MATH-6620 PERTURBATION METHODS SPRING 2016

Homework-4

Assigned Friday March 18, 2016

Due Wednesday March 30, 2016

PROBLEMS

1. Consider the signaling problem

$$\epsilon(u_{xx} - u_{tt}) = u_t + 2u_x, \quad 0 < x < \pi, \ t > 0,$$

with auxiliary conditions

$$u(x,0) = u_t(x,0) = 0, \ \ u(0,t) = -\sin t, \ u(\pi,t) = 0.$$

Construct a leading-order solution for $0 < \epsilon \ll 1$, paying due attention to the location of any layers.

Outer solution. Let $u(x,t;\epsilon) \sim u_0(x,t)$. Then, u_0 satisfies

$$u_{0_t} + 2u_{0_x} = 0.$$

This reduced PDE has characteristics (which are subcharacteristics of the full PDE)

$$x = 2t + \xi,$$

and hence the general solution

$$u_0(x,t) = f_0(x - 2t).$$

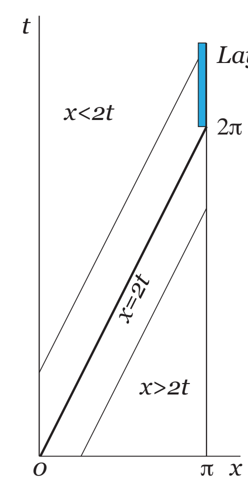

Figure 1: Subcharacteristics and layers for Problem 1.

The regions $x > 2t$ and $x < 2t$ have to be treated differently; see Figure 1. In each region the outer solution has the option of satisfying one of two boundary conditions. In the region $x > 2t$ the two conditions are $u(x,0) = 0$ and $u(\pi,t) = 0$, requiring $f_0(x) = 0$ and $f_0(\pi - 2t) = 0$ respectively. Both are satisfied by choosing $f_0 \equiv 0$, leading to

$$u_0(x,t) = 0, \quad x > 2t. \tag{1}$$

In the region $x < 2t$ we choose, subject to confirmation later, to satisfy the boundary condition $u(0,t) = -\sin t$. Then $f_0(-2t) = -\sin t$, or $f_0(T) = \sin(T/2)$. Then the solution is

$$u_0(x,t) = \sin\left(\frac{x}{2} - t\right), \quad x < 2t. \tag{2}$$

This solution does not satisfy the condition $u_0(\pi,t) = 0$, thereby suggesting a boundary layer at $x = \pi$, $t > 2\pi$.

Layer L_1. This layer is transverse to the subcharacteristics, and therefore, $O(\epsilon)$ thick. In it we set $x = \pi + \epsilon\xi$, $\xi \le 0$, and $u(x,t;\epsilon) = v(\xi,t;\epsilon)$. Then the PDE becomes

$$v_{\xi\xi} - 2v_\xi = \epsilon v_t + \epsilon^2 v_{tt}.$$

Let $v \sim v_0(\xi,t)$. Then v_0 satisfies

$$v_{0_{\xi\xi}} - 2v_{0_\xi} = 0,$$

with general solution

$$v_0 = A_0(t) + B_0(t)e^{2\xi}.$$

The boundary condition $v_0(0,t) = 0$ implies $B_0(t) = A_0(t)$, so that

$$v_0(\xi,t) = A_0(t)\left[1 - e^{2\xi}\right].$$

We now match the inner and outer solutions to order unity. The inner solution in the outer variable is

$$v_0 = A_0(t)\left[1 - e^{-2(\pi-x)\epsilon}\right] \to A_0(t) \text{ as } \epsilon \to 0,\ 0 < x < \pi \text{ fixed}.$$

The outer solution in the inner variable is

$$u_0 = \sin\left(\frac{\pi}{2} - t + \epsilon\frac{\xi}{2}\right) \to \sin\left(\frac{\pi}{2} - t\right) = \cos t \text{ as } \epsilon \to 0,\ \xi \text{ fixed}.$$

Matching finds $A_0(t) = \cos t$, so that the solution in layer L_1 is

$$v_0(\xi,t) = (1 - e^{2\xi})\cos t.$$

Had we let the outer solution satisfy the boundary condition at $x = \pi$, thereby locating the layer at $x = 0$, we would have found that the layer solution grows exponentially as it exits the layer and hence is unmatchable with the outer region.

