

LoadDef Tutorials

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Setup Anaconda Environment

```
conda config --prepend channels conda-forge
```

```
conda config --set channel_priority strict
```

```
conda create -n LDEF python=3.7 basemap mpi4py scipy netCDF4 pyshp shapely numba jupyter
```

Getting started

- Download LoadDef as a zip file from GitHub
 - <https://github.com/hrmartens/LoadDef>
 - Look for the green-colored button that says “Code”
- Unpack the zip folder
- Activate the conda environment:
 - >> conda activate LDEF
- Launch a Jupyter Notebook:
 - >> jupyter notebook

LOAD GREEN'S FUNCTIONS

★ Vertical and Horizontal
Displacement, Gravity,
Tilt, and Strain ★

LOVE NUMBERS

★ h, l, k ★
★ Potential, Load,
and Shear ★

Main program:
`run_ln.py`
Basic input:
SNREI planetary-
structure model

Main program: `run_gf.py`
Basic input: Load Love numbers

LOADDEF

PARTIAL DERIVATIVES OF LOVE NUMBERS

★ h, l, k ★
★ Potential, Load, and Shear ★

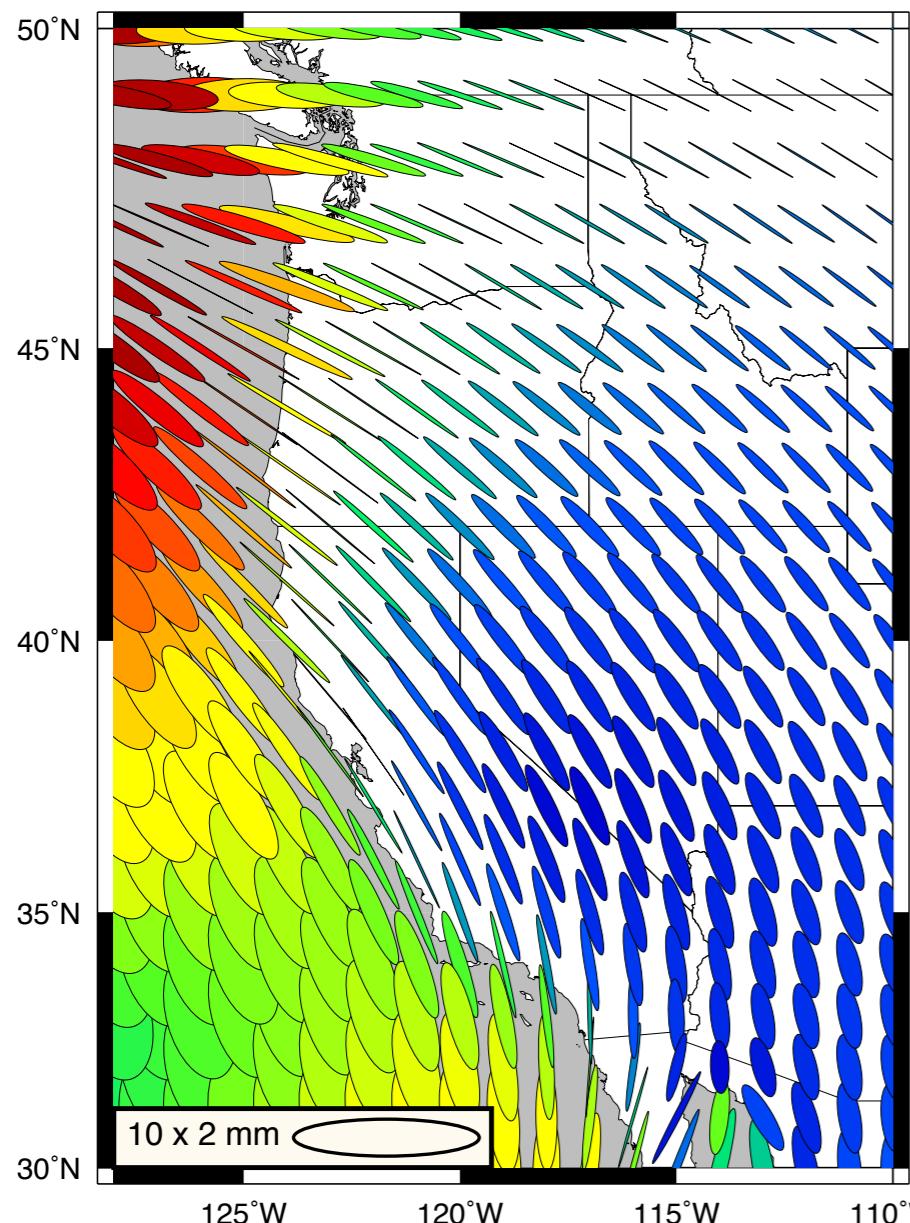
Main program: `run_pl.py`
Basic input: SNREI planetary-
structure model

