MATH-6620 PERTURBATION METHODS SPRING 2016

Homework-3

Assigned Wednesday February 24, 2016

Due Monday March 7, 2016

NOTES

1. Writing solutions in LaTeX is strongly recommended but not required.
2. Show all work. Illegible or undecipherable solutions will be **returned without grading**.
3. Figures, if any, should be neatly drawn (either by hand or by a drawing program), properly labelled and captioned.
4. Please make sure that the pages are stapled together.
5. The assignment can be submitted in the labelled box in Amos Eaton 301, at my office, or in class.

PROBLEMS

1. Consider the initial-value problem

$$\epsilon \frac{dy}{dt} = ty, \quad y(-1) = 1.$$

(a) Find the exact solution and discuss its qualitative character. Plot it on the interval $t \in [-1, 1]$ for $\epsilon = 0.25$. Construct a leading-order asymptotic solution for $\epsilon > 0$ and small, and discuss whether it is able to capture all significant features of the exact solution.

(b) Repeat part (a) for the slightly altered differential equation

$$\epsilon \frac{dy}{dt} = ty + \epsilon, \quad y(-1) = 1.$$

Discuss what you find. Any surprises?

(a) The exact solution is

$$y = \exp\left(\frac{t^2 - 1}{2\epsilon}\right).$$

In the interval $t \in [-1, 1]$, the solution is symmetric about $t = 0$. It is exponentially small for $\epsilon \to 0$ except in narrow boundary layers at either end where it rises up to the value unity. The exact solution is plotted in Figure 3.

Outer solution. Let $y \sim y_0(t)$. Then $ty_0 = 0$, suggesting $y_0 = 0$. The boundary condition at $t = -1$ is not satisfied.

Inner solution. Let $t = -1 + \epsilon\tau$, $y(t; \epsilon) = Y(\tau; \epsilon)$. Then the problem becomes

$$\frac{dY}{d\tau} = (-1 + \epsilon\tau)Y, \quad Y(0; \epsilon) = 1.$$

With $Y \sim Y_0(\tau)$, Y_0 satisfies

$$Y_0' = -Y_0, \quad Y_0(0) = 1,$$

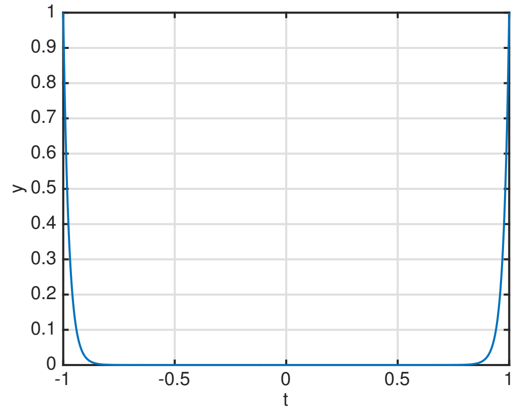

Figure 1: Exact solution for Problem 1(a).

yielding

$$Y_0 = e^{-\tau}.$$

This solution matches with the zero solution in the outer region, and the composite expansion becomes

$$y_c(t;\epsilon) \sim y_0 + Y_0 - 0 = e^{-(1+t)/\epsilon}.$$

Unlike the exact solution, this leading-order approximation remains exponentially small at the right boundary $t = 1$, and thus does not capture the boundary layer there.

(b) The leading-order asymptotic solution is the same as in part (a) above. The exact solution is

$$y = \exp\left(\frac{t^2 - 1}{2\epsilon}\right) + \sqrt{\frac{\pi\epsilon}{2}} e^{t^2/(2\epsilon)} \left(\operatorname{erf} \frac{1}{\sqrt{2\epsilon}} + \operatorname{erf} \frac{t}{\sqrt{2\epsilon}}\right).$$

This solution is plotted in Figure 2.

