Navier–Stokes Equations – Stationary case

Find $u : \Omega \to \mathbb{R}^d$, $p : \Omega \to \mathbb{R}$ s.t.

$$(NS)\quad \begin{cases} -\nu \Delta u + (u\cdot\nabla)u + \nabla p = f & \text{in } \Omega \\ \nabla\cdot u = 0 & \text{in } \Omega \\ u = g & \text{on } \Gamma_D \\ -\nu(\nabla u)\hat{n} + p\hat{n} = d & \text{on } \Gamma_N \end{cases}$$

weak formulation:

find $(u,p) \in V\times Q$ with $u = g$ on Γ_D s.t.

$$\begin{cases} a(u,v) + c(u,u,v) + b(v,p) = F(v) & \forall v \in V_0 \\ b(u,q) = 0 & \forall q \in Q \end{cases}$$

where $c : V\times V\times V \to \mathbb{R}$ s.t. $c(w,u,v) = \int_\Omega (w\cdot\nabla)u\cdot v \, d\Omega$

Good posedness is given by the following conditions:

1) $a(\cdot,\cdot)$ continuous and coercive on V_0
2) $b(\cdot,\cdot)$ continuous on $V\times Q$
3) inf-sup holds
4) $c(\cdot,\cdot,\cdot)$ continuous on V_0:

$$c(w,u,v) \leqslant C \|\nabla w\|_{L^2}\cdot\|\nabla u\|_{L^2}\cdot\|\nabla v\|_{L^2}$$

__Proposition__:

$\forall w \in V$ s.t. $\nabla\cdot w = 0$, $c(\cdot,\cdot,\cdot)$ is anti-symmetric if $\Gamma_D = \phi$ and $g = 0$

__Recall__:

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \Rightarrow \tilde{u} = u + w \text{ s.t. } w|_{\partial\Omega} = -g$$

$$\Rightarrow Lu = L\tilde{u} - Lw = f \Rightarrow L\tilde{u} = f + Lw$$

$$\Rightarrow \tilde{u}|_{\partial\Omega} = (u+w)|_{\partial\Omega} = g - g = 0 \Rightarrow \tilde{u} = 0 \text{ on } \partial\Omega$$

__Proof__:

We need to show that:

$$\forall w \in V \quad \nabla\cdot w = 0 \wedge \Gamma_N = \phi,\ c(w,u,v) = -c(w,v,u)$$

we have:

$$\Rightarrow c(w,u,v) = \int_\Omega (w\cdot\nabla)u\cdot v\, d\Omega = \int_\Omega \sum_{i,j} w_j \partial u_j \cdot v_j \, d\Omega$$

$$= \sum_{i,j}\int_\Omega w_j \partial_{x_j} u_j \cdot v_j \, d\Omega \underset{\text{by parts}}{=} -\sum_{i,j}\int_\Omega u_j \partial_{x_i}(w_i\cdot v_j)\, d\Omega \; +$$

$$+\sum_{i,j}\int_{\partial\Omega}\cancel{u_j w_i v_j} n_i \, d\Gamma = -\sum_{i,j}\int_\Omega u_j \partial_{x_i} w_i v_j \, d\Omega$$

$u = v = 0$ on $\partial\Omega$

$$-\sum_{i,j}\int_\Omega u_j w_i \partial_{x_i} v_j \, d\Omega = -\int_\Omega \underbrace{(\nabla\cdot w)}_{=0} v\cdot u \, d\Omega - \int_\Omega (w\cdot\nabla) v\cdot u \, d\Omega$$

(w is divergent free)

$$= -c(w,v,u)$$

$\square$

$\Rightarrow$ good posedness is given by:

$\forall f \in H^{-1}(\Omega)$ $\exists (u,p)$ sol. of NS with

$$\|\nabla u\|_{L^2} + \|p\|_{L^2} \leq C\|f\|_{H^{-1}},$$

where $\|f\|_{H^{-1}} = \sup_{v\in H_0^1(\Omega)} \frac{F(v)}{\|\nabla u\|_{L^2}}$