LOAD-INDUCED SURFACE DISPLACEMENTS

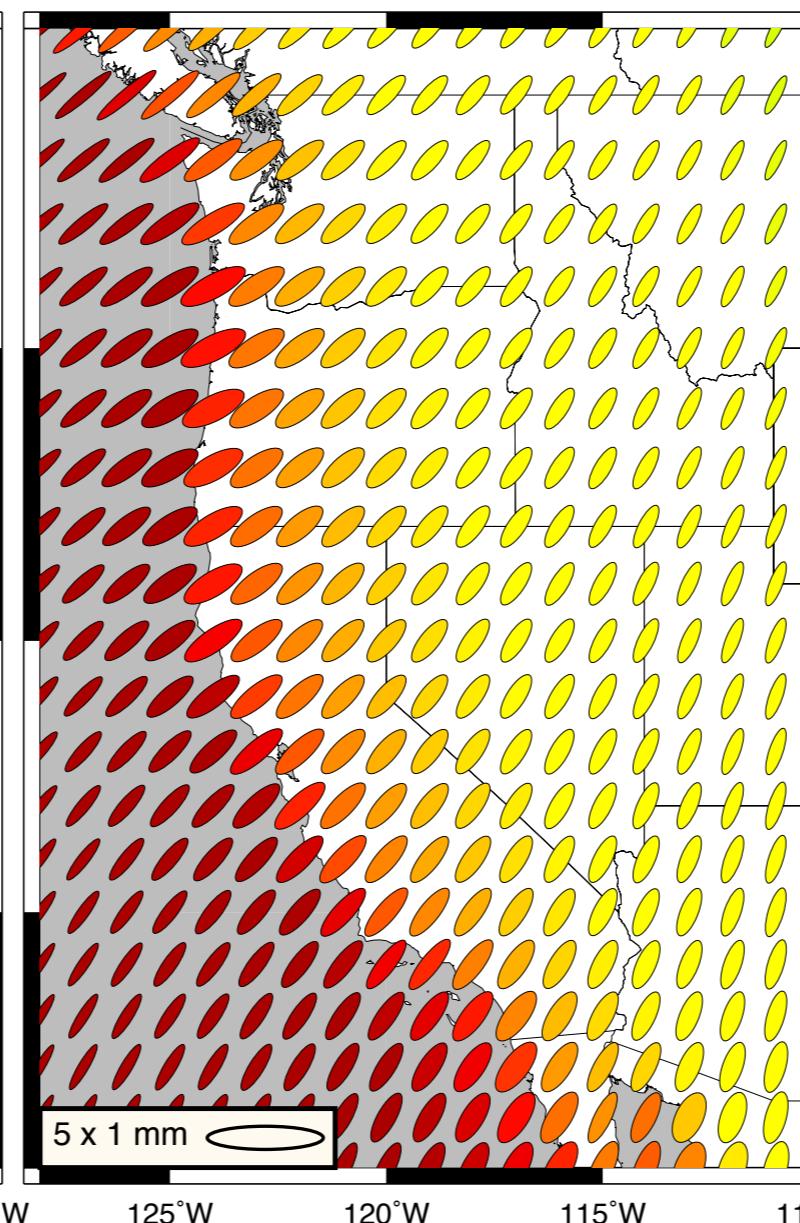
★ East, North, Up ★

Main program:
`run_cn.py`
Basic inputs:
(1) Load Green's
functions; (2) Surface-
load distribution

