Lecture 4

Polynomial-time formula classes

Horn-SAT, 2-SAT, X-SAT, Walk-SAT

Introduction to Logic for Computer Science

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These slides are minor variants of those made by Prof Worrell and Dr Haase for their logic course at Oxford.

Recap and some additional notation

- A **literal** is a propositional variable or the negation of a propositional variable:

$$x \text{ or } \neg x.$$

- We call x a **positive literal** and $\neg x$ a **negative literal**.
- A disjunction of literals is a **clause**.
- A formula F is in **conjunctive normal form** (**CNF**) if it is a conjunction of disjunctions of literals $L_{i,j}$:

$$F = \bigwedge_{i=1}^{n} \Big(\bigvee_{j=1}^{m_i} L_{i,j} \Big).$$

- Convention: *true* is CNF with no clauses, *false* is CNF with a single empty clause without literals.

Agenda

1. Polynomial-time fragments of propositional logic
2. Walk-SAT: A randomised algorithm for satisfiability

The satisfiability problem

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But, we can often do better for formulas of special form:

- **Horn formulas**: SAT can be decided in polynomial time.
- **2-CNF formulas**: SAT can be decided in polynomial time.
- **X-CNF formulas**: SAT can be decided in polynomial time.

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- Horn formulas can be rewritten in a more intuitive way as conjunctions of implications, called **implication form**. E.g.:

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- Many applications in computer science. Prolog and Datalog are based on Horn formulas.

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Unsat.:
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Horn-SAT algorithm

Idea:

- Maintain an assignment $\mathcal{A}$ for propositional variables, starting with $p \mapsto 0$.
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Ex: Why correct?

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- Order assignments by $\mathcal{A} \leq \mathcal{B}$ when $\mathcal{A}(p) \leq \mathcal{B}(p)$ for every p.

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- If UNSAT returned, then F is unsatisfiable. Really?
- **Loop invariant**: If $\mathcal{B}$ satisfies F, then $\mathcal{A} \leq \mathcal{B}$.
- Ex1: Prove that this is a loop invariant.
- Ex2: Prove that this loop invariant gives the desired result.

2-CNF formulas

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- The **implication graph** of a 2-CNF formula F is a directed graph $\mathcal{G} = (V, E)$, where

$$V := \{p_1, \ldots, p_n\} \cup \{\neg p_1, \ldots, \neg p_n\},$$

  with $p_1, \ldots, p_n$ prop. variables mentioned in F.

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- Ex: If (L, M) is an edge, is $(\overline{M}, \overline{L})$ also an edge?

2-CNF formulas: example

$$(p_0 \vee p_2) \wedge (p_0 \vee \neg p_3) \wedge (p_1 \vee \neg p_3) \wedge (p_1 \vee \neg p_4) \wedge (p_2 \vee \neg p_4)$$
$$\wedge (p_0 \vee \neg p_5) \wedge (p_1 \vee \neg p_5) \wedge (p_2 \vee \neg p_5) \wedge (p_3 \vee p_6) \wedge (p_4 \vee p_6) \wedge (p_5 \vee p_6)$$

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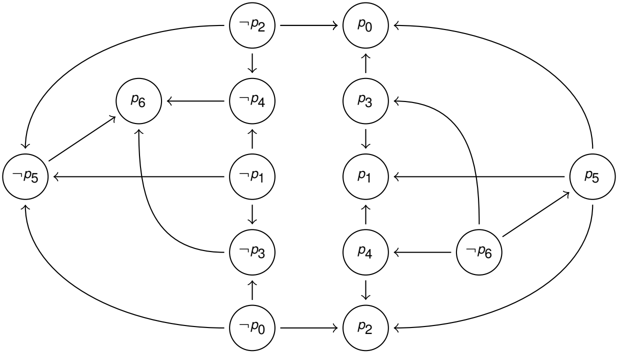

- Paths in $\mathcal{G}$ correspond to chains of implications.

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- Can reduce **satisfiability** for 2-CNF formulas to *reachability* problem of implication graph, which is solvable in *poly.* time.
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If $\mathcal{G}$ is not consistent, there are paths $\neg p \to p$, $p \to \neg p$. So $\mathcal{A} \models F$ would imply $\mathcal{A}(\neg p) \leq \mathcal{A}(p) \leq \mathcal{A}(\neg p)$. Contradiction.

