MATH-6620 PERTURBATION METHODS SPRING 2016

Test 2

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I have abided by the ground rules of this test

PROBLEMS

1. An oscillation is described by the pair of equations

$$\frac{du}{dt} + v = -\epsilon u,$$

$$\frac{dv}{dt} - u = -\epsilon v^3, \;\; t \geq 0,$$

with initial conditions $u(0) = 1, \;\; v(0) = 0$. We introduce a fast time scale $T = (1 + \epsilon^2\omega_2 + ...)t$ and a slow scale $\tau = \epsilon t$. Then, but letting $u(t;\epsilon) = u(T, \tau; \epsilon)$ and $v(t;\epsilon) = v(T, \tau; \epsilon)$, the equations then become

$$(1 + \epsilon^2\omega_2)u_T + \epsilon u_\tau + v = -\epsilon u$$

$$(1 + \epsilon^2\omega_2)v_T + \epsilon v_\tau - u = -\epsilon v^3,$$

with initial conditions $u(0,0) = 1, \;\; v(0,0) = 0$. Now we look for asymptotic solutions by allowing $u(T,\tau;\epsilon) \sim u_0(T,\tau) + \epsilon u_1(T,\tau) + \epsilon^2 v_2(T,\tau)$ and $v(T,\tau;\epsilon) \sim v_0(T,\tau) + \epsilon v_1(T,\tau) + \epsilon^2 v_2(T,\tau)$, and grouping the resulting terms in powers of ϵ:

$$u_{0T} + v_0 + \epsilon[u_{1T} + u_{0\tau} + v_1 + u_0] + \epsilon^2[u_{2T} + u_{1\tau} + \omega_2 u_{0T} + v_2 + u_1] = 0$$

$$v_{0T} - u_0 + \epsilon[v_{1T} + v_{0\tau} - u_1 + v_0^3] + \epsilon^2[v_{2T} + v_{1\tau} + \omega_2 v_{0T} - u_2 + 3v_0^2 v_1] = 0.$$

The initial conditions here are $u_0(0,0) = 1, \;\; u_1(0,0) = u_2(0,0) = v_0(0,0) = v_1(0,0) = v_2(0,0) = 0$. We now inspect the leading order equation

$$\epsilon^0 : \;\; u_{0T} + v_0 = 0$$

$$v_{0T} - u_0 = 0.$$

By setting $u_0 = v_{0T}$ and combining the equations, we get the second order equation

$$v_{0TT} + v_{0T} = 0,$$

with solutions

$$v_0 = f_0(\tau)\sin(T + \phi(\tau)), \;\; u_0 = f_0(\tau)\cos(T + \phi(\tau)).$$

Applying the initial conditions, we find that $\phi(0) = 0$, and $f_0(0) = 1$. We now look at the $O(\epsilon)$ equation to determine the amplitude and phase shift functions and, subsequently, u_1 and v_1.

$$\epsilon^1 : \;\; u_{1T} + u_{0\tau} + v_1 + u_0 = 0$$

$$v_{1T} + v_{0\tau} - u_1 + v_0^3 = 0,$$

with initial conditions as above. If we let $u_1 = v_{1T} + v_{0\tau} + v_0^3$, we combine the equations to get

$$v_{1TT} + v_1 = -[2u_{0\tau} + u_0 + 3v_0^2 u_0].$$

Expanding out the right hand side of the equation leads to the differential equations for the amplitude and phase shift of u_0:

$$\phi' = 0 \implies \phi = 0.$$

$$4f_0 + 3f_0^3 + 8f_0' = 0 \implies f_0 = \frac{2}{\sqrt{7e^\tau - 3}}.$$

Then the $O(\epsilon)$ equation reduces to

$$v_{1TT} + v_1 = \frac{6\cos(3T)}{(7e^\tau - 3)^{3/2}}.$$

This equation has the solution

$$v_1 = f_1(\tau)\cos(T) + g_1(\tau)\sin(T) - \frac{3(2\cos(T) + \cos(3T))}{4(7e^\tau - 3)^{3/2}}.$$

Then,

$$u_1 = -f_1(\tau)\sin(T) + g_1(\tau)\cos(T) + \frac{(30 - 28e^\tau)\sin(T) - \sin(3T)}{4(7e^\tau - 3)^{3/2}}.$$