The solution is now complete. We do not need a layer across the subcharacteristic $x = 2t$ as the two outer solutions are continuous across it, both taking the value zero there. The derivatives of the outer solutions are discontinuous across $x = 2t$, to be sure, but smoothing them would require a higher-order correction.

2. Consider the elliptic problem

$$\epsilon(u_{xx} + u_{yy}) + u_x + u_y + u = 0, \quad 0 < x < 1,\ 0 < y < 1,$$

with boundary conditions

$$u(x,0) = 0,\ u(x,1) = 1 - x,\ \ u(0,y) = e^{-y},\ u(1,y) = 1 - y.$$

(a) Construct a leading-order solution for $0 < \epsilon \ll 1$, paying due attention to the location of the layers.

(b) Repeat the problem if the second boundary condition above is changed to $u(x,1) = 1$.

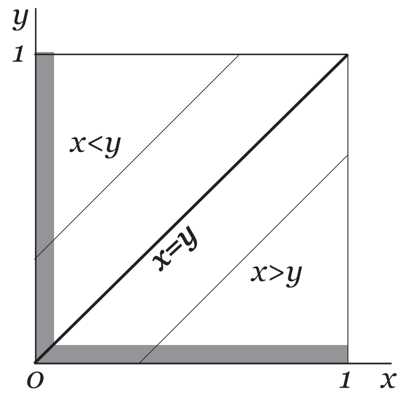

Figure 2: Subcharacteristics and layers for Problem 2(a).

(a) **Outer solution.** We seek the outer solution as $u(x,y;\epsilon) \sim u_0(x,y)$. Then u_0 satisfies the reduced PDE

$$u_{0_x} + u_{0_y} + u_0 = 0.$$

The subcharacteristics are the lines $y = x + \xi$ and hence the general solution is

$$u_0(x,y) = A_0(y-x)e^{-x}.$$

The regions $x > y$ and $x < y$ need to be considered separately. In the region $x > y$ the correct boundary condition to satisfy, subject to verification later, is $u_0(1,y) = 1 - y$. Then

$$e^{-1}A_0(y-1) = 1 - y, \quad \text{or,} \quad A_0(Y) = -eY.$$

Therefore

$$u_0(x,y) = -e(y-x)e^{-x} = (x-y)e^{1-x}, \ x > y. \tag{3}$$

In the region $x < y$ we apply the boundary condition $u_0(x,1) = 1-x$, again subject to revision. Then

$$e^{-x}A_0(1-x) = 1-x, \quad \text{or,} \quad A_0(Y) = Ye^{(1-Y)}.$$

Therefore,

$$u_0(x,y) = (y-x)e^{1-y+x}e^{-x} = (y-x)e^{1-y}, \ x < y. \tag{4}$$

Inner solutions. In the region $x > y$ the outer solution (3) does not satisfy the zero boundary condition on the line $y = 0$, thus requiring an $O(\epsilon)$ thick boundary layer there. In this layer let $y = \epsilon\eta$, $\eta \geq 0$, $u(x,y;\epsilon) = v(x,\eta;\epsilon)$. Then v satisfies the PDE

$$v_{\eta\eta} + v_\eta + \epsilon^2 v_{xx} + \epsilon v_x + \epsilon v = 0.$$

We seek the solution as $v(x,\eta;\epsilon) \sim v_0(x,\eta)$. Then,

$$v_{0_{\eta\eta}} + v_{0_\eta} = 0,$$

with general solution

$$v_0(x,\eta) = B_0(x) + C_0(x)e^{-\eta}.$$

The boundary condition $v_0(x,0) = 0$ requires $C_0 = -B_0$ so that

$$v_0(x,\eta) = B_0(x)[1 - e^{-\eta}].$$

Matching between the inner and outer solutions, carried out as in Problem 1 above, provides to the condition $v_0(x,\infty) = u_0(x,0)$, which leads to

$$B_0(x) = xe^{1-x}.$$

Then the layer solution becomes

$$v_0(x,\eta) = xe^{1-x}[1 - e^{-\eta}]. \tag{5}$$

Had we elected the outer solution to satisfy the boundary condition on $y = 0$, a layer at $x = 1$ would have been unmatchable.

