01) Find the Fourier transform of $f(x) = \begin{cases} 1 & , \ |x| \leq a \\ 0 & , \ |x| > a \end{cases}$

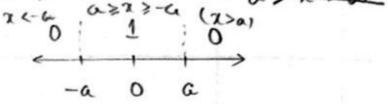
Hence evaluate $\int_0^\infty \frac{\sin x}{x} \, dx$.

$|x| \leq a$ | $|x| > a$
$x \leq a$ | $x > a$
$-x \leq a$ | $-x > a$
$x \geqslant -a$ | $x < -a$

$\Rightarrow a \geqslant x \geqslant -a$ ~~$a > x > a$~~

$$F(S) = \int_{-\infty}^{\infty} f(x) \, e^{isx} \, dx$$

$$F(S) = \int_{-\infty}^{-a} \cancelto{0}{f(x)} \, e^{isx} \, dx + \int_{-a}^{a} f(x) \, e^{isx} \, dx + \int_{a}^{\infty} \cancelto{0}{f(x)} \, e^{isx} \, dx$$



$$F(S) = \int_{-a}^{a} (1) \, e^{isx} \, dx = \left[\frac{e^{isx}}{is} \right]_{-a}^{a} = \frac{1}{is} \left[e^{isa} - e^{-isa} \right]$$

$$F(S) = \frac{1}{is} \left[\cos ax + i \sin as - \cos ax + i \sin as \right] = \frac{1}{is} \left[2i \sin as \right] = \frac{2 \sin as}{s}$$

Now by inverse fourier transform,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(S) \, e^{-isx} \, ds = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2 \sin as}{s} \cdot e^{-isx} \, ds$$

$$f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin as}{s} \cdot e^{-isx} \, ds$$

put $x = 0$, we have $f(0) = 1$.

$$1 = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin as}{s} \, (1) \, ds$$

$$\pi = \int_{-\infty}^{\infty} \frac{\sin as}{s} \, ds$$

$\frac{\sin as}{s}$ is an even function, so,

$$\pi = 2 \int_{0}^{\infty} \frac{\sin as}{s} \, ds$$

put $a = 1$ and $s = x$,

$$\int_0^\infty \frac{\sin x}{x} \, dx = \frac{\pi}{2}$$

Q2. If $f(x) = \begin{cases} 1-x^2 , & |x| < 1 \\ 0 , & |x| \geq 1 \end{cases}$, find $\int_0^\infty \frac{x\cos x - \sin x}{x^3} dx$

$|x|<1$ | $|x| \geq 1$

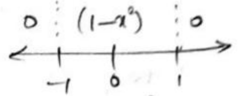
$x<1$ ✓ | $x \geq 1$ ✓

$-x<1$ | $-x \geq 1$

$x>-1$ | $x \leq -1$

$$F(s) = \int_{-\infty}^{\infty} f(x)\, e^{isx}\, ds$$

$$= \int_{-\infty}^{-1} f(x) e^{isx} dx + \int_{-1}^{1} f(x) e^{isx} dx + \int_{+1}^{\infty} f(x)\, e^{isx} dx$$



$$= \int_{-1}^{1} (1-x^2) e^{isx}\, dx$$

$$= \left[(1-x^2) \frac{e^{isx}}{is} - (-2x) \frac{e^{isx}}{(is)^2} + (-2) \frac{e^{isx}}{(is)^3} \right]_{-1}^{1}$$

$$= (0-0) + 2(1) \frac{e^{is}}{i^2 s^2} - 2(-1) \frac{e^{-is}}{i^2 s^2} - 2 \frac{e^{is}}{i^3 s^3} + \frac{2 e^{-is}}{i^3 s^3}$$

$$= 2\left[\frac{e^{is}}{-s^2} + \frac{e^{-is}}{-s^2} - \frac{e^{is}}{-is^3} + \frac{e^{-is}}{-is^3} \right]$$

$$= 2\left[\frac{e^{is} + e^{-is}}{-s^2} - \left[\frac{e^{is} - e^{-is}}{-is^3} \right] \right]$$

$$\left(\because \sin x = \frac{e^{ix} - e^{-ix}}{2i} \qquad \cos x = \frac{e^{ix} + e^{-ix}}{2} \right)$$

$$= 2\left[-2 \frac{\cos s}{s^2} + \frac{2 \sin s}{s^3} \right]$$

$$F(s) = \frac{-4}{s^3} \left[s\cos x - \sin x \right]$$

Applying inverse fourier tranform,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{-4}{s^3} \left[s\cos x - \sin x \right] e^{-isx}\, ds$$

put $x=0$, $f(0) = 1 - 0^2 = 1$.

$$1 = \frac{4}{\pi} \int_0^\infty \frac{-[s\cos x - \sin s]}{s^3}\, ds$$

(since the function is even)

put $s = x$,

$$\int_0^\infty \frac{x\cos x - \sin x}{x^3}\, dx = \frac{-\pi}{4}$$

Q3) Find Fourier transform of $f(x) = \begin{cases} 1-|x| & , |x| \le 1 \\ 0 & , |x| > 1 \end{cases}$ and hence deduce that $\int_0^{\infty} \frac{\sin^2 t}{t^2} dt = \frac{\pi}{2}$

here, $f(x) = \begin{cases} 1+x & , -1 \le x < 0 \\ 1-x & , 0 \le x \le 1 \\ 0 & , x > 1 \text{ and } x < -1. \end{cases}$

$$F(s) = \int_{-\infty}^{\infty} f(x) e^{isx} dx = \int_{-1}^{0} (1+x) e^{-isx} dx + \int_{0}^{1} (1-x) e^{isx} dx$$

$$= \left[(1+x) \frac{e^{isx}}{is} - 1 \frac{e^{isx}}{(is)^2} \right]_{-1}^{0} + \left[(1-x) \frac{e^{isx}}{is} - (-1) \frac{e^{isx}}{(is)^2} \right]_{0}^{1}$$

