es. 3) Sia $\gamma(t) = \left(e^{\cos t}, \cos t, \sin t\right)$, $t \in \mathbb{R}$

una curva in $\mathbb{R}^3$

1) Stabilire se γ è un'immersione

$\Rightarrow \dot{\gamma}(t) = \left(-\sin t \, e^{\cos t}, -\sin t, \cos t\right)$

$\Rightarrow \dot{\gamma}(t) \neq \vec{0} \ \forall t \in \mathbb{R}$ ($\sin t$ e $\cos t$ non si annullano mai per uno stesso $t \in \mathbb{R}$)

$\Rightarrow \gamma(t)$ è iniettiva $\Rightarrow \gamma$ è immersione

2) Determinare il versore tangente e il versore binormale $\forall t \in \mathbb{R}$:

$$\vec{t} = \frac{\dot{\gamma}}{\|\dot{\gamma}\|} = \frac{\left(-\sin t \, e^{\cos t}, -\sin t, \cos t\right)}{\sqrt{\sin^2 t \, e^{2\cos t} + \sin^2 t + \cos^2 t}}$$

$$= \boxed{\frac{\left(-\sin t \, e^{\cos t}, -\sin t, \cos t\right)}{\sqrt{1 + \sin^2 t \, e^{2\cos t}}}}$$

$\Rightarrow \vec{b} = \dfrac{\dot{\gamma} \times \ddot{\gamma}}{\|\dot{\gamma} \times \ddot{\gamma}\|}$ $\Rightarrow \ddot{\gamma} = \left(-\cos t \, e^{\cos t} + \sin^2 t \, e^{\cos t}, -\cos t, -\sin t\right)$

$= \left(e^{\cos t}(\sin^2 t - \cos t), -\cos t, -\sin t\right)$

$\Rightarrow \dot{\gamma} \times \ddot{\gamma} = ?$

$$\Rightarrow \det \begin{pmatrix} i & j & k \\ -\sin t\, e^{\cos t} & -\sin t & \cos t \\ e^{\cos t}(\sin^2 t - \cos t) & -\cos t & -\sin t \end{pmatrix}$$

$$= \Big(1, -\big(\sin^2 t\, e^{\cos t} - \cos t\, e^{\cos t}(\sin^2 t - \cos t)\big),$$
$$\sin t \cos t\, e^{\cos t} + \sin t\, e^{\cos t}(\sin^2 t - \cos t)\Big)$$

$$= \Big(1, -\sin^2 t\, e^{\cos t} + \sin^2 t \cos t\, e^{\cos t} - \cos^2 t\, e^{\cos t},$$
$$\sin t \cancel{\cos t}\, e^{\cos t} + \sin^3 t\, e^{\cos t} - \sin t \cancel{\cos t}\, e^{\cos t}\Big)$$

$$= \left(1, e^{\cos t}(\sin^2 t \cos t - 1), \sin^3 t\, e^{\cos t}\right)$$

$$\Rightarrow \|\cdot\| = \sqrt{1 + e^{2\cos t}(\sin^2 t \cos t - 1)^2 + \sin^6 t\, e^{2\cos t}}$$

$$\Rightarrow \vec{b} = \boxed{\frac{\left(1, e^{\cos t}(\sin^2 t \cos t - 1), \sin^3 t\, e^{\cos t}\right)}{\sqrt{1 + e^{2\cos t}(\sin^2 t \cos t - 1)^2 + \sin^6 t\, e^{2\cos t}}}}$$

3) Calcolare la curvatura di γ per $t = \frac{\pi}{2}$

$$\Rightarrow \|\dot{\gamma}(\tfrac{\pi}{2}) \times \ddot{\gamma}(\tfrac{\pi}{2})\| = \sqrt{1 + 1 + 1} = \sqrt{3}$$

$$\Rightarrow \|\dot{\gamma}(\tfrac{\pi}{2})\|^3 = (\sqrt{2})^3 = 2\sqrt{2}$$

$$\Rightarrow k(\tfrac{\pi}{2}) = \frac{\sqrt{3}}{2\sqrt{2}} = \boxed{\frac{\sqrt{6}}{4}}$$

es. 4)

