$\mathcal{L}$

1) $\mathcal{L}(\tau_c u) = e^{-cs}\cdot\mathcal{L}(u) \quad (c>0)$

2) $\mathcal{L}(e^{ct}\cdot u) = \tau_c\,\mathcal{L}(u) \quad (c\in\mathbb{C})$

3) $\mathcal{L}'(u) = \mathcal{L}(-t\cdot u(t))$

4) $\mathcal{L}(u*v) = \mathcal{L}(u)\cdot\mathcal{L}(v)$

5) $\mathcal{L}(U)(s) = \dfrac{\mathcal{L}(u)}{s}$

6) $\mathcal{L}(u')(s) = s\,\mathcal{L}(u)(s) - u(0)$

7) $\mathcal{L}(u'')(s) = s^2\mathcal{L}(u) - su(0) - u'(0)$

$\mathcal{L}^{-1}$

1) $\mathcal{L}^{-1}(\tau_c v) = e^{-ct}\cdot\mathcal{L}^{-1}(v) \quad (c\in\mathbb{C})$

2) $\mathcal{L}^{-1}(e^{-cs}\cdot v) = \tau_c\,\mathcal{L}^{-1}(v) \quad (c>0)$

3) RIEMANN - FOURIER:

se $\mathcal{L}^{-1}(v)\in C^0(\mathbb{R})$:

$$u \longrightarrow \mathcal{L}(u)(s) = \int_0^{+\infty} u(t)\,e^{-st}\,dt$$

$$v \longrightarrow \mathcal{L}^{-1}(v) = \frac{1}{2\pi i}\int_{\substack{x+2\pi i\mathbb{R},\\ x>\lambda(\mathcal{L}^{-1}(v))}} v(s)\,e^{st}\,ds$$

4) HEAVISIDE:

se $v = \frac{p}{q}$ con $p,q\in\mathbb{C}[s]$:

$$\mathcal{L}^{-1}(v) = \sum_{q(s_k)=0} Res\left(v(s)e^{st}, s_k\right)\cdot H(t)$$

---

1)

$$\begin{cases} \ddot{y} - 3\dot{y} + 2y = \delta_1 = \delta_0(t-1) \\ y(0) = \dot{y}(0) = 0 \end{cases}, \quad t>0$$

$\Rightarrow \mathcal{L}(y) = Y \Rightarrow \mathcal{L}(\dot{y}) = sY - \cancel{y(0)} \wedge \mathcal{L}(\ddot{y}) = s^2Y - \cancel{y(0)} - \cancel{\dot{y}(0)}$

$\Rightarrow \mathcal{L}(\delta_1) = e^{-s} \Rightarrow Y\cdot(s^2 - 3s + 2) = e^{-s}$

$\Rightarrow Y = \dfrac{e^{-s}}{s^2 - 3s + 2} = \dfrac{e^{-s}}{(s-1)(s-2)} = e^{-s}\cdot v(s)$ con $v(s) = \dfrac{1}{(s-1)(s-2)}$

$\Rightarrow \mathcal{L}^{-1}(Y) = \mathcal{L}^{-1}(e^{-s}v(s)) = \tau_1\mathcal{L}^{-1}(v) = \mathcal{L}^{-1}(v)(t-1)$

Ci sono 2 metodi per calcolare $\mathcal{L}^{-1}(v)$:

1) Formula di Heaviside (v é razionale !!!)

$$\mathcal{L}^{-1}(v) = \sum_{\substack{s_k \text{ poli}\\ \text{di } e^{st}v(s)}} Res\left(v(s)e^{st}, s_k\right)\cdot H(t)$$

