9. CALCOLO DI INTEGRALI MEDIANTE CALCOLO DEI RESIDUI

1) Integrali di funzioni razionali di $\sin x$, $\cos x$:

$$\Rightarrow I = \int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta \quad \text{con } F \text{ razionale}$$

Posto $z = \cos\theta + i\sin\theta = e^{i\theta}$ si ha:

$$|z|^2 = z\cdot\bar{z} = 1 \Rightarrow \bar{z} = z^{-1}$$

$$\cos\theta = \frac{z+\bar{z}}{2} = \frac{z+z^{-1}}{2}, \quad \sin\theta = \frac{z-\bar{z}}{2i} = \frac{z-z^{-1}}{2i}$$

$$dz = i e^{i\theta} d\theta \Rightarrow d\theta = \frac{dz}{ie^{i\theta}} = \frac{z^{-1}dz}{i}$$

N.B.

In generale si ha:

$$\cos(n\theta) = \frac{z^n + \bar{z}^n}{2} = \frac{z^n + z^{-n}}{2}, \quad \sin(n\theta) = \frac{z^n - \bar{z}^n}{2i} = \frac{z^n - z^{-n}}{2i}$$

$\Rightarrow$ L'integrale diventa quindi:

$$\boxed{I = \oint_{\partial B_1} F\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right)\frac{dz}{iz}}$$

esempio:

$$I = \int_0^{2\pi} \frac{1}{\cos\theta - a}\, d\theta \quad \text{con } a > 1 \xrightarrow[z,dz]{} I = \int_{\partial B_1} \frac{1}{\frac{z+z^{-1}}{2} - a}\frac{dz}{iz}$$

$$= \int_{\partial B_1} \frac{2}{i}\frac{dz}{z^2 - 2az + 1} \Rightarrow \text{gli zeri di } f(z) = \frac{1}{z^2-2az+1}$$

sono:

$$z_{1,2} = a \pm \sqrt{a^2-1} \Rightarrow |z_1| > a > 1, \quad |z_2| = \frac{1}{|z_1|} < 1$$

quindi, applicando il Teorema dei Residui si ha:

$$I = 2\pi \cancel{i} \cdot \frac{2}{\cancel{i}} \operatorname{Res}\left(\frac{1}{z^2-2az+1}, z_2\right)$$

$$\Rightarrow \operatorname{Res}\left(\frac{1}{z^2-2az+1}, z_2\right) = \left.\frac{1}{2z-2a}\right|_{z=z_2} = -\frac{1}{2\sqrt{a^2-1}}$$

$$\Rightarrow I = -\frac{2\pi}{\sqrt{a^2-1}}$$

2) Integrali del tipo $\int_0^{+\infty} \frac{f(x)}{x^\alpha}\,dx$, $\alpha \in \mathbb{R}, |\alpha| < 1$:

esempio:

$$I = \int_0^{+\infty} \frac{1}{\sqrt{x}\,(x^2-1)}\,dx \Rightarrow \alpha = \frac{1}{2} < 1,\ g(z) = \frac{1}{\sqrt{z}\,(z^2-1)},$$
$$f(z) = \frac{1}{x^2-1}$$

$\Rightarrow$ g ha un PUNTO DI RAMIFICAZIONE in $z=0$ (a causa di $\sqrt{z}$)

e 2 POLI in $z = \pm 1$

N.B.

Ricordiamo che il dominio massimale di $\sqrt{z}$ è la sua superficie di Riemann (spazio a 2 fogli incollati lungo la semiretta positiva)

