Navier-Stokes Equations - Non Stationary case

Find $u : \Omega \to \mathbb{R}^d$, $p : \Omega \to \mathbb{R}$ s.t.

$$\begin{cases} \partial_t u - \nu \Delta u + (u \cdot \nabla) u + \nabla p = f & \text{in } \Omega \times (0,T) \\ \nabla \cdot u = 0 & \text{in } \Omega \times (0,T) \\ u = g & \text{on } \Gamma_D \times (0,T) \\ -\nu (\nabla u) \cdot \hat{n} + p \cdot \hat{n} = d & \text{on } \Gamma_N \times (0,T) \\ u = u_0 & \text{in } \bar{\Omega} \times (0,T) \end{cases}$$

Suppose $\Gamma_N = \emptyset$, $g = 0$.

<u>Energy estimates:</u>

$$\int_\Omega \partial_t u \cdot u \, d\Omega = \int_\Omega \frac{1}{2} \partial_t |u|^2 d\Omega = \frac{1}{2} \frac{d}{dt} \|u(t)\|_{L^2}^2$$

$$\int_\Omega -\nu \Delta u \cdot u \, d\Omega = \nu \int_\Omega \nabla u : \nabla u \, d\Omega = \nu \|\nabla u(t)\|_{L^2}^2$$

$$\int_\Omega (u \cdot \nabla) u \cdot u \, d\Omega = c(u,u,u) = 0 \quad (\text{skew-symmetry of } c(\cdot,\cdot,\cdot))$$

$$\int_\Omega \nabla p \cdot u \, d\Omega = -\int_\Omega p \underbrace{\nabla u}_{=0} \, d\Omega = 0$$

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|_{L^2}^2 + \nu \|\nabla u(t)\|_{L^2}^2 = \int_\Omega f u \, d\Omega$$

N.B.

recall the Young's ineq:

$\forall a, b > 0$, $\forall p, q \in [1, +\infty]$ then:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

$$\Rightarrow \forall \varepsilon > 0 \quad ab \leq \frac{a^2}{2\varepsilon} + \frac{\varepsilon b^2}{2}$$

$$\Rightarrow \int_\Omega f \cdot u \, d\Omega \overset{CS}{\leq} \|f\|_{H^{-1}} \|u\|_{H^1} \underset{(H_0^1)}{=} \|f\|_{H^{-1}} \cdot \|\nabla u\|_{L^2} \overset{Y}{\leq} \frac{\|f\|_{H^{-1}}^2}{2\nu} + \nu \frac{\|\nabla u\|_{L^2}^2}{2}$$

$$\Rightarrow \frac{1}{2} \frac{d}{dt} \|u(t)\|_{L^2}^2 + \nu \|\nabla u(t)\|_{L^2}^2 \leq \frac{\|f\|_{H^{-1}}^2}{2\nu} + \frac{\nu}{2} \|\nabla u(t)\|_{L^2}^2$$

$$\Rightarrow \frac{d}{dt} \|u(t)\|_{L^2}^2 + \nu \|\nabla u(t)\|_{L^2}^2 \leq \frac{1}{\nu} \|f(t)\|_{H^{-1}}^2$$

$$\Rightarrow \|u(T)\|_{L^2}^2 + \nu \int_0^T \|\nabla u(t)\|_{L^2}^2 dt \leq \|u_0\|_{L^2}^2 + \frac{1}{\nu} \int_0^T \|f(t)\|_{H^{-1}}^2 dt$$

but this estimate is not useful: it could explode !!!

$\Rightarrow$ we look for another estimate:

__Lemma__ (Gronwall):

Let $f \in L^1(0,T)$ s.t. $f \geq 0$ a.e. in $(0,T)$. Let $\phi, g \in C^0(0,T)$ with g non decreasing a.e. in $(0,T)$. Then:

$$\phi(t) \leq g(t) + \int_0^t f(s)\phi(s)\,ds \quad \forall t \in (0,T) \Rightarrow \phi(t) \leq g(t)\, e^{\int_0^t f(s)ds} \quad \forall t \in (0,T)$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2} + \nu\|\nabla u(t)\|^2_{L^2} = \int_\Omega f\, u\, d$$

