Perturbation Methods
Test 2, Spring '16

Due: Tuesday May 3, by 5:00 PM.

NOTES.

- Do all three problems.
- Read the statement of each problem carefully prior to attempting a solution. Answers with appropriate supporting work should be written out legibly. In all the problems, asymptotic solution is sought in the limit $\epsilon \to 0+$.
- You may consult your class notes and books, but consultation with another individual is not allowed. Please write and sign the following statement on the first page of your answer book:

  I have abided by the ground rules of this test.

PROBLEMS

1. An oscillation is described by the pair of equations

$$\begin{aligned}\frac{du}{dt} + v &= -\epsilon u,\\ \frac{dv}{dt} - u &= -\epsilon v^3, \quad t \geq 0.\end{aligned}$$

The initial conditions are $u = 1,\ v = 0$ at $t = 0$. Introduce a fast time scale $T = (1 + \epsilon^2\omega_2 + \cdots)t$ and a slow scale $\tau = \epsilon t$. Use the method of multiple scales and find, completely, the leading terms of uniformly valid asymptotic expansions. From the equations that define the third term in the expansions, show that the solution is uniformly valid as $\tau \to \infty$ only if $\omega_2 = 1/8$.

This problem can be treated either as a system of two first-order equations, or as a single second-order equation. Choosing the latter alternative, elimination of v yields the ODE

$$\frac{d^2u}{dt^2} + u + \epsilon\frac{du}{dt} + \epsilon\left(\frac{du}{dt} + \epsilon u\right)^3 = 0,$$

with initial conditions $u = 1,\ du/dt = -\epsilon,$ at $t = 0$. With $T = (1 + \epsilon^2\omega_2 + \cdots)t,\ \ \tau = \epsilon t,$ and $u(t;\epsilon) = x(t, \tau; \epsilon),$ we have

$$\begin{aligned}\frac{du}{dt} &= (1 + \epsilon^2\omega_2 + \cdots)\frac{\partial x}{\partial T} + \epsilon\frac{\partial x}{\partial \tau},\\ \frac{d^2u}{dt^2} &= (1 + \epsilon^2\omega_2 + \cdots)^2\frac{\partial^2 x}{\partial T^2} + 2\epsilon(1 + \epsilon^2\omega_2 + \cdots)\frac{\partial^2 x}{\partial T\partial \tau} + \epsilon^2\frac{\partial^2 x}{\partial \tau^2}.\end{aligned}$$

Then the ODE for u transforms into

$$(1 + \epsilon^2\omega_2 + \cdots)^2\frac{\partial^2 x}{\partial T^2} + 2\epsilon(1 + \epsilon^2\omega_2 + \cdots)\frac{\partial^2 x}{\partial T\partial \tau} + \epsilon^2\frac{\partial^2 x}{\partial \tau^2} + x + \epsilon\left\{(1 + \epsilon^2\omega_2 + \cdots)\frac{\partial x}{\partial T} + \epsilon\frac{\partial x}{\partial \tau}\right\}$$
$$+\epsilon\left\{(1 + \epsilon^2\omega_2 + \cdots)\frac{\partial x}{\partial T} + \epsilon\frac{\partial x}{\partial \tau} + \epsilon x\right\}^3 = 0.$$

We now seek the expansion

$$x \sim x_0(T,\tau) + \epsilon x_1(T,\tau) + \epsilon^2 x_2(T,\tau).$$

At leading order the reduced problem is

$$x_{0_{TT}} + x_0 = 0, \quad x_0 = 1,\ x_{0_T} = 0 \ \text{ at } \ T = \tau = 0.$$

The solution, conveniently written in complex form, is

$$x_0 = A_0(\tau)e^{iT} + \bar{A}_0(\tau)e^{-iT},$$

and therefore,

$$x_{0_T} = i\left\{A_0(\tau)e^{iT} - \bar{A}_0(\tau)e^{-iT}\right\}.$$

Overbar denotes complex conjugate. The initial conditions require

$$A_0(0) = \bar{A}_0 = \frac{1}{2}. \tag{1}$$

At $O(\epsilon)$ the solution satisfies

$$\begin{aligned}
x_{1_{TT}} + x_1 &= -2x_{0_{T\tau}} - x_{0_T} - \{x_{0_T}\}^3 \\
&= -2i\left\{A_0'(\tau)e^{iT} - \bar{A}_0'(\tau)e^{-iT}\right\} - i\left\{A_0(\tau)e^{iT} - \bar{A}_0(\tau)e^{-iT}\right\} \\
&\quad + i\left\{A_0(\tau)e^{iT} - \bar{A}_0(\tau)e^{-iT}\right\}^3 \\
&= -i\{2A_0' + A_0 + 3A_0^2\bar{A}_0\}e^{iT} + iA_0^3e^{3iT} + cc,
\end{aligned} \tag{2}$$

