PERTURBATION METHODS
SPRING 2016

LESSON 2: Singular-perturbation problems for differential equations: Matched Asymptotic Expansions

The method of matched asymptotic expansions is examined via examples. This lesson supplements Chapter 2 of the Text.

1 Singular Perturbations in ODEs

A regular perturbation problem depends upon ϵ in such a way that its solution $f(x, \epsilon)$ converges as $\epsilon \to 0$ (uniformly with respect to the independent variable x in the relevant domain $\mathcal{D}_x$) to the solution $f(x, 0)$ of the unperturbed problem. In this case a single asymptotic expansion of Poincaré type suffices for $f(x, \epsilon)$ uniformly in D_x. A singular perturbation occurs when the regular perturbation limit $f(x, \epsilon) \to f(x, 0)$ fails to exist. We now examine some ways in which singular perturbation problems arise in ODEs.

Example 1.1. Infinite domain. Consider the IVP for the Duffing equation,

$$x'' + x + \epsilon x^3 = 0,\ x(0) = 1,\ x'(0) = 0.$$

This is the model of an oscillator with a weakly nonlinear spring. We expect the solution to be periodic, and the attached graph indicates so. The exact solution cannot be written in terms of elementary functions, but can be represented implicitly as an integral or in terms of elliptic functions.

A straightforward expansion leads to

$$x \sim \cos t + \epsilon \left(\frac{1}{32}(\cos 3t - \cos t) - \frac{3}{8} t \sin t \right) + \cdots .$$

In the range $0 \le t \le T < \infty$ the expansion holds. However, it does not hold for t arbitrarily large because of the linearly growing term in the $O(\epsilon)$ correction. The nonuniformity occurs when $t = O(1/\epsilon)$. Furthermore, the growing (or secular) term prevents the solution from being periodic. The situation is illustrated in the plots displayed in Figure 1.

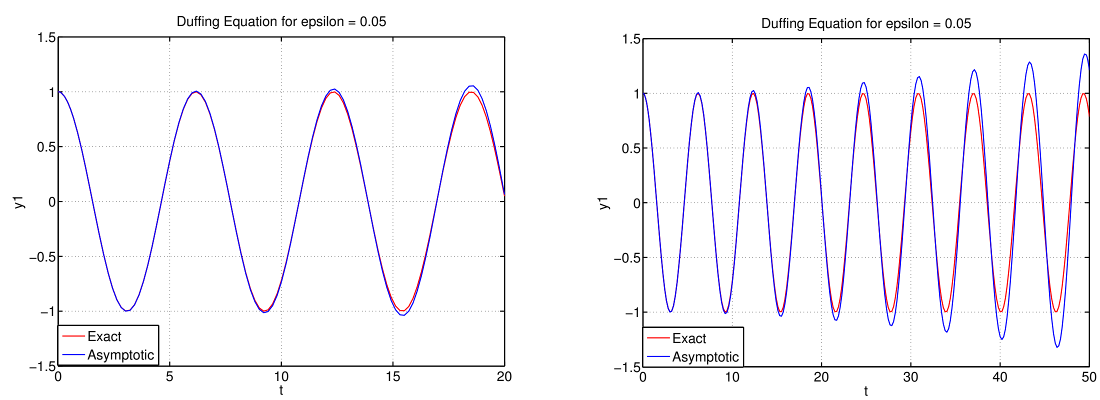

Figure 1: Exact and perturbation solutions for Example 1. Deterioration of the perturbation solution for long times is evident.

The nonlinear problem has a period that is slightly different from 2π. This difference is not adequately captured by the straightforward expansion. Similar difficulty arises in a slightly damped linear oscillator, modelled by the equation $x'' + \epsilon x' + x = 0$. Problems of this sort are handled by the multiple-scale procedure which explicitly recognizes the fact that the solution has two characteristic time scales; one corresponding to the unperturbed oscillation and the other to the amplitude and/or frequency modulation introduced by the perturbation. Over long times the errors accumulate to vitiate the solution unless their effect is explicitly built into the asymptotic ansatz.

Example 1.2. The small parameter multiplying the highest derivative; problem of layer type. Consider the IVP

$$\epsilon x' + x = e^{-t},\ x(0) = 2.$$

The straightforward expansion yields

$$x \sim e^{-t}(1+\epsilon+\cdots).$$

Clearly, it fails to satisfy the initial condition, and is therefore not valid in some region abutting $t=0$. The exact solution is

$$x = \frac{1-2\epsilon}{1-\epsilon}e^{-t/\epsilon} + \frac{e^{-t}}{1-\epsilon}.$$

The exact and perturbation solutions are displayed in Figure 2. For any $t>0$ the first term in the

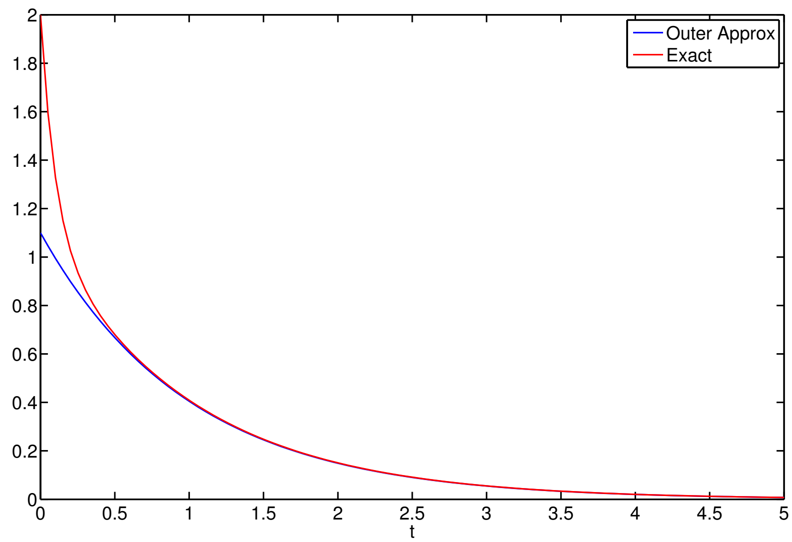

Figure 2: Exact and perturbation solutions for Example 2, plotted for $\epsilon = 0.1$. The perturbation solution fails to capture the layer.

exact solution is exponentially small as $\epsilon \to 0$ while the second term expands to yield the straightforward asymptotic series developed above. However, in a thin region $t = O(\epsilon)$, the second term is essentially a constant, 1 to leading order, while the first term varies over an amount of order unity, and is responsible for the satisfaction of the initial condition. This layer provides a region of rapid transition and in it the derivatives are large ($x'(0) = 1/\epsilon$). The straightforward expansion fails in this region because its derivation did not recognize the unboundedness of the derivative there. As we shall see later, the remedy is to develop a separate expansion of Poincaré type in the layer, and to then combine the two expansions in an appropriate way (matching) to produce a composite expansion that is uniformly valid.

Example 1.3. The small parameter multiplying the highest derivative; multiple layers. Consider the BVP

$$\epsilon^2 x'' - x = 0, \; 0 \le t \le 1.$$

The exact solution is

$$x = ae^{-t/\epsilon} + be^{-(1-t)/\epsilon},$$

where

$$a = \frac{x(0) - x(1)e^{-1/\epsilon}}{1-e^{-2/\epsilon}}, \; b = \frac{x(1) - x(0)e^{-1/\epsilon}}{1-e^{-2/\epsilon}}.$$

As $\epsilon \to 0$, the solution is exponentially small in $0 < t < 1$ (agreeing with the straightforward expansion $x = 0$ to all orders) with layers of thickness ϵ at each end in which the boundary conditions are accommodated.

Example 1.4. The small parameter multiplying the highest derivative; global rapid variation. Consider the BVP

$$\epsilon^2 x'' + x = 0,\ 0 \le t \le 1,\ x(0) = 0,\ x(0) = 1.$$

The exact solution is

$$x = \frac{\sin(t/\epsilon)}{\sin(1/\epsilon)}.$$

It exhibits a rapid oscillation for small ϵ, while the straightforward expansion yields $x = 0$ to all orders. The exact solution is displayed in Figure 3. Problems of this kind need an expansion procedure that, as we shall see later, explicitly recognizes the globally rapidly-varying character of the solution (the WKBJ procedure).

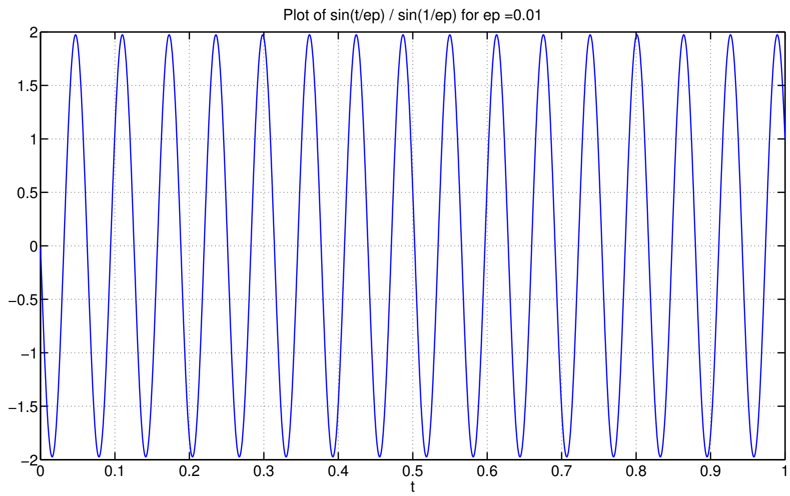

Figure 3: The exact solution for Example 4 for $\epsilon = 0.01$.

A Note on Patching and Matching

As already suggested, layer-type problems are handled by a procedure (the method of matched asymptotic expansions) in which the region D_x is broken into two (or more) subintervals, and separate Poincare type expansions are developed in each by applying an appropriate limit process. The crucial point is that these expansions have domains of validity that overlap, and in each overlap region we require that the expansions from either side agree in an appropriate way so that they 'blend smoothly together.' The agreement is to hold over an interval, which is more than requiring that the adjacent expansions are 'patched together' to a requisite degree of smoothness at the boundary between abutting regions.

Example of patching. Consider the BVP

$$y'' - y = e^{-|x|},\ y(\pm\infty) = 0.$$

The solution satisfying the left BC and valid in $x < 0$ is

$$y_- = ae^x + \frac{1}{2}xe^x,$$

while that valid in $x > 0$ and satisfying the right BC is

$$y_+ = be^{-x} - \frac{1}{2}xe^{-x}.$$

The constants a and b are found by patching the two solutions together at $x = 0$, *i.e.*, requiring that y and y' be continuous at $x = 0$. The result is

$$y = -\frac{1}{2}e^{-|x|} - \frac{1}{2}|x|e^{-|x|}.$$

Example of matching. Consider the function

$$f(x,\epsilon) = 1 + x + \epsilon + e^{-x/\epsilon}, \quad 0 \le x \le 1, \tag{1.1}$$

which exhibits a layer behavior in a region of thickness ϵ at $x = 0$. Outside the layer an asymptotic approximation to order unity is

$$f_0(x) = 1 + x. \tag{1.2}$$

This approximation corresponds to the limit process $\epsilon \to 0$, with $x > 0$ and fixed. We note that the last term in f is then exponentially small. In the layer the appropriate coordinate is $\xi = x/\epsilon$. The limit process $\epsilon \to 0$ with ξ fixed then yields the leading approximation

$$g_0(\xi) = 1 + e^{-\xi}. \tag{1.3}$$

Note the behavior of the errors. The error

$$f(x,\epsilon) - f_0(x) = \epsilon + e^{-x/\epsilon} \tag{1.4}$$

in the outer approximation is $o(1)$ in intervals of form $0 < x_0 \le x \le 1$, while the error

$$f(x,\epsilon) - g_0(\xi) = \epsilon + \epsilon\xi \tag{1.5}$$

in the layer approximation is $o(1)$ in intervals of form $0 \le \xi \le \xi_0$, where x_0 and ξ_0 are constants independent of ϵ. The second interval above can also be written as $0 \le x \le \epsilon\xi_0$. It is $O(\epsilon)$ in width on the x-scale and shrinks towards 0 as $\epsilon \to 0$. There would thus seem to be a gap between the two domains.

Closer inspection reveals that the domains of validity are conservative. The error in (4) remains $o(1)$ over a wider interval of form $0 < \mu(\epsilon) \le x \le 1$, in which the left end point $\mu(\epsilon)$ shrinks to zero with ϵ provided $O(\mu) > \epsilon$. The error increases to be sure, but remains $o(\epsilon)$. Similarly, the domain of validity of the layer solution can be extended to $0 \le \xi \le \eta(\epsilon)$, or $0 \le x \le \epsilon\eta(\epsilon)$, where $O(\eta) < 1/\epsilon$. Furthermore, $\mu(\epsilon)$ and $\eta(\epsilon)$ may be chosen such that the two domains continue to overlap as $\epsilon \to 0$. One possible pair of choices are $\mu = \epsilon^{1/2}$ and $\epsilon\eta = \epsilon^{1/4}$. Then the overlap domain is $O(\epsilon^{1/2}) \le x \le \epsilon^{1/4}$. In particular, $x = \epsilon^{1/3}\zeta$, where $\zeta > 0$, lies in the overlap domain.

Matching to leading order is now achieved by requiring that under the limit process $\epsilon \to 0$ and ζ fixed, the outer and layer approximations agree to order unity. We have, under this limit process,

$$f_0(x) = 1 + x = 1 + \epsilon^{1/3}\zeta = 1 + o(1)$$

and

$$g_0(\xi) = 1 + e^{-\xi} = 1 + \exp(-\epsilon^{-2/3}\zeta) = 1 + o(1).$$

The leading-order composite expansion is then constructed according to the recipe

$$\begin{aligned} f_c(x,\epsilon) &= f_0 + g_0 - \text{the matched part} && (1.6)\\ &= (1 + x) + (1 + e^{-\xi}) - 1 && (1.7)\\ &= 1 + x + e^{-x/\epsilon}. && (1.8) \end{aligned}$$

This is clearly not of Poincaré type but is uniformly valid since

$$f(x,\epsilon) - f_c(x,\epsilon) = \epsilon = o(1) \text{ for } 0 \le x \le 1.$$

Figure 4 shows the exact function and its outer, inner and composite expansions.

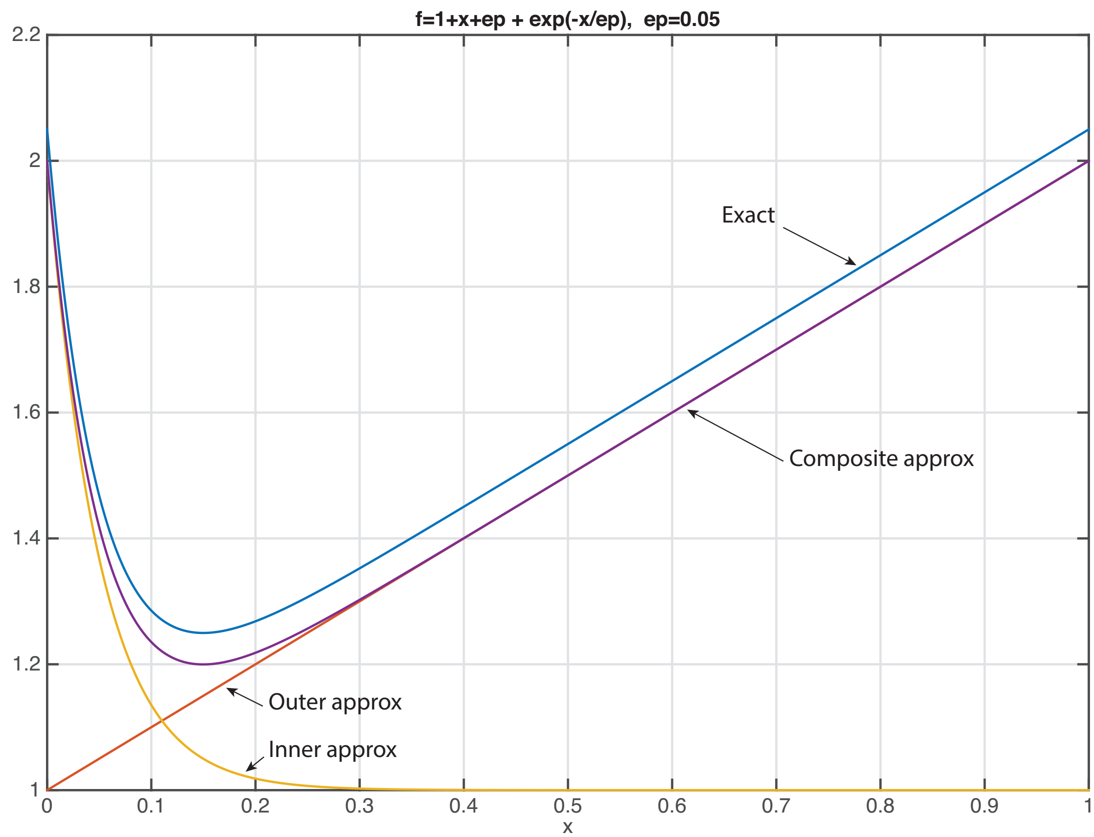

Figure 4: Plots of the exact function $f(x;\epsilon)$, the outer approximation $f_0(x)$, the inner approximation $g_0(\epsilon x)$ and the composite approximation $f_c(x;\epsilon)$ for $\epsilon = 0.05$. Note the smooth blending of the outer and inner approximations into the composite approximation.

2 Layer-type problems: matched asymptotic expansions

2.1 Introduction

We now focus on layer-type problems which, as we have already seen, are characterized by the presence of one or more thin regions in which the solution undergoes a rapid variation. These regions, or layers, shrink in thickness as $\epsilon \to 0$. A prototypical example, and one that gave birth to the subject more than a century ago, is the viscous boundary layer present in the flow of a fluid of small viscosity past a solid wall first examined by L. Prandtl in 1905. Outside the layer the fluid is essentially inviscid, while within the layer viscosity serves to rapidly adjust the fluid velocity to that of the wall. Smaller the viscosity, the thinner is the layer. It is the solution in the layer that determines the viscous drag exerted on the wall by the motion of the fluid.

Matched asymptotic expansions is a technique designed specifically to handle layer-type problems. It uses a set of complementary expansions to approximate the solution $f(x, \epsilon)$ in different parts of D_x. The expansion appropriate for the layer is generally called an *inner* expansion, while that appropriate for the region outside the layer is called an *outer* expansion. When one is attempting to solve an ODE or a PDE by this procedure, the reduced problems defining these expansions are incompletely posed due to loss of auxiliary (initial or boundary) conditions outside the regions in question. The missing information is retrieved by a *matching* procedure, which reflects the fact that different asymptotic expansions of the same function in overlapping regions should be related. We illustrate the procedure by examining a number of examples. You are strongly encouraged to also work through the examples discussed in Chapter 2 of the Holmes text.