Turning now to the region $x < y$, the outer solution (4) does not satisfy the boundary condition at $x = 0$, thus requiring an $O(\epsilon)$ thick layer there. In this layer let $x = \epsilon\xi$, $\xi \geq 0$, $u(x,y;\epsilon) = w(\xi,y;\epsilon)$. Then w satisfies the PDE

$$w_{\xi\xi} + w_\xi + \epsilon^2 w_{yy} + \epsilon w_y + \epsilon w = 0.$$

We seek the solution as $w(\xi,y;\epsilon) \sim w_0(\xi,y)$. Then,

$$w_{0_{\xi\xi}} + w_{0_\xi} = 0,$$

with general solution

$$w_0(\xi,y) = D_0(y) + E_0(y)e^{-\xi}.$$

This solution satisfies the boundary condition $w_0(0,y) = e^{-y}$ and $w_0(\infty,y) = ye^{1-y}$, the matching condition provided by the outer solution (4). These conditions lead to

$$D_0(y) = ye^{1-y}, \quad E_0(y) = (1 - ey)e^{-y}.$$

Then the layer solution becomes

$$w_0(\xi,y) = ye^{1-y} + (1 - ey)e^{-\xi-y}. \tag{6}$$

Again, as in Problem 1, the outer solutions (3) and (4) are continuous across the subcharacteristic $y = x$ and no layer is required there at order unity.

(b) Proceeding as above the outer solution (4) is now replaced by

$$u_0(x,y) = e^{1-y},\ x < y, \tag{7}$$

while the inner solution (6) changes to

$$w_0(\xi,y) = e^{1-y} + (1 - e)e^{-y-\xi}.$$

The solutions in the region $x > y$ are not changed.

The two outer solutions (7) and (3) are now discontinuous across the line $y = x$, so that a layer of thickness $O(\sqrt{\epsilon})$ will need to be inserted there. In this layer let $y = x + \sqrt{\epsilon}\sigma$, $|\sigma| < \infty$, and $u(x,y;\epsilon) = \phi(x,\sigma;\epsilon)$. Then

$$u_x = \phi_x - \frac{1}{\sqrt{\epsilon}}\phi_\sigma,\ \ u_{xx} = \phi_{xx} - \frac{2}{\sqrt{\epsilon}}\phi_{x\sigma} + \frac{1}{\epsilon}\phi_{\sigma\sigma},\ \ u_y = \frac{1}{\sqrt{\epsilon}}\phi_\sigma,\ \ u_{yy} = \frac{1}{\epsilon}\phi_{\sigma\sigma},$$

and the PDE for u transforms into

$$2\phi_{\sigma\sigma} + \phi_x + \phi - 2\sqrt{\epsilon}\phi_{x\sigma} + \epsilon\phi_{xx} = 0.$$

On setting $\phi(x,\sigma;\epsilon) \sim \phi_0(x,\sigma)$, we obtain the reduced PDE

$$-\phi_{0_x} = 2\phi_{0_{\sigma\sigma}} + \phi_0.$$

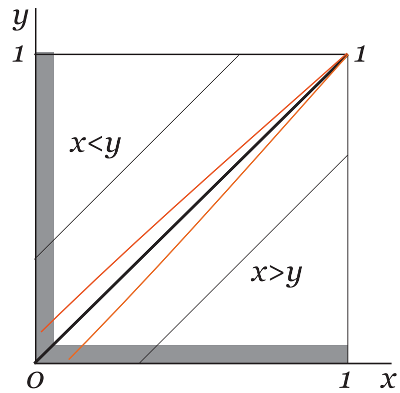

Figure 3: Subcharacteristics and layers for Problem 2(b).