where cc stands for *complex conjugate* of the terms explicitly displayed above on the RHS. The first term on the RHS has the same frequency as the homogeneous solution and will lead to secularity. To prevent such an event we let

$$2A_0' + A_0 + 3A_0^2\bar{A}_0 = 0.$$

On setting $A_0(\tau) = R_0(\tau)e^{i\phi_0(\tau)}$, where R_0 and ϕ_0 are real, the above equation transforms into

$$\left[2R_0' + 2iR_0\phi_0' + R_0 + 3R_0^3\right]e^{i\phi_0} = 0.$$

We are led to two equations,

$$\phi_0' = 0, \quad \text{and} \quad 2R_0' + R_0 + 3R_0^3 = 0. \tag{3}$$

The initial conditions (1) translate into $R_0(0) = 1/2$ and $\phi_0(0) = 0$, leading to the solutions

$$\phi_0(\tau) = 0, \quad R_0(\tau) = \sqrt{\frac{e^{-\tau}}{7 - 3e^{-\tau}}}. \tag{4}$$

Since $A_0 = R_0$, equation (2) now reduces to

$$x_{1_{TT}} + x_1 = iR_0^3(e^{3iT} - e^{-3it}). \tag{5}$$

The general solution is

$$x_1 = A_1(\tau)e^{iT} + \bar{A}_1(\tau)e^{-iT} - \frac{i}{8}R_0^3(e^{3iT} - e^{-3iT}). \tag{6}$$

The initial conditions will provide $A_1(0)$ and $\bar{A}_1(0)$. The following derivatives of this solution will be useful in the sequel.

$$\begin{aligned}
x_{1_T} &= i\left(A_1e^{iT} - \bar{A}_1e^{-iT}\right) + \frac{3}{8}R_0^3(e^{3iT} + e^{-3iT}), \\
x_{1_\tau} &= A_1'e^{iT} + \bar{A}_1'e^{-iT} - \frac{i}{8}3R_0^2R_0'(e^{3iT} - e^{-3iT}), \\
x_{1_{T\tau}} &= i\left(A_1'e^{iT} - \bar{A}_1'e^{-iT}\right) + \frac{9}{8}R_0^2R_0'(e^{3iT} + e^{-3iT}).
\end{aligned}$$

At $O(\epsilon^2)$ the reduced equation is

$$x_{2_{TT}} + x_2 = 2\omega_2 x_0 - x_{0_{\tau\tau}} - x_{0_\tau} - 3x_0 x_{0_T}^2 - 3x_{0_\tau} x_{0_T}^2 - 2x_{1_{T\tau}} - x_{1_T} - 3x_{0_T}^2 x_{1_T}.$$

Upon substituting the expressions found above for x_0 and x_1 the above equation becomes

$$\begin{aligned} x_{2_{TT}} + x_2 &= (2\omega_2 R_0 - R_0'' - R_0')(e^{iT} + e^{-iT}) + 3R_0^2(R_0 + R_0')(e^{iT} + e^{-iT})(e^{2iT} + e^{-2iT} - 2) \\ &\quad -2i\left(A_1' e^{iT} - \bar{A}_1' e^{-iT}\right) - \frac{9}{4} R_0^2 R_0' (e^{3iT} + e^{-3iT}) \\ &\quad -i\left(A_1 e^{iT} - \bar{A}_1 e^{-iT}\right) - \frac{3}{8} R_0^3 (e^{3iT} + e^{-3iT}) \\ &\quad +3R_0^2(e^{2iT} + e^{-2iT} - 2)\left(i\left(A_1 e^{iT} - \bar{A}_1 e^{-iT}\right) + \frac{3}{8} R_0^3 (e^{3iT} + e^{-3iT})\right). \end{aligned}$$

