MATH-6620 PERTURBATION METHODS SPRING 2016

Homework-6 (OPTIONAL)

Assigned Thursday May 5, 2016

Due Monday May 16, 2016.

NOTES

1. Writing solutions in LaTeX is strongly recommended but not required.
2. Show all work. Illegible or undecipherable solutions will be **returned without grading**.
3. Figures, if any, should be neatly drawn (either by hand or by a drawing program), properly labelled and captioned.
4. Please make sure that the pages are stapled together.
5. The assignment can be submitted in the labelled box in Amos Eaton 301, at my office, or in class.

PROBLEMS

1. Consider the equation

$$y'' + \lambda^2(x^2 - 1)y = 0, \qquad 0 \le x \le 2,$$

as $\lambda \to \infty$ with $y(0) = 0$, $y'(0) = 1$. First obtain asymptotic forms for $y(x, \lambda)$ for x not near the transition point $x = 1$. Then find the appropriate transition solution, and hence a leading-order uniformly valid asymptotic solution to the problem, carefully noting the connection between the three parts of the solution.

According to the WKBJ procedure, the leading-order solution is sought in the form

$$y \sim e^{\lambda\omega(x)}\phi(x). \tag{1.2}$$

Substitution into the ODE leads to

$$\lambda^2 {\omega'}^2\phi + \lambda(\omega''\phi + 2\omega'\phi') + \phi'' = \lambda^2(1 - x^2)\phi.$$

At $O(\lambda^2)$, the balance is

$${\omega'}^2 = (1 - x^2)$$

whence

$$\omega = \pm \int^x \sqrt{1 - u^2}\, du.$$

At $O(\lambda)$,

$$\omega''\phi + 2\omega'\phi' = 0,$$

which integrates to

$$\phi = (1 - x^2)^{-1/4}.$$

This leads to the leading-order solution

$$y \sim (1 - x^2)^{-1/4} \exp\left(\pm\lambda \int^x \sqrt{1 - u^2}\, du\right), \tag{1.3}$$

valid away from the singularity $x = 1$. (The singularity $x = -1$ is not relevant as the domain of interest is $0 < x < 2$.)

For $0 < x < 1$, we can combine the two solutions above to construct the following solution that satisfies the initial data $y(0) = 0$, $y'(0) = 1$.

$$y \sim \frac{1}{\lambda}(1-x^2)^{-1/4}\sinh\{\lambda\Phi(x)\}, \quad \Phi(x) = \int_0^x \sqrt{1-u^2}\,du. \tag{1.4}$$

For $1 < x < 2$, we can write down the general solution as

$$y \sim A(x^2-1)^{-1/4}\sin\{\lambda\Psi(x) + B\}, \quad \Psi(x) = \int_1^x \sqrt{u^2-1}\,du. \tag{1.5}$$

We must now find A and B such that the two solutions written above connect with an inner solution valid near $x = 1$.

In this inner region we set

$$x = 1 + \delta X, \; y(x) = Y(X).$$

where

$$\delta = \left(\frac{1}{2\lambda^2}\right)^{1/3}. \tag{1.6}$$

Then the ODE (1.1) transforms into

$$2\lambda^2\frac{d^2Y}{dX^2} + \lambda^2\{2 + (2\lambda^2)^{-1/3}X\}XY = 0.$$

At leading order it reads

$$Y'' + XY = 0.$$

Its solutions are the Airy functions, $\mathrm{Ai}\,(-X)$ and $\mathrm{Bi}\,(-X)$, and the general solution is a linear combination of these. We anticipate that matching will require discarding the Bi solution, as it grows exponentially and is thus unmatchable. So we take the inner solution to be

$$Y \sim C\mathrm{Ai}(-X), \qquad (1.7).$$

where C, possibly dependent upon λ, is to be determined. As shown in class, this solution has the following asymptotic behavior:

$$\mathrm{Ai}(-X) \sim \begin{cases} \frac{C}{\sqrt{\pi}}|X|^{-1/4}\sin\left(\frac{2}{3}|X|^{3/2} + \frac{\pi}{4}\right) & \text{as } X \to \infty, \\ \\ \frac{C}{2\sqrt{\pi}}|X|^{-1/4}\exp\left(-\frac{2}{3}|X|^{3/2}\right) & \text{as } X \to -\infty. \end{cases} \tag{1.8}$$

