frac{d^2u}{dt^2} &= v_{tt} + 2\epsilon v_{t\tau} + \epsilon^2 v_{\tau\tau}. \end{aligned}$$

Then the ODE and the initial conditions transform as

$$v_{tt} + 2\epsilon v_{t\tau} + \epsilon^2 v_{\tau\tau} + v + \epsilon\left(v - av_t - \epsilon a v_\tau\right)^3 = 0,$$

with

$$v = 0, \quad v_t + \epsilon v_\tau = 0 \quad \text{at} \quad t = \tau = 0.$$

We seek the expansion

$$v(t,\tau;\epsilon) \sim v_0(t,\tau) + \epsilon v_1(t,\tau).$$

At order unity the reduced problem is

$$v_{0_{tt}} + v_0 = 0, \quad v_0(0,0) = 1, \ v_{0_t}(0,0) = 0.$$

The solution is

$$v_0 = A_0(\tau)\cos\theta, \quad \theta = t + \phi_0(\tau), \quad A_0(0) = 1, \ \phi_0(0) = 0.$$

At $O(\epsilon)$ we have

$$v_{1tt} + v_1 + 2v_{0_{t\tau}} + (v_0 - av_{0_t})^3 = 0.$$

On substituting for v_0 the above becomes

$$\begin{aligned} v_{1tt} + v_1 &= 2[A_0'\sin\theta + A_0\phi_0'\cos\theta] - [A_0\cos\theta + aA_0\sin\theta]^3 \\ &= 2[A_0'\sin\theta + A_0\phi_0'\cos\theta] - A_0^3[\cos^3\theta + 3a\cos^2\theta\sin\theta + 3a^2\cos\theta\sin^2\theta + a^3\sin^3\theta] \\ &= 2[A_0'\sin\theta + A_0\phi_0'\cos\theta] \\ &\quad -A_0^3[(1-3a^2)\cos^3\theta + a(a^2-3)\sin^3\theta + 3a\sin\theta + 3a^2\cos\theta]. \end{aligned} \tag{1}$$

Standard trigonometric identities,

$$\cos^3\theta = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta, \quad \sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta, \tag{2}$$

reduce (1) to

$$\begin{aligned} v_{1tt} + v_1 &= 2[A_0' \sin\theta + A_0\phi_0' \cos\theta] \\ &\quad -\frac{1}{4}A_0^3[(1-3a^2)(\cos 3\theta + 3\cos\theta) + a(a^2-3)(3\sin\theta - \sin 3\theta) + 12a\sin\theta + 12a^2\cos\theta]. \end{aligned}$$

The terms $\sin\theta$ and $\cos\theta$ on the RHS give rise to secular terms in the solution. Elimination of these terms leads to the equations

$$\begin{aligned} 2A_0' &= \frac{1}{4}A_0^3[3a(a^2-3)+12a], \\ 2A_0\phi_0' &= \frac{1}{4}A_0^3[3(1-3a^2)+12a^2]. \end{aligned}$$

These simplify to

$$\begin{aligned} A_0' &= \frac{3a}{8}(a^2+1)A_0^3, \\ A_0\phi_0' &= \frac{3}{8}(a^2+1)A_0^3. \end{aligned}$$

Subject to the initial conditions $A_0(0)=1,\ \phi_0=0,$ the solutions are

$$A_0(\tau) = \frac{1}{\sqrt{1-3a(a^2+1)\tau/4}}, \qquad \phi_0 = -\frac{1}{2a}\ln[1-3a(a^2+1)\tau/4].$$

For $a>0$ the amplitude and phase both exhibit finite time blowup. For $a<0$ the amplitude decays to zero while the phase is logarithmically unbounded for τ large.

