PERTURBATION METHODS
SPRING 2016

LESSON 3: Singular-perturbation problems for differential equations: Multiple Scales

The method of multiple scales is examined via some ODE and PDE examples. This lesson supplements Chapter 3 of the Text.

1 Introduction

Simultaneous evolution on more than one time scales arises in many contexts. Stock markets fluctuate daily but exhibit yearly trends. Weather changes daily but change in climate is seasonal. Signals are transmitted by slow modulations of the amplitude or the frequency of rapid oscillations. Figure 1 plots the function

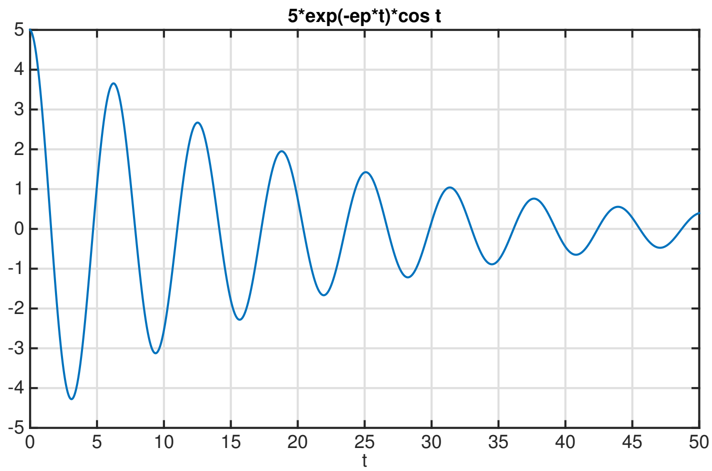

Figure 1: Plot of $5e^{-\epsilon t}\cos t$.

$f(t) = 5e^{-\epsilon t}\cos t$ on the interval $t \in [0, 50]$ for $\epsilon = 0.05$. The function is a rapid oscillation with a fixed period 2π whose amplitude decays gradually; during a single period the relative decay in the amplitude is $e^{-2\pi\epsilon}$.

Figure 2 is a plot of the function $g(t) = 5\sin\epsilon t\cos t$ on the interval $t \in [0, 200]$ for $\epsilon = 0.05$. The function represents an oscillation with period 2π whose amplitude oscillates with period $2\pi/\epsilon$.

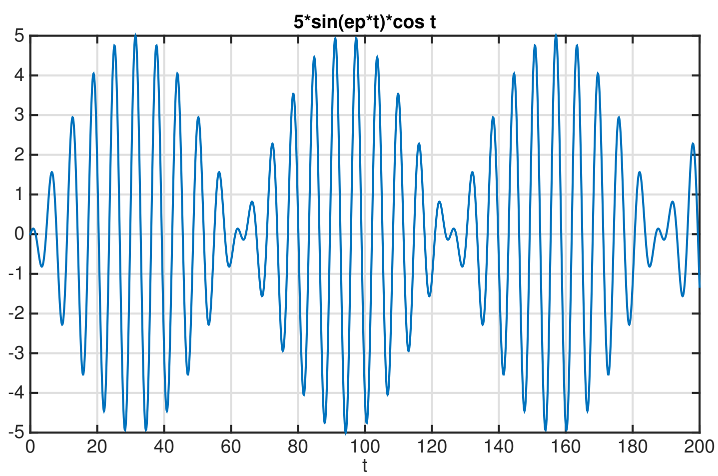

Figure 2: Plot of $5\sin\epsilon t\cos t$.

The aim of the method of multiple scales is to explicitly recognize the existence of more than one time scales and to develop a perturbation procedure that captures the multi-scale features. We begin with an example of slowly-modulated oscillations where an exact solution is available.

Example 1. Consider the damped oscillator

$$\frac{d^2x}{dt^2} + 2\epsilon\frac{dx}{dt} + x = 0, \quad t \geq 0, \tag{1.1}$$

with initial conditions

$$x = 0, \quad \frac{dx}{dt} = 1 \text{ at } t = 0. \tag{1.2}$$

The exact solution

$$x_e(t;\epsilon) = \frac{e^{-\epsilon t}}{\sqrt{1-\epsilon^2}} \sin(\sqrt{1-\epsilon^2}\, t) \tag{1.3}$$

is an oscillation with a fixed period and an exponentially decaying amplitude. The oscillation is controlled by the *fast* time $T = \sqrt{1-\epsilon^2}\, t$ and the decay by the *slow* time $\tau = \epsilon t$. In terms of these times the solution can be written as

$$x_e(t;\epsilon) = \frac{e^{-\tau}}{\sqrt{1-\epsilon^2}} \sin T. \tag{1.4}$$

A naive asymptotic expansion obtained by seeking the solution of the ODE (1.1) in the form $x(t;\epsilon) \sim x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t)$ leads to the result (you may wish to fill in the details)

$$x \sim \sin t - \epsilon t \sin t + \epsilon^2 \left(\frac{t^2+1}{2} \sin t - \frac{1}{2} t \cos t \right). \tag{1.5}$$

The quality of the approximation is evident from Figure 3, where the above expansion is plotted against the exact solution for $\epsilon = 0.05$ on the interval $t \in [0, 50]$. For times upto about $t = 15$ the expansion is quite accurate, but worsens as t is increased, with the amplitude of the approximation growing steadily.

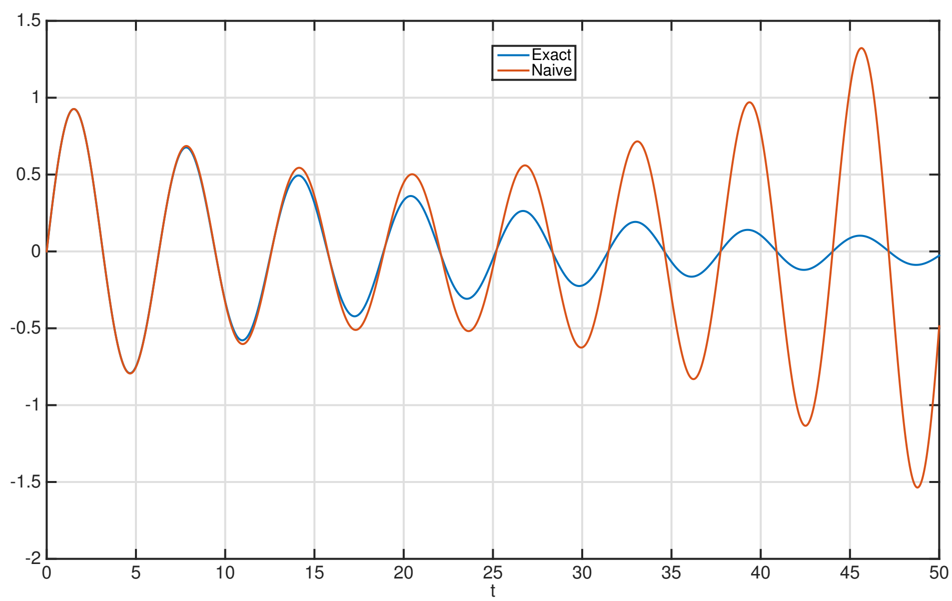

Figure 3: Comparison of the exact solution and the naive asymptotic expansion for Example 1, for $\epsilon = 0.05$.