$= \left(Res(v(s)e^{st}, 1) + Res(v(s)e^{st}, 2)\right)\cdot H(t)$

$= \left(-e^{t} + e^{2t}\right)H(t)$

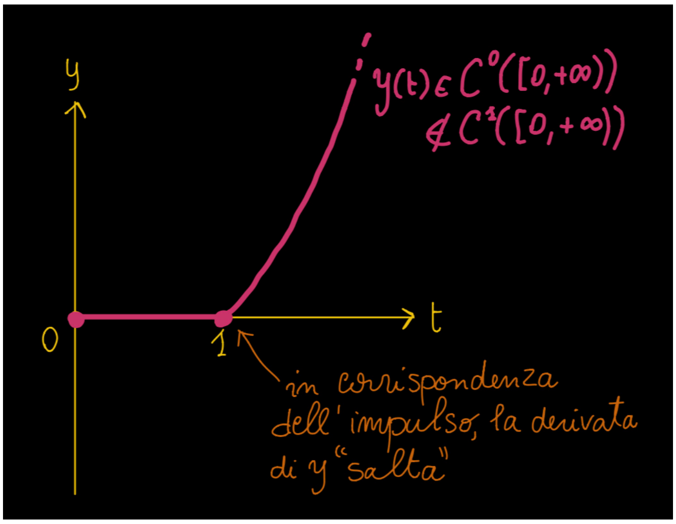
2) Fratti semplici (solo se v ammette UNICAMENTE POLI SEMPLICI):

$v(s) = \dfrac{A}{s-1} + \dfrac{B}{s-2} = \dfrac{-1}{s-1} + \dfrac{1}{s-2}$ con $A = Res(v,1)$, $B = Res(v,2)$

$\Rightarrow \mathcal{L}^{-1}(s) = \mathcal{L}^{-1}\left(-\frac{1}{s-1}\right) + \mathcal{L}^{-1}\left(\frac{1}{s-2}\right) = -\mathcal{L}^{-1}\left(\tau_1\left(\frac{1}{s}\right)\right) + \mathcal{L}^{-1}\left(\tau_2\left(\frac{1}{s}\right)\right)$

$= \left(-e^{t} + e^{2t}\right)H(t)$

$$\Rightarrow y(t) = \left(-e^{(t-1)} + e^{2(t-1)}\right)\cdot H(t-1) = \begin{cases} -e^{(t-1)} + e^{2(t-1)} & t\geq 1 \\ 0 & t<1 \end{cases}$$



2) $\begin{cases} \ddot{y} + \dot{y} + y = \delta_1 & t > 0 \\ y(0) = \dot{y}(0) = 0 \end{cases}$

$\Rightarrow \mathcal{L}(y) = Y \Rightarrow \mathcal{L}(\dot{y}) = sY - \cancel{y(0)}, \; \mathcal{L}(\ddot{y}) = s^2 Y - s\cancel{y(0)} - \cancel{\dot{y}(0)}$

$\Rightarrow \mathcal{L}(\delta_1) = e^{-s}$

$\Rightarrow Y \cdot (s^2 + s + 1) = e^{-s} \Rightarrow Y = \dfrac{e^{-s}}{s^2+s+1} = \dfrac{e^{-s}}{(s-s_+)(s-s_-)}$

Il polinomio caratteristico non è riducibile in $\mathbb{R}$!!!

$\Rightarrow s_\pm = \frac{1}{2}(-1 \pm i\sqrt{3})$ MA NON È NECESSARIO CALCOLARLE !!!

$\Rightarrow y = \mathcal{L}^{-1}(Y) = \mathcal{L}^{-1}(v)(t-1)$ con $v(s) := \dfrac{1}{(s-s_+)(s-s_-)}$

Calcolo di y:

1) Formula di Heaviside (v è razionale) ✓
2) Fratti semplici (v ha solo 2 poli semplici)
3) Metodo dei quadrati (solo se il denominatore di v è un trinomio di 2° grado IRRIDUCIBILE in $\mathbb{R}$!!!):

$$v(s) = \frac{1}{(s+\frac{1}{2})^2 + \frac{3}{4}} = \frac{1}{(s+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} = \underbrace{\frac{\sqrt{3}}{2}}_{=:\,w} \frac{1}{(s+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} \cdot \frac{2}{\sqrt{3}}$$

$$= \frac{1}{w} \frac{w}{(s+\frac{1}{2})^2 + w^2} = \frac{1}{w} \mathcal{L}(\sin wt)(s - (-\tfrac{1}{2})) = \frac{1}{w} \tau_{-\frac{1}{2}} \mathcal{L}(\sin wt)$$