This is a diffusion equation with a linear source, in which σ is the space-like and $-x$ the time-like variable. As indicated above, $|\sigma| < \infty$, and $0 < x < 1$. Matching with the outer solutions provides the boundary conditions

$$\phi(x,\sigma) \to 0 \;\text{ as }\; \sigma \to \infty, \quad \phi(x,\sigma) \to e^{1-x} \;\text{ as }\; \sigma \to -\infty.$$

The initial condition comes from the top right corner, yielding

$$\phi_0(1,\sigma) = \begin{cases} 0 & \sigma > 0 \\ 1 & \sigma < 0. \end{cases}$$

The problem is simplified by setting $x = 1 - t$ and $\phi_0 = e^{1-x}\psi(t,\sigma)$. Then the problem for ϕ_0 transforms into

$$\psi_t = 2\psi_{\sigma\sigma}, \quad 0 < t < 1, \; |\sigma| < \infty,$$

with

$$\psi(t,\sigma) \to 0 \;\text{ as }\; \sigma \to \infty, \quad \psi(t,\sigma) \to 1 \;\text{ as }\; \sigma \to -\infty,$$

and

$$\psi(0,\sigma) = \begin{cases} 0 & \sigma > 0 \\ 1 & \sigma < 0. \end{cases}$$

The solution is

$$\psi(t,\sigma) = \frac{1}{2}\left(1 - \operatorname{erf}\frac{\sigma}{2\sqrt{t}}\right),$$

or,

$$\phi_0(x,\sigma) = \frac{1}{2}e^{1-x}\left(1 - \operatorname{erf}\frac{\sigma}{2\sqrt{1-x}}\right).$$

This completes the solution to order unity.

3. In class we had examined heat transfer on a flat plate in a uniform stream. Now consider heat transfer from a cylinder of radius unity and center at the origin, placed in an otherwise uniform stream. The flow velocity is given by $\boldsymbol{u} = \nabla\phi$, where the potential ϕ is given in polar coordinates as

$$\phi = \left(r + \frac{1}{r}\right)\cos\theta.$$

The temperature T satisfies the PDE

$$\boldsymbol{u}\cdot\nabla T = \epsilon\nabla^2 T, \quad r \geq 1,$$

with boundary conditions

$$T = 1 \text{ on } r = 1, \quad T \to 0 \text{ as } r \to \infty.$$

Find the leading-order solution for $0 < \epsilon \ll 1$. Sketch a graph of the isotherms (lines of constant T). Also, find an expression for $\partial T/\partial r$, the heat flux, from the cylinder surface.

Hint: Look for a similarity solution of the PDE for the inner problem.

We shall use polar coordinates. Symmetry allows us to consider only the upper half of the geometry, so that $\theta \in [0, \pi]$.

If $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$ are the unit vectors in the radial and angular directions, then the velocity $\boldsymbol{u}$ is given by

$$\boldsymbol{u} = \nabla\phi = \frac{\partial \phi}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial \phi}{\partial \theta}\boldsymbol{e}_\theta = \left(1 - \frac{1}{r^2}\right)\cos\theta\, \boldsymbol{e}_r - \left(1 + \frac{1}{r^2}\right)\sin\theta\, \boldsymbol{e}_\theta.$$

Far from the cylinder, $r = \infty$ and the velocity reduces to $\cos\theta\, \boldsymbol{e}_r - \sin\theta\, \boldsymbol{e}_\theta$, which in Cartesian coordinates is simply the unit vector $\boldsymbol{i}$, *i.e.*, in the far field the velocity field is uniform, with unit magnitude and directed along the positive x-direction. On the cylinder, $r = 1$ and the velocity is $-2\sin\theta\, \boldsymbol{e}_\theta$. The radial component is zero and the flow is tangential to the cylinder, *i.e.*, the cylinder is a streamline. At the front and back of the cylinder ($\theta = 0$ and π), the velocity vanishes; these are *stagnation points*.

In polar coordinates the governing PDE is

$$\left(1 - \frac{1}{r^2}\right)\cos\theta\frac{\partial T}{\partial r} - \left(1 + \frac{1}{r^2}\right)\frac{\sin\theta}{r}\frac{\partial T}{\partial \theta} = \epsilon\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}\right).$$

Outer solution. We expand the solution as $T(r, \theta; \epsilon) \sim T_0(r, \theta)$. Then at leading order the PDE reduces to

$$\boldsymbol{u} \cdot \nabla T_0 = 0.$$

The above equation states that the rate of change of T_0 along the direction of flow is zero. Thus

