MATH-6620 PERTURBATION METHODS SPRING 2016

Homework-2

Assigned Tuesday February 9, 2016

Due Friday February 19, 2016

NOTES

1. Writing solutions in LaTeX is strongly recommended but not required.
2. Show all work. Illegible or undecipherable solutions will be **returned without grading**.
3. Figures, if any, should be neatly drawn (either by hand or by a drawing program), properly labelled and captioned.
4. Please make sure that the pages are stapled together.
5. The assignment can be submitted in the labelled box in Amos Eaton 301, at my office, or in class.

PROBLEMS

1. Consider the function

$$f(x;\epsilon) = \frac{1+\epsilon x+\sqrt{x+\epsilon}}{1+\sqrt{x+\epsilon e^{-x/\epsilon}}}, \quad x \in [0,1].$$

(a) Explain, analytically, why this function has a layer of thickness ϵ at $x=0$.

(b) Compute $F_0(x)+\cdots+\epsilon F_1(x)$, the outer expansion of f to order ϵ. This expansion corresponds to the outer limit process $\epsilon \to 0$, x fixed.

(c) Let $\xi = x/\epsilon$. Compute $G_0(\xi)+\cdots+\epsilon G_1(\xi)$, the inner expansion of f to order ϵ. This expansion corresponds to the inner limit process $\epsilon \to 0$, ξ fixed.

(d) Let $x=\mu\eta$ define an intermediate variable, where $\mu(\epsilon)$ is to be determined. Find the restrictions on μ so that F_0 and G_0 match to order unity (in the intermediate limit), *i.e.,*

$$\lim_{\epsilon\to 0}[F_0(\mu\eta) - G_0(\mu\eta/\epsilon] = 0.$$

(e) Find the restrictions on μ so that $F_0+\cdots+\epsilon F_1$ and $G_0+\cdots+\epsilon G_1$ match to order ϵ (in the intermediate limit), *i.e.,*

$$\lim_{\epsilon\to 0}\frac{[\{F_0(\mu\eta)+\cdots+\epsilon F_1(\mu\eta)\} - \{G_0(\mu\eta/\epsilon)+\cdots+\epsilon G_1(\mu\eta/\epsilon)\}]}{\epsilon} = 0.$$

If the above limit does not exist, then consider relaxing the order of ϵ to which the two expansions match, and explain the consequences and meaning of such a relaxation.

(a) The term $e^{-x/\epsilon}$ in the denominator is transcendentally small in the limit $x=O(1)$, $\epsilon\to 0$. However, it is reinstated as $e^{-\xi}$ in the limit $\xi = O(1)$, $\epsilon \to 0$. Hence a layer of thickness ϵ at $x=0$.

(b) Consider the outer limit $x=O(1)$, $\epsilon\to 0$. Then,

$$\begin{aligned} f(x;\epsilon) &= \frac{1+\epsilon x+\sqrt{x}(1+\epsilon/x)^{1/2}}{1+\sqrt{x}(1+(\epsilon/x)e^{-x/\epsilon})^{1/2}} \\ &= \frac{1+\epsilon x+\sqrt{x}(1+\epsilon/(2x)+\cdots)}{1+\sqrt{x}+TST} \end{aligned}$$

Here, TST stands for Transcendentally Small Term. Continuing with the above expansion,

$$f(x;\epsilon) \sim F_0+\epsilon F_1 = 1+\frac{\epsilon}{1+\sqrt{x}}\left(x+\frac{1}{2\sqrt{x}}\right). \tag{1}$$

(c) With $x = \epsilon\xi$, consider the inner limit $\xi = O(1)$, $\epsilon \to 0$. Then,

$$\begin{aligned} f(\epsilon\xi;\epsilon) &= \frac{1+\epsilon^2\xi+\sqrt{\epsilon}\sqrt{\xi+1}}{1+\sqrt{\epsilon}\sqrt{\xi+e^{-\xi}}} \\ &= [1+\sqrt{\epsilon}\sqrt{\xi+1}+\epsilon^2\xi][1-\sqrt{\epsilon}\sqrt{\xi+e^{-\xi}}+\epsilon(\xi+e^{-\xi})+\cdots] \\ &\sim G_0+\sqrt{\epsilon}G_{1/2}+\epsilon G_1 = 1+\sqrt{\epsilon}\{\sqrt{\xi+1}-\sqrt{\xi+e^{-\xi}}\} \\ &\quad +\epsilon\{\xi+e^{-\xi}-\sqrt{(\xi+1)(\xi+e^{-\xi})}\}. \end{aligned} \tag{2}$$

(d) Equations (1) and (2) show that $F_0 = G_0 = 1$, so matching is automatic.