Remark:

The sol. could be NOT unique, sufficient cond. for uniqueness is given by the following:

$$(SD) \quad \frac{C}{\sqrt{2}}\|f\|_{H^{-1}} < 1 \quad \text{(Small data hypothesis)}$$

In Galerkin approximations, it could happen that $\nabla\cdot w \neq 0$. In this case we use a modified trilinear form $\tilde{c}$:

$$\tilde{c}(w,u,v) = c(w,u,v) + \frac{1}{2}\int_\Omega (\nabla\cdot w)\, u\cdot v \, d\Omega$$

$\Rightarrow$ $\tilde{c}$ is strongly consistent w.r.t. the continuum problem:

1) $\tilde{c}(u,u,v) = \underbrace{c(u,u,v)}_{=0} + \frac{1}{2}\int_\Omega \underbrace{(\nabla\cdot u)}_{=0} u\cdot v \, d\Omega = 0$

2) $\tilde{c}(\cdot,\cdot,\cdot)$ is anti-symmetric

If $\Gamma_N \neq \emptyset$, $c(\cdot,\cdot,\cdot)$ and $\tilde{c}(\cdot,\cdot,\cdot)$ are NOT anti-symmetric anymore !!! In this case we have:

$$\Rightarrow \tilde{c}(u,u,v) = \int_{\Gamma_N} |u|^2 u\cdot\hat{n} \, d\Gamma \neq 0$$

$\Rightarrow$ the energy estimate for u holds:

$$\|u\|_{L^2} + \int_{\Gamma_N} |u|^2 u\cdot\hat{n} \, d\Gamma \leq C\|f\|_{H^{-1}}$$

$\Rightarrow$ it's required for the boundary conditions and for f to be compatible the energy estimate !!! Otherwise the sol. does NOT have any physical sense!

GALERKIN APPROXIMATION

$V_h = \text{span}\{\phi_i\}_{i=1}^{N_u} \subseteq V, \quad Q_h = \text{span}\{\psi_k\}_{k=1}^{N_p} \subseteq Q$

$\Rightarrow$ the weak formulation becomes:

find $u_h = \sum_{i=1}^{N_u} \hat{u}_i \phi_i, \; p_h = \sum_{k=1}^{N_p} \hat{p}_k \psi_k, \; (u_h, p_h) \in V_h \times Q_h$ with $u_h|_{\Gamma_D} = g_h$ s.t.:

$$\begin{cases} \sum_{i=1}^{N_u} \hat{u}_i \, a(\phi_i, \phi_j) + \sum_{m,i=1}^{N_u} \hat{u}_m \hat{u}_i \cdot c(\phi_m, \phi_i, \phi_j) + \sum_{k=1}^{N_p} \hat{p}_k \cdot b(\phi_j, \psi_k) = F(\phi_j) \\ \sum_{i=1}^{N_u} \hat{u}_i \, b(\phi_i, \psi_l) = 0 \qquad \forall j = 1, \dots, N_u, \; \forall l = 1, \dots, N_p \end{cases}$$

A, B^T, F, B

$\Rightarrow$ introduce $C(\hat{w}) \in \mathbb{R}^{N_u}$ with $C(\hat{w})_j = \sum_{m,i=1}^{N_u} \hat{w}_m \hat{u}_i \, c(\phi_m, \phi_i, \phi_j)$

$\Rightarrow$ the problem then becomes:

$$\begin{cases} A\hat{u} + C(\hat{u}) + B^T \hat{p} = F \\ B\hat{u} = \vec{0} \end{cases} \quad \text{NON linear !!!}$$