Sia $S \subseteq \mathbb{R}^3$ superficie parametrizzata da:

$$\varphi(u,v) = (\cos u + v \sin u,\ 1 + v\cos u,\ v + \sin u)$$

con $(u,v) \in \mathbb{R}^2$

1) Determinare l'esistenza di eventuali punti singolari di S

$$\Rightarrow X_1 = \partial_u \varphi(u,v) = (v\cos u - \sin u,\ -v\sin u,\ \cos u)$$

$$\Rightarrow X_2 = \partial_v \varphi(u,v) = (\sin u,\ \cos u,\ 1)$$

$$\Rightarrow X_1 \times X_2 = \det \begin{pmatrix} i & j & k \\ v\cos u - \sin u & -v\sin u & \cos u \\ \sin u & \cos u & 1 \end{pmatrix}$$

$$= \Big(-v\sin u - \cos^2 u,\ -\big((v\cos u - \sin u) - \sin u \cos u\big),$$
$$\cos u\,(v\cos u - \sin u) + v\sin^2 u\Big)$$

$$= \Big(-v\sin u - \cos^2 u,\ -(v\cos u - \sin u - \sin u\cos u),$$
$$v - v\sin u\cos u\Big)$$

$$= \Big(-v\sin u - \cos^2 u,\ \sin u + \sin u\cos u - v\cos u,$$
$$v - v\sin u\cos u\Big)$$

$$= \left(-v\sin u - \cos^2 u,\ \cos u(\sin u - v) + \sin u,\ v(1-\sin u\cos u)\right)$$

$\Rightarrow$ deve essere $X_1 \times X_2 = \vec{0} \in \mathbb{R}^3$

$$\begin{cases} -v\sin u - \cos^2 u = 0 \\ \cos u(\sin u - v) + \sin u = 0 \\ v(1 - \sin u\cos u) = 0 \end{cases}$$

$\Rightarrow$ dall'ultima ottengo $v = 0 \lor \sin u\cos u = 1$

$\Rightarrow$ se $v = 0$ si ha:

$$\begin{cases} -\cos^2 u = 0 \\ \sin u\cos u + \sin u = 0 \end{cases} \Rightarrow \begin{cases} u = \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z} \\ \sin u(\cos u + 1) = 0 \end{cases}$$

$\Rightarrow \sin u = 0$ $\lightning$ impossibile se $\cos^2 u = 0$

$\Rightarrow$ se $\sin u\cos u = 1$ si ha:

$$\begin{cases} \sin u\cos u = 1 \\ -v\sin u - \cos^2 u = 0 \\ \cos u(\sin u - v) + \sin u = 0 \end{cases} \Rightarrow \begin{cases} \sin u\cos u = 1 \\ -v\sin u - \cos^2 u = 0 \\ 1 - \cos u\, v + \sin u = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \sin u = \frac{1}{\cos u} \land \cos u = \frac{1}{\sin u} \Rightarrow \sin u \neq 0 \neq \cos u \\ 1 - \frac{v}{\sin u} + \sin u = 0 \land -\frac{v}{\cos u} - \cos^2 u = 0 \end{cases}$$

$$\Rightarrow \frac{v}{\cos u} + \cos^2 u = 0 \Rightarrow v + \cos^3 u = 0$$

$\Rightarrow \quad v = -\cos^3 u \quad \Rightarrow \quad \cos u = -\sqrt[3]{v} \quad \Rightarrow \quad \sin u = -\frac{1}{\sqrt[3]{v}}$

$\Rightarrow \quad 1 + v \cdot \sqrt[3]{v} - \frac{1}{\sqrt[3]{v}} = 0$

$\Rightarrow \quad \sqrt[3]{v} + v \cdot \sqrt[3]{v^2} - 1 = 0$

$\Rightarrow \quad \sqrt[3]{v} + \sqrt[3]{v^5} - 1 = 0$ $\Rightarrow$ le radici di questo polinomio $\neq 0$ restituiscono le coordinate (u, v) dei punti singolari di S

$\Rightarrow$ $\boxed{\text{Non tutti i punti di } S \text{ sono regolari}}$

2) Determinare l'equazione cartesiana del piano tangente in $(u, v) = \left(\frac{\pi}{2}, 1\right)$