(e) In the intermediate region let $x = \mu\eta$, so that $\xi = \mu\eta/\epsilon$. We have

$$\begin{aligned} F_0+\epsilon F_1-[G_0+\sqrt{\epsilon}G_{1/2}+\epsilon G_1] &= 1+\frac{\epsilon}{1+\sqrt{x}}\left(x+\frac{1}{2\sqrt{x}}\right) \\ &\quad -\left\{1+\sqrt{\epsilon}\{\sqrt{\xi+1}-\sqrt{\xi+e^{-\xi}}\}\right\} \\ &\quad -\epsilon\{\xi+e^{-\xi}-\sqrt{(\xi+1)(\xi+e^{-\xi})}\} \\ &= \frac{\epsilon}{1+\sqrt{\mu\eta}}\left(\mu\eta+\frac{1}{2\sqrt{\mu\eta}}\right) \\ &\quad -\sqrt{\epsilon}\left\{\sqrt{1+\mu\eta/\epsilon}-\sqrt{\mu\eta/\epsilon+e^{-\mu\eta/\epsilon}}\right\} \\ &\quad -\epsilon\left\{\mu\eta/\epsilon+e^{-\mu\eta/\epsilon}-\sqrt{(1+\mu\eta/\epsilon)(\mu\eta/\epsilon+e^{-\mu\eta/\epsilon})}\right\} \end{aligned}$$

We expand for $\eta > 0$ fixed and $\epsilon \to 0$, keeping in mind that provisionally, $\epsilon \ll \mu \ll 1$. Then the RHS of the above expression expands as

$$\begin{aligned} \text{RHS} &= \frac{\epsilon}{2\sqrt{\mu\eta}}(1+\sqrt{\mu\eta})^{-1}\big(1+2(\mu\eta)^{3/2}\big) \\ &\quad -\sqrt{\mu\eta}\left\{\sqrt{1+\epsilon/\mu\eta}\;-\sqrt{1+(\epsilon/\mu\eta)e^{-\mu\eta/\epsilon}}\right\} \\ &\quad -\mu\eta\left\{1+(\epsilon/\mu\eta)e^{-\mu\eta/\epsilon}-\sqrt{(1+\epsilon/\mu\eta)\big(1+(\epsilon/\mu\eta)e^{-\mu\eta/\epsilon}\big)}\right\} \\ &\sim \frac{\epsilon}{2\sqrt{\mu\eta}}(1-\sqrt{\mu\eta}+O(\mu))-\sqrt{\mu\eta}\left(\frac{\epsilon}{2\mu\eta}+O(\epsilon^2/\mu^2)\right)-\mu\eta\left(-\frac{\epsilon}{2\mu\eta}+O(\epsilon^2/\mu^2)\right) \\ &= O(\epsilon\sqrt{\mu})+O(\epsilon^2/\mu^{3/2})+O(\epsilon^2/\mu). \end{aligned}$$

Therefore,

$$\frac{F_0+\epsilon F_1-[G_0+\sqrt{\epsilon}G_{1/2}+\epsilon G_1]}{\epsilon} = O(\sqrt{\mu})+O(\epsilon/\mu^{3/2})+O(\epsilon/\mu) \to 0 \quad \text{as} \quad \epsilon \to 0$$

provided $\epsilon^{2/3} \ll \mu \ll 1$.

In each of the following problems, anticipate (if possible) the location of the inner region(s). You may use the Van Dyke Principle for matching.

2. For the BVP $\epsilon y'' - y' + \epsilon x^2 y = 2x$, $0 < x < 1$, $y(0;\epsilon) = 2$, $y(1;\epsilon) = 2+\epsilon$, find the first two terms in the outer and the inner solutions, and the composite approximation.