$$= \left[(1) \frac{(1)}{is} - \frac{1}{i^2 s^2} + \frac{(+1) e^{-is}}{i^2 s^2} \right] + \left[\frac{e^{is}}{i^2 s^2} - \frac{e^{is(0)}}{is} - \frac{1}{i^2 s^2} \right]$$

$$= \frac{1}{is} \left[1 - 1 \right] + \frac{1}{i^2 s^2} \left[-1 + e^{-is} + e^{is} - 1 \right]$$

$$= \frac{2[-1]}{i^2 s^2} + \frac{(2) e^{-is} + e^{is}}{i^2 s^2 (2)} = \frac{+2}{+s^2} + \frac{2 \cos s}{-s^2} = \frac{1}{s^2} \left[2 - 2\cos s \right] = \frac{2}{s^2} \left[1 - \cos s \right]$$

$$F(s) = \frac{2}{s^2} \left[\frac{2 \sin^2 s/2}{1} \right] = \frac{4}{s^2} \left[\sin^2 s/2 \right]$$

Applying inverse fourier transform, $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) e^{-isx} ds$

$$\therefore f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{4}{s^2} \left[\sin^2 \frac{s}{2} \right] e^{-isx} ds$$

put $x = 0$, $f(0) = 1.$ $\Rightarrow$ $1 = \frac{\cancel{2}}{\cancel{2}\pi} \int_{0}^{\infty} \frac{4}{s^2} \left[\sin^2 \frac{s}{2} \right] ds \Rightarrow \int_0^{\infty} \frac{\sin^2 s/2}{s^2} ds = \frac{\pi}{4}$

putting $\frac{s}{2} = t$, $\Rightarrow s = 2t.$

$\frac{1}{2} ds = dt$

$ds = 2dt$

$$\Rightarrow \int_0^{\infty} \frac{\sin^2 t}{4t^2} 2dt = \frac{\pi}{4}$$

$$\rightarrow \int_0^{\infty} \frac{\sin^2 t}{t^2} dt = \frac{\pi}{2}$$

Q4) Find the Fourier transform of $f(x) = \begin{cases} 1 + x/a , & -a < x < 0 \\ 1 - x/a , & 0 < x < a \\ 0 , & \text{otherwise.} \end{cases}$

$$F(S) = \int_{-\infty}^{\infty} f(x) \cdot e^{isx} \, dx$$

$$= \int_{-a}^{0} f(x) e^{isx} dx + \int_{0}^{a} f(x) e^{isx} dx = \int_{-a}^{0} \left(1 + \frac{x}{a}\right) e^{isx} dx + \int_{0}^{a} \left(1 - \frac{x}{a}\right) e^{isx} dx$$

$$= \left[\left(1 + \frac{x}{a}\right) \frac{e^{isx}}{is} - \left(\frac{1}{a}\right) \cdot \frac{e^{isx}}{(is)^2} \right]_{-a}^{0} + \left[\left(1 - \frac{x}{a}\right) \frac{e^{isx}}{is} - \left(\frac{-1}{a}\right) \frac{e^{isx}}{(is)^2} \right]_{0}^{a}$$

$$= \left[(1) \frac{(1)}{is} - \left(\frac{1}{a}\right) \frac{(1)}{(is)^2} - \left[(0) - \left(\frac{1}{a}\right) \frac{e^{-isa}}{(is)^2} \right] \right] + \left[0 - \left(\frac{-1}{a}\right) \frac{e^{ias}}{(is)^2} - \left[(1) \frac{(1)}{is} - \left(\frac{-1}{a}\right) \frac{(1)}{(is)^2} \right] \right]$$

$$= \left[\frac{1}{is} - \frac{1}{a \cdot (is)^2} + \frac{e^{-isa}}{a (is)^2} \right] + \left[\frac{1}{a} \frac{e^{isa}}{(is)^2} - \frac{1}{is} - \frac{1}{a} \left(\frac{1}{(is)^2} \right) \right]$$

$$= \frac{1}{is} \left[1 - 1 \right] + \frac{1}{a (is)^2} \left[-1 + e^{-isa} + e^{isa} - 1 \right]$$

$$= \frac{1}{-a s^2} \left[-2 + e^{isa} + e^{-isa} \right] = \frac{2}{as^2} + \frac{2 \left(e^{isa} + e^{-isa} \right)}{-as^2 (2)} = \frac{2}{as^2} + \frac{2 \cos as}{-as^2}$$

$$F(S) = \frac{2}{as^2} \left[1 - \cos as \right] = \frac{2}{as^2} \left[2 \sin^2 \frac{as}{2} \right] = \frac{4 \sin^2 \frac{as}{2}}{as^2}$$

Q5) Find the inverse Fourier transform of e^{-u^2}.

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(u) \, e^{-iux} \, du$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-u^2} \cdot e^{-iux} \, du = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-u^2 - iux} \, du = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-(u^2 + ui x)} \, du$$

$$u^2 + iux = \left[u^2 + 2(u) \left(\frac{ix}{2} \right) + \left(\frac{ix}{2} \right)^2 \right] - \left(\frac{ix}{2} \right)^2 = \left(u + \frac{ix}{2} \right)^2 + \frac{x^2}{4}$$

$$\therefore f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\left((u + ix/2)^2 + x^2/4 \right)} du = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\left((u + ix/2)^2 \right)} \circ e^{-x^2/4} \, du = \frac{e^{-\frac{x^2}{4}}}{2\pi} \int_{-\infty}^{\infty} e^{-\left(u + \frac{ix}{2}\right)^2} du$$

put $u + ix/2 = t \Rightarrow du = dt.$

at $u = -\infty$, $t = -\infty$, $\Rightarrow$ $f(x) = \frac{e^{-\frac{x^2}{4}}}{2\pi} \int_{-\infty}^{\infty} e^{-t^2} dt = \frac{e^{-x^2/4}}{\pi} \int_{0}^{\infty} e^{-t^2} dt = \frac{e^{-x^2/4}}{\pi} \times \frac{\sqrt{\pi}}{2} = \frac{e^{-x^2/4}}{2\sqrt{\pi}}$

