Lecture 5

Resolution

Resolution proof calculus, Davis-Putnam procedure

Introduction to Logic for Computer Science

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These slides are minor variants of those made by Prof Worrell and Dr Haase for their logic course at Oxford.

Overview

SAT is bad.

- Truth tables: exponential time usually.
- Horn-SAT, 2-SAT and X-SAT require special formulas.
- **Resolution**: still worst case exponential time.

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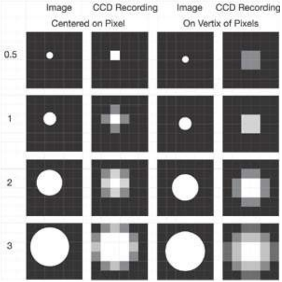
- **Resolution**: still worst case exponential time.

But, resolution has merits.

- Can exploit the structure of a given formula.
- Only takes polynomial time on Horn and 2-CNF formulas.
- Very easy to automate.
- Very easy to analyse theoretically.
- Still sound and complete.

Resolution



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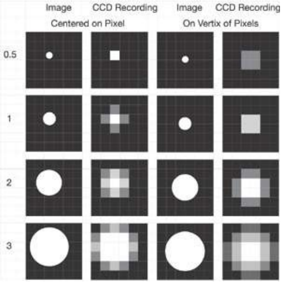
Resolution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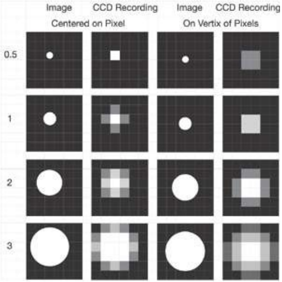

The latter is more related to resolution in logic.

Proof calculus

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- A proof calculus consists of rules of inference.
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- Mechanical.

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Resolution is a **proof calculus** for propositional logic.

- A proof calculus consists of rules of inference.
- Enables to derive series of conclusions from series of hypothesis.
- Mechanical.
- Resolution has only one rule of inference.
- Sound and complete.
- Soundness: Anything proved is valid.
- Completeness: Anything valid can be proved.

Set representation of CNF formulas

Resolution only works on CNF formulas.

Handy representation:

- Clause $\rightarrow$ set of literals.
- CNF formula $\rightarrow$ set of clauses.

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Example

$$(p_1 \vee \neg p_2) \wedge (p_3 \vee \neg p_4 \vee p_5) \wedge (\neg p_2)$$

is represented as

$$\{\{p_1, \neg p_2\}, \{p_3, \neg p_4, p_5\}, \{\neg p_2\}\}.$$

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Ex1: Write $(p_1 \vee \neg p_2 \vee p_1) \wedge (\neg p_2 \vee p_1) \wedge (p_3 \vee \neg p_4) \wedge (\neg p_2)$ as a set.

Ex2: What is good about this set representation?

Set representation of CNF formulas

Elements have no order or multiplicity, so set representation is only normal form modulo associativity, commutativity, and idempotence:

$$(p_3 \wedge (p_1 \vee p_1 \vee \neg p_2) \wedge p_3),$$
$$((\neg p_2 \vee p_1 \vee \neg p_2) \wedge (p_3 \vee p_3)), \text{ and}$$
$$(p_3 \wedge (\neg p_2 \vee p_1))$$

all have representation $\{\{p_3\}, \{p_1, \neg p_2\}\}$.

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Empty clause (empty set of literals) is equivalent to *false*. Denoted $\Box$.

If a CNF formula contains $\Box$, it is unsatisfiable.

If a CNF formula is the empty set, it is equivalent to *true*.

(Compare: the sum of empty set of natural numbers is 0, but the product of empty set of natural numbers is 1.)