Example 2.1. Consider the boundary-value problem

$$\epsilon \frac{d^2y}{dx^2} + \frac{dy}{dx} = 1 + 2x, \ 0 < x < 1, \ y(0) = 1, \ y(1) = 4. \tag{2.1}$$

Although the exact solution is available, we aim to develop an asymptotic approximation to it for small ϵ. As $\epsilon \to 0$ we ignore $\epsilon y''$, expecting this approximation to hold in that portion of the interval (0,1) where the derivatives are bounded. The reduced equation

$$\frac{dy}{dx} = 1 + 2x$$

is only a first-order ODE and has the general solution

$$y_0(x) = x^2 + x + a_0.$$

The integration constant a_0 can be chosen to satisfy at most one of the two boundary conditions. Thus y_0 can approximate y only in some *outer* region, departing significantly from it in some *inner* region where $\epsilon y''$ cannot be ignored. In this inner region, therefore, y'' must grow without bound as $\epsilon \to 0$.

The location of the inner region, or layer, is yet to be determined. It may be near either boundary, in the interior, and indeed, there may be more than one layers. In this problem the possibilities can be narrowed by recognizing that if an interior layer were to exist, it would contain points at which $y' > 1 + 2x$ and $y'' > 0$, which is incompatible with (2.1). The same argument rules out a boundary layer at $x = 1$, leaving that at $x = 0$ as the only possibility. Thus it is appropriate for y_0 to satisfy the boundary condition at $x = 1$, whence

$$y_0 = x^2 + x + 2, \tag{2.2}$$

and we note that this solution does not satisfy the boundary condition at $x = 0$. We now turn to the inner region at the left boundary where a rapid accommodation must take place. The boundary-layer thickness must shrink as $\epsilon \to 0$ because in the layer, y'' must become (negatively) unbounded, with y' remaining small compared with $|y''|$. The nature of the rapid transition can be understood by magnifying the layer so that its shrinkage, as $\epsilon \to 0$, is compensated. We employ the *stretching* transformation

$$x = \delta(\epsilon)\xi, \ \ y(x) = Y(\xi),$$

Then, the ODE and the left boundary condition in (2.1) become

$$\frac{\epsilon}{\delta^2}\frac{d^2Y}{d\xi^2} + \frac{1}{\delta}\frac{dY}{d\xi} = 1 + 2\epsilon\xi, \quad Y(0) = 1. \tag{2.3}$$

The right boundary condition is abandoned in favor of the requirement that the inner and outer approximations match.

We now assume that in the stretched variable ξ, the derivatives are bounded as $\epsilon \to 0$. The magnification factor δ is chosen so that there is a dominant balance in the ODE, a balance in which the second derivative participates. Here the only choice is $\delta = \epsilon$. Then, (2.3) becomes

$$\frac{d^2Y}{d\xi^2} + \frac{dY}{d\xi} = \epsilon + 2\epsilon^2\xi. \tag{2.4}$$

At first approximation one obtains the redued equation

$$\frac{d^2Y}{d\xi^2} + \frac{dY}{d\xi} = 0$$

whose solution satisfying $Y_0(0) = 1$ is

$$Y_0 = b_0 + (1 - b_0)e^{-\xi}, \tag{2.5}$$

where the arbitrary constant b_0 is to be determined by matching.

Now we expect the outer approximation $y_0(x)$ to hold in a region $0 < x_* \le x \le 1$ bounded away from the left end, and the inner approximation in a region $0 \le x/\epsilon \le \xi_* < \infty$ adjacent to the left end. For fixed x_* and ξ_* these two regions will be separated by a gap as $\epsilon \to 0$. However, it was revealed in class (following consideration of an explicit example) that the above estimates of the regions of validity are conservative. In particular, it is possible to extend the region on which $y_0(x)$ is valid to intervals whose left end point goes to zero with ϵ, but at a rate slower than that at which the boundary layer shrinks. Then there exists an *intermediate* domain on which the two approximations overlap; this is the basic hypothesis underlying the method of matched asymptotic expansions. Matching simply asserts that in the overlap region the two aproximations must agree since they represent the same function.

In order to carry out the mechanics of matching one defines an intermediate variable η via the relation

$$x = \mu(\epsilon)\eta,$$

where μ satisfies

$$\mu(\epsilon) = \epsilon^\lambda, \quad 0 < \lambda < 1, \tag{2.6}$$

thereby ensuring that the intermediate region, defined by $\eta = O(1)$, lies between the outer region, $x = O(1)$, and the inner region, $x = O(\epsilon)$. With the intermediate region one can associate an intermediate limit process, $\epsilon \to 0$ with η fixed, in analogy with the outer limit process, $\epsilon \to 0$ with x fixed, and the inner limit process, $\epsilon \to 0$ with ξ fixed. Observe that in the intermediate limit, $x = \mu\eta \to 0$ while $\xi = \mu\eta/\epsilon \to \infty$. Now, matching to $O(1)$ requires that the outer and inner approximations agree to $O(1)$ in the intermediate limit, i.e.,

$$R_0 \equiv y_0(\mu\eta) - Y_0(\mu\eta/\epsilon) \to 0 \text{ as } \epsilon \to 0, \ \eta \text{ fixed}.$$

In the present problem we have

$$\begin{aligned} R_0 &= 2 + \mu\eta + \mu^2\eta^2 - [b_0 + (1 - b_0)e^{-\mu\eta/\epsilon}] \\ &= 2 - b_0 + O(\mu) + O(e^{-\mu\eta/\epsilon}) \text{ as } \epsilon \to 0, \eta \text{ fixed}, \end{aligned}$$

so that we must choose

$$b_0 = 2. \tag{2.7}$$

The remaining terms in R_0 above are vanishingly small in view of our choice of $\mu = \epsilon^\lambda$. In fact, we could have left μ unspecified, allowing it to be determined by the vanishing requirement in the above step. The inner approximation (2.5) is now fully determined, as

$$Y_0 = 2 - e^{-\xi}. \tag{2.8}$$

We can now onstruct an additive *composite approximation* by means of the following prescription:

$$\text{composite approximation} = \text{outer approximation} + \text{inner approximation - matched part.} \tag{2.9}$$

In the present case the matched part, i.e., the part common to the inner and outer approximations in the intermediate limit, is 2. Therefore the leading-order composite approximation is

$$\begin{aligned} y_{c0}(x) &= (2 + x + x^2) + (2 - e^{-x/\epsilon}) - 2 \\ &= 2 + x + x^2 - e^{-x/\epsilon}. \end{aligned}$$

Remarks.

1. In the above example the location of the layer was deduced on the basis of an argument about the structure of the solution. This is not always possible. An alternative is to assume a location and see if a coherent construction emerges. Otherwise one must start with a different assumption and try again. *A priori* knowledge of the qualitative character of the solution, based on mathematical or physical arguments, is often helpful. In some problems, position of the layer may be indicated by the appearance of a singularity in the outer solution, or by the disordering of the outer expansion.

2. The choice of the stretching function $\delta(\epsilon)$ is guided by the requirement that the inner problem capture the essential elements omitted in the outer problem, and that the inner and the outer approximations match.

3. Once the outer and the inner approximations have been obtained, a natural tendency may be to "patch" the two approximations together by having them agree at a certain location. While such a procedure may not be unreasonable numerically, it suffers from two shortcomings: the patching location is not unique, and more importantly, the patched solution will have a kink. Matching, as described above, produces a smooth solution.

4. The composite soluion is uniformly valid across the entire domain. However, it is not a Poincaré-type expansion. Also, it does not satisfy all the boundary conditions exactly, but does so at least to the approximation of the solution. (Check in the present case.)

We now turn to the construction of the higher-order terms in our approximation to the solution of (2.1). It is natural to assume that the outer and inner expansions proceed, respectively, as

$$y \sim \sum_0^\infty \epsilon^n y_n(x), \tag{2.10}$$

$$Y \sim \sum_0^\infty \epsilon^n Y_n(\xi). \tag{2.11}$$

On substituting the outer expansion (2.10) into the ODE and the right boundary condition in (2.1), and setting the coefficients of ϵ^n $(n = 0, 1, 2, \cdots)$ to zero, we obtain an array of problems for the y_n. Of these the first has already been solved. The second is

$$y_1' = -y_0'', \;\; y_1(1) = 0,$$

with the solution

$$y_1 = 2 - 2x. \tag{2.12}$$

Higher-order terms may be computed in a similar way. Analogously, substitution of the inner expansion (2.11) into (2.3) generates a sequence of problems for the Y_n. We already know Y_0 and Y_1 satisfies

$$Y''(\xi) + Y'(\xi) = 1, \;\; Y_1(0) = 0,$$

whose solution is

$$Y_1 = b_1(1 - e^{-\xi}) + \xi. \tag{2.13}$$

The constant b_1 is to be determined by matching.

The inner and outer expansions have now each been carried to $O(\epsilon)$. Prior to matching, we apply the intermediate limit, $\epsilon \to 0$ with η fixed, to each. Thus,

$$\begin{aligned} y(\mu\eta) &\sim y_0(\mu\eta) + \epsilon y_1(\mu\eta) \\ &= 2 + \mu\eta + \mu^2\eta^2 + \epsilon(2 - 2\mu\eta) \\ &= 2 + \mu\eta + 2\epsilon + O(\mu^2) + O(\epsilon\mu), \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} Y(\mu\eta/\epsilon) &\sim Y_0(\mu\eta/\epsilon) + \epsilon Y_1(\mu\eta/\epsilon) \\ &= 2 - e^{-\mu\eta/\epsilon} + \epsilon\{b_1(1 - e^{-\mu\eta/\epsilon}) + \mu\eta/\epsilon\} \\ &= 2 + \mu\eta + \epsilon b_1 + \text{TST}, \end{aligned} \tag{2.15}$$

where TST stands for such transcendentally small terms as $e^{-\mu\eta/\epsilon}$. We now match to $O(\epsilon)$ by requiring that (2.14) and (2.15) agree to $O(\epsilon)$, i.e.,

$$R_1 \equiv \frac{1}{\epsilon}[\{y_0(\mu\eta) + \epsilon y_1(\mu\eta)\} - \{Y_0(\mu\eta/\epsilon) + \epsilon Y_1(\mu\eta/\epsilon)\}] \to 0 \text{ as } \epsilon \to 0.$$

The expressions (2.14) and (2.15) show that

$$R_1 = 2 - b_1 + O(\mu) + O(\mu^2/\epsilon) + \text{ TST as } \epsilon \to 0, \ \eta \text{ fixed},$$

whence

$$b_1 = 2, \tag{2.16}$$

provided the (μ^2/ϵ) term vanishes in R_1 above. For that to happen, the size of the intermediate region can no longer be given by (2.6) but must shrink to $\mu = \epsilon^\lambda$, $1/2 < \lambda < 1$. *This is a general feature of the matching procedure; as the order of approximation increases, the overlap region shrinks.*

A 2-term composite expansion may now be obtained by appealing to the formula (2.9), once we recognize that the matched part (common to (2.14) and (2.15)) is now $2 + x + 2\epsilon$. The result is

$$y_{c1} \sim 2 + x + x^2 + \epsilon(2 - 2x) - (1 + 2\epsilon)e^{-x/\epsilon}. \tag{2.17}$$

The above procedure extends to higher orders in a straightforward way.

Remarks.

1. In this problem the obvious choice ϵ^n for the asymptotic sequence turned out to be the correct one. In general, however, the same asymptotic sequence may not suffice for both the expansions. Also, the expansions may involve fractional powers of ϵ, and even logarithms. In such cases the asymptotic sequences are determined one term at a time, by matching the two expansions at each stage as fresh terms are computed. Occasionally one needs to go back and inset a term at an earlier stage in order to ensure a consistent development of the expansions. In practice, more than a handful of terms are seldom computed, and often one is content with just the leading term.

2. The composite solution (2.17) can easily be checked to find that it is a uniform expansion, to $O(\epsilon)$, of the exact solution
$$y_{\text{exact}} = \frac{2 + 2\epsilon - e^{-1/\epsilon} - (1 + 2\epsilon)e^{-x/\epsilon}}{1 - e^{-1/\epsilon}} + x^2 + x(1 - 2\epsilon).$$
For $\epsilon = 0.1$ Figure 5 plots the exact solution along with the composite expansions y_{c0} and y_{c1} derived above. The exact solution and y_{c1} are indistinguishable.

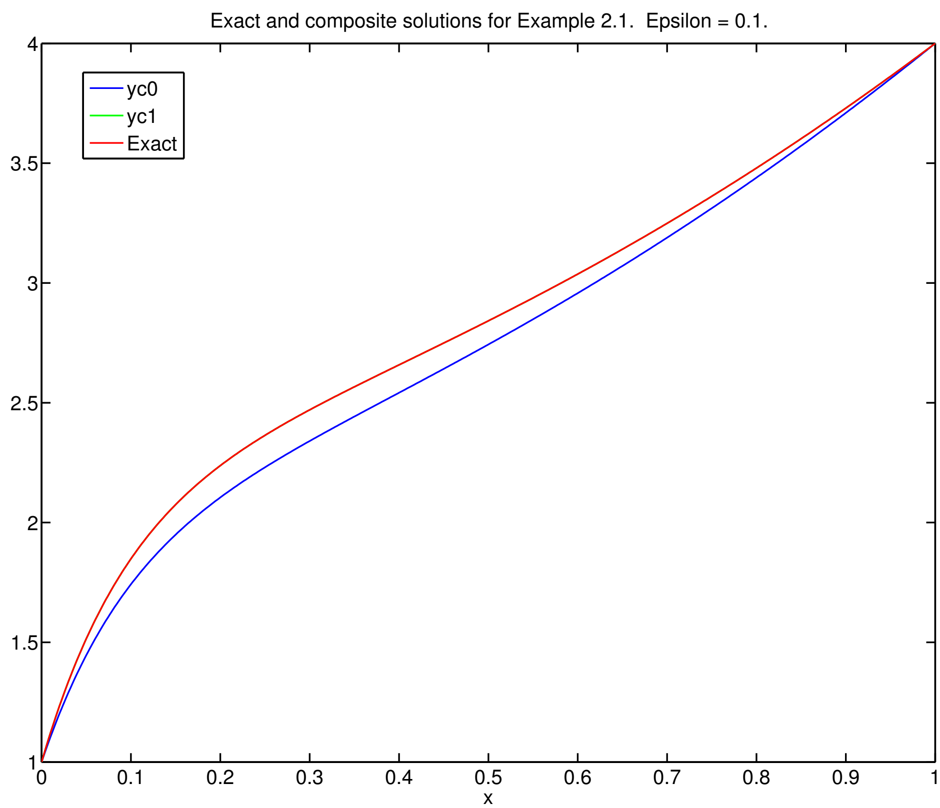

Figure 5: Exact and composite solutions for Example 2.1.

3. In this example we were able to secure an $O(\epsilon)$ match between the inner and outer expansions which themselves had been computed to $O(\epsilon)$ each. On occasions one finds that in order to obtain a match at a specified order, say δ, one or both of the expansions being matched need to be computed to orders higher than δ.

4. An alternative approach to matching, which does not explicitly employ an intermediate variable, was enunciated by Van Dyke and is embodied in the following principle:

   Van Dyke's Matching Principle: The inner expansion to order Δ (of the outer expansion to order δ) = the outer expansion to order δ (of the inner expansion to order Δ), where δ and Δ are any two gauge functions.

2.2 Linear equations with variable coefficients

2.2.1 Theoretical considerations

In the linear problem considered in Example 2.1 the differential operator had constant coefficients. We now consider the following, general boundary-value problem with variable coefficients.

$$\epsilon y''(x) + a(x)y'(x) + b(x)y = 0, \quad y(0) = A,\ y(1) = B.$$

Before we begin the perturbation procedure, it is useful to ascertain that the problem has a solution and that it is unique. Existence and uniqueness are guaranteed if the corresponding problem with homogeneous boundary conditions has only the null solution.

> You can check that the problem $y''(x) + \pi^2 y(x) = 0,\ y(0) = y(1) = 0$ has the nontrivial solution $y = \sin \pi x$, and that the solution to the inhomogeneous problem $y''(x) + \pi^2 y(x) = 0,\ y(0) = 0,\ y(1) = 1$ does not exist.

The above ODE can be put in a canonical form by setting

$$y(x) = \phi(x)w(x),$$

where

$$\phi(x) = \exp\left(-\frac{1}{2\epsilon}\int_0^x a(t)\, dt\right),$$

and w satisfies

$$\epsilon^2 w'' - \left(\frac{a^2}{4} - \epsilon b + \frac{1}{2}\epsilon a'\right) w = 0.$$

It can be shown that existence and uniqueness are guaranteed by the requirement

$$\frac{a^2}{4} - \epsilon b + \frac{1}{2}\epsilon a' > 0.$$

For small ϵ the above holds if

$$a(x) \neq 0,\ |a'(x)| < \infty,\ b(x) < \infty, \quad \text{for} \quad x \in [0, 1].$$

Three cases can be distinguished.

Case A. When $a(x) \neq 0)$ there is an exponential boundary layer at $x = 0$ for $a > 0$ and at $x = 1$ for $a < 0$. Example 2.1 discussed above corresponds to this case.

Case B. $a(x)$ vanishes at one end point but is nonzero elsewhere. Let $a(0) = 0$ and $a(x) > 0,\ x \in (0, 1]$. Then the reduced ODE $a(x)y' + b(x)y = 0$ is singular at $x = 0$ and its solution will not, in general, satisfy the boundary condition at $x = 0$. Therefore a boundary layer will occur at $x = 0$. The nature of the zero at $x = 0$ is crucial. We shall only consider zeros of the form $a(x) \sim x^p$ as $x \to 0$. The cases $0 < p < 1$ and $p > 1$ are significantly different. Example 2.2 below corresponds to this case.

Case C. $a(x)$ changes sign in the interval. Such problems are known as *turning-point problems*. Let the interval be $[-1, 1]$ and let $a(x)$ have a simple zero at $x = 0$. For $a' > 0$ there are no boundary layers but an interior layer at $x = 0$ may occur. For $a' < 0$ an exponential boundary layer is possible at each end, but the solution may be under-determined. Examples 2.3 - 2.5 below fall in this category.

