MATH-6620 PERTURBATION METHODS SPRING 2016

Homework-4

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PROBLEMS

1. Consider the signaling problem

$$\epsilon(u_{xx} - u_{tt}) = u_t + 2u_x, \quad 0 < x < \pi,\ t > 0,$$

with auxiliary conditions

$$u(x,0) = u_t(x,0) = 0, \ \ u(0,t) = -\sin t,\ u(\pi,t) = 0.$$

Construct a leading-order solution for $0 < \epsilon \ll 1$, paying due attention to the location of any layers.

Solution:

We begin by finding the outer solution of the PDE by letting $u(x,t;\epsilon) \sim u_0(x,t)$ and collecting only the $O(1)$ terms. This gives the linear transport equation

$$u_{0t} + 2u_{0x} = 0.$$

This equation has characteristics

$$\gamma = x - 2t$$

which implies the solution to the PDE is then

$$u_0 = A(\gamma) = A(x - 2t).$$

We now apply the auxiliary conditions to find a reasonable outer solution. Clearly, if we apply the boundary conditions at $t = 0$, we get the zero solution. Similarly, we get the zero solution if we apply the condition at $x = \pi$. However, if we apply the boundary condition at $x = 0$ we get an interesting solution.

$$u_0(0,t) = A(-2t) = -\sin t \implies u_0(x,t) = \sin(\frac{x-2t}{2}).$$

Thus if we let u_0 be a piecewise continuous function defined

$$u_0(x,t) = \begin{cases} 0, & x \geq 2t \\ \sin(\frac{x-2t}{2}), & x < 2t \end{cases}$$

we satisfy every boundary condition but the one at $x = \pi$ and remove any possibility of a leading-order layer along $x = 2t$. However, ince the line $x = \pi$ is transverse to the subcharacteristics of the equation, we do find that there is an ϵ-thick layer at $x = \pi, t > \pi/2$. Note we are only concerned with a layer at $t > \pi/2$ for the boundary condition at $x = \pi$ is only dissatisfied on the right side of the line $x = 2t$. Thus we let

$$x = \pi + \epsilon\xi, \ \ t = T + \frac{\pi}{2},\ T > 0, \ \ u(x,t) = U(\xi, T).$$

Then the original signalling problem becomes

$$U_{0\xi\xi} = 2U_{0\xi}, \quad U(0,T) = 0$$

in the layer. The inner solution is then

$$U_0(\xi, T) = \frac{1}{2}B_0(T)(e^{2\xi} - 1).$$

We then match to the outer solution in this region. We let $\xi \to \infty$ in U_0 to find

$$U_0(\xi, T) \to -\frac{1}{2}B_0(T),$$

and let $x \to \pi$ in u_0 to find

$$u_0(x,t) \to \sin(\frac{\pi - 2t}{2}).$$

Thus we express the outer solution in the inner variable and we find that

$$u_0 \to -\sin(T) = -\frac{1}{2}B_0(T) \implies B_0(T) = 2\sin(T).$$

Then the inner solution is

$$U_0(x,t) = \cos(t)\left(1 - e^{2(x-\pi)/\epsilon}\right).$$

To then write the composite solution, we first note that the common part found above

$$-\sin(T) = \cos(t).$$

Now we simply add the inner and outer solutions together and subtract off a $\cos(t)$ to find

$$u_C(c,t) = \left\{ \begin{array}{cc} 0, & x \geq 2t \\ \sin\left(\frac{x-2t}{2}\right) - \cos(t)e^{2(x-\pi)/\epsilon}, & x < 2t \end{array} \right. .$$

2. Consider the elliptic problem

$$\epsilon(u_{xx} + u_{yy}) + u_x + u_y + u = 0, \quad 0 < x < 1, \ 0 < y < 1,$$

with boundary conditions

$$u(x,0) = 0, \ u(x,1) = 1 - x, \ u(0,y) = e^{-y}, \ u(1,y) = 1 - y.$$

(a) Construct a leading-order solution for $0 < \epsilon \ll 1$, paying due attention to the location of the layers.