We note that the first term in the expansion (1.5) is periodic but the subsequent terms have nonperiodic components that grow without bound with t, the so-called *secular* terms. The same expansion is obtained when the exact solution $x_e(t;\epsilon)$ above is expanded to $O(\epsilon^2)$ for $\epsilon \to 0$, t fixed, and now the origin of the secular terms is revealed. They arise because, in obtaining (1.5) from (1.3), Taylor series are used to approximate the exponential and the sine,

$$\begin{aligned} e^{-\epsilon t} &= 1 - \epsilon t + \frac{1}{2}\epsilon^2 t^2 + \cdots, \\ \sin(\sqrt{1-\epsilon^2}\,t) &= \sin t - \frac{1}{2}\epsilon^2 t \cos t + \cdots. \end{aligned}$$

Partial sums of this kind are poor approximations indeed when t is large, no matter how many terns are retained; such sums do not capture the decaying property of the exponential or the periodic nature of the sine. A better approximation of the exact solution is

$$x \sim e^{-\epsilon t} \sin\left\{\left(1 - \frac{1}{2}\epsilon^2\right) t\right\}. \tag{1.6}$$

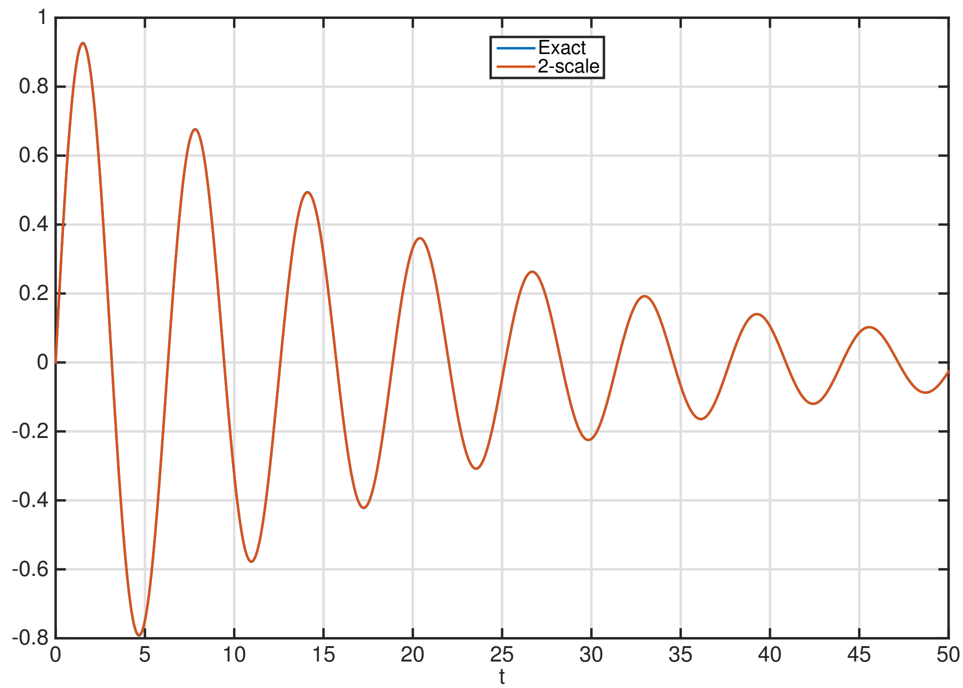

Figure 4: Comparison of the exact solution and the two-time asymptotic expansion for Example 1, for $\epsilon = 0.05$.

This is derived from the exact solution by employing the result

$$\sin(\sqrt{1-\epsilon^2}\,t) \sim \sin\left\{\left(1 - \frac{1}{2}\epsilon^2\right) t\right\} \tag{1.7}$$

to approximate the oscillatory component of (1.3) while keeping the decay factor $e^{-\epsilon t}$ intact. The comparison of the expansion (1.6) with the exact solution on the time interval $t \in [0, 50]$ is remarkably good, as shown in Figure 4.

The above construction is, of course, based on a knowledge of the exact solution. However, it is highly suggestive as to how one might proceed in attempting to obtain an expansion such as (1.7) from the governing ODE. The procedure recognizes explicitly that the solution varies simultaneously on two different time scales

T and τ; hence the term *two-timing* or *two-variable* approach. The original problem is reformulated in terms of these two scales by treating the solution as a function of two variables, *i.e.*, $x(t;\epsilon) = X(T,\tau;\epsilon)$. This changes the governing ODE into a PDE. Of course, the two times are not independent as they are both related to t. How this apparent inconsistency is resolved is quite straightforward, as we shall see.

Returning to the initial-value problem (1.1) - (1.2) we introduce a fast time T and a slow time τ, defined by

$$T = \omega t = (1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots)t, \quad \tau = \epsilon t, \tag{1.8}$$

where the constants ω_i characterize the shift in the period and are to be determined during the course of the solution. The time derivatives are related as follows,

$$\begin{aligned} \frac{d}{dt} &= (1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots)\frac{\partial}{\partial T} + \epsilon\frac{\partial}{\partial \tau}, \\ \frac{d^2}{dt^2} &= (1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots)^2\frac{\partial^2}{\partial T^2} + 2\epsilon(1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots)\frac{\partial^2}{\partial T\partial \tau} + \epsilon^2\frac{\partial^2}{\partial \tau^2}. \end{aligned}$$

Then, with $x(t;\epsilon) = X(T,\tau;\epsilon)$ the ODE (1.1) transforms into the PDE

$$\begin{aligned} (1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots)^2\frac{\partial^2 X}{\partial T^2} &+ 2\epsilon(1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots)\frac{\partial^2 X}{\partial T\partial \tau} + \epsilon^2\frac{\partial^2 X}{\partial \tau^2} \\ &+ 2\epsilon\left((1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots)\frac{\partial X}{\partial T} + \epsilon\frac{\partial X}{\partial \tau}\right) + X = 0. \end{aligned} \tag{1.9}$$

We assume the expansion

$$X(T,\tau;\epsilon) \sim f_0(T,\tau) + \epsilon f_1(T,\tau) + \epsilon^2 f_2(T,\tau) + \cdots. \tag{1.10}$$

When the above expansion is substituted into the PDE (1.9) we obtain a sequence of equations for the f_i; the first three are given below.

$$\frac{\partial^2 f_0}{\partial T^2} + f_0 = 0, \tag{1.11}$$

$$\frac{\partial^2 f_1}{\partial T^2} + f_1 = -2\omega_1\frac{\partial^2 f_0}{\partial T^2} - 2\frac{\partial^2 f_0}{\partial T\partial \tau} - 2\frac{\partial f_0}{\partial T}, \tag{1.12}$$

$$\begin{aligned} \frac{\partial^2 f_2}{\partial T^2} + f_2 &= -(2\omega_2 + \omega_1^2)\frac{\partial^2 f_0}{\partial T^2} - 2\omega_1\left(\frac{\partial^2 f_1}{\partial T^2} + \frac{\partial^2 f_0}{\partial T\partial \tau}\right) - \frac{\partial^2 f_0}{\partial \tau^2} - 2\frac{\partial^2 f_1}{\partial T\partial \tau} \\ &\quad -2\left(\frac{\partial f_0}{\partial \tau} + \frac{\partial f_1}{\partial T} + \omega_1\frac{\partial f_0}{\partial T}\right). \end{aligned} \tag{1.13}$$

Similarly we find that the initial conditions transform as follows. (Note that $T = \tau = 0$ at $t = 0$.)

$$f_0(0,0) = 0, \qquad \frac{\partial f_0}{\partial T}(0,0) = 1, \tag{1.14}$$

$$f_1(0,0) = 0, \qquad \frac{\partial f_1}{\partial T}(0,0) + \omega_1\frac{\partial f_0}{\partial T}(0,0) + \frac{\partial f_0}{\partial \tau}(0,0) = 0, \tag{1.15}$$

$$f_2(0,0) = 0, \qquad \frac{\partial f_2}{\partial T}(0,0) + \omega_1\frac{\partial f_1}{\partial T}(0,0) + \omega_2\frac{\partial f_0}{\partial T}(0,0) + \frac{\partial f_1}{\partial \tau}(0,0) = 0. \tag{1.16}$$