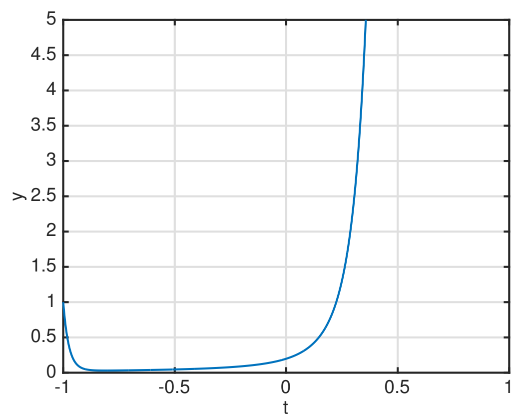

Figure 2: Exact solution for Problem 1(b).

It differs from that in part (a) by the second term, which is induced by the presence of the additional ϵ on the RHS of the ODE. In order to estimate the second term let us recall that

$$\operatorname{erf} x \sim \begin{cases} 1 - \frac{1}{\sqrt{\pi} x} e^{-x^2} + O\left(\frac{e^{-x^2}}{x^3}\right), & x \to \infty, \\ -1 + \frac{1}{\sqrt{\pi}|x|} e^{-x^2} + O\left(\frac{e^{-x^2}}{x^3}\right), & x \to -\infty. \end{cases}$$

Therefore,

$$\operatorname{erf}\left(\frac{t}{\sqrt{2\epsilon}}\right) \sim \begin{cases} 1-\sqrt{\frac{2\epsilon}{\pi}}\frac{1}{t}e^{-t^2/(2\epsilon)}+O\left(\frac{\epsilon^{3/2}}{t^3}e^{-t^2/(2\epsilon)}\right), & t\to\infty, \\ -1+\sqrt{\frac{2\epsilon}{\pi}}\frac{1}{|t|}e^{-t^2/(2\epsilon)}+O\left(\frac{\epsilon^{3/2}}{t^3}e^{-t^2/(2\epsilon)}\right), & t\to-\infty. \end{cases}$$

Now for $-1<t<0$,

$$\begin{aligned} \sqrt{\frac{\pi\epsilon}{2}}e^{t^2/(2\epsilon)}\left(\operatorname{erf}\frac{1}{\sqrt{2\epsilon}}+\operatorname{erf}\frac{t}{\sqrt{2\epsilon}}\right) &\sim \sqrt{\frac{\pi\epsilon}{2}}e^{t^2/(2\epsilon)}\Bigg[1-\sqrt{\frac{2\epsilon}{\pi}}e^{-1/(2\epsilon)}+O\left(\epsilon^{3/2}e^{-1/(2\epsilon)}\right) \\ &\qquad -1+\sqrt{\frac{2\epsilon}{\pi}}\frac{1}{|t|}e^{-t^2/(2\epsilon)}+O\left(\frac{\epsilon^{3/2}}{t^3}e^{-t^2/(2\epsilon)}\right)\Bigg] \\ &= O(\epsilon/|t|). \end{aligned}$$

For $0<t<1$,

$$\sqrt{\frac{\pi\epsilon}{2}}e^{t^2/(2\epsilon)}\left(\operatorname{erf}\frac{1}{\sqrt{2\epsilon}}+\operatorname{erf}\frac{t}{\sqrt{2\epsilon}}\right)=O\left(\sqrt{\epsilon}e^{t^2/(2\epsilon)}\right).$$

Thus, while the solution is $O(\epsilon)$ in the left half of the interval $[-1,1]$, it begins to grow as $t\to 0-$ and in the right half of the interval the solution is exponentially large, a feature that is missed entirely by the leading-order asymptotic solution.

The reason why the asymptotic outer solution in the region $t\in(0,1]$ fails to agree with the exact solution is that the latter assumes the derivative $\epsilon dy/dt$ to vanish as $\epsilon\to 0$. This automatically disallows a solution growing exponentially as $e^{t^2/(2\epsilon)}$; because for such a solution $\epsilon dy/dt$ would be included in a dominant balance.

2. Consider the initial-value problem

$$\epsilon\frac{dy}{dt}+ty=te^{-t},\quad y(0)=2.$$

For $\epsilon>0$ and small, find the leading-order composite solution.