(b) Repeat the problem if the second boundary condition above is changed to $u(x,1) = 1$.

Solution:

(a) We first determine the subcharacteristic of the PDE by looking at the outer $O(1)$ equation

$$u_x + u_y + u = 0 \implies y = x + \gamma \implies u = A(\gamma)e^{-x} = A(y-x)e^{-x}.$$

We then satisfy the boundary conditions on the right and upper boundaries to get the outer solution

$$u_0(x,t) = \left\{ \begin{array}{ll} (y-x)e^{1-y}, & x < y \\ (x-y)e^{1-x}, & y < x \end{array} \right.$$

We choose to satisfy these boundary conditions (and therefore get backward flowing characteristics) because this outer solution is continuous along $x = y$ suggesting that there is no leading-order layer at $x = y$. This outer solution does lead us to beleive that there are $O(\epsilon)$ thick layers at $x = 0$ and $y = 0$. Thus we derive and solve the inner PDEs:

i. We let $x = \epsilon\xi$, and $u(x, y; \epsilon) = U(\xi, y; \epsilon)$. Collecting the $O(1)$ terms and letting $U(\xi, y; \epsilon) \sim U_0(\xi, y)$ gives the PDE

$$U_{0\xi\xi} = -U_{0\xi},$$

with solution

$$U_0(\xi, y) = -A_0(y)e^{-\xi} + B_0(y).$$

Since we must satisfy $u(0, y) = e^{-y}$ we have

$$U_0(0, y) = -A_0(y) + B_0(y) = e^{-y} \implies B_0 = A_0(y) + e^{-y}$$

giving us the inner solution

$$U_0(\xi, y) = A_0(y)(1 - e^{-\xi}) + e^{-y}.$$

To determine $A_0(y)$ we must match the inner and outer solutions in the layer. We let $\xi \to \infty$ and $x \to 0$ in the inner and outer solutions respectively to find

$$U_0(\xi, y) \to A_0(y) + e^{-y} \quad u_0(x, y) \to ye^{1-y}.$$

Hence we determine that

$$A_0(y) = ye^{1-y} - e^{-y} \implies U_0(x, y) = ye^{1-y} - ye^{1-y-x/\epsilon} + e^{-y-x/\epsilon}.$$

We now form the left composite solution:

$$u_{CL} = (y - x)e^{1-y} - ye^{1-y-x/\epsilon} + e^{-y-x/\epsilon}.$$

ii. Now we determine the equation in the layer at $y = 0$. We let $y = \epsilon\eta$, and $u(x, y; \epsilon) = v(x, \eta; \epsilon)$ and find that the leading order equation in the layer is

$$v_{0\eta\eta} = -v_{0\eta}, \quad v_0(x, 0) = 0.$$

This equation has solution

$$v_0(x, \eta) = C_0(x)(1 - e^{-\eta}).$$

We then match to determine $C_0(x)$. We let $\eta \to \infty$ in the inner solution and $y \to 0$ in the outer solution to find

$$v_0 \to C_0(x), \quad u_0 \to xe^{1-x}.$$

Thus we have

$$C_0(x) = xe^{1-x}.$$

The inner solution can then be written

$$v_0(x, y) = xe^{1-x}(1 - e^{-y/\epsilon}).$$

We then form the composite solution on the right of the center characteristic line

$$u_{CR} = (x - y)e^{1-x} - xe^{1-x-y/\epsilon}.$$

Note that we do see a transcendentally small error at $u(1, y)$ with this composite. The full composite can be written

$$u_c = \begin{cases} (y - x)e^{1-y} - ye^{1-y-x/\epsilon} + e^{-y-x/\epsilon}, & x < y \\ (x - y)e^{1-x} - xe^{1-x-y/\epsilon}, & y < x \end{cases}.$$