$\Rightarrow \quad \varphi\left(\frac{\pi}{2}, 1\right) = (1, 1, 2)$

$\Rightarrow \quad X_1 \times X_2 \left(\frac{\pi}{2}, 1\right) = (-1, 1, 1)$

$\Rightarrow \quad T(S) = -x + y + z + d = 0$ con $d \in \mathbb{R}$

$\Rightarrow$ deve essere : $-1 + 1 + 2 + d = 0$

$\Rightarrow \quad d = -2$

$\Rightarrow \quad \boxed{T_p(S) : \{ \, z = x - y + 2}$

3) Determinare la 1^a forma quadratica fondamentale

$$X_1 = (v\cos u - \sin u, -v\sin u, \cos u)$$

$$X_2 = (\sin u, \cos u, 1)$$

$$\Rightarrow (X_1 | X_1) = (v\cos u - \sin u)^2 + v^2\sin^2 u + \cos^2 u$$

$$= v^2\cos^2 u + \sin^2 u - 2v\cos u \sin u + v^2\sin^2 u + \cos^2 u$$

$$= v^2 + 1 - 2v\cos u \sin u$$

$$= v(v - 2\cos u \sin u) + 1$$

$$\Rightarrow (X_1 | X_2) = \sin u (v\cos u - \sin u) - v\sin u \cos u + \cos u$$

$$= \cos u - \sin^2 u$$

$$\Rightarrow (X_2 | X_2) = 2$$

$\Rightarrow$ La 1^a forma quadratica fondamentale è:

$$\boxed{du^2(v^2 + 1 - 2v\cos u \sin u) + 2\,du\,dv(\cos u - \sin^2 u) + 2dv^2}$$

4) Determinare se tra i punti regolari di S sono presenti punti ellittici, parabolici, iperbolici:

$\Rightarrow$ P punto ellittico $\Leftrightarrow$ $\ell_p > 0$

$$\Rightarrow \quad \ell = \det L = \det \begin{pmatrix} \ell_{11} & \ell_{12} \\ \ell_{12} & \ell_{22} \end{pmatrix}$$

$$\Rightarrow \quad \ell_{11} = \frac{\det \begin{pmatrix} X_{11} \\ X_1 \\ X_2 \end{pmatrix}}{\sqrt{g}}, \quad \ell_{12} = \frac{\det \begin{pmatrix} X_{12} \\ X_1 \\ X_2 \end{pmatrix}}{\sqrt{g}}, \quad \ell_{22} = \frac{\det \begin{pmatrix} X_{22} \\ X_1 \\ X_2 \end{pmatrix}}{\sqrt{g}}$$

$$\Rightarrow \quad g = \det \begin{pmatrix} v(v - 2\cos u \sin u) + 1 & \cos u - \sin^2 u \\ \cos u - \sin^2 u & 2 \end{pmatrix}$$

$$= 2v(v - 2\cos u \sin u) + 2 - (\cos u - \sin^2 u)^2$$

$$\Rightarrow \quad X_{11} = \partial_u X_1 = (-v \sin u - \cos u, -v\cos u, -\sin u)$$

$$\Rightarrow \quad X_{12} = \partial_v X_1 = (\cos u, -\sin u, 0)$$

$$\Rightarrow \quad X_{22} = \partial_v X_2 = (0, 0, 0) \Rightarrow \ell_{22} = 0$$

$$\Rightarrow \quad \ell = -\ell_{12}^2 = -\frac{\det \begin{pmatrix} X_{12} \\ X_1 \\ X_2 \end{pmatrix}^2}{g}$$

$$\Rightarrow \quad \det \begin{pmatrix} X_{12} \\ X_1 \\ X_2 \end{pmatrix}^2 \geq 0 \quad \forall (u,v), \quad g > 0 \quad \forall (u,v)$$

$$\Rightarrow \quad \ell \leq 0 \quad \forall (u,v) \Rightarrow \boxed{\nexists \text{ punti ellittici di } S}$$

$\Rightarrow$ calcolo $\det \begin{pmatrix} X_{12} \\ X_1 \\ X_2 \end{pmatrix}$:

$$\det \begin{pmatrix} \cos u & -\sin u & 0 \\ v\cos u - \sin u & -v \sin u & \cos u \\ \sin u & \cos u & 1 \end{pmatrix}$$

$$= \cos u \left(-v \sin u - \cos^2 u \right) + \sin u \left(v \cos u - \sin u - \cos u \sin u \right)$$

$$= -v \cancel{\sin u} \cos u - \cos^3 u + v \cancel{\sin u} \cos u - \sin^2 u - \cos u \sin^2 u$$

