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NONLINEAR BUCKLING CHARACTERISTICS OF REGULAR ASYMMETRIC LAMINATES

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Abstract

The paper concerns the nonlinear buckling behavior of asymmetric laminated columns subjected to axial compression. The formulation of the problem is based on large deformation theory, accordingly, the obtained solution affords the exact analysis of the postbuckling structural response. In the study, attention is focused on the behavior of regular asymmetric laminated columns composed of two layers of different materials. The effects of bending-stretching coupling and the imperfection sensitivity of such laminates are examined in detail. The analysis leads to a number of general conclusions. It is shown, in particular, that asymmetric laminated columns with no imperfections have buckling characteristics similar to those of their homogeneous counterparts subjected to eccentric compressive loads. In general, asymmetric laminated columns exhibit stable postbuckling behavior, however, under certain conditions, they undergo bifurcation. In this regard, the concepts of "critical" and "bifurcation" loads for asymmetric laminates are discussed. The results obtained from the analysis are illustrated in graphical form.

Nomenclature

a	load eccentricity
A	area
E	effective elastic modulus
E_k	elastic modulus of k-th layer
h	thickness of composite column
h_k	thickness of k-th layer
I	interia moment
l	length
M	bending moment
P	axial compressive load
P_{cr}	elastic critical load
κ	curvature
γ	moduli ratio

Introduction

In the past decades, the dramatic increase in the use of composites has motivated consistent efforts to enhance the understanding of their behavior and properties. A large number of studies concern, in particluar, the buckling response of laminated structures with emphasis on the analysis of symmetric laminates, see Refs. [1]-[4]. The response of laminated structures with asymmetric material lay-ups has attracted relatively limited attention partly because their performance is complicated by the effects of mechanical couplings that cause, for example, bending due to membrane forces, elongation or contraction due to shear flow, or twisting due to bending moments. Typically, these factors tend to reduce the load carrying capacity of asymmetric composites depending on the stacking sequence, number, thickness and orientation of the individual layers of the laminates.

Based on the studies concerning the buckling behavior of asymmetric laminated structures, Refs. [5]-[15], it is clear that, in general, the buckling characteristics of asymmetric laminates are different than those of their homogeneous counterparts. For example, asymmetric composite plates with no imperfections tend to produce out-of-plane deflections under inplane loading conditions. Moreover, the experimental results reported in Refs. [12] and [13] indicate that asymmetric laminated plates subjected to in-plane compression may suddenly change their initial buckling direction and, at increasing loads, follow an entirely new deformation path. An in-depth understanding of such unparalleled patterns of behavior requires rigorous analyses that, according to the studies in Refs. [8]-[10], [13], [15] and [16]-[18], must be free from the limitations of linear deformation theory.

This paper deals with the nonlinear buckling problem of regular asymmetric laminated columns subjected to an eccentrically applied compressive force. The term "regular", first introduced in Ref. [7], implies that the layers of the laminates are of the same thickness. The objectives of the study are to examine the typical nonlinear buckling characteristics of the structures under consideration, review the concept of the "critical load" in application to asymmetric composites, and analyze the combined effects of bending-stretching couplings and load eccentricity.

Problem Formulation

Consider a laminated composite column with rectanglar cross-section ($b \times h$) and length l. The column is subjected to axial compression by a force P applied to the free end of the column, the other end of which is rigidly fixed. The load is applied eccentrically at a distance a from the geometric center of the column's cross-section.