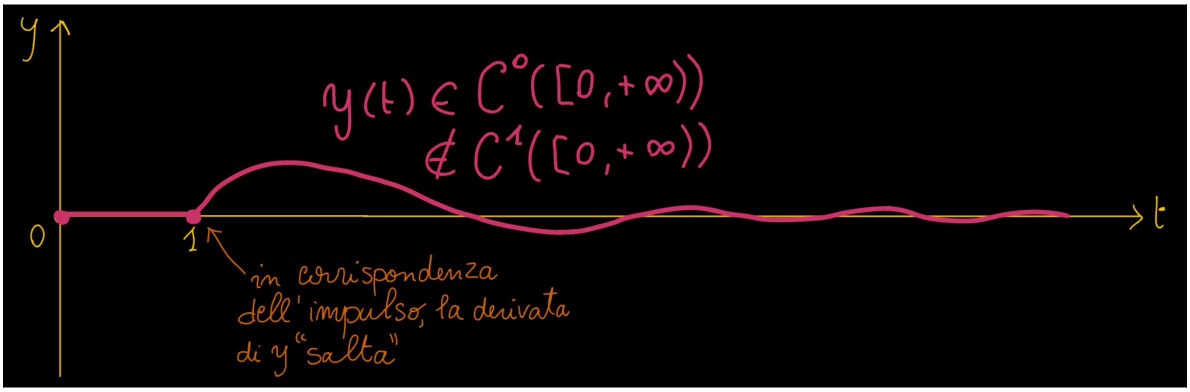
$$= \frac{1}{w} \mathcal{L}(e^{-\frac{1}{2}t} \cdot \sin wt)$$

$\Rightarrow \mathcal{L}^{-1}(v)(t) = \frac{1}{w} e^{-\frac{1}{2}t} \cdot \sin\left(\frac{\sqrt{3}}{2} t\right) \cdot H(t)$

$\Rightarrow y(t) = \frac{2}{\sqrt{3}} e^{-\frac{1}{2}(t-1)} \sin\left(\frac{\sqrt{3}}{2}(t-1)\right) \cdot H(t-1)$

Quindi:

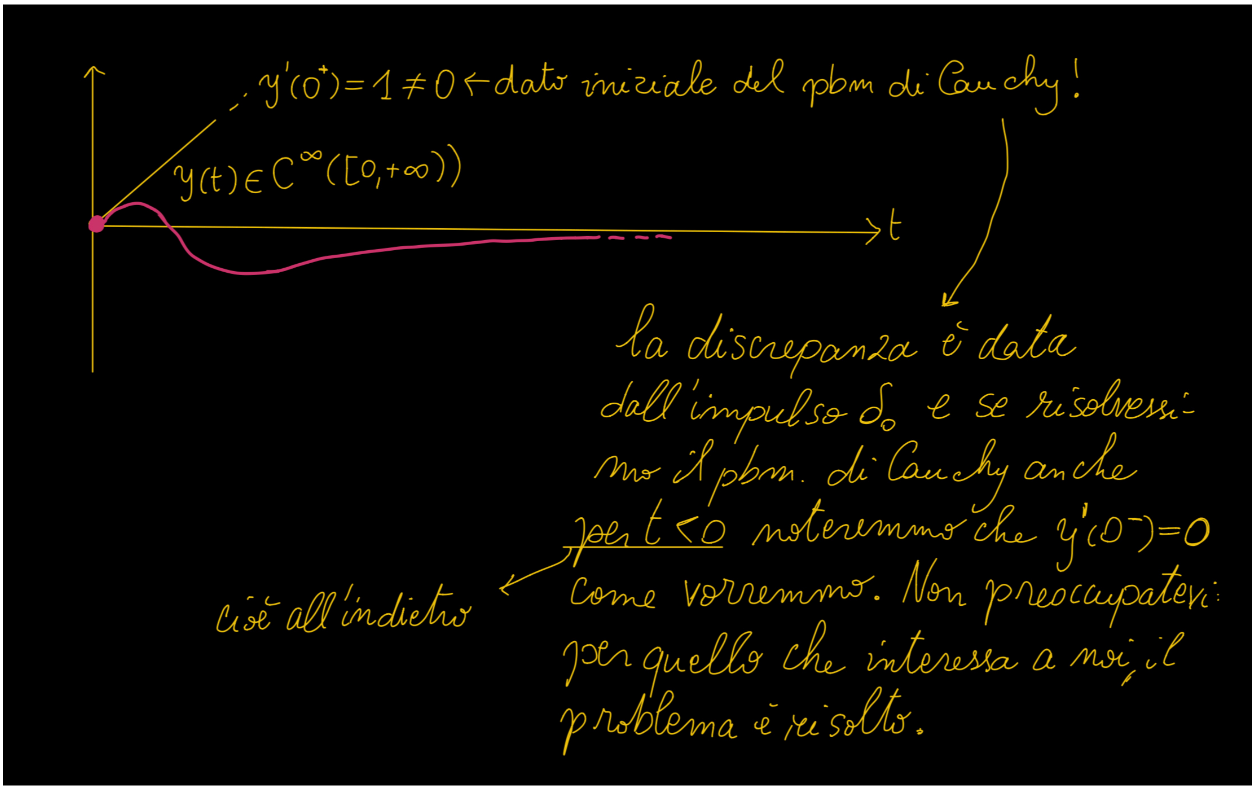
$$y(t) = \begin{cases} \frac{2}{\sqrt{3}} e^{-\frac{1}{2}(t-1)} \sin\left(\frac{\sqrt{3}}{2}(t-1)\right) & t \geq 1 \\ 0 & t < 1 \end{cases}$$