$$\Rightarrow \underbrace{\frac{1}{2}\|u(t)\|^2_{L^2} + \nu\int_0^t \|\nabla u(s)\|^2_{L^2}ds}_{\phi} = \frac{1}{2}\|u_0\|^2_{L^2} + \int_0^t\int_\Omega f\cdot u\, d\Omega\, dt$$

$$\underset{CS\,(L^2)}{\leq} \frac{1}{2}\|u_0\|^2_{L^2} + \int_0^t \|f\|_{L^2}\|u\|_{L^2}ds \overset{Y}{\leq} \underbrace{\frac{1}{2}\|u_0\|^2_{L^2} + \frac{1}{2}\int_0^t \|f\|^2_{L^2}ds}_{g}$$

$$+ \frac{1}{2}\int_0^t \|u\|^2_{L^2}ds \leq g + \int_0^t \underset{\underset{f(s)}{\parallel}}{1}\cdot\phi(s)\,ds$$

$$\Rightarrow \|u(t)\|^2_{L^2} + 2\nu\int_0^t \|\nabla u(s)\|^2_{L^2}ds \leq \left(\|u_0\|^2_{L^2} + \int_0^t \|f(s)\|^2_{L^2}ds\right)e^t$$

(Gronwall)

GALERKIN APPROXIMATION

Find $(u,p) \in V \times Q$ with $u|_{t=0} = u_0$, $u|_{\Gamma_D} = g$ s.t.

$$\begin{cases} \int_\Omega \partial_t u \cdot v\, d\Omega + a(u,v) + c(u,u,v) + b(v,p) = F(v) & \forall v \in V_0 \\ b(u,q) = 0 & \forall q \in Q \end{cases}$$

$\Rightarrow V_h = \mathrm{span}\{\phi_i\}_{i=1}^{N_u}$, $Q_h = \mathrm{span}\{\psi_k\}_{k=1}^{N_p}$, introduce the coefficients for the separation of the variables:

$$u_h = \sum_{i=1}^{N_u} \hat{u}_i(t)\phi_i, \quad p_h = \sum_{k=1}^{N_p} \hat{p}_k(t)\psi_k$$

$$\begin{cases} \overset{M}{\sum_{i=1}^{N_u}\int_\Omega \frac{d}{dt}\hat{u}_i \cdot \phi_i \cdot \phi_j\, d\Omega} + \overset{A}{\sum_{i=1}^{N_u}\hat{u}_i\, a(\phi_i,\phi_j)} + \overset{C(\hat{u})}{\sum_{m,i=1}^{N_u}\hat{u}_m\hat{u}_i\, c(\phi_m,\phi_i,\phi_j)} \\ + \overset{B^T}{\sum_{k=1}^{N_p}\hat{p}_k\, b(\phi_j,\psi_k)} = F(\phi_j) & \forall j = 1,\dots,N_u \\ \underset{B}{\sum_{i=1}^{N_u}\hat{u}_i\, b(\phi_i,\psi_l)} = \vec{0} & \forall l = 1,\dots,N_p \end{cases}$$

$$\Rightarrow \begin{cases} M\frac{d}{dt}\hat{u} + A\hat{u} + C(\hat{u}) + B^T\hat{p} = F \\ B\hat{u} = \vec{0} \end{cases} \quad \text{with} \quad \begin{aligned} &M \in \mathbb{R}^{N_u \times N_u}, \\ &M_{ij} = \int_\Omega \phi_i \cdot \phi_j\, d\Omega \end{aligned}$$

Resolution of the Galerkin problem

Temporal schemes for non-stationary Stokes:

$$\begin{cases} \partial_t u - \nu \Delta u + \nabla p = f & \text{in } \Omega \times (0,T) \\ \nabla \cdot u = 0 & \text{in } \Omega \times (0,T) \end{cases}$$
$+ B.C. + I.C.$

$(0,T) = \bigcup_{n=0}^{N-1} (t^n, t^{n+1})$

$u_i^n = u(x_i, t^n)$,

$\Delta t = t^{n+1} - t^n$

1) Explicit Euler:

$$\partial_t u \approx \frac{u^{n+1} - u^n}{\Delta t} \Rightarrow \frac{u^{n+1} - u^n}{\Delta t} = f^{n+1} + \nu \Delta u^n - \nabla p^n$$

Remark:

We can't force $\nabla \cdot u^{n+1} = 0$ in this way!!!