Equation (1.11) is just the linearized pendulum equation. Its general solution is

$$f_0 = A_0(\tau)\sin[T + \phi_0(\tau)], \tag{1.17}$$

where the 'constants' of integration A_0 and ϕ_0 are functions of the slow variable τ. Then

$$f_{0_T} = A_0(\tau)\cos[T + \phi_0(\tau)].$$

The initial conditions (1.14) lead to

$$A_0(0) = 1, \quad \phi_0(0) = 0, \tag{1.18}$$

and this is the extent of information that can be gleaned from the f_0 - problem. We shall determine A_0 and ϕ_0 by requiring that secular components do not appear at the $O(\epsilon)$ stage. On inserting (1.17) into (1.12) the equation for f_1 becomes

$$\frac{\partial^2 f_1}{\partial T^2} + f_1 = 2\omega_1 A_0 \sin\theta_0 - 2\{A_0' \cos\theta_0 - A_0\phi_0' \sin\theta_0\} - 2A_0\cos\theta_0, \tag{1.19}$$

where

$$\theta_0 = T + \phi_0(\tau). \tag{1.20}$$

Here one must recall how the general solution of the above problem is computed (see an elementary text on differential equations if need be) and note that in order for the solution for f_1 not to involve any resonant or secular terms (terms growing with T), the coefficients of $\sin\theta_0$ and $\cos\theta_0$ on the RHS above must vanish, *i.e.*,

$$\begin{aligned} A_0' + A_0 &= 0, \\ (\omega_1 + \phi_0')A_0 &= 0. \end{aligned}$$

Under the initial conditions (1.18) the above system has the unique solution

$$A_0 = -e^{-\tau}, \quad \phi_0 = -\omega_1\tau.$$

Then (1.17) reduces to

$$f_0 = e^{-\tau}\sin(T - \omega_1\tau).$$

We have thus obtained the leading-order term in the expansion (1.10). We note that with T and τ as defined in (1.8),

$$T - \omega_1\tau = (1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots)t - \epsilon\omega_1 t = t(1 + \epsilon\omega^2 + \cdots).$$

Thus ω_1 plays no role in the definition of $T,$ so we may as well set it to zero here, and in all that is to follow. This is not surprising; the term ϵt is already accounted for in $\tau = \epsilon t$ and including in T will just be redundant. With this choice, $\phi_0 = -\omega_1\tau = 0,$ and therefore, $\theta_0 = T,$ so the leading-order solution is simply

$$f_0 = e^{-\tau}\sin T. \tag{1.21}$$

With the RHS of (1.19) now zero, the general solution for f_1 is

$$f_1 = A_1(\tau)\sin[T + \phi_1(\tau)] = A_1(\tau)\sin\theta_1, \quad \theta_1 = T + \phi_1(\tau). \tag{1.22}$$

Upon inserting (1.21) and (1.22) into (1.13), and simplifying, we have

$$\frac{\partial^2 f_2}{\partial T^2} + f_2 = [2\omega_2 A_0 - 2A_0' - A_0'']\sin T - 2[A_1'\cos\theta_1 - A_1\phi_1'\sin\theta_1] - 2A_1\cos\theta_1.$$

Upon replacing θ_1 by $T + \phi_1$ and using standard trigonometric identities on the RHS, the above equation can be rewritten as

$$\begin{aligned} \frac{\partial^2 f_2}{\partial T^2} + f_2 &= [2\omega_2 A_0 - 2A_0' - A_0'']\sin T - 2(A_1' + A_1)[\cos T\cos\phi_1 - \sin T\sin\phi_1] \\ &+ 2A_1\phi_1'[\sin T\cos\phi_1 + \cos T\sin\phi_1]. \end{aligned}$$

The requirement that no secular terms appear in f_2 leads to the equations

$$A_0'' + 2A_0' - 2\omega_2 A_0 - 2(A_1' + A_1)\sin\phi_1 - 2A_1\phi_1'\cos\phi_1 = 0, \tag{1.23}$$

$$(A_1' + A_1)\cos\phi_1 - A_1\phi_1'\sin\phi_1 = 0. \tag{1.24}$$

The initial conditions for these equations follow from (1.15) and (1.22) and are

$$A_1(0)\sin\phi_1(0) = 0, \quad A_1(0)\cos\phi_1(0) = 0,$$

so that

$$A_1(0) = 0, \quad \text{with} \quad \phi_1(0) \text{ arbitrary.} \tag{1.25}$$

Equation (1.24) can be written as $(A_1 \cos\phi_1)' + A_1 \cos\phi_1 = 0$, which, in view of the initial condition above, integrates to

$$A_1(\tau)\cos\phi_1(\tau) = 0. \tag{1.26}$$

Similarly, equation (1.23) can be expressed as

$$\begin{aligned}(A_1 \sin\phi_1)' + A_1 \sin\phi_1 &= \frac{1}{2}A_0'' + A_0' - \omega_2 A_0 \\ &= -\left(\frac{1}{2} + \omega_2\right)e^{-\tau}.\end{aligned}$$

The solution for $A_1 \sin\phi_1$ will contain the secular (in τ) term $\tau e^{-\tau}$ unless one sets $\omega_2 = -1/2$. (The secular term, if not eliminated, will render the expansion (1.10) nonuniform at large τ.) Then the solution subject to the initial conditions above is

$$A_1(\tau)\sin\phi_1(\tau) = 0. \tag{1.27}$$

From (1.26) and (1.27) we conclude that $A_1(\tau) = 0$. That ϕ_1 remains arbitrary is then unimportant. Thus we have $f_1 = 0$

With f_0 and f_1 determined, the solution to order ϵ is

$$X \sim f_0 + \epsilon f_1 + O(\epsilon^2) = e^{-\tau}\sin T + O(\epsilon)^2 = e^{-\epsilon t}\sin\left\{\left(1 - \frac{1}{2}\epsilon^2\right)t\right\} + O(\epsilon)^2.$$

This agrees with the expansion (1.6) derived from the exact solution.

Example 2. In this example we explore the effect of weak damping on the solution of an initial-value problem for the wave equation. Intuitively we expect damping to have a dissipative effect on the waves, but as we had seen for the ODEs, the ordinary perturbation procedure fails to capture this feature and a multi-scale approach lis needed. The PDE is

$$u_{tt} + \epsilon u_t - u_{xx} = 0, \quad -\infty < x < \infty,\ t > 0, \tag{1.28}$$

with initial conditions

$$u(x,0) = F(x),\ u_t(x,0) = 0. \tag{1.29}$$

It is convenient to recast the problem in terms of the characteristic variables $x - t = r,\ x + t = s$. Then,

$$\begin{aligned} u_t &= -u_r + u_s, \\ u_{tt} &= u_{rr} - 2u_{rs} + u_{ss}, \\ u_x &= u_r + u_s, \\ u_{xx} &= u_{rr} + 2u_{rs} + u_{ss} \end{aligned}$$

and the PDE reads

$$u_{rs} = \frac{1}{4}\epsilon(-u_r + u_s).$$

As $t = 0$ corresponds to $r = s = x$, the initial conditions become

$$u = F(r), \quad -u_r + u_s = 0, \quad \text{at} \quad r = s.$$

First we show that a straightforward perturbation expansion results is the appearance of secular terms. With $u \sim u_0 + u_1$, the leading-order problem is

$$u_{0rs} = 0,$$

so that

$$u_0 = f_0(r) + g_0(s).$$

The initial conditions $u_0 = F(r)$ and $u_{0r} = u_{0s}$ at $r = s$ yield

$$f_0(r) + g_0(r) = F(r), \quad f_0'(r) = g_0'(r).$$

so that

$$f_0(r) = \frac{F(r) + A}{2}, \quad g_0(r) = \frac{F_0(r) - A}{2},$$

where A is a constant. The leading-order solution becomes

$$u_0 = \frac{1}{2}[F_0(r) + F_0(s)].$$

At the next order the PDE is

$$\begin{aligned} u_{1rs} &= \frac{1}{4}[-u_{0r} + u_{0s}] \\ &= \frac{1}{8}[-F_0'(r) + F_0'(s)]. \end{aligned}$$

The general solution is

$$u_1 = \frac{1}{8}[-sF_0(r) + rF_0(s)] + f_1(r) + g_1(s).$$

The functions $f_1(r)$ and $g_1(s)$ are determined by initial conditions but the other terms clearly exhibit linear growth as $r = x - t$ or $s = x + t$ becomes large, thereby disordering the asymptotic expansion.