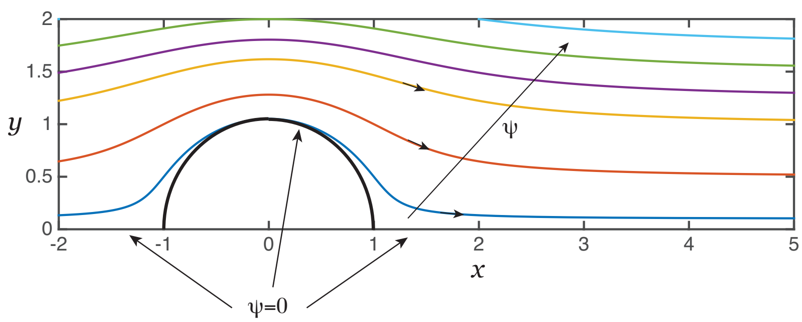

Figure 4: Plot of streamlines (sub-characteristics). The streamline label $\psi = 0$ on the circle $r = 1$ and on the streamlines $\theta = 0$ and $\theta = \pi$.

T_0 is constant along streamlines, which are curves locally tangent to the direction of flow and are also sub-characteristics. The above equation has the expanded form

$$\left(1 - \frac{1}{r^2}\right)\cos\theta\frac{\partial T_0}{\partial r} - \left(1 + \frac{1}{r^2}\right)\frac{\sin\theta}{r}\frac{\partial T_0}{\partial \theta} = 0.$$

Thereforet the sub-characteristics satisfy the ODE

$$\frac{dr}{d\theta} = -\frac{r(r^2-1)\cos\theta}{(r^2+1)\sin\theta},$$

which integrates to yield the streamlines

$$\left(r - \frac{1}{r}\right)\sin\theta = \psi,$$

where the constant ψ is the stream-function label. A sketch of the streamlines is shown in Figure 4. Since the streamlines (except the cylinder itself, $r = 1$, and the streamline $\theta = 0$ originating at the rear stagnation point) start at $x = -\infty$ where $T_0 = 0$, the outer solution is simply $T_0 = 0$.

Boundary layer. The above solution does not satisfy the boundary condition on the cylinder where a boundary layer of thickness $\sqrt{\epsilon}$ must be inserted. (The layer thickness is $\sqrt{\epsilon}$ since the cylinder is a streamline and therefore a sub-characteristic.) With $r = 1+\sqrt{\epsilon}\rho$ and $T(r,\theta;\epsilon) = F(\rho,\theta;\epsilon)$, the PDE transforms into

$$\left(1 - \frac{1}{(1+\sqrt{\epsilon}\rho)^2}\right)\frac{\cos\theta}{\sqrt{\epsilon}}\frac{\partial F}{\partial\rho} - \left(1 + \frac{1}{(1+\sqrt{\epsilon}\rho)^2}\right)\frac{\sin\theta}{1+\sqrt{\epsilon}\rho}\frac{\partial F}{\partial\theta} = \frac{\partial^2 F}{\partial\rho^2} + \frac{\sqrt{\epsilon}}{1+\sqrt{\epsilon}\rho}\frac{\partial F}{\partial\rho} + \frac{\epsilon}{(1+\sqrt{\epsilon}\rho)^2}\frac{\partial^2 F}{\partial\theta^2}.$$

We seek the expansion $F(\rho,\theta;\epsilon) \sim F_0(\rho,\theta)$. Then F_0 satisfies the reduced PDE

$$2\rho\cos\theta\frac{\partial F_0}{\partial\rho} - 2\sin\theta\frac{\partial F_0}{\partial\theta} = \frac{\partial^2 F_0}{\partial\rho^2}, \quad \rho > 0,\ 0 < \theta < 2\pi. \tag{8}$$

The boundary condition is $F_0 = 1$ at $\rho = 0$ and matching with the outer solution provides the condition $F_0 \to 0$ as $\rho \to \infty$.

