

Lecture 2

Propositional logic

syntax and semantics, the satisfiability problem, constraint problems

Introduction to Logic for Computer Science

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These slides are minor variants of those made by Prof Worrell and Dr Haase for their logic course at Oxford.

Agenda

- 1 Propositional logic
- 2 Syntax and semantics of propositional logic
- 3 Encoding constraint problems into satisfiability problems

Propositional logic

- Informally, a study on a type of boolean expressions in PLs, called sentences, formulas or propositions.
- The most basic kind of sentences are *atomic propositions*, which can be true, or false, or variables.
- Sentences are combined using *logical connectives*.
- Propositional logic analyses how the truth values of compound sentences depend on their constituents.
- A prime concern: *given a compound sentence, determine which truth values of its atoms make it true.*
- Key to formulate the notions of *logical consequence* and *valid argument*.

An example

- Atomic propositions:

a “Alice is an architect”

b “Bob is a builder”

c “Charlie is a cook”

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- Ex: Assume the three propositions. What can you say about a ?
- Answer: Alice is an architect. That is, $\{\neg c, a \vee b, b \rightarrow c\} \models a$.
- The correctness of this entailment is *independent* of the meaning of the atomic propositions!

1 Propositional logic

2 **Syntax and semantics of propositional logic**

3 Encoding constraint problems into satisfiability problems

Syntax of propositional logic

Definition (Syntax of propositional logic)

Let $X = \{x_1, x_2, x_3, \dots\}$ be a countably infinite set of **propositional variables**. **Formulas** of propositional logic are inductively defined as follows:

- 1 *true* and *false* are formulas.
- 2 Every propositional variable x_i is a formula.
- 3 If F is a formula, then $\neg F$ is a formula.
- 4 If F and G are formulas, then $(F \wedge G)$ and $(F \vee G)$ are formulas.

Additional notation

- We often write x, y, z or p to denote propositional variables.
- We call $\neg F$ the **negation** of F .
- Given formulas F and G , $(F \wedge G)$ is the **conjunction** of F and G , and $(F \vee G)$ is the **disjunction** of F and G .
- We call \neg, \wedge and \vee **logical connectives**.
- We denote by $\mathcal{F}(X)$ the **set of all formulas** built from propositional variables in X .

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- **Indexed Conjunction:** $\bigwedge_{i=1}^n F_i := ??$
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- Ex1: Fill in ??.
- Ex2: Find a connective that can be used to define all the others.
- **Ex3: Prove that \wedge is not an answer for Ex2. By Mono.**

Convention on bracketing

- We drop brackets, unless doing so causes big confusion.
- No outer brackets usually.
- Use the standard precedence of connectives.
- Example: $\neg x \wedge y \rightarrow z$ means $((\neg x) \wedge y) \rightarrow z$.
- Lecture notes for the detail. Ask me when confused.

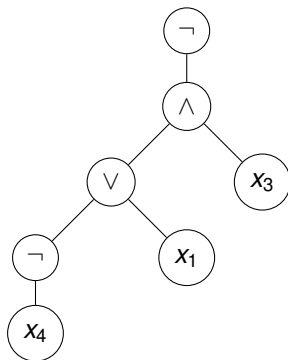
Syntax trees

- Every formula F can be represented by a *syntax tree* whose nodes are labelled with connectives or propositional variables.
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Example: syntax tree of $\neg((\neg x_4 \vee x_1) \wedge x_3)$:



Inductive definitions

Inductive definition of formulas allows us to define functions on formulas by **structural induction**, by defining the function

- for the base cases *true*, *false* and x_i , and
- for the induction steps $\neg F$, $F \wedge G$ and $F \vee G$.

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Example

The function $size : \mathcal{F}(X) \rightarrow \mathbb{N}$ returning the number of symbols in a given formula can be defined by:

- $size(true) = size(false) = size(x) = 1$;
- $size(\neg F) = 1 + size(F)$;
- $size(F \wedge G) = size(F \vee G) = size(F) + size(G) + 1$.

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-
- Ex1: What is $size(\neg((\neg x_4 \vee x_1) \wedge x_3))$?
 - Ex2: Define a function $sub : \mathcal{F}(X) \rightarrow 2^{\mathcal{F}(X)}$ that returns the set of all subformulas of a given formula.

Syntax vs semantics

The *syntax* tells us how we write something down, the *semantics* what it means:

- syntax: some formal *language*.
- semantics: some mathematical *model*.
- semantics should capture the 'essence' of what's going on.
- have to have semantics to prove anything *about* syntax.