The way in which ϵ appears in the ODE suggests an outer expansion of the form

$$y(x;\epsilon) \sim y_0(x) + \epsilon y_1(x).$$

Then y_0 satisfies

$$-y_0' = 2x, \quad y_0(0) = 2.$$

We have elected to have the outer solution satisfy the left BC as the negative sign of the y' term in the ODE suggests a layer at the right end. The solution is

$$y_0 = 2 - x^2.$$

The problem satisfied by y_1 is

$$y_1' = y_0'' + x^2 y_0 = -2 + 2x^2 - x^4, \quad y_1(0) = 0.$$

The solution is

$$y_1 = -2x + \frac{2}{3}x^3 - \frac{x^5}{5}.$$

Thus the 2-term outer solution is

$$y \sim 2 - x^2 + \epsilon\left(-2x + \frac{2}{3}x^3 - \frac{x^5}{5}\right). \tag{3}$$

This solution does not satisfy the right boundary condition, thus requiring a boundary layer at $x = 1$. Dominant balance shows (provide details) that the layer is $O(\epsilon)$ thick. We set $x = 1 + \epsilon\xi$, $\xi \le 0$, and $y(x;\epsilon) = Y(\xi;\epsilon)$. Then Y satisfies

$$Y'' - Y' - 2\epsilon(1+\epsilon\xi) + \epsilon^2(1+\epsilon\xi)^2 Y = 0, \quad Y(0;\epsilon) = 2 + \epsilon.$$

We seek the inner expansion

$$Y \sim Y_0 + \epsilon Y_1,$$

suggested by the way in which ϵ appears in the ODE. Then the problem for Y_0 is

$$Y_0'' - Y_0' = 0, \quad Y_0(0) = 2,$$

with solution

$$Y_0 = A_0 + (2 - A_0)e^{\xi}.$$

The problem for Y_1 is

$$Y_1'' - Y_1' - 2 = 0, \quad Y_1(0) = 1.$$

The solution is

$$Y_1 = -2\xi + A_1 + (1 - A_1)e^{\xi}.$$

Thus the 2-term inner solution is

$$Y \sim A_0 + (2 - A_0)e^{\xi} + \epsilon\left\{-2\xi + A_1 + (1 - A_1)e^{\xi}\right\}. \tag{4}$$

We match (3) and (4) according to Van Dyke's Principle, with $\delta = \Delta = \epsilon$.

Outer expansion expanded in the inner variable to $O(\epsilon)$:

$$\begin{aligned}
y(x) &\sim 2 - x^2 + \epsilon\left(-2x + \frac{2}{3}x^3 - \frac{x^5}{5}\right) \\
&= 2 - (1+\epsilon\xi)^2 + \epsilon\left(-2(1+\epsilon\xi) + \frac{2}{3}(1+\epsilon\xi)^3 - \frac{(1+\epsilon\xi)^5}{5}\right) \\
&\sim 2 - 1 - 2\epsilon\xi + \epsilon\left(-2 + \frac{2}{3} - \frac{1}{5}\right) \\
&= 1 - \epsilon\left(2\xi + \frac{23}{15}\right) = 3 - 2x - \frac{23}{15}\epsilon.
\end{aligned}$$

Inner expansion expanded in the outer variable to $O(\epsilon)$:

$$\begin{aligned} Y &\sim A_0 + (2 - A_0)e^{\xi} + \epsilon\left\{-2\xi + A_1 + (1 - A_1)e^{\xi}\right\} \\ &= A_0 + (2 - A_0)e^{-(1-x)/\epsilon} + \epsilon\left\{2\frac{1-x}{\epsilon} + A_1 + (1 - A_1)e^{-(1-x)/e}\right\} \\ &\sim A_0 + 2(1 - x) + \epsilon A_1. \end{aligned}$$

Matching leads to $A_0 + 2 = 3$, or $A_0 = 1$ and $A_1 = -23/15$. Then the inner expansion is

$$Y \sim 1 + e^{\xi} + \epsilon\left(-2\xi - \frac{23}{15} + \frac{38}{15}e^{\xi}\right).$$

The composite expansion is the sum of the outer and inner expansions, from which the matched part has been deleted. The result, eventually written in terms of x, is

$$\begin{aligned} y_c(x;\epsilon) &\sim 2 - x^2 + \epsilon\left(-2x + \frac{2}{3}x^3 - \frac{x^5}{5}\right) + 1 + e^{\xi} + \epsilon\left(-2\xi - \frac{23}{15} + \frac{38}{15}e^{\xi}\right) - \left(3 - 2x - \frac{23}{15}\epsilon\right) \\ &= 2 - x^2 + e^{-(1-x)/\epsilon} + \epsilon\left(-2x + \frac{2}{3}x^3 - \frac{x^5}{5} + \frac{38}{15}e^{-(1-x)/\epsilon}\right). \end{aligned}$$

3. For the BVP $\epsilon y'' + x^{1/3}y' + y^2 = 0,\ -1 < x < 1,\ y(-1;\epsilon) = 2/9,\ \ y(1) = 1/3,$ find the leading-order outer and inner solutions, and the composite approximation.