Example 2.2. Consider the boundary-value problem

$$\epsilon y'' + x^2 y' - y = 0, \quad y(0) = 1,\ y(1) = 2. \tag{2.18}$$

Here, $a(x) = x^2$ vanishes at the left boundary $x = 0$ but is positive elsewhere. Thus we expect a boundary layer at $x = 0$. We seek the outer solution $y \sim y_0(x)$ which satisfies

$$x^2 y_0' - y_0 = 0, \quad y(1) = 2.$$

The solution is

$$y_0(x) = 2\exp\left(1 - \frac{1}{x}\right).$$

Note that $y_0 \to 0$ as $x \to 0+$, and therefore does not satisfy the boundary condition there.
In the inner region we set $x = \delta\xi,\ y(x) = Y(\xi)$, so that the ODE transforms into

$$\frac{\epsilon}{\delta^2} Y''(\xi) + \delta\xi^2 Y'(\xi) - Y = 0, \quad Y(0) = 1,$$

and we expect the solution to satisfy the left boundary condition. Dominant balance will occur between the first and third terms of the ODE, yielding $\delta = \sqrt{\epsilon}$, so that the ODE becomes

$$Y''(\xi) - Y = -\sqrt{\epsilon}\, Y'(\xi).$$

The solution satisfying $Y_0(0) = 1$ is

$$y_0 = (1 - A_0)e^{-\xi} + A_0 e^{\xi}.$$

We now perform matching by using Van Dyke's Matching Principle enunciated at the end of Section 2.1, by taking $\delta = \Delta = O(1)$.
Outer expansion expanded in the inner variable to $O(1)$:

$$\begin{aligned} y_0(x) &= 2\exp\left(1 - \frac{1}{x}\right) \\ &= 2\exp\left(1 - \frac{1}{\sqrt{\epsilon}\xi}\right) \\ &\sim 0. \end{aligned}$$

Inner expansion expanded in the outer variable to $O(1)$:

$$\begin{aligned} Y_0(\xi) &= (1 - A_0)e^{-\xi} + A_0 e^{\xi} \\ &= (1 - A_0)e^{-x/\sqrt{\epsilon}} + A_0 e^{x/\sqrt{\epsilon}} \\ &\sim 0, \end{aligned}$$

provided $A_0 = 0$. Expansion for $\epsilon \to 0$ and ξ fixed reduces the first term to zero and the second to infinity unless $A_0 = 0$. Thus matching has determined the unknown coefficient A_0 in the inner expansion. Then,

$$Y_0(\xi) = e^{-\xi}.$$

As the matched part is zero, the composite expansion is just the sum of the inner and outer expansions:

$$y_c = y_0(x) + Y_0(\xi) = y_0(x) + Y_0(\sqrt{\epsilon}x) = 2\exp\left(1 - \frac{1}{x}\right) + e^{-x/\sqrt{\epsilon}}.$$

The outer, inner and composite expansions are shown in Figure 6. Note the smooth blending of the outer and inner expansions to form the composite.

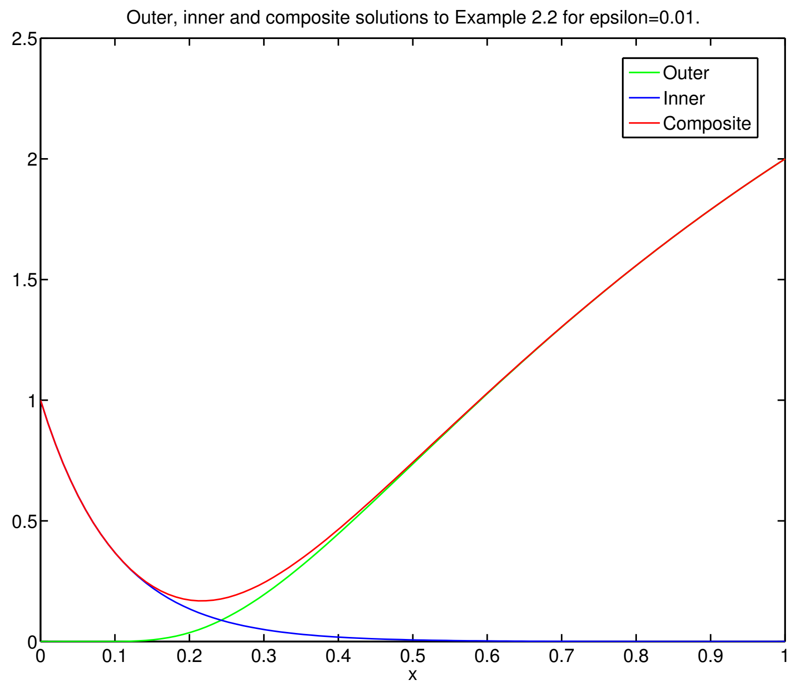

Figure 6: Outer, inner and composite expansions for $\epsilon = 0.01$ in Example 2.2.

2.2.2 Examples of turning-point problems

Example 2.3. Consider the boundary-value problem

$$\epsilon y'' + xy' + xy = 0, \quad -1 < x < 1, \quad y(-1) = e, \; y(1) = 2/e.$$

Here $a(x)$, the coefficient of y' in the ODE, equals x, and as such is negative for $x < 0$ and positive for $x > 0$, with a simple zero at $x = 0$. We cannot have a boundary layer at either end because, as can be seen with a little bit of work, inner solutions in such boundary layers will have exponential growth as they exit the layer and will therefore be unmatchable with an outer solution. An interior layer at a location other than $x = 0$ must be discarded for the same reason, leaving $x = 0$ as the only site for an interior layer.

Outer solution. Let

$$y \sim y_0(x) + \cdots$$

in the outer region. Then y_0 satisfies

$$xy_0' + xy_0 = 0, \quad \text{or} \quad y_0' + y_0 = 0 \quad \text{for} \quad x \neq 0.,$$

with solution

$$y_0 = a_0 e^{-x}.$$

In accordance with the arguments made above this solution will hold for both $-1 < x < 0$ and $0 < x < 1$, satisfying the appropriate boundary condition in each case. The result is

$$y_0(x) = \begin{cases} e^{-x}, & -1 < x < 0, \\ 2e^{-x}, & 0 < x < 1. \end{cases}$$

This solution exhibits a discontinuity at $x = 0$.

Inner solution. Let the inner region be defined by $x = \delta(\epsilon)\xi$, $y(x) = Y(\xi)$. Then the ODE becomes

$$\frac{\epsilon}{\delta^2}\frac{d^2Y}{d\xi^2} + \frac{\delta\xi}{\delta}\frac{dY}{d\xi} + \delta\xi Y = 0.$$

The second term being $O(1)$ dominates the third which is $O(\delta)$ and must therefore be in dominant balance with the first, thereby determining $\delta = \sqrt{\epsilon}$. The ODE then reads

$$Y'' + \xi Y + \sqrt{\epsilon}\,\xi Y = 0.$$

We now seek the inner expansion

$$Y \sim Y_0 + \cdots,$$

where Y_0 satisfies

$$Y_0'' + \xi Y_0' = 0.$$

The first integral, easily found by using an integrating factor, is

$$Y_0'(\xi)e^{\xi^2/2} = \text{constant} = \sqrt{\frac{2}{\pi}}\,b_0,\ \text{say},$$

which, following a second integration, leads to the inner solution

$$\begin{aligned}
Y_0(\xi) &= \sqrt{\frac{2}{\pi}}\,b_0\int_0^{\xi} e^{-s^2/2}ds + c_0 \\
&= b_0\frac{2}{\sqrt{\pi}}\int_0^{\xi/\sqrt{2}} e^{-t^2}dt + c_0 \\
&= b_0\ \mathrm{erf}(\xi/\sqrt{2}) + c_0.
\end{aligned}$$

The integration constants b_0 and c_0 will be determined by matching.

Matching. We must match the inner solution separately with the outer solution on the left and on the right, and will do so using the Van Dyke Principle, applied to order unity.

From the left, the inner expansion to $O(1)$ of the outer expansion to $O(1)$ is given by

$$\begin{aligned}
y_0(x) &= e^{-x} \\
&= e^{-\sqrt{\epsilon}\xi} && \text{(written in the inner variable } \xi) \\
&\sim 1 && \text{(expanded to } O(1) \text{ in the `inner' limit } \epsilon \to 0,\ \xi \text{ fixed).}
\end{aligned}$$

To the left, the outer expansion to $O(1)$ of the inner expansion to $O(1)$ is given by

$$\begin{aligned}
Y_0(\xi) &= b_0\ \mathrm{erf}(\xi/\sqrt{2}) + c_0 \\
&= b_0\ \mathrm{erf}(x/\sqrt{2\epsilon}) + c_0 && \text{(written in the outer variable } x) \\
&= b_0\ \mathrm{erf}(-\infty) + c_0 && \text{(expanded to } O(1) \text{ in the `outer' limit } \epsilon \to 0,\ x < 0 \text{ and fixed)} \\
&= -b_0 + c_0.
\end{aligned}$$

Here we have used the result $\mathrm{erf}(-\infty) = -1$. Matching yields

$$-b_0 + c_0 = 1. \tag{2.19}$$

In an entirely analogous way we match to the right.

From the left, the inner expansion to $O(1)$ of the outer expansion to $O(1)$ is given by

$$\begin{aligned}
y_0(x) &= 2e^{-x} \\
&= 2e^{-\sqrt{\epsilon}\xi} && \text{(written in the inner variable } \xi) \\
&\sim 2 && \text{(expanded to } O(1) \text{ in the `inner' limit } \epsilon \to 0,\ \xi \text{ fixed).}
\end{aligned}$$

To the right, the outer expansion to $O(1)$ of the inner expansion to $O(1)$ is given by

$$\begin{aligned} Y_0(\xi) &= b_0 \operatorname{erf}(\xi/\sqrt{2}) + c_0 \\ &= b_0 \operatorname{erf}(x/\sqrt{2\epsilon}) + c_0 && \text{(written in the outer variable } x) \\ &= b_0 \operatorname{erf}(\infty) + c_0 && \text{(expanded to } O(1) \text{ in the `outer' limit } \epsilon \to 0,\ x > 0 \text{ and fixed)} \\ &= b_0 + c_0. \end{aligned}$$

Here we have used the result $\operatorname{erf}(\infty) = 1$. Matching yields

$$b_0 + c_0 = 2. \tag{2.20}$$

Equations (2.19) and (2.20) determine $b_0 = 1/2,\ c_0 = 3/2$.

Composite expansion. A single additive composite expansion is not possible when the outer solution is discontinuous across the inner layer. It is, however, possible to construct a multiplicative composite expansion in the present case:

$$y_{c0} \sim \left(\frac{1}{2}\operatorname{erf}(x/\sqrt{2\epsilon}) + \frac{3}{2}\right) e^{-x}.$$

In the layer, $x = O(\sqrt{\epsilon})$. Then $e^{-x} = 1 + O(\sqrt{\epsilon})$ and y_{c0} reduces to the inner solution to leading order. In the outer region $x > 0$, $x/\sqrt{2\epsilon} \to \infty$ so that $\operatorname{erf}(x/\sqrt{2\epsilon}) \to 2$ and therefore, $y_{c0} \to 2e^{-x}$, the right outer solution. In the outer region $x < 0$, $x/\sqrt{2\epsilon} \to -\infty$ so that $\operatorname{erf}(x/\sqrt{2\epsilon}) \to 1$ and therefore, $y_{c0} \to e^{-x}$, the left outer solution. A graph of the composite solution is drawn in Figure 7 for $\epsilon = 0.001$.

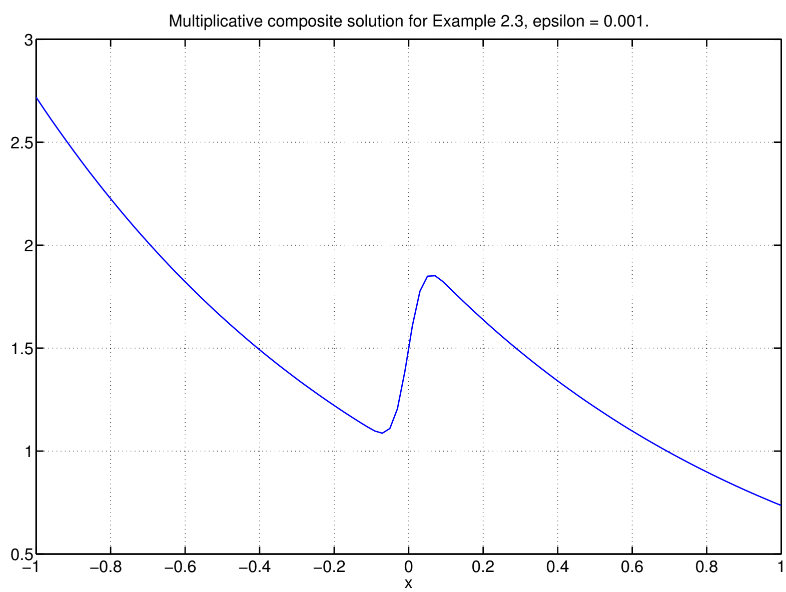

Figure 7: Graph of the multiplicative composite solution for Example 2.3, for $\epsilon = 0.001$.

Example 2.4. Consider the boundary-value problem

$$\epsilon y'' + 2xy' - 2y = \epsilon e^{y}, \quad -1 < x < 1, \quad y(-1;\epsilon) = -1,\ y(1;\epsilon) = 2.$$

Here $a(x)$, the coefficient of y' in the ODE, equals $2x$, and as in the preceding example, is negative for $x < 0$, positive for $x > 0$, and has a simple zero in the interior of the domain at $x = 0$. Again we dismiss a boundary layer at either end and expect an interior layer at $x = 0$. However, the layer now has a different character, as we shall see.

Outer solution. Let

$$y \sim y_0(x)$$

in the outer region. Then y_0 satisfies

$$2(xy_0' - y_0) = 0,$$

with solution

$$y_0 = a_0 x.$$

Application of the boundary conditions yields

$$y_0(x) = \begin{cases} x, & -1 < x < 0, \\ 2x, & 0 < x < 1. \end{cases}$$

From both sides the outer solution approaches zero as $x \to 0$. Thus y_0 is continuous there but the slope y_0' is discontinuous. To resolve this discontinuity we shall need a *corner layer*, or *derivative layer*, at $x = 0$, across which the solution stays close to zero but the derivative changes by order unity to accommodate the discontinuity in the derivative of the outer solution.

Inner solution. Let the inner region be defined by $x = \delta(\epsilon)\xi$, $y(x) = Y(\xi)$. Then the ODE becomes

$$\frac{\epsilon}{\delta^2}\frac{d^2Y}{d\xi^2} + \frac{2\delta\xi}{\delta}\frac{dY}{d\xi} - 2Y = \epsilon e^Y.$$

The second and third terms on the LHS are both $O(1)$ and must therefore be retained in a dominant balance with the first, thereby determining $\delta = \sqrt{\epsilon}$. The ODE then reads

$$Y'' + 2\xi Y - 2Y = \epsilon e^Y.$$

We now seek the inner expansion

$$Y \sim 0 + \sqrt{\epsilon}Y_1.$$

This choice is dictated by the requirement that in the $O(\sqrt{\epsilon})$ thick layer, the solution remains zero to leading order but changes at $O(\sqrt{\epsilon})$ at the next order, so that the derivatives remain of order unity. Now Y_1 satisfies

$$Y_1'' + 2\xi Y_1' - 2Y_1 = 0.$$

This is an example of a second-order ODE with variable coefficients, and such equations are not amenable to analytical solutions in general. However, $Y_0 = \xi$ is one solution in this case, and the second can then be found easily by the method of Reduction of Order. The result is the general solution

$$\begin{aligned} Y_1(\xi) &= b_0\xi + c_0\left[e^{-\xi^2} + 2\xi\int_0^\xi e^{-s^2}\,ds\right] \\ &= b_0\xi + c_0\left(e^{-\xi^2} + \sqrt{\pi}\,\xi\,\mathrm{erf}\,\xi\right). \end{aligned}$$

The integration constants b_0 and c_0 will be determined by matching.

Matching. As in the preceding example, we must match the inner solution separately with the outer solution on the left and on the right, and will again do so with the aid of the Van Dyke Principle, keeping in mind that the outer solution to $O(1)$ is to be matched with the inner solution to $O(\sqrt{\epsilon})$.

From the left, the inner expansion to $O(\sqrt{\epsilon})$ of the outer expansion to $O(1)$ is given by

$$\begin{aligned} y_0(x) &= x \\ &= \sqrt{\epsilon}\xi \qquad \text{(written in the inner variable } \xi\text{)} \\ &\sim \sqrt{\epsilon}\xi \qquad \text{(expanded to } O(\sqrt{\epsilon}) \text{ in the `inner' limit } \epsilon \to 0,\ \xi \text{ fixed).} \end{aligned}$$

To the left, the outer expansion to $O(1)$ of the inner expansion to $O(\sqrt{\epsilon})$ is given by

$$\begin{aligned}
0+\sqrt{\epsilon}Y_1(\xi) &= \sqrt{\epsilon}b_0\xi+\sqrt{\epsilon}c_0\left(e^{-\xi^2}+\sqrt{\pi}\,\xi\ \mathrm{erf}\,\xi\right)\\
&= b_0x+\sqrt{\epsilon}c_0\left[e^{-x^2/\epsilon}+\sqrt{\frac{\pi}{\epsilon}}\,x\ \mathrm{erf}(x/\sqrt{\epsilon})\right] \qquad \text{(written in the outer variable } x)\\
&= b_0x+c_0\sqrt{\pi}x\ \mathrm{erf}(-\infty) \qquad \text{(expanded to } O(1) \text{ in the `outer' limit } \epsilon\to 0,\ x<0 \text{ and fixed)}\\
&= \left(b_0-\sqrt{\pi}c_0\right)\sqrt{\epsilon}\xi. \qquad \text{(rewritten in the inner variable).}
\end{aligned}$$

Here we have used the result $\mathrm{erf}(-\infty)=-1$. Matching yields

$$b_0-\sqrt{\pi}c_0=1. \tag{2.21}$$

In an entirely analogous way we match to the right.