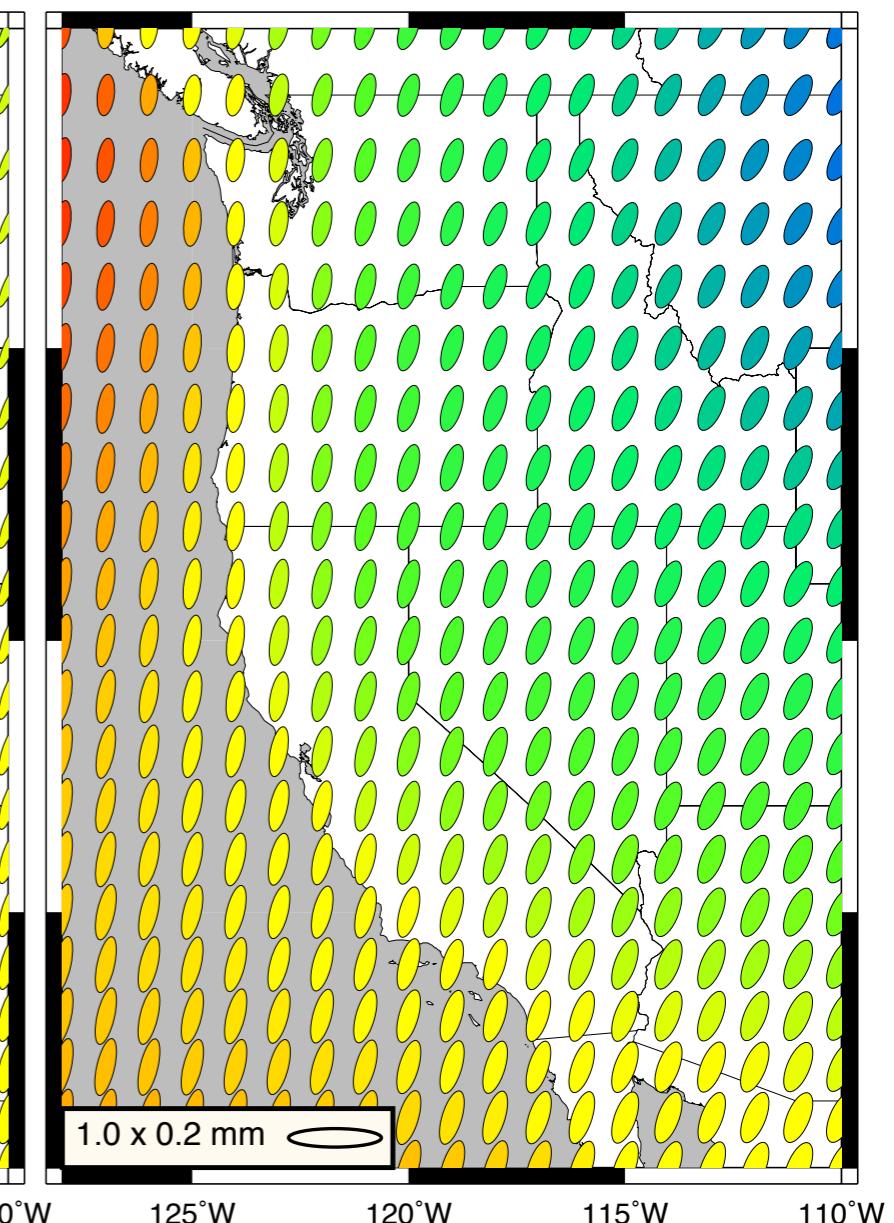
a M_2 OTL Displacement | CM Frame



b O_1 OTL Displacement | CM Frame



c M_f OTL Displacement | CM Frame



0 5 10 15 20 25 30
Vertical displacement (mm)

0 3 6 9 12 15
Vertical displacement (mm)

0.00 0.25 0.50 0.75 1.00
Vertical displacement (mm)

