MATH-6620 PERTURBATION METHODS SPRING 2016

Homework-3

Assigned Wednesday February 24, 2016

Due Monday March 7, 2016

NOTES

1. Writing solutions in LaTeX is strongly recommended but not required.
2. Show all work. Illegible or undecipherable solutions will be **returned without grading**.
3. Figures, if any, should be neatly drawn (either by hand or by a drawing program), properly labelled and captioned.
4. Please make sure that the pages are stapled together.
5. The assignment can be submitted in the labelled box in Amos Eaton 301, at my office, or in class.

PROBLEMS

1. Consider the initial-value problem
$$\epsilon \frac{dy}{dt} = ty, \quad y(-1) = 1.$$
   (a) Find the exact solution and discuss its qualitative character. Plot it on the interval $t \in [-1, 1]$ for $\epsilon = 0.25$. Construct a leading-order asymptotic solution for $\epsilon > 0$ and small, and discuss whether it is able to capture all significant features of the exact solution.
   (b) Repeat part (a) for the slightly altered differential equation
$$\epsilon \frac{dy}{dt} = ty + \epsilon, \quad y(-1) = 1.$$
   Discuss what you find. Any surprises?
2. Consider the initial-value problem
$$\epsilon \frac{dy}{dt} + ty = te^{-t}, \quad y(0) = 2.$$
   For $\epsilon > 0$ and small, find the leading-order composite solution.
3. Consider the initial-value problem for the system of equations
$$\begin{aligned} \frac{dx}{dt} &= xy, \\ \epsilon \frac{dy}{dt} &= y - y^3, \end{aligned}$$
   with initial conditions
$$x(0) = \alpha, \quad y(0) = \beta.$$
   (a) For $\epsilon > 0$ and small, seek an outer solution of the form $x(t; \epsilon) \sim x_0(t)$, $y(t; \epsilon) \sim y_0(t)$. Consider all possibilities, and note that the initial conditions will not be met, in general.
   (b) Consider an initial layer by using the stretching $t = \delta(\epsilon)\tau$, $x(t; \epsilon) = X(\tau; \epsilon)$, $y(t; \epsilon) = Y(\tau; \epsilon)$, where δ is to be found by a suitable argument. Seek an inner solution of the form $X(\tau; \epsilon) \sim X_0(\tau)$, $Y(t; \epsilon) \sim Y_0(\tau)$. Construct the inner solution for (i) $\beta > 0$, (ii) $\beta < 0$ and (iii) $\beta = 0$. For each case determine the leading-order composite solution.

4. Consider the problem

$$(x^2 + \epsilon y)y' + 2xy = \frac{3\epsilon}{2y}, \quad y(1;\epsilon) = 1.$$

Of interest is the domain $0 \le x \le 1$. Find the first two terms of an outer expansion, for $\epsilon > 0$ and small, satisfying the boundary condition at $x = 1$. Show that this expansion is not uniformly valid as $x \to 0$. Find the breakdown, rescale and hence find the first term of the inner expansion valid near $x = 0$. Find the dominant asymptotic behavior of y as $x \to 0$.