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- semantics should capture the 'essence' of what's going on.
- have to have semantics to prove anything *about* syntax.
- our syntax: propositional formulas.
- our semantics: **truth values** $\{0, 1\}$.

Semantics of propositional logic

Definition

An **assignment** is a function $\mathcal{A}: X \rightarrow \{0, 1\}$. It induces a function $\hat{\mathcal{A}}: \mathcal{F}(X) \rightarrow \{0, 1\}$, called **assignment** again, by structural induction:

- 1 $\hat{\mathcal{A}}(false) := 0, \hat{\mathcal{A}}(true) := 1.$
- 2 For every $x \in X, \hat{\mathcal{A}}(x) := \mathcal{A}(x).$
- 3 $\hat{\mathcal{A}}(\neg F) := \begin{cases} 1 & \text{if } \hat{\mathcal{A}}(F) = 0, \\ 0 & \text{otherwise.} \end{cases}$
- 4 $\hat{\mathcal{A}}(F \wedge G) := \begin{cases} 1 & \text{if } \hat{\mathcal{A}}(F) = 1 \text{ and } \hat{\mathcal{A}}(G) = 1, \\ 0 & \text{otherwise.} \end{cases}$
- 5 $\hat{\mathcal{A}}(F \vee G) := \begin{cases} 1 & \text{if } \hat{\mathcal{A}}(F) = 1 \text{ or } \hat{\mathcal{A}}(G) = 1, \\ 0 & \text{otherwise.} \end{cases}$

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Ex: Let \mathcal{A} be an assignment s.t. $\mathcal{A}(x) = 1$ and $\mathcal{A}(y) = \mathcal{A}(z) = 0$.
What are $\hat{\mathcal{A}}((x \wedge \neg y) \vee z)$ and $\hat{\mathcal{A}}(x \wedge (x \vee y) \wedge (y \vee \neg z))$?

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From now on we will not write the hat on top of \mathcal{A} .

Semantics via truth tables

Example

The semantics of logical connectives via **truth tables**:

$\mathcal{A}(F)$	$\mathcal{A}(G)$	$\mathcal{A}(F \wedge G)$	$\mathcal{A}(F)$	$\mathcal{A}(G)$	$\mathcal{A}(F \vee G)$
0	0	0	0	0	0
1	0	0	1	0	1
0	1	0	0	1	1
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Formalising natural language: an example

A device consists of a thermostat, a pump, and a warning light.
Suppose we are told the following four facts about the pump:

- The thermostat or the pump (or both) are broken.
- If the thermostat is broken then the pump is also broken.
- If the pump is broken and the warning light is on then the thermostat is not broken.
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Ex: Is it possible for all four to be true at the same time? Express this question using a formula, and answer it using semantics.

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Answer:

$$F := (t \vee p) \wedge (t \rightarrow p) \wedge (p \wedge w \rightarrow \neg t) \wedge w$$

So, yes under \mathcal{A} with $\mathcal{A}(t) = 0$ and $\mathcal{A}(p) = \mathcal{A}(w) = 1$.

Truth table

$$F := (t \vee p) \wedge (t \rightarrow p) \wedge (p \wedge w \rightarrow \neg t) \wedge w$$

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There is a unique assignment that makes F true. We can think of each assignment as describing a *possible world*, and there is only one world in which F is true.

Models, satisfiability and validity

Definition

Let $F \in \mathcal{F}(X)$ and $\mathcal{A}: X \rightarrow \{0, 1\}$ be an assignment.

- 1 If $\mathcal{A}(F) = 1$ then we write $\mathcal{A} \models F$ (" F **holds under** \mathcal{A} ", or " \mathcal{A} is a **model** of F ", or " \mathcal{A} **satisfies** F ").
- 2 If F has at least one model, then F is **satisfiable**. Otherwise, F is **unsatisfiable**.
- 3 If F holds under any assignment $\mathcal{A}: X \rightarrow \{0, 1\}$, then F is called **valid** or a **tautology**, written $\models F$.

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The **Boolean satisfiability problem (SAT)** is to decide whether a given formula $F \in \mathcal{F}(X)$ is satisfiable.

Ex: Suppose that we have a program for solving SAT. How to convert it to a checker for validity?

Models, satisfiability and validity

Example

The subsequent first two tautologies are known as the *distributive laws*, the last two as *de Morgan's laws*:

$$\models (F \vee (G \wedge H)) \leftrightarrow ((F \vee G) \wedge (F \vee H))$$

$$\models (F \wedge (G \vee H)) \leftrightarrow ((F \wedge G) \vee (F \wedge H))$$

$$\models \neg(F \wedge G) \leftrightarrow \neg F \vee \neg G$$

$$\models \neg(F \vee G) \leftrightarrow \neg F \wedge \neg G.$$

Ex: Prove the last two.

