MATH-6620 PERTURBATION METHODS SPRING 2016

Homework-6 (OPTIONAL)

Assigned Thursday May 5, 2016

Due Monday May 16, 2016.

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PROBLEMS

1. Consider the equation
$$y'' + \lambda^2(x^2-1)y = 0, \qquad 0 \le x \le 2,$$
as $\lambda \to \infty$ with $y(0) = 0$, $y'(0) = 1$. First obtain asymptotic forms for $y(x, \lambda)$ for x not near the transition point $x = 1$. Then find the appropriate transition solution, and hence a leading-order uniformly valid asymptotic solution to the problem, carefully noting the connection between the three parts of the solution.

Solution:

We begin by transforming the equation to move the turning point to 0 by letting $\xi = 1 - x$, and $y(x;\lambda) = u(\xi, \lambda)$. Then the ODE becomes
$$u'' = \lambda^2(2\xi - \xi^2)u \quad -1 \le \xi \le 1$$
as $\lambda \to \infty$, with $u(1) = 0$ and $u'(1) = -1$. We then make the standard transformation
$$u(\xi;\lambda) = \phi(\xi)e^{\lambda\omega(\xi)}$$
to transform the ODE into
$$\phi'' + 2\lambda\omega'\phi' + \lambda\omega''\phi + \lambda^2\omega'^2\phi = \lambda^2(2\xi - \xi^2)\phi.$$
We match the powers of λ to find
$$\lambda^2: \quad \omega'^2 = 2\xi - \xi^2 \implies \omega = \pm\int \sqrt{2\xi - \xi^2}\,d\xi.$$
$$\lambda^1: \quad 2\omega'\phi' + \omega''\phi = 0 \implies \phi = (2\xi - \xi^2)^{-1/4}.$$
Then the leading order asymptotic form of the outer solution is
$$u \sim \begin{cases} |2\xi - \xi^2|^{-1/4}\left[c_1\cos\left(\lambda\int_\xi^0 |2s - s^2|^{1/2}ds\right) + c_2\sin\left(\lambda\int_\xi^0 |2s - s^2|^{1/2}ds\right)\right], & -1 \le \xi < 0 \\ (2\xi - \xi^2)^{-1/4}\left[b_1 e^{\lambda\int_0^\xi \sqrt{2s - s^2}ds} + b_2 e^{-\lambda\int_0^\xi \sqrt{2s-s^2}ds}\right], & 0 < \xi \le 1 \end{cases}.$$
We now determine the solution in the layer at $\xi = 0$. We let $\xi = \lambda^{-p}\zeta$, $(2\xi - \xi^2) \sim 2\xi \sim 2\lambda^{-p}\zeta$, and $w(\zeta;\lambda) = u(\xi;\lambda)$. Then the ODE becomes
$$w''\lambda^{2p} = 2\lambda^{2-p}\zeta w, \quad -\infty < \zeta < \infty.$$
To match the powers of λ on each side of the equation, we take
$$2p = 2 - p \implies p = \frac{2}{3}.$$
Then we have a near-Airy equation
$$w'' = 2\zeta w.$$

Here we approximate $w(\zeta;\lambda) \sim \mu_0(\lambda) w_0(\zeta)$. Note this does not change the equation above. If we take $\eta = B\zeta$, and $v_0(\eta) = w_0(\zeta)$, we get a new ODE:

$$B^2 v_0'' = \frac{2\eta}{B} v_0.$$

Then to transform to the Airy equation, we want

$$\frac{2}{B^3} = 1 \implies B = 2^{1/3}.$$

Thus we have the Airy equation in $v_0(\eta)$ with solution

$$v_0 = a_1 \mathrm{Ai}(\eta) + a_2 \mathrm{Bi}(\eta).$$

We can transform back to w easily:

$$w \sim \mu_0(\lambda)[a_1 \mathrm{Ai}(2^{1/3}\zeta) + a_2 \mathrm{Bi}(2^{1/3}\zeta)].$$