Resolvents

Recall: for a literal L, its **complementary** literal $\overline{L}$ is defined by

$$\overline{L} := \begin{cases} \neg p & \text{if } L = p, \\ p & \text{if } L = \neg p. \end{cases}$$

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Definition

Let C_1 and C_2 be clauses. A clause R is called a **resolvent** of C_1 and C_2 if there are complementary literals $L \in C_1$ and $\overline{L} \in C_2$ such that

$$R = (C_1 \setminus \{L\}) \cup (C_2 \setminus \{\overline{L}\}).$$

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We say R **is derived from** C_1 **and** C_2 **by resolution**, and write

$$\frac{C_1 \quad C_2}{R}$$

Resolvents: example

Example

$\{p_1, p_3, \neg p_4\}$ resolves $\{p_1, p_2, \neg p_4\}$ and $\{\neg p_2, p_3\}$.

The empty clause is a resolvent of $\{p_1\}$ and $\{\neg p_1\}$.

$$\frac{\{p_1, p_2, \neg p_4\} \quad \{\neg p_2, p_3\}}{\{p_1, p_3, \neg p_4\}} \qquad \frac{\{p_1\} \quad \{\neg p_1\}}{\Box}$$

Resolvents: example

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Ex: Compute a resolvent of each of the following pairs of clauses:

1. $\{p_1, \neg p_2, p_4\}$ and $\{p_1, \neg p_4, \neg p_5\}$.
2. $\{p_1, \neg p_2, p_4\}$ and $\{\neg p_1, \neg p_4, \neg p_5\}$.
3. $\{p_1, \neg p_2, p_4\}$ and $\{\neg p_1\}$.

Derivations and refutations

Definition

A **derivation** (or **proof**) of a clause C from a set of clauses F is a sequence $C_1, C_2, \ldots, C_m$ of clauses where

- $C_m = C$; and
- for each $i = 1, 2, \ldots, m$, either $C_i \in F$ or C_i is a resolvent of C_j and C_k for some $j, k < i$.

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A derivation of the empty clause $\square$ from a formula F is called a **refutation** of F.

Derivations: example

A resolution refutation of the CNF formula

$$\{\{x, \neg y\}, \{y, z\}, \{\neg x, \neg y, z\}, \{\neg z\}\}$$

is as follows:

1.	$\{x, \neg y\}$	(Assumption)	5.	$\{\neg x, z\}$	(2,4 Resolution)
2.	$\{y, z\}$	(Assumption)	6.	$\{\neg z\}$	(Assumption)
3.	$\{x, z\}$	(1,2 Resolution)	7.	$\{z\}$	(3,5 Resolution)
4.	$\{\neg x, \neg y, z\}$	(Assumption)	8.	$\Box$	(6,7 Resolution)

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Graphically represented by the following **proof tree**:

$$\dfrac{\dfrac{\dfrac{\{x, \neg y\} \qquad \{y, z\}}{\{x, z\}} \qquad \dfrac{\{y, z\} \qquad \{\neg x, \neg y, z\}}{\{\neg x, z\}}}{\{z\}} \qquad \{\neg z\}}{\square}$$

Refutations: comments

- A resolution refutation of a formula F can be seen as a proof that F is unsatisfiable.
- Resolution can be used to prove entailments by transforming them to refutations.
- For example, the refutation in previous example can be used to show that

$$(x \vee \neg y) \wedge (y \vee z) \wedge (\neg x \vee \neg y \vee z) \models z.$$

- Intuitively, proof by contradiction.

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- Intuitively, proof by contradiction.
- Ex: Suppose that we would like to prove $F \models G$ for CNF formulas F and G. How to do this using resolution?

Set of resolvents

Given a set F of clauses, we are interested in the set of all clauses derivable from F by resolution.

Definition

For a set F of clauses, $Res(F)$ is defined as

$$Res(F) = F \cup \{R \mid \ R \text{ is a resolvent of two clauses in } F\}.$$

Furthermore, define

$$Res^0(F) = F, \qquad Res^{n+1}(F) = Res(Res^n(F)) \quad \text{for } n \geq 0$$

and write

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Theorem

$C \in Res^*(F)$ iff there is a derivation of C from F.

Ex: Prove the theorem.