A multiscale approach would seem to be in order. We note that secularity is caused by both r and s becoming large, and that this can happen when either x or t or both are large. To retain sufficient generality at this stage we introduce two slow scales, $\tau = \epsilon t$ and $\xi = \epsilon x$, and assume that $u = u(r, s, \xi, \tau)$. Then the derivatives transform as follows.

$$\begin{aligned} u_t &= -u_r + u_s + \epsilon u_\tau, \\ u_x &= u_r + u_s + \epsilon u_\xi, \\ u_{tt} &= u_{rr} - 2u_{rs} + u_{ss} - 2\epsilon u_{r\tau} + 2\epsilon u_{s\tau} + \epsilon^2 u_{\tau\tau}, \\ u_{xx} &= u_{rr} + 2u_{rs} + u_{ss} + 2\epsilon u_{r\xi} + 2\epsilon u_{s\xi} + \epsilon^2 u_{\xi\xi}. \end{aligned}$$

The PDE transforms into

$$-4u_{rs} - 2\epsilon u_{r\tau} + 2\epsilon u_{s\tau} + \epsilon^2 u_{\tau\tau} - [2\epsilon u_{r\xi} + 2\epsilon u_{s\xi} + \epsilon^2 u_{\xi\xi}] - \epsilon u_r + \epsilon u_s + \epsilon^2 u_\tau = 0.$$

Again we seek an expansion of form $u \sim u_0(r, s, \xi, \tau) + \epsilon u_1(r, s, \xi, \tau)$.

The leading-order PDE is the same as before, *i.e.*,

$$u_{0rs} = 0,$$

so that

$$u_0 = f_0(r, \xi, \tau) + g_0(s, \xi, \tau).$$

Since $t = 0$ corresponds to $r = s$ and $\tau = 0$, the initial conditions (1.29) can be written as

$$u_0(r, r, \xi, 0) = F(r), \quad -u_{0r}(r, r, \xi, 0) + u_{0s}(r, r, \xi, 0) = 0.$$

As the initial conditions do not contain ξ, it makes sense to assume that the solution u_0 is also independent of ξ. Thus we can write

$$u_0(r, s, \tau) = f_0(r, \tau) + g_0(s, \tau). \tag{1.30}$$

Then the initial conditions lead to

$$\begin{aligned} f_0(r,0) + g_0(r,0) &= F(r), \\ -f_{01}(r,0) + g_{01}(r,0) &= 0. \end{aligned}$$

In the second equation above the suffix 1 indicates partial differentiation with respect to the first argument, which is r for f_0 and s for g_0. This equation can be integrated to yield

$$f_0(r,0) - g_0(r,0) = A,$$

where A is a constant. Upon solving this equation with the first of the pair above we get the initial conditions

$$f_0(r,0) = \frac{F(r)+A}{2}, \quad g_0(r,0) = \frac{F(r)-A}{2}. \tag{1.31}$$

At the next order the PDE is

$$\begin{aligned} u_{1rs} &= \frac{1}{2}[-u_{0r\tau} + u_{0s\tau}] + \frac{1}{4}[-u_{0r} + u_{0s}] \\ &= \frac{1}{2}[-f_{0r\tau} + g_{0s\tau}] + \frac{1}{4}[-f_{0r} + g_{0s}]. \end{aligned}$$

In writing the RHS of this PDE we have made use of the fact that u_0 is independent of ξ. The solution is

$$\begin{aligned} u_1 &= \frac{1}{2}[-sf_{0\tau} + rg_{0\tau}] + \frac{1}{4}[-sf_0 + rg_0] + f_1(r,\tau) + g_1(s,\tau) \\ &= \frac{1}{4}[r\{2g_{0\tau} + g_0\} - s\{2f_{0\tau} + f_0\}] + f_1(r,\tau) + g_1(s,\tau). \end{aligned}$$

To avoid secularity, caused by the linear growth in r and s, we set

$$\begin{aligned} 2f_{0\tau} + f_0 &= 0, \\ 2g_{0\tau} + g_0 &= 0. \end{aligned}$$

The solutions, subject to the initial conditions (1.31), are

$$\begin{aligned} f_0(r,\tau) &= \frac{F(r)+A}{2} e^{-\tau/2}, \\ g_0(s,\tau) &= \frac{F(s)-A}{2} e^{-\tau/2}. \end{aligned}$$

Then the leading-order solution (1.30) becomes

$$u_0 = \frac{1}{2}[F(x-t) + F(x+t)]e^{-\epsilon t/2}.$$

The solution is essentially what one would intuitively expect. The multiscale approach has uncovered the long-time dissipative effect of the weak damping on the left- and right-traveling waves.

Dispersive waves. We now consider some perturbation problems associated with nonlinear dispersive waves in the limit of weak nonlinearity. Consider the model equation

$$u_{tt} - u_{xx} + V'(u) = 0, \tag{1.32}$$

where $V(u)$ is a nonlinear function representing a potential. This model arises in several physical contexts, for a variety of choices for the potential, some of which are listed below.

- $V = 0$ leads to the standard wave equation, $u_{tt} - u_{xx} = 0$.

- $V = u^2/2$ leads to the equation
$$u_{tt} - u_{xx} + u = 0, \tag{1.33}$$
known as the linear Klein-Gordon equation.
- $V = u^2/2 + \epsilon u^4/4$ leads to the equation
$$u_{tt} - u_{xx} + u + \epsilon u^3 = 0, \tag{1.34}$$
known as the nonlinear Klein-Gordon equation.
- $V = -\cos u$ leads to the equation $u_{tt} - u_{xx} + \sin u = 0$, known as the sine-Gordon equation.