(b) The problem changes considerably when we replace the boundary condition as stated above. We no longer have a piecewise continuous outer solution, which therefore implies we have a boundary layer on the characteristic line $x = y$. In determining the outer solution, we satisfy the same boundary conditions as before, except the new boundary condition results in the solution

$$u(x,y) = \begin{cases} e^{1-y}, & x < y \\ (x-y)e^{1-}, & x > y \end{cases}.$$

Thus we only satisfy the boundary conditions along the top of the domain and the right of the domain. We can see along the line $x = y$ this outer solution is not continuous. Thus we now have a third layer. We will begin by finding the inner solutions along the left and bottom sides of the domain.

i. First we let $x = \epsilon\xi$ and $u(x, y; \epsilon) = U(\xi, y; \epsilon)$ then collect the resulting $O(1)$ terms in the PDE given by the transformed variables to get

$$U_{0\xi\xi} = -U_{0\xi}, \;\; U_0(0,y) = e^{-y}.$$

This equation has the solution

$$U_0(\xi, y) = A_0(y)(1 - e^{-\xi}) + e^{-y}.$$

We then match the inner and outer solutions as $\epsilon \to 0$ to find that

$$e^{1-y} = A_0(y) + e^{-y} \implies A_0(y) = e^{1-y} - e^{-y}.$$

Thus we know the inner solution is

$$U_0(x,y) = e^{1-y} - e^{1-y-x/\epsilon} + e^{-y-x/\epsilon}.$$

The inner solution is also the composite solution on the left:

$$u_{cl} = e^{1-y} + e^{1-y} - e^{1-y-x/\epsilon} + e^{-y-x/\epsilon} - e^{1-y} = e^{1-y} - e^{1-y-x/\epsilon} + e^{-y-x/\epsilon}.$$

The solution near the layer on the bottom of the domain is the same as before

$$u_{cr} = (x-y)e^{1-x} - xe^{1-x-y/\epsilon}.$$

ii. Now we inspect the layer along $x = y$. As the layer is parallel to the characteristics, we know it has thickness $O(\sqrt{\epsilon})$. Thus we let $x = y + \sqrt{\epsilon}\eta$, and $u(x, y; \epsilon) = v(\eta, y; \epsilon)$ and we get the PDE

$$2v_{\eta\eta} - 2\sqrt{\epsilon}v_{\eta y} + \epsilon v_{yy} + v_y + v = 0.$$

Then if we let $v(\eta, y; \epsilon) \sim v_0(\eta, y)$ we find the $O(1)$ PDE in the layer behaves like a reverse diffusion with source:

$$2v_{0\eta\eta} + v_{0y} + v_0 = 0.$$

We can use the tranformation $v_0(\eta, y) = e^{1-y}w_0(\eta, y)$ to get a simpler equation

$$w_{0\eta\eta} = \frac{-1}{2}w_{0y}.$$

We solve this equation by letting

$$w = f(\gamma) = f\left(\frac{-i\eta}{\sqrt{y}}\right).$$

This results in the differential equation for γ

$$f''(\gamma) = -\frac{\gamma}{4}f'(\gamma).$$

This equation has solution

$$f\left(\frac{-i\eta}{\sqrt{y}}\right) = \sqrt{2\pi}c_1 erf\left(\frac{-i\eta}{2\sqrt{2y}}\right) + c_2 = w_0(\eta, y).$$

$$\implies v_0(\eta, y) = e^{1-y}\sqrt{2\pi}c_1 erf\left(\frac{-i\eta}{2\sqrt{2y}}\right) + c_2 e^{1-y}.$$

We then write the inner solution in terms of the outer variable to find

$$v_0(x, y) = e^{1-y}\sqrt{2\pi}c_1 erf\left(\frac{i(y-x)}{2\sqrt{2y\epsilon}}\right) + c_2 e^{1-y}.$$

Now if we let $\epsilon \to 0$ we get

$$v_0 \to \begin{cases} c_1\infty + c_2 e^{1-y}, & x < y \\ -c_1\infty + c_2 e^{1-y}, & \end{cases}$$

