Introduction to Logic for Computer Science Spring 2023

Assignment 3 (Deadline: 6:00pm on 12 May 2023)

Prof Hongseok Yang

Submit your solutions in KLMS. (Reminder: We adopt a very strict policy for handling dishonest behaviours. If a student is found to copy answers from fellow students or other sources in his or her homework submission, she or he will get F.)

1. Consider the following structures over a signature with a single binary relation symbol R:

   $U_{\mathcal{A}} = \mathbb{N}$ and $R_{\mathcal{A}} = \{(n, m) \in \mathbb{N} \times \mathbb{N} : n < m\}$
   $U_{\mathcal{B}} = \mathbb{Z}$ and $R_{\mathcal{B}} = \{(n, m) \in \mathbb{Z} \times \mathbb{Z} : n < m\}$
   $U_{\mathcal{C}} = \mathbb{Q}$ and $R_{\mathcal{C}} = \{(n, m) \in \mathbb{Q} \times \mathbb{Q} : n < m\}$

   Give a formula that is satisfied by $\mathcal{B}$ but not by $\mathcal{A}$, and a formula that is satisfied by $\mathcal{C}$ but not by $\mathcal{B}$. **(20 points)**

2. Translate the following formula to rectified form, then to prenex form, and finally to Skolem form:

   $$\forall z\, \exists y\, (Q(x, g(y), z) \vee \neg\forall x\, P(x)) \wedge \neg\forall z\, \exists x\, \neg R(f(x, z), z).$$

   (20 points)

3. Are the following claims correct? Justify your answers.

   (a) For any formula F and term t, if F is valid then $F[t/x]$ is valid. **(6 points)**

   (b) $\exists x\, (P(x) \rightarrow \forall y\, P(y))$ is valid. **(7 points)**

   (c) For any formula F and constant symbol c, if $F[c/x]$ is valid and c does not appear in F then $\forall x\, F$ is valid. **(7 points)**

4. Fix a signature σ. Consider a relation $\sim$ on σ-assignments that satisfies the following two properties:

   (P1) If $\mathcal{A} \sim \mathcal{B}$, then for every atomic formula F we have $\mathcal{A} \models F$ iff $\mathcal{B} \models F$.

   (P2) If $\mathcal{A} \sim \mathcal{B}$, then for each variable x we have (i) for each $a \in U_{\mathcal{A}}$ there exists $b \in U_{\mathcal{B}}$ such that $\mathcal{A}_{[x \mapsto a]} \sim \mathcal{B}_{[x \mapsto b]}$, and (ii) for all $b \in U_{\mathcal{B}}$ there exists $a \in U_{\mathcal{A}}$ such that $\mathcal{A}_{[x \mapsto a]} \sim \mathcal{B}_{[x \mapsto b]}$.

   Prove that if $\mathcal{A} \sim \mathcal{B}$ then for any formula F, $\mathcal{A} \models F$ if and only if $\mathcal{B} \models F$. You may assume that F is built from atomic formulas using the connectives $\wedge$ and $\neg$ and the quantifier $\exists$. **(20 points)**

5. In this question we work with first-order logic without equality.

   (a) Consider a signature σ containing only a binary relation symbol R. For each positive integer n show that there is a satisfiable σ-formula F_n such that every model $\mathcal{A}$ of F_n has at least n elements. **(5 points)**

(b) Consider a signature σ containing only unary predicate symbols $P_1, \ldots, P_k$. Using Question 4, or otherwise, show that any satisfiable σ-formula has a model with at most 2^k elements. **(15 points)**