2. Use the multiple-scales technique, with t as the fast time and $\tau=\epsilon t$ the slow time, to compute the leading-order approximation, valid for long time, to the problem

$$\frac{d^2u}{dt^2} + \{1+\epsilon\sin(\epsilon t)\}u + \epsilon\left(\frac{du}{dt}\right)^3 = 0, \quad t>0; \qquad u(0)=1,\ \frac{du}{dt}(0)=0.$$

We proceed much as in the previous problem. On setting $u(t;\epsilon)=v(t,\tau;\epsilon),$ and noting that

$$\begin{aligned} \frac{du}{dt} &= v_t + \epsilon v_\tau, \\ \frac{d^2u}{dt^2} &= v_{tt} + 2\epsilon v_{t\tau} + \epsilon^2 v_{\tau\tau}, \end{aligned}$$

the ODE and the initial conditions transform as

$$v_{tt} + 2\epsilon v_{t\tau} + \epsilon^2 v_{\tau\tau} + (1+\epsilon\sin\tau)v + \epsilon\left(v_t + \epsilon a v_\tau\right)^3 = 0,$$

with

$$v=0, \quad v_t + \epsilon v_\tau = 0 \quad \text{at} \quad t=\tau=0.$$

We seek the expansion

$$v(t,\tau;\epsilon) \sim v_0(t,\tau) + \epsilon v_1(t,\tau).$$

At order unity the reduced problem is

$$v_{0_{tt}} + v_0 = 0, \quad v_0(0,0)=1,\ v_{0_t}(0,0)=0.$$

The solution is

$$v_0 = A_0(\tau)\cos\theta, \quad \theta = t+\phi_0(\tau), \quad A_0(0)=1,\ \phi_0(0)=0.$$

At $O(\epsilon)$ we have

$$\begin{aligned} v_{1tt} + v_1 &= -2v_{0_{t\tau}} - v_0 \sin\tau - v_{0_t}^3 \\ &= 2[A_0' \sin\theta + A_0\phi_0' \cos\theta] - A_0 \cos\theta \sin\tau + \frac{{A_0}^3}{4}(3\sin\theta - \sin 3\theta). \end{aligned}$$

Here the RHS has been modified by using the second of the trigonometric identities in (2). Elimination of secularity requires setting the coefficients of $\sin\theta$ and $\cos\theta$ to zero. We are led to the equations

$$\begin{aligned} A_0' + \frac{3}{8}A_0^3 &= 0, \\ A_0\phi_0' - \frac{1}{2}A_0 \sin\tau &= 0. \end{aligned}$$

Subject to the initial conditions $A_0(0) = 1,\ \phi_0 = 0,$ the solutions are

$$A_0(\tau) = \frac{1}{\sqrt{1 + 3\tau/4}}, \quad \phi_0(\tau) = \frac{1}{2}(1 - \cos\tau).$$

The leading-order solution valid for long times is thus determined.

3. Consider the weakly nonlinear equation

$$u_{tt} - u_{xx} + u + \epsilon u^2 = 0, \quad |x| < \infty,\ t > 0.$$

Note that we considered a similar equation in class but with a cubic nonlinearity. Find a multi-scale expansion for the right-running wave that removes the secular term coming from the nonlinearity. Note also that the scales needed to remove the secular term in this problem are different from those needed for the problem with cubic nonlinearity. Comment on what scales would be needed if the nonlinear term were u^n, n being a positive integer.

Following the discussion in class on weakly nonlinear wave equations, we first consider a regular perturbation procedure to discover at what stage of perturbation the nonlinearity contributes to the appearance of secularity. This will allow us to determine the necessary slow scales to suppress secularity. As we are interested in a right-going wave, we let $\theta = kx - \omega t$ and look for a solution in the form of the traveling wave, $u(\theta)$. Then,

$$u_t = -\omega u_\theta,\ u_{tt} = \omega^2 u_{\theta\theta},\ u_x = k u_\theta,\ u_{xx} = k^2 u_{\theta\theta},$$

and the PDE reduces to the ODE

$$(\omega^2 - k^2)u_{\theta\theta} + u + \epsilon u^2 = 0.$$

For the linear equation (corresponding to $\epsilon = 0$), the solutions are harmonic in θ, *i.e.*, $u = e^{\pm i\theta}$, provided ω and k obey the dispersion relation

$$\omega^2 - k^2 = 1.$$

We retain this assumption for the weakly nonlinear equation as well, which allows the ODE above to be written as

$$u_{\theta\theta} + u + \epsilon u^2 = 0. \tag{3}$$

We now seek the expansion

$$u \sim u_0(\theta) + \epsilon u_1(\theta) + \epsilon^2 u(\theta). \tag{4}$$

At $O(1)$ the ODE yields

$$u_{0\theta\theta} + u_0 = 0,$$

whose solution can be written in the complex form as

$$u_0 = A_0 e^{i\theta} + A_0^* e^{-i\theta},$$

where A_0^* is the complex conjugate of the complex constant A_0. We could have written the solution in real form in terms of trigonometric functions, but the complex form makes the subsequent algebra simpler.