$$= -\cos^3 u - \sin^2 u - \cos u \sin^2 u$$

$\Rightarrow$ trova u t.c. $-\cos^3 u - \sin^2 u - \cos u \sin^2 u = 0$

$$\Rightarrow -\cos^3 u - (1 - \cos^2 u) - \cos u (1 - \cos^2 u) = 0$$

$$\Rightarrow -\cancel{\cos^3 u} - 1 + \cos^2 u - \cos u + \cancel{\cos^3 u} = 0$$

$$\Rightarrow \cos^2 u - \cos u - 1 = 0$$

$\Rightarrow \cos u_{1,2} = \dfrac{1 \pm \sqrt{1+4}}{2} = \dfrac{1 \pm \sqrt{5}}{2} \Rightarrow$ non può essere $\dfrac{1+\sqrt{5}}{2} > 1$

$$\Rightarrow \cos u = \frac{1-\sqrt{5}}{2} < 1 \Rightarrow \sin u = \sqrt{1 - \cos^2 u} \simeq 0{,}786$$

$\Rightarrow$ per $u = \arccos\left(\dfrac{1-\sqrt{5}}{2}\right)$ si ha che $l = 0$

$\Rightarrow$ $\boxed{\text{per } u \neq \arccos\left(\frac{1-\sqrt{5}}{2}\right),\ \varphi(u,v) \text{ è punto iperbolico}}$

$\Rightarrow$ se $u = \arccos\left(\dfrac{1-\sqrt{5}}{2}\right)$ devo verificare se $\varphi(u,v)$ è punto piatto o parabolico

$\Rightarrow$ Verifico che $l_{11} \neq 0$ per alcuni (u,v) t.c. $\cos u = \frac{1-\sqrt{5}}{2}$

$$\Rightarrow \det \begin{pmatrix} X_{11} \\ X_1 \\ X_2 \end{pmatrix} = \det \begin{pmatrix} -v\sin u - \cos u & -v\cos u & -\sin u \\ v\cos u - \sin u & -v\sin u & \cos u \\ \sin u & \cos u & 1 \end{pmatrix}$$

$$= -\sin u\left(v\cos^2 u - \cos u \sin u + v\sin^2 u\right) - \cos u\left(-v\cancel{\sin u}\cos u - \cos^2 u + v\cancel{\cos u}\sin u\right) + v\sin^2 u + v\cancel{\sin u}\cos u + v^2\cos^2 u - v\cancel{\sin u}\cos u$$

$$= -\sin u\left(v - \cos u \sin u\right) + \cos^3 u + v\sin^2 u + v^2\cos^2 u$$

$$= -v\sin u + \cos u \sin^2 u + \cos^3 u + v\sin^2 u + v^2\cos^2 u$$

$$= -v\sin u + \cos u\left(1 - \cos^2 u\right) + \cos^3 u + v\left(1 - \cos^2 u\right) + v^2\cos^2 u$$

$$= -v\sin u + \cos u - \cancel{\cos^3 u} + \cancel{\cos^3 u} + v - v\cos^2 u + v^2\cos^2 u$$

$$= v^2\cos^2 u + v\left(1 - \sin u - \cos^2 u\right) + \cos u$$

$\Rightarrow$ affinché sia 0 deve essere:

$$v_{1,2} = \frac{\cos^2 u + \sin u - 1 \pm \sqrt{(1 - \sin u - \cos^2 u)^2 - 4\cos^3 u}}{2\cos^2 u}$$

$\Rightarrow$ Studio $\Delta = (1 - \sin u - \cos^2 u)^2 - 4\cos^3 u$

$\Rightarrow$ se $\cos u = \frac{1 - \sqrt{5}}{2} \approx -0{,}62$ si ha $\sin u \approx 0{,}786$

$\Rightarrow \cos^3 u \approx -0{,}24 \Rightarrow (1 - \sin u - \cos^2 u)^2 \approx 0{,}03$

$\Rightarrow \ (1-\sin u-\cos^2 u)^2-4\cos^3 u \approx -0{,}93<0$

$\Rightarrow$ per $u=\arccos\left(\frac{1-\sqrt{5}}{2}\right)$, NON può essere $l_{11}=0$,

$\Rightarrow \ \nexists$ punti piatti di S

$\Rightarrow$ $\boxed{S \text{ possiede solo punti parabolici e iperbolici}}$