Now we rewrite the leading order outer solution in terms of the inner variable:

$$u_0(\eta, y) = \begin{cases} e^{1-y}, & x < y \\ \sqrt{\epsilon}\eta e^{1-y-\sqrt{\epsilon}\eta}, & x > y \end{cases}$$

then we take the limit as $\epsilon \to 0$ to find

$$u_0 \to \begin{cases} e^{1-y}, & x < y \\ 0 & x > y \end{cases}.$$

Then clearly matching fails in this case, unless we take the trivial inner solution

$$v_0 = e^{1-y}.$$

Instead, we must find some other way to match. The only method I could find that works is to take the modulus of the argument of the error function located in the inner solution. Perhaps there was a mistake in our derivation of the PDEs in the inner layer or their solutions. The other possibility is that the backwards diffusion equation's initial conditions require the solution be a real error function instead of an imaginary error function. Either way, we proceed by adjusting our inner solution to

$$v_0 = e^{1-y}\sqrt{2\pi}c_1 erf\left(\frac{(y-x)}{2\sqrt{2y\epsilon}}\right) + c_2 e^{1-y}.$$

Then when we limit $\epsilon \to 0$ we get

$$v_0 \sim \begin{cases} e^{1-y}(\sqrt{2\pi}c_1 + c_2), & x < y \\ e^{1-y}(c_2 - \sqrt{2\pi}c_1), & x > y \end{cases}.$$

Then matching gives us $c_1 = 1/(2\sqrt{2\pi})$ and $c_2 = 1/2$. Thus we have the matched inner solution

$$v_0(x, y)\frac{1}{2}e^{1-y}\left(erf\left(\frac{y-x}{2\sqrt{2\epsilon y}}\right) + 1\right).$$

Now we form the composite solution:

$$u_c = \begin{cases} -e^{1-y-x/\epsilon} + e^{-y-x/\epsilon} + \frac{1}{2}\left(erf\left(\frac{y-x}{2\sqrt{2\epsilon y}}\right) + 1\right), & x < y \\ (x-y)e^{1-x} - xe^{1-x-y/\epsilon} + \frac{1}{2}e^{1-y}\left(erf\left(\frac{y-x}{2\sqrt{2\epsilon y}}\right) + 1\right), & x > y \end{cases}.$$

3. In class we had examined heat transfer on a flat plate in a uniform stream. Now consider heat transfer from a cylinder of radius unity and center at the origin, placed in an otherwise uniform stream. The flow velocity is given by $\boldsymbol{u} = \nabla\phi$, where the potential ϕ is given in polar coordinates as

$$\phi = \left(r + \frac{1}{r}\right)\cos\theta.$$

The temperature T satisfies the PDE

$$\boldsymbol{u}\cdot\nabla T = \epsilon\nabla^2 T, \quad r \geq 1,$$

with boundary conditions

$$T = 1 \text{ on } r = 1, \quad T \to 0 \text{ as } r \to \infty.$$

Find the leading-order solution for $0 < \epsilon \ll 1$. Sketch a graph of the isotherms (lines of constant T). Also, find an expression for $\partial T/\partial r$, the heat flux, from the cylinder surface.

Hint: Look for a similarity solution of the PDE for the inner problem.

Solution:

We first derive the PDE governing the heat transference:

$$\nabla\phi\cdot\nabla T = (1 - \frac{1}{r^2})\cos\theta T_r - (\frac{1}{r} + \frac{1}{r^3})\sin\theta T_\theta$$

$$\nabla^2 T = T_{rr} + \frac{1}{r}^2 T_{\theta\theta} + \frac{1}{r}T_r$$

$$\implies (1 - \frac{1}{r^2})\cos\theta T_r - (\frac{1}{r} + \frac{1}{r^3})\sin\theta T_\theta = \epsilon(T_{rr} + \frac{1}{r}^2 T_{\theta\theta} + \frac{1}{r}T_r).$$

Now we look at the $O(1)$ outer problem by letting $T(r,\theta;\epsilon) \sim T_0(r,\theta)$ and collecting the appropriate terms:

$$(1 - \frac{1}{r^2})\cos\theta T_{0r} - (\frac{1}{r} + \frac{1}{r^3})\sin\theta T_{0\theta} = 0.$$