As the above problem for the PDE has no natural 'length' or 'time' scales, we expect it to have a similarity solution. So we look for a solution of form $F_0(\rho,\theta) = f(\eta)$, where η is the similarity variable which we expect to be of form $\eta = \rho g(\theta)$, where $g(\theta)$ is to be found under the prescription that the PDE (8) transforms into an ODE. With

$$\frac{\partial F_0}{\partial\rho} = f'(\eta)g(\theta), \quad \frac{\partial^2 F_0}{\partial\rho^2} = f''(\eta)\{g(\theta)\}^2, \quad \frac{\partial F_0}{\partial\theta} = f'(\eta)\rho g'(\theta) = \eta f'(\eta)\frac{g'(\theta)}{g(\theta)},$$

the PDE (8) becomes

$$2\rho f'(\eta)[g\cos\theta - g'\sin\theta] = f'' g^2,$$

or, with $\rho = \eta/g$,

$$2\eta f'(\eta)[g\cos\theta - g'\sin\theta] = f'' g^3. \tag{9}$$

This equation reduces to the ODE

$$f'' + 2\eta f = 0, \tag{10}$$

if $g(\theta)$ satisfies

$$g'\sin\theta - g\cos\theta = g^3.$$

Dividing both sides by $\sin^2\theta$,

$$\frac{g'}{\sin\theta} - \frac{g\cos\theta}{\sin^2\theta} = \frac{g^3}{\sin^2\theta},$$

or,

$$\left(\frac{g}{\sin\theta}\right)' = \frac{g^3}{\sin^2\theta}.$$

Let $g(\theta)/\sin\theta = h(\theta)$. Then $h(\theta)$ satisfies

$$h' = h^3\sin\theta,$$

whose solution is

$$h(\theta) = \frac{1}{\sqrt{2(K + \cos\theta)}},$$

where the constant K remains to be chosen. We choose $K = 1$, subject to justification later. Then

$$\eta = \rho\, g(\theta) = \rho \sin\theta\, h(\theta) = \frac{\rho\, \sin\theta}{\sqrt{2(1 + \cos\theta)}}.$$

We note that for fixed $\theta \in (0, \pi)$, η is proportional to ρ, with $\eta = 0$ when $\rho = 0$ and $\eta = \infty$ when $\rho = \infty$. Therefore the boundary and matching conditions on F_0 translate into the boundary conditions $f(0) = 1$, $f(\infty) = 0$ on the function f. Subject to these boundary conditions the solution to (10) is

$$f_0(\eta) = 1 - \operatorname{erf} \eta.$$

The solution is constant on the curves $\eta =$ constant, which are therefore the isotherms. These are plotted in Figure 5. The layer solution is also the leading-order composite solution.

A word about the choice $K = 1$ in the definition of h above. Without this choice,

$$\eta = \frac{\rho\, \sin\theta}{\sqrt{2(K + \cos\theta)}} = 0 \quad \text{for } \theta = \pi.$$

This implies that the streamline $\theta = \pi$ originating at $x = -\infty$ and terminating in the stagnation point is also an isotherm within the boundary layer, with $T \sim f_0(0) = 1$. This does not match with the solution $T = 0$ in the outer region. For $K = 1$, $\theta = \pi$ corresponds to $\eta = \rho$, which allows η to vary from zero at the circle $(\rho = 0)$ to infinity as $\rho \to \infty$, allowing f_0 to decay from the value unity at the circle to zero towards the outer region.

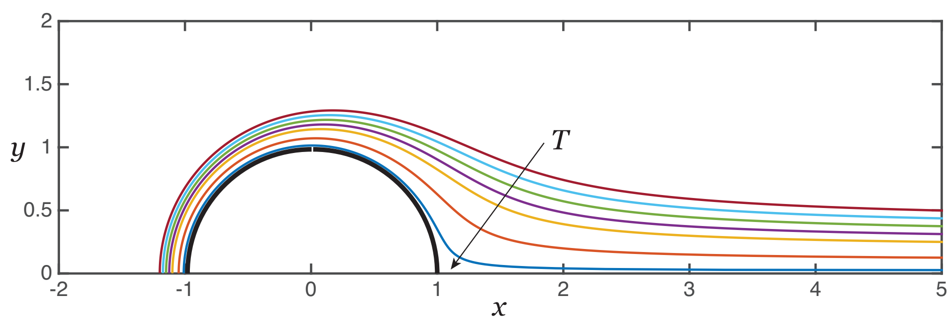

Figure 5: Isotherms (contours of temperature). The arrow indicates the direction of increasing T. The boundary of the unit circle, and the line $\theta = 0$, $r \geq 1$, are contours of $T = 1$.

It is worth noting that the boundary layer starts at the front stagnation point, wraps around the circle, and extends down the line $\theta = 0$.