Applying initial conditions tells us $g_1(0) = 0$, and $f_1(0) = 9/32$. To determine g_1, f_1, and ω_2, we now look at the $O(\epsilon^2)$ equation:

$$\epsilon^2: \quad u_{2T} + u_{1\tau} + \omega_2 u_{0T} + v_2 + u_1 = 0$$

$$v_{2T} + v_{1\tau} + \omega_2 v_{0T} - u_2 + 3v_0^2 v_1 = 0.$$

Now substituting with $u_2 = v_{2T} + v_{1\tau} + \omega_2 v_{0T} + 3v_0^2 v_1$, we combine the equations to get

$$v_{2TT} + v_2 = -v_{1T\tau} - \omega_2 v_{0TT} - 6v_0 v_{0T} v_1 - 3v_0^2 v_{1T} - u_{1\tau} - \omega_2 u0T - u_1.$$

The coefficients of the secular terms that result on the right hand side of the above equation must be set to zero, thus we have two more differential equations:

$$\cos(T): -1512e^\tau g_1'(\tau) + 3528e^{2\tau} g_1'(\tau) - 2744e^{3\tau} g_1'(\tau) + 216g_1'(\tau) - 4\left(7e^\tau + 6\right)\left(7e^\tau - 3\right)^2 g_1(\tau) = 0$$

$$\sin(T): 1512\exp(\tau)f_1'(\tau) - 3528\exp(2\tau)f_1'(\tau) + 2744\exp(3\tau)f_1'(\tau) + 28\exp(\tau)(7\exp(\tau) - 3)^2 f_1(\tau)$$
$$-672\omega_2\exp(\tau)\sqrt{7\exp(\tau) - 3} + 144\omega_2\sqrt{7\exp(\tau) - 3} - 784\omega_2\exp(2\tau)\sqrt{7\exp(\tau) - 3} + 98\exp(2\tau)\sqrt{7\exp(\tau) - 3}$$
$$+81\sqrt{7\exp(\tau) - 3} - 216f_1'(\tau) = 0$$

The solution of the first, is $g_1 = 0$, with the boundary conditions applied. The second equations solution is difficult, but we can see that it does have a solution. Instead of solving the equation, since we only want a condition on ω_2, we look at the two inhomogeneous $e^{2\tau}$ terms. If $\omega_2 = 1/8$, these terms vanish, which eliminates the possibility that the amplitude of the second order term grows exponentially (and thus can become disordered as $\tau \to \infty$.) Thus the leading order solutions are

$$u \sim \frac{2}{\sqrt{7e^{\epsilon t}}}\cos((1 + \epsilon^2/8)t)$$

$$v \sim \frac{2}{\sqrt{7e^{\epsilon t}}}\sin((1 + \epsilon^2/8)t).$$

If one wishes to compute the next term, all that must be done is solve the $f_1(\tau)$ differential equation.

2. Here we model a child's swing by treating it as a pendulum which changes its length by a small amount in a periodic manner. The relevant ODE is

$$\frac{d^2u}{dt^2} + \left(\frac{2\epsilon\omega\cos(\omega t)}{1 + \epsilon\sin(\omega t)}\right)\frac{du}{dt} + u = 0.$$

By using the fast scale $T = t$ and slow scale $\tau = \epsilon t$, we look for a multi-scale expansion $u \sim u_0 + \epsilon u_1$. We then have the PDE

$$(u_{TT} + 2\epsilon u_{T\tau} + \epsilon^2 u_{\tau\tau})(1 + \epsilon\sin(\omega T)) + 2\epsilon\omega\cos(\omega T)(u_T + \epsilon u_\tau) + u = 0.$$

If we look at the $O(1)$ equation that results from the multi-scale expansion, we get

$$\epsilon^0: \quad u_{0TT} + u_0 = 0$$

with solution

$$u_0 = f_0(\tau)\cos(T) + g_0(\tau)\sin(T).$$

We then inspect the $O(\epsilon)$ equation to determine f_0 and ϕ_0:

$$\epsilon^1: \quad u_{1TT} + u_1 = -(2u_{0T\tau} - 2\omega\cos(\omega T)u_{0T} - \sin(\omega T)u_{0TT}.$$

The right hand side of the above equation can be expressed as

$$\frac{1}{2}\left(4\sin(T)f_0'(\tau) - 2\omega f_0(\tau)\sin(T\omega + T) - f_0(\tau)\sin(T\omega + T) - 2\omega f_0(\tau)\sin(T - T\omega) + f_0(\tau)\sin(T - T\omega)\right.$$

$$\left.-4\cos(T)g_0'(\tau) + 2\omega g_0(\tau)\cos(T\omega + T) + g_0(\tau)\cos(T\omega + T) - g_0(\tau)\cos(T - T\omega) + 2\omega g_0(\tau)\cos(T - T\omega)\right).$$