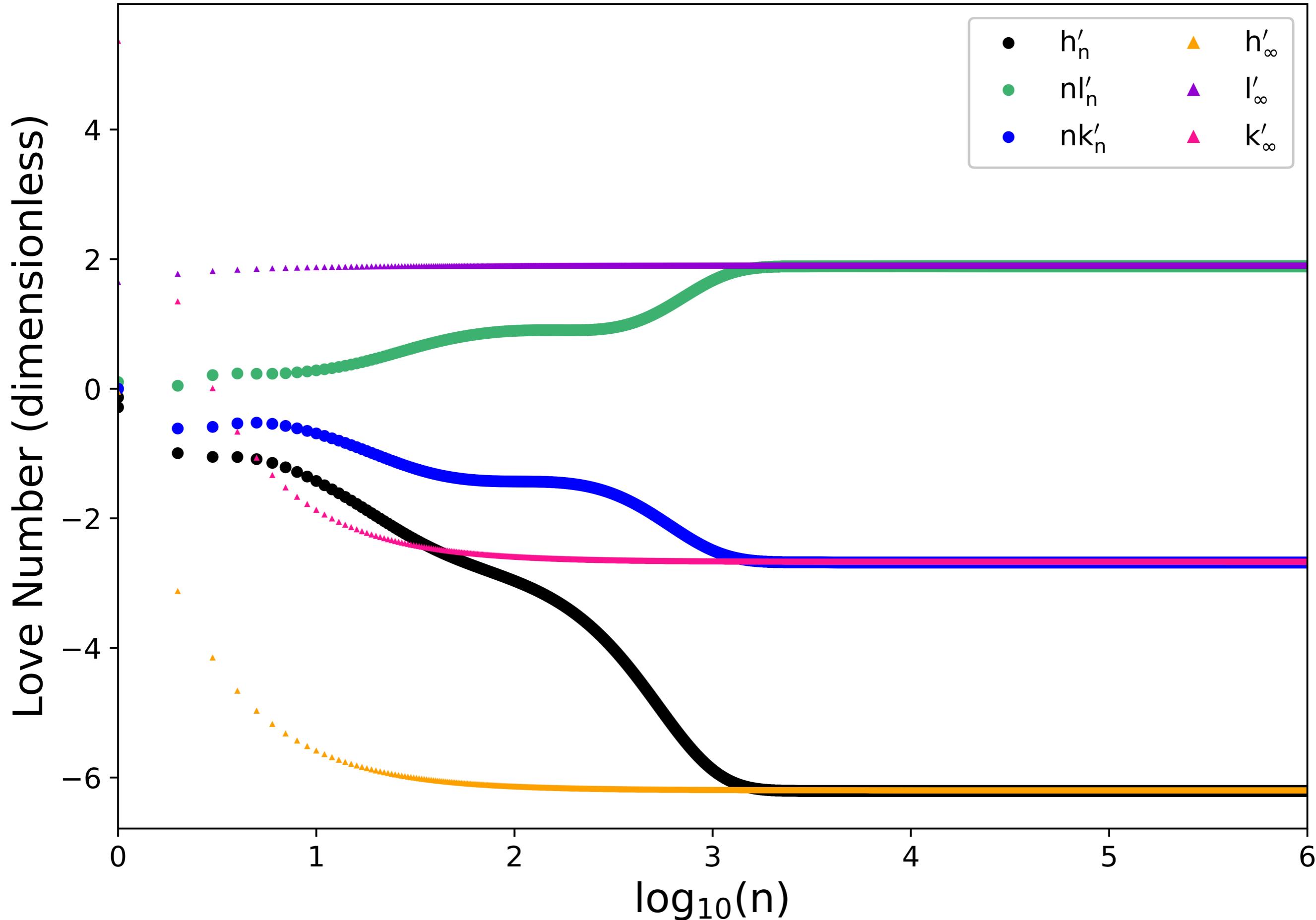
Computing Love Numbers

- **Option A – From Jupyter Notebook:**
 - Navigate to the web browser
 - Launch “working/run_ln.ipynb”
 - Click “run”
- **Option B – From command line:**
 - >> cd working
 - >> mpirun -np 1 python run_ln.py

Plotting Love Numbers

- **Option A** – From Jupyter Notebook:
 - Launch “utility/plots/run_ln.ipynb”
 - Click “run”
- **Option B** – From command line:
 - >> cd ..utility/plots
 - >> python plot_ln.py