The column is composed of n different elastic layers bonded together such that the thickness of the laminate

$$h = \sum_{k=1}^{n} h_k \tag{1}$$

where h_k ($k = 1, 2, 3, \ldots, n$) is the thickness of an individual layer. The material properties of the k-th layer are defined by

$$\sigma_k = E_k \epsilon_k, \qquad k = 1, 2, 3, \ldots, n \tag{2}$$

where E_k is the respective elastic modulus. Compatibility between the layers requires that

$$\epsilon_1 = \epsilon_2 = \cdots = \epsilon_n = \epsilon \tag{3}$$

According to the Bernoulli-Euler theory

$$\epsilon = \epsilon_0 - y\kappa \tag{4}$$

where $-\frac{h}{2} \le y \le \frac{h}{2}$; the axial strain ϵ_0 and curvature κ are functions of the curvilinear coordinate s along the deflected axis of the column.

Note that Eq. (4) is defined with respect to the coordinate system Oxy with the origin O at the free end of the column and x directed vertically down as considered in Ref. [19].

In the case of large deformations

$$\kappa = \frac{d\phi}{ds} \tag{5}$$

where ϕ is the slope of the deflected axis of the column with respect to x.

Equations (2)-(4) combined with the equations of equilibrium

$$P = \sum_{k=1}^{n} \int_{A_k} \sigma_k \, dA \tag{6}$$

$$M = \sum_{k=1}^{n} \int_{A_k} \sigma_k z \, dA \tag{7}$$

result in

$$\begin{Bmatrix} P \\ M \end{Bmatrix} = \begin{bmatrix} Q_1 & Q_2 \\ Q_2 & Q_3 \end{bmatrix} \begin{Bmatrix} \epsilon_0 \\ -\kappa \end{Bmatrix} \tag{8}$$

where

$$Q_1 = \sum_{k=1}^{n} q_{1k} E_k \tag{9}$$

$$Q_2 = \sum_{k=1}^{n} q_{2k} E_k \tag{10}$$

$$Q_3 = \sum_{k=1}^{n} q_{3k} E_k \tag{11}$$

in which

$$q_{1k} = \int_{A_k} dA \tag{12}$$

$$q_{2k} = \int_{A_k} y \, dA \tag{13}$$

$$q_{3k} = \int_{A_k} y^2 \, dA \tag{14}$$

and A_k is the cross-sectional area of the k-th layer of the laminate. Two simultaneous Eqs. (8) can be reduced to a single differential equation

$$-\kappa \left[Q_1 Q_3 - (Q_2)^2 \right] = Q_1 M - Q_2 P \tag{15}$$

in which

$$M = P \left[y + a (\cos\phi)_{s=0} \right] \tag{16}$$

Substituting Eqs. (5) and (16) into Eq. (15), differentiating by s and introducing the relation

$$\frac{dy}{ds} = \sin\phi \tag{17}$$

results in the governing equation

$$\frac{d^2\phi}{ds^2} + P \frac{Q_1}{Q_1 Q_3 - (Q_2)^2} \sin\phi = 0 \tag{18}$$

in which the function ϕ must satisfy the boundary conditions

$$\phi|_{s=l} = 0 \tag{19}$$

and

$$\left. \frac{d\phi}{ds} \right|_{s=0} = P \frac{Q_2 - Q_1 a (\cos\phi)_{s=0}}{Q_1 Q_3 - (Q_2)^2} \tag{20}$$

Note that Eqs. (18)-(20) describe as special cases the nonlinear buckling responses of homogeneous and symmetrically laminated columns for which the problem becomes uncoupled because $Q_2 = 0$. Accordingly, Eqs. (18) and (20) reduce to

$$\frac{d^2\phi}{ds^2} + \frac{P}{Q_3} \sin\phi = 0 \tag{21}$$

$$\left. \frac{d\phi}{ds} \right|_{s=0} = \frac{-a (\cos\phi)_{s=0}}{Q_3} \tag{22}$$

In the case of homogeneous columns, $Q_3 = EI$, where E and I denote, respectively, the elastic modulus of the material and the inertia moment of the cross-section of the column.