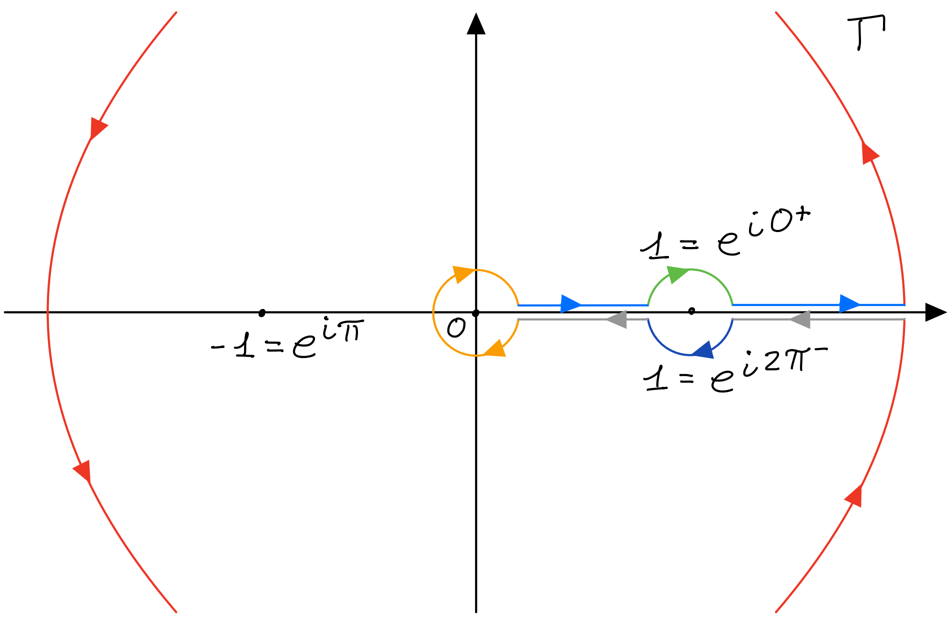
$\Rightarrow$ $z=0$ è il punto in comune ai 2 fogli che definiscono l'estensione analitica di $\sqrt{z}$.

Scegliamo un dominio in cui si possa definire univocamente una delle radici di z:

$$z = \rho e^{i\theta} \Rightarrow z^{\frac{1}{2}} = \rho^{\frac{1}{2}} e^{\frac{i}{2}\theta} \text{ su } \mathbb{C}\setminus\{z = x : x > 0\} =: D$$

e con $\theta = \overset{\text{scegliamo}}{\mathrm{Arg}}(z) \in (0, 2\pi)$ è ben definita

Costruiamo dunque il circuito "Keyhole" $\Gamma \subseteq D$:



Si ha quindi:

$$\oint_{\Gamma} \frac{dz}{\sqrt{z}\,(z^2-1)} = \operatorname{Res}\left(\frac{1}{\sqrt{z}\,(z^2-1)},\ z = e^{i\pi} \right)$$

$$= \int_{\substack{\{z=\rho e^{i0^+}\} \\ \varepsilon \leqslant \rho \leqslant 1+\varepsilon \\ 1+\varepsilon \leqslant \rho \leqslant R}} - \int_{\substack{\{z=e^{i0^+}+\varepsilon e^{i\theta}\} \\ 0 \leqslant \theta \leqslant \pi}} + \int_{\substack{\{z=Re^{i\phi}\} \\ 0<\phi<2\pi}}$$

$$- \int_{\substack{\{z=\rho e^{i2\pi^-}\} \\ \varepsilon \leqslant \rho \leqslant 1-\varepsilon \\ 1+\varepsilon \leqslant \rho \leqslant R}} - \int_{\substack{\{z=e^{i2\pi^-}+\varepsilon e^{i\theta}\} \\ \pi \leqslant \theta \leqslant 2\pi}} - \int_{\substack{\{z=\varepsilon e^{i\phi}\} \\ 0<\phi<2\pi}}$$

$\Rightarrow$ Si ha che $\bullet \xrightarrow[\varepsilon\to 0,\, R\to+\infty]{} I$!!!!