Load Love Numbers

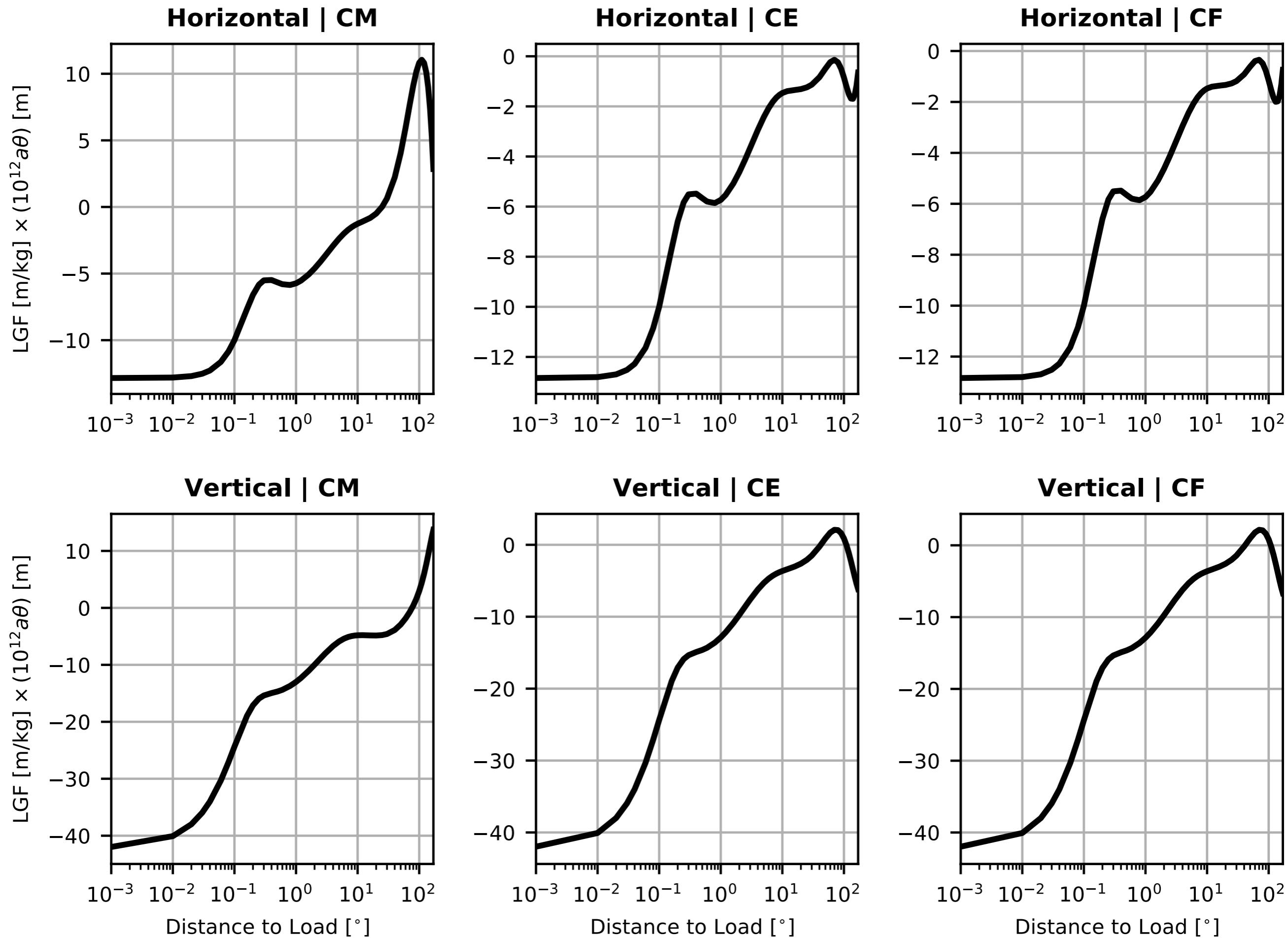


Computing Green's Functions

- Option A – From Jupyter Notebook:
 - Launch “working/run_gf.ipynb”
 - Click “run”
- Option B – From command line:
 - >> cd ../../working
 - >> mpirun -np 1 python run_gf.py

Plotting Green's Functions

- **Option A** – From Jupyter Notebook:
 - Launch “utility/plots/plot_gf.ipynb”
 - Click “run”
- **Option B** – From command line:
 - >> cd ..utility/plots
 - >> python plot_gf.py