From the right, the inner expansion to $O(\sqrt{\epsilon})$ of the outer expansion to $O(1)$ is given by

$$\begin{aligned}
y_0(x) &= 2x\\
&= 2\sqrt{\epsilon}\xi \qquad \text{(written in the inner variable } \xi)\\
&\sim 2\sqrt{\epsilon}\xi \qquad \text{(expanded to } O(\sqrt{\epsilon}) \text{ in the `inner' limit } \epsilon\to 0,\ \xi \text{ fixed).}
\end{aligned}$$

To the right, the outer expansion to $O(1)$ of the inner expansion to $O(\sqrt{\epsilon})$ is given by

$$\begin{aligned}
0+\sqrt{\epsilon}Y_1(\xi) &= \sqrt{\epsilon}b_0\xi+\sqrt{\epsilon}c_0\left(e^{-\xi^2}+\sqrt{\pi}\,\xi\ \mathrm{erf}\,\xi\right)\\
&= b_0x+\sqrt{\epsilon}c_0\left[e^{-x^2/\epsilon}+\sqrt{\frac{\pi}{\epsilon}}\,x\ \mathrm{erf}(x/\sqrt{\epsilon})\right] \qquad \text{(written in the outer variable } x)\\
&= b_0x+c_0\sqrt{\pi}x\ \mathrm{erf}(\infty) \qquad \text{(expanded to } O(1) \text{ in the `outer' limit } \epsilon\to 0,\ x>0 \text{ and fixed)}\\
&= \left(b_0+\sqrt{\pi}c_0\right)\sqrt{\epsilon}\xi. \qquad \text{(rewritten in the inner variable).}
\end{aligned}$$

Here we have used the result $\mathrm{erf}(\infty)=1$. Matching yields

$$b_0+\sqrt{\pi}c_0=2. \tag{2.22}$$

Equations (2.21) and (2.22) determine

$$b_0=\frac{3}{2},\quad c_0=\frac{1}{2}\sqrt{\frac{1}{\pi}}.$$

Composite expansion. With b_0 and c_0 determined as above the inner solution becomes

$$\begin{aligned}
Y(\xi;\epsilon) &\sim \frac{3}{2}\sqrt{\epsilon}\,\xi+\frac{1}{2\sqrt{\pi}}\sqrt{\epsilon}\left(e^{-\xi^2}+\sqrt{\pi}\,\xi\ \mathrm{erf}\,\xi\right)\\
&= \frac{3}{2}x+\frac{1}{2}x\ \mathrm{erf}(x/\sqrt{\epsilon})+\frac{1}{2}\sqrt{\frac{\epsilon}{\pi}}e^{-x^2/\epsilon}.
\end{aligned}$$

The outer limit of this solution to the right is $2x$ and to the left, x. Thus the inner expansion captures the full outer solution to order unity, and therefore, is itself the composite expansion. The outer and composite solutions are displayed in Figure 8 for $\epsilon=0.1$.

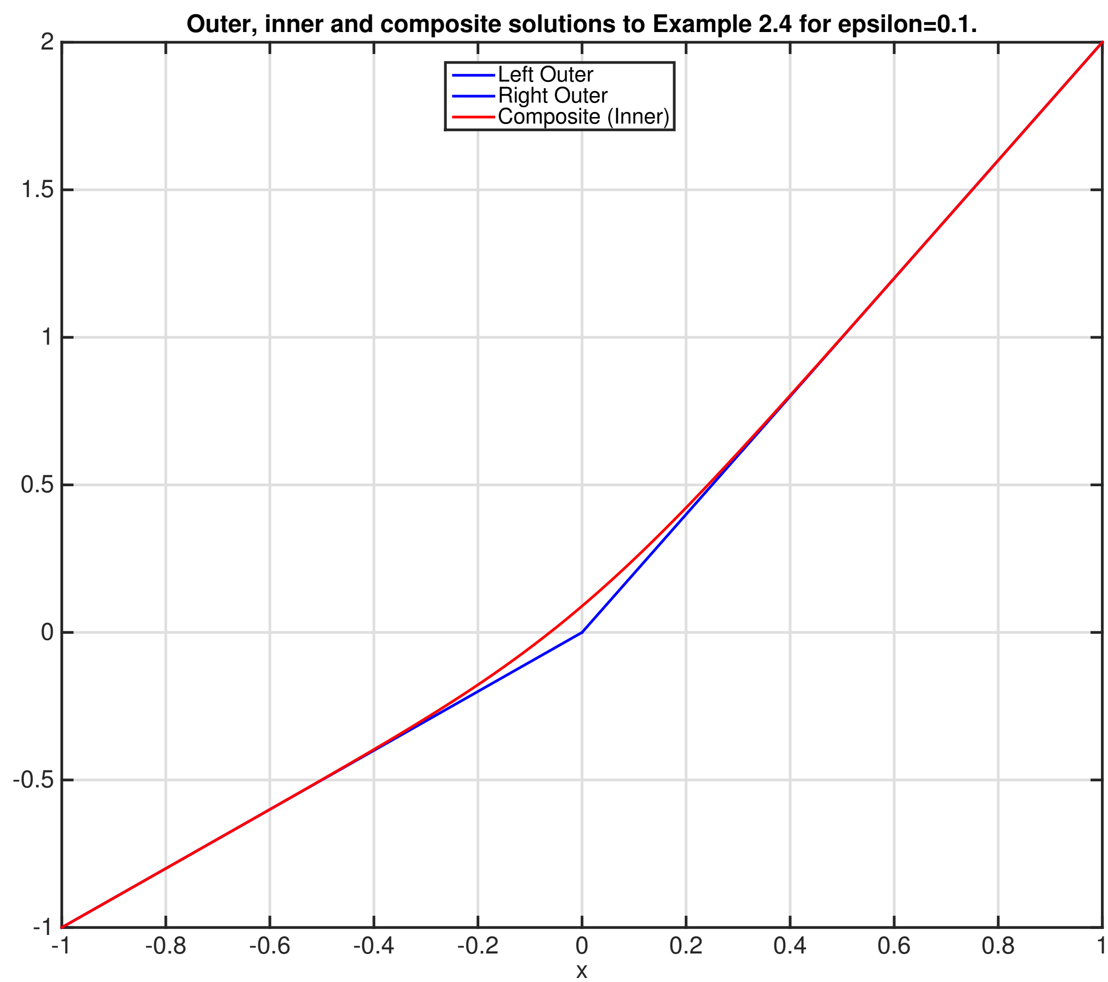

Figure 8: Outer and composite solutions for Example 2.4, for $\epsilon = 0.1$.

Example 2.5. Consider the boundary-value problem

$$\epsilon y'' - 2xy' = 0, \quad -1 < x < 1, \quad y(-1) = \alpha, \; y(1) = \beta.$$

Here $a(x)$, the coefficient of y' in the ODE, equals $-2x$, and as such is positive for $x < 0$, negative for $x > 0$, and has a simple zero in the interior of the domain at $x = 0$. Therefore we can have a boundary layer at either end or boundary layers at both ends. Before constructing the perturbation solution, let us write down the exact solution which is easily computed by using an integrating factor. First, the general solution is

$$y_{\text{exact}}(x) = \sqrt{\epsilon} A \int_0^{x/\sqrt{\epsilon}} e^{t^2}\, dt + B.$$

The boundary conditions require A and B to satisfy

$$\begin{aligned} A\sqrt{\epsilon} \int_0^{-1/\sqrt{\epsilon}} e^{t^2}\, dt + B &= \alpha, \\ A\sqrt{\epsilon} \int_0^{1/\sqrt{\epsilon}} e^{t^2}\, dt + B &= \beta. \end{aligned}$$

Since the integrand is an even function, the first of the above equations can be written as

$$-A\sqrt{\epsilon} \int_0^{1/\sqrt{\epsilon}} e^{t^2}\, dt + B = \alpha.$$

Then A and B are determined to be

$$\begin{aligned} A &= \frac{\beta - \alpha}{2\sqrt{\epsilon} \int_0^{1/\sqrt{\epsilon}} e^{t^2}\, dt}, \\ B &= \frac{\alpha + \beta}{2}, \end{aligned}$$

allowing the exact solution to be written as

$$y_{\text{exact}}(x) = \frac{\alpha + \beta}{2} + \frac{\beta - \alpha}{2} \frac{\int_0^{x/\sqrt{\epsilon}} e^{t^2}\, dt}{\int_0^{1/\sqrt{\epsilon}} e^{t^2}\, dt}.$$

Let us examine the asymptotic behavior of the solution as $\epsilon \to 0$. First, we note that

$$\int_0^z e^{t^2}\, dt \sim \frac{e^{z^2}}{2z} \quad \text{as} \quad z \to \infty,$$

a result that can be easily proven by using integration-by-parts. Then, as $\epsilon \to 0$, we have

$$\begin{aligned} \int_0^{x/\sqrt{\epsilon}} e^{t^2}\, dt &\sim \frac{\sqrt{\epsilon}}{2x}\, e^{x^2/\epsilon}, \\ \int_0^{1/\sqrt{\epsilon}} e^{t^2}\, dt &\sim \frac{\sqrt{\epsilon}}{2}\, e^{1/\epsilon}. \end{aligned}$$

In view of these results, the exact solution has the expansion

$$y_{\text{exact}}(x) \sim \frac{\alpha + \beta}{2} + \frac{\beta - \alpha}{2x}\, e^{(x^2-1)/\epsilon} \quad \text{as} \quad \epsilon \to 0.$$

A plot of this solution is displayed in Figure 9. It shows a flat solution with a boundary layer at each end. The boundary-layer behavior can be revealed analytically by writing the above expansion as

$$y_{\text{exact}}(x) \sim \frac{\alpha + \beta}{2} + \frac{\beta - \alpha}{2x}\, e^{(x-1)(x+1)/\epsilon}. \tag{2.23}$$

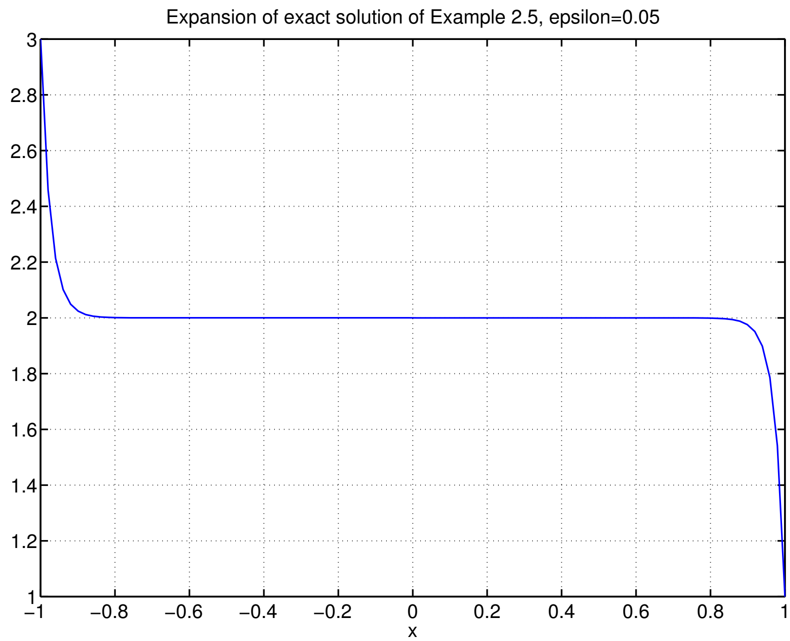

Figure 9: Plot of the leading-order expansion of the exact solution for Example 2.5, for $\epsilon = 0.05$.

We note that in the outer limit where x is fixed in the interval $-1 < x < 1$ and $\epsilon \to 0$, the solution simplifies to the constant value

$$\frac{\alpha+\beta}{2}. \tag{2.24}$$

In the left boundary layer where $x = -1 + O(\epsilon)$ the simplified form is

$$\frac{\alpha+\beta}{2} - (\beta-\alpha)\, e^{-2(x+1)/\epsilon},$$

and in the right boundary layer where $x = 1 + O(\epsilon)$ the simplified form is

$$\frac{\alpha+\beta}{2} + (\beta-\alpha)\, e^{-2(1-x)/\epsilon}.$$

Now let us see whether the perturbation procedure captures the above features.

Outer solution. In the outer region let the solution be expanded as $y \sim y_0(x)$. Then y_0 satisfies

$$2xy_0' = 0,$$

with general solution

$$y_0 = a_0, \tag{2.25}$$

with the possible exception of $x = 0$ where the reduced ODE is singular. As already mentioned, the ODE will admit a layer at either end and the above solution may not satisfy either boundary condition.
(One may argue that one must select different constant solutions for $x > 0$ and $x < 0$, with a layer at $x = 0$. However, such a layer is disallowed. Can you reason why?)

Layer at $x = -1$. We define the stretched coordinate ξ in the layer by $x = -1 + \delta\xi$, $\xi \geq 0$. With $y(x) = Y(\xi)$, the ODE transforms into

$$\frac{\epsilon}{\delta^2} Y''(\xi) - 2(-1+\delta\xi)\frac{1}{\delta} Y'(\xi) = 0.$$

Dominate balance finds $\delta = \epsilon$, whence the above equation becomes

$$Y''(\xi) + 2(1 - \epsilon\xi)Y'(\xi) = 0.$$

The expansion $Y \sim Y_0(\xi)$ leads to

$$Y_0''(\xi) + 2Y_0'(\xi) = 0,$$

whose solution satisfying the left boundary condition $Y_0(0) = \alpha$ is

$$Y_0(\xi) = \alpha - b_0 + b_0 e^{-2\xi}. \tag{2.26}$$

The unknown coefficient b_0 will need to be determined by matching.

Layer at $x = 1$. Now the stretched coordinate η is defined by $x = 1 + \epsilon\eta$, $\eta \leq 0$, where, in analogy with the left boundary layer, we have anticipated that the right layer will also be $O(\epsilon)$ thick. Then, with $y(x) = Z(\eta)$, the ODE transforms into

$$Z''(\eta) - 2(1 + \epsilon\eta)Z'(\eta) = 0.$$

The expansion $Z \sim Z_0(\eta)$ leads to the reduced equation

$$Z''(\eta) - 2Z_0'(\eta) = 0,$$

whose solution satisfying the right boundary condition $Z_0(0) = \beta$ is

$$Z_0(\eta) = \beta - c_0 + c_0 e^{2\eta}. \tag{2.27}$$

Again, the constant c_0 will be determined by matching.

Matching. We must match the inner solution separately with the outer solution on the left and on the right, and will again do so with the aid of the Van Dyke Principle, matching the outer solution to $O(1)$ with the inner solution to $O(\sqrt{\epsilon})$.

From the left, the inner expansion to $O(1)$ of the outer expansion to $O(1)$ is given by

$$y_0 \quad = \quad a_0.$$

The outer expansion to $O(1)$ of the inner expansion to $O(1)$ is

$$\begin{aligned} Y_0 &= \alpha - b_0 + b_0 e^{-2\xi} \\ &= \alpha - b_0 + b_0 e^{-2(x+1)/\epsilon} && \text{(written in the outer variable } x\text{)} \\ &\sim \alpha - b_0. && \text{(expanded to } O(1) \text{ in the `outer' limit } \epsilon \to 0,\ x > -1 \text{ and fixed)} \end{aligned}$$

Matching yields

$$a_0 = \alpha - b_0. \tag{2.28}$$

From the right, the inner expansion to $O(1)$ of the outer expansion to $O(1)$ is again given by

$$y_0 \quad = \quad a_0,$$

while the outer expansion to $O(1)$ of the inner expansion to $O(1)$ is

$$\begin{aligned} Y_0 &= \beta - c_0 + c_0 e^{-2\xi} \\ &= \beta - c_0 + c_0 e^{-2(x-1)/\epsilon} && \text{(written in the outer variable } x\text{)} \\ &\sim \beta - c_0. && \text{(expanded to } O(1) \text{ in the `outer' limit } \epsilon \to 0,\ x < 1 \text{ and fixed)} \end{aligned}$$

Matching now yields

$$a_0 = \beta - c_0. \tag{2.29}$$

The matching conditions (2.28) and (2.29) lead to

$$b_0 = \alpha - a_0, \quad c_0 = \beta - a_0.$$

However, a_0 is left undetermined. A comparison with the outer expansion (2.24) of the exact solution reveals that $\alpha_0 = (\alpha+\beta)/2$ but there is no indication in the perturbation procedure as to how to resolve the issue. This is a deficiency of the process.

One possible resolution is to seek some global property of the solution that may provide additional information. For example, if one can prove that the solution is an odd function then that would provide an extra condition. Other options that have been successful in special cases include recasting the BVP for the ODE as a variational problem, where the solution to the BVP is the function that satisfies the boundary conditions and renders stationary a certain functional (in the form of an integral) whose Euler-Lagrange equation is the given ODE. Details of the procedure can be found in Grasman & Matkowsky, SIAM J Appl. Math., 32, pp. 588-597 (1977). Appropriate introduction of exponentially small terms in the expansion has also salvaged the classical technique uncertain situations. For additional discussion of this intriguing question, consult the paper by R. E. O'Malley, Jr. in SIAM Review, 50, pp. 459-482 (2008). A copy of each of the two papers paper is posted in the Lesson 2 folder on LMS.

2.2.3 Miscellaneous boundary-value problems

Example 2.6. A linear problem involving logarithmic gauge functions. Consider the BVP

$$(\epsilon + x)y'' + y' = 1, \quad 0 < x < 1, \;\; y(0) = 0, \; y(1) = 2.$$

Her the highest-derivative term is not multiplied by ϵ. Rather, the reduced equation corresponding to $\epsilon = 0$ displays singular behavior at $x = 0$, where a boundary layer is therefore expected.

Outer solution. With $y \sim y_0(x)$, y_0 satisfies

$$xy_0'' + y_0' = 1.$$

The solution satisfying the right boundary condition $y_0(1) = 2$ is

$$y_0 = 1 + x + c_0 \ln x. \tag{2.30}$$

This solution does not satisfy the left boundary condition. In fact, it is logarithmically singular at $x = 0$ for $c_0 \neq 0$.

Inner solution. The inner solution must recover the nonsingular nature of the exact solution near $x = 0$. This suggests the stretching (also confirmed by dominant balancing) $x = \epsilon\xi$. Let $y(x) = Y(\xi)$. Then Y satisfies

$$(1+\xi)Y'' + Y' = \epsilon.$$

The expansion $Y \sim Y_0$ leads to the following ODE at leading order:

$$(1+\xi)Y_0'' + Y_0' = 0.$$

The solution, subject to the left boundary condition $Y_0(0) = 0$, is

$$Y_0 = a_0 \ln(1+\xi). \tag{2.31}$$

Matching to order unity. We apply the Van Dyke procedure and expand the $O(1)$ outer solution to $O(1)$ in the inner region and the $O(1)$ inner solution to $O(1)$ in the outer region. The details are as follows. An important cautionary note: for the purpose of Van Dyke matching, a term of order $\epsilon^n[\ln(1/\epsilon)]^m$ is treated as being of order ϵ^n. In other words, this matching procedure does not recognize the modification of an algebraic gauge function by a logarithm.