At $O(\epsilon)$ the ODE leads to

$$\begin{aligned} u_{1\theta\theta} + u_1 &= -u_0^2 \\ &= -A_0^2 e^{2i\theta} - 2A_0A_0^* - A_0^{*2} e^{-2i\theta}. \end{aligned}$$

The solution is

$$u_1 = -2A_0A_0^* + \left(A_1 e^{i\theta} + \frac{1}{3}A_0^2\, e^{2i\theta} + CC \right),$$

where CC refers to complex conjugate terms. We note that the solution is periodic and that secularity has not raised its head thus far.

In an analogous way, the problem at $O(\epsilon^2)$ is

$$\begin{aligned} u_{2\theta\theta} + u_2 &= -2u_0u_1 \\ &= -2[A_0e^{i\theta} + A_0^* e^{-i\theta}] \left[-2A_0A_0^* + \left(A_1e^{i\theta} + \frac{1}{3}A_0^2\, e^{2i\theta} + CC \right) \right] \\ &= -2e^{i\theta} \left(-2A_0^2A_0^* + \frac{1}{3}A_0^2A_0^* \right) + CC + OT. \end{aligned}$$

Here we have explicitly displayed the $e^{i\theta}$ term on the RHS. Also, CC denotes the complex conjugate of the first term and OT denotes all other terms, whose frequencies are different from the natural frequency. The solution of the above ODE will contain a term proportional to $\theta e^{i\theta}$, which is a secular term that becomes of the same order as the leading term u_0 when $\epsilon^2\theta = \epsilon^2(kx - \omega t)$ is of order unity. This disordering of the regular-perturbation expansion suggests introducing the slow scales

$$\tau = \epsilon^2 t, \quad \xi = \epsilon^2 x.$$

[One may argue that the expansion (4) becomes disordered much earlier, at $\epsilon\theta = O(1)$ in fact, when the third term in the expansion becomes of the same order as the second. This is a reasonable point of view and it is recommended that you follow it to see where it leads, and why it does not by itself resolve the problem.]

With $u = u(\theta, \xi, \tau)$, we have

$$\begin{aligned} u_t &= -\omega u_\theta + \epsilon^2 u_\tau, \\ u_x &= k u_\theta + \epsilon^2 u_\xi, \\ u_{tt} &= \omega^2 u_{\theta\theta} - 2\epsilon^2 \omega u_{\theta\tau} + \epsilon^4 u_{\tau\tau}, \\ u_{xx} &= k^2 u_{\theta\theta} + 2\epsilon^2 k u_{\theta\xi} + \epsilon^4 u_{\xi\xi}. \end{aligned}$$

Then the PDE becomes

$$u_{\theta\theta} + u + \epsilon u^2 - 2\epsilon^2[\omega u_{\theta\tau} + k u_{\theta\xi}] + \epsilon^4[u_{\tau\tau} - u_{\xi\xi}] = 0.$$

We now seek the multiple-scale expansion

$$u(\theta, \xi, \tau) \sim u_0 + \epsilon u_1 + \epsilon^2 u_2. \tag{5}$$

At $O(1)$, the PDE yields

$$u_{0\theta\theta} + u_0 = 0,$$

whose solution is

$$u_0 = A_0 e^{i\theta} + A_0^* e^{-i\theta},$$

Here, $A_0 = A_0(\xi, \tau)$.

At $O(\epsilon)$,

$$
\begin{aligned}
u_{1\theta\theta} + u_1 &= -u_0^2 \\
&= -A_0^2 e^{2i\theta} - 2A_0A_0^* - A_0^{*2} e^{-2i\theta}.
\end{aligned}
$$

The solution is

$$
u_1 = -2A_0A_0^* + \left(A_1 e^{i\theta} + \frac{1}{3} A_0^2 \, e^{2i\theta} + CC \right),
$$

and as for the regular perturbation, still periodic.