06) Obtain the Fourier cosine transform of $f(x) = \begin{cases} 4x, & 0 < x < 1 \\ 4-x, & 1 < x < 4 \\ 0, & x > 4 \end{cases}$

$$F_c(S) = \int_0^\infty f(x)\cos sx \, dx$$

$$= \int_0^1 (4x)\cos sx \, dx + \int_1^4 (4-x)\cos sx \, dx + \int_4^\infty (0)\cos sx \, dx$$

$$= \left[(4x)\frac{\sin sx}{s} - 4\left[\frac{-\cos sx}{s^2}\right]\right]_0^1 + \left[(4-x)\frac{\sin sx}{s} - (-1)\frac{-\cos sx}{s^2}\right]_1^4$$

$$= \left[4\frac{\sin s}{s} + 4\frac{\cos s}{s^2} + \left[-\frac{(4)}{s^2}\right]\right] + \left[\frac{-\cos 4s}{s^2} - \left[3\frac{\sin 3x}{s} - \frac{\cos 3s}{s^2}\right]\right]$$

$$= \frac{4\sin s}{s} + \frac{4\cos s}{s^2} - \frac{4}{s^2} \frac{-\cos 4s}{s^2} - \frac{3\sin 3x}{s} + \frac{\cos 3s}{s^2}$$

$$= \frac{1}{s}\left[4\sin s - 3\sin 3s\right] + \frac{1}{s^2}\left[4\cos s - 4 - \cos 4s + \cos 3s\right]$$

$$= \frac{1}{s}\left[\sin s\right] + \frac{1}{s^2}\left[(4+1)\cos s - 4 - \cos 4s\right]$$

$$F_c(S) = \frac{\sin s}{s} + \frac{5\cos s - 4 - \cos 4s}{s^2}$$

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07) Find the Fourier sine transform of $f(x) = e^{-|x|}$.

Hence evaluate $\int_0^\infty \frac{x\sin mx}{1+x^2}\,dx$, $m > 0$.

$$F_s(S) = \int_0^\infty f(x)\sin sx \, dx$$

$$F_s(S) = \int_0^\infty e^{-|x|}\sin sx \, dx$$

In the interval $[0, \infty)$, x is positive $\Rightarrow e^{-|x|} = e^{-x}$.

$$\therefore \quad F_s(s) = \int_0^\infty e^{-x} \sin sx \, dx$$

$$= \left[\frac{e^{-x}}{1+s^2} \left[-\sin sx - s \cos sx \right] \right]_0^\infty \qquad \left(\because \int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2} \left[a \sin bx - b \cos bx \right] \right)$$

$$= \frac{0 - 1(-0-s)}{1+s^2}$$

$$F_s(s) = \frac{s}{1+s^2}$$

Applying inverse Fourier sine transform

$$f(x) = \frac{2}{\pi} \int_0^\infty F_s(s) \sin sx \, ds$$

$$e^{-x} = \frac{2}{\pi} \int_0^\infty \left(\frac{s}{1+s^2} \right) \sin sx \, ds$$

put $x = m$, $$e^{-m} = \frac{2}{\pi} \int_0^\infty \frac{s}{1+s^2} \sin sm \, ds$$

$$\Rightarrow \int_0^\infty \left(\frac{s}{1+s^2} \right) \sin sm \, ds = \frac{\pi e^{-m}}{2}$$

putting $s = x$, we have

$$\int_0^\infty \frac{x \sin xm}{(1+x^2)} \, dx = \frac{\pi e^{-m}}{2}$$

Q8 Find the inverse Fourier sine transform of $\hat{f}_s(\alpha) = \frac{1}{\alpha} e^{-a\alpha}$, $a > 0$

$$f(x) = \frac{2}{\pi} \int_0^\infty F_s(s) \sin sx \, dx$$

$$= \frac{2}{\pi} \int_0^\infty f_s(\alpha) \sin \alpha x \, d\alpha$$

$$f(x) = \frac{2}{\pi} \int_0^\infty \frac{e^{-a\alpha}}{\alpha} \sin \alpha x \, d\alpha \longrightarrow (1)$$

Consider the integral. $I = \int_0^\infty \frac{e^{-a\alpha}}{\alpha} \sin \alpha x \, d\alpha$

diff wrt x, $I' = \int_0^\infty \frac{e^{-a\alpha}}{\alpha} \alpha \cos \alpha x \, d\alpha$

$$I' = \int_0^\infty e^{-a\alpha} \cos \alpha x \, d\alpha$$

Using the formula, $\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2} [a \cos bx + b \sin bx]$,

$$I' = \left[\frac{e^{-a\alpha}}{(-a)^2 + x^2} [-a \cos \alpha x + x \sin \alpha x] \right]_0^\infty$$

$$I' = \left[0 - \frac{1}{a^2+x^2} (0 - a) \right] = \frac{a}{a^2+x^2}$$

$$\therefore \quad I = \int \frac{a}{a^2+x^2} dx = \frac{1}{a} \times \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c = \tan^{-1}\left[\frac{x}{a}\right] + c$$

put $x = 0$, $I = \tan^{-1}(0) + c \Rightarrow \underline{\underline{0 = c}}$

$$\therefore \quad I = \tan^{-1}(x/a)$$

$\Rightarrow$ $f(x)$ from (1) becomes. $f(x) = \frac{2}{\pi} \times \tan^{-1}\left(\frac{x}{a}\right)$