Now we inspect the asymptotic behavior of the inner and outer solutions in an attempt to match across the turning point. For the inner solution, the asymptotics are known:

$$w \sim \begin{cases} \mu_0(\lambda) \frac{1}{2^{7/12}\pi^{1/2}|\zeta|^{1/4}} \left[(a_1 - a_2)\sin\left(\frac{2\sqrt{2}}{3}|\zeta|^{3/2}\right) + (a_1 + a_2)\cos\left(\frac{2\sqrt{2}}{3}|\zeta|^{3/2}\right)\right], & \zeta \to -\infty \\ \mu_0(\lambda) \frac{1}{2^{1/12}\pi^{1/2}\zeta^{1/4}} \left[\frac{a_1}{2} e^{-2\sqrt{2}\zeta^{3/2}/3} + a_2 e^{2\sqrt{2}\zeta^{3/2}/3}\right], & \zeta \to \infty \end{cases}.$$

To look at the outer solution we first make a few approximations. We begin by letting $\xi = \lambda^{-2/3}\zeta$. Then we approximate $2\xi - \xi^2 \sim 2\xi \sim 2\lambda^{-2/3}\zeta$. We then take

$$(2\xi - \xi^2)^{-1/4} \sim 2^{-1/4}\lambda^{1/6}\zeta^{-1/4},$$

$$\int_\xi^0 |2s - s^2|^{1/2} ds \sim \int_{\lambda^{-2/3}\zeta}^0 |2s|^{1/2} ds = -\frac{2\sqrt{2}}{3}\lambda^{-1}|\zeta|^{3/2},$$

$$\int_0^\xi \sqrt{2s - s^2}\, ds \sim \int_0^{\lambda^{-2/3}\zeta} \sqrt{2s}\, ds = \frac{2\sqrt{2}}{3}\lambda^{-1}\zeta^{3/2}.$$

Hence the asymptotic behavior of the outer solution is:

$$u \sim \begin{cases} \frac{\lambda^{1/6}}{2^{1/4}|\zeta|^{1/4}} \left[c_1 \cos\left(\frac{2\sqrt{2}}{3}|\zeta|^{3/2}\right) - c_2 \sin\left(\frac{2\sqrt{2}}{3}|\zeta|^{3/2}\right)\right], & \zeta \to -\infty \\ \frac{\lambda^{1/6}}{2^{1/4}\zeta^{1/4}} \left[b_1 e^{2\sqrt{2}\zeta^{3/2}/3} + b_2 e^{-2\sqrt{2}\zeta^{3/2}/3}\right], & \zeta \to \infty \end{cases}.$$

Matching the solutions as $\zeta \to -\infty$ gives

$$\mu_0(\lambda) = 2^{1/3}\lambda^{1/6}\pi^{1/2}, \quad \text{and } c_2 = a_2 - a_1,\ c_1 = a_1 + a_2.$$

Then using these values, we find from the matching on the other side

$$b_2 = \frac{a_1}{\sqrt{2}},\ b_1 = \sqrt{2} a_2.$$

We can then express all coefficients in terms of the b_i as

$$c_1 = \sqrt{2} b_2 + \frac{b_1}{\sqrt{2}},\ c_2 = \frac{b_1}{\sqrt{2}} - \sqrt{2} b_2.$$

We now apply the boundary conditions at $\xi = 1$ to determine the b_i's.

$$u(1) \sim b_1 + b_2 = 0 \implies b_1 = -b_2.$$

$$u'(1) \sim 2b_1\lambda = -1 \implies b_1 = -\frac{1}{2\lambda}.$$

Then the coefficients are

$$b_2 = \frac{1}{2\lambda},\ a_1 = \frac{\sqrt{2}}{2\lambda},\ a_2 = -\frac{\sqrt{2}}{4\lambda},\ c_1 = \frac{\sqrt{2}}{4\lambda},\ c_2 = -\frac{3\sqrt{2}}{4\lambda}.$$