We now proceed with matching, by expanding each outer solution in the inner variable. First, consider the solution (1.4). We shall need to expand the argument of the hyperbolic sine. This expansion proceeds by replacing x by $1 + \delta X$, recognizing that $X < 0$ and then manipulating as follows.

$$\begin{aligned} \lambda\Phi(x) = \lambda\int_0^x \sqrt{1-u^2}\,du &= \lambda\int_0^{1+\delta X}\sqrt{1-u^2}\,du \\ &= \lambda\Phi(1) + \lambda\int_1^{1+\delta X}\sqrt{1-u^2}\,du \\ &= \lambda\frac{\pi}{4} - \lambda\int_0^{-\delta X}\sqrt{2v-v^2}\,dv \\ &= \lambda\frac{\pi}{4} - \lambda\int_0^{-\delta X}\sqrt{2v}(1 - v/4 + \cdots)\,dv \\ &= \lambda\frac{\pi}{4} - \lambda\left(\frac{1}{3}(2v)^{3/2} + \cdots\right)_0^{-\delta X} \\ &= \lambda\frac{\pi}{4} - \lambda\frac{1}{3}\left(2\delta|X|)^{3/2} + \cdots\right) \\ &= \lambda\frac{\pi}{4} - \frac{2}{3}|X|^{3/2} + \cdots. \end{aligned}$$

In carrying out the last step above we have simply substituted for δ its definition in terms of λ from (1.6). (It is worth pointing out that the integral for $\Phi(x)$, as defined in (1.4), can be explicitly evaluated as $\Phi(x) = (1/2)[\sin^{-1} x + x\sqrt{1-x^2}]$, from which the expansion computed above can be obtained more directly. It is instructive, however, to expand the integral itself, as the procedure is more general.)

We also need to expand the pre-exponential factor appearing in (1.4). Thus,

$$(1-x^2)^{-1/4} = \big((2+\delta X)(-\delta X)\big)^{-1/4} \sim (2\delta|X|)^{-1/4} = 2^{-1/6}\lambda^{1/6}|X|^{-1/4}.$$

Substituting the last two results in (1.4), and expanding the hyperbolic sine, we obtain

$$\begin{aligned} y &\sim \lambda^{-5/6} 2^{-7/6} |X|^{-1/4} \left\{ \exp\left(\lambda\frac{\pi}{4} - \frac{2}{3}|X|^{3/2}\right) - \exp\left(-\lambda\frac{\pi}{4} + \frac{2}{3}|X|^{3/2}\right) \right\} \\ &\sim \lambda^{-5/6} 2^{-7/6} |X|^{-1/4} \exp\left(\lambda\frac{\pi}{4} - \frac{2}{3}|X|^{3/2}\right), \end{aligned}$$

where we have ignored the exponentially small term in the expansion. Matching with the second result in (1.8) then determines C as

$$C = 2^{-1/6}\sqrt{\pi}\lambda^{-5/6}e^{\lambda\pi/4}.$$

We now turn to (1.5), set $x = 1 + \delta X$, $X > 0$, and expand the various terms appearing there as follows.

$$\begin{aligned} \lambda\Psi(x) = \lambda\int_1^x \sqrt{u^2-1}\,du &= \lambda\int_1^{1+\delta X} \sqrt{u^2-1}\,du \\ &\sim \lambda\int_0^{\delta X} \sqrt{2v}\,dv \\ &= \lambda\frac{1}{3}(2\delta X)^{3/2}\,dv \\ &= \frac{2}{3}X^{3/2}. \end{aligned}$$

Also,

$$(x^2-1)^{-1/4} = \big((2+\delta X)(\delta X)\big)^{-1/4} \sim (2\delta X)^{-1/4} = 2^{-1/6}\lambda^{1/6}X^{-1/4}.$$

In view of these results, (1.5) expands as

$$y \sim A\, 2^{-1/6}\lambda^{1/6}X^{-1/4}\sin\left(\frac{2}{3}X^{3/2} + B\right).$$

Matching with the first result in (1.8) then determines the following:

$$A = 2^{1/6}\lambda^{-1/6}C/\sqrt{\pi}, \quad B = \pi/4,$$

so that the solution (1.5) becomes

$$y \sim \frac{1}{\lambda}(x^2-1)^{-1/4}e^{\lambda\pi/4}\sin\{\lambda\Psi(x) + \pi/4\}.$$