Inner expansion of the outer expansion:

$$\begin{aligned} y_0 &= 1 + x + c_0 \ln x \\ &= 1 + \epsilon\xi + c_0 \ln(\epsilon\xi) && \text{(written in the inner variable } \xi) \\ &\sim 1 + c_0 \ln\epsilon + c_0 \ln\xi. && \text{(expanded to } O(1) \text{ in the `inner' limit } \epsilon \to 0,\ \xi > 0 \text{ and fixed)} \end{aligned}$$

Outer expansion of the inner expansion:

$$\begin{aligned} Y_0 &= a_0 \ln(1+\xi) \\ &= a_0 \ln\left((1+\frac{x}{\epsilon}\right) = a_0\left[\ln\frac{x}{\epsilon}\left(1+\frac{\epsilon}{x}\right)\right] && \text{written in the outer variable} \\ &= a_0 \ln x + a_0 \ln(1/\epsilon) && \text{expanded to } O(1) \text{ in the outer limit} \\ &= a_0 \ln \xi. \end{aligned}$$

Note that in the expansion above, certain terms have been dropped to make sure that only terms to $O(1)$ are retained. Matching now leads to

$$a_0 = c_0 = -1/\ln\epsilon = 1/\ln(1/\epsilon).$$

At this stage then, the outer and inner solutions at leading order are

$$\begin{aligned} y_0(x) &= 1 + x + \delta\ln x, \\ Y_0(\xi) &= \delta\ln(1+\xi), \end{aligned}$$

where

$$\delta = \frac{1}{\ln(1/\epsilon)} = -\frac{1}{\ln\epsilon}.$$

<u>Determination of higher-order terms.</u>

First, the outer expansion. Note that the ODE in the outer variables is

$$xy'' + y' - 1 = -\epsilon y''.$$

The leading-order term y_0, when substituted into the RHS, generates an $O(\epsilon\delta)$ correction. Therefore the outer expansion must proceed as

$$y \sim y_0 + \epsilon\delta y_1.$$

Then y_1 satisfies

$$xy_1'' + y_1' = y_0'' = \frac{1}{x^2}, \quad y_1(1) = 0.$$

The solution is

$$y_1 = \frac{1}{x} + a_1\ln x - 1.$$

Now the inner expansion. The ODE in the inner variables is

$$(1+\xi)Y'' + Y' = \epsilon.$$

Now the RHS introduces an $O(\epsilon)$ correction, thereby determining the form of the inner expansion as

$$Y \sim Y_0 + \epsilon Y_1.$$

Notice that the gauge sequence for the inner expansion is not the same as that for the outer expansion. Now Y_1 satisfies

$$(1+\xi)Y_1'' + y_1' = 1, \quad Y_1(0) = 0,$$

leading to the solution

$$Y_1 = \xi + b_1\ln(1+\xi).$$

<u>Higher-order matching.</u>

We apply the Van Dyke procedure and expand the $O(\epsilon)$ outer solution to $O(\epsilon)$ in the inner region and the $O(\epsilon)$ inner solution to $O(\epsilon)$ in the outer region. Again, recall that terms of order $(\epsilon\delta)$ and $O(\epsilon/\delta)$ will be treated as $O(\epsilon)$ for purposes of matching.

Inner expansion of the outer expansion:

$$
\begin{aligned}
y_0 + \epsilon\delta y_1 &\sim 1 + x + \delta \ln x + \epsilon\delta \left(\frac{1}{x} - 1 + a_1 \ln x \right) \\
&= 1 + \epsilon\xi + \delta \ln \epsilon + \delta \ln \xi + \epsilon\delta \left(\frac{1}{\epsilon\xi} - 1 + a_1 \ln \epsilon + a_1 \ln \xi \right) \\
&\qquad\qquad\qquad\qquad \text{(written in the inner variable } \xi \text{ and expanded to } O(\epsilon)\text{)} \\
&= 1 + x + \delta \ln x + \epsilon\delta \left(\frac{1}{x} - 1 + a_1 \ln x \right) \qquad \text{(re-expressed in the outer variable).}
\end{aligned}
$$

Outer expansion of the inner expansion:

$$
\begin{aligned}
Y_0 + \epsilon Y_1 &\sim \delta \ln(1+\xi) + \epsilon[\xi + b_1 \ln(1+\xi)] \\
&= \delta \ln \left((1 + \frac{x}{\epsilon} \right) + \epsilon \left[\frac{x}{\epsilon} + b_1 \ln \left(1 + \frac{x}{\epsilon} \right) \right] \quad \text{(written in the outer variable)} \\
&= \delta \ln \frac{x}{\epsilon} + \delta \ln \left(1 + \frac{\epsilon}{x} \right) + x + \epsilon b_1 \left[\ln \frac{x}{\epsilon} + \ln \left(1 + \frac{\epsilon}{x} \right) \right] \\
&\sim \delta \ln x + \delta \ln \frac{1}{\epsilon} + \frac{\epsilon\delta}{x} + x + \epsilon b_1 \ln x + \epsilon b_1 \ln \frac{1}{\epsilon} \qquad \text{(expanded to } O(\epsilon) \text{ in the outer limit)} \\
&= 1 + x + \delta \ln x + \frac{\epsilon\delta}{x} + \frac{\epsilon}{\delta} b_1 + \epsilon b_1 \ln x.
\end{aligned}
$$

Note that in the expansion above certain terms have been dropped to make sure that only terms to $O(\epsilon)$ are retained. Matching now leads to

$$
b_1 = -\delta^2, \; a_1 = -\delta.
$$

At this stage then, the outer and inner solutions are

$$
\begin{aligned}
y &\sim 1 + x + \delta \ln x + \epsilon\delta \left(\frac{1}{x} - 1 \right) - \epsilon\delta^2 \ln x, \\
Y &\sim \delta \ln(1+\xi) + \epsilon\xi - \epsilon\delta^2 \ln(1+\xi).
\end{aligned}
$$

Since the matched part is the outer expansion itself, the composite expansion is just the inner expansion. It can be rearranged, to $O(\epsilon)$, as

$$
\begin{aligned}
y_c &\sim x + \delta(1 - \epsilon\delta) \ln \left(1 + \frac{x}{\epsilon} \right) \\
&\sim x + \frac{\delta}{1 + \epsilon\delta + \cdots} \ln \left(1 + \frac{x}{\epsilon} \right) \\
&\sim x + \frac{1}{(1/\delta) + \epsilon + \cdots} \ln \left(1 + \frac{x}{\epsilon} \right) \\
&\sim x + \frac{1}{-\ln \epsilon + \epsilon + \cdots} \ln \left(1 + \frac{x}{\epsilon} \right).
\end{aligned}
$$

The above result can be checked against the exact solution

$$
y_{\text{exact}} = x + \frac{1}{\ln(1+\epsilon) - \ln \epsilon} \ln \left(1 + \frac{x}{\epsilon} \right).
$$

Example 2.7. A nonlinear problem. Consider the BVP

$$\epsilon y'' + yy' = 0, \quad y(0) = \alpha,\ y(1) = \beta.$$

Exact solution is available, but we shall build the perturbation solution for all possible real values of α and β. We note immediately that if $y = f(x;\alpha,\beta)$ is a solution, then so is $y = f(1-x;-\beta,-\alpha)$. Thus it is enough to consider only one half of the $\alpha\beta$-parameter plane, $\alpha \geq -\beta$, shown in Figure 10.

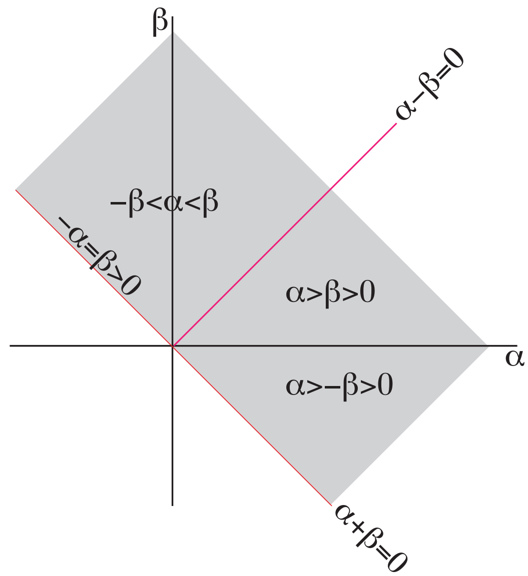

Figure 10: The $\alpha\beta$-plane. It is enough to consider the shaded half plane $\alpha + \beta = 0$ due to symmetry.

<u>Outer solution.</u> With $y \sim y_0(x)$, the leading-order term satisfies $y_0 y_0' = 0$, so that

$$y_0 = 0 \quad \text{or} \quad y = \text{a constant}.$$

Leaving the zero solution aside for a moment, we take the outer solution to satisfy the boundary conditions by setting

$$y_0 = \begin{cases} \alpha, & x < x_0 \text{ (left solution)}, \\ \beta, & x > x_0 \text{ (right solution)}. \end{cases} \tag{2.32}$$

For $\alpha \neq \beta$ we shall seek to resolve the discontinuity by an internal boundary layer at the yet unknown location x_0.

<u>Inner solution.</u> In the layer let the stretched variable ξ defined by $x = x_0 + \delta\xi$. With $y(x) = Y(\xi)$, the ODE transforms into

$$\frac{\epsilon}{\delta^2} Y''(\xi) + \frac{1}{\delta} YY'(\xi) = 0.$$

Dominant balance requires $\delta = \epsilon$, which reduces the equation to

$$Y'' + YY' = 0.$$

(This is the full differential equation, and if we are to solve this then we may as well solve the original problem exactly. However, we shall pretend that the original problem is too difficult to solve exactly.) The first integral is

$$Y' + \frac{1}{2} Y'^2 = \text{constant, say } \frac{1}{2} c^2.$$

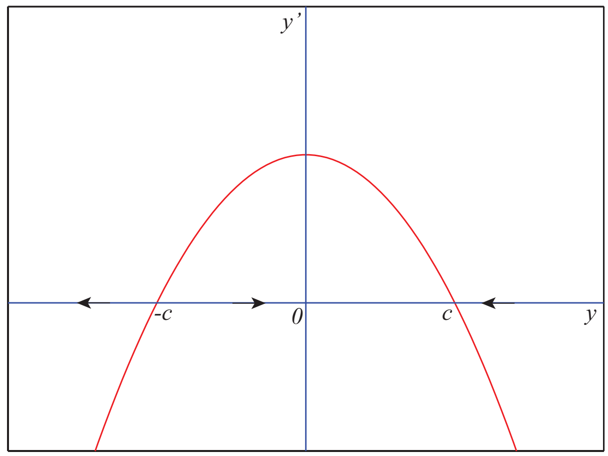

Figure 11: The phase plot of Y' against Y for the ODE (2.33). The arrows indicate the behavior of $Y(\xi)$ with increasing ξ.

We have taken the constant of integration to be non-negative, anticipating that as the inner solution exits the layer, *i.e.*, as $\xi \to \pm\infty$, the solution must level out to match the constant outer solutions, *i.e.*, $Y' \to 0$. Therefore,

$$Y' = \frac{c^2 - Y^2}{2}, \tag{2.33}$$

whose solution has the implicit form

$$\int^Y \frac{dZ}{c^2 - Z^2} = \frac{1}{2}(\xi - \xi_0). \tag{2.34}$$

Here ξ_0 is the second integration constant. The solution can be written down explicitly, but it is enough to invoke its geometric structure without referring to the actual formulas. As the graph of Y' against Y displayed in Figure 11 indicates, three different types of solution are possible.

- If $Y > c$, then the solution approaches c as $\xi \to \infty$. We shall refer to this solution as a solution of Type I.
- If $-c < Y < c$, then the solution approaches c as $\xi \to \infty$ and $-c$ as $\xi \to -\infty$. We shall refer to this solution as a solution of Type II.
- If $Y < -c$, then the solution approaches $-c$ as $\xi \to -\infty$. We shall refer to this solution as a solution of Type III.

The graphs of these solutions are shown in Figure 12. Note that at $\xi = \xi_0$, $Y = \infty$ for the solution of type I, $Y = 0$ for the solution of type II and $Y = \infty$ for the solution of type III.

We now show that these solutions can provide four different kinds of layers in the parameter space $\alpha + \beta > 0$.

1. Type I solution providing a layer at the left boundary ($x_0 = 0$). In this case the layer solution starts at a value higher than c at $\xi = 0$, decreases monotonically, and approaches c asymptotically as it

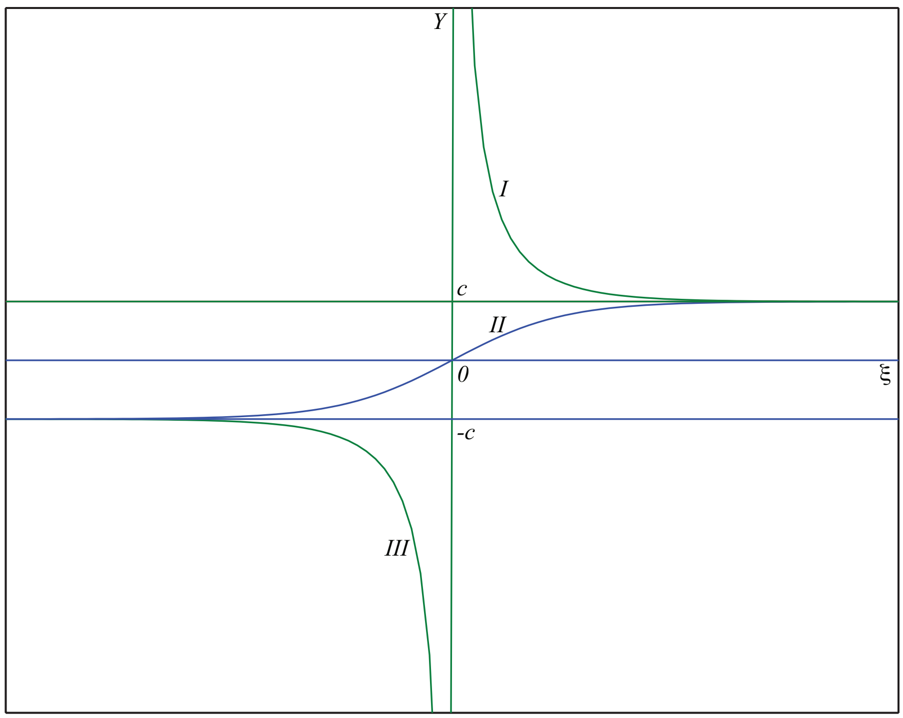

Figure 12: Three possible solutions of equation (2.34).

exits the layer, *i.e.*, as $\xi \to \infty$. Since it must match with the right outer solution $y_0 = \beta$, it requires $c = \beta > 0$. The integration constant ξ_0 is determined by the condition $Y(0) = \alpha$. Monotonic decay requires $Y(0) > Y(\infty)$, thus restricting the solution to the case $\alpha > \beta > 0$.

2. Type II solution providing a layer at the left boundary. In this case $-c < Y(0) = \alpha < c$ and $Y \to c$ as $\xi \to \infty$. Matching requires $c = \beta$. Thus this solution applies when $\beta > 0$ and $-\beta < \alpha < \beta$. The condition $Y(0) = \alpha$ determines ξ_0.

3. Type II solution providing a layer in the interior ($0 < x_0 < 1$). In this layer, $-\infty < \xi < \infty$, and $Y \to \mp c$ as $\xi \to \mp\infty$. Matching shows that this solution is valid only when $-\alpha = \beta > 0$.

   In this case $-c < Y(0) = \alpha < c$ and $Y \to c$ as $\xi \to \infty$. Matching requires $c = \beta$. Thus this solution applies when $\beta > 0$ and $-\beta < \alpha < \beta$. The condition $Y(0) = \alpha$ determines ξ_0. Matching fails to determine the location of the layer, but one may conjecture, based on considerations of symmetry, that $x_0 = 1/2$.

4. A layer at each end. In this case the only feasible outer solution is $y = 0$ as the left layer (type I solution) must decay to $c > 0$ and the right layer (type III solution) must rise to $c < 0$ to match the outer solution. This solution applies when $\alpha > 0$ and $\beta < 0$.

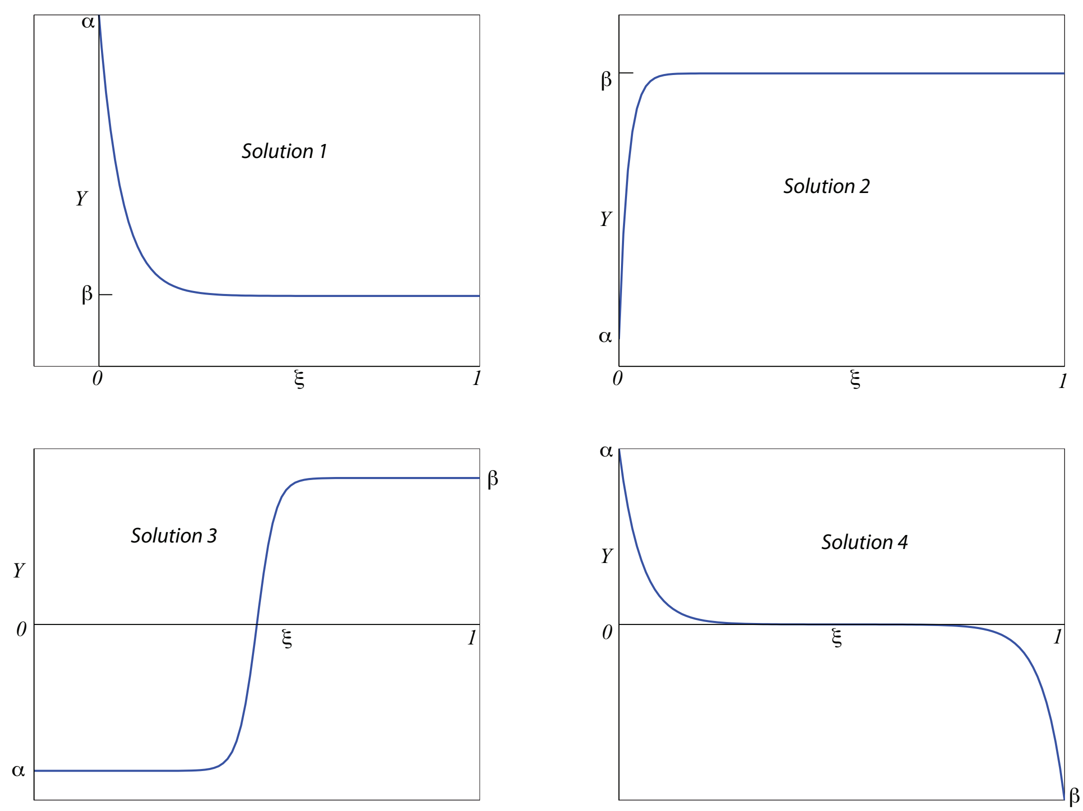

Figure 13: Possible layer solutions for $\alpha + \beta > 0$.

The plots of the various layer solutions are displayed in Figure 13. Among themselves these solutions cover the entire half plane $\alpha + \beta > 0$, shown in Figure 10.

2.2.4 Initial-value problems

Example 2.8. This example is due to Cole. We omit some of the details as the procedure is now straightforward.