At $O(\epsilon^2)$,

$$
\begin{aligned}
u_{2\theta\theta} + u_2 &= -2u_0u_1 + 2[\omega u_{0\theta\tau} + k u_{0\theta\xi}] \\
&= -2[A_0 e^{i\theta} + A_0^* e^{-i\theta}] \left[-2A_0A_0^* + \left(A_1 e^{i\theta} + \frac{1}{3} A_0^2 \, e^{2i\theta} + CC \right) \right] \\
&\quad + 2[\omega i A_{0\tau} e^{i\theta} + k i A_{0\xi} e^{i\theta}] \\
&= -2e^{i\theta} \left(-2A_0^2 A_0^* + \frac{1}{3} A_0^2 A_0^* + 2\omega i A_{0\tau} + 2ki A_{0\xi} \right) + CC + OT.
\end{aligned}
$$

As before, the terms with frequencies other than the natural frequency are contained in OT. Suppression of secularity requires that A_0 must satisfy

$$
-\frac{5}{3} A_0^2 A_0^* + 2\omega i A_{0\tau} + 2ki A_{0\xi} = 0.
$$

We let

$$
A_0 = R_0 e^{i\phi_0},
$$

where $R_0(\xi, \tau)$ is the amplitude and $\phi_0(\xi, \tau)$ the phase of the leading-order solution. Then the above equation becomes

$$
-\frac{5}{6} R_0^3 e^{i\phi_0} + \omega i [R_{0\tau} + i R_0 \phi_{0\tau}] e^{i\phi_0} + ki [R_{0\xi} + i R_0 \phi_{0\xi}] e^{i\phi_0} = 0.
$$

On dividing by $e^{i\phi_0}$ and separating into real and imaginary parts, we get the evolution equations

$$
\begin{aligned}
\omega R_{0\tau} + k R_{0\xi} &= 0, \\
\omega \phi_{0\tau} + k \phi_{0\xi} &= -\frac{5}{6} R_0^2.
\end{aligned}
$$

The method of characteristics finds the solutions to these PDEs as

$$
\begin{aligned}
R_0 &= R_0(\omega\xi - k\tau), \\
\phi_0 &= -\frac{5}{6\omega} R_0^2 (\omega\xi - k\tau)\tau + \hat{\phi}_0(\omega\xi - k\tau).
\end{aligned}
$$

Then the leading-order solution is

$$
u_0 = A_0 e^{i\theta} + CC = R_0 e^{i(\theta + \phi_0)} + CC = 2R_0(\omega\xi - k\tau) \cos \left(-\frac{5}{6\omega} R_0^2 (\omega\xi - k\tau)\tau + \hat{\phi}_0(\omega\xi - k\tau) \right).
$$

The initial conditions will determine R_0 and $\hat{\phi}_0$.

We now turn to the case of the nonlinearity ϵu^n. With $u = u_0 = Ae^{i\theta} + A^* e^{-i\theta}$, u^n generates the terms $e^{in\theta}, e^{i(n-2)\theta}, e^{i(n-4)\theta}, \cdots, e^{-in\theta}$. Only for n odd will the term $e^{i\theta}$ appear in this list, leading to secularity at the $O(\epsilon)$ level and hence the scalings ϵx and ϵt. For n even the scalings will be delayed to the $O(\epsilon^2)$ stage.

4. Consider the weakly nonlinear oscillator with forcing,

$$\frac{d^2u}{dt^2} + u + \epsilon u^3 = \alpha \cos \omega t.$$

Here α and ω are constants independent of ϵ.

(a) Find the general form of the first term such that the solution is valid for a long time, given that $\omega \neq 0,\ \pm 1,\ \pm 3,\ \pm 1/3$.

(b) Explain the consequences of (a) $\omega = 0$; (b) $\omega = \pm 1$.