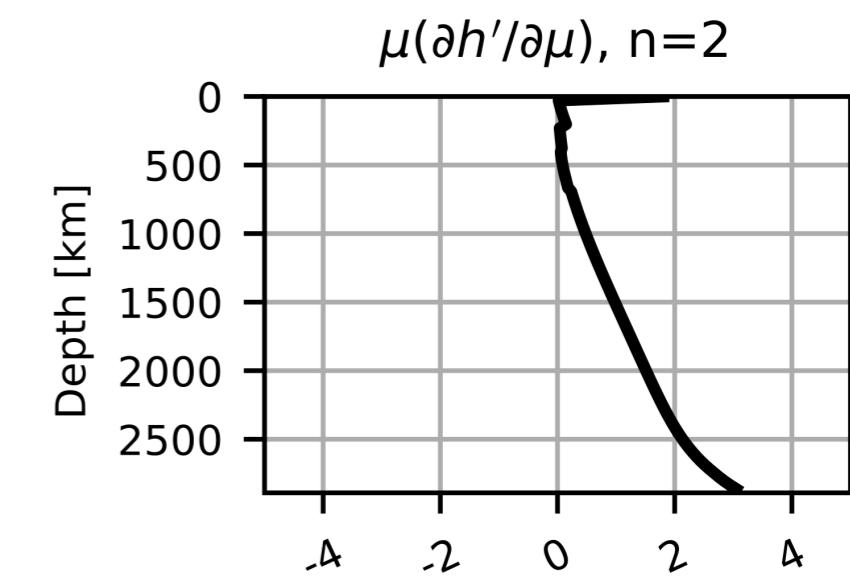
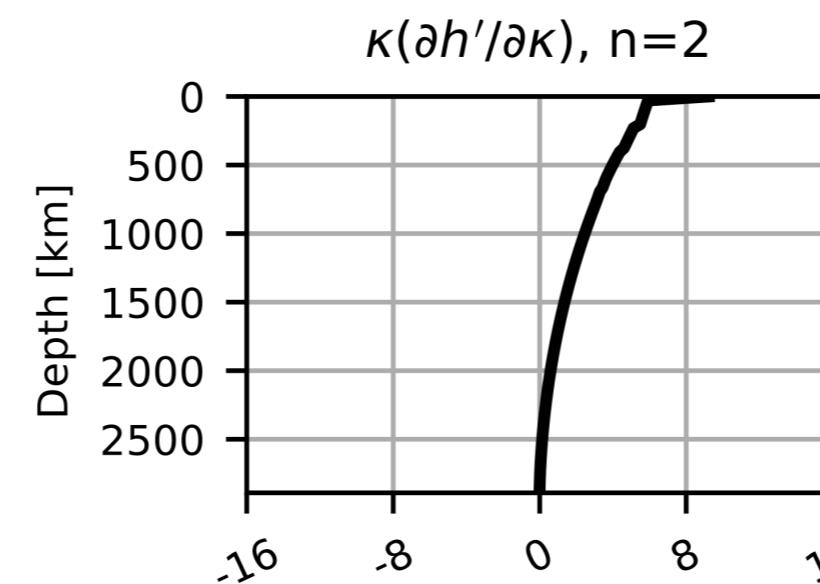
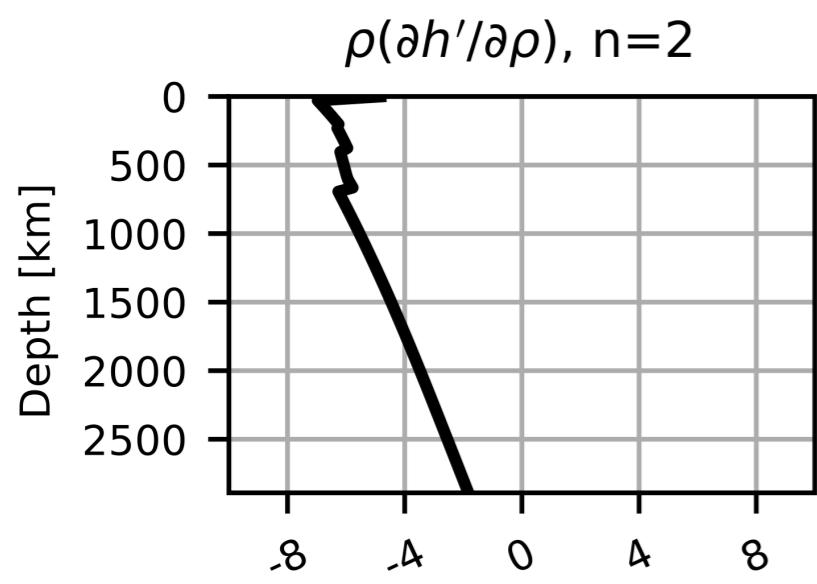