Soundness and completeness

- **Soundness**: Anything proved is valid.
- **Completeness**: Anything valid can be proved.
- For every F, there is a resolution refutation of F iff $\neg F$ is valid.

The resolution lemma

Lemma

Let F be a CNF formula represented as a set of clauses. If R is a resolvent of clauses C_1 and C_2 of F, then $F \equiv F \cup \{R\}$.

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Proof.

We focus on proving $F \models F \cup \{R\}$. Suppose

- $\mathcal{A} \models F$, and
- $R = (C_1 \setminus \{L\}) \cup (C_2 \setminus \{\overline{L}\})$ for some literal $L \in C_1$ with $\overline{L} \in C_2$.

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We will have to show $\mathcal{A} \models F \cup \{R\}$.

- If $\mathcal{A} \models L$, then since $\mathcal{A} \models C_2$, it follows that $\mathcal{A} \models C_2 \setminus \{\overline{L}\}$, and thus $\mathcal{A} \models R$.
- If $\mathcal{A} \models \overline{L}$, then since $\mathcal{A} \models C_1$, it follows that $\mathcal{A} \models C_1 \setminus \{L\}$, and thus $\mathcal{A} \models R$.

□

Soundness of resolution

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If we can derive $\Box$ *from* F*, then* F *is unsatisfiable.*

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Theorem

If we can derive $\square$ from F, then F is unsatisfiable.

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Proof.

Suppose $C_1, C_2, \ldots, C_m = \square$ is a resolution proof of $\square$ from F.

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Proof.

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The repeated application of the Resolution Lemma shows

$$F \equiv F \cup \{C_1, C_2, \ldots, C_m\}.$$

But the latter set of clauses includes the empty clause. Thus, F is unsatisfiable. $\Box$

Completeness

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By induction on the number n of variables in F.

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- Ex: Prove the inductive case.
- Suppose variables $p_0, \ldots, p_n$. Since F is unsatisfiable, so is $F_0 := F[\mathit{false}/p_n]$. Induction hypothesis gives a resolution proof $C_0, C_1, \ldots, C_m = \Box$ that derives $\Box$ from F_0.

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- Suppose variables $p_0, \ldots, p_n$. Since F is unsatisfiable, so is $F_0 := F[\mathit{false}/p_n]$. Induction hypothesis gives a resolution proof $C_0, C_1, \ldots, C_m = \Box$ that derives $\Box$ from F_0. Each C_i from F_0 is either already in F or $C_i \cup \{p_n\}$ is in F. Re-introducing p_n and propagating it gives a res. proof $C'_0, C'_1, \ldots, C'_m$ from F where either $C'_m = \Box$ or $C'_m = \{p_n\}$.

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Proof.

By induction on the number n of variables in F.

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- Apply similar reasoning to $F_1 := F[\mathit{true}/p_n]$. Get a proof of $C''_l = \Box$ or $C''_l = \{\neg p_n\}$ from F.

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- Ex: Prove the inductive case.
- Suppose variables $p_0, \ldots, p_n$. Since F is unsatisfiable, so is $F_0 := F[\mathit{false}/p_n]$. Induction hypothesis gives a resolution proof $C_0, C_1, \ldots, C_m = \Box$ that derives $\Box$ from F_0. Each C_i from F_0 is either already in F or $C_i \cup \{p_n\}$ is in F. Re-introducing p_n and propagating it gives a res. proof $C'_0, C'_1, \ldots, C'_m$ from F where either $C'_m = \Box$ or $C'_m = \{p_n\}$.
- Apply similar reasoning to $F_1 := F[\mathit{true}/p_n]$. Get a proof of $C''_l = \Box$ or $C''_l = \{\neg p_n\}$ from F.
- If $C'_m = \Box$ or $C''_l = \Box$, done. Otherwise, glue together these two proofs and apply one more resolution step to $\{p_n\}$ and $\{\neg p_n\}$.

$\Box$

Completeness: example

Example

Consider $F = \{\{p, r\}, \{\neg p, q\}, \{\neg q, r\}, \{\neg q, \neg r\}, \{p, \neg r\}\}$.