If V_h, Q_h are inf-sup compatible, then the following convergence estimates hold:

$$\|\nabla(u - u_h)\|_{L^2} + \|p - p_h\|_{L^2} \leq C_1 \cdot \inf_{v_h \in V_h} \|\nabla(u - v_h)\| + C_2 \cdot \inf_{\pi_h \in Q_h} \|p - \pi_h\|_{L^2}$$

$$\text{with } C_1, C_2 \propto \frac{1}{\beta_h}$$

RESOLUTION OF THE GALERKIN PROBLEM

1) FIXED POINT ITERATION METHOD:

Consider the Oseen problem:

$w : \Omega \to \mathbb{R}^d$ with $\nabla \cdot w = 0$. Find $u : \Omega \to \mathbb{R}^d, \; p : \Omega \to \mathbb{R}$ s.t.

$$\begin{cases} -\nu \Delta u + (w \cdot \nabla) u + \nabla p = f & \text{in } \Omega \\ \nabla \cdot u = 0 & \text{in } \Omega \\ u = g & \text{on } \Gamma_D \\ -\nu (\nabla u) \cdot \hat{n} + p \cdot \hat{n} = 0 & \text{on } \Gamma_N \end{cases}$$

$\Rightarrow$ let $w \in V$ with $\nabla \cdot w = 0$. Find $(u, p) \in V_h \times Q_h$ with $u|_{\Gamma_D} = g$ s.t.:

$$\begin{cases} a(u,v) + c(w,u,v) + b(v,p) = F(v) & \forall v \in V_0 \\ b(v,q) = 0 & \forall q \in Q \end{cases}$$

$\Rightarrow$ this problem is well posed and the sol. is unique $\forall w \in V$ s.t. $\nabla \cdot w = 0$

Remark:

In the Oseen problem we can also use $\tilde{c}(\cdot,\cdot,\cdot)$ instead of $c(\cdot,\cdot,\cdot)$

$\Rightarrow$ we can define the functional $T: V \longrightarrow V$ s.t. $Tw = u$

$\Rightarrow$ then, the sol. of (NS) is a fixed point of T!!!

$$Tu = u$$

It can be proven that for small data (i.e. $\frac{C}{\nu^2}\|f\|_{H^{-1}} \leq 1$) T is a contraction in $H^1(\Omega)$:

$$\|T(w_1 - w_2)\|_{H^1} \leq \rho \|w_1 - w_2\|_{H^1} \quad \forall w_1, w_2 \in V$$

$\Rightarrow$ by the Contraction Thm. $\exists!\, u \in V$ fixed point of T

$\Rightarrow$ the succession $u^{k+1} = Tu^k$ is s.t. $\{u^k\}_{k\in\mathbb{N}} \xrightarrow[H^1(\Omega)]{} u \quad \forall u^0 \in V$

$\Rightarrow \|u - u^k\|_{H^1} \xrightarrow[k\to+\infty]{} 0$

Moreover, convergence is linear:

$$\|u^k - u^{k+1}\|_{H^1} \leq \rho \|u - u^k\|_{H^1} \leq \dots \leq \rho^k \|u - u^0\|_{H^1}$$

ρ is called convergence rate.

Let $u^0 \in V$ s.t. $\nabla \cdot u^0 = 0 \wedge u^0|_{\Gamma_D} = g$. Find $\forall k \in \mathbb{N}$ $(u^{k+1}, p^{k+1}) \in V \times Q$ with $u^{k+1}|_{\Gamma_D} = g$ s.t.:

$$\begin{cases} \overset{A}{a(u^{k+1}, v)} + c(u^k, u^{k+1}, v) + \overset{B^T}{b(v, p^{k+1})} = \overset{F}{F(v)} & \forall v \in V \\ \underset{B}{b(u^{k+1}, q)} = 0 & \forall q \in Q \end{cases}$$

$\Rightarrow$ as before, we introduce $C(\hat{w}) \in \mathbb{R}^{N_u \times N_u}$ s.t. $C(\hat{w})_{is} = \tilde{c}(w, \phi_i, \phi_s)$ the problem becomes:

$$\begin{cases} A\hat{u}^{k+1} + C(\hat{u}^k)\hat{u}^{k+1} + B^T\hat{p}^{k+1} = F \\ B\hat{u}^{k+1} = \vec{0} \end{cases}$$