3)
$$\begin{cases} \ddot{y} + 6\dot{y} + 9y = \delta_0 - 3e^{-3t} \\ y(0) = \dot{y}(0) = 0 \end{cases} \qquad t>0$$

$\Rightarrow$ trasformo: $\mathcal{L}(y)(s)\cdot(s^2+6s+9) = 1 - \frac{3}{s+3}$

$$\Rightarrow \mathcal{L}(y)(s) = \frac{1}{(s+3)^2} - \frac{3}{(s+3)^3} = \tau_{-3}\left(\frac{1}{s^2}\right) - 3\tau_{-3}\left(\frac{1}{s^3}\right)$$
$$= \tau_{-3}\mathcal{L}(t)(s) - 3\tau_{-3}\mathcal{L}\left(\frac{t^2}{2}\right)(s)$$
$$= \tau_{-3}\mathcal{L}\left(t - \frac{3}{2}t^2\right)(s) = \mathcal{L}\left(e^{-3t}\left(t-\frac{3}{2}t^2\right)\right)(s)$$
$$\Rightarrow y(t) = e^{-3t}\left(t - \frac{3}{2}t^2\right)\cdot H(t)$$



4)
$$\begin{cases} \ddot{y} - 4\dot{y} + 4y = 4\cdot \mathbb{1}_{(0,2)} \\ y(0) = 1, \ \dot{y}(0) = 0 \end{cases}$$

$$\Rightarrow \mathcal{L}(y)(s) = Y \Rightarrow \mathcal{L}(\dot{y}) = sY - y(0) = sY - 1, \ \mathcal{L}(\ddot{y}) = s^2Y - s$$
$$\Rightarrow \mathcal{L}(4\cdot\mathbb{1}_{(0,2)}) = 4\mathcal{L}(\mathbb{1}_{(0,2)}) = 4\mathcal{L}(H - \tau_2 H) = 4(\mathcal{L}(H) - \mathcal{L}(\tau_2 H))$$
$$= \frac{4}{s} - \frac{4e^{-2s}}{s}$$

$$\Rightarrow s^2 Y - s - 4(sY - 1) + 4Y = \frac{4}{s} - \frac{4e^{-2s}}{s}$$

$$\Rightarrow Y(s^2 - 4s + 4) = \frac{4}{s} - \frac{4e^{-2s}}{s} + s - 4$$

$$\Rightarrow Y = \frac{s-4}{(s-2)^2} + \frac{4}{s(s-2)^2} - \frac{4e^{-2s}}{s(s-2)^2} \Rightarrow y = \mathcal{L}^{-1}(Y)$$