We can then write our leading-order asymptotic solution as an outer solution

$$u \sim \begin{cases} |2\xi - \xi^2|^{-1/4}\left[\frac{\sqrt{2}}{4\lambda}\cos\left(\lambda\int_\xi^0 |2s-s^2|^{1/2}ds\right) - \frac{3\sqrt{2}}{4\lambda}\sin\left(\lambda\int_\xi^0 |2s-s^2|^{1/2}ds\right)\right], & -1 \le \xi < 0 \\ (2\xi-\xi^2)^{-1/4}\left[-\frac{1}{2\lambda}e^{\lambda\int_0^\xi \sqrt{2s-s^2}ds} + \frac{1}{2\lambda}e^{-\lambda\int_0^\xi \sqrt{2s-s^2}ds}\right], & 0 < \xi \le 1 \end{cases},$$

and an inner solution

$$u \sim 2^{1/3}\lambda^{1/6}\pi^{1/2}\left[\frac{\sqrt{2}}{2\lambda}\mathrm{Ai}(2^{1/3}\lambda^{2/3}\xi) - \frac{\sqrt{2}}{4\lambda}\mathrm{Bi}(2^{1/3}\lambda^{2/3}\xi)\right],$$

where ξ is in a neighborhood of 0. We then transform back to the original variable:

$$y \sim \begin{cases} |1-x^2|^{-1/4}\frac{\sqrt{2}}{4\lambda}\left[\cos\left(\lambda\int_1^x |1-s^2|ds\right) - 3\sin\left(\lambda\int_1^x |1-s^2|ds\right)\right], & 1 < x \le 2 \\ (1-x^2)^{-1/4}\frac{1}{2\lambda}\left[e^{-\lambda\int_x^1 \sqrt{1-s^2}ds} - e^{\lambda\int_x^1 \sqrt{1-s^2}ds}\right], & 0 \le x < 1 \end{cases}$$

for the outer solution, and

$$y \sim 2^{1/3}\lambda^{1/6}\pi^{1/2}\frac{\sqrt{2}}{2\lambda}\left[\mathrm{Ai}(2^{1/3}\lambda^{2/3}(1-x)) - \frac{1}{2}\mathrm{Bi}(2^{1/3}\lambda^{2/3}(1-x))\right],$$

where x is in a neighborhood of 1.

2. The Schrödinger equation describing the quantum mechanics of a particle in a potential field V has the form

$$y''(x) + [E - V(x)]y = 0,\ \ y(\pm\infty) = 0.$$

Take $V(x) = x^4$. Then $x = \pm E^{1/4}$ are the two turning points. Find an appropriate expression for the eigenvalues (energy levels) E_n as $n \to \infty$, for which a nontrivial solution exists.

Solution:

We let $x = E^{1/4}\xi$, and $u(\xi;\lambda) = y(x;\lambda)$. With $V(x) = x^4$, we get the differential equation for u:

$$u'' + E^{3/2}[1-\xi^4]u = 0$$

We then use the WKBJ ansatz, with $\lambda = E^{3/4}$, $u = \phi(\xi)e^{\lambda\omega(\xi)}$, to get the differential equation

$$\phi'' + \lambda(2\omega'\phi' + \omega''\phi) + \lambda^2(\omega'^2 + 1 - \xi^4)\phi = 0.$$

If we look at the powers of λ we can solve for ω and ϕ:

$$\lambda^2 :\ \ \omega'^2 = \xi^4 - 1 \implies \omega = \pm\int\sqrt{\xi^4-1}d\xi,$$

$$\lambda^1 :\ \ 2\omega'\phi' + \omega''\phi = 0 \implies \phi = (\xi^4-1)^{-1/4}.$$

Then u has the general form

$$u = (\xi^4-1)^{-1/4}e^{\pm\lambda\int\sqrt{\xi^4-1}d\xi}.$$

In the region between the two transition points at $\xi = \pm 1$, we argue that our outer solution will have the form

$$u = (1-\xi^4)^{-1/4} \sin\left(\lambda \int \sqrt{1-\xi^4} d\xi + \frac{\pi}{4}\right),$$

as matching in the layer at the turning points will be done with the first Airy function Ai. Then, if we enter the region through the left transition point, $\xi = -1$, we see

$$u = \frac{C}{(1-\xi^4)^{1/4}} \sin\left(\lambda \int_{-1}^{\xi} \sqrt{1-s^4} ds + \frac{\pi}{4}\right).$$

And if we enter through the right at $\xi = 1$, we see

$$u = \frac{D}{(1-\xi^4)^{1/4}} \sin\left(\lambda \int_{\xi}^{1} \sqrt{1-s^4} ds + \frac{\pi}{4}\right).$$