Contributo sul cerchio grande:

$$\left| \int_{\{z=Re^{i\phi}\}} \frac{dz}{z^{\frac{1}{2}}(z^2-1)} \right| \leqslant \int_{\{z=Re^{i\phi}\}} \frac{|dz|}{R^{\frac{1}{2}}(R^2-1)} \leqslant \frac{2\pi R}{R^{\frac{1}{2}}(R^2-1)} \xrightarrow[R\to+\infty]{} 0$$

Contributo sul cerchio piccolo attorno a $z=0$:

$$\left| \int_{\{z=\varepsilon e^{i\phi}\}} \frac{dz}{z^{\frac{1}{2}}(z^2-1)} \right| \leqslant \int_{\{z=\varepsilon e^{i\phi}\}} \frac{|dz|}{\varepsilon^{\frac{1}{2}}(1-\varepsilon^2)} \leqslant \frac{2\pi R}{\varepsilon^{\frac{1}{2}}(\varepsilon^2-1)} \xrightarrow[\varepsilon\to 0]{} 0$$

Contributo sul cerchio piccolo attorno a $z=e^{i0^+}$:

$$\int_{\substack{\{z=e^{i0^+}+\varepsilon e^{i\theta}\} \\ 0<\theta<\pi}} = i\pi \cdot \operatorname{Res}\left(\frac{1}{z^{\frac{1}{2}}(z^2-1)},\ z=e^{i0^+} \right)$$

$$= \frac{i\pi}{e^{i\frac{0^+}{2}}(e^{i0^+}-1)} = i\,\frac{\pi}{2}$$

Contributo sul cerchio piccolo attorno a $z=e^{i2\pi^-}$:

$$\int_{\substack{\{z=e^{i2\pi^-}+\varepsilon e^{i\theta}\} \\ \pi<\theta<2\pi}} = i\pi \cdot \operatorname{Res}\left(\frac{1}{z^{\frac{1}{2}}(z^2-1)},\ z=e^{i2\pi^-} \right)$$

$$= \frac{i\pi}{e^{i\pi^-}(e^{i4\pi^-}-1)} = -i\,\frac{\pi}{2}$$

Contributo sulla semiretta:

$$\int_{\{z=\rho e^{i2\pi^-}\}} \xrightarrow[\varepsilon\to 0,\, R\to+\infty]{\rho,\, d\rho} \int_0^{+\infty} \frac{d\rho}{\rho^{\frac{1}{2}} e^{i\frac{2\pi^-}{2}}(\rho^2 e^{4i\pi}-1)} = -\int_0^{+\infty}\frac{d\rho}{\sqrt{\rho}(\rho^2-1)} = -I$$

$$\Rightarrow \operatorname{Res}\left(g(z),\, z=e^{i\pi}\right) = \frac{1}{e^{i\frac{\pi}{2}}(e^{i\pi}-1)} = \frac{i}{2}, \text{ quindi si ha:}$$

$$\cancel{2}\pi i\cdot\frac{i}{\cancel{2}} = I - i\cancel{\frac{\pi}{2}} - (-I) - \left(-i\cancel{\frac{\pi}{2}}\right) \Rightarrow I = -\frac{\pi}{2}$$

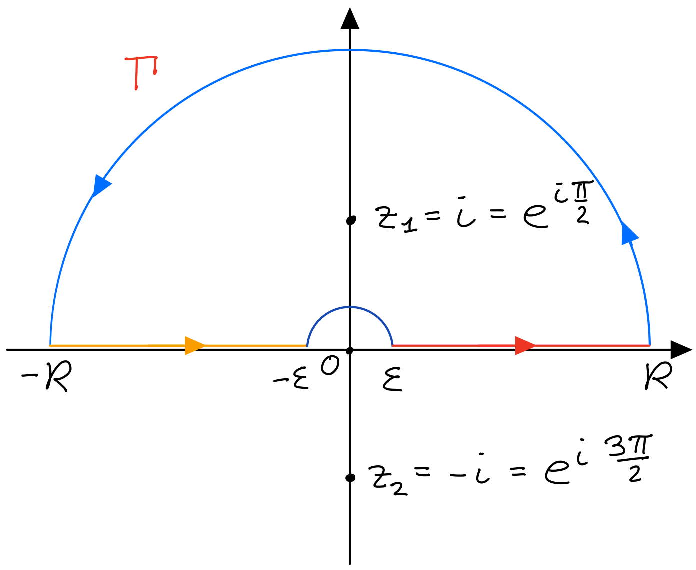
3) Integrali del tipo $\int_0^{+\infty} f(x)\log x\, dx$ con f pari:

esempio:

$$\int_0^{+\infty}\frac{\log x}{x^2+1}dx \Rightarrow f(x) = \frac{1}{x^2+1} \text{ é pari e ha zeri } z_{1,2}=\pm i$$

$$\Rightarrow \text{sia } \Pi \subseteq \mathbb{C}\setminus\{z=x:\ x>0\},\ z=\rho e^{i\theta} \text{ con } 0<\underset{=\operatorname{Arg} z}{\theta}<2\pi$$

$$\Rightarrow \log z = (\text{su } \Pi)\ \operatorname{Log} z = \log\rho + i\theta$$



$\Rightarrow$ si ha:

$$\oint_{\Pi}\frac{\log z}{z^2+1} = 2\pi i\operatorname{Res}\left(\frac{\log z}{z^2+1},\ z=e^{i\frac{\pi}{2}}\right) = \int_{\substack{\{z=\rho e^{i0^+}\}\\ \varepsilon<\rho<R}}$$

$$+\int_{\substack{\{z=Re^{i\phi}\}\\ 0<\phi<\pi}} - \int_{\substack{\{z=\rho e^{i\pi}\}\\ \varepsilon<\rho<R}} - \int_{\substack{\{z=\varepsilon e^{i\theta}\}\\ 0<\theta<\pi}}$$

$$\left|\int_{\{z=Re^{i\phi}\}}\frac{\log z}{z^2+1}\right| \leqslant \pi R\,\frac{\log R+\pi}{R^2} \xrightarrow[R\to+\infty]{} 0$$

$$\left|\int_{\{z=\varepsilon e^{i\theta}\}}\frac{\log z}{z^2+1}\right| \leqslant \pi\varepsilon\,\frac{\log\varepsilon+\pi}{1-\varepsilon^2} \xrightarrow[\varepsilon\to 0]{} 0$$

$$\int_{\substack{\{z=\rho e^{i\pi}\} \\ \varepsilon<\rho<+\infty}} \frac{\log z}{z^2+1} \xrightarrow{\quad dz = e^{i\pi}d\rho = -d\rho \quad} -\int_0^{+\infty} \frac{\log(\rho e^{i\pi})}{1+\rho^2(e^{i2\pi})}\, d\rho = -i\pi\int_0^{+\infty}\frac{d\rho}{1+\rho^2} - \int_0^{+\infty}\frac{\log\rho}{1+\rho^2}\,d\rho$$

$\Rightarrow$ si ha quindi:

$$\mathrm{Res}\left(\frac{\log z}{z^2+1},\ z=e^{i\frac{\pi}{2}}\right) = \frac{\mathrm{Log}(e^{i\frac{\pi}{2}})}{2e^{i\frac{\pi}{2}}} = \frac{\pi}{4}$$

$$\Rightarrow \cancel{2}\pi i \cdot \frac{\pi}{\cancel{4}_2} = I - (-I) + i\pi\int_0^{+\infty}\frac{d\rho}{1+\rho^2} \Rightarrow \text{si tratta}$$

di un'identità tra numeri complessi, quindi si ha:

$$\begin{cases} I = 0 \\ \displaystyle\int_0^{+\infty}\frac{d\rho}{1+\rho^2} = \frac{\pi}{2} \end{cases}$$

SOMME DI SERIE

Oss.:

La funzione $g(z) = \pi\cot(\pi z) = \pi\dfrac{\cos(\pi z)}{\sin(\pi z)}$ ha poli semplici $z_n = n$, $n \in \mathbb{Z}$ e $\mathrm{Res}(g(z), z=n) = \ldots = 1$

Sia $f(x)$ una funzione t.c. il suo prolungamento analitico $f(z)$ verifichi la proprietà di decadimento all'infinito $|f(z)\cdot z| \xrightarrow[|z|\to+\infty]{} 0$ $\left(\Leftrightarrow f(z) = o\left(\frac{1}{|z|}\right)\right)$