In the outer region, let $y\sim y_0(t)$. Then,

$$y_0=e^{-t}.$$

This solution does not satisfy the initial condition, thereby suggesting a layer at $t=0$. Dominant balance forces us to conclude that the layer is $O(\sqrt{\epsilon})$ thick. With $t=\sqrt{\epsilon}\tau$ and $y(t;\epsilon)=Y(\tau;\epsilon)$, the ODE transforms into

$$\frac{dY}{d\tau}+\tau Y=\tau e^{-\sqrt{\epsilon}\tau}.$$

We seek the expansion $Y\sim Y_0(\tau)$, where Y_0 satisfies

$$Y_0'+\tau Y_0=\tau,\quad Y_0(0)=2,$$

yielding the solution

$$Y_0=1+e^{-\tau^2/2}.$$

To match, we expand the inner solution in the outer variable, retaining only $O(1)$ terms. The result is

$$Y_0(t/\sqrt{\epsilon})=1+e^{-t^2/(2\epsilon)}\sim 1\quad\text{as}\quad \epsilon\to 0,\ \ t\text{ fixed}.$$

Similarly, the inner expansion of the outer expansion to order unity is

$$y_0(\sqrt{\epsilon}\tau)=e^{-\sqrt{\epsilon}\tau}\sim 1\quad\text{as}\quad \epsilon\to 0,\ \ \tau\text{ fixed}.$$

Thus matching is confirmed. The composite expansion is

$$y_c(t;\epsilon)\sim y_0+Y_0-1=e^{-t^2/(2\epsilon)}+e^{-t}.$$

3. Consider the initial-value problem for the system of equations

$$\begin{aligned}\frac{dx}{dt} &= xy,\\ \epsilon\frac{dy}{dt} &= y-y^3,\end{aligned}$$

with initial conditions

$$x(0)=\alpha,\quad y(0)=\beta.$$

(a) For $\epsilon>0$ and small, seek an outer solution of the form $x(t;\epsilon)\sim x_0(t)$, $y(t;\epsilon)\sim y_0(t)$. Consider all possibilities, and note that the initial conditions will not be met, in general.

(b) Consider an initial layer by using the stretching $t=\delta(\epsilon)\tau$, $x(t;\epsilon)=X(\tau;\epsilon)$, $y(t;\epsilon)=Y(\tau;\epsilon)$, where δ is to be found by a suitable argument. Seek an inner solution of the form $X(\tau;\epsilon)\sim X_0(\tau)$, $Y(t;\epsilon)\sim Y_0(\tau)$. Construct the inner solution for (i) $\beta>0$, (ii) $\beta<0$ and (iii) $\beta=0$. For each case determine the leading-order composite solution.

Let $y\sim y_0(t)$, $x\sim x_0(t)$ in an outer region. Then $y_0-y_0^3=0$ so that

$$y_0=1,\ 0,\ \text{or}\ -1.$$

With

$$\frac{dx_0}{dt}=x_0y_0,$$

the corresponding solutions for x_0 are

$$x_0=a_0e^t,\ a_0,\ \text{or}\ a_0e^{-t}.$$

The outer solutions for y will not satisfy the initial condition in general. To reinstate the y-derivative in the inner region we let $t=\epsilon\tau$, $x(t,\epsilon)=X(\tau;\epsilon)$ and $y(t;\epsilon)=Y(\tau;\epsilon)$. Then the ODE system becomes

$$\begin{aligned}\frac{dX}{d\tau} &= \epsilon XY,\\ \frac{dY}{d\tau} &= Y-Y^3.\end{aligned}$$

We seek the inner expansions $X\sim X_0(\tau)$, $Y\sim Y_0(\tau)$, and find that at leading order the ODEs reduce to the uncoupled system

$$\begin{aligned}\frac{dX_0}{d\tau} &= 0,\quad X_0(0)=\alpha,\\ \frac{dY_0}{d\tau} &= Y_0-Y_0^3,\quad Y_0(0)=\beta.\end{aligned}$$

The solution for X_0 is simply $X_0(\tau)=\alpha$. The solution for Y_0 is given by

$$Y_0(\tau)=\begin{cases}\frac{\beta}{\sqrt{\beta^2+(1-\beta^2)e^{-2\tau}}} & \text{for } \beta\neq 0,\\ 0 & \text{for } \beta=0.\end{cases}$$

Keeping in mind that $\sqrt{\beta^2}=|\beta|$, we note that $Y_0(\tau)\to\pm1$ as $\tau\to\infty$ according as $\beta\gtrless 0$.