$\Rightarrow$ we proceed as follows:

$$\begin{cases} \dfrac{u^{n+1} - u^n}{\Delta t} + \nabla p^{n+1} = f^{n+1} + \nu \Delta u^n \\ \nabla \cdot u^{n+1} = 0 \end{cases}$$

$\Rightarrow$ this method is unstable if $\Delta t > C \frac{h^2}{\nu}$!!!

2) Implicit Euler:

$$\begin{cases} \dfrac{u^{n+1} - u^n}{\Delta t} - \nu \Delta u^{n+1} + \nabla p^{n+1} = f^{n+1} \\ \nabla \cdot u^{n+1} = 0 \end{cases} \Longleftrightarrow \begin{pmatrix} C & B^T \\ B & \vec{0} \end{pmatrix} \begin{pmatrix} \hat{u}^{n+1} \\ \hat{p}^{n+1} \end{pmatrix} = \begin{pmatrix} \tilde{F}^{n+1} \\ \vec{0} \end{pmatrix}$$

$$\Rightarrow C = A + \frac{M}{\Delta t} = \nu K + \frac{M}{\Delta t} \quad \Big| \quad \tilde{F}^{n+1} = F^{n+1} + \frac{M}{\Delta t} \hat{u}^n$$

$\Rightarrow$ Schur: $\Sigma = B C^{-1} B^T = B \left(\nu K + \frac{M}{\Delta t} \right)^{-1} B^T$

If $\nu \gg \frac{1}{\Delta t}$ (High diffusion $\Leftrightarrow$ High viscosity):

$\Rightarrow \Sigma \sim B \frac{K^{-1}}{\nu} B^T = B A^{-1} B^T$

$\Rightarrow$ preconditioner: $P = \frac{M_p}{\nu}$

If $\nu \ll \frac{1}{\Delta t}$ (High fluid velocity):

$\Rightarrow \Sigma \sim B \Delta t M^{-1} B^T$

$\Rightarrow$ if V_h, Q_h are inf-sup compatible, the swapped inf-sup condition holds as well:

$$\exists \tilde{\beta}_h > 0 \text{ s.t. } \inf_{q_h \in Q_h} \sup_{v_h \in V_h} \frac{b(v_h, q_h)}{\|v_h\|_{L^2} \cdot \|\nabla q_h\|_{L^2}} \geq \tilde{\beta}_h$$

$$\Rightarrow 0 < \tilde{\beta}_h^2 \leq \frac{\hat{p}^T B M^{-1} B^T \hat{p}}{\Delta t\, \hat{p}^T K_p \hat{p}}, \quad K_p \in \mathbb{R}^{N_p \times N_p}, \quad (K_p)_{ij} = \int_\Omega \nabla \psi_i : \nabla \psi_j \, d\Omega$$

$\Rightarrow$ preconditioner : $P = \Delta t\, K_p$

If $\nu \approx \frac{1}{\Delta t}$:

$\Rightarrow$ preconditioner : $P_{CC} = \left(\nu M_p^{-1} + \frac{K_p}{\Delta t}^{-1} \right)^{-1}$

<u>Remark</u>

For E. E., the method is unconditionally stable $(\forall \Delta t)$

TREATMENT OF THE NON LINEAR TERM:

$$\begin{cases} \frac{u^{n+1} - u^n}{\Delta t} - \nu \Delta u^{n+1} + (u^* \cdot \nabla) u^{**} + \nabla p^{n+1} = f^{n+1} \\ \nabla \cdot u^{n+1} = 0 \end{cases}$$