The linear model. We begin with a brief discussion of the linear Klein-Gordon equation (1.33), which may be considered to be a reduced form of the nonlinear equation (1.34) in the limit $\epsilon = 0$. As this equation has constant coefficients, we seek exponential solutions of the product type,
$$u(x,t) = Ae^{i(kx-\omega t)}, \tag{1.35}$$
where A is a complex constant. In real form this solution is a linear combination of $\sin(kx-\omega t)$ and $\cos(kx-\omega t)$, and is therefore a harmonic wave. It can also be represented as $C\sin(kx-\omega t+\phi)$, where the real constant C is the amplitude and ϕ the phase shift. (The quantity $\theta = kx - \omega t$ is referred to as the phase.) The wavelength is $2\pi/k$ and the period $2\pi/\omega$. We refer to k as the *wave number* and ω as the *frequency.* The solution (1.35) satisfies the PDE (1.33) if $\omega^2 = k^2 + 1$, or
$$\omega = \pm W(k) = \pm\sqrt{1+k^2}. \tag{1.36}$$
The above equation, connecting the wave number and the frequency, is known as the *dispersion relation.*

As the wave travels, suppose that an observer wishes to follow a crest, a trough, or some other constant value of the phase $\theta = kx - \omega t$. Then the observer's speed $c = dx/dt$ must satisfy $d\theta/dt = 0$, *i.e.*, $kdx/dt - \omega = 0$, yielding $c = dx/dt = \omega/k$. This speed is known as the *phase speed* of the wave. We note that for the ordinary wave equation $u_{tt} - u_{xx} = 0$, (1.35) is a solution if $\omega^2 = k^2$, *i.e.*, $\omega = \pm k$. Then the phase speed is a constant, *i.e.*, $c = \omega/\kappa = \pm 1$. For equation (1.33) on the other hand, the phase speed is a function of k, *i.e.*,
$$c = \omega/k = \pm\frac{\sqrt{1+k^2}}{k}.$$
Waves of this type, where the phase speed depends upon the wave number, are called *dispersive waves.* Unlike the solutions of the ordinary wave equation which propagate at a fixed speed and without change in shape, dispersive waves distort as they propagate because different wave numbers travel at different wave speeds.

A powerful property of linear equations is superposition, allowing us to construct the general solution as a superposition of elementary solutions. Thus the general solution of (1.33) can be expressed as a superposition of left- and right-traveling waves, in the form
$$u(x,t) = \int_{-\infty}^{\infty} A(k)e^{kx-W(k)t}\,dk + \int_{-\infty}^{\infty} B(k)e^{kx+W(k)t}\,dk.$$

The nonlinear model. We now turn to the model (1.32) and show that like its linear counterpart, this nonlinear PDE also has periodic solutions but they are not harmonic. We set
$$u(x,t) = F(\theta), \quad \theta = kx - \omega t,$$
and find that F satisfies
$$(\omega^2 - k^2)F''(\theta) + V'(F) = 0.$$

If θ is viewed as time and F as displacement, the above equation can be recognized as the equation of a spring-mass system for which the restoring force is $V'(F)/(\omega^2 - k^2)$ and the mass is unity. It has the first integral

$$\frac{1}{2}(\omega^2 - k^2)(F')^2 + V(F) = A,$$

where A can be viewed as the total energy. Consider the linear case for which $V(F) = F^2/2$. Then the equation becomes

$$\frac{1}{2}(\omega^2 - k^2)(F')^2 + \frac{1}{2}F^2 - A = 0. \tag{1.37}$$

We take $\omega^2 - k^2 > 0$. Then the graph of the above equation in the FF'- phase plane, shown in Figure 5, is a closed orbit, indicative of periodic solutions. The solution F oscillates between $\pm\sqrt{2A}$, the two zeros

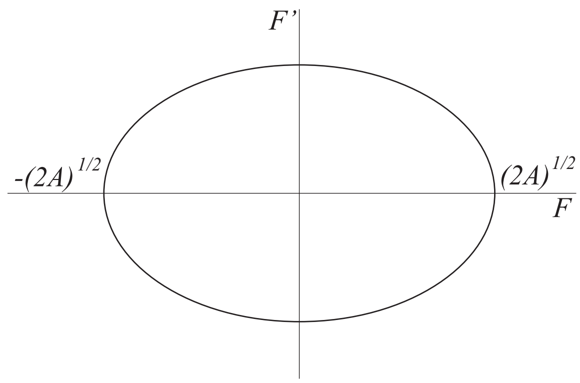

Figure 5: Plot of equation (1.37).

of $A - V(F) = A - F^2/2$. It integrates to yield

$$\theta = \left[\frac{1}{2}(\omega^2 - k^2)\right]^{1/2} \int \frac{dF}{\sqrt{A - F^2/2}}. \tag{1.38}$$

If the integral is carried out over a complete oscillation, changing the sign of the square root in the integrand as needed over the two halves of the cycle, then one obtains

$$2\pi = \left[\frac{1}{2}(\omega^2 - k^2)\right]^{1/2} \oint \frac{dF}{\sqrt{A - F^2/2}}, \tag{1.39}$$

where the period of the oscillation has been normalized to 2π. Then the periodic solution is

$$F = a \sin\theta, \quad a = \sqrt{2A}.$$

When substituted into (1.39) the amplitude parameter A cancels out and the equation reduces to

$$\omega^2 - k^2 = 1,$$

which is the already-found dispersion relation of the linear equation.

For the case of the nonlinear Klein-Gordon equation, corresponding to $V(F) = F^2/2 + \epsilon F^2/4$, the integral

$$\theta = \left[\frac{1}{2}(\omega^2 - k^2)\right]^{1/2} \int \frac{dF}{\sqrt{A - V(F)}}$$

can be evaluated in terms of elliptic functions. Periodic solutions are obtained again as $A - V(F)$ has two simple zeros F_+ and F_- given by

$$F_\pm^2 = \frac{-2 \pm \sqrt{4 + 16\epsilon A}}{2\epsilon}.$$

Again we have

$$2\pi = \left[\frac{1}{2}(\omega^2 - k^2)\right]^{1/2} \oint \frac{dF}{\sqrt{A - V(F)}},$$

but now the parameter A does not drop out and the above relation yields a dispersion relation of the form

$$\omega = W(k, A, \epsilon).$$

Additionally, even though the solution is periodic, superposition is no longer available to employ this special solution to construct more general solutions. We now consider two examples that explore how a simple solution of the linear Klein-Gordon equation is distorted by the weak nonlinearity as it propagates.

Example 3. We seek a solution of the nonlinear equation (1.34) that is a small perturbation of the harmonic wave $\cos(kx - \omega t)$, where

$$\omega^2 = 1 + k^2$$

in accordance with the reduced linear problem. Our experience indicates that if the expansion is to remain valid over long times, then it must be of the multi-scale variety. Therefore we introduce the phase coordinate $\theta = kx - \omega t$, and the two slow coordinates $\xi = \epsilon x$ and $\tau = \epsilon t$. We note that

$$\begin{aligned}
u_t &= -\omega u_\theta + \epsilon u_\tau, \\
u_{tt} &= \omega^2 u_{\theta\theta} - 2\epsilon\omega u_{\theta\tau} + \epsilon^2 u_{\tau\tau}, \\
u_x &= k u_\theta + \epsilon u_\xi, \\
u_{xx} &= k^2 u_{\theta\theta} + 2\epsilon k u_{\theta\xi} + \epsilon^2 u_{\xi\xi}.
\end{aligned}$$

With $u = u(\theta, \xi, \tau)$, the PDE (1.34) transforms into

$$u_{\theta\theta} + u - 2\epsilon\omega u_{\theta\tau} - 2\epsilon k u_{\theta\xi} + \epsilon u^3 - \epsilon^2 u_{\tau\tau} - \epsilon^2 u_{\xi\xi} = 0.$$

where we have used $\omega^2 - k^2 = 1$. Now we expand the solution as $u \sim u_0(\theta, \xi, \tau) + \epsilon u_1(\theta, \xi, \tau)$.