To eliminate secular terms we set the coefficient of e^{iT} on the RHS of the above equation to zero, and get

$$2\omega_2 R_0 - R_0'' - R_0' - 3R_0^2(R_0 + R_0') - 2iA_1' - iA_1 + 3R_0^2\left(-2iA_1 - i\bar{A}_1 + \frac{3}{8} R_0^3\right) = 0,$$

or, upon rearranging,

$$\begin{aligned} i\left\{2A_1' + (1 + 6R_0^2)A_1 + 3R_0^2 \bar{A}_1\right\} &= 2\omega_2 R_0 - R_0'' - R_0' - 3R_0^2(R_0 + R_0') + \frac{9}{8} R_0^5 \\ &= \left(2\omega_2 + \frac{1}{4}\right) R_0 - 3R_0^3 - \frac{9}{8} R_0^5. \end{aligned} \tag{7}$$

Here we have used the second equation from (3) to replace the derivatives of R_0 by terms involving R_0. Now, let

$$A_1 = S_1 - iR_1,$$

where $S_1(\tau)$ and $R_1(\tau)$ are real. Then the complex equation (7) yields two real equations,

$$2S_1' + (1 + 9R_0^2)S_1 = 0, \tag{8}$$

$$2R_1' + (1 + 3R_0^2)R_1 = \left(2\omega_2 + \frac{1}{4}\right) R_0 - 3R_0^3 - \frac{9}{8} R_0^5. \tag{9}$$

Upon substituting for R_0 from (4), the first of the above equations becomes

$$2S_1' + S_1\left(1 + \frac{9e^{-\tau}}{7 - 3e^{-\tau}}\right) = 0,$$

which integrates to yield

$$S_1(\tau) = k_1 \frac{e^{-\tau/2}}{(7 - 3e^{-\tau})^{3/2}}.$$

Here the constant k is determined by initial conditions on A_1. We note that S_1 decays exponentially as $\tau \to \infty$.

Since

$$R_0' = -\frac{1}{2} R_0 (1 + 3R_0^2),$$

the ODE (9) for R_1 can be rewritten as

$$R_1' - \frac{R_0'}{R_0} R_1 = \left(\omega_2 + \frac{1}{8}\right) R_0 - \frac{3}{2} R_0^3 - \frac{9}{16} R_0^5.$$

Dividing both sides by R_0 we get

$$\frac{d}{d\tau}\left(\frac{R_1}{R_0}\right) = \left(\omega_2 + \frac{1}{8}\right) - \frac{3}{2} R_0^2 - \frac{9}{16} R_0^4.$$

Upon integration the first term on the RHS will generate a linear term in τ while the other forcing terms, exhibiiting exponential decay as $\tau \to \infty$, will produce integrals that are bounded. To avoid growth in τ we must choose

$$\omega_2 = -\frac{1}{8}.$$

2. We are all familiar with a child's swing, and the technique for increasing the arc (*i.e.*, the amplitude) of the swing. The process of swinging the legs causes the center of gravity of the body to be raised and lowered periodically. This can be modeled by treating the swing as a pendulum which changes its length by a small amount in a periodic manner. We further simplify the problem by assuming that the amplitude is not too large. Then the relevant ODE is

$$\frac{d^2u}{dt^2} + \left(\frac{2\epsilon\omega\cos\omega t}{1+\epsilon\sin\omega t}\right)\frac{du}{dt} + u = 0.$$

By using fast scale $T = t$ and slow scale $\tau = \epsilon t$, look for a multi-scale expansion $u \sim u_0 + \epsilon u_1$. Select a value of ω such that the leading-order solution is periodic in T but has an amplitude that grows on the time scale τ. (A child instinctively knows what value of ω to use for a thrilling ride on the swing.)

Let us prescribe the initial condition $u = 1$, $du/dt = 0$ at $t = 0$.

With $u = u(t, \tau; \epsilon)$, the governing equation becomes

$$u_{tt} + 2\epsilon u_{t\tau} + \epsilon^2 u_{\tau\tau} + u + 2\epsilon\omega\cos\omega t(1 - \epsilon\sin\omega t + \cdots)(u_t + \epsilon u_\tau) = 0.$$

Let $u \sim u_0 + \epsilon u_1$. Then u_0 satisfies

$$u_{0_{tt}} + u_0 = 0,$$

with solution

$$u_0 = A_0(\tau)e^{it} + \bar{A}_0(\tau)e^{-it}, \quad u_{0_t} = i(A_0 e^{it} - \bar{A}_0 e^{-it}),$$

where $A_0(\tau)$ and $\bar{A}_0(\tau)$ are complex conjugates. The initial conditions select $A_0(0) = \bar{A}_0(0) = 1/2$.