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($\Leftarrow$) Construct a satisfying assignment. Ex: How? $\square$

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Ex: Why correct? Invariants for the outer and inner loops?

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- After updating $\mathcal{A}(L) := 1$, the inner invariant holds. Ex: Why?

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- Inner loop maintains the invariant. So when it terminates, every node reachable from a true node is true.

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Proof.

List all subformulas of F: $F_1 = p_1, \ldots, F_m = p_m, F_{m+1}, \ldots, F_n$.

Introduce new variables $p_{m+1}, \ldots, p_n$.

Associate formulas G_i asserting $p_i \leftrightarrow F_i$.

Take $G = G_{m+1} \wedge \cdots \wedge G_n \wedge p_n$. □

X-CNF formulas

- An **XOR-clause** is an exclusive-or of literals.
- An **X-CNF** formula is a conjunction of XOR-clauses.

$$F = (p_1 \oplus p_3) \wedge (\neg p_1 \oplus p_2) \wedge (p_1 \oplus p_2 \oplus \neg p_3).$$

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- Because we can use linear algebra over $\{0, 1\}$.
- Rewrite a formula as a system of equations over $\mathbb{Z}_2$, and solve.

$$\begin{array}{ccccccc}
p_1 & & & + & p_3 & = & 1 \\
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Can solve these equations using **Gaussian elimination**.

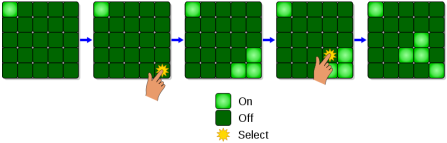
Lights out

Given: An $N \times N$ grid, each button coloured black or white.

Move: Pressing a button inverts colours of it and its neighbours.

Goal: The colours of all buttons are black.

Ex: Translate this to X-SAT.



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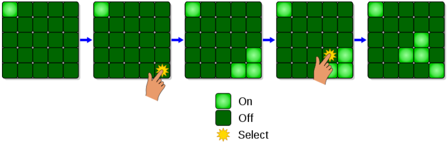

Hint1: Even number of same moves doesn't do anything.

Hint2: Let $p_{i,j}$ denote whether the button (i, j) is pressed, and $c_{i,j}$ be the initial colour of the button (i, j), where $c_{i,j} = true$ means black.

1. Polynomial-time fragments of propositional logic

2. **Walk-SAT: A randomised algorithm for satisfiability**

Walk-SAT: overview

Randomised algorithm for solving SAT for CNF formulas F.

- Guess an assignment for F uniformly at random.
- While there is an unsatisfied clause of F, pick a literal in the clause and flip its truth value.
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Theorem: Walk-SAT on n-variable satisfiable 2-CNF formula for $r = 2mn^2$ succeeds with probability $\geq 1 - 2^{-m}$.

Walk-SAT: algorithm precisely

Input: CNF formula F with n variables, repetition parameter r

pick an assignment (to the n variables) uniformly at random
if F is satisfied **then** return the current assignment
repeat r times
    pick an unsatisfied clause
    pick a literal in the clause uniformly at random, and flip value
    if F is satisfied **then** return the current assignment
return UNSAT

By assignments, we mean maps from the variables in F to $\{0, 1\}$.

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$$T_0 = 0, \qquad T_n \leq 1 + T_{n-1}, \qquad T_i \leq 1 + (T_{i+1} + T_{i-1})/2.$$

Ex: Why do these relationships hold?

Walk-SAT: analysis

- Replacing inequalities by equalities gives bound $H_i \geq T_i$:

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  Proof: Divide $2mn^2$ iterations of the main loop into m phases.
  Markov: not finding a satisfying assignment in a phase has probability $\leq n^2/2n^2 = 1/2$.

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- Walk-SAT: one-dimensional random walk on line $\{0, \ldots, n\}$ with absorbing barrier 0 and reflecting barrier n.

  Similar trick for 3-CNF formulas with probability $2/3$ of going right and $1/3$ of going left

  However, then r needs to be exponential in n.

Summary

- SAT is bad, but we can do better in special cases.
- Horn-SAT, 2-SAT and X-SAT can be solved by polynomial-time algorithms.
- But 3-SAT is as “bad” as the satisfiability of the entire propositional logic.