09) Find the Fourier sine and cosine transform of $f(x) = \begin{cases} x , & 0<x<2 \\ 0 , & \text{Otherwise} \end{cases}$

$$F_S(S) = \int_0^\infty f(x) \sin sx \, dx$$

$$= \int_0^{\cancel{\infty}2} (x) \sin sx \, dx + \int_2^\infty (0) \sin sx \, dx$$

$$= \left[x\left[-\frac{\cos sx}{s}\right] - 1\left[-\frac{\sin sx}{s^2}\right] \right]_0^2 = \left[\frac{-2\cos 2s}{s} + \frac{\sin 2s}{s^2} + 0 \right]$$

$$\therefore F_S(S) = \frac{1}{s^2}[\sin 2s] - \frac{1}{s}[2\cos 2s]$$

$$F_C(S) = \int_0^\infty f(x) \cos sx \, dx$$

$$= \int_0^2 (x) \cos sx \, dx + \int_2^\infty (0) \cos sx \, dx$$

$$= \left[x\left[\frac{\sin sx}{s}\right] - 1\left[\frac{-\cos sx}{s^2}\right] \right]_0^2 = \left[\frac{2\sin 2s}{s} + \frac{\cos 2s}{s^2} - \frac{1}{s^2} \right]$$

$$\therefore F_C(S) = \frac{1}{s}\left[2\sin 2s\right] + \frac{1}{s^2}\left[\cos 2s - 1\right]$$

10) Find the Z transform of the following:

a) i) $a^{-n} \cos n\theta$

We know that $Z_T(a^n) = \frac{Z}{Z-a}$

Consider $Z_T(e^{in\theta}) = Z_T[(e^{i\theta})^n]$, here $a = e^{i\theta} = \cos\theta + i\sin\theta$

$$= \frac{Z}{Z - e^{i\theta}}$$

$$= \frac{Z}{Z - [\cos\theta + i\sin\theta]}$$

$$= \frac{Z}{[Z-\cos\theta] + i[\sin\theta]} \times \frac{[Z-\cos\theta] - i\sin\theta}{[Z-\cos\theta] - i\sin\theta}$$

$$Z_T[e^{in\theta}] = \frac{z(z-\cos\theta) + i z\sin\theta}{(z-\cos\theta)^2 + \sin^2\theta}$$

$$Z_T[e^{in\theta}] = \frac{z[z-\cos\theta] + i z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

$$Z_T[\cos n\theta + i\sin n\theta] = \frac{z[z-\cos\theta] + i z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

$$Z_T[\cos n\theta] + i\, Z_T[\sin n\theta] = \frac{z[z-\cos\theta]}{z^2-2z\cos\theta+1} + i\,\frac{z\sin\theta}{z^2-2z\cos\theta+1}$$

On comparing, we have,

$$Z_T[\cos n\theta] = \frac{z[z-\cos\theta]}{z^2-2z\cos\theta+1}$$ and

$$Z_T[\sin n\theta] = \frac{z\sin\theta}{z^2-2z\cos\theta+1}$$

Now, Damping rule says that, If $Z_T(u_n) = U(z)$, then $Z_T[a^{-n}u_n] = U[az]$

So, $$Z_T[a^{-n}\cos n\theta] = \frac{az[az-\cos\theta]}{(az)^2 - 2(az)\cos\theta + 1}$$

ii) $a^n\sin n\theta$

Also, $$Z_T[a^n\sin n\theta] = \frac{a^{-1}z\sin\theta}{(a^{-1}z)^2 - 2a^{-1}z\cos\theta + 1} = \frac{\frac{z}{a}\sin\theta}{\frac{z^2}{a^2} - \frac{2za}{a^2}\cos\theta + \frac{a^2}{a^2}}$$

$$= \frac{\frac{z}{a}\sin\theta}{\frac{z^2-2az\cos\theta+a^2}{a^2}} = \frac{z\sin\theta}{a} \times \frac{a^2}{z^2-2az\cos\theta+a^2}$$

$$= \frac{za\sin\theta}{z^2-2az\cos\theta+a^2} \quad //$$

b) $2n + \sin\left(n\frac{\pi}{4}\right) + 1$

$$Z_T\left[2n + \sin\left(n\frac{\pi}{4}\right) + 1\right] = Z_T[2n] + Z_T\left[\sin\left(n\frac{\pi}{4}\right)\right] + Z_T[1]$$

$$= 2 Z_T[n] + Z_T\left[\sin\left(n\frac{\pi}{4}\right)\right] + Z_T(1)$$

$$= 2\left[\frac{Z}{(Z-1)^2}\right] + \frac{Z\sin\pi/4}{Z^2 - 2Z\cos\pi/4 + 1} + \frac{Z}{Z-1}$$

$$= \frac{2Z}{(Z-1)^2} + \frac{Z\left(\frac{1}{\sqrt{2}}\right)}{Z^2 - 2Z\left(\frac{1}{\sqrt{2}}\right) + 1} + \frac{Z}{Z-1}$$

$$= \frac{2Z}{(Z-1)^2} + \frac{Z\left(\frac{1}{\sqrt{2}}\right)}{Z^2 - \sqrt{2}Z + 1} + \frac{Z(Z-1)}{(Z-1)(Z-1)}$$

$$= \frac{2Z + Z^2 - Z}{(Z-1)^2} + \frac{Z\left(\frac{1}{\sqrt{2}}\right)}{Z^2 + (-\sqrt{2})Z + 1}$$

$$= \frac{Z^2 + Z}{(Z-1)^2} + \frac{Z\left(\frac{1}{\sqrt{2}}\right)}{Z^2 - \sqrt{2}Z + 1}$$