We begin by seeking a straightforward expansion without invoking multiple scales. With $u \sim u_0 + u_1$, the leading-order problem is

$$u_{0_{tt}} + u_0 = \alpha \cos \omega t = \frac{\alpha}{2}[e^{i\omega t} + e^{-i\omega t}].$$

We have employed the complex representation for algebraic ease. The general solution is

$$u_0 = A_0 e^{it} + A_0^* e^{-it} + \frac{\alpha}{2(1-\omega^2)}(e^{i\omega t} + e^{-i\omega t}).$$

At $O(\epsilon)$,

$$\begin{aligned}
u_{1_{tt}} + u_1 &= -u_0^3 \\
&= -\left[(A_0 e^{it} + A_0^* e^{-it}) + \frac{\alpha}{2(1-\omega^2)}(e^{i\omega t} + e^{-i\omega t})\right]^3 \\
&= -(A_0 e^{it} + A_0^* e^{-it})^3 - 3(A_0 e^{it} + A_0^* e^{-it})^2 \frac{\alpha}{2(1-\omega^2)}(e^{i\omega t} + e^{-i\omega t}) \\
&\quad -3(A_0 e^{it} + A_0^* e^{-it})\left(\frac{\alpha}{2(1-\omega^2)}(e^{i\omega t} + e^{-i\omega t})\right)^2 - \left(\frac{\alpha}{2(1-\omega^2)}(e^{i\omega t} + e^{-i\omega t})\right)^3 \\
&= -(A_0^3 e^{3it} + 3A_0^2 A_0^* e^{it} + 3A_0 {A_0^*}^2 e^{-it} + {A_0^*}^3 e^{-3it}) \\
&\quad -\frac{3\alpha}{2(1-\omega^2)}(A_0^2 e^{2it} + 2A_0 A_0^* + {A_0^*}^2 e^{-2it})(e^{i\omega t} + e^{-i\omega t}) \\
&\quad -\frac{3\alpha^2}{4(1-\omega^2)^2}(A_0 e^{it} + A_0^* e^{-it})(e^{2i\omega t} + 2 + e^{-2i\omega t}) \\
&\quad -\frac{\alpha^3}{8(1-\omega^2)^3}(e^{3i\omega t} + 3e^{i\omega t} + 3e^{-i\omega t} + e^{-3i\omega t})
\end{aligned}$$

The terms $e^{\pm it}$ appearing on the RHS will resonate with the homogenous solution and result in linear growth. Terms on the RHS generated by the forcing function are $e^{i\omega t}$, $e^{(2+\omega)it}$, $e^{(2-\omega)it}$, $e^{(2\omega+1)it}$, $e^{(2\omega-1)it}$, $e^{3\omega it}$ and their complex conjugates. One or more of these will resonate with $e^{\pm it}$ for ω having the critical values

$$\omega = 0,\ \pm 1,\ \pm 3,\ \pm 1/3. \tag{6}$$

(a) Let us now consider the case where ω takes none of these special values. Then $e^{\pm it}$ is the sole contributor to resonance. To prevent secularity due to this term we need to introduce two times, t and $\tau = \epsilon t$. With $u(t;\epsilon) = v(t,\tau;\epsilon)$, in the by-now-familiar way we obtain the PDE

$$v_{tt} + 2\epsilon v_{t\tau} + \epsilon^2 v_{\tau\tau} + v + \epsilon v^3 = \frac{\alpha}{2}[e^{i\omega t} + e^{-i\omega t}],$$

and seek the expansion

$$v \sim v_0(t,\tau) + \epsilon v_1(t,\tau).$$

Then, at order unity the problem is

$$v_{0_{tt}} + v_0 = \frac{\alpha}{2}[e^{i\omega t} + e^{-i\omega t}],$$

with solution

$$v_0 = A_0(\tau)e^{it} + A_0^*(\tau)e^{-it} + \frac{\alpha}{2(1-\omega^2)}(e^{i\omega t} + e^{-i\omega t}). \tag{7}$$