Clearly, removal of secularity demands that

$$g_0'(\tau) = 0 \implies g_0 = constant.$$

$$f_0'(\tau) = 0 \implies f_0 = constant.$$

Then, to find u_1, we must solve

$$u_{1TT} + u_1 = \frac{1}{2}\left(-f_0(2\omega + 1)\sin(T + \omega T) + f_0(1 - 2\omega)\sin(T - \omega T)\right.$$

$$\left.+g_0(2\omega + 1)\cos(T + \omega T) + g_0(2\omega - 1)\cos(T - \omega T)\right)$$

Fortunately, this equation has a simple solution,

$$u_1 = f_1(\tau)\cos(T) + g_1(\tau)\sin(T) + \frac{g_0(1 - 2\omega)}{2\omega(\omega - 2)}\cos(T - \omega T) - \frac{g_0(1 + 2\omega)}{2\omega(\omega + 2)}\cos(T + \omega T)$$

$$+\frac{f_0(2\omega - 1)}{2\omega(\omega - 2)}\sin(T - \omega T) + \frac{f_0(1 + 2\omega)}{2\omega(\omega + 2)}\sin(T + \omega T).$$

Now we look at the reduced $O(\epsilon^2)$ equation (as all τ derivatives of u_0 are 0,):

$$u_{2TT} + u_2 + 2u_{1T\tau} + u_{1TT}\sin(\omega T) + 2\omega u_{1T}\cos(\omega T) = 0.$$

The secular terms that arise result in the differential equations

$$f_1' = \frac{3g_0\omega(\omega^2 - 1)}{4\omega(4 - \omega^2)},$$

$$g_1' = \frac{3f_0\omega(\omega^2 - 1)}{4\omega(\omega^2 - 4)}.$$

These equations have obvious solutions

$$f_1 = \frac{3g_0\omega(\omega^2 - 1)}{4\omega(4 - \omega^2)}\tau,$$

$$g_1 = \frac{3f_0\omega(\omega^2 - 1)}{4\omega(\omega^2 - 4)}\tau.$$

We then write our multi-scale approximation

$$u_0+\epsilon u_1 = f_0\cos(T)+g_0\sin(T)+\epsilon\left(\frac{3g_0(\omega^2-1)}{4(4-\omega^2)}\tau\cos(T)+\frac{3f_0(\omega^2-1)}{4(\omega^2-4)}\tau\sin(T)\right.$$

$$\left.+\frac{g_0(1-2\omega)}{2\omega(\omega-2)}\cos(T-\omega T)-\frac{g_0(1+2\omega)}{2\omega(\omega+2)}\cos(T+\omega T)+\frac{f_0(2\omega-1)}{2\omega(\omega-2)}\sin(T-\omega T)+\frac{f_0(1+2\omega)}{2\omega(\omega+2)}\sin(T+\omega T)\right).$$

We can then see that amplitude growth can result from the linear growth of τ in the u_1 term. This growth becomes $O(1)$ when $t=O(\epsilon^{-2})$, for any choice of ω. However, some amplitude growth can result from the other terms of u_1 when $|\omega|<\epsilon f_0/4$. We can see this more clearly, when we look at the coefficients on the last line of the multi-scale approximation. Each of these coefficients is approximately equal to

$$\pm\frac{\epsilon f_0}{4\omega}.$$

If we want amplitude to grow in $O(1)$, we let

$$\pm\frac{\epsilon f_0}{4\omega}>1 \implies |\omega|<\frac{\epsilon f_0}{4}.$$

If we want to be more exact, we can solve the coefficients for ϵ without assuming $(\omega-2)\sim -2$ and $(1-2\omega)\sim 1$. Solving one of these coefficients gives

$$\omega=\frac{2\epsilon-4\pm\sqrt{(2\epsilon-4)^2+8\epsilon}}{4}.$$

3. Here we determine a first-term approximation to the solution of a wave equation with slowly-varying phase speed

$$u_{tt}=c^2(\epsilon t)u_{xx},\quad |x|<\infty,\ t>0.$$

We begin by taking the naive approach and letting $\tau=\epsilon t$, $u(x,t;\epsilon)=u(x,t,\tau;\epsilon)$. Then we get

$$u_{tt}+2\epsilon u_{t\tau}+\epsilon^2 u_{\tau\tau}=c^2(\tau)u_{xx}.$$

We then apply the standard expansion to $u(x,t,\tau;\epsilon)\sim u_0(x,t,\tau)+\epsilon u_1(x,t,\tau)+...$ If we look at specifically the resultant $O(\epsilon)$ equation, we have

$$u_{0tt}=c^2(\tau)u_{0xx},$$

which can be solved using the method of separation of variables, by letting $u_0(x,t,\tau)=T(t,\tau)X(x)$. In an effort to increase clarity, we only consider only a single term of the resulting solution, set $\lambda=1$, and state that the following efforts of removing singularities will hold for every term in the series solution. The solution, then for u_0 is

$$u_0=a_0(\tau)\cos[x-c(\tau)t]+a_1(\tau)\cos[x+c(\tau)t]+b_0(\tau)\sin[x-c(\tau)t]+b_1(\tau)\sin[x+c(\tau)t].$$