At order unity the PDE reduces to

$$u_{0\theta\theta} + u_0 = 0.$$

We write the solution in the form

$$u_0 = A(\xi, \tau)\cos(\theta + \phi(\xi, \tau)). \tag{1.40}$$

At $O(\epsilon)$ the PDE is

$$u_{1\theta\theta} + u_1 = 2\omega u_{0\theta\tau} + 2k u_{0\theta\xi} - u_0^3.$$

Upon substituting for u_0, we get

$$\begin{aligned}
u_{1\theta\theta} + u_1 = \; & -2\omega[A_\tau \sin(\theta + \phi(\xi, \tau)) + A\phi_\tau \cos(\theta + \phi(\xi, \tau))] \\
& -2k[A_\xi \sin(\theta + \phi(\xi, \tau)) + A\phi_\xi \cos(\theta + \phi(\xi, \tau))] \\
& -A^3\left(\frac{1}{4}\cos[3(\theta + \phi(\xi, \tau))] + \frac{3}{4}\cos[(\theta + \phi(\xi, \tau))]\right).
\end{aligned}$$

Here we have used the identity $\cos 3x = -3\cos x + 4\cos^3 x$. Removal of secular growth requires the coefficients of $\cos(\theta + \phi(\xi, \tau))$ and $\sin(\theta + \phi(\xi, \tau))$ to vanish. Thus,

$$\begin{aligned}
2\omega A_\tau + 2k A_\xi &= 0, \\
-2\omega A\phi_\tau - 2k A\phi_\xi - \frac{3}{4}A^3 &= 0.
\end{aligned}$$

These equations simplify to

$$\begin{aligned}\omega A_\tau + kA_\xi &= 0,\\ \omega\phi_\tau + k\phi_\xi + \frac{3}{8}A^2 &= 0.\end{aligned}$$

These first-order PDEs can be solved by introducing the characteristics

$$\frac{d\xi}{d\tau} = \frac{k}{\omega},$$

which integrate to $\omega\xi - k\tau = s$, say. Then the above pair of PDEs reduces to the following ODE pair along the characteristics,

$$\begin{aligned}\frac{dA}{d\tau} &= 0,\\ \frac{d\phi}{d\tau} &= -\frac{3A^2}{8\omega}.\end{aligned}$$

The solution is

$$\begin{aligned}A &= A(s),\\ \phi &= -\frac{3A^2}{8\omega}\tau + \phi_0(s).\end{aligned}$$

With A and ϕ determined, the leading-order solution to the nonlinear problem is

$$u \sim A(\omega\xi - k\tau)\cos\left(kx - \omega t - \frac{3A^2}{8\omega}\tau + \phi_0(\omega\xi - k\tau)\right). \tag{1.41}$$

For later use, note that

$$u_\theta \sim -A(\omega\xi - k\tau)\sin\left(kx - \omega t - \frac{3A^2}{8\omega}\tau + \phi_0(\omega\xi - k\tau)\right).$$

It remains to determine the functions A and ϕ, for which the initial conditions are yet to be specified. Suppose that the initial conditions are consistent with a right-going harmonic wave $u(x,t) = a\cos(kx - \omega t)$, *i.e.*,

$$u(x,0) = a\cos kx \quad \text{and} \quad u_t(x,0) = \omega a \sin kx.$$

In the multiple-scale setup employed here, the above conditions imply that $u_0(\theta,\xi,\tau)$ satisfies

$$u_0(\theta,\xi,0) = a\cos\theta \quad \text{and} \quad -\omega u_{0\theta}(\theta,\xi,0) = \omega a\sin\theta.$$

On applying these conditions to the solution (1.41) one finds that

$$A(\omega\xi) = a, \quad \text{and} \quad \phi_0(\omega\xi) = 0.$$

Then (1.41) becomes

$$u \sim a\cos\left(kx - \omega t - \frac{3a^2\epsilon t}{8\omega}\right).$$

An observer following a crest or a trough will now have to travel at a speed dx/dt given by

$$k\frac{dx}{dt} - \omega - \frac{3a^2\epsilon}{8\omega} = 0.$$

Thus the phase speed is

$$c = \frac{dx}{dt} = \frac{\omega}{k}\left(1 + \frac{3a^2\epsilon}{8\omega^2}\right).$$

Thus the effect of the nonlinearity is to enhance the phase speed by the factor $\left(1 + \frac{3a^2\epsilon}{8\omega^2}\right)$ which depends upon the amplitude a of the wave.

Waves in reactive-diffusive systems. A key aspect of many phenomena in developmental biology is the appearance of a traveling wave, either of chemical concentration or of mechanical deformation. Many wavelike events are found in a developing embryo subsequent to fertilization. Waves of epidemics spreading across interacting populations is another example. In the laboratory, the target patterns exhibited by the Belousov-Zhabotinsky reaction provide a dramatic example of chemical waves. Waves in a reactive-diffusive system are mathematically quite different from those in hyperbolic systems studied earlier. We begin with some preliminary discussion.

<u>Diffusion alone.</u> Let us see if the diffusion equation

$$u_t = Du_{xx}$$

can support a traveling wave by itself. With $u(x,t) = U(z)$, where $z = x - ct$, the diffusion equation transforms into the ODE $DU_{zz} + cU_z = 0$, with general solution $U(z) = A + Be^{-cz/D}$. Boundedness for all z requires $B = 0$, leaving only the constant solution $U = A$, which is not a wave.

<u>Reaction alone.</u> Now consider a purely reactive equation,

$$u_t = u(1-u).$$

This is the well-known logistic equation. The plot of du/dt against u, displayed in Figure 6, shows that

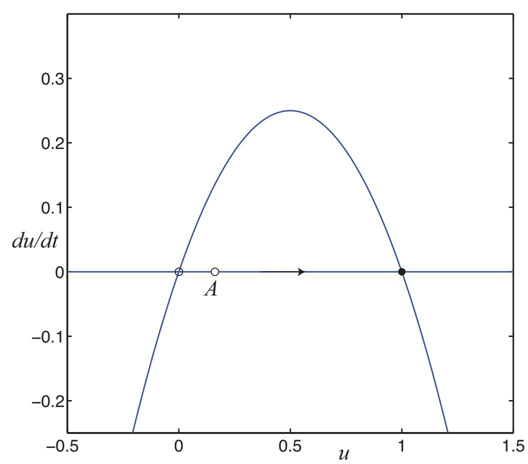

Figure 6: Plot of du/dt against u for the logistic equation.

there are two equilibria, $u = 0$ which is unstable, and $u = 1$ which is stable. Any initial condition such as $u = A$ in the figure will approach $u = 1$ as $t \to \infty$ (and $u = 0$ as $t \to -\infty$). One may therefore expect a wave whose leading edge advances into the unstable state $u = 0$ and the trailing edge leaves behind the stable state $u = 1$. Consider the spatially nonuniform initial condition

$$u(0;x) = \frac{1}{1+e^{\lambda x}}, \quad \lambda > 0.$$

Note that $u(0;\infty) = 0$ and $u(0;-\infty) = 1$. The logistic equation can be integrated as

$$\int_{1/2}^{u} \frac{dr}{r(1-r)} = t + \theta(x), \tag{1.42}$$

where $\theta(x)$ is a function of integration. The integral on the left can be evaluated explicitly, and the result is

$$u = \frac{1}{1+e^{\theta(x)-t}}. \tag{1.43}$$

The initial condition finds $\theta(x) = \lambda x$, and leads to the traveling-wave solution

$$u(t;x) = \frac{1}{1 + e^{\lambda x - t}}. \tag{1.44}$$

The situation is unstable as the speed of the wave, $c = 1/\lambda$, depends crucially upon the initial condition, and the wave will be distorted if the initial condition is perturbed.