$$= \frac{Z(Z+1)}{(Z-1)^2} + \frac{Z/\sqrt{2}}{Z^2 - \sqrt{2}Z + 1}$$

c) $\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)$

$$Z_T\left[\cos\left(n\frac{\pi}{2}+\frac{\pi}{4}\right)\right] = Z_T\left[\cos n\frac{\pi}{2}\cos\frac{\pi}{4} - \sin n\frac{\pi}{4}\sin\frac{\pi}{4}\right]$$

$$= \cos\frac{\pi}{4}\cdot Z_T\left[\cos n\frac{\pi}{2}\right] - \sin\frac{\pi}{4}\cdot Z_T\left[\sin n\frac{\pi}{2}\right].$$

$$= \frac{1}{\sqrt{2}}\left[\frac{z(z-\cos\pi/2)}{z^2+2z\cos\frac{\pi}{2}+1}\right] - \frac{1}{\sqrt{2}}\left[\frac{z\sin\pi/2}{z^2-2z\cos\frac{\pi}{2}+1}\right]$$

$$= \frac{1}{\sqrt{2}}\left[\frac{z(z-0) - z}{z^2+2z(0)+1}\right]$$

$$= \frac{1}{\sqrt{2}}\left[\frac{z^2-z}{z^2+1}\right].$$

$$= \frac{1}{\sqrt{2}}\,\frac{z(z-1)}{(z^2+1)}$$

d) $\sin(3n+5)$

$$Z_T[\sin(3n+5)] = Z_T[\sin 3n\cos 5 + \cos 3n\sin 5]$$

$$= \cos 5\; Z_T[\sin 3n] + \sin 5\; Z_T[\cos 3n]$$

$$= \cos 5\cdot\left[\frac{z\sin 3}{z^2-2z\cos 3+1}\right] + \sin 5\left[\frac{z(z-\cos 3)}{z^2-2z\cos 3+1}\right]$$

$$= \frac{z\cdot\cos 5\sin 3 + z^2\sin 5 - z\cos 3\sin 5}{z^2-2z\cos 3+1}$$

$$= \frac{z\cdot[\cos 5\sin 3 - \cos 3\sin 5) + z^2\sin 5}{z^2-2z\cos 3+1}$$

$$Z_T[\sin(3n+5)] = z\cdot\frac{(z\sin 5 - \sin 2)}{(z^2-2z\cos 3+1)}$$

e) i) cosh nθ

$$Z_T[\cosh n\theta] = Z_T\left[\frac{e^{n\theta}+e^{-n\theta}}{2}\right] = \frac{1}{2}\left[Z_T[e^{n\theta}] + Z_T[e^{-n\theta}]\right]$$

$$= \frac{1}{2}\left[Z_T[(e^{\theta})^n\cdot(1)] + Z_T[(e^{-\theta})^n\cdot(1)]\right]$$

It is of the form. $Z_T[k^n u_n]$ and it is found by using Damping rule. i.e. If $Z_T[u_n] = U(z)$ then $Z_T[k^n u_n] = U(\frac{z}{k})$.
here $k = e^{\theta}$.

$$\Rightarrow Z_T[\cosh n\theta] = \frac{1}{2}\left[\frac{z/e^{\theta}}{\frac{z}{e^{\theta}}-1}\right] + \frac{1}{2}\left[\frac{z/e^{-\theta}}{\frac{z}{e^{-\theta}}-1}\right] \qquad \left(\because Z_T(1) = \frac{z}{z-1}\right)$$

$$= \frac{1}{2}\left[\frac{ze^{-\theta}}{ze^{-\theta}-1}\right] + \frac{1}{2}\left[\frac{ze^{\theta}}{ze^{\theta}-1}\right]$$

$$= \frac{1}{2}\left[\frac{ze^{-\theta}(ze^{\theta}-1) + ze^{\theta}(ze^{-\theta}-1)}{(ze^{-\theta}-1)(ze^{\theta}-1)}\right]$$

$$= \frac{1}{2}\left[\frac{z^2 - ze^{-\theta} + z^2 - ze^{\theta}}{z^2 - z[e^{\theta}+e^{-\theta}]+1}\right]$$

$$= \frac{1}{2}\left[\frac{2z^2 - z[e^{-\theta}+e^{\theta}]}{z^2 - z[e^{\theta}+e^{-\theta}]+1}\right]$$

$$= \frac{z^2 - z\cosh \cancel{n}\theta}{z^2 - 2z\cosh \cancel{n}\theta + 1}$$

(ii) sinh nθ.

Similarly, $Z_T[\sinh n\theta] = \dfrac{z \sinh \cancel{n}\theta}{z^2 - 2z\cosh \cancel{n}\theta + 1}$

f) $\cosh\left(n\frac{\pi}{2}+\theta\right)$

since $Z_T(a^n) = \frac{z}{z-a}$; $\therefore$ $Z_T\left[e^{n\pi/2}\right] = Z_T\left[(e^{\pi/2})^n\right] = \frac{z}{z-e^{\pi/2}}$

and $Z_T\left[e^{-n\pi/2}\right] = Z_T\left[(e^{-\pi/2})^n\right] = \frac{z}{z-e^{-\pi/2}}$

Thus, $Z_T\left[\cosh\left(n\frac{\pi}{2}+\theta\right)\right] = \frac{1}{2}\left\{e^{\theta}\cdot\frac{z}{z-e^{\pi/2}} + e^{-\theta}\cdot\frac{z}{z-e^{-\pi/2}}\right\}$

$$= \frac{z}{2}\left[\frac{z[e^{\theta}+e^{-\theta}] - [e^{(\pi/2-\theta)} + e^{-(\pi/2-\theta)}]}{z^2 - z(e^{\pi/2}+e^{-\pi/2}) + 1}\right]$$

$$= \frac{z^2\cosh\theta - z\cosh(\pi/2-\theta)}{z^2 - 2z\cosh\left(\frac{\pi}{2}\right) + 1}$$