We then note that secularities may only arise in the $O(\epsilon)$ equation from the term $2u_{0t\tau}$. When we expand out this term, and attempt to remove the singularities, we find 2 identical systems of 2 differential equations,:

$$c(\tau)a_0'+a_0c'(\tau)-ta_1c(\tau)c'(\tau)=0$$
$$c(\tau)a_1'+a_1c'(\tau)-ta_0c(\tau)c'(\tau)=0.$$
$$c(\tau)b_0'+b_0c'(\tau)-tb_1c(\tau)c'(\tau)=0$$
$$c(\tau)b_1'+b_1c'(\tau)-tb_0c(\tau)c'(\tau)=0.$$

Clearly then, the solutions are the same for each set of equations, though they may differ by some multiplicative constant:

$$a_0,b_1=\frac{c_1\cos[c(\tau)t]+c_2\sin[c(\tau)t]}{c(\tau)}.$$

$$a_1, b_0 = \frac{c_1 \cos[c(\tau)t] - c_2 \sin[c(\tau)t]}{c(\tau)}.$$

Then substituting these into the solution u_0, we get a solution independent of time that does not satisfy the $O(1)$ equation. Thus we should assume our fast-time scale must also change (we suspect that $\tau = \epsilon t$ is a good slow time scale as it appears explicitly in the governing equation.) We let $f(t, \epsilon) = T$ be the fast time scale, $u(x,t) = u(x, T, \tau)$ and derive a new multi-scale equation:

$$f_t^2 u_{TT} + f_{tt} u_T + 2\epsilon f_t u_{T\tau} + \epsilon^2 u_{\tau\tau} = c^2(\tau) u_{xx}.$$

For the solution to be wavelike, we assume balance occurs between the first term on the left hand side and the term on the right hand side:

$$f_t^2 u_{TT} \sim c^2(\tau) u_x x \implies f_t = c(\tau) \implies f = \int_0^t c(\epsilon s) ds.$$

Using this fast time scale leads to the multi-scale equation

$$c^2(\tau) u_{TT} + \epsilon c'(\tau) u_T + 2\epsilon c(\tau) u_{T\tau} + \epsilon^2 u_{\tau\tau} = c^2(\tau) u_{xx}.$$

Now letting $u(x, T, \tau; \epsilon) \sim u_0(x, T, \tau) + \epsilon u_1(x, T, \tau) + \dots$, we can inspect the resulting equations in terms of powers of ϵ. We first look at the leading order equation:

$$\epsilon^0 : \quad u_{0TT} = u_{0xx}.$$

This equation has solutions:

$$u_0 = a_0(\tau) \sin(T - x) + a_1 \sin(T + x) + b_0(\tau) \cos(T - x) + b_1(\tau) \cos(T + x).$$

Now if we move to the $O(\epsilon)$ equation, we find

$$u_{1TT} + \frac{c'(\tau)}{c^2(\tau)} u_{0T} + \frac{2}{c(\tau)} u_{0T\tau} = u_{1xx}.$$

Singularities may only result from

$$\frac{c'(\tau)}{c^2(\tau)} u_{0T} + \frac{2}{c(\tau)} u_{0T\tau}.$$

To remove these singularities, we get a system of 4 independent, identical equations:

$$2c(\tau) a_0' + c'(\tau) a_0 = 0$$

$$2c(\tau) a_1' + c'(\tau) a_1 = 0$$

$$2c(\tau) b_0' + c'(\tau) b_0 = 0$$

$$2c(\tau) b_1' + c'(\tau) b_1 = 0.$$

With solutions

$$a_0, a_1, b_0, b_1 = \frac{1}{\sqrt{c(\tau)}} (\alpha_0, \alpha_1, \beta_0, \beta_1), \quad \alpha_{0,1} \beta_{0,1} \in \mathbb{R}.$$

Then the first-term approximation valid for large t is

$$u_0 = \frac{\alpha_0}{\sqrt{c(\tau)}} \sin\left[\int_0^t c(\epsilon s) ds - x\right] + \frac{\alpha_1}{\sqrt{c(\tau)}} \sin\left[\int_0^t c(\epsilon s) ds + x\right]$$

$$+ \frac{\beta_0}{\sqrt{c(\tau)}} \cos\left[\int_0^t c(\epsilon s) ds - x\right] + \frac{\beta_1}{\sqrt{c(\tau)}} \cos\left[\int_0^t c(\epsilon s) ds + x\right].$$