$\Rightarrow$ C is anti-symmetric:

$$C(\hat{w})_{is} = \tilde{c}(w_i, \phi_i, \phi_s) = -\tilde{c}(w_i, \phi_s, \phi_i) = -C(\hat{w})_{si}$$

If we take $V_h = \text{span}\left\{ \begin{pmatrix}\phi_i\\0\\0\end{pmatrix}, \begin{pmatrix}0\\\phi_i\\0\end{pmatrix}, \begin{pmatrix}0\\0\\\phi_i\end{pmatrix} \right\}$ then:

$$\tilde{c}\left(w, \begin{pmatrix}\phi_i\\0\\0\end{pmatrix}, \begin{pmatrix}0\\\phi_j\\0\end{pmatrix}\right) = 0$$

$$\Rightarrow C = \begin{pmatrix} C_{11} & 0 & 0 \\ 0 & C_{22} & 0 \\ 0 & 0 & C_{33} \end{pmatrix}, \quad A\ (+C) = \begin{pmatrix} \nu K + C_{11} & & \mathbb{O} \\ & \nu K + C_{22} & \\ \mathbb{O} & & \nu K + C_{33} \end{pmatrix}$$

2) NEWTON'S METHOD:

Recap:

$f: \mathbb{R}^n \to \mathbb{R}^n$ s.t. $f(u) = 0$. Newton's Method consists in:

$$\begin{cases} J(\hat{u}^k)\, \delta u = -f(u^k) \\ u^{k+1} = u^k + \delta u \end{cases}$$

where J is the Jacobian of f ($J_{ij} = \partial_{u_j} f_i$) and $J(u^k)\delta u$ is the directional derivative of f in the δu direction at $\hat{u}^k$
$\Rightarrow \exists\, \mathcal{U}(u) \subseteq \mathbb{R}^n$ s.t. $\forall u^0 \in \mathcal{U}(u)$ the succession $\{u^k\}_{k \in \mathbb{N}} \xrightarrow{k \to +\infty} u$ quadratically:

$$\begin{cases} \|u - u^k\| \xrightarrow[k \to +\infty]{} 0 \\ \|u - u^{k+1}\| \leqslant C \|u - u^k\|^2 \end{cases}$$

For (NS) we have:

$$\mathcal{L}(u,p) - F = 0$$

$$\text{with} \quad \mathcal{L}(u,p) = \begin{pmatrix} -\nu \Delta u + (u \cdot \nabla) u + \nabla p \\ \nabla \cdot u \end{pmatrix}, \quad F = \begin{pmatrix} f \\ 0 \end{pmatrix}$$

$\Rightarrow$ the tangent problem is:

$$(TNS) \begin{cases} D\mathcal{L}_{(u^k, p^k)}(\delta u, \delta p) = F - \mathcal{L}(u^k, p^k) \\ (u^{k+1}, p^{k+1}) = (u^k, p^k) + (\delta u, \delta p) \end{cases}$$

where $D\mathcal{L}_{(u^k,p^k)}(\delta u, \delta p)$ is the directional derivative of $\mathcal{L}(\cdot,\cdot)$ in the direction $(\delta u, \delta p)$ at (u^k, p^k), i.e. it is a Gateaux derivative:

$$D\mathcal{L}_{(u^k,p^k)}(\delta u, \delta p) = \lim_{\varepsilon \to 0} \frac{\Delta \mathcal{L}}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{\mathcal{L}(u^k + \varepsilon \delta u, p^k + \delta p) - \mathcal{L}(u^k, p^k)}{\varepsilon}$$

$$\Rightarrow \Delta\mathcal{L} = \begin{pmatrix} \varepsilon\left(-\nu\Delta\delta u + (\delta u\cdot\nabla)u^k + (u^k\cdot\nabla)\delta u + \nabla\delta p\right) + \varepsilon^2(\delta u\cdot\nabla)\delta u \\ \varepsilon\,\nabla\cdot\delta u \end{pmatrix}$$

$$\Rightarrow D\mathcal{L}_{(u^k,p^k)}(\delta u,\delta p) = \begin{pmatrix} -\nu\Delta\delta u + (\delta u\cdot\nabla)u^k + (u^k\cdot\nabla)\delta u + \nabla\cdot\delta p \\ \nabla\cdot\delta u \end{pmatrix}$$