Transform the following derivation of $\square$ from $F[\mathit{false}/r]$

$$\dfrac{\dfrac{\{p\} \qquad \{\neg p, q\}}{\{q\}} \qquad \{\neg q\}}{\square}$$

to the following derivation of $\{r\}$ from F:

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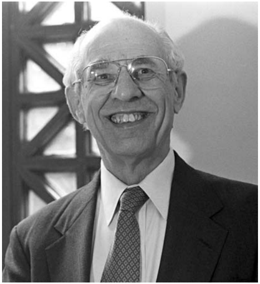
Ex: Handle the other case $F[\mathit{true}/r]$. Construct the derivation of $\Box$ from F.

The Davis–Putnam procedure

Can turn resolution into a **SAT solver**.

Davis–Putnam procedure.



Basic idea: Use resolution to perform **variable elimination** when searching for a satisfying assignment.

Variable elimination

Eliminate p from a CNF formula F to get a new formula G:

1. If p occurs only positively in F,
   delete all clauses containing p, so $G := F[\mathit{true}/p]$.
2. If p occurs only negatively in F,
   delete all clauses containing $\overline{p}$, so $G := F[\mathit{false}/p]$.
3. Suppose p occurs both positively and negatively in F.
   For every pair of clauses C, D in F with $p \in C$ and $\overline{p} \in D$,
   add the resolvant of C and D to F.
   Delete all clauses containing p or $\overline{p}$ from F to get G.

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Example

Eliminating p from $\{\{p\}, \{\neg p, q\}, \{\neg q, r\}, \{\neg r, s, t\}, \{r, s\}, \{\neg r, t\}\}$
gives $\{\{q\}, \{\neg q, r\}, \{\neg r, s, t\}, \{r, s\}, \{\neg r, t\}\}$.

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gives $\{\{q\}, \{\neg q, r\}, \{\neg r, s, t\}, \{r, s\}, \{\neg r, t\}\}$.

Ex: Eliminate r from the above set of clauses.

Variable elimination: correctness

Lemma (Elimination Lemma)

If eliminating a variable p from F gives G then

- *F and G are equisatisfiable; and*
- *if $\mathcal{A} \models G$ then $\mathcal{A}_{[p \mapsto a]} \models F$ for some $a \in \{0, 1\}$ that can be determined from $\mathcal{A}$ and F.*

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Ex: Prove the lemma.

The Davis–Putnam algorithm

Davis–Putnam(F)
begin
remove all valid clauses from F
if $F = \{\square\}$ **then** return UNSAT
if $F = \emptyset$ **then** return the 0 assignment
let G arise by eliminating a variable p from F
if Davis–Putnam(G) = UNSAT **then** return UNSAT
if Davis–Putnam(G) = $\mathcal{A}$ **then** return $\mathcal{A}_{[p \mapsto a]}$,
  with a chosen as in the Elimination Lemma
end

Davis–Putnam: example

First eliminate variables (p, q, r, s):

$$
\begin{aligned}
&\text{Davis–Putnam}(\{\{p\}, \{\neg p, \neg q\}, \{q, r\}, \{\neg r, s, \neg t\}\}) \\
&= \text{Davis–Putnam}(\{\{\neg q\}, \{q, r\}, \{\neg r, s, \neg t\}\}) \\
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Then recurse back up to get satisfying assignment:

$$\begin{aligned}
t &\mapsto 0 \\
s &\mapsto 1 \\
r &\mapsto 1 \\
q &\mapsto 0 \\
p &\mapsto 1
\end{aligned}$$

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(unsurprising: intermediate clauses can become big).

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Actually, I don't know the answers. But I recommend you to think about this type of questions.

Summary

Resolution:

- A proof calculus.
- Sound and complete.
- Very simple.

Davis–Putnam:

- Decision algorithm for SAT.
- Basis of SAT solvers.
- Polynomial time on nice formulas.
- Worst case exponential time.
- Depend on the order of variable elimination.