The results can now be collected as follows.

$$y \sim \begin{cases} \frac{1}{\lambda}(1-x^2)^{-1/4}\sinh\{\lambda\Phi(x)\}, & 0 \le x < 1, \\ 2^{-1/6}\sqrt{\pi}\lambda^{-5/6}e^{\lambda\pi/4}\mathrm{Ai}\{2^{1/3}\lambda^{2/3}(1-x)\}, & x \approx 1, \\ \frac{1}{\lambda}(x^2-1)^{-1/4}e^{\lambda\pi/4}\sin\{\lambda\Psi(x) + \pi/4\}, & 1 < x < 2. \end{cases}$$

2. The Schrödinger equation describing the quantum mechanics of a particle in a potential field V has the form

$$y''(x) + [E - V(x)]y = 0, \quad y(\pm\infty) = 0.$$

Take $V(x) = x^4$. Then $x = \pm E^{1/4}$ are the two turning points. Find an appropriate expression for the eigenvalues (energy levels) E_n as $n \to \infty$, for which a nontrivial solution exists.

Hint: Use $x = E^{1/4}\xi$ to convert the problem into one to which the WKBJ method can be applied in the limit $E \to \infty$.

With $x = E^{1/4}\xi$ the problem becomes

$$\frac{d^2y}{d\xi^2} = E^{3/2}(\xi^4 - 1)y, \quad y(\pm\infty) = 0. \tag{2.1}$$

The turning points are $A(\xi = -1)$ and $B(\xi = 1)$. We use the WKBJ procedure to examine the solution of (2.1) asymptotically as $E \to \infty$, making sure that the solution passes smoothly through the turning points.

With the work done in class in mind, and considering the turning point B, a solution that decays as $\xi \to \infty$ can be written as

$$y \sim \frac{1}{(\xi^4 - 1)^{1/4}} \exp\left(-\int_1^\xi E^{3/4}(x^4 - 1)^{1/2} dx\right), \quad \xi > 1. \tag{2.2}$$

There behavior of this solution in the interval $\xi \in (-1, 1)$ (see class notes again) is

$$y \sim \frac{2}{(1 - \xi^4)^{1/4}} \sin\left(-\int_1^\xi E^{3/4}(1 - x^4)^{1/2} dx + \frac{\pi}{4}\right), \quad -1 < \xi < 1. \tag{2.3}$$

According to the turning point A, if this solution is to decay as $\xi \to -\infty$, it must have the form

$$y \sim \frac{K}{(\xi^4 - 1)^{1/4}} \exp\left(-\int_1^\xi E^{3/4}(x^4 - 1)^{1/2} dx\right), \quad \xi < -1, \tag{2.4}$$

and in $(-1, 1)$, the form

$$y \sim \frac{2K}{(1 - \xi^4)^{1/4}} \sin\left(\int_{-1}^\xi E^{3/4}(1 - x^4)^{1/2} dx + \frac{\pi}{4}\right), \quad -1 < \xi < 1. \tag{2.5}$$

For (2.3) and (2.5) to agree we must have, with the shorthand notation $\phi(x) \equiv E^{3/4}(1 - x^4)^{1/2}$,

$$\begin{aligned}
\sin\left(-\int_1^\xi \phi(x)\, dx + \frac{\pi}{4}\right) &= 2K \sin\left(\int_{-1}^\xi \phi(x)\, dx + \frac{\pi}{4}\right) \\
&= 2K \sin\left[\int_{-1}^1 \phi(x)\, dx + \frac{\pi}{2} + \left\{\int_1^\xi \phi(x)\, dx - \frac{\pi}{4}\right\}\right].
\end{aligned}$$

With

$$\alpha \equiv \int_{-1}^1 \phi(x)\, dx + \frac{\pi}{2} \tag{2.6}$$

the above expression becomes

$$\sin\left(-\int_1^\xi \phi(x)\, dx + \frac{\pi}{4}\right) = 2K \sin\alpha \cos\left(\int_1^\xi \phi(x)\, dx - \frac{\pi}{4}\right) + 2K \cos\alpha \sin\left(\int_1^\xi \phi(x)\, dx - \frac{\pi}{4}\right), \tag{2.7}$$

which can only be true if $\sin\alpha = 0$, *i.e.*, if

$$\alpha \equiv \int_{-1}^{1} \phi(x)\, dx + \frac{\pi}{2} = n\pi, \quad n = 1, 2, \cdots . \tag{2.8}$$