Note that for small deformations, Eqs. (18)-(20) can be linearized using the assumptions $\sin\phi \simeq \phi$, $\cos\phi \simeq 1$, and $s \simeq x$. As a result, the problem under consideration is defined by the linear governing equation

$$\frac{d^2y}{dx^2} + P \frac{Q_1 y}{Q_1 Q_3 - (Q_2)^2} = P \frac{Q_2 - a Q_1}{Q_1 Q_3 - (Q_2)^2} \tag{23}$$

and the boundary conditions

$$y|_{x=l} = 0 \tag{24}$$

$$\left. \frac{dy}{dx} \right|_{x=l} = 0 \tag{25}$$

Analysis of Two Layer Column

Consider a regular laminated column composed of two layers of different materials, i.e.

$$h_1 = h_2 = \frac{h}{2}, \text{ and } E_1 \neq E_2 \tag{26}$$

In this case, the coefficients $Q_i (i = 1, 2, 3)$ defined by Eqs. (9)-(11) are of the form

$$Q_1 = EA \tag{27}$$

$$Q_2 = \frac{EAh}{4}\left(\frac{1-\gamma}{1+\gamma}\right) \tag{28}$$

$$Q_3 = EI \tag{29}$$

where

$$\gamma = \frac{E_1}{E_2} \tag{30}$$

E represents the effective modulus of the column,

$$E = \frac{E_1 + E_2}{2} \tag{31}$$

It follows from Eqs. (30) and (31) that

$$E_1 = \frac{2\gamma}{1+\gamma}E \tag{32}$$

and

$$E_2 = \frac{2}{1+\gamma}E \tag{33}$$

Substituting Eqs. (27)-(29) into Eqs. (18)-(20) leads to the problem formulation in the form

$$\frac{d^2\phi}{d\xi^2} + \frac{k_e^2}{\psi_2(\gamma)}\sin\phi = 0 \tag{34}$$

$$\phi|_{\xi=1} = 0 \tag{35}$$

$$\left.\frac{d\phi}{d\xi}\right|_{\xi=0} = \frac{k_e^2}{\psi_2(\gamma)}\frac{h}{2l}\left[\psi_1(\gamma) - \mu\cos(\phi)_{\xi=0}\right] \tag{36}$$

where $\xi = s/l$, $0 \leq \mu = 2a/h \leq 1$, $k_e^2 = \pi^2\lambda/4$, and $\lambda = P/P_{cr}$, in which $P_{cr} = \pi^2 EI/4l^2$ denotes the critical load of identical homogeneous column with the elastic modulus E. In Eqs. (34)-(36), the functions ψ_1 and ψ_2 are defined by

$$\psi_1 = \frac{1-\gamma}{2(1+\gamma)} \tag{37}$$

$$\psi_2 = 1 - \frac{3}{4}\left(\frac{1-\gamma}{1+\gamma}\right)^2 \tag{38}$$

In the present study, the solution of Eq. (34) with the boundary conditions, Eqs. (35) and (36), is obtained using software for bifurcation analysis, *AUTO*, Ref. [20].

It is clear from Eqs. (34)-(36) that the buckling behavior of the column depends on the material composition, the slenderness ratio of the laminate, and the load eccentricity, as defined by $\gamma, h/l$, and μ, respectively.

First, consider a composite column characterized by $h/l = 0.1$ and subjected to axial compression by a perfectly centered force, i.e., $\mu = 0$.

The buckling response of such a column for γ ranging from 0.01 to 100 is depicted in Fig. 1. The special case of $\gamma = 1$ represents the symmetrical bifurcation path of a geometrically identical homogeneous column with the same elastic modulus as the effective modulus of the laminate.