Diffusion with reaction. We now consider the combined effects of reaction and diffusion by examining the equation

$$u_t = u_{xx} + u(1-u). \tag{1.45}$$

This equation was suggested by Fisher (1937) as the model for the spread of a favored gene. We look for traveling wave solutions by setting

$$u(x,t) = U(z), \quad z = x - ct.$$

As (1.45) is invariant if x is replaced by $-x$, the wave speed c can have either sign. For definiteness we shall take c to be positive. The function U satisfies

$$U'' + cU' + U(1-U) = 0.$$

We let $V = dU/dz$ and write the above equation as the system

$$\begin{aligned} \frac{dU}{dz} &= V, \\ \frac{dV}{dz} &= -cV - U(1-U), \end{aligned}$$

or equivalently as the single ODE representing trajectories in the phase plane,

$$\frac{dV}{dU} = \frac{-cV - U(1-U)}{V}.$$

The singular points of this equation are $U = V = 0$ and $U = 1,\ V = 0$. A straightforward linear stability analysis shows that for the singular point $U = V = 0$, the eigenvalues are

$$\lambda = \frac{-c \pm \sqrt{c^2 - 4}}{2}.$$

Thus the singular point is a stable node for $c \geq 2$ and a stable focus for $0 < c < 2$. Similarly, the eigenvalues at the singular point $U = 1,\ V = 0$ are

$$\lambda = \frac{-c \pm \sqrt{c^2 + 4}}{2},$$

indicating that the singular point is a saddle.

Figure 7 displays the phase portrait. We see that there is a unique trajectory marked W that connects the two singular points and lies entirely in the physically relevant region $U \geq 0$. The profile of the numerically-computed solution is displayed in Figure 8.

Perturbation-theory treatment.

To apply perturbation theory to the problem we consider small diffusion. Thus,

$$u_t = \epsilon u_{xx} + u(1-u). \tag{1.46}$$

A perturbation solution $u(x,t)$ subject to the initial condition

$$u(x,0) = \frac{1}{1 + e^{\lambda x}} \tag{1.47}$$

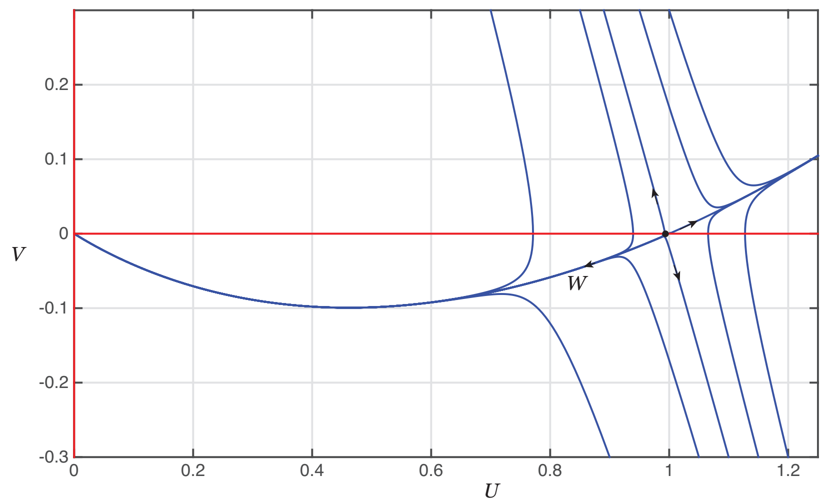

Figure 7: Phase portrait of equation (1.45) for $c = 2.5$.

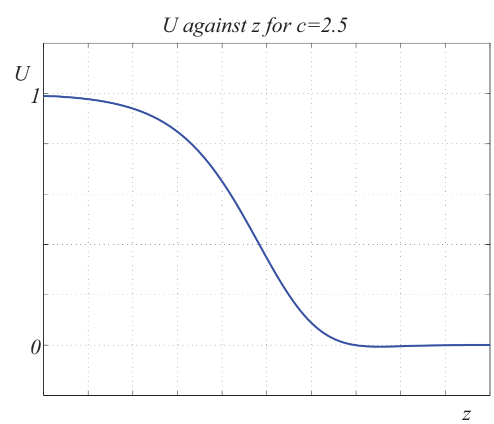

Figure 8: Wave profile for $c = 2.5$.

is sought. First, consider a regular perturbation expansion,

$$u(x,t) \sim u_0(x,t) + \epsilon u_1(x,t).$$

At order unity,

$$u_{0t} = u_0(1-u_0), \;\; u_0(x,0) = \frac{1}{1+e^{\lambda x}}. \tag{1.48}$$

This problem was just considered above and has the solution (1.44), *i.e.*,

$$u_0(x,t) = \frac{1}{1+e^{\lambda x - t}}. \tag{1.49}$$

At order ϵ,

$$u_{1t} - u_1(1-2u_0) = u_{0xx}, \;\; u_1(x,0) = 0. \tag{1.50}$$

We note that

$$\begin{aligned}
u_{0x} &= -\lambda \, \frac{e^{\lambda x - t}}{(e^{\lambda x - t}+1)^2}, \\
u_{0xx} &= \lambda^2 \, \frac{e^{\lambda x - t}\,(e^{\lambda x - t}-1)}{(e^{\lambda x - t}+1)^3}, \\
1-2u_0 &= \frac{e^{\lambda x - t}-1}{e^{\lambda x - t}+1}.
\end{aligned}$$

The problem for u_1 now becomes

$$u_{1t} - \frac{e^{\lambda x - t}-1}{e^{\lambda x - t}+1} u_1 = \lambda^2 \, \frac{e^{\lambda x - t}\,(e^{\lambda x - t}-1)}{(e^{\lambda x - t}+1)^3}, \;\; u_1(x,0) = 0.$$

For convenience we define a new variable s by $\lambda x - t = s,$ which allows the above ODE to be written as

$$u_{1s} + \frac{e^s - 1}{e^s + 1} u_1 = -\lambda^2 \frac{e^s(e^s-1)}{(e^s+1)^3}. \tag{1.51}$$

The integrating factor for this first-order ODE is

$$\begin{aligned}
\phi(s) &= \exp\left(\int \frac{e^s-1}{e^s+1}\, ds\right) \\
&= \exp\left(2\int \frac{(e^{s/2}-e^{-s/2})/2}{e^{s/2}+e^{-s/2}}\, ds\right) \\
&= \exp\left(2\ln(e^{s/2}+e^{-s/2})\right) \\
&= (e^{s/2}+e^{-s/2})^2 = \frac{(e^s+1)^2}{e^s}.
\end{aligned} \tag{1.52}$$

Multiplication by the integrating factor leads to

$$\frac{\partial}{\partial s}\left(\frac{(e^s+1)^2}{e^s} u_1\right) = -\lambda^2 \frac{e^s-1}{e^s+1},$$

and then, an integration yields

$$\begin{aligned}
\frac{(e^s+1)^2}{e^s} u_1 &= -\lambda^2 \int \frac{e^s-1}{e^s+1}\, ds \\
&= -\lambda^2 \left(\ln \frac{(e^s+1)^2}{e^s} + G(x)\right).
\end{aligned}$$

Here $G(x)$ is a function of integration. On reverting to the t variable via $s = \lambda x - t$, one obtains

$$u_1 = -\lambda^2 \frac{e^{\lambda x - t}}{(e^{\lambda x - t}+1)^2}\left(\ln\frac{(e^{\lambda x - t}+1)^2}{e^{\lambda x - t}} + G(x)\right).$$

The initial condition $u_1(x,0) = 0$ finds

$$G(x) = -\ln\frac{(e^{\lambda x}+1)^2}{e^{\lambda x}}.$$

Then

$$u_1(x,t) = -\lambda^2 \frac{e^{\lambda x - t}}{(e^{\lambda x - t}+1)^2}\left(\ln\frac{(e^{\lambda x - t}+1)^2}{e^{\lambda x - t}} - \ln\frac{(e^{\lambda x}+1)^2}{e^{\lambda x}}\right).$$

The solution for u to $O(\epsilon)$ has now been found,

$$u(x,t) \sim \frac{1}{1+e^{\lambda x - t}} - \epsilon\lambda^2 \frac{e^{\lambda x - t}}{(e^{\lambda x - t}+1)^2}\left(\ln\frac{(e^{\lambda x - t}+1)^2}{e^{\lambda x - t}} - \ln\frac{(e^{\lambda x}+1)^2}{e^{\lambda x}}\right).$$

As $t \to \infty$, u has the asymptotic form

$$u \sim (1 - e^{\lambda x - t}) - \epsilon\lambda^2 e^{\lambda x - t}(t - \lambda x),$$

and the expansion remains well-ordered in ϵ. We note that in this limit, $u \to 1$. As $t \to -\infty$,

$$u \sim e^{t - \lambda x} - \epsilon\lambda^2 e^{t - \lambda x}(\lambda x - t).$$

In this limit, $u \to 0$. We see that now the expansion breaks down when $t = O(1/\epsilon)$. This suggests a multi-scale approach.