11) If $\bar{u}(z) = \frac{2z^2+3z+12}{(z-1)^4}$, find the values of u_0, u_1, u_2, u_3.

$$u_0 = \lim_{z\to\infty} \bar{u}(z) = \lim_{z\to\infty} \frac{2z^2+3z+12}{(z-1)^4} = \operatorname{lt}_{z\to\infty} \frac{z^2(2+3z^{-1}+12z^{-2})}{z^4(1-z^{-1})^4}$$

$$u_0 = \lim_{z\to\infty} \frac{2+3z^{-1}+12z^{-2}}{z^2(1-z^{-1})^4} = \underline{\underline{0}}$$

$$u_1 = \lim_{z\to\infty} z\left[\bar{u}(z) - u_0\right] = \lim_{z\to\infty} z\left[\frac{2z^2+3z+12}{(z-1)^4} - 0\right] = \operatorname{lt}_{z\to\infty} \frac{2z^3+3z^2+12z}{(z-1)^4}$$

$$= \lim_{z\to\infty} \frac{z^3(2+3z^{-1}+12z^{-2})}{z^4(1-z^{-1})^4} = \lim_{z\to\infty} \frac{2+3z^{-1}+12z^{-2}}{z(1-z^{-1})^4} = \underline{\underline{0}}$$

$$u_2 = \lim_{z\to\infty} z^2\left[\bar{u}(z) - u_0 - u_1 z^{-1}\right] = \lim_{z\to\infty} z^2\left[\frac{2z^2+3z+12}{(z-1)^4}\right] = \operatorname{lt}_{z\to\infty} \frac{z^4[2+3z^{-1}+12z^{-2}]}{z^4[1-z^{-1}]^4}$$

$$= \lim_{z\to\infty} \frac{2+\frac{3}{z}+\frac{12}{z^2}}{\left(1-\frac{1}{z}\right)^4} = \underline{\underline{2}}$$

$$u_3 = \lim_{z\to\infty} z^3\left[\bar{u}(z) - u_0 - u_1 z^{-1} - u_2 z^{-2}\right]$$

$$= \lim_{z\to\infty} z^3\left[\frac{2z^2+3z+12}{(z-1)^4} - \frac{2}{z^2}\right]$$

$$= \lim_{z\to\infty} z^3\left[\frac{(2z^2+3z+12)z^2 - 2(z-1)^4}{z^2(z-1)^4}\right]$$

$$= \lim_{z\to\infty} z\left[\frac{2z^4+3z^3+12z^2 - 2[z^4-4z^3+6z^2-4z+1]}{(z-1)^4}\right]$$

$$= \lim_{z\to\infty} \frac{z\left[\cancel{2z^4}+3z^3+\cancel{12z^2}-\cancel{2z^4}+8z^3-\cancel{12z^2}+8z-2\right]}{(z-1)^4}$$

$$= \lim_{z\to\infty} \frac{z\,[11z^3+8z-2]}{(z-1)^4}$$

$$= \lim_{z\to\infty} \frac{z(z^3)\,[11+8z^{-2}-2z^{-3}]}{z^4[1-z^{-1}]^4}$$

$$= \lim_{z\to\infty} \frac{11+8z^{-2}-2z^{-3}}{[1-z^{-1}]^4}$$

$$= \underline{\underline{11}}$$

12) If $Z_T(u_n) = \frac{z}{z-1} + \frac{z}{z^2+1}$, then find the Z-transform of u_{n+2}

We know that, $Z_T(u_{n+k}) = z^k\left[\bar{u}(z) - \sum_{k=0}^{k-1} u_r z^{-r}\right]$

here, $k=2$.

$$\therefore Z_T[u_{n+2}] = z^2\left[\bar{u}(z) - \sum_{r=0}^{1} u_r z^{-r}\right]$$

$$Z_T(u_{n+2}) = z^2\left[\bar{u}(z) - (u_0 z^0 + u_1 z^{-1})\right] = z^2\left[\bar{u}(z) - u_0 - \frac{u_1}{z}\right]$$

$$u_0 = \lim_{z\to\infty} \bar{u}(z) = \lim_{z\to\infty}\left[\frac{z}{z-1} + \frac{z}{z^2+1}\right] = \lim_{z\to\infty}\left[\frac{1}{1-z^{-1}} + \frac{1}{z(1+z^{-2})}\right] = 1+0 = \underline{\underline{1}}$$

$$u_1 = \lim_{z\to\infty} z[\bar{u}(z) - u_0] = \lim_{z\to\infty} z\left[\frac{z}{z-1} + \frac{z}{z^2+1} - 1\right] = \lim_{z\to\infty}\left[\frac{z^2}{z-1} + \frac{z^2}{z^2+1} - z\right]$$

$$= \lim_{z\to\infty}\left[\frac{z^2(z^2+1)(1) + z^2(z-1)(1) + (-z)(z-1)(z^2+1)}{(z-1)(z^2+1)}\right]$$

$$= \lim_{z\to\infty}\left[\frac{z^4+z^2+z^3-z-[z^2-z][z^2+1]}{(z-1)(z^2+1)}\right]$$

$$= \lim_{z\to\infty}\left[\frac{\cancel{z^4}+\cancel{z^2}+z^3-\cancel{z}-\cancel{z^4}+\cancel{z}-\cancel{z^2}+z^3}{(z-1)(z^2+1)}\right]$$

$$= \lim_{z\to\infty}\left[\frac{2z^3}{z^3(1-1/z)(1+1/z^2)}\right]$$

$$= \underline{\underline{2}}$$

Hence, $$Z_T(u_{n+2}) = z^2\left[\bar{u}(z) - u_0 - \frac{u_1}{z}\right]$$

$$= z^2\left[\frac{z}{z-1} + \frac{z}{z^2+1} - 1 - \frac{2}{z}\right]$$

$$= z^2\left[\frac{z(z^2+1)(z) + z(z-1)(z) - 1(z^2+1)(z)(z-1) - 2(z-1)(z^2+1)}{z(z-1)(z^2+1)}\right]$$