So the problem becomes:

$$\begin{cases} -\nu\Delta\delta u + (\delta u\cdot\nabla)u^k + (u^k\cdot\nabla)\delta u + \nabla\cdot\delta p = f + \nu\Delta u^k - (u^k\cdot\nabla)u^k - \nabla p^k \\ \nabla\cdot\delta u = -\nabla\cdot u^k \end{cases}$$

$\Rightarrow$ since $(u^{k+1},p^{k+1}) = (u^k,p^k) + (\delta u,\delta p)$ we have:

$$\begin{cases} -\nu\Delta u^{k+1} + (u^k\cdot\nabla)u^{k+1} + (u^{k+1}\cdot\nabla)u^k + \nabla p^{k+1} = f + (u^k\cdot\nabla)u^k \\ \nabla\cdot u^{k+1} = 0 \end{cases}$$

We now pass to the weak formulation:
$u^0 \in U(u)$ with $\nabla\cdot u^0 = 0 \wedge u^0|_{\Gamma_D} = g$. Find $\forall k\in\mathbb{N}$ $(u^{k+1},p^{k+1}) \in V\times Q$ s.t. $u^{k+1}|_{\Gamma_D} = g$ and:

$$\begin{cases} \overset{A}{a(u^{k+1},v)} + \overset{M}{\tilde{c}(u^{k+1},u^k,v)} + \overset{N}{\tilde{c}(u^k,u^{k+1},v)} + \overset{B^T}{b(v,p^{k+1})} = F(v) + \tilde{c}(u^k,u^k,v) \\ \underset{B}{b(u^{k+1},q)} = 0 \quad \forall v\in V_0,\ \forall q\in Q \end{cases}$$

$\Rightarrow$ Introduce the following matrices:

1) $N(\hat{\omega}) \in \mathbb{R}^{N_u\times N_u}$ s.t. $N(\hat{\omega})_{is} = \tilde{c}(\hat{\omega},\phi_i,\phi_s)$
2) $M(\hat{\omega}) \in \mathbb{R}^{N_u\times N_u}$ s.t. $M(\hat{\omega})_{is} = \tilde{c}(\phi_i,\hat{\omega},\phi_s)$
3) $\tilde{F}(\hat{\omega}) \in \mathbb{R}^{N_u}$ s.t. $\tilde{F}(\hat{\omega})_s = F(\phi_s) + \tilde{c}(\hat{\omega},\hat{\omega},\phi_s)$

$\Rightarrow$ the problem becomes:

$$\begin{cases} \left[A + N(\hat{u}^k) + M(\hat{u}^k)\right]\hat{u}^{k+1} + B^T\hat{p}^{k+1} = \tilde{F}(\hat{u}^k) \\ B\hat{u}^{k+1} = \vec{0} \end{cases}$$

where A is symmetric, $N(\hat{\omega})$ is anti-symmetric $\forall\hat{\omega}\in\mathbb{R}^{N_u}$, $M(\hat{\omega})$ doesn't have any symmetric property nor any sparsity pattern, indeed:

$$N(\hat{\omega})_{is} = \tilde{c}\left(\hat{\omega}, \begin{pmatrix}\phi_1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\\phi_1\\0\end{pmatrix}\right) = 0$$

$$M(\hat{\omega})_{is} = \tilde{c}\left(\begin{pmatrix}\phi_1\\0\\0\end{pmatrix}, \hat{\omega}, \begin{pmatrix}0\\\phi_1\\0\end{pmatrix}\right) \neq 0$$