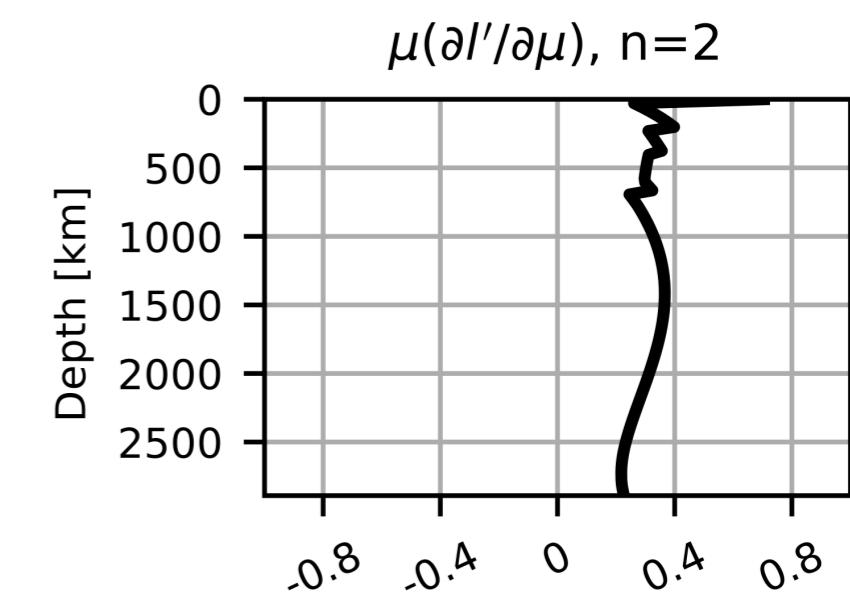
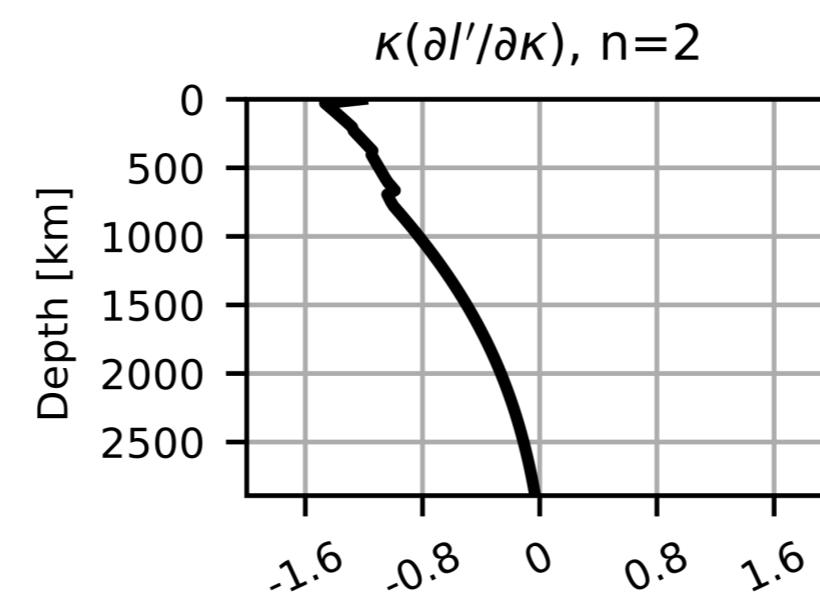