$$\Rightarrow y = \mathcal{L}^{-1}\left(\frac{s-4}{(s-2)^2}\right) + 4\mathcal{L}^{-1}\left(\frac{1}{s(s-2)^2}\right) - 4\tau_2\mathcal{L}^{-1}\left(\frac{1}{s(s-2)^2}\right)$$

$$= \operatorname{Res}\left(e^{st}\frac{s-4}{(s-2)^2}, s=2\right) + 4\mathcal{L}^{-1}\left(\frac{1}{s} - \frac{1}{s-2} + \frac{2}{(s-2)^2}\right) - 4\tau_2\mathcal{L}^{-1}\left(\frac{1}{s(s-2)^2}\right)$$

$$= \lim_{s\to 2}\frac{d}{ds}e^{st}\cdot(s-4) + 4\left(1 - e^{2t} + 2te^{2t}\right)H(t) - 4\left(1 - e^{2(t-2)} + 2(t-2)e^{2(t-2)}\right)H(t-2)$$

$$= e^{2t}(1-2t)H(t) + 4\left(1 - e^{2t} + 2te^{2t}\right)H(t) - 4\left(1 - e^{2(t-2)} + 2(t-2)e^{2(t-2)}\right)H(t-2)$$

$$= \left(4 - 3e^{2t} + 6te^{2t}\right)H(t) - \left(4 - 4e^{2(t-2)} + 2(t-2)e^{2(t-2)}\right)H(t-2)$$

5)

$$\begin{cases} \dot{y} + \int_0^t (t-x)\,y(x)\,dx = t + \frac{t^2}{2} + \frac{t^4}{24} & t > 0 \\ y(0) = 0 \end{cases}$$

$$\Rightarrow \int_0^t (t-x)\,y(x)\,dx =: \text{antiderivata di } (t-x)\,y(x)$$

$$\Rightarrow \mathcal{L}(y) = Y \Rightarrow \mathcal{L}(\dot{y}) = sY$$

$$\Rightarrow \mathcal{L}\left(\int_0^t (t-x)\,y(x)\,dx\right) = \mathcal{L}\Big(\int_{\mathbb{R}} \underbrace{(t-x)\,H(t-x)}_{f(t-x)}\cdot\underbrace{y(x)\,H(x)}_{g(x)}\,dx\Big)$$

$$= \mathcal{L}(f * g)$$

$$= \mathcal{L}(f)\,\mathcal{L}(g) = \mathcal{L}(tH)\,\mathcal{L}(yH) = \mathcal{L}(t)\,\mathcal{L}(y) = \frac{1}{s^2}Y$$

$$\Rightarrow sY + \frac{1}{s^2}Y = \frac{1}{s^2} + \frac{1}{\cancel{2}}\frac{\cancel{2!}}{s^3} + \frac{1}{\cancel{24}}\cdot\frac{\cancel{4!}}{s^5}$$

$$\Rightarrow Y = \frac{s^2}{s^3+1}\cdot\left(\frac{1}{s^2} + \frac{1}{s^3} + \frac{1}{s^5}\right) = \frac{1}{s^3+1} + \frac{1}{s(s^3+1)} + \frac{1}{s^3(s^3+1)}$$

3 poli semplici — polo semplice — 3 poli semplici — polo semplice — 3 poli semplici

$\Rightarrow$ Fratti semplici ... si trova:

$$Y = \frac{1}{s} + \frac{1}{s^3} - \frac{1}{3(s+1)} - \frac{2s-1}{3(s^2-s+1)}$$

$$= \frac{1}{s} + \frac{1}{s^3} + \frac{1}{3(s+1)} - \frac{2}{3}\,\frac{s-\frac{1}{2}}{(s-\frac{1}{2})^2 + \underbrace{\left(\frac{\sqrt{3}}{2}\right)}_{=:\,\omega}{}^2}$$

$\mathcal{L}(\cos \omega t)(s - \tfrac{1}{2})$

e, per calcolare y, si opera come in precedenza.