Further, (2.7) implies the equality $2K \cos n\pi = -1$, *i.e.*,

$$K = \frac{1}{2}(-1)^{n+1}.$$

On substituting for ϕ in (2.8) we get the final result

$$E_n \sim \left(\frac{(n - 1/2)\pi}{\int_{-1}^{1} (1 - x^4)^{1/2}\, dx} \right)^{4/3}, \quad \text{as } n \to \infty.$$

3. Consider the homogeneous ODE

$$y''(x) - \frac{x}{(x+1)^4} y(x) = 0.$$

Find the first three terms of the asymptotic expansion of each of the two linearly independent solutions for large x.

With $y = e^{\phi}$, the ODE transforms into

$$\phi'' + \phi'^2 - \frac{x}{(x+1)^4} = 0.$$

For $x \to \infty$,

$$\phi'' + \phi'^2 - \frac{1}{x^3}\left(1 - \frac{4}{x} + \cdots\right) = 0. \tag{3.1}$$

Solution 1

Since $\alpha = -3 < -2$, one solution corresponds to the dominant balance

$$\phi'' \sim \frac{1}{x^3}$$

whence

$$\phi \sim \frac{1}{2x}.$$

For higher-order terms let

$$\phi = \frac{1}{2x} + \phi_1, \quad \phi_1 = o(1/x) \text{ as } x \to \infty.$$

Then (3.1) leads to

$$\left(\frac{1}{x^3} + \phi_1''\right) + \left(-\frac{1}{2x^2} + \phi_1'\right) - \frac{1}{x^3} + \frac{4}{x^4} + O(x^{-5}) = 0,$$

or,

$$\phi_1'' + \frac{1}{4x^4} - \frac{1}{x^2}\phi_1' + \phi_1'^2 + \frac{4}{x^4} + O(x^{-5}) = 0.$$

Now the dominant balance is

$$\phi_1'' \sim -\frac{17}{4x^4},$$

whence

$$\phi_1 \sim -\frac{17}{24x^2}.$$

Therefore,

$$\phi \sim \frac{1}{2x} - \frac{17}{24x^2},$$

so that the first solution has the expansion

$$y = e^{\phi} \sim \exp\left(\frac{1}{2x} - \frac{17}{24x^2}\right) \sim 1 + \frac{1}{2x} - \frac{7}{12x^2} + \cdots \quad \text{as} \quad x \to \infty. \tag{3.2}$$

Solution 2

A second solution is found from (4.1) by using the dominant balance $\phi'' \sim -\phi'^2$, leading to $\phi \sim \ln x$. For higher-order terms let

$$\phi = \ln x + \phi_1, \quad \phi_1 = o(\ln x) \quad \text{as} \quad x \to \infty.$$

Then (3.1) leads to

$$-\frac{1}{x^2} + \phi_1'' + \left(\frac{1}{x} + \phi_1'\right)^2 - \frac{1}{x^3} + \frac{4}{x^4} + O(x^{-5}) = 0,$$

or,

$$\phi_1'' + \frac{2}{x}\phi_1' + \phi_1'^2 - \frac{1}{x^3} + \frac{4}{x^4} + O(x^{-5}) = 0.$$

Now the dominant balance is

$$\phi_1'' + \frac{2}{x}\phi_1' \sim \frac{1}{x^3},$$

whose general solution is

$$\phi_1 \sim -\frac{1}{x}\ln x + \frac{B_1}{x} + A_1.$$

There is no loss of generality in taking $A_1 = 0$. Then the second solution is

$$y = e^{\phi} \sim \exp\left(\ln x - \frac{1}{x}\ln x + \frac{B_1}{x} + \cdots\right) \sim x - \ln x + B_1 + \cdots. \tag{3.3}$$

The general solution is a linear combination of (3.2) and (3.3). The coefficient B_1 in (3.3) can also be set to zero as it merely repeats the solution (3.2).