So, finally, the problem in 3D can be written as follows:

$$\left[\begin{array}{ccc|c} \nu K + N_{11} + M_{11} & M_{12} & M_{13} & B_1^T \\ M_{21} & \nu K + N_{11} + M_{11} & M_{23} & B_2^T \\ M_{31} & M_{32} & \nu K + N_{11} + M_{11} & B_3^T \\ \hline B_1 & B_2 & B_3 & \vec{0} \end{array}\right] \cdot \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \\ \hat{p} \end{bmatrix} = \begin{bmatrix} \tilde{F}_1 \\ \tilde{F}_2 \\ \tilde{F}_3 \\ \vec{0} \end{bmatrix}$$

<u>Remark</u>:
For the (NS) tangent problem $\exists\, U(u) \subseteq V$ s.t. $\{u^k\}_{k\in\mathbb{N}}$ (defined by the tangent problem) converges quadratically to u sol. of (NS) $\forall u^0 \in U(u)$.

$$\|u-u^k\|_{H^1} \xrightarrow[k\to+\infty]{} 0 \qquad (\text{if } u^0 \in U(u))$$

$$\|u-u^{k+1}\|_{H^1} \leq C \cdot \|u-u^k\|^2$$

3) SUPG (Streamline Upwind Petrov-Galerkin):

<u>Recap</u>:

$$\begin{cases} Lu = f \\ + \text{B.C.} \end{cases} \quad \text{with } L \text{ a differential operator}$$

$$(\text{e.g. } Lu = -\nu\Delta u + \beta \cdot \nabla u)$$

We study the Péclet local number and stabilize the problem:

$$\mathbb{P}e_h = \frac{h|\beta|}{2\nu} > 1 \Rightarrow \text{the problem is UNSTABLE}$$

$\Rightarrow$ F.E. UNSTABLE: $\langle Lu, v\rangle = \langle f, v\rangle \quad \forall v \in V$

$\Rightarrow$ F.E. STABILIZED: $\langle Lu, v\rangle + \langle Lu - f, L_{SS}v\rangle = \langle f, v\rangle \; \forall v \in V$

where L_{SS} is the antisymmetric part of L

<u>Def.</u> (Symmetric / Anti-Symmetric Operator):
L is symmetric if $\langle Lu, v\rangle = \langle u, Lv\rangle \quad \forall u, v \in V$.
L is anti-symmetric if $\langle Lu, v\rangle = -\langle u, Lv\rangle \; \forall u, v \in V$.

Oseen Problem:

$$\mathcal{L}(w)(u,p) = F$$

where $\mathcal{L}(w)(u,p) = \begin{pmatrix} -\nu\Delta u + (u\cdot\nabla)u + \nabla p \\ \nabla \cdot u \end{pmatrix}, \; F = \begin{pmatrix} f \\ 0 \end{pmatrix}$

__Def.__ (Scalar Product in $V \times Q$):
We define the following:

$$\langle (a,b),(c,d)\rangle := \int_\Omega a\cdot c \, d\Omega + \int_\Omega b\cdot d \, d\Omega$$
$$\forall (a,b),(c,d) \in V\times Q$$

$\Rightarrow$ find $(u,p)\in V\times Q$ s.t.:

$$\langle \mathcal{L}(w)(u,p),(v,q)\rangle = \langle F,(v,q)\rangle \quad \forall (v,q)\in V\times Q$$

$\Rightarrow a(u,v) + \tilde{c}(w,u,v) + b(v,p) - b(u,q) = F(v) \quad \forall (v,q)\in V\times Q$