At the next order,

$$u_{1_{tt}} + u_1 + 2u_{0_{t\tau}} + 2\omega u_{0_t}\cos\omega t = 0.$$

The above equation can be written as

$$u_{1_{tt}} + u_1 = -2i(A_0' e^{it} - \bar{A}_0' e^{-it}) - \omega i(A_0 e^{it} - \bar{A}_0 e^{-it})(e^{i\omega t} + e^{-i\omega t})$$

The choice $\omega = 2$ will generate secular terms in u_1. Avoidance of these, for $\omega = 2$, leads to the constraint

$$A_0' - \bar{A}_0 = 0,$$

obtained by setting the coefficient of e^{it} to zero. The solution, subject to $A_0(0) = 1/2$, is

$$A_0(\tau) = \frac{1}{2}e^{\tau}.$$

Then the leading-order solution is

$$u_0 = e^{\tau}\cos t,$$

indicating a slow growth in amplitude.

3. The wave equation with a slowly-varying phase speed is

$$u_{tt} = c^2(\epsilon t)\, u_{xx}, \quad |x| < \infty,\ t > 0.$$

Find a first-term approximation to the solution that is valid for large t.

We shall use the initial conditions

$$u(x,0) = F(x),\ u_t(x,0) = 0. \tag{10}$$

Based on past experience with problems involving slowly-varying coefficients, we distort the time such that the speed measured by the new time is a constant. Let the new time scale T be defined by

$$T = f(t).$$

Then

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial T} f'(t),\ \ \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial T^2}\{f'(t)\}^2 + \frac{\partial u}{\partial T} f''(t),$$

and the PDE transforms into

$$u_{TT}\{f'(t)\}^2 + u_T f''(t) = c^2(\epsilon t) u_{xx},$$

or,

$$u_{TT} + \frac{f''(t)}{\{f'(t)\}^2} u_T = \frac{c^2(\epsilon t)}{\{f'(t)\}^2} u_{xx}. \tag{11}$$

We choose

$$f'(t) = c(\epsilon t), \quad \text{so that} \quad T = f(t) = \int_0^t c(\epsilon s)\, ds = \frac{1}{\epsilon}\int_0^\tau c(\sigma)\, d\sigma, \tag{12}$$

where

$$\tau = \epsilon t. \tag{13}$$

Then the PDE (11) becomes

$$u_{TT} + \epsilon \frac{c'(\tau)}{c^2(\tau)} u_T = u_{xx}. \tag{14}$$

The initial conditions remain

$$u = F(x),\ u_T = 0 \quad \text{at} \quad T = 0. \tag{15}$$

We now apply the coordinate transformation

$$r = x - T,\ s = x + T,$$

and introduce the slow time scale τ explicitly, by letting $u = u(r,s,\tau)$. (A slow length scale ϵx is not introduced as the initial conditions do not display such a variation.) According to the chain rule the derivatives transform as

$$\begin{aligned}
u_T &= -u_r + u_s + u_\tau \frac{d\tau}{dT} \\
&= -u_r + u_s + u_\tau \frac{d\tau}{dt}\frac{dt}{dT} \\
&= -u_r + u_s + \frac{\epsilon}{c(\tau)} u_\tau, \\
u_{TT} &= \left(-\frac{\partial}{\partial r} + \frac{\partial}{\partial s} + \frac{\epsilon}{c(\tau)}\frac{\partial}{\partial \tau}\right)\left(-u_r + u_s + \frac{\epsilon}{c(\tau)} u_\tau\right) \\
&= u_{rr} - 2u_{rs} + u_{ss} - \frac{2\epsilon}{c(\tau)} u_{r\tau} + \frac{2\epsilon}{c(\tau)} u_{s\tau} + \frac{\epsilon^2}{\{c(\tau)\}^2}\left(u_{\tau\tau} - \frac{c'(\tau)}{c(\tau)} u_\tau\right), \\
u_x &= u_r + u_s, \\
u_{xx} &= u_{rr} + 2u_{rs} + u_{ss}.
\end{aligned}$$