At $O(\epsilon)$,

$$\begin{aligned}
v_{1_{tt}} + v_1 &= -v_0^3 - 2v_{0_{t\tau}} \\
&= -(A_0e^{it} + A_0^*e^{-it})^3 - 3(A_0e^{it} + A_0^*e^{-it})^2\left(\frac{\alpha}{2(1-\omega^2)}(e^{i\omega t} + e^{-i\omega t})\right) \\
&\quad -3(A_0e^{it} + A_0^*e^{-it})\left(\frac{\alpha}{2(1-\omega^2)}(e^{i\omega t} + e^{-i\omega t})\right)^2 - \left(\frac{\alpha}{2(1-\omega^2)}(e^{i\omega t} + e^{-i\omega t})\right)^3 \\
&\quad -2i[A_0'e^{it} - A_0^{*\prime}e^{-it}] \\
&= -(A_0^3e^{3it} + 3A_0^2A_0^*e^{it} + 3A_0A_0^{*2}e^{-it} + A_0^{*3}e^{-3it}) \\
&\quad -\frac{3\alpha}{2(1-\omega^2)}(A_0^2e^{2it} + 2A_0A_0^* + A_0^{*2}e^{-2it})(e^{i\omega t} + e^{-i\omega t}) \\
&\quad -\frac{3\alpha^2}{4(1-\omega^2)^2}(A_0e^{it} + A_0^*e^{-it})(e^{2i\omega t} + 2 + e^{-2i\omega t}) \\
&\quad -\frac{\alpha^3}{8(1-\omega^2)^3}(e^{3i\omega t} + 3e^{i\omega t} + 3e^{-i\omega t} + e^{-3i\omega t}) \\
&\quad -2i[A_0'e^{it} - A_0^{*\prime}e^{-it}].
\end{aligned} \tag{8}$$

As long as ω does not equal any of the critical values identified in (6), the secularity-avoidance requirement, obtained by setting the coefficient of e^{it} to zero, leads to

$$3A_0^2A_0^* + 2iA_0' + \frac{3\alpha^2}{2(1-\omega^2)^2}A_0 = 0. \tag{9}$$

Let $A_0 = \frac{1}{2}R_0(\tau)e^{i\phi_0(\tau)}$. Then the above equation transforms into

$$\frac{3}{8}R_0^3 - R_0\phi_0' + iR_0' + \frac{3\alpha^2}{4(1-\omega^2)^2}R_0 = 0,$$

yielding the ODEs

$$R_0'(\tau) = 0, \quad \text{and} \quad \phi_0'(\tau) = \frac{3}{8}R_0^2 + \frac{3\alpha^2}{4(1-\omega^2)^2}.$$

If the initial conditions are $R_0 = a$ and $\phi_0 = K$, then the above equations integrate to give

$$R_0 = a, \quad \phi_0 = \left(\frac{3}{8}a^2 + \frac{3\alpha^2}{4(1-\omega^2)^2}\right)\tau + K.$$

The leading-order solution (7) then takes the form

$$v_0 = a\cos\left\{t + \left(\frac{3}{8}a^2 + \frac{3\alpha^2}{4(1-\omega^2)^2}\right)\tau + K\right\} + \frac{\alpha}{1-\omega^2}\cos\omega t. \tag{10}$$

(b) First, consider $\omega = 0$. Then we revisit (8) and re-derive the secularity-avoidance condition, which now appears as

$$3A_0^2A_0^* + 2iA_0' + 3\alpha^2A_0 = 0.$$

With $A_0 = \frac{1}{2}R_0(\tau)e^{i\phi_0(\tau)}$ we obtain

$$\frac{3}{8}R_0^3 - R_0\phi_0' + iR_0' + \frac{3\alpha^2}{2}R_0 = 0,$$

and proceeding as above, are led to the ODEs

$$R_0'(\tau) = 0, \quad \text{and} \quad \phi_0'(\tau) = \frac{3}{8}R_0^2 + \frac{3\alpha^2}{2},$$

and then to the integrals

$$R_0 = a, \quad \phi_0 = \left(\frac{3}{8}a^2 + \frac{3\alpha^2}{2}\right)\tau + K,$$

so that the leading-order solution is

$$v_0 = a\cos\left\{t + \left(\frac{3}{8}a^2 + \frac{3\alpha^2}{2}\right)\tau + K\right\} + \alpha.$$

Note that this solution is not the same as the limit of (10) for $\omega \to 0$. The difference is in the expression for the phase shift.