In tal caso è possibile calcolare $\displaystyle\sum_{\substack{n\in\mathbb{Z} \\ n\notin \mathrm{Sing}(f)}} f(n)$ mediante la formula:

$$\boxed{\sum_{\substack{n\in\mathbb{Z} \\ n\notin \mathrm{Sing}(f)}} f(n) = -\sum_{z_s\in\mathrm{Sing}(f)} \mathrm{Res}\left(f(z)\pi\cot(\pi z),\ z=z_s\right)}$$

esempio:

$$f(z) = \frac{1}{z^2} \Rightarrow \sum_{n=1}^{+\infty}\frac{1}{n^2} = \frac{1}{2}\sum_{\substack{n\in\mathbb{Z}, \\ n\neq 0}}\frac{1}{n^2} = -\frac{1}{2}\mathrm{Res}\left(\frac{\pi\cot(\pi z)}{n^2},\ z=0\right)$$

$$\frac{\pi\cot(\pi z)}{n^2} = \frac{\pi\cos(\pi z)}{n^2\sin(\pi z)} = \frac{\pi - \frac{\pi^3 z^2}{2} + \frac{\pi^5 z^4}{4!} - \dots}{\pi z^3 - \frac{\pi^3 z^5}{3!} + \frac{\pi^5 z^7}{5!} - \dots} = \dots$$

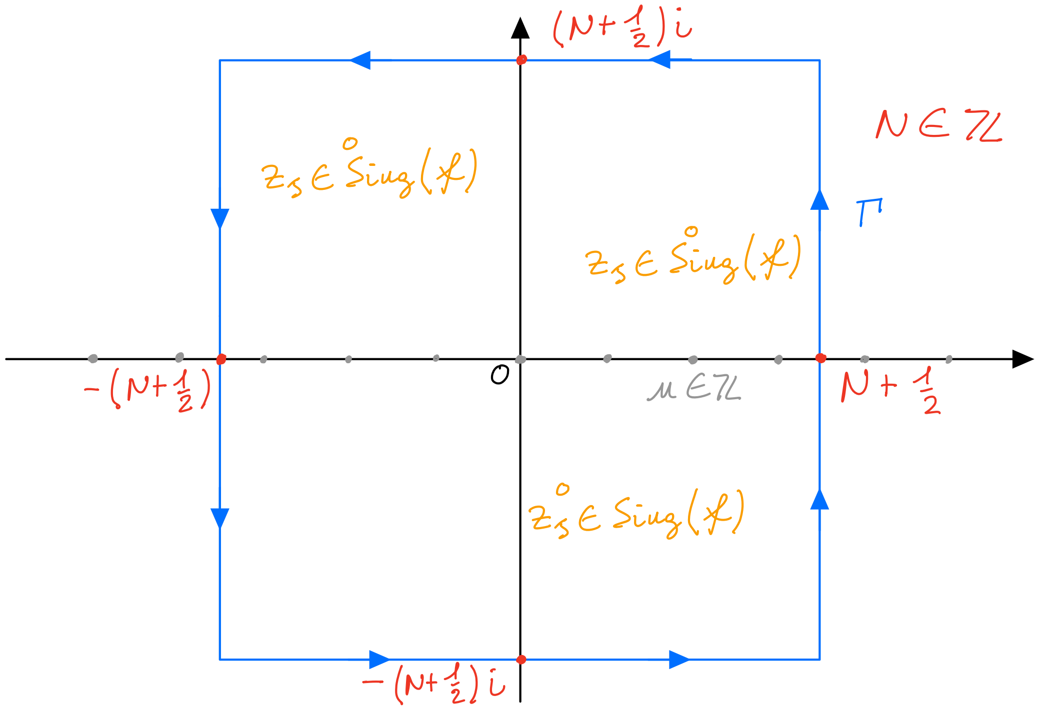
$$= \frac{1}{z^3}\,\frac{1 - \frac{\pi^2 z^2}{2} + \frac{\pi^4 z^4}{4!} - \dots}{1 - \left(\frac{\pi^2 z^2}{3!} + \frac{\pi^4 z^4}{5!} - \dots\right)} = \frac{1}{z^3}\left[1 - \frac{\pi^2 z^2}{2} + \frac{\pi^4 z^4}{4!} - \dots\right]\cdot\Big[\underset{1+}{1+}$$