Matching of y and Y selects the outer solution $y_0=0,\ 1$ or -1 according as $\beta=0,\ >0$ or <0. Matching of x and X then selects $a_0=\alpha$, and $x_0=\alpha e^t$, α or αe^{-t} according as $\beta=0,\ >0$ or <0.

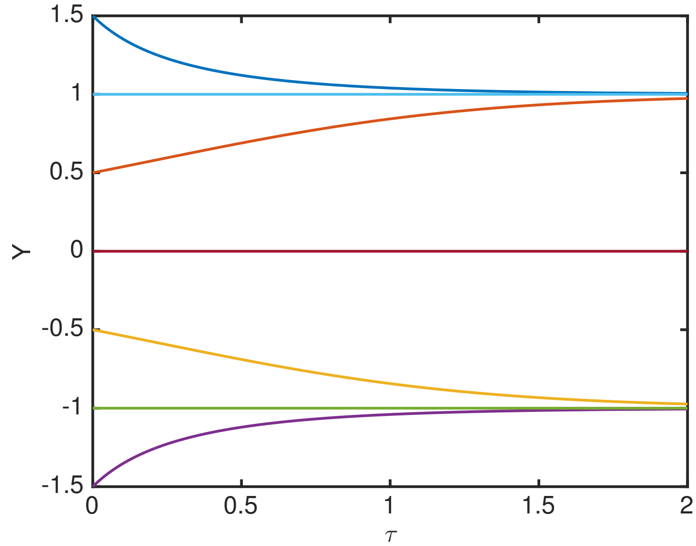

Figure 3: Plots of the leading-order inner solution $Y_0(\tau)$ for $\beta = 1.5,\ 1,\ 0.5,\ 0,\ -0.5,\ -1$ and -1.5. This is also the leading-order composite solution.

The composite solution is

$$x_c(t) \sim \begin{cases} \alpha & \text{for } \beta = 0, \\ \alpha e^{t} & \text{for } \beta > 0, \\ \alpha e^{-t} & \text{for } \beta < 0, \end{cases}, \qquad y_c(t) \sim \begin{cases} \frac{\beta}{\sqrt{\beta^2 + (1-\beta^2)e^{-2t}}} & \text{for } \beta \neq 0, \\ 0 & \text{for } \beta = 0. \end{cases}$$

The solution for y is displayed in Figure 3.

4. Consider the problem

$$(x^2 + \epsilon y)y' + 2xy = \frac{3\epsilon}{2y}, \quad y(1;\epsilon) = 1.$$

Of interest is the domain $0 \le x \le 1$. Find the first two terms of an outer expansion, for $\epsilon > 0$ and small, satisfying the boundary condition at $x = 1$. Show that this expansion is not uniformly valid as $x \to 0$. Find the breakdown, rescale and hence find the first term of the inner expansion valid near $x = 0$. Find the dominant asymptotic behavior of y as $x \to 0$.

Let the outer expansion be $y(x;\epsilon) \sim y_0(x) + \epsilon y_1(x)$. Then, the reduced problems at $O(1)$ and $O(\epsilon)$ are

$$\begin{aligned} x^2 y_0' + 2xy_0 &= 0, \quad y_0(1) = 1, \\ x^2 y_1' + 2xy_1 + y_0 y_0' &= \frac{3}{2y_0}, \quad y_1(1) = 0. \end{aligned}$$

The solutions are

$$\begin{aligned} y_0 &= \frac{1}{x^2}, \\ y_1 &= \frac{1}{2}\left(-\frac{1}{x^6} + x\right). \end{aligned}$$