Turning now to $\omega = 1$ it is best to start afresh as resonance changes the character of the solution of the unperturbed problem ($\epsilon = 0$). The governing equation is

$$\frac{d^2u}{dt^2} + u + \epsilon u^3 = \alpha\cos t.$$

To keep the algebra simple let us choose null initial conditions, $u = du/dt = 0$ at $t = 0$.
We begin with a straightforward expansion, $u \sim u_0 + \epsilon u_1$. Then u_0 satisfies

$$u_{0_{tt}} + u_0 = \alpha\cos t.$$

We have elected to represent the solution in real form (just to be different from the preceding discussion),

$$u_0 = \frac{\alpha}{2}t\sin t,$$

where the initial conditions have forced the complementary function to be zero. The ODE for u_1 is

$$\begin{aligned} u_{1_{tt}} + u_1 &= -u_0^3 \\ &= -\frac{\alpha^3}{8}t^3\sin^3 t \\ &= -\frac{\alpha^3}{32}t^3(3\sin t - \sin 3t). \end{aligned}$$

The particular solution generated by the first forcing term on the RHS is proportional to t^4 for large t. Therefore the expansion for u has the behavior

$$u \sim u_0 + \epsilon u_1 = O(t) + O(\epsilon t^4) \quad \text{for} \quad t \to \infty,$$

indicating a non-uniformity at $t = O(\epsilon^{-1/3})$, when the first two terms in the expansion are both of the same order, $O(\epsilon^{-1/3})$. This non-uniformity suggests the need for a slow time scale $\tau = \epsilon^{-1/3}t$, and an expansion in which the solution to leading order is not $O(1)$ but $O(\epsilon^{-1/3})$. We set $\epsilon = \delta^3$, and let $u(t;\delta) = (1/\delta)v(t,\tau;\delta)$. Then v satisfies the PDE

$$v_{tt} + v + 2\delta v_{t\tau} + \delta^2 v_{\tau\tau} + \delta v^3 = \delta\alpha\cos t.$$

The null initial conditions lead to

$$v = 0, \quad v_t + \delta v_\tau = 0 \quad \text{at} \quad t = \tau = 0.$$

We now seek the expansion $v \sim v_0 + \delta v_1$. Then v satisfies the unperturbed, unforced equation

$$v_{0_{tt}} + v_0 = 0,$$

with solution

$$v_0 = A_0(\tau)\cos t + B_0(\tau)\sin t, \quad v_{0_t} = -A_0(\tau)\sin t + B_0(\tau)\cos t.$$

The initial conditions require

$$A_0(0) = B_0(0) = 0.$$

At $O(\delta)$ the governing equation is

$$\begin{aligned}
v_{1_{tt}} + v_1 &= -2v_{0_{t\tau}} - v_0^3 + \alpha\cos t \\
&= -2[-A_0'\sin t + B_0'\cos t] - [A_0\cos t + B_0\sin t]^3 + \alpha\cos t \\
&= -2[-A_0'\sin t + B_0'\cos t] \\
&\quad -[A_0^3\cos^3 t + 3A_0^2B_0\cos^2 t\sin t + 3A_0B_0^2\cos t\sin^2 t + B_0^3\sin^3 t] + \alpha\cos t \\
&= -2[-A_0'\sin t + B_0'\cos t] \\
&\quad -[A_0^3\cos^3 t + 3A_0^2B_0\{\sin t - \sin^3 t\} + 3A_0B_0^2\{\cos t - \cos^3 t\} + B_0^3\sin^3 t] + \alpha\cos t \\
&= -2[-A_0'\sin t + B_0'\cos t] - \frac{1}{4}A_0(A_0^2 - 3B_0^2)(3\cos t + \cos 3t) - \frac{1}{4}B_0(B_0^2 - 3A_0^2)(3\sin t - \sin 3t) \\
&\quad -3A_0^2B_0\sin t - 3A_0B_0^2\cos t + \alpha\cos t,
\end{aligned}$$

where standard trigonometric identities have been employed. Avoidance of secularity requires the vanishing of the coefficients of $\sin t$ and $\cos t$ on the RHS, thereby leading to the ODE pair

$$\begin{aligned}
A_0' - \frac{3}{8}B_0(A_0^2 + B_0^2) &= 0, \\
B_0' + \frac{3}{8}A_0(A_0^2 + B_0^2) &= \frac{\alpha}{2}.
\end{aligned}$$

The solution to these equations (subject to the null initial conditions above) can only be obtained numerically. The resulting plots of $u(t)$ and of the $u - du/dt$ phase plane are shown in Figure 1 for $\epsilon = 0.05$. They show that the amplitude of the oscillations remains bounded over long times, but the oscillations do not reach a steady state.