We introduce the SUPG method for the Oseen problem:

$$\langle \mathcal{L}(w)(u,p),(v,q)\rangle + \sum_{\Omega_i\in\mathcal{T}_i} \delta_{\Omega_i} \langle \mathcal{L}(w)(u,p) - F, \mathcal{L}(w)_{SS}(v,q)\rangle_{\Omega_i}$$
$$= \langle F,(v,q)\rangle \quad \forall (v,q)\in V\times Q$$

__Remark__:
The stabilizer is strongly consistent W.R.T. the sol. (u,p) for the residual:

$$\mathcal{L}(w)(u,p) - F = 0$$

Consider now a fully Dirichlet B.C. (i.e. $\Gamma_N = \phi$). Then:

$$\mathcal{L}(w)(u,p) = \underbrace{\begin{bmatrix} -\nu\Delta u \\ 0 \end{bmatrix}}_{\mathcal{L}_1(w)} + \underbrace{\begin{bmatrix} (w\cdot\nabla)u \\ 0 \end{bmatrix}}_{\mathcal{L}_2(w)} + \underbrace{\begin{bmatrix} \nabla p \\ \nabla\cdot u \end{bmatrix}}_{\mathcal{L}_3(w)}$$

$\Rightarrow \mathcal{L}_2(w)$ is trivially anti-symmetric.

$\Rightarrow \mathcal{L}_1(w)$ is symmetric:

$$\langle \mathcal{L}_1(w)(u,p),(v,q)\rangle = \int_\Omega -\nu\Delta u\cdot v\, d\Omega = \int_\Omega \nu \nabla u : \nabla v \, d\Omega$$
$$= \int_\Omega -\nu\Delta v\cdot u \, d\Omega = \langle (u,p), \mathcal{L}_1(w)(v,q)\rangle$$

$\Rightarrow \mathcal{L}_2(w)$ is anti-symmetric:

$$\langle \mathcal{L}_3(w)(u,p),(v,q)\rangle = \int_\Omega \nabla p\cdot v \, d\Omega + \int_\Omega \nabla\cdot u \, q \, d\Omega$$
$$= \int_\Omega p\nabla v \, d\Omega - \int_\Omega u\cdot\nabla q \, d\Omega = -\langle (u,p), \mathcal{L}_3(w)(v,q)\rangle$$

$\Rightarrow$ so we have $\mathcal{L}_{SS}(w) = \mathcal{L}_2(w) + \mathcal{L}_3(w)$, then:

$$\sum_{\Omega_i\in\mathcal{T}_i} \delta_K \langle \mathcal{L}(w)(u,p) - F, \mathcal{L}_{SS}(w)(v,q)\rangle =$$

$$= \sum_{\Omega_i \in \mathcal{T}_i} \delta_K \int_{\Omega_i} (-\nu \Delta u + (w \cdot \nabla) u + \nabla p - f)((w \cdot \nabla) v + \nabla q) \, d\Omega$$

$$+ \sum_{\Omega_i \in \mathcal{T}_i} \delta_K \int_{\Omega_i} (\nabla \cdot u)(\nabla \cdot v) \, d\Omega$$

Stabilizer for very high Reynolds numbers

Remarks:

1) For (NS) problem the stabilizer is the same one as above exept for the fact that one needs to replace u instead of w !!!

2) $\delta_K \int_{\Omega_i} \nabla p \cdot \nabla q \, d\Omega$ is the Brezzi-Pitkeraste stabilizer, so SUPG can also use discrete NON inf-sup compatible spaces

3) $Re_h = \dfrac{u_{\Omega_i} \cdot h}{\nu}$

4) For inf-sup compatible spaces we have:

$$\delta_K = \delta_{\Omega_i} = \begin{cases} \dfrac{\delta h_i}{u_i} & Re_h \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

For NON inf-sup compatible spaces we have:

$$\delta_K = \delta_{\Omega_i} = \begin{cases} \dfrac{\delta h_i}{u_i} & Re_h \geq 1 \\ \dfrac{\delta h_i^2}{\nu} & \text{otherwise} \end{cases} = \frac{\delta h_i}{u_K} \min\{1, Re_h\}$$