$$\underbrace{\left(\frac{\pi^2 z^2}{3!} - \frac{\pi^4 z^4}{5!} + \dots\right)}_{t} + \underbrace{\left(\frac{\pi^2 z^2}{3!} - \frac{\pi^4 z^4}{5!} + \dots\right)^2}_{t^2} + \dots\Big]$$

$$= \frac{1}{z^3} + \frac{1}{z}\underbrace{\left(\frac{\pi^2}{3!} - \frac{\pi^2}{2}\right)}_{=\mathrm{Res}} + \dots$$

$$\Rightarrow \sum_{n=1}^{+\infty} \frac{1}{n^2} = -\frac{1}{2}\,\mathrm{Res}\left(\frac{\pi\cot\pi z}{z^2}, z=0\right) = -\frac{1}{2}\left(\frac{\pi^2}{3!} - \frac{\pi^2}{2}\right) = \frac{\pi^2}{6}$$

Derivazione della formula:



Sia $N \longrightarrow +\infty$ t.c. su $\Gamma \equiv \Gamma_N$ non vi siano singolarità di f:

$$\underset{\underset{0}{\parallel}}{\lim_{N\to+\infty}} \oint_{\Gamma_N} f(z)\pi\cot(\pi z)\,dz = 2\pi i \sum_{z_s \in \mathrm{Sing}(f)} \mathrm{Res}\big(f(z)\,\pi\cot(\pi z), z=z_s\big)$$

$$+ \underbrace{2\pi i \sum_{\substack{n\in\mathbb{Z}\\ n\notin \mathrm{Sing}(f)}} \mathrm{Res}\big(f(z)\,\pi\cot(\pi z), z=n\big)}_{2\pi i \sum_{\substack{n\in\mathbb{Z}\\ n\notin \mathrm{Sing}(f)}} f(n) =}$$

Mostriamo che $\oint_{T_N} f(z)\pi\cot(\pi z)\,dz \longrightarrow 0$:

ricordiamo che abbiamo supposto $|f(z)|\cdot|z| \xrightarrow[|z|\to+\infty]{} 0$

$$\Rightarrow \pi\cot(\pi z) = \pi\frac{\cos(\pi z)}{\sin(\pi z)} = \pi i\,\frac{e^{i\pi z}+e^{-i\pi z}}{e^{i\pi z}-e^{-i\pi z}}$$

se $z = x + i(N+\frac{1}{2})$ con $|x| < N+\frac{1}{2}$, si ha:

$$\Rightarrow \pi\cot(\pi z) = \pi i\,\frac{e^{i\pi x}\cdot e^{-N\pi-\frac{\pi}{2}} + e^{-i\pi x}\cdot e^{N\pi+\frac{\pi}{2}}}{e^{i\pi x}\cdot e^{-N\pi-\frac{\pi}{2}} - e^{-i\pi x}\cdot e^{N\pi+\frac{\pi}{2}}}$$

$$\Rightarrow \pi|\cot(\pi z)| \leqslant \pi\,\frac{e^{\pi(N+\frac{1}{2})}+e^{-\pi(N+\frac{1}{2})}}{e^{\pi(N+\frac{1}{2})}-e^{-\pi(N+\frac{1}{2})}} \leqslant \pi\cdot 2 \xrightarrow[N\to+\infty]{} \pi$$

se $z = N+\frac{1}{2}+iy$ con $|y| < N+\frac{1}{2}$, si ha:

$$\Rightarrow \pi\cot(\pi z) = \pi i\,\frac{e^{i\pi N}\cdot \overset{i}{\overset{\shortparallel}{e^{i\frac{\pi}{2}}}} e^{-y} + e^{-i\pi N}\cdot e^{-i\frac{\pi}{2}}{=}{-i}\; e^{y}}{e^{i\pi N}\cdot \underset{i}{\underset{\shortparallel}{e^{i\frac{\pi}{2}}}} e^{-y} - e^{-i\pi N}\cdot \underset{-i}{\underset{\shortparallel}{e^{-i\frac{\pi}{2}}}} e^{y}} = \pi i\,\frac{e^{-y}-e^{y}}{e^{-y}+e^{y}}$$

$$\Rightarrow \pi|\cot(\pi z)| \leqslant \pi$$

Analoghe stime per $\pi\cot(\pi z)$ si ottengono sui rimanenti lati di T_N. Quindi:

$$\left|\oint_{T_N} f(z)\pi\cot(\pi z)\,dz\right| \leqslant \sup_{|z|>N}|f(z)|\cdot 2\pi\cdot(8N+4)$$

$$\leqslant 2\pi\cdot\sup_{|z|>N}|f(z)|\,(8\underset{N<|z|}{|z|}+4) \xrightarrow[N\to+\infty]{} 0$$

$\square$

Oss.:

$\frac{\pi}{\sin(\pi z)}$ ha poli semplici in $n\in\mathbb{Z}$.

$$\mathrm{Res}\left(\frac{\pi}{\sin(\pi z)},\ z=n\right) = \left.\frac{1}{\cos(\pi z)}\right|_{z=n} = (-1)^n$$

$$\Rightarrow \sum(-1)^n f(n) = -\sum_{z_s\in \mathrm{Sing}(f)} \mathrm{Res}\left(\frac{f(z)\cdot\pi}{\sin(\pi z)},\ z=z_s\right)$$

RESIDUI DERIVATA LOGARITMICA DI FUNZIONI MEROMORFE

Sia $f(z)$ MEROMORFA (es. $f(z) = \frac{P(z)}{Q(z)}$ razionale)
$\Rightarrow$ le sue singolarità sono poli

Consideriamo la DERIVATA LOGARITMICA DI f:

$$\boxed{\frac{d}{dz}\log(f(z)) = \frac{f'(z)}{f(z)}}$$

le sue singolarità, oltre ai poli di f, saranno gli zeri di f. Sia z_0 uno zero di f di molteplicità m:

$\Rightarrow f(z) = (z-z_0)^m h(z)$ con $h(z_0) \neq 0$, $h \in \mathcal{O}$

$\Rightarrow f'(z) = m(z-z_0)^{m-1} \cdot h(z) + (z-z_0)^m \cdot h'(z)$

quindi:

$$\frac{f'(z)}{f(z)} = \frac{m}{z-z_0} + \frac{h'(z)}{h(z)} \quad \text{con } \frac{h'(z)}{h(z)} \in \mathcal{O}(U_\varepsilon(z_0))$$

$\Rightarrow$ z_0 è un polo semplice per $\frac{d}{dz}\log(f(z))$ e si ha:

$$\boxed{\mathrm{Res}\left(\frac{f'(z)}{f(z)}, z = z_0\right) = m = \text{molteplicità dello zero } z_0}$$

Sia ora z_1 un polo di ordine m per $f(z)$. Si ha:

$$f(z) = \frac{h(z)}{(z-z_0)^m} \quad \text{con } h(z_0) \neq 0, \ h \in \mathcal{O}(U_\varepsilon(z_0))$$

$\Rightarrow f'(z) = -m(z-z_0)^{-m-1} h(z) + (z-z_0)^{-m} h'(z)$

$\Rightarrow \frac{f'(z)}{f(z)} = -\frac{m}{z-z_0} + \frac{h'(z)}{h(z)}$

$\Rightarrow$ z_1 è polo semplice per $\frac{d}{dz}\log(f(z))$ e si ha:

$$\boxed{\mathrm{Res}\left(\frac{f'(z)}{f(z)}, z = z_1\right) = -m = -\text{ordine del polo } z_1}$$