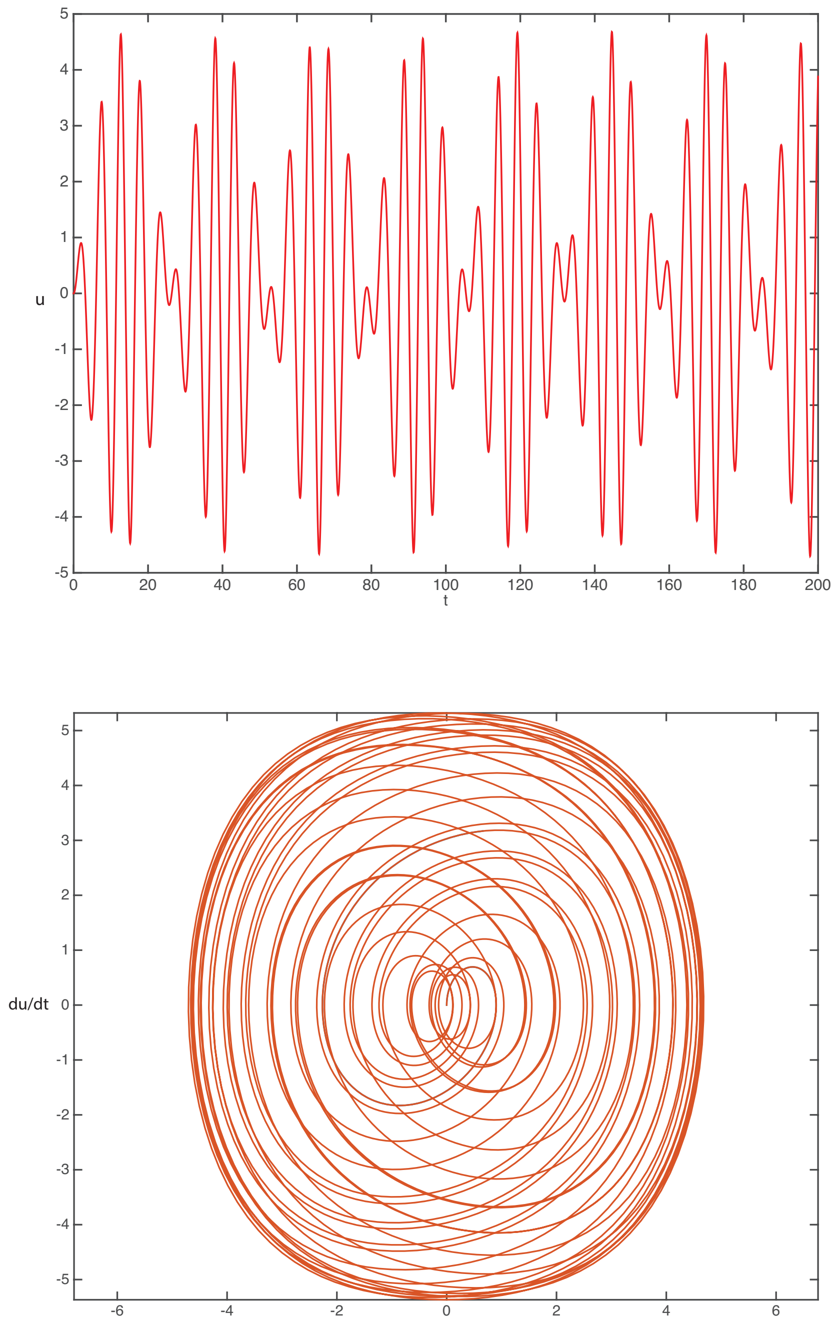

Figure 1: Plot of $u(t)$, top, and of du/dt against u, bottom, for $\epsilon = 0.05$.