$$= z^2\left[\frac{z^4+z^2+z^3-z^2-1[(z^2-z)(z^2+1)] - 2[z^3-z^2+z+1(-1)]}{(z^2-z)(z^2+1)}\right]$$

$$= z^2\left[\frac{z^4+z^2+z^3-z^2-z^4-z^2+z^3+z-2z^3-2z+2z^2+2}{(z^4+z^2-z^3-z)}\right]$$

$$= \frac{z^2(z^2-z+2)}{z^4+z^2-z^3+z} = \frac{z^3-1z^2+2z}{z^3-z^2+z-1}$$

$$Z_T(u_{n+2}) = \underline{\underline{\frac{z(z^2-z+2)}{(z-1)(z^2+1)}}}$$

13) Find the inverse Z-transforms of the following.

iii) $\dfrac{3z^2+2z}{(5z-1)\ (5z+2)}$

Consider, $\dfrac{(3z+2)}{(5z-1)(5z+2)} = \dfrac{A}{(5z-1)} + \dfrac{B}{(5z+2)}$

$$3z+2 = A(5z+2) + B(5z-1)$$

$$3z+2 = (5A+5B)z + 2A - B$$

$\therefore\ A+B = \frac{3}{5}$ and $2A-B=2$

$2A = 2+B$

$A = \frac{2+B}{2}$

$$\frac{2+B}{2} + B = \frac{3}{5}$$

$$1 + \frac{3B}{2} = \frac{3}{5} \Rightarrow \frac{3B}{2} = \frac{-2}{5}$$

$$\underline{\underline{B = \frac{-4}{15}}}$$

hence,

$$\underline{\underline{A = 13/15}}$$

$$\bar{u}(z) = \frac{z\ (3z+2)}{(5z-1)\ (5z+2)}$$

$$\frac{\bar{u}(z)}{z} = \frac{(3z+2)}{(5z-1)(5z+2)}$$

$$\frac{\bar{u}(z)}{z} = \frac{A}{(5z-1)} + \frac{B}{(5z+2)}$$

$$\therefore\ \frac{\bar{u}(z)}{z} = \frac{13/15}{(5z-1)} + \frac{-4/15}{(5z+2)}$$

$$\bar{u}(z) = \frac{z\left[13/15\right]}{(5z-1)} + \frac{z\left[-4/15\right]}{(5z+2)}$$

$$Z_T^{-1}\left[\bar{u}(z)\right] = \frac{13}{15}\left[\frac{1}{5} Z_T^{-1}\left[\frac{z}{(z-1/5)}\right]\right] - \frac{4}{15}\left[\frac{1}{5} Z_T^{-1}\left[\frac{z}{z-(\frac{-2}{5})}\right]\right]$$

$$\underline{\underline{Z_T^{-1}\left[\bar{u}(z)\right] = \frac{13}{75}\left(\frac{1}{5}\right)^n - \frac{4}{75}\left(\frac{-2}{5}\right)^n}} \qquad \left(\because\ Z_T^{-1}\left(\frac{z}{z-a}\right) = a^n\right)$$

v) $z \log\left(\frac{z}{z+1}\right)$ $\left(\because \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right)$

$$\log\left(\frac{Z}{Z+1}\right) = \log\left(\frac{1}{1+z^{-1}}\right) = \log\left((1+z^{-1})^{-1}\right) = -\log(1+z^{-1})$$

$$= -\left[\frac{1}{z} + \frac{-1/z^2}{2} + \frac{1/z^3}{3} + \cdots\cdots\right]$$

$$\log\left(\frac{Z}{Z+1}\right) = \left[\frac{-1}{z} + \frac{Z^{-2}}{2} - \frac{Z^{-3}}{3} + \cdots\right]$$

so, $\bar{u}(z) = z \log\left(\frac{z}{z+1}\right)$

$$= z\left[\frac{-1}{z} + \frac{Z^{-2}}{2} + \frac{-Z^{-3}}{3} + \cdots\right]$$

$$= -1 + \frac{Z^{-1}}{2} - \frac{Z^{-2}}{3} + \cdots$$

$$Z_T^{-1}[\bar{u}(z)] = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n+1)} Z^{-n}$$

$$\therefore u(n) = u_n = \frac{(-1)^{n+1}}{(n+1)}.$$

vii) show that $Z_T^{-1}\left[z\left[e^{1/z}-1\right]\right] = \frac{1}{(n+1)!}$

We know that, $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$$\Rightarrow e^{1/z} = 1 + \frac{1}{z} + \frac{1/z^2}{2!} + \frac{1/z^3}{3!} + \cdots\cdots$$

$$e^{1/z} = 1 + z^{-1} + \frac{z^{-2}}{2!} + \frac{z^{-3}}{3!} + \cdots$$

$$e^{1/z} - 1 = z^{-1} + \frac{z^{-2}}{2!} + \frac{z^{-3}}{3!} + \cdots$$

$$z\left(e^{1/z}-1\right) = 1 + \frac{z^{-1}}{2!} + \frac{z^{-2}}{3!} + \cdots\cdots = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} z^{-n} \Rightarrow u_n = \frac{1}{(n+1)!}$$

$\therefore \cancel{Z_T^{-1}\left[z(e^{1/2}-1)\right]} =$

14) Solve $u_{n+2} + (-3)u_{n+1} + 2u_n = 1$ by using Z transformations.

$$u_{n+2} - 3u_{n+1} + 2u_n = 1$$

Taking Z - transforms.

