Proofs for Euler Problem 451

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1 Sufficient condition for I(n) = 1

By Gauss we know that the group $\mathbb{Z}/n\mathbb{Z}^{\times}$ is cyclic if and only if n is $1, 2, 4, p^k, 2p^k$ where p is an odd prime and $k > 0$. For all other values of n the group is not cyclic. We also know that a cyclic group has a unique group of order d for each d dividing the order. In the case of cyclic $\mathbb{Z}/n\mathbb{Z}^{\times}$, this means that the unique subgroup is $\langle -1 \rangle$. However, the conditions of problem 451 do not include $n-1$, so $I(n)$ must be 1 as a result of the uniqueness of the cyclic subgroup of order 2.