The diagrams shown in Fig. 1 also indicate that the buckling behavior of asymmetric laminated columns subjected to a perfectly centered axial compressive force is similar to that of eccentrically loaded homogeneous columns. The "ideal"

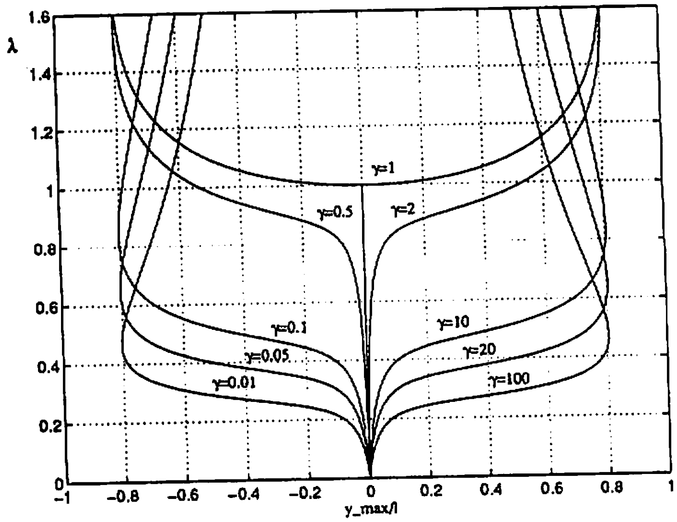

Figure 1: Nonlinear Buckling Response of Two Layer Regular Laminated Column for Different Values of γ ($h/l = 0.1, \mu = 0$).

asymmetric composites do not follow the bifurcation path of their homogeneous counterparts, hence, their buckling response cannot be characterized using the concept of the classical critical load. This conclusion confirms previous results reported in Ref. [12] regarding the buckling behavior of asymmetric composite plates.

A more general interpretation of the critical load for asymmetric laminates can be deduced from the following observation. According to the diagrams in Fig. 1, at a certain magnitude of the load, the lateral deflections of asymmetric columns begin to increase at accelerating rates. Although the columns appear capable of sustaining greater loads, intensified buckling deformations may produce undesirable effects leading to the loss of controlled behavior of the structures. Respectively, the magnitude of the load associated with the latter type of behavior can be defined as "critical". Coincidentally, as shown in Fig. 2, the magnitude of this critical load can be readily determined from the linear buckling analysis since, at this point, the respective lateral deflections of the structures tend to infinity.

In general, according to Fig. 1, asymmetric composite columns with no imperfections exhibit stable postbuckling behavior. The structures tend to deform toward the side having the weaker layer, i.e., the layer with the relatively smaller elastic modulus as defined by the parameter γ. Specifically, the deflections are positive when $\gamma > 1$, or $E_1 > E_2$, and vice versa. The magnitude of the critical buckling load as defined above tends to decrease for increasing γ; however, at relatively large values of γ, the buckling response of the columns becomes insensitive to further increases of this parameter.

Consider further the response of asymmetric composite columns subjected to an eccentrically applied compressive force. In this case, $\mu \neq 0$ and, according to Eq. (36), the behavior of the laminates is governed by the combined effects of bending-stretching coupling and load eccentricity. As an example, the buckling characteristics of a column with $\gamma = 2$ are shown in Fig. 3 for various values of μ. These diagrams indicate that the deflections of the column are either amplified or reduced depending on the sign and magnitude of μ. It is of interest that, when $\mu = -1/6$, the asymmetric composite column exhibits

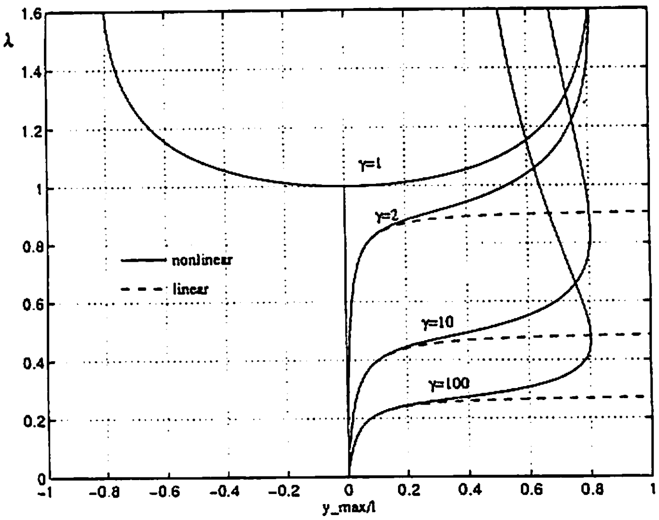

Figure 2: Comparison of Buckling Characteristics Obtained from Nonlinear and Linear Analyses.