This is a turning-point problem. Since the sign of $x^{1/3}$, the coefficient of the y' term in the ODE, is negative for $x < 0$ and positive for $x > 0$, $x = 0$ is the only possibility for the location of the layer. Let the outer solution be $y \sim y_0(x)$. Then the outer problem at leading order is

$$x^{1/3}y_0' + y_0^2 = 0,$$

with general solution

$$y_0(x) = \frac{2}{3(C_0 + x^{2/3})}.$$

The left boundary condition $y_0(-1) - 2/9$ determines $C_0 = 2$ and the right boundary condition finds $C_0 = 1$. Therefore,

$$y_0(x) = \begin{cases} \frac{2}{3(2+x^{2/3})} & -1 \le x < 0, \\ \frac{2}{3(1+x^{2/3})} & 0 < x \le 1. \end{cases}$$

The solution is discontinuous at $x = 0$, thus necessitating a layer there. In the layer we let $x = \delta\xi$, $y(x;\epsilon) = Y(\xi;\epsilon)$, so that the ODE transforms into

$$\frac{\epsilon}{\delta^2}Y'' + \frac{\delta^{1/3}\xi^{1/3}}{\delta}Y' + Y^2 = 0.$$

Dominant balance requires $\epsilon/\delta = \delta^{1/3}$ si that $\delta = \epsilon^{3/4}$. Then the above ODE becomes

$$Y'' + \xi^{1/3}Y' + \sqrt{\epsilon}Y^2 = 0.$$

Assuming the inner expansion to be $Y(\xi;\epsilon) \sim Y_0(\xi)$, the reduced ODE at leading-order is

$$Y_0'' + \xi^{1/3}Y_0' = 0.$$

The first integral is $Y_0' = A_0 \exp\left(-\frac{3}{4}\xi^{4/3}\right)$, and a second integration yields the general solution

$$Y_0 = B_0 + A_0F_0(\xi),$$

where

$$F_0(\xi) = \int_0^\xi \exp\left(-\frac{3}{4}s^{4/3}\right)\, ds.$$

We note that $F_0(\xi)$ is an odd function, defined for all ξ, and approaches constant values $\pm F_0(\infty)$ as $\xi \to \pm\infty$. We are now ready to match at order unity, using the Van Dyke Principle.

Matching from the left.

$$\begin{aligned}
\text{Inner expansion of outer solution:} \quad y(x;\epsilon) &\sim \frac{2}{3(2 + x^{2/3})} \\
&= \frac{2}{3(2 + (\epsilon^{3/4}\xi)^{2/3})} \\
&\sim \frac{1}{3} \quad \text{as} \quad \epsilon \to 0, \quad \xi < 0 \text{ and fixed.}
\end{aligned}$$

$$\begin{aligned}
\text{Outer expansion of inner solution:} \quad Y(\xi;\epsilon) &\sim B_0 + A_0 F_0(\xi) \\
&= B_0 + A_0 F_0(\epsilon^{-3/4}x) \\
&\sim B_0 - A_0 F_0(\infty) \quad \text{as} \quad \epsilon \to 0, \quad x < 0 \text{ and fixed.}
\end{aligned}$$

Matching from the right.

$$\begin{aligned}
\text{Inner expansion of outer solution:} \quad y(x;\epsilon) &\sim \frac{2}{3(2 + x^{2/3})} \\
&= \frac{2}{3(1 + (\epsilon^{3/4}\xi)^{2/3})} \\
&\sim \frac{2}{3} \quad \text{as} \quad \epsilon \to 0, \quad \xi > 0 \text{ and fixed.}
\end{aligned}$$

$$\begin{aligned}
\text{Outer expansion of inner solution:} \quad Y(\xi;\epsilon) &\sim B_0 + A_0 F_0(\xi) \\
&= B_0 + A_0 F_0(\epsilon^{-3/4}x) \\
&\sim B_0 + A_0 F_0(\infty) \quad \text{as} \quad \epsilon \to 0, \quad x > 0 \text{ and fixed.}
\end{aligned}$$