Consider the motion of a linear spring-mass-damper system where the mass is small and is subject to a large impulse. The dimensionless problem can be posed as

$$\epsilon y''(t) + y'(t) + y(t) = 0, \quad y(0) = 0,\ y'(0) = \frac{1}{\epsilon}.$$

The effect of the impulse is to impart a large initial velocity to the mass. In the outer region, where the $\epsilon y''$ term is neglected to leading order, the principal balance is between the damping force and the restoring force in the spring, the inertial force being negligible. It is a simple matter to show that the outer expansion to $O(\epsilon)$ is

$$y \sim A_0 e^{-t} + \epsilon(A_1 - A_0 t)e^{-t}.$$

This solution cannot satisfy two initial conditions. We do not force it to satisfy even one, leaving that to matching. The inertial term will be reinstated in a thin initial layer, whose thickness, as revealed by dominant balance, is ϵ. With $t = \epsilon\tau$ and $y(t) = Y(\tau)$, the inner differential equation is

$$Y''(\tau) + Y'(\tau) + \epsilon Y(\tau) = 0,$$

with initial conditions

$$Y(0) = 0,\ Y'(0) = 1.$$

The inertia force and the damping force now balance to leading order, and the inner expansion is found to

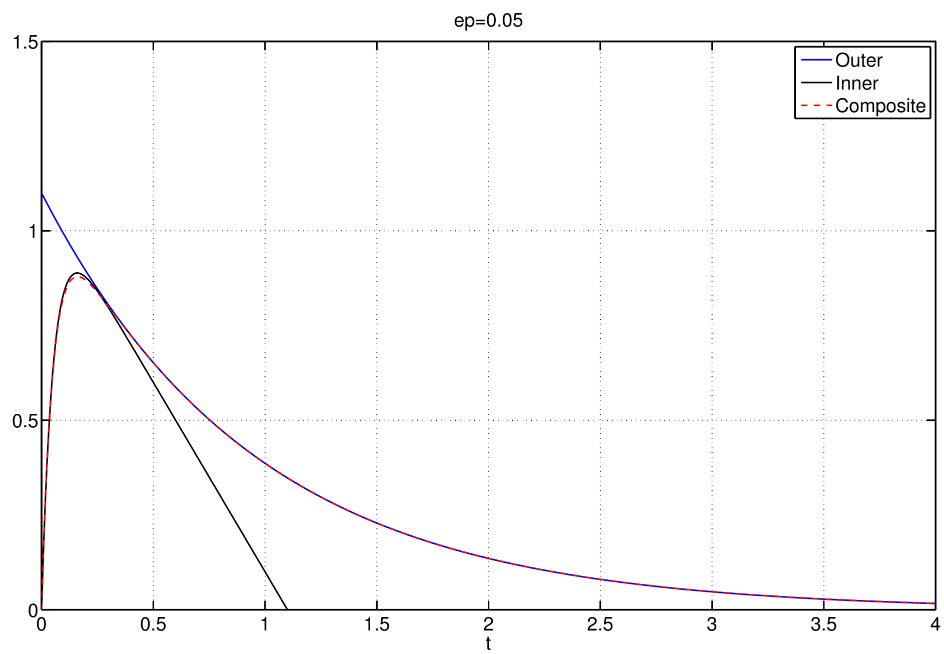

Figure 14: Plots of the outer, inner and composite expansions for $\epsilon = 0.05$ for Example 2.8.

be

$$Y \sim 1 - e^{-\tau} + \epsilon\left(2 - \tau - (2+\tau)e^{-\tau}\right).$$

Matching finds $A_0 = 1,\ A_1 = 2,$ and leads to the composite expansion

$$y_c \sim e^{-t} - e^{-t/\epsilon} + \epsilon(2-t)e^{-t} - (t + 2\epsilon)e^{-t/\epsilon}.$$

The expansion is not uniformly valid for large t and breaks down when $t = O(1/\epsilon)$. This is not a serious issue in this case as by then the solution has decayed to an exponentially small amplitude. The graphs of the outer, inner and composite expansions are displayed in Figure 14.

Example 2.9. A nonlinear initial-value problem due to Dahlquist.

$$\epsilon y'(t) = (1-t)y - y^2, \quad y(0) = \alpha > 0.$$

Outer expansion.

Consider an outer expansion of the standard form,

$$y \sim y_0(t) + \epsilon y_1(t) + \epsilon^2 y_2(t). \tag{2.35}$$

The leading-order term satisfies

$$(1-t)y_0 - y_0^2 = 0,$$

yielding the solutions

$$y_0 = 0 \quad \text{and} \quad y_0 = 1 - t.$$

Unless $y(0) = 0$ or 1, one expects a layer at $t = 0$, of which the thickness through dominant balance is easily seen to be ϵ.

Inner expansion.

With $t = \epsilon\tau$ and $y(t) = Y(\tau)$, the inner problem is

$$Y'(\tau) - Y(\tau)(1 - Y(\tau)) = -\epsilon\tau Y(\tau), \quad Y(0) = \alpha.$$

At leading order,

$$Y_0' = Y_0(1 - Y_0).$$

This is the logistic equation, with two constant solutions $Y_0 = 0$ and $Y_0 = 1$. All solutions with $Y_0 > 0$ will approach the stable solution $Y_0 = 1$ monotonically as $\xi \to \infty$, *i.e.*, as one exits the initial layer. Thus Y_0 will match the outer solution $y_0 = 1 - t$.

Let us compute additional terms in the outer expansion (2.35). The equations for y_0 and y_1 are

$$\begin{aligned} y_0' &= (1-t)y_1 - 2y_0y_1, \\ y_1' &= (1-t)y_2 - (y_1^2 + 2y_0y_2). \end{aligned}$$

These can be solved for y_1 and y_2 and the results are

$$\begin{aligned} y_1 &= \frac{1}{1-t}, \\ y_2 &= -\frac{2}{(1-t)^3}. \end{aligned}$$

Thus we have the outer expansion

$$y \sim 1 - t + \frac{\epsilon}{1-t} - \frac{2\epsilon^2}{(1-t)^3}. \tag{2.36}$$

This expansion is not valid all the way out to $t = \infty$. Rather, it is disordered (*i.e.*, breaks down) when $1 - t = O(\sqrt{\epsilon})$. Then each term in the expansion is $O(\sqrt{\epsilon})$, suggesting that we should consider a new layer of thickness $\sqrt{\epsilon}$ at $t = 1$ in which the size of the solution is also $O(\sqrt{\epsilon})$. Note that *to this point we had employed the concept of dominant balance to discover the thickness of an inner region. Here we have computed an outer expansion to higher orders and exploited its breakdown to discover a new layer.*

Layer at $t = 1$.

We now set $t = 1 + \sqrt{\epsilon}\, s$ and $y(t) = \sqrt{\epsilon}\, w(s)$. Then the layer equation becomes

$$w'(s) + sw(s) = -w^2(s), \quad -\infty < s < \infty.$$

The doubly infinite domain of s reflects the fact that this is an internal layer that must match with outer solutions to the left and to the right. The appropriate outer solution to the right cannot be $y_0 = 1 - t$, so it must be the other outer solution, $y_0 = 0$. Upon using an integrating factor the ODE can be rewritten as

$$\frac{d}{ds}\left(e^{s^2/2} w\right) = -e^{s^2/2} w^2 = -e^{-s^2/2}\left(e^{s^2/2} w\right)^2.$$

With

$$u = e^{s^2/2} w,$$

the ODE transforms into the separated version

$$-\frac{1}{u^2} du = e^{-s^2/2} ds$$

which integrates to

$$\frac{1}{u} = \sqrt{\frac{\pi}{2}}\, \mathrm{erf}\, \frac{s}{\sqrt{2}} + k,$$

where k is an integration constant. Reverting back to w one obtains

$$w = \frac{e^{-s^2/2}}{k + \sqrt{\frac{\pi}{2}}\, \mathrm{erf}\, \frac{s}{\sqrt{2}}} = \left(k e^{s^2/2} + \sqrt{\frac{\pi}{2}}\, e^{s^2/2}\, \mathrm{erf}\, \frac{s}{\sqrt{2}}\right)^{-1}. \tag{2.37}$$

<u>Matching between the layer at $t = 1$ and the outer solution for $0 < t < 1$.</u>

To match with the solution upstream, we shall need the asymptotic form of the layer solution $\sqrt{\epsilon}\, w(s)$ as we exit the layer to the left, *i.e.*, for $s \to -\infty$. That, in turn, will require the expansion of the error function. This has been done in the Aside below, and the use of the result (2.41) allows us to write w above as

$$w = \left[(k - \sqrt{\pi/2}) e^{s^2/2} - \frac{1}{s} + \frac{1}{s^3} - \frac{3}{s^5} + O\left(\frac{1}{s^7}\right)\right]^{-1}, \quad \text{as} \quad s \to -\infty.$$

If we choose

$$k = \sqrt{\pi/2} \tag{2.38}$$

(we shall justify the choice in a moment), then the above result simplifies and expands as follows.

$$\begin{aligned} w &= -\left[\frac{1}{s} - \frac{1}{s^3} + \frac{3}{s^5} + O\left(\frac{1}{s^7}\right)\right]^{-1} \\ &= -s\left[1 - \frac{1}{s^2} + \frac{3}{s^4} + O\left(\frac{1}{s^6}\right)\right]^{-1} \\ &= -s\left[1 + \frac{1}{s^2} - \frac{3}{s^4} + \frac{1}{s^4} + \cdots\right] \\ &\sim -s - \frac{1}{s} + \frac{2}{s^3}, \quad \text{as} \quad s \to -\infty. \end{aligned} \tag{2.39}$$

We are now ready to match in the usual way.

Inner expansion to $O(\sqrt{\epsilon})$ of the outer solution to $O(\epsilon^2)$:

$$\begin{aligned} y_0 + \epsilon y_1 + \epsilon^2 y_2 &= 1 - t + \frac{\epsilon}{1-t} - \frac{2\epsilon^2}{(1-t)^3} \\ &= \sqrt{\epsilon}\left(s + \frac{1}{s} - \frac{2}{s^3}\right) \qquad \text{(expressed in the inner variable } s.\text{)} \end{aligned}$$

No expansion is needed, as the above result is correct to $O(\sqrt{\epsilon})$.

Outer expansion to $O(\epsilon^2)$ of the inner solution to $O(\sqrt{\epsilon})$:

$$\begin{aligned}
\sqrt{\epsilon}\, w(s) &= \sqrt{\epsilon}\left(ke^{s^2/2} + \sqrt{\frac{\pi}{2}}\, e^{s^2/2}\, \mathrm{erf}\,\frac{s}{\sqrt{2}}\right)^{-1} \\
&= \sqrt{\epsilon}\left(ke^{(1-t)^2/2\epsilon} + \sqrt{\frac{\pi}{2}}\, e^{(1-t)^2/2\epsilon}\, \mathrm{erf}\,\frac{-(1-t)}{\sqrt{2\epsilon}}\right)^{-1} && \text{(expressed in the outer variable } t\text{)} \\
&= \sqrt{\epsilon}\left(\frac{(1-t)}{\sqrt{\epsilon}} + \frac{\sqrt{\epsilon}}{1-t} - 2\frac{\epsilon^{3/2}}{(1-t)^3}\right) && \text{(expanded for } \epsilon \to 0,\ t < 1 \text{ fixed)} \\
&= 1 - t + \frac{\epsilon}{1-t} - \frac{2\epsilon^2}{(1-t)^3}.
\end{aligned}$$

The penultimate step above draws on the expansion (2.39) found above. We see that the inner and outer solutions match. One purpose of matching, of course, is to determine the constant k that appears in the layer solution (2.37). Without the choice (2.38) anticipated above, matching would not have occurred, as the expansion (2.39) would not have held. With k determined we can now write the layer solution (2.37) as

$$w = \sqrt{\frac{2}{\pi}}\frac{e^{-s^2/2}}{1 + \mathrm{erf}\,(s/\sqrt{2})} \tag{2.40}$$

As $s \to \infty$, *i.e.*, as one exits the inner layer to the right, $w \to 0$ exponentially. Thus it must match up with the outer solution $y_0 = 0$ on the right.

This completes the asymptotic description of the solution to the IVP. A graph of the solution for two different values of α and for $\epsilon = 0.01$ is shown in Figure 15. The new feature in this problem is the observation that the need for an inner region can also be assessed by computing several terms in the outer expansion and checking whether this expansion breaks down in some inner domain.

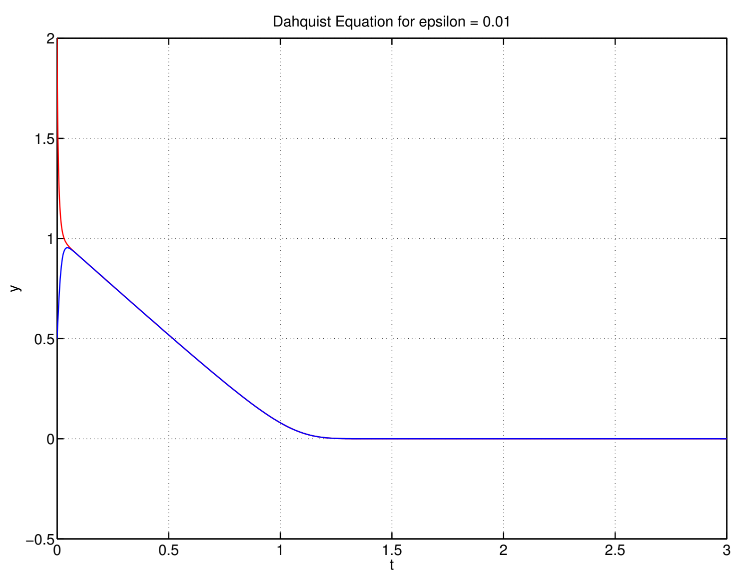

Figure 15: Solution of the IVP in Example 2.9 for $\alpha = 1.5$ and $\alpha = 0.5$. Note the thin initial layer at $t = 0$ and the thicker inner layer (corner layer) at $t = 1$.

Aside. Asymptotic expansion of $\operatorname{erf} z$ as $z \to -\infty$. The result is obtained by repeated integration by parts. We have

$$
\begin{aligned}
\operatorname{erf} z &= \frac{2}{\sqrt{\pi}} \int_0^z e^{-x^2}\, dx \\
&= \operatorname{erf}(-\infty) - \frac{2}{\sqrt{\pi}} \int_z^{-\infty} e^{-x^2}\, dx \\
&= -1 - \frac{2}{\sqrt{\pi}} \int_z^{-\infty} (-2x)\, e^{-x^2} \left(\frac{-1}{2x}\right) dx.
\end{aligned}
$$

Writing the integrand as above allows us to integrate by parts. Doing so repeatedly leads to

$$
\begin{aligned}
\operatorname{erf} z + 1 &= -\frac{2}{\sqrt{\pi}} \left[\left(\frac{-1}{2x}\right) e^{-x^2} \Big|_z^{-\infty} - \int_z^{-\infty} e^{-x^2} \left(\frac{1}{2x^2}\right) dx \right] \\
&= -\frac{2}{\sqrt{\pi}} \left[\frac{1}{2z} e^{-z^2} - \int_z^{-\infty} (-2x) e^{-x^2} \left(\frac{-1}{4x^3}\right) dx \right] \\
&= -\frac{2}{\sqrt{\pi}} \left[\frac{1}{2z} e^{-z^2} - \left\{ \left(\frac{-1}{4x^3}\right) e^{-x^2} \Big|_z^{-\infty} - \int_z^{-\infty} e^{-x^2} \left(\frac{3}{4x^4}\right) dx \right\} \right] \\
&= -\frac{2}{\sqrt{\pi}} \left[\frac{1}{2z} e^{-z^2} - \frac{1}{4z^3} e^{-z^2} + \int_z^{-\infty} e^{-x^2} \left(\frac{3}{4x^4}\right) dx \right] \\
&= -\frac{2}{\sqrt{\pi}} \left[\frac{1}{2z} e^{-z^2} - \frac{1}{4z^3} e^{-z^2} + \int_z^{-\infty} (-2x) e^{-x^2} \left(\frac{-3}{8x^5}\right) dx \right] \\
&= -\frac{2}{\sqrt{\pi}} \left[\frac{1}{2z} e^{-z^2} - \frac{1}{4z^3} e^{-z^2} + \left(\frac{-3}{8x^5}\right) e^{-x^2} \Big|_z^{-\infty} - \int_z^{-\infty} e^{-x^2} \left(\frac{15}{8x^6}\right) dx \right] \\
&= -\frac{2}{\sqrt{\pi}} \left[\frac{1}{2z} e^{-z^2} - \frac{1}{4z^3} e^{-z^2} + \frac{3}{8z^5} e^{-z^2} - \int_z^{-\infty} e^{-x^2} \left(\frac{15}{8x^6}\right) dx \right] \\
&= -\frac{2}{\sqrt{\pi}} \left[\frac{1}{2z} e^{-z^2} - \frac{1}{4z^3} e^{-z^2} + \frac{3}{8z^5} e^{-z^2} + O\left(\frac{e^{-z^2}}{z^7}\right) \right].
\end{aligned}
$$

In the last step we have simply recognized the pattern of the expansion to infer the order of the last term. With $z = s/\sqrt{2}$ the above result becomes

$$
\operatorname{erf}\left(\frac{s}{\sqrt{2}}\right) = -1 - \sqrt{\frac{2}{\pi}} e^{-s^2/2} \left[\frac{1}{s} - \frac{1}{s^3} + \frac{3}{s^5} + O\left(\frac{1}{s^7}\right) \right], \quad \text{as} \quad s \to -\infty. \tag{2.41}
$$

Example 2.10. A nonlinear problem modeling thermal explosion in a stirred-tank environment. Consider a well-stirred insulated container filled with a gaseous reactive mixture. Let the reaction be exothermic (*i.e.*, be accompanied by release of heat) and one in which a reactant A is converted into a product B according to Arrhenius kinetics. The governing equations are

$$
\begin{aligned}
\frac{dY}{dt} &= -AY \exp(-E/RT), \\
\frac{dT}{dt} &= \frac{Q}{c} AY \exp(-E/RT),
\end{aligned}
$$

with initial conditions

$$
Y(0) = Y_0, \quad T(0) = T_0.
$$

Here t denotes time, Y the mass fraction of the reacting species and T the temperature. The parameters appearing above are the heat of reaction Q, the specific heat c, the activation energy E, the Universal Gas Constant R and the pre-exponential reaction-rate factor A. Our interest is in exploring the dynamics of the system when the activation energy E is large.