$$Z_T[u_{n+2}] - 3\,Z_T[u_{n+1}] + 2\,Z_T[u_n] = Z_T[1]$$

$$z^2\left[\bar{u}(z) - u_0 - u_1 z^{-1}\right] - 3z\left[\bar{u}(z) - u_0\right] + 2\,\bar{u}(z) = \frac{z}{z-1}$$

$$z^2\bar{u}(z) - z^2u_0 - z^2u_1z^{-1} - 3z\bar{u}(z) + 3zu_0 + 2\bar{u}(z) = \frac{z}{z-1}$$

$$\bar{u}(z)\left[z^2 - 3z + 2\right] + u_0\left[-z^2 + 3z\right] + u_1[-z] = \frac{z}{z-1}$$

$$\bar{u}(z)\left[z^2 - 3z + 2\right] = \frac{z}{z-1} + zu_1 - u_0\left[-z^2 + 3z\right]$$

$$\bar{u}(z)\left[z^2 - 3z + 2\right] = \frac{z}{z-1} + zu_1 + (z^2 - 3z)u_0$$

$$\bar{u}(z) = \frac{z}{(z-1)(z-2)(z-1)} + \frac{zu_1}{\cancel{(z-1)}(z-2)(z-1)} + \frac{z(z-3)\cdot u_0}{(z-2)(z-1)}$$

$$\Rightarrow \bar{u}(z) = \frac{z}{(z-2)(z-1)^2} + u_1\left[\frac{z\,\cancel{(z-1)}}{(z-2)(z-1)^{\cancel{2}}}\right] + u_0\left[\frac{z(z-3)\cancel{(z-1)}}{(z-2)\,(z-1)^{\cancel{2}}}\right]$$

$$Z_T^{-1}[\bar{u}(z)] = Z_T^{-1}\left[\frac{z}{(z-2)(z-1)^2}\right] + u_1\, Z_T^{-1}\left[\frac{z}{(z-2)(z-1)}\right] + u_0\, Z_T^{-1}\left[\frac{z(z-3)}{(z-2)(z-1)}\right]$$

Using method of partial fractions,

$$Z_T[\bar{u}(z)] = Z_T^{-1}\left[\frac{z}{(z-2)} + \frac{-z}{(z-1)} + \frac{-z}{(z-1)^2}\right] + u_1\left[Z_T^{-1}\left[\frac{z}{(z-2)} - \frac{z}{(z-1)}\right]\right] +$$

$$u_0\left[Z_T^{-1}\left[\frac{2z}{(z-1)} - \frac{z}{(z-2)}\right]\right]$$

$$u_n = \left[2^n - 1 - n\underset{\substack{\downarrow \\ a=1}}{a^n}\right] + u_1\left[2^n - 1\right] + u_0\left[2(1) - 2^n\right]$$

$$u_n = (2^n - 1 - n) + u_1[2^n - 1) + u_0[2 - 2^n]$$

$$u_n = 2^{\check{n}} - \check{1} - \check{n} + u_1\check{2}^n - \check{u}_1 + 2\check{u}_0 - \check{2}^n u_0$$

$$u_n = 2^n[1 + u_1 - u_0] - n + 2u_0 - u_1 - 1$$

$u_n = 2^n[1 + c_1] - n + c_2 - 1$ where $c_1 = u_1 - u_0$, $c_2 = 2u_0 - u_1$

or $u_n = 2^n[k] - n + m$ where $k = c_1 + 1$, $m = c_2 - 1$

15) Solve the difference equation by using z-transform

$y_{n+2} + 2y_{n+1} + y_n = n$, provided $y_0 = y_1 = 0$.

$$y_{n+2} + 2y_{n+1} + y_n = n$$

Taking z-transforms,

$$Z_T[y_{n+2}] + 2Z_T[y_{n+1}] + Z_T[y_n] = Z_T[n]$$

$$z^2[\bar{y}(z) - y_0 - y_1 z^{-1}] + 2z[\bar{y}(z) - y_0] + \bar{y}(z) = \frac{z}{(z-1)^2}$$

since $y_0 = y_1 = 0$,

$$z^2[\bar{y}(z)] + 2z[\bar{y}(z)] + \bar{y}(z) = \frac{z}{(z-1)^2}$$

$$\bar{y}(z)[z^2 + 2z + 1] = \frac{z}{(z-1)^2}$$

$$\bar{y}(z) = \frac{z}{(z-1)^2(z^2+2z+1)} = \frac{z}{(z-1)^2(z+1)^2} = \cancel{\frac{z}{[(z-1)(z+1)]^2}}$$

$$\bar{y}(z) = \cancel{\frac{z}{[z^2-1]^2}}$$

$$\frac{\bar{y}(z)}{z} = \frac{1}{(z+1)^2 (z-1)^2}$$

Applying partial fractions method, we get

$$\frac{\bar{y}(z)}{z} = \frac{1/4}{(z+1)} + \frac{1/4}{(z+1)^2} + \frac{-1/4}{(z-1)} + \frac{1/4}{(z-1)^2}$$

or, $\bar{y}(z) = \frac{1}{4}\left[\frac{z}{z+1}\right] + \frac{-1}{4}\left[\frac{-z}{(z+1)^2}\right] - \frac{1}{4}\left[\frac{z}{(z-1)}\right] + \frac{1}{4}\left[\frac{z}{(z-1)^2}\right]$

Taking inverse z-transforms, we get

$$y_n = \frac{1}{4}\left[(-1)^n\right] + \frac{-1}{4}\left[n(-1)^n\right] - \frac{1}{4}\left[(1)\right] + \frac{1}{4}\left[n\right]$$

$$y_n = \frac{1}{4}\left[(-1)^n - n(-1)^n - 1 + n\right]$$

$y_n = \frac{1}{4}\left[(-1)^n(1-n) - 1 + n\right]$ ~~$= \frac{1}{4}\left[(-1)^n(1-n) - ((-1)(-n+1))\right]$~~