Matching leads to the conditions

$$\begin{aligned}
B_0 - A_0 F_0(\infty) &= \frac{1}{3}, \\
B_0 + A_0 F_0(\infty) &= \frac{2}{3}.
\end{aligned}$$

Therefore,

$$B_0 = \frac{1}{2}, \qquad A_0 = \frac{1}{6F_0(\infty)}.$$

The inner solution can therefore be written, interns of the variable x, as

$$Y \sim \frac{1}{2} + \frac{F_0(\epsilon^{-3/4}x)}{6F_0(\infty)}.$$

Composite expansions. Consider the composite solution based on the inner and the right outer solution, appropriate for $x > 0$,

$$\begin{aligned}
y_{c_+}(x) &\sim \frac{2}{3(1 + x^{2/3})} + \frac{1}{2} + \frac{F_0(\epsilon^{-3/4}x)}{6F_0(\infty)} - \frac{2}{3} \\
&= \frac{2}{3(1 + x^{2/3})} + \frac{F_0(\epsilon^{-3/4}x)}{6F_0(\infty)} - \frac{1}{6}.
\end{aligned}$$

Similarly, we have the composite solution based on the inner and the left outer solution,

$$\begin{aligned} y_{c_-}(x) &\sim \frac{2}{3(2+x^{2/3})} + \frac{1}{2} + \frac{F_0(\epsilon^{-3/4}x)}{6F_0(\infty)} - \frac{1}{3} \\ &= \frac{2}{3(2+x^{2/3})} + \frac{F_0(\epsilon^{-3/4}x)}{6F_0(\infty)} + \frac{1}{6}, \end{aligned}$$

which is appropriate for $x < 0$.

Remark. Composite expansions involving inner layers should be treated with care. In general, with two outer expansions and one inner expansion, the usual procedure of adding the outer and inner expansions and subtracting out the (now two) common parts simply does not yield an expansion that is valid over the entire domain. (There are exceptions; in some problems a one-sided composite expansion holds over the entire domain, see Problem 4 below for example.) In such cases it is often suggested in the literature that two one-sided composite expansions should be constructed, joined at the layer location. However, this is simply an example of *patching*, there is no guarantee that these expansions would have the required degree of continuity at the point where they are joined.

A recommended procedure is to use the two composite expansions only in the regions away from the layer location.

4. For the BVP $\epsilon y'' + e^x(xy' - y) = x^2$, $-1 < x < 1$, $y(-1) = 1$, $y(1) = -1$, find the leading-order outer and inner solutions, and the composite approximation. How does the situation alter if the ODE is changed to $\epsilon y'' - e^x(xy' - y) = -x^2$ while the boundary conditions remain the same?

We have

$$\epsilon y'' + e^x(xy' - y) = x^2, \quad -1 < x < 1, \quad y(-1) = 1,\ y(1) = -1.$$

Since $a(x) = xe^x < 0$ for $x < 0$ and > 0 for $x > 0$, a layer can only exist at $x = 0$.

Outer solution.

With $y \sim y_0(x)$, the reduced ODE at leading order is

$$xy_0' - y_0 = x^2 e^{-x},$$

so that $y_0 = c_0 x - xe^{-x}$. Application of the boundary conditions yields

$$y_0 = \begin{cases} y_0^- = (e-1)x - xe^{-x} & : \quad x < 0 \\ y_0^+ = (e^{-1}-1)x - xe^{-x} & : \quad x > 0 \end{cases}$$

Note that at $x = 0$, the solution is continuous (and has the value 0), while the slope has a jump. Therefore a corner layer is needed.

Corner layer.

