Lecture 10
Herbrand's theorem and ground resolution

Introduction to Logic for Computer Science

Prof Hongseok Yang
KAIST

These slides are minor variants of those made by Prof Worrell and Dr Haase for their logic course at Oxford.

Recap

Let F be a quantifier-free formula.

Prenex form: $Q_1 x_1 \; Q_2 x_2 \; \cdots \; Q_n x_n \; F$, where $Q_i \in \{\forall, \exists\}$.

Skolem form: $\forall x_1 \; \forall x_2 \; \cdots \forall x_n \; F$.

Every first-order formula can be translated into an equi-satisfiable formula in Skolem form in poly. time.

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Example

Let $\sigma = \langle c, d,\ f, g,\ P, Q\rangle$ with unary f and binary g, the set of ground terms is

$$\{c, d, f(c), f(d), g(c, c), g(c, d), g(d, c), g(d, d), f(f(c)), f(f(d)), \ldots\}.$$

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Ex: Find a signature σ such that its set of ground terms is isomorphic to $\mathbb{N}$.

Herbrand structures

Definition

Let σ be a signature with at least one constant symbol. A σ-structure $\mathcal{H}$ is a **Herbrand structure** if the following hold:

- The universe $U_{\mathcal{H}}$ is the set of ground terms over σ.
- For every constant symbol c, we have $c_{\mathcal{H}} = c$.
- For every function symbol f, $f_{\mathcal{H}}(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$.

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Ex2: Prove the lemma.

Jaques Herbrand (1908 – 1931)

Herbrand's theorem

Theorem

Let $F = \forall x_1 \forall x_2 \ldots \forall x_n F^*$ *be a formula in Skolem form. Then* F *is satisfiable if and only if* F *has a Herbrand model.*

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Suppose $\mathcal{A} \models F$.

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Suppose $\mathcal{A} \models F$. Define a Herbrand model $\mathcal{H}$ by setting the interpretation of each predicate symbol P as follows:

$$(t_1, \ldots, t_k) \in P_{\mathcal{H}} \text{ iff } \mathcal{A} \models P(t_1, \ldots, t_k).$$

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Now show that for all closed G in Skolem form, if $\mathcal{A} \models G$, then $\mathcal{H} \models G$. Use induction on the number of $\forall$ in G.

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Let $F = \forall x_1 \forall x_2 \ldots \forall x_n F^*$ be a formula in Skolem form. Then F is satisfiable if and only if F has a Herbrand model.

Proof.

Assume that F is closed. Suppose $\mathcal{A} \models F$.

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We show that for all *closed* $G = \forall x_1 \, \forall x_2 \, \ldots \forall x_m \, G^*$ with quantifier free G^*, if $\mathcal{A} \models G$, then $\mathcal{H} \models G$. We use induction on m.

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For $m = 0$, G is closed. Thus, it is a Boolean combination of $P(t_1, \ldots, t_k)$. By definition, $\mathcal{H} \models G$.

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But then $\mathcal{H} \models \forall x \, G'$. $\square$

Ground resolution

Herbrand expansion of $F = \forall x_1 \forall x_2 \ldots \forall x_n F^*$:

$$E(F) := \{F^*[t_1/x_1][t_2/x_2]\ldots[t_n/x_n] : t_1, \ldots, t_n \text{ ground terms}\}.$$

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Ex1: What does this imply in terms of developing a validity checker for first-order formulas?

Ex2: Prove the theorem.

Hint: Prove that Herbrand' theorem implies the following theorem. Then, use the Compactness theorem for propositional logic.

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Proof.

By Herbrand's theorem, F is satisfiable if and only if F has a Herbrand model. Now

$$\begin{aligned}
\mathcal{H} \models F \text{ iff } & \mathcal{H}_{[x_1 \mapsto t_1]\cdots[x_n \mapsto t_n]} \models F^* \text{ for all ground terms } t_1, \ldots, t_n \\
& \text{iff } \mathcal{H} \models F^*[t_1/x_1]\ldots[t_n/x_n] \text{ for all ground } t_i \text{ (by Trans. Lemma)} \\
& \text{iff } \mathcal{H} \models E(F) \\
& \text{iff } E(F) \text{ as a set of prop. formulas is satisfied by } \mathcal{H}.
\end{aligned}$$

In the last line, we view $\mathcal{H}$ as an assignment to prop. variables of the form $P(t_1, \ldots, t_k)$. $\square$

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Proof.

By the Compactness theorem, $E(F)$ is unsatisfiable if and only if some finite subset of $E(F)$ is unsatisfiable. The latter happens if and only if $\square$ can be derived from $E(F)$ using resolution. $\square$