bifurcation at a load slightly less than the Euler critical load of the respective homogeneous column with the same elastic modulus as the effective modulus of the laminate.

It is important to note that the buckling problem defined by Eqs. (34)-(36) reduces to an eigenvalue problem when the boundary conditions are homogeneous. Thus, assuming that, in Eq. (36), $\cos\phi = 1$, one obtains the bifurcation condition for eccentrically loaded asymmeytric laminated columns in the form

$$\psi_1(\gamma) = \bar{\mu} \tag{39}$$

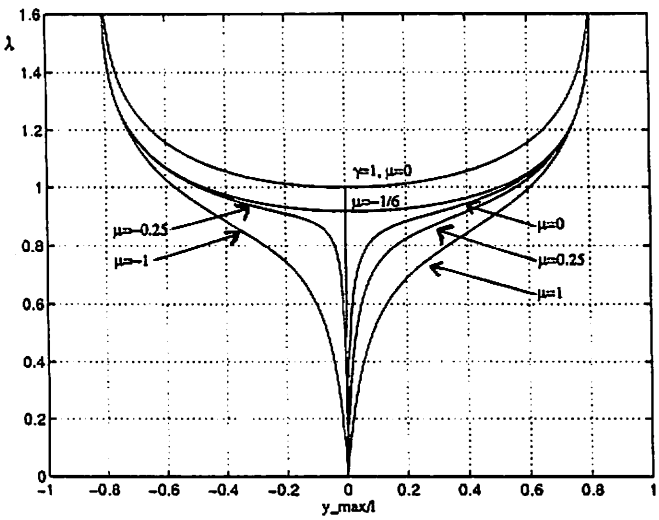

Figure 3: Nonlinear Buckling Response of Eccentrically Loaded Column ($\gamma = 2, h/l = 0.1$).

where γ is defined by Eq. (30), and $\bar{\mu}$ denotes the value of μ which, for a given γ, satisfies Eq. (39). Combining Eqs. (30) and (39) leads to the relation

$$\bar{\mu} = \frac{1-\gamma}{2(1+\gamma)} \tag{40}$$

the diagram of which is shown in Fig. 4. Note that, in the case of a homogeneous column, $\gamma = 1$ and $\bar{\mu} = 0$. As $\gamma \to \infty$, $\bar{\mu} \to -0.5$, which implies that for any value of γ, on can determine the respective parameter $\bar{\mu}$ within the limits $0 \leq \bar{\mu} \leq 1$ such that the bifurcation condition for asymmetric composite columns, Eq. (40), is satisfied.

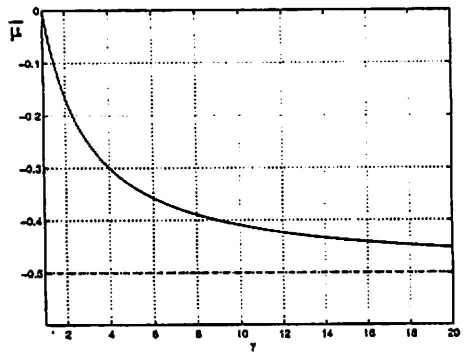

Figure 4: Bifurcation Condition as Defined by Eq. (40).