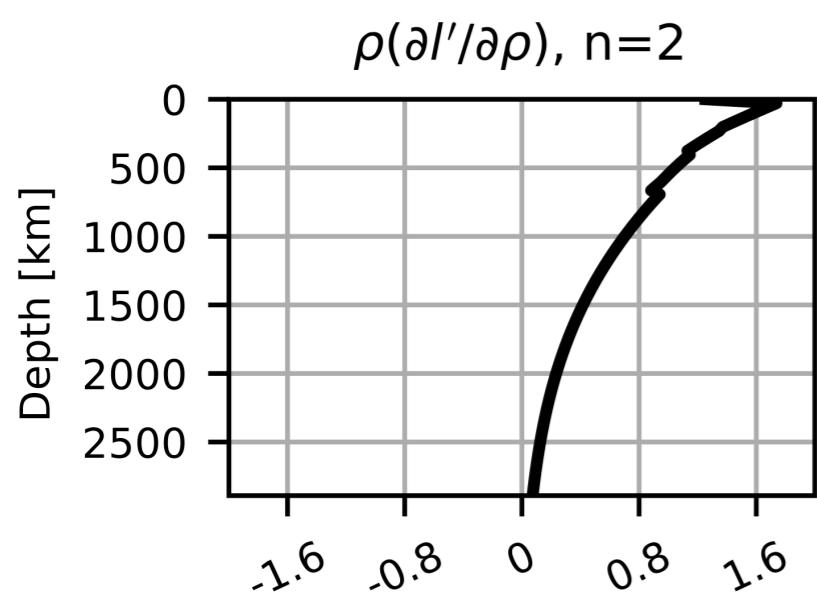
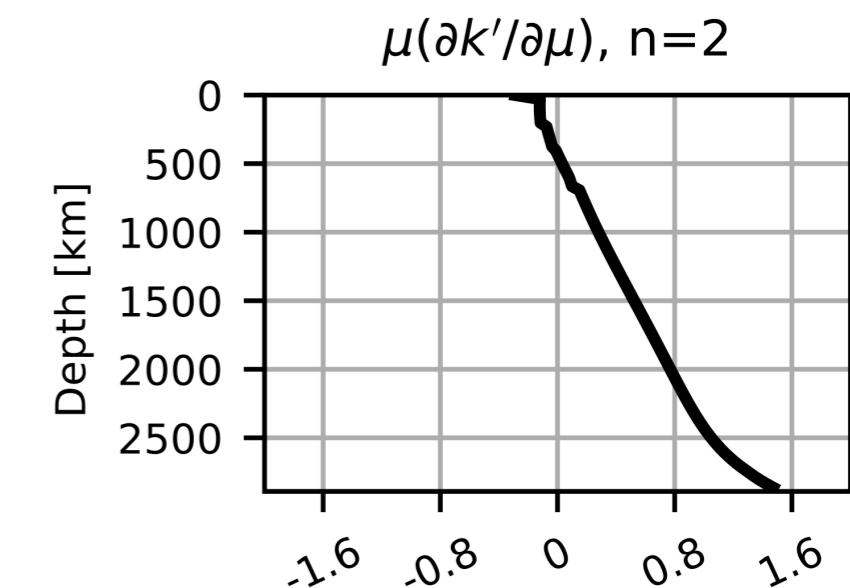
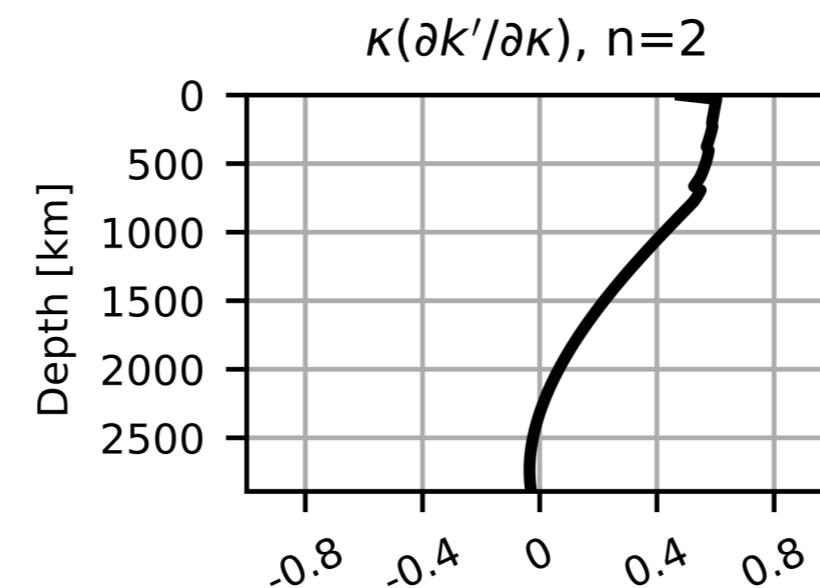