The corner layer is $\sqrt{\epsilon}$ thick (justify), so we set $x = \sqrt{\epsilon}\xi$. and let $y(x) = Y(\xi)$. Then the ODE becomes

$$Y'' + e^{\sqrt{\epsilon}\xi}\,(\xi Y' - Y) = \epsilon\,\xi^2.$$

With $Y \sim \sqrt{\epsilon} Y_0(\xi) + \cdots$ (justify the $\sqrt{\epsilon}$ scaling of the leading term), Y_0 satisfies

$$Y_0'' + \xi Y_0' - Y_0 = 0.$$

One solution is ξ, and a second may be found by reduction of order. The general solution is

$$Y_0 = b_1\xi + c_1 e^{-\xi^2/2} + c_1\sqrt{\frac{\pi}{2}}\,\xi\,\mathrm{erf}(\xi/\sqrt{2}).$$

Matching of the inner solution to $O(\sqrt{\epsilon})$ with the left outer solution to $O(1)$ (provide details) finds

$$b_1 - c_1\sqrt{\frac{\pi}{2}} = e - 2,$$

while a similar matching to the right yields

$$b_1 + c_1\sqrt{\frac{\pi}{2}} = e^{-1} - 2,$$

so that

$$b_1 = \cosh 1 - 2 \quad \text{and} \quad c_1 = -\sqrt{\frac{2}{\pi}}\,\sinh 1.$$

Thus the layer solution is known:

$$Y_0 = (\cosh 1 - 2)\xi - \sinh 1\left(\sqrt{\frac{2}{\pi}}e^{-\xi^2/2} + \xi\,\mathrm{erf}(\xi/\sqrt{2})\right).$$

Consider the composite approximation constructed by adding $\sqrt{\epsilon}Y_0$ and y_0^+ and subtracting the common part, $(e^{-1}-2)x$. The result, when written in terms of x, is

$$y_c = \left(\cosh 1 - 2 - \sinh 1\,\mathrm{erf}(x/\sqrt{2\epsilon})\right)x - \sinh 1\sqrt{\frac{2}{\pi}}\,\sqrt{\epsilon}\,e^{-x^2/2\epsilon} + x(1-e^{-x}),$$

and is found to be uniformly valid across the entire domain (check).

We now consider the altered ODE,

$$\epsilon y'' - e^x(xy' - y) = -x^2, \quad -1 < x < 1, \quad y(-1) = 1,\ y(1) = -1.$$

Since $a(x) = -xe^x > 0$ for $x < 0$ and < 0 for $x > 0$, a layer can exist at $x = \pm 1$ and possibly at $x = 0$.

<u>Outer solution</u>.

As before, $y \sim y_0(x) + \cdots$ leads to

$$xy_0' - y_0 = x^2 e^{-x},$$

so that $y_0 = c_0 x - xe^{-x}$. It is not clear, however, whether either of the boundary conditions will apply.

<u>Internal layer</u>.

Let us consider the possibility of a layer at $x = 0$. It must again be $\sqrt{\epsilon}$ thick, so that in it, $x = \sqrt{\epsilon}\xi$ and $y(x) = Y(\xi)$. With $Y \sim Y_0 + \cdots$, we get

$$Y_0'' - \xi Y_0' + Y_0 = 0.$$

This equation is similar to the one found above for the corner layer, but the signs of the second and third terms are reversed. The general solution is

$$Y_0 = b_1\xi + c_1\left[-e^{\xi^2/2} + \int_0^{\xi} e^{s^2/2}\,ds\right].$$

The (unmatchable) exponential growth of this solution as $\xi \to \pm\infty$ suggests that there cannot be a layer at $x = 0$.

<u>Boundary layers</u>

It is now easy to deduce that there are $O(\epsilon)$ layers at $x = \pm 1$. The solutions in these layers can be shown to be

$$y \sim a_0^- + b_0^- e^{-(x+1)/e\epsilon}$$

in the left layer, and

$$y \sim a_0^+ + b_0^+ e^{-(1-x)e/\epsilon}$$

in the right layer. The boundary conditions at $x = \pm 1$ require

$$a_0^- + b_0^- = 1, \quad a_0^+ + b_0^+ = -1,$$

while the matching requirements yield

$$a_0^- = 1 - c_0 \quad \text{and} \quad a_0^+ = c_0 - 1.$$

These are only four conditions for the five constants a_0^-, b_0^-, a_0^+, b_0^+ and c_0. As we had seen for a similar situation in class, the solution remains undetermined.