Activation energy measures the sensitivity of the reaction rate to change in temperature. It is the minimum energy the colliding molecules must possess to undergo a chemical reaction. The Arrhenius factor $e^{-E/RT}$ is the fraction of collisions with energy higher than E. Suppose that T_r is a reference temperature in a system. Then a normalized Arrhenius factor is

$$\frac{e^{-E/RT}}{e^{-E/RT_r}} = \exp\left[\frac{E}{RT_r}\left(1 - \frac{1}{T/T_r}\right)\right].$$

Figure 16 displays the graphs of the normalized Arrhenius factor as a function of the normalized temperature

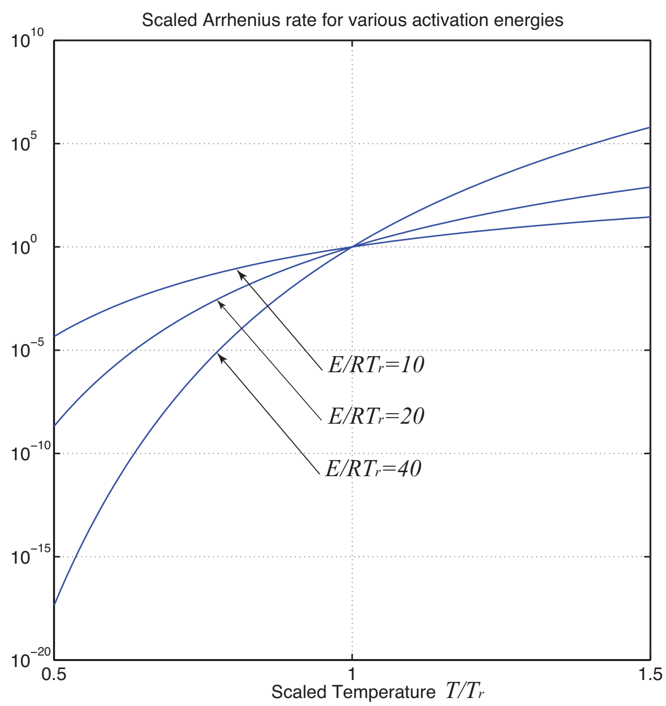

Figure 16: Scaled Arrhenius factor as a function of scaled temperature. Large activation energy shows a high sensitivity to temperature changes.

T/T_r for various values of the normalized activation energy E/RT_r. We see that the Arrhenius factor, and hence the reaction rate, is a strongly sensitive function of temperature at higher activation energies, *i.e.*, small temperature variations correspond to large changes in the reaction rate. High activation energies are widely prevalent, and our goal in the current exercise is to explore how the system evolves under such conditions.

We begin by noting that the governing ODEs combine into

$$\frac{d}{dt}(T + QY/c) = 0,$$

yielding the first integral

$$T + \frac{QY}{c} = T_0 + \frac{QY_0}{c}.$$

Then the problem can be reduced to an initial-value problem for T, *i.e.*,

$$\frac{dT}{dt} = A\left(T_0 + \frac{QY_0}{c} - T\right)e^{-E/RT}, \quad T(0) = T_0.$$

We render the problem dimensionless by choosing the reference scale T_0 for T and t_0 for t where

$$1/t_0 = \frac{E}{RT_0}\frac{AQY_0}{cT_0}e^{-E/RT_0}.$$

We note that the inverse reference time is the same as the reaction rate at the initial temperature, except for the factor E/RT_0. Then there IVP takes the form

$$\frac{dT}{dt} = \frac{\epsilon}{\beta}(1+\beta-T)\exp\left(\frac{T-1}{\epsilon T}\right), \quad T(0)=1. \tag{2.42}$$

Here, t and T now denote dimensionless quantities and the parameters appearing above are

$$\begin{aligned} \beta &= \frac{QY_0}{cT_0}, \text{ dimensionless heat release, and} \\ \epsilon &= \frac{RT_0}{E}, \text{ dimensionless inverse activation energy.} \end{aligned}$$

We seek the solution to (2.42) when ϵ is small, *i.e.*, when the dimensionless activation energy is large. The qualitative form of the solution is easily established. We note that at $t=0$, $T=1$ and therefore, from (2.42), $dT/dt>0$. Thus T increases with t, and as it does, dT/dt remains positive and approaches zero as $T\to 1+\beta$. Thus the solution rises monotonically from 1 to $1+\beta$ with time t.

It is possible to write down the exact solution. It is given, implicitly, by

$$t = t_e + \frac{\beta}{e}\left[e^{-1/\epsilon}\,\mathrm{Ei}\left(\frac{1}{\epsilon T}\right) - \exp\left(-\frac{1}{\epsilon}+\frac{1}{\epsilon(1+\beta)}\right)\mathrm{Ei}\left(\frac{1}{\epsilon T}-\frac{1}{\epsilon(1+\beta)}\right)\right],$$

where

$$t_e = -\frac{\beta}{e}\left[e^{-1/\epsilon}\,\mathrm{Ei}\left(\frac{1}{\epsilon}\right) - \exp\left(-\frac{1}{\epsilon}+\frac{1}{\epsilon(1+\beta)}\right)\mathrm{Ei}\left(\frac{1}{\epsilon}-\frac{1}{\epsilon(1+\beta)}\right)\right].$$

The function Ei appearing above is the *exponential integral*, defined by

$$\mathrm{Ei}\,(x) = \mathrm{PV}\int_{-\infty}^{x}\frac{e^y}{y}\,dy,$$

where PV stands for Principal Value. The above solution is distinguished only by its opaqueness; it gives little insight into how the temperature evolves with time. We now proceed directly to obtain a perturbation solution. In such a solution, an outer region will be characterized by bounded derivative dT/dt in the limit $\epsilon\to 0$. We now show that there are two such regions.

Outer region O_I

As already mentioned, the solution is expected to increase with t monotonically from 1 to $1+\beta$. Thus there exists a time interval in which $T-1$ is positive and of order unity. Equation (2.42) shows that in such an interval the Arrhenius factor $\exp\left(\frac{T-1}{\epsilon T}\right)$ on the RHS is exponentially large as $\epsilon\to 0$. For the LHS to stay bounded, the factor $1+\beta-T$ must therefore compensate by becoming exponentially small. The result is the approximation

$$T = 1+\beta+\mathrm{TST}. \tag{2.43}$$

Here TST stands for Transcendentally Small Terms. As the above approximation does not satisfy the initial condition, it is valid in a time interval bounded away from $t=0$.

Outer region O_{II}

For $T-1=O(\epsilon)$ the RHS of (2.42) is bounded, and is in fact $O(\epsilon)$. This suggests a second outer approximation, near $t=0$, of the form

$$T\sim 1+\epsilon T_1(t).$$

Substitution into (2.42) finds the leading-order problem to be

$$\frac{dT_1}{dt} = e^{T_1}, \quad T_1(0) = 0.$$

The solution

$$T_1 = -\ln(1-t)$$

determines the outer expansion in region O_{II} as

$$T \sim 1 - \epsilon \ln(1-t). \tag{2.44}$$

We note that the above solution is valid only in the interval $[0, 1)$ as the logarithmic term is unbounded at $t = 1$. The monotonicity of the solution suggests a transition from region O_{II} ($T \sim 1$) to the region O_I ($T \sim 1 + \beta$) in a thin layer at $t = 1$ in which the derivative dT/dt is large for small ϵ.

Layer L_I

In earlier examples we have come across two different ways of determining the thickness of the layer region. One is the application of dominant balance to the ODE and the other is to exploit the breakdown of the outer expansion that has been carried to more than one term. If we were to employ the first strategy, we would look for a layer of thickness μ around $t = 1$ by setting $t = 1 + \mu\sigma$ and $T(t) = \phi(\sigma)$. Anticipating matching with the two outer solutions developed above, we would expect ϕ to increase from 1 to $1 + \beta$ as σ ranges from $-\infty$ to ∞. In the new variables (2.42) becomes

$$\frac{1}{\mu}\frac{d\phi}{d\sigma} = \frac{\epsilon}{\beta}(1 + \beta - \phi) \exp\left(\frac{\phi - 1}{\epsilon\phi}\right).$$

Now the layer thickness $\mu(\epsilon)$ should be selected to be such that the two sides of the equation above balance for $\phi \in (1, 1 + \beta)$ and $\epsilon \to 0$. However, such a balance is not possible because for a given ϵ the size of the LHS, determined by $1/\mu(\epsilon)$, is fixed while the size of the RHS, determined primarily by $\epsilon \exp\left(\frac{\phi-1}{\epsilon\phi}\right)$, varies with ϕ. This suggests that the layer thickness must be allowed to evolve dynamically with the solution. Falling back on the second strategy, we examine the outer expansion (2.44) and note that it breaks down when $1-t$ is small enough to cause the second term to become comparable to the first term, *i.e.*, when $-\epsilon \ln(1-t)$ is of order unity. Accordingly we define the inner variable σ by means of the *nonlinear* transformation

$$-\epsilon \ln(1-t) = \sigma, \quad \text{or} \quad t = 1 - e^{-\sigma/\epsilon}, \quad \sigma > 0, \tag{2.45}$$

with

$$T(t) = \phi(\sigma).$$

Then (2.42) transforms into

$$\frac{d\phi}{d\sigma} = \frac{1}{\beta}(1 + \beta - \phi) \exp\left[\frac{1}{\epsilon}\left(1 - \sigma - \frac{1}{\phi}\right)\right]. \tag{2.46}$$

Now, the requirement that $d\phi/d\sigma$ is bounded as $\epsilon \to 0$ forces $1-\sigma-1/\phi$ to be $O(\epsilon)$ in the above equation, thereby suggesting the expansion

$$\phi \sim \frac{1}{1-\sigma} + \epsilon\phi_1(\sigma). \tag{2.47}$$

Substitution of the above expansion into (2.46) shows that to leading order the latter is reduced to an algebraic equation for ϕ_1, which has the solution

$$\phi_1 = -\frac{1}{(1-\sigma)^2} \ln\left[(1-\sigma)\left(1 - \frac{1+\beta}{\beta}\sigma\right)\right].$$

The solution in the layer can now be written as

$$\phi \sim \frac{1}{1-\sigma} - \epsilon\frac{1}{(1-\sigma)^2} \ln\left[(1-\sigma)\left(1 - \frac{1+\beta}{\beta}\sigma\right)\right]. \tag{2.48}$$

As σ increases from 0 to $\beta/(1+\beta)$, the leading term in the above expansion shows that ϕ increases from $\phi \sim 1$ to $\phi \sim 1+\beta$. This would seem to support the conjecture, made earlier, that the layer provides the necessary transition between O_I and O_{II}. However, the second term in (2.48) becomes singular at $\sigma = \beta/(1+\beta)$. This singularity suggests the presence of another layer L_{II}, a corner layer in which the solution is not expected to stray far from $1+\beta$.

Layer L_{II}

We try a linear magnification $t = 1 + \delta s$, and let $T(t) = \psi(s)$. In terms of the new variables (2.42) reads

$$\frac{d\psi}{ds} = \frac{\epsilon\delta}{\beta} \exp\left(\frac{\beta}{\epsilon(1+\beta)}\right)(1+\beta-\psi)\exp\left(\frac{\psi-(1+\beta)}{\epsilon\psi(1+\beta)}\right).$$

In the limit $\epsilon \to 0$ a balance is achieved for $1+\beta-\psi = O(\epsilon)$ if we select

$$\delta = \frac{\beta}{\epsilon}\exp\left(-\frac{\beta}{\epsilon(1+\beta)}\right), \tag{2.49}$$

whence the equation for ψ becomes

$$\frac{d\psi}{ds} = (1+\beta-\psi)\exp\left(\frac{\psi-(1+\beta)}{\epsilon\psi(1+\beta)}\right). \tag{2.50}$$

With the above choice, σ and s are related by the expressions

$$\sigma = \frac{\beta}{1+\beta} - \epsilon\ln(-\beta s)/\epsilon), \quad i.e., \quad s = -\frac{\epsilon}{\beta}\exp\left\{\frac{1}{\epsilon}\left(\frac{\beta}{1+\beta}-\sigma\right)\right\}. \tag{2.51}$$

In the layer L_{II} we assume the expansion

$$\psi \sim 1+\beta+\epsilon(1+\beta)^2\psi_1(s), \tag{2.52}$$

where ψ_1, by virtue of the monotonicity of the solution, is nonpositive. Substitution of the above expansion into (2.50) yields, to leading order,

$$\frac{d\psi_1}{ds} = -\psi_1 e^{\psi_1},$$

which has the implicit solution

$$s - s_1 = F(\psi) \equiv -\int_1^{-\psi_1}\frac{e^x}{x}\,dx. \tag{2.53}$$

Here s_1 is an integration constant. For matching purposes we collect the following properties of the solution. (Details of how these expansions are obtained are given at the end of the exercise.)

$$s - s_1 = \frac{e^{-\psi_1}}{\psi_1}\left\{1 - \frac{1}{\psi_1} + O\left(\frac{1}{\psi_1^2}\right)\right\} \quad \text{as} \quad \psi_1 \to -\infty \quad (i.e., \quad s \to -\infty), \tag{2.54}$$

$$s - s_1 = -\ln(-\psi_1) + \ln K_1 + O(\psi_1) \quad \text{as} \quad \psi_1 \to 0 \quad (i.e., \quad s \to \infty), \tag{2.55}$$

where

$$K_1 = \exp\left(\int_0^1 \frac{e^x-1}{x}\,dx\right).$$

The above expansions can be inverted to yield

$$\psi_1 = -\ln(-s) - \ln\{\ln(-s)\} + O\left(\frac{\ln\{\ln(-s)\}}{\ln(-s)}\right), \quad \text{as} \quad s \to -\infty, \tag{2.56}$$

$$\psi_1 = -K_1 e^{s_1 - s} + O(e^{-2s}), \quad \text{as} \quad s \to -\infty. \tag{2.57}$$

Note that s_1 does not appear in the expansion (2.56) to the order to which it has been taken. In fact, s_1 will first appear at the $O(1/s)$ level, which is transcendentally small when compared to the logarithmic gauge sequence of the expansion (2.56). We are now ready to match by using the Van Dyke Principle.

Matching of O_{II} and L_I

L_I - expansion to $O(\epsilon)$, equation (2.48), written in the O_{II} - variable t :

$$\frac{1}{1+\epsilon\ln(1-t)} - \frac{\epsilon}{[1+\epsilon\ln(1-t)]^2}\ln\left[\{1+\epsilon\ln(1-t)\}\left\{1+\frac{1+\beta}{\beta}\epsilon\ln(1-t)\right\}\right].$$

Expanded for $\epsilon \to 0$, t fixed, to $O(\epsilon)$:

$$1-\epsilon\ln(1-t). \tag{2.58}$$

O_{II} - expansion to $O(\epsilon)$, equation (2.44), written in the L_I - variable σ :

$$1+\sigma.$$

Expanded for $\epsilon \to 0$, σ fixed, to $O(\epsilon)$:

$$1+\sigma.$$

Written in the O_{II} variable t :

$$1-\epsilon\ln(1-t). \tag{2.59}$$

Matching is confirmed by the agreement of (2.58) and (2.59).

Matching of L_I and L_{II}

L_I - expansion to $O(\epsilon)$, equation (2.48), written in the L_{II} - variable s :

$$\frac{1}{\frac{1}{1+\beta}+\epsilon\ln(-\beta s/\epsilon)} - \frac{\epsilon}{\left[\frac{1}{1+\beta}+\epsilon\ln(-\beta s/\epsilon)\right]^2}\ln\left[\left\{\frac{1}{1+\beta}+\epsilon\ln(-\beta s/\epsilon)\right\}\left\{\frac{1+\beta}{\beta}\epsilon\ln(-\beta s/\epsilon)\right\}\right].$$

Expanded for $\epsilon \to 0$, s fixed, to $O(\epsilon)$ (without cutting between logarithms):

$$1+\beta-\epsilon(1+\beta)^2\ln(-\beta s/\epsilon)-\epsilon(1+\beta)^2\ln[(\epsilon/\beta)\ln(-\beta s/\epsilon)].$$

Further simplified to:

$$1+\beta-\epsilon(1+\beta)^2[\ln(-s)+\ln\{\ln(-\beta s/\epsilon)\}]. \tag{2.60}$$

L_{II} - expansion to $O(\epsilon)$, equation (2.52), written in the L_I - variable σ :

$$1+\beta+\epsilon(1+\beta)^2\psi_1\left[-\frac{\epsilon}{\beta}\exp\left\{\frac{1}{\epsilon}\left(\frac{\beta}{1+\beta}-\sigma\right)\right\}\right].$$

Expanded for $\epsilon \to 0$, σ fixed, to $O(\epsilon)$, without cutting between logarithms:

$$1+\beta+\epsilon(1+\beta)^2\left[-\ln(\epsilon/\beta)-\frac{1}{\epsilon}\left(\frac{\beta}{1+\beta}-\sigma\right)-\ln\left\{\frac{1}{\epsilon}\left(\frac{\beta}{1+\beta}-\sigma\right)\right\}\right].$$

(Here, (2.56) has been used.) Rewritten in the L_{II} variable s :

$$1+\beta-\epsilon(1+\beta)^2[\ln(-s)+\ln\{\ln(-\beta s/\epsilon)\}]. \tag{2.61}$$

Matching is confirmed by the agreement of (2.60) and (2.61). However, s_1 is left undetermined due to reasons explained earlier. (Incidentally, the exact solution reveals that $s_1 = -\mathrm{Ei}(1)$.)

Matching of L_{II} and O_I

L_{II} - expansion to $O(\epsilon)$, equation (2.52), written in the O_I - variable t :

$$1+\beta+\epsilon\psi_1[(t-1)/\delta].$$

Expanded for $\epsilon \to 0$, $t>1$ fixed, to $O(\epsilon)$:

$$1+\beta.$$

(Here, (2.57) has been used.) The above result agrees with the $O_!$ - expansion (2.43) when the transcendentally small terms are ignored in the latter. Thus matching is confirmed, and this completes the solution to the problem.

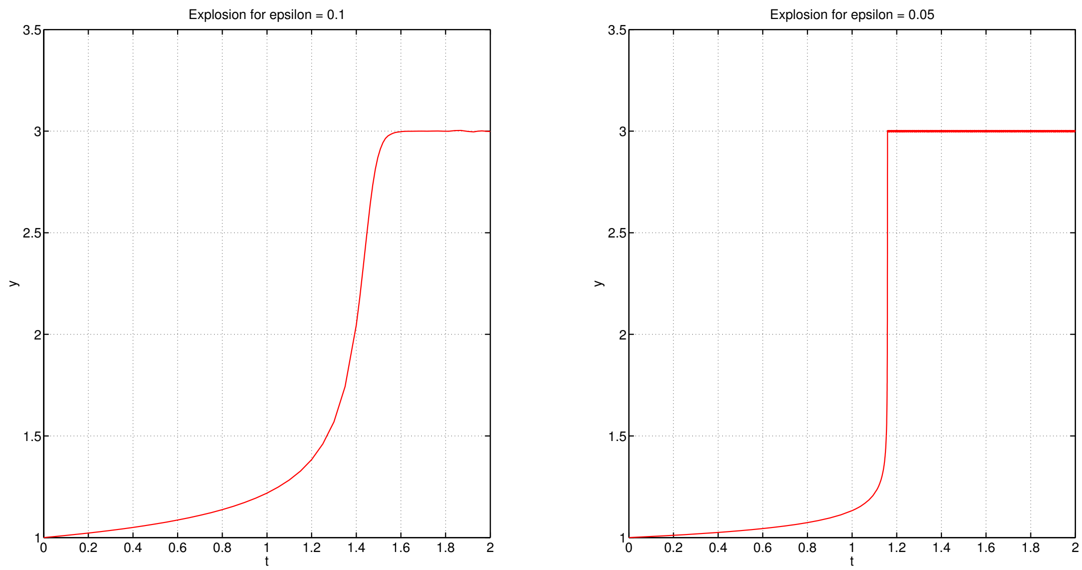

Figure 17: Exact solution plotted for $\epsilon = 0.1$ (left) and $\epsilon = 0.05$ (right) for $\beta = 2$.