This equation has the solution

$$T_0(r,\theta) = f\left(\log\left[\frac{1-r^2}{r}\sin\theta\right]\right).$$

The conditions on T imply that

$$T_0(\infty,\theta) = f(\infty) = 0.$$

We note here that we want the outer solution to satisfy the condition at ∞ because having a layer at infinity for gradual heat dissipation would be very strange. We then find that we have a boundary layer that is $O(\sqrt{\epsilon})$ thick along the perimeter of the cylinder in the stream (as the characteristics run right around the cylinder). Thus we let $r = 1 + \sqrt{\epsilon}\rho$ and $T(r,\theta;\epsilon) = \tau(\rho,\theta;\epsilon)$ and we find the leading order problem in the layer

$$2(\rho\cos\theta\tau_{0\rho} - \sin\theta\tau_{0\theta}) = \tau_{0\rho\rho}$$

where $\tau(\rho,\theta;\epsilon) \sim \tau_0(\rho,\theta)$. To solve this PDE we let

$$\tau_0(\rho,\theta) = f(\eta) = f\left(\frac{\rho}{\sqrt{g(\theta)}}\right).$$

This results in the eigenvalue problem

$$2\cos\theta g(\theta) + \sin\theta g'(\theta) = \frac{f''(\eta)}{\eta f'(\eta)} = \lambda,$$

thereby giving us two equations:

$$g' + 2\cot\theta g = \lambda\csc\theta,$$

and

$$f'' = \lambda\eta f'.$$

The first equation has solution

$$g = c_1\csc^2\theta - \lambda\cot\theta\csc\theta.$$

while the second equation has the solution

$$f = k\sqrt{\frac{\pi}{2|\lambda|}}erf\left(\frac{\sqrt{|\lambda|}\eta}{\sqrt{2}}\right) + c_2.$$

We note that this solution results from choosing $\lambda < 0$ to aid in matching later on. We then let $\lambda = -1$ for ease of computation and $c_1 = 0$ as we are not interested in the homogeneous solution to the $g(\theta)$ equation. This gives the inner solution

$$\tau_0(\rho,\theta) = k\sqrt{\frac{\pi}{2}}erf\left(\frac{\rho\sin\theta}{\sqrt{2\cos\theta}}\right) + c_2.$$

We then apply the boundary condition at $r = 1$ to find

$$\tau_0(0,\theta) = c_2 = 1.$$

Hence the inner solution is

$$\tau_0(\rho,\theta) = k\sqrt{\frac{\pi}{2}}erf\left(\frac{\rho\sin\theta}{\sqrt{2\cos\theta}}\right) + 1.$$

To perform the matching, we let $\rho \to \infty$ in the inner solution and $r \to 1$ in the inner solution and find

$$\tau_0 \to k\sqrt{\frac{\pi}{2}} + 1,$$

$$T_0 \to f(-\infty).$$

Therefore we let

$$k = \sqrt{\frac{2}{\pi}}(f(-\infty) - 1).$$

The inner solution is then (in terms of the outer variable)

$$\tau_0(r,\theta) = (f(-\infty) - 1)erf\left(\frac{(r-1)\sin\theta}{\sqrt{2\epsilon\cos\theta}}\right) + 1.$$

The composite solution is then

$$T_c = -f(-\infty) + (f(-\infty) - 1)erf\left(\frac{(r-1)\sin\theta}{\sqrt{2\epsilon\cos\theta}}\right) + 1 + f\left(\log\left[\frac{1-r^2}{r}\sin\theta\right]\right),$$

with the condition that $f(\infty) = 0$. Now to find $\partial T/\partial r$ we simply differentiate the composite solution to find

$$\frac{\partial T}{\partial r} = -\sqrt{\frac{2}{\pi}}\frac{\sin\theta\exp\left[\frac{-(r-1)^2\sin^2\theta}{2\epsilon\cos\theta}\right]}{\sqrt{\epsilon\cos\theta}} + f'\left(\log\left[\frac{1-r^2}{r}\sin\theta\right]\right)\frac{r^2+1}{r^3-r}.$$

I do not understand how to draw the graph of the isotherms, so that part is not included here.