We define a new time scale $\tau = \epsilon t$. With $u = u(x,t,\tau)$, the time derivative transforms as

$$u_t = u_t + \epsilon u_\tau,$$

and the PDE becomes

$$u_t + \epsilon u_\tau = \epsilon u_{xx} + u(1-u),$$

with initial condition

$$u(x,0,0) = \frac{1}{e^{\lambda x}+1}.$$

We seek an expansion in the form

$$u \sim u_0 + \epsilon u_1.$$

At order unity,

$$u_{0t} = u_0(1-u_0), \quad u_0(x,0,0) = \frac{1}{1+e^{\lambda x}}.$$

This problem is the same as (1.48), but u now depends upon the additional variable τ. Proceeding as before we arrive at the solution

$$u_0(x,t,\tau) = \frac{1}{e^{\theta(x,\tau)-t}+1}, \quad \text{with} \quad \theta(x,0) = \lambda x. \tag{1.53}$$

At order ϵ,

$$u_{1t} - u_1(1-2u_0) = -u_{0\tau} + u_{0xx}.$$

The sole purpose of considering this $O(\epsilon)$ problem is to identify the criterion for the absence of secularity. It is convenient to replace the variable t by

$$\sigma = \theta(x,\tau) - t. \tag{1.54}$$

Then the above PDE for u_1 becomes

$$u_{1\sigma} + u_1(1-2u_0) = u_{0\tau} - u_{0xx}. \tag{1.55}$$

From equation (1.53), $u_0 = 1/(e^\sigma + 1)$. Then,

$$\begin{aligned}
u_{0x} &= -\theta_x \frac{e^\sigma}{(e^\sigma+1)^2}, \\
u_{0xx} &= -\theta_{xx}\,\frac{e^\sigma}{(e^\sigma+1)^2} + \theta_x^2\,\frac{e^\sigma(e^\sigma-1)}{(e^\sigma+1)^3}, \\
u_{0\tau} &= -\theta_\tau\,\frac{e^\sigma}{(e^\sigma+1)^2}, \\
1-2u_0 &= \frac{e^\sigma-1}{e^\sigma+1}.
\end{aligned}$$

In view of the above relations, (1.55) becomes

$$u_{1\sigma} + \frac{e^\sigma-1}{e^\sigma+1}u_1 = -\theta_x^2\,\frac{e^\sigma(e^\sigma-1)}{(e^\sigma+1)^3} - (\theta_\tau - \theta_{xx})\frac{e^\sigma}{(e^\sigma+1)^2}. \tag{1.56}$$

This equation is a slightly more general version of (1.51), with s in that equation replaced by σ here. The integrating factor from (1.52), now written as

$$\phi(\sigma) = \frac{(e^\sigma+1)^2}{e^\sigma},$$

applies again, to transform (1.56) into

$$\frac{\partial}{\partial\sigma}\left(\frac{(e^\sigma+1)^2}{e^\sigma}u_1\right) = -\theta_x^2\,\frac{e^\sigma-1}{e^\sigma+1} - (\theta_\tau - \theta_{xx}). \tag{1.57}$$

An integration yields

$$\frac{(e^\sigma+1)^2}{e^\sigma}u_1 = -\theta_x^2\,\ln\frac{(e^\sigma+1)^2}{e^\sigma} - (\theta_\tau-\theta_{xx})\sigma + G(x,\tau),$$

where $G(x,\tau)$ is a function of integration. A rearrangement, combined with the use of the leading-order solution (1.53), leads to the following 2-term expansion.

$$u \sim u_0 + \epsilon u_1 = \frac{1}{e^\sigma+1} + \epsilon\frac{e^\sigma}{(e^\sigma+1)^2}\left[-\theta_x^2\,\ln\frac{(e^\sigma+1)^2}{e^\sigma} - (\theta_\tau-\theta_{xx})\sigma + G(x,\tau)\right].$$

To uncover secularity we now examine this solution as $t \to \pm\infty$. For $t \to \infty$, which corresponds to $\sigma = \theta(x,\tau) - t \to -\infty$, the asymptotic form of the above expansion is

$$u \sim 1 - e^\sigma + \epsilon e^\sigma[\sigma\theta_x^2 - (\theta_\tau - \theta_{xx})\sigma],$$

indicating no breakdown in the expansion. For $t \to -\infty$, or $\sigma \to \infty$, the asymptotic form

$$u \sim e^{-\sigma} + \epsilon e^{-\sigma}[-\theta_x^2\sigma - (\theta_\tau - \theta_{xx})\sigma]$$

signals a disordering of the expansion when $\epsilon\sigma$, and therefore ϵt, is of order unity. Suppression of secularity requires

$$\theta_\tau + \theta_x^2 - \theta_{xx} = 0.$$

The relevant initial condition, supplied by (1.53), is $\theta(x,0) = \lambda x$. The substitution $\theta = -\ln\phi$ transforms the above PDE into the diffusion equation

$$\phi_\tau = \phi_{xx}, \quad \phi(x,0) = e^{-\lambda x}.$$

The solution to this initial-value problem can be written as a convolution integral involving the unit source kernel, *i.e.*,

$$\phi(x,\tau) = \frac{1}{\sqrt{4\pi\tau}}\int_{-\infty}^{\infty} e^{-\lambda y}\, e^{-(x-y)^2/4\tau}dy.$$

A rearrangement leads to

$$\phi(x,\tau) = \frac{1}{\sqrt{4\pi\tau}} \int_{-\infty}^{\infty} \exp\left(-\frac{(y-x-2\lambda\tau)^2}{4\tau} - \lambda x + \lambda^2\tau\right)\, dy.$$

The substitutton $y - x + 2\lambda\tau = \sqrt{4\tau}\, p$ simplifies the RHS so that

$$\phi(x,\tau) = \frac{1}{\sqrt{\pi}} e^{-\lambda^2\tau-\lambda x} \int_{-\infty}^{\infty} e^{-p^2}\, dp = e^{\lambda^2\tau-\lambda x}.$$

Then $\theta(x,\tau) = -\ln\phi = -\lambda^2\tau + \lambda x,$ and finally, from (1.53),

$$u \sim \frac{1}{1+e^{\lambda x - t - \epsilon\lambda^2 t}}. \tag{1.58}$$

Noting that the regular perturbation expansion had led to equation (1.49), *i.e.*,

$$u \sim \frac{1}{1+e^{\lambda x - t}},$$

the corrected solution (1.58) indicates an $O(\epsilon)$ change in phase caused by diffusion. The corrected phase speed is

$$c = \frac{dx}{dt} = \frac{1}{\lambda} + \epsilon\lambda.$$

Thus diffusion has caused the wave to be slightly accelerated.