We rewrite this equation as

$$u = \frac{D}{(1-\xi^4)^{1/4}} \sin\left(\lambda \int_{-1}^{1} \sqrt{1-s^4} ds + \frac{\pi}{2} - \lambda \int_{\xi}^{1} \sqrt{1-s^4} ds - \frac{\pi}{4}\right).$$

If we let

$$\theta = \lambda \int_{\xi}^{1} \sqrt{1-s^4} ds + \frac{\pi}{4}, \; Q = \lambda \int_{-1}^{1} \sqrt{1-s^4} ds + \frac{\pi}{2},$$

then for the solution in the region between the transition points to be equivalent, we enforce

$$C \sin\theta = D \sin(Q-\theta) = -D\sin(\theta - Q).$$

This only occurs when $Q = n\pi$ and $D = (-1)^n C$. Thus we have

$$\lambda \int_{-1}^{1} \sqrt{1-s^4} ds + \frac{\pi}{2} = n\pi.$$

Therefore,

$$E_n = \left(\frac{\pi(n-\frac{1}{2})}{\int_{-1}^{1} \sqrt{1-s^4} ds}\right)^{4/3}.$$

3. Consider the homogeneous ODE

$$y''(x) - \frac{x}{(x+1)^4} y(x) = 0.$$

Find the first three terms of the asymptotic expansion of each of the two linearly independent solutions for large x.

Solution:

Since our equation is of the form

$$y'' + r(x)y = 0,$$

we can make the substitution $y = e^{\phi(x)}$, to get the equation

$$\phi'' + \phi'^2 = \frac{x}{(x+1)^4}.$$

We can approximate this equation by Taylor expanding the term on the right hand side of the equation

$$\phi'' + \phi'^2 \sim \frac{1}{x^3} - \frac{4}{x^4} + \frac{10}{x^5}.$$

Then, since the leading order term is of the form $x^{-\alpha}$, where $\alpha > 2$, we take $\phi \sim \phi_0$ to get

$$\phi_0'' \sim \frac{1}{x^3} \implies \phi_0 = \frac{1}{2x} + \phi_1.$$

If we substitute this into the ODE for ϕ, we find

$$\left[\frac{1}{x^3} + \phi_1''\right] + \left(-\frac{1}{2x^2} + \phi_1'\right)^2 = \frac{1}{x^3} - \frac{4}{x^4} + \frac{10}{x^5}.$$

Then, expanding the square on the left hand side of the equation and combining the x^{-4} terms, we see

$$\phi_1'' \sim -\frac{17}{4x^4} \implies \phi_1 = -\frac{17}{24x^2}.$$

Thus our first solution for y has the asymptotic expansion

$$y \sim e^{1/2x}\left[1 - \frac{17}{24x^2} + \frac{289}{1152x^4} + \ldots\right].$$

To find the second solution, we try letting $\phi_0 = A\ln x$. Then, substituting this into our ODE for ϕ, we have

$$-\frac{A}{x^2} + \frac{A^2}{x^2} = \frac{1}{x^3} - \frac{4}{x^4} + \frac{10}{x^5}.$$

Since there are no x^{-2} terms on the right side of the equation, we enforce

$$A^2 - A = 0 \implies A = 0, 1.$$

We note that the first solution for y corresponds to the 0 solution here. We take, then, $\phi = \ln(x) + \phi_1$. Thus our ODE becomes

$$\left[-\frac{1}{x^2} + \phi_1''\right] + \left[\frac{1}{x} + \phi_1'\right]^2 = \frac{1}{x^3} - \frac{4}{x^4} + \frac{10}{x^5}.$$

If we make a leading-order approximation for ϕ_1, we have

$$\phi_1'' + \frac{2}{x}\phi_1' \sim \frac{1}{x^3}.$$

Then $\phi_1 = -\frac{1}{x} - \frac{\ln x}{x}$. We write

$$\phi = \ln(x) - \frac{1}{x} - \frac{\ln(x)}{x} \implies y \sim x^{1-1/x}\left[1 - \frac{1}{x} + \frac{1}{2x^2} + \ldots\right].$$