Since, as a special case, asymmetric laminated columns exhibit bifurcation due to the combined effects of bending-stretching coupling and load eccentricity, it is appropriate to determine the magnitude of the respective bifurcation load, $\bar{\lambda}$. This load is obtained in the form

$$\bar{\lambda} = \psi_2(\gamma) = 1 - \frac{3}{4}\left(\frac{1-\gamma}{1+\gamma}\right)^2 \tag{41}$$

which represents the first eigenvalue of Eq. (34) with $\psi_2(\gamma)$ defined by Eq. (38). The diagram of Eq. (41) shown in

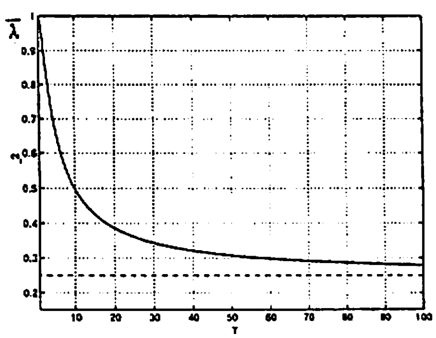

Figure 5: Bifurcation Load as Defined by Eq. (41).

Fig. 5 indicates that, for $\gamma = 1$, $\bar{\lambda} = 1$, and as γ increases, the magnitude of the critical load tends to decrease. It follows from Eq. (41) that as $\gamma \to \infty$, $\bar{\lambda} \to 0.25$. The bifurcation response of asymmetric laminated columns for different values of γ and μ is illustrated in Fig. 6.

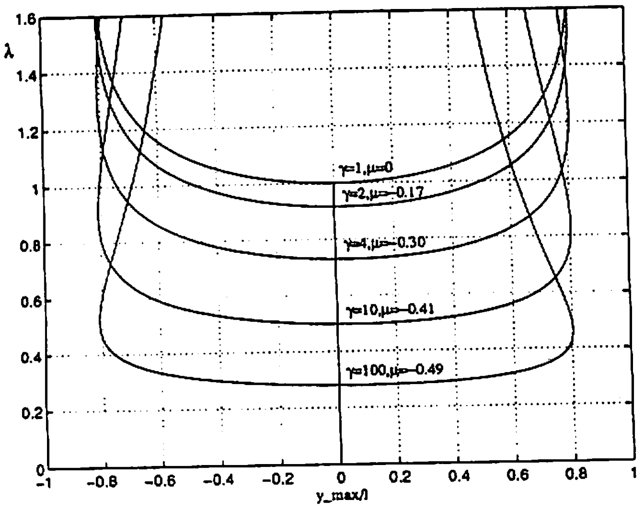

Figure 6: Bifurcation Response of Eccentrically Loaded Asymmetric Laminated Columns.

Conclusions

In this paper, the buckling and postbuckling behavior of eccentrically compressed asymmetric laminated columns is examined. The formulation of the problem is based on geometrically nonlinear deformation theory. Numerical results are obtained for regular asymmetric laminated columns composed of two layers of different elastic materials bonded together. The respective nonlinear buckling characteristics of the laminates are examined in detail leading to a number of general conclusions.

It is shown that asymmetric composite columns tend to produce lateral deflections under the action of a perfectly centered compressive force. This type of behavior of asymmetric laminates is similar to that of their homogeneous counterparts with imperfections. Hence, the concept of classical "critical load" can no longer be applied to characterize the buckling behavior of asymmetrically laminated structures. A more general definition of the critical load for asymmetric laminates is suggested to be the load associated with the onset of large postbuckling deformations.

Further, the analysis concerns the combined effects of bending-stretching coupling and load eccentricity on the buckling response of asymmetric laminates. The results indicate that, under certain conditions, regular asymmetric composite columns tend to exhibit bifurcation at a load less than the Euler critical load of homogeneous columns. In this paper, this bifurcation condition and the respective bifurcation load are determined analytically as functions of the ratio between the elastic moduli of the individual layers of the laminate. In a number of special cases, the nonlinear bifurcation characteristics of asymmetric composite columns are computed and presented in graphical form.

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