1) EXPLICIT:

$$(u^* \cdot \nabla) u^{**} = (u^n \cdot \nabla) u^n$$

Stable if CFC condition holds: $\Delta t < CFL \cdot \frac{h}{\|\bar{u}\|}$

2) FULLY IMPLICIT:

$$(u^* \cdot \nabla) u^{**} = (u^{n+1} \cdot \nabla) u^{n+1}$$

$$\Rightarrow \int_\Omega \frac{u^{n+1} - u^n}{\Delta t} \cdot u^{n+1} \, d\Omega = \frac{1}{2\Delta t} \left(\|u^{n+1}\|^2_{L^2} - \|u^n\|^2_{L^2} + \|u^{n+1} - u^n\|^2_{L^2} \right)$$

$$\Rightarrow -\int \nu \Delta u^{n+1} \, u^{n+1} \, d\Omega = \nu \int_\Omega \nabla u^{n+1} : \nabla u^{n+1} d\Omega = \nu \|\nabla u^{n+1}\|^2_{L^2}$$

$$\Rightarrow \int_\Omega (u^{n+1} \cdot \nabla) u^{n+1} \cdot u^{n+1} \, d\Omega = c(u^{n+1}, u^{n+1}, u^n) = 0 \quad (\text{sk. sy.})$$

$$\Rightarrow \int_\Omega \nabla \cdot p^{n+1} \cdot u^{n+1} \, d\Omega = -\int_\Omega p^{n+1} \underbrace{\nabla \cdot u^{n+1}}_{=0} d\Omega = 0$$

$$\Rightarrow \int_\Omega f^{n+1} \cdot u^{n+1} \, d\Omega \leq \frac{1}{2\nu} \|f^{n+1}\|^2_{H^{-1}} + \frac{\nu}{2} \|\nabla u^{n+1}\|^2_{L^2}$$

so we have :

$$\frac{1}{\Delta t} \|u^{n+1}\|^2_{L^2} - \frac{1}{\Delta t} \|u^n\|^2_{L^2} + \underbrace{\frac{1}{\Delta t} \|u^{n+1} - u^n\|^2_{L^2}}_{\geq 0} + \nu \|\nabla u^{n+1}\|^2_{L^2} \leq \frac{1}{\nu} \|f\|^2_{H^{-1}}$$

$$\Rightarrow \frac{1}{\Delta t} \left(\|u^{n+1}\|^2_{L^2} - \|u^n\|^2_{L^2} \right) + \nu \|\nabla u^{n+1}\|^2_{L^2} \leq \frac{1}{\nu} \|f\|^2_{H^{-1}}$$

$$\Rightarrow \|u^N\|^2_{L^2} + \nu \Delta t \sum_n \|\nabla u^n\|^2_{L^2} \leq \|u^0\|^2_{L^2} + \frac{\Delta t}{\nu} \sum_n \|f^n\|^2_{H^{-1}}$$

3) SEMI-IMPLICIT:

$$(u^* \cdot \nabla) u^{**} = (u^n \cdot \nabla) u^{n+1}$$

$\Rightarrow$ the discrete energy estimate of the implicit case holds

$\Rightarrow$ it also is unconditionally stable.

Remark:

All of these schemes have rate of convergence 1

2ND ORDER SCHEMES:

1) BDF-2 (Backward Difference Formula - order 2)

$$\partial_t u\big|_{t=t^n} \approx \frac{1}{\Delta t}\left(\frac{3}{2}u^{n+1} - 2u^n + \frac{1}{2}u^{n-1}\right)$$

$$\Rightarrow \begin{cases} \dfrac{3u^{n+1} - 4u^n + u^{n-1}}{2\Delta t} - \nu\Delta u^{n+1} + (u^* \cdot \nabla)u^{**} + \nabla p^{n+1} = f^{n+1} \\ \nabla \cdot u^{n+1} = 0 \end{cases}$$

$$\Rightarrow (u^* \cdot \nabla)u^{**} = \begin{cases} (u^{n+1} \cdot \nabla)u^{n+1} & \text{implicit} \\ ((2u^n - u^{n-1}) \cdot \nabla)u^n & \text{semi-implicit} \\ ((2u^n - u^{n-1}) \cdot \nabla)(2u^n - u^{n-1}) & \text{explicit} \end{cases}$$