5. Find the leading-order outer and inner solutions and the composite approximation to the solution of the boundary-value problem

$$\epsilon y'' - (1 + 3x^2)y - x = 0, \;\; 0 < x < 1, \;\; y(0; \epsilon) = y(1; \epsilon) = 1.$$

In the by-now usual way we seek the outer solution $y \sim y_0(x)$, which satisfies the reduced equation

$$-(1 + 3x^2)y_0 - x = 0.$$

This equation is just an algebraic equation, with solution

$$y_0(x) = -\frac{x}{1 + 3x^2}.$$

It does not satisfy either boundary condition, suggesting a layer at each end.

In the layer at $x = 0$ we set $x = \sqrt{\epsilon}\xi$ (justify) and $y(x; \epsilon) = Y(\xi; \epsilon)$. Then Y satisfies

$$Y'' - (1 + 3\epsilon\xi^2)Y - \sqrt{\epsilon}\xi = 0, \quad Y(0; \epsilon) = 1.$$

With $Y \sim Y_0(\xi)$, Y_0 satisfies

$$Y_0'' - Y_0 = 0, \quad Y_0(0) = 1,$$

with solution

$$Y_0(\xi) = A_0 e^{\xi} + (1 - A_0)e^{-\xi}.$$

Similarly, in the layer at $x = 1$ we set $x = 1 + \sqrt{\epsilon}\eta$, $\eta \le 0$, $y(x; \epsilon) = Z(\eta; \epsilon)$. Then the ODE transforms into

$$Z'' - [1 + 3(1 + \sqrt{\epsilon}\eta)^2]Z - 1 - \sqrt{\epsilon}\eta = 0, \quad Z(0; \epsilon) = 1.$$

With $Z(\eta; \epsilon) \sim Z_0(\eta)$, the leading-order problem is

$$Z_0'' - 4Z_0 - 1 = 0, \quad Z_0(0) = 1.$$

The solution is

$$Z_0 = \left(\frac{5}{4} - C_0\right) e^{2\eta} + C_0 e^{-2\eta} - \frac{1}{4}.$$

Matching from the left.

$$\text{Inner expansion of outer solution:} \quad y(x; \epsilon) \;\sim\; -\frac{x}{1 + 3x^2} \;=\; -\frac{\sqrt{\epsilon}\xi}{1 + 3\epsilon\xi^2} \;\sim\; 0 \quad \text{as} \quad \epsilon \to 0, \;\; \xi > 0 \text{ and fixed.}$$

$$\text{Outer expansion of inner solution:} \quad Y(\xi;\epsilon) \sim A_0 e^{\xi} + (1 - A_0)e^{-\xi}$$
$$= A_0 e^{x/\sqrt{\epsilon}} + (1 - A_0)e^{-x/\sqrt{\epsilon}}$$
$$\sim 0 \quad \text{as} \quad \epsilon \to 0, \quad x > 0 \text{ and fixed,}$$

provided $A_0 = 0$. Therefore the inner solution becomes $Y_0 = e^{-\xi}$.

Matching from the right.

$$\text{Inner expansion of outer solution:} \quad y(x;\epsilon) \sim -\frac{x}{1+3x^2}$$
$$= -\frac{1+\sqrt{\epsilon}\eta}{1+3(1+\sqrt{\epsilon}\eta)^2}$$
$$\sim -\frac{1}{4} \quad \text{as} \quad \epsilon \to 0, \quad \eta < 0 \text{ and fixed.}$$

$$\text{Outer expansion of inner solution:} \quad Z(\eta;\epsilon) \sim \left(\frac{5}{4} - C_0\right) e^{2\eta} + C_0 e^{-2\eta} - \frac{1}{4}$$
$$= \left(\frac{5}{4} - C_0\right) e^{-2(1-x)/\sqrt{\epsilon}} + C_0 e^{2(1-x)/\sqrt{\epsilon}} - \frac{1}{4}$$
$$\sim -\frac{1}{4} \quad \text{as} \quad \epsilon \to 0, \quad x < 1 \text{ and fixed,}$$

provided $C_0 = 0$. Then the inner solution becomes $Z_0 = (5e^{2\eta} - 1)/4$.

The composite solution is given by the sum of the outer and two inner solutions minus the two common parts. The result, expressed in the x-variable, is

$$y_c(x;\epsilon) \sim -\frac{x}{1+3x^2} + e^{-x/\sqrt{\epsilon}} + \frac{5}{4}e^{-2(1-x)/\sqrt{\epsilon}}.$$