Entailment and equivalence

Definition (Entailment)

A formula G is a **consequence** of (or is **entailed** by) a set of formulas \mathcal{S} if every assignment that satisfies all formulas in \mathcal{S} also satisfies G . In this case, we write $\mathcal{S} \models G$.

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Two formulas F and G are said to be **logically equivalent** if $\mathcal{A}(F) = \mathcal{A}(G)$ for every assignment \mathcal{A} . We write $F \equiv G$ to denote that F and G are equivalent.

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Sudoku

	2		5		1		9	
8			2		3			6
	3			6			7	
		1				6		
5	4						1	9
		2				7		
	9			3			8	
2			8		4			7
	1		9		7		6	

How to encode an instance of Sudoku into the satisfiability of a propositional formula?

Sudoku

For each $i, j, k \in \{1, \dots, 9\}$, we have a propositional variable $x_{i,j,k}$ expressing that *grid position i, j contains number k* .

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Build formula F as the conjunction of the following *constraints*:

- Each number appears in each row and in each column:
- Each number appears in each 3×3 block:
- No square contains two numbers:

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- Each number appears in each 3×3 block:

$$F_3 := \bigwedge_{u=0}^2 \bigwedge_{v=0}^2 \bigwedge_{k=1}^9 \bigvee_{i=1}^3 \bigvee_{j=1}^3 x_{3u+i, 3v+j, k}$$

- No square contains two numbers:

$$F_4 := \bigwedge_{i=1}^9 \bigwedge_{j=1}^9 \bigwedge_{1 \leq k < k' \leq 9} \neg (x_{i,j,k} \wedge x_{i,j,k'}) .$$

Sudoku

- Certain numbers appear in certain positions: we assert

$$F_5 := x_{1,2,2} \wedge x_{2,1,8} \wedge x_{3,2,3} \wedge \dots \wedge x_{9,8,6}.$$

	2		5		1		9	
8			2		3			6
	3			6			7	
		1				6		
5	4						1	9
		2				7		
	9			3			8	
2			8		4			7
	1		9		7		6	

Sudoku

- Missing constraints? What about: no number appears twice in the same row?

$$F_6 := \bigwedge_{i=1}^9 \bigwedge_{k=1}^9 \bigwedge_{1 \leq j < j' < 9} \neg (x_{i,j,k} \wedge x_{i,j',k})$$

Sudoku

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- Entailed by the existing formulas: adding F_6 as an extra constraint would not change the set of satisfying assignments.
- But adding logically redundant constraints may help a computer search for a satisfying assignment.
- The number of variables $x_{i,j,k}$ is $9^3 = 729$. Thus a truth table for the corresponding formula would have $2^{729} > 10^{200}$ lines! Nevertheless a modern SAT-solver can find a satisfying assignment in milliseconds.

Hamiltonian path

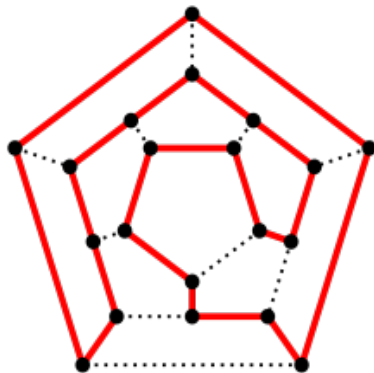


Figure: Example of a Hamiltonian path in an undirected graph.

How to encode an instance of the Hamiltonian path problem into the satisfiability of a propositional formula?

Hamiltonian path for an undirected graph $G = (V, E)$

For each vertex $i, j \in \{1, \dots, n\}$, we have propositional variables

- $x_{i,j}$ expressing that *i is the j th vertex in the Hamiltonian path*;
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$$F_3 := \bigwedge_{i=1}^n \bigwedge_{j=1}^n \bigwedge_{k=1}^{n-1} x_{i,k} \wedge x_{j,k+1} \rightarrow e_{i,j}.$$

- $e_{i,j}$ encodes E :

$$F_4 := \bigwedge_{(i,j) \in E} e_{i,j} \wedge \bigwedge_{(i,j) \notin E} \neg e_{i,j}.$$

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- Can do better for special formula classes: Horn formulas, 2-CNF formulas, XOR-clauses, ...
- Reductions of constraint problems to SAT should run in polynomial-time!