The PDE (14) then transforms into

$$4u_{rs} + \frac{2\epsilon}{c} u_{r\tau} - \frac{2\epsilon}{c} u_{s\tau} - \frac{\epsilon^2}{c^2}\left(u_{\tau\tau} - \frac{c'(\tau)}{c(\tau)} u_\tau\right) - \frac{\epsilon c'}{c^2}\left(-u_r + u_s + \frac{\epsilon}{c(\tau)} u_\tau\right) = 0,$$

which simplifies to

$$4u_{rs} + \frac{2\epsilon}{c}u_{r\tau} - \frac{2\epsilon}{c}u_{s\tau} - \frac{\epsilon^2}{c^2}u_{\tau\tau} - \frac{\epsilon c'}{c^2}(-u_r + u_s) = 0. \tag{16}$$

We seek the expansion $u \sim u_0(r, s, \tau) + \epsilon u_1(r, s, \tau)$. At leading-order,

$$u_{0rs} = 0,$$

so that

$$u_0(r, s, \tau) = f_0(r, \tau) + g_0(s, \tau). \tag{17}$$

Since $t = 0$ corresponds to $T = 0$ and hence to $r = s$ and $\tau = 0$, the initial conditions (10) can be written as

$$u_0(r, r, 0) = F(r), \quad -u_{0r}(r, r, 0) + u_{0s}(r, r, 0) = 0.$$

When applied to u_0 these conditions lead to

$$\begin{aligned} f_0(r, 0) + g_0(r, 0) &= F(r), \\ -f_{01}(r, 0) + g_{01}(r, 0) &= 0. \end{aligned}$$

In the second equation above the suffix 1 indicates partial differentiation with respect to the first argument, which is r for f_0 and s for g_0. This equation can be integrated to yield

$$f_0(r, 0) - g_0(r, 0) = A,$$

where A is a constant. Upon solving this equation with the first of the pair above we get the initial conditions

$$f_0(r, 0) = \frac{F(r) + A}{2}, \quad g_0(r, 0) = \frac{F(r) - A}{2}. \tag{18}$$

At the next order the PDE is

$$\begin{aligned} u_{1rs} &= \frac{1}{2c}[-u_{0r\tau} + u_{0s\tau}] + \frac{c'}{4c^2}[-u_{0r} + u_{0s}] \\ &= \frac{1}{2c}[-f_{0r\tau} + g_{0s\tau}] + \frac{c'}{4c^2}[-f_{0r} + g_{0s}]. \end{aligned}$$

It can be integrated to yield

$$\begin{aligned} u_1 &= \frac{1}{2c}[-sf_{0\tau} + rg_{0\tau}] + \frac{c'}{4c^2}[-sf_0 + rg_0] + f_1(r, \tau) + g_1(s, \tau) \\ &= \frac{1}{4c}\left[r\left\{2g_{0\tau} + \frac{c'}{c}g_0\right\} - s\left\{2f_{0\tau} + \frac{c'}{c}f_0\right\}\right] + f_1(r, \tau) + g_1(s, \tau). \end{aligned}$$

To avoid the linear growth in r and s, we set

$$\begin{aligned} 2f_{0\tau} + \frac{c'}{c}f_0 &= 0, \\ 2g_{0\tau} + \frac{c'}{c}g_0 &= 0. \end{aligned}$$

The solutions, subject to the initial conditions (18), are

$$\begin{aligned} f_0(r, \tau) &= \frac{F(r) + A}{2}\sqrt{\frac{c(0)}{c(\tau)}}, \\ g_0(s, \tau) &= \frac{F(s) - A}{2}\sqrt{\frac{c(0)}{c(\tau)}}. \end{aligned}$$

Then the leading-order solution (17) becomes

$$u_0 = \frac{1}{2}[F(x-T) + F(x+T)]\sqrt{\frac{c(0)}{c(\tau)}}.$$

We note that the wave speed is given by

$$\frac{dx}{dt} = \frac{dT}{dt} = f'(t) = c(\epsilon t),$$

and that the slowly-varying wave speed also slowly modulates the waveform by virtue of the factor

$$\sqrt{\frac{c(0)}{c(\tau)}}.$$