Therefore the outer expansion is

$$y \sim \frac{1}{x^2} + \frac{\epsilon}{2}\left(-\frac{1}{x^6} + x\right).$$

As $x \to 0$, this expansion breaks down when $x = O(\epsilon^{1/4})$. Correspondingly, $y = O(\epsilon^{-1/2})$. This non-uniformity suggests the inner scalings $x = \epsilon^{1/4}\xi$, $y(x;\epsilon) = \epsilon^{-1/2}Y(\xi;\epsilon)$, which transform the ODE into

$$(\xi^2 + Y)Y' + 2\xi Y = \frac{3\epsilon^2}{2Y}.$$

Let the inner expansion be $Y \sim Y_0(\xi)$. Then Y_0 satisfies the reduced ODE

$$(\xi^2 + Y_0)Y_0' + 2\xi Y_0 = 0,$$

which has the integral

$$\xi^2 Y_0 + \frac{1}{2}Y_0^2 = \frac{K}{2},$$

where K is an arbitrary constant. This quadratic has the solutions

$$Y_0^{(\pm)}(\xi) = -\xi^2 \pm \sqrt{\xi^4 + K}.$$

For matching we shall need the asymptotic behavior of Y_0 as $\xi \to \infty$. A simple expansion shows that

$$\begin{aligned} Y_0^{(+)}(\xi) &\sim -\xi^2 + \xi^2\left(1 + \frac{K}{2\xi^4} - \frac{K^2}{8\xi^8} + \cdots\right) && (1)\\ &= \frac{K}{2\xi^2} - \frac{K^2}{8\xi^6} + \cdots, && (2)\\ Y_0^{(-)}(\xi) &\sim -\xi^2 - \xi^2\left(1 + \frac{K}{2\xi^4} - \frac{K^2}{8\xi^8} + \cdots\right) && (3)\\ &= -2\xi^2 - \frac{K}{2\xi^2} + \cdots. \end{aligned}$$

We expect the inner solution to decay as it leaves the boundary layer. Based on the above asymptotic behavior, therefore, we select

$$Y_0 = Y_0^{(+)} = -\xi^2 + \sqrt{\xi^4 + K} \tag{4}$$

as the relevant inner solution.

Matching. The $O(\epsilon)$ outer expansion expanded to $O(\epsilon^{-1/2})$ in the inner variable:

$$\begin{aligned} y &\sim y_0(\epsilon^{1/4}\xi) + \epsilon y_1(\epsilon^{1/4}\xi)\\ &= \frac{1}{\sqrt{\epsilon}\xi^2} + \frac{\epsilon}{2}\left(-\frac{1}{\epsilon^{3/2}\xi^6} + \epsilon^{1/4}\xi\right)\\ &\sim \frac{1}{\sqrt{\epsilon}}\left(\frac{1}{\xi^2} - \frac{1}{2\xi^6}\right). && (5) \end{aligned}$$

The $O(\epsilon^{-1/2})$ inner expansion expanded to $O(\epsilon)$ in the outer variable:

$$\begin{aligned} \frac{1}{\sqrt{\epsilon}}Y &\sim \frac{1}{\sqrt{\epsilon}}Y_0^{(+)}(\xi)\\ &= \frac{1}{\sqrt{\epsilon}}Y_0^{(+)}\left(\frac{x}{\epsilon^{1/4}}\right). \end{aligned}$$

We need to expand the RHS above to $O(\epsilon)$ as $\epsilon \to 0$ for ξ fixed. Upon using the expansion (1) above we get

$$\begin{aligned} \frac{1}{\sqrt{\epsilon}}Y &\sim \frac{1}{\sqrt{\epsilon}}\left(\frac{K\sqrt{\epsilon}}{2x^2} - \frac{K^2\epsilon^{3/2}}{8x^6}\right)\\ &= \frac{K}{2x^2} - \epsilon\frac{K^2}{8x^6}\\ &= \frac{1}{\sqrt{\epsilon}}\left(\frac{K}{2\xi^2} - \frac{K^2}{8\xi^6}\right). \end{aligned}$$

Matching with (5) yields $K = 2$. We can now write the inner solution (4) as

$$y \sim \frac{1}{\sqrt{\epsilon}} Y_0(\xi) = \frac{1}{\sqrt{\epsilon}}[-\xi^2 + \sqrt{\xi^4 + 2}].$$

The dominant asymptotic behavior of y as $x \to 0$ is given by

$$\frac{1}{\sqrt{\epsilon}} Y_0(0) = \sqrt{\frac{2}{\epsilon}}.$$