Figure 17 shows the exact solution for two different values of ϵ. The plots confirm the general trend uncovered by the perturbation solutions.

Details of expansions. We now provide the details behind expansions (2.56) and (2.57). The starting point is equation (2.53),

$$s - s_1 = F(\psi_1),$$

where

$$F(\psi_1) = -\int_1^{-\psi_1} \frac{e^x}{x}\, dx. \tag{2.62}$$

(i) First, we describe the computations leading to expansions (2.54) and (2.55).

The expansion (2.54), for $\psi_1 \to -\infty$, is obtained via repeated integration by parts. Thus,

$$\begin{aligned}
F(\psi_1) &= -\left[\frac{e^x}{x}\Big|_1^{-\psi_1} + \int_1^{-\psi_1} \frac{e^x}{x^2}\, dx\right] \\
&= -\left[\frac{e^x}{x}\Big|_1^{-\psi_1} + \frac{e^x}{x^2}\Big|_1^{-\psi_1} + 2\int_1^{-\psi_1} \frac{e^x}{x^3}\, dx\right] \\
&= -\left[\frac{e^x}{x}\Big|_1^{-\psi_1} + \frac{e^x}{x^2}\Big|_1^{-\psi_1} + 2\frac{e^x}{x^3}\Big|_1^{-\psi_1} + 3!\int_1^{-\psi_1} \frac{e^x}{x^4}\, dx\right] \\
&= -\left[-\frac{e^{-\psi_1}}{\psi_1} - e + \frac{e^{-\psi_1}}{\psi_1^2} - e - 2\frac{e^{-\psi_1}}{\psi_1^3} - 2e + 3!\int_1^{-\psi_1} \frac{e^x}{x^4}\, dx\right] \\
&= \frac{e^{-\psi_1}}{\psi_1} - \frac{e^{-\psi_1}}{\psi_1^2} + 2\frac{e^{-\psi_1}}{\psi_1^3} + 4e - 3!\int_1^{-\psi_1} \frac{e^x}{x^4}\, dx.
\end{aligned}$$

The pattern is now evident. For $\psi_1 \to -\infty$ the above expression has the asymptotic form

$$s - s_1 = F(\psi_1) = \frac{e^{-\psi_1}}{\psi_1} - \frac{e^{-\psi_1}}{\psi_1^2} + O\left(\frac{e^{-\psi_1}}{\psi_1^3}\right),$$

thereby confirming the result in (2.54).

To obtain the expansion (2.55) we return to (2.62) and proceed as follows.

$$\begin{aligned}
F(\psi_1) &= -\int_1^{-\psi_1} \frac{e^x - 1}{x}\, dx - \int_1^{-\psi_1} \frac{dx}{x} \\
&= -\ln(-\psi_1) - \int_1^{-\psi_1} \frac{e^x - 1}{x}\, dx \\
&= -\ln(-\psi_1) - \int_1^{0} \frac{e^x - 1}{x}\, dx - \int_0^{-\psi_1} \frac{e^x - 1}{x}\, dx \\
&= -\ln(-\psi_1) - \int_1^{0} \frac{e^x - 1}{x}\, dx - \int_0^{-\psi_1} \left(1 + \frac{x}{2} + \frac{x^2}{3!} + \cdots\right) dx.
\end{aligned}$$

The second term above is a constant, which we denote by

$$\ln K_1 = \int_0^1 \frac{e^x - 1}{x}\, dx,$$

while the last term, in the limit $\psi_1 \to 0$, is $O(\psi_1)$. Therefore, we can write

$$s - s_1 = F(\psi_1) = -\ln(-\psi_1) + \ln K_1 + O(\psi_1) \quad \text{as} \quad \psi_1 \to 0.$$

Thus (2.55) is recovered.

(ii) Now consider the details behind the inversion of (2.54). We begin by rewriting this equation as

$$-s = F(\psi_1) = \frac{e^{-\psi_1}}{-\psi_1} + \frac{e^{-\psi_1}}{\psi_1^2} + O\left(\frac{e^{-\psi_1}}{\psi_1^3}\right) - s_1, \quad \psi_1 \to -\infty, \;\; s \to -\infty.$$

The constant s_1 on the right-hand side can be ignored as it is exponentially small compared to the terms retained. Then we can write

$$-s = \frac{e^{-\psi_1}}{-\psi_1}\left[1 - \frac{1}{\psi_1} + O\left(\frac{1}{\psi_1^2}\right)\right],$$

or as

$$\ln(-s) = -\psi_1 - \ln(-\psi_1) + O\left(\frac{1}{\psi_1}\right).$$

Therefore,

$$\begin{aligned}
\psi_1 &= -\ln(-s) - \ln(-\psi_1) + O\left(\frac{1}{\psi_1}\right) \\
&= -\ln(-s) - \ln\left[\ln(-s) + \ln(-\psi_1) + O\left(\frac{1}{\psi_1}\right)\right] + O\left(\frac{1}{\psi_1}\right) \\
&= -\ln(-s) - \ln\{\ln(-s)\} - \ln\left[1 + \frac{\ln(-\psi_1)}{\ln(-s)} + O\left(\frac{1}{\psi_1 \ln(-s)}\right)\right] + O\left(\frac{1}{\psi_1}\right) \\
&= -\ln(-s) - \ln\{\ln(-s)\} + O\left(\frac{\ln(-\psi_1)}{\ln(-s)}\right) \\
&= -\ln(-s) - \ln\{\ln(-s)\} + O\left(\frac{\ln\{\ln(-s)\}}{\ln(-s)}\right), \quad \text{as} \quad s \to -\infty.
\end{aligned}$$

Thus the result (2.56) is confirmed.

We now turn to the inversion of (2.55), and begin by writing it as

$$\ln(-\psi_1) = s_1 - s + \ln K_1 + O(\psi_1) \quad \text{as} \quad s \to \infty, \;\; \psi_1 \to 0.$$

Then,

$$
\begin{aligned}
-\psi_1 &= K_1 e^{s_1 - s + O(\psi_1)} \\
&= K_1 e^{s_1 - s}\ e^{O(\psi_1)} \\
&= K_1 e^{s_1 - s}[1 + O(\psi_1)] \\
&= K_1 e^{s_1 - s} + O(e^{s_1 - s}\psi_1) \\
&= K_1 e^{s_1 - s} + O(e^{-2s}), \quad \text{as} \quad s \to \infty.
\end{aligned}
$$

Thus equation (2.57) is recovered.

3 Layer-type problems in PDES

We shall consider PDEs of type

$$\epsilon \mathcal{L}_2(u) + \mathcal{L}_1(u) = 0,$$

where $\mathcal{L}_1$ and $\mathcal{L}_2$ are first- and second-order partial differential operators respectively, acting on the function $u(x, y)$. We shall consider the three basic forms of $\mathcal{L}_2$, *i.e.*, elliptic: $\mathcal{L}_2(u) = u_{xx} + u_{yy}$, hyperbolic: $\mathcal{L}_2(u) = u_{xx} - u_{yy}$, and parabolic: $\mathcal{L}_2(u) = u_{xx}$. The first-order operator $\mathcal{L}_1(u)$ will be linear in most cases, of the form $a(x, y)u_x + b(x, y)u_y + c(x, y)u$, but may also be quasilinear of the form $a(x, y, u)u_x + b(x, y, u)u_y + c(x, y, u)$. Since the reduced equation obtained in the limit $\epsilon \to 0$ is the first-order equation $\mathcal{L}_1(u) = 0$, we begin with a brief discussion of such equations.

3.1 First-order, linear PDEs

The general form is

$$a(x, y)u_x + b(x, y)u_y + c(x, y)u = d(x, y). \tag{3.1}$$

This equation can be solved by using the method of characteristics. The characteristics are curves in the xy - plane along which the PDE reduces to an ODE. These curves are solutions of the differential equation

$$\frac{dy}{dx} = \frac{b(x, y)}{a(x, y)}.$$

Along a characteristic the PDE (3.1) can be written as

$$u_x + \frac{dy}{dx}u_y + \frac{c(x, y)}{a(x, y)}u = \frac{d(x, y)}{a(x, y)}.$$

According to the chain rule the first two terms above define the total derivative of u with respect to x along the characteristic, so that the PDE can be rewritten as the ODE

$$\frac{du}{dx} + \frac{c(x, y)}{a(x, y)}u = \frac{d(x, y)}{a(x, y)}.$$

This ODE can be integrated in principle to yield u, provided u is specified along some nowhere characteristic curve.

Example 3.1. Consider the IVP

$$u_x + 2xu_y + u = 0, \quad u(0, y) = \sin y.$$

The characteristics satisfy the ODE

$$\frac{dy}{dx} = 2x$$

which integrates to $y = x^2 + c$. Here c is the parameter that labels the characteristics, and is the intercept of the characteristic on the y - axis. The characteristics, displayed in Figure 18 are parabolas opening upwards. On the characteristics, *i.e.*, at fixed c, the PDE transforms into the ODE

$$\frac{du}{dx} + u = 0,$$

with general solution

$$u = A(c)\, e^{-x}.$$

Note that the integration constant is a function of the characteristic label c. On replacing c by $y - x^2$ from the equation of the characteristics we obtain

$$u(x, y) = A(y - x^2)\, e^{-x}.$$

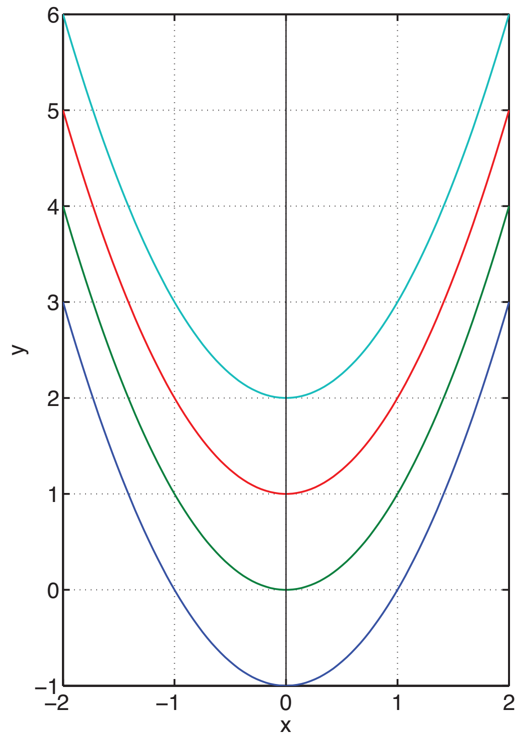

Figure 18: Characteristics in Example 3.1.

The initial condition leads to

$$\sin y = A(y),$$

whence the solution

$$u(x, y) = \sin(y - x^2)\, e^{-x}.$$

3.2 Singularly-perturbed PDEs: elliptic problems

Example 3.2 Consider the elliptic problem

$$u_y = \epsilon(u_{xx} + u_{yy}), \quad 0 < x < 1,\ 0 < y < 1,$$

with Dirichlet boundary conditions

$$u(x, 0) = f_1(x),\ u(x, 1) = f_2(x),\ u(0, y) = g_1(y),\ u(1, y) = g_2(y).$$

This PDE corresponds to steady-state but weak diffusion in a stream moving with unit velocity, and also arises in mangetohydrodynamics (where ϵ measures the strength of the viscous force relative to the electromagnetic force) and in the theory of plate membranes under tension in the y - direction (where ϵ measures the bending stiffness). We seek a perturbation solution in the limit of small ϵ. The configuration is shown in Figure 19.

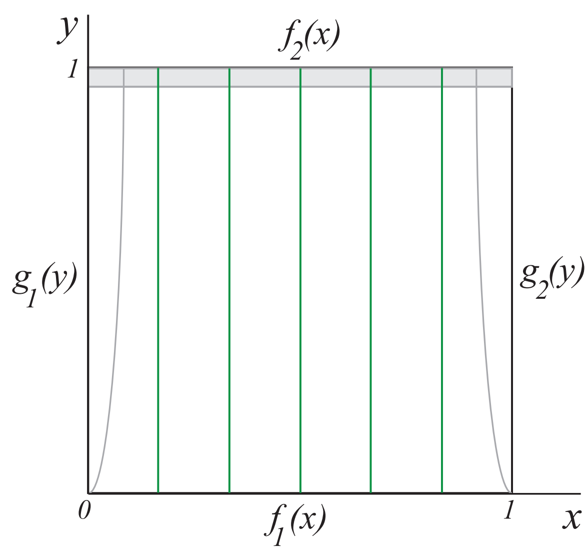

Figure 19: Example 3.2.

<u>Outer solution</u>

With $u \sim u_0$, the reduced equation is $u_{0y} = 0$. The subcharacteristics are lines parallel to the y - axis, and the solution is simply $u_0 = F(x)$.

<u>Inner solution - I</u>

The outer solution cannot satisfy both $u_0(x, 0) = f_1(x)$ and $u_0(x, 1) = f_2(x)$. Let us suppose that u_0 satisfies the lower boundary condition for $y < a$ and the upper boundary condition for $y > a$, with $0 < a < 1$. Then there is a layer at $y = a$. We examine the layer by setting $y = a + \delta\eta$, and $u(x, y) = v(x, \eta)$. Then the PDE transforms into

$$\frac{1}{\delta} v_\eta = \epsilon v_{xx} + \frac{\epsilon}{\delta^2} v_{\eta\eta}.$$

Dominant balance finds $\delta = \epsilon$, thereby reducing the PDE to

$$v_{\eta\eta} - v_\eta = -\epsilon v_{xx}.$$

With $v \sim v_0$ the above equation yields, at leading order,

$$v_{0\eta\eta} - v_{0\eta} = 0.$$

The general solution is

$$v_0 = a_0(x) + b_0(x)e^{\eta}.$$

Exponential growth is avoided only if η is not allowed to grow to ∞, suggesting that the layer can only occur on the upper boundary, *i.e.*, $a = 1$, so that $-\infty < \eta \leq 0$. Then the outer solution must satisfy the lower boundary condition, *i.e.*, $u_0 = f_1(x)$, and matching along with the application of the upper boundary condition determines the layer solution to be

$$v_0 = f_1(x) + [f_2(x) - f_1(x)]e^{\eta}.$$

Additional layers are needed along the side walls to satisfy the boundary conditions there.

Inner solution - II: Layers at the side walls

In the layer at the left sidewall, $x = 0,\ 0 < y < 1$, let $x = \mu\xi$, $u(x, y) = w(\xi, y)$. Then the PDE transforms into

$$w_y = \frac{\epsilon}{\mu^2} w_{\xi\xi} + \epsilon w_{yy}.$$

Dominant balance selects $\mu = \sqrt{\epsilon}$, and then the PDE becomes

$$w_y - w_{\xi\xi} = \epsilon w_{yy}.$$

With $w \sim w_0$, PDE reduces at leading order to

$$w_{0y} - w_{0\xi\xi} = 0. \tag{3.2}$$

This is a diffusion equation, with y the time-like direction and $\xi \in [0, \infty)$. The boundary conditions are

$$w_0(0, y) = g_1(y), \quad w_0(\infty, y) = f_1(0), \ \ 0 < y < 1. \tag{3.3}$$

The second of the above conditions is provided by matching with the outer solution. The initial condition at $y = 0$ can be found by recognizing that in the layer variable, the boundary condition at the lower wall reads

$$u(\sqrt{\epsilon}\xi, 0) = f_1(\sqrt{\epsilon}\xi) = f_1(0) + O(\sqrt{\epsilon}).$$

Therefore, the layer solution satisfies

$$w_0(\xi, 0) = f_1(0). \tag{3.4}$$

The solution to the layer problem, consisting of equations (3.2) - (3.4), can be obtained in a variety of ways, by employing the Laplace transform for example, or by using the method of reflection. Here we use the former method which yields the solution as

$$w_0(\xi, y) = f_1(0)\ \mathrm{erf}\ (\xi/2\sqrt{y}) + \frac{\xi}{\sqrt{4\pi}} \int_0^y \frac{g_1(y - \eta)}{\eta^{3/2}} e^{-\xi^2/4\eta} d\eta. \tag{3.5}$$

In particular, if g_1 is a constant, the solution simplifies to

$$w_0(\xi, y) = g_1 + [f_1(0) - g_1]\ \mathrm{erf}\ (\xi/2\sqrt{y}).$$

Similarly, on the sidewall $x = 1$, the leading-order layer solution $z_0(\chi, y)$ is given by

$$z_0(\chi, y) = f_1(1)\ \mathrm{erf}\ (\chi/2\sqrt{y}) + \frac{\chi}{\sqrt{4\pi}} \int_0^y \frac{g_2(y - \eta)}{\eta^{3/2}} e^{-\chi^2/4\eta} d\eta, \tag{3.6}$$

where now,

$$\chi = \frac{1 - x}{\sqrt{\epsilon}}.$$

Composite solution

The composite expansion is given by

$$u(x, y) \sim v_0(x, \eta) + w_0(\xi, y) + z_0(\chi, y) - f_1(0) - f_1(1). \tag{3.7}$$

Note that the solution v_0 in the upper layer contains the outer solution u_0.

Remarks

- For more general convex regions the *upper* boundary may be curved, $y = y_u(x)$, say. Then the inner variable in the layer at the upper boundary is defined as $y = y_u(x) + \epsilon\eta$.
- Here the outer solution was constant along the straight subcharacteristics. For more general first-order operators such as $a(x, y)u_x + b(x, y)u_y + c(x, y)u = 0$, the subcharacteristics will be curved and the solution will evolve along the subcharacteristics.
- Two types of layers exist. On boundaries nowhere subcharacteristic the layers are exponential and $O(\epsilon)$ thick. On bounding walls along a subcharacteristic the layers are $O(\sqrt{\epsilon})$ thick and parabolic. Discontinuity on a nowhere subcharacteristic boundary will produce a parabolic layer along the subcharacteristic emerging from the discontinuity.
- The solution determined above is uniformly valid except in $O(\epsilon)$ neighborhoods of the four corners, where the full elliptic PDE applies. For details see Cook & Ludford (1971, 1973). These papers are posted on LMS.