2) Crank-Nicolson:

$$\partial_t u\big|_{t^{n+\frac{1}{2}}} \approx \frac{u^{n+1} - u^n}{\Delta t}$$

$$\begin{cases} \dfrac{u^{n+1} - u^n}{\Delta t} - \nu\Delta\left(\dfrac{u^{n+1} + u^n}{2}\right) + (u^* \nabla)u^{**} + \nabla\left(\dfrac{p^{n+1} + p^n}{2}\right) = f^{n+\frac{1}{2}} \\ \nabla \cdot u^{n+1} = 0 \end{cases}$$

$$\Rightarrow (u^* \cdot \nabla)u^{**} = \begin{cases} \left(\left(\dfrac{u^{n+1} - u^n}{2}\right) \cdot \nabla\right)\left(\dfrac{u^{n+1} + u^n}{2}\right) & \text{implicit} \\ \left(\left(\dfrac{3u^n - u^{n-1}}{2}\right) \cdot \nabla\right)\left(\dfrac{u^{n+1} + u^n}{2}\right) & \text{semi-implicit} \\ \left(\left(\dfrac{3u^n - u^{n-1}}{2}\right) \cdot \nabla\right) \cdot \left(\dfrac{3u^n - u^{n+1}}{2}\right) & \text{explicit} \end{cases}$$

PROJECTION METHODS:

$$\begin{cases} \partial_t u + L_1 u + L_2 u = f & \text{in } \Omega \\ + \text{B.C.} + \text{I.C.} \end{cases}$$

L_1, L_2 differential operators

$\Rightarrow$ fractional methods:

$$1)\quad \frac{\tilde{u}^{n+1} - u^n}{\Delta t} + L_1 \tilde{u} = f$$

$$2)\quad \frac{u^{n+1} - \tilde{u}^{n+1}}{\Delta t} + L_2 \tilde{u} = f$$

1) CHORIN-TEMAM METHOD:

Def (Trace):

$u: \Omega \to \mathbb{R}$ function, its TRACE is $u|_{\partial\Omega}$

1) $$\frac{\tilde{u}^{n+1}-u^n}{\Delta t} - \nu \Delta \tilde{u}^{n+1} + (\tilde{u}^{*}\cdot\nabla)\tilde{u}^{**} = f^{n+1}$$

2) $$\begin{cases} \dfrac{u^{n+1}-\tilde{u}^{n+1}}{\Delta t} + \nabla p^{n+1} = 0 \\ \nabla\cdot u^{n+1} = 0 \end{cases}$$

$\Rightarrow$ for (1), the problem is closed with B.C. described in the original NS problem.

$\Rightarrow$ for (2), velocity is $H_{div} = \{ v \in (L^2(\Omega))^d : \nabla v = 0 \}$. But on H_{div} there is no trace operator!!! We can only define the usual trace $u\cdot\hat{n}|_{\partial\Omega}$. This introduces a splitting error $\approx \mathcal{O}(\Delta t^{\frac{1}{2}})$

As usual, take $\Gamma_N = \phi$. We have:

$$\phi \in H^0_{div} = \{ v \in H_{div} : v\cdot\hat{n}|_{\partial\Omega} = 0 \}$$

$$\Rightarrow \int_\Omega \frac{u^{n+1}-\tilde{u}^{n+1}}{\Delta t}\phi \, d\Omega = -\int_\Omega \nabla p^{n+1}\cdot\phi \, d\Omega$$
$$= \int_\Omega p^{n+1}\underbrace{\nabla\cdot\phi^{n+1}}_{=0} d\Omega - \int_{\partial\Omega} p^{n+1} \underbrace{\phi\cdot\hat{n}}_{=0} d\Gamma = 0 \quad \forall \phi \in H^0_{div}$$

$$\Rightarrow \int_\Omega u^{n+1}\cdot\phi \, d\Omega = \int_\Omega \tilde{u}^{n+1}\cdot\phi \, d\Omega \quad \forall\phi \in H^0_{div}$$

$\Rightarrow$ this means that u^{n+1} is the projection of $\tilde{u}^{n+1}$ on H^0_{div}. This is also due to the following thm.:

Thm. (Helmotz Decomposition):

Let $\Omega \subset \mathbb{R}^d$ simply connected. We have:

$\forall v \in (L^2(\Omega))^d$ $\exists!$ decomposition of v in a sum of a soleinoidal function and a irrotational function

$$\Rightarrow v = w + \nabla\phi, \quad w \in H^0_{div}, \ \phi \in H^1(\Omega)$$

$$\Rightarrow \frac{u^{n+1}-\tilde{u}^{n+1}}{\Delta t} + \nabla p^{n+1} = 0 \Leftrightarrow \tilde{u}^{n+1} = \underbrace{u^{n+1}}_{\text{soleinoidal}} + \Delta t\cdot \underbrace{\nabla p^{n+1}}_{\text{irrotational}}$$

$$\Rightarrow \nabla\cdot\left(\frac{u^{n+1}-\tilde{u}^{n+1}}{\Delta t} + \nabla p^{n+1}\right) = 0 \Rightarrow \frac{1}{\Delta t}\nabla\cdot u^{n+1} - \frac{1}{\Delta t}\nabla\cdot\tilde{u}^{n+1} + \Delta p^{n+1} = 0$$

$$\Rightarrow \Delta p^{n+1} = \frac{1}{\Delta t}\nabla\cdot\tilde{u}^{n+1} \Rightarrow \left(\frac{u^{n+1}-\tilde{u}^{n+1}}{\Delta t} + \nabla p^{n+1}\right)\cdot\hat{n} = 0 \text{ on } \partial\Omega$$

$$\Rightarrow \underbrace{\frac{1}{\Delta t}u^{n+1}\cdot\hat{n}}_{=0\ (H^0_{div})} - \underbrace{\frac{1}{\Delta t}\tilde{u}^{n+1}\cdot\hat{n}}_{=0} + \nabla p^{n+1}\cdot\hat{n} = 0 \Rightarrow \partial_{\hat{n}} p^{n+1}|_{\partial\Omega} = 0$$

B.C. on $\partial\Omega$

The method has 3 steps:

1) Prediction:

$$\begin{cases} \dfrac{\tilde{u}^{n+1} - u^n}{\Delta t} - \nu \Delta \tilde{u}^{n+1} + (\tilde{u}^* \cdot \nabla)\tilde{u}^* = f \\ + \text{B.C.} \end{cases}$$

2) Projection:

$$\begin{cases} \Delta p^{n+1} = \dfrac{1}{\Delta t} \nabla \cdot \tilde{u}^{n+1} & \text{in } \Omega \\ \nabla p^{n+1} \cdot \hat{n} = 0 & \text{on } \partial\Omega \end{cases}$$

3) Correction:

$$u^{n+1} = \tilde{u}^{n+1} - \Delta t \, \nabla p^{n+1}$$

<u>Remark</u>:

Chorin-Temam doesn't solve stationary problems properly

$\Rightarrow$ we adopt the Incremental Chorin-Temam, which has the following steps:

$$p^{n+1} = p^n + \delta p$$

1) Prediction:

$$\begin{cases} \dfrac{\tilde{u}^{n+1} - u^n}{\Delta t} - \nu \Delta \tilde{u}^{n+1} + (\tilde{u}^* \cdot \nabla)\tilde{u}^{**} = f^{n+1} - \nabla \cdot p^n \\ + \text{B.C.} \end{cases}$$

2) Projection:

$$\begin{cases} \dfrac{u^{n+1} - \tilde{u}^{n+1}}{\Delta t} + \nabla \delta p = 0 \\ \nabla u^{n+1} = 0 \end{cases} \Longleftrightarrow \begin{cases} \Delta \delta p = \dfrac{1}{\Delta t} \nabla \tilde{u}^{n+1} & \text{in } \Omega \\ \nabla \delta p \cdot \hat{n} = 0 & \text{on } \partial\Omega \end{cases}$$

3) Correction:

$$\begin{cases} u^{n+1} = \tilde{u}^{n+1} - \Delta t \, \nabla \delta p \\ p^{n+1} = p^n + \delta p \end{cases}$$

Semi-implicit weak formulation:

$$\begin{cases} \dfrac{1}{\Delta t} \displaystyle\int_\Omega u^{n+1} \cdot v \, d\Omega + a(u^{n+1}, v) + c(u^n, u^{n+1}, v) + b(v, p^{n+1}) = F(v) \\ \qquad + \dfrac{1}{\Delta t} \displaystyle\int_\Omega u^n \cdot v \, d\Omega \\ b(u^{n+1}, q) = 0 \end{cases}$$

Chorin-Temam

$$\begin{cases} \partial_{\hat{n}} \hat{p}^{n+1} = 0 & \text{on } \Gamma_D \\ p = 0 & \text{on } \Gamma_N \end{cases} \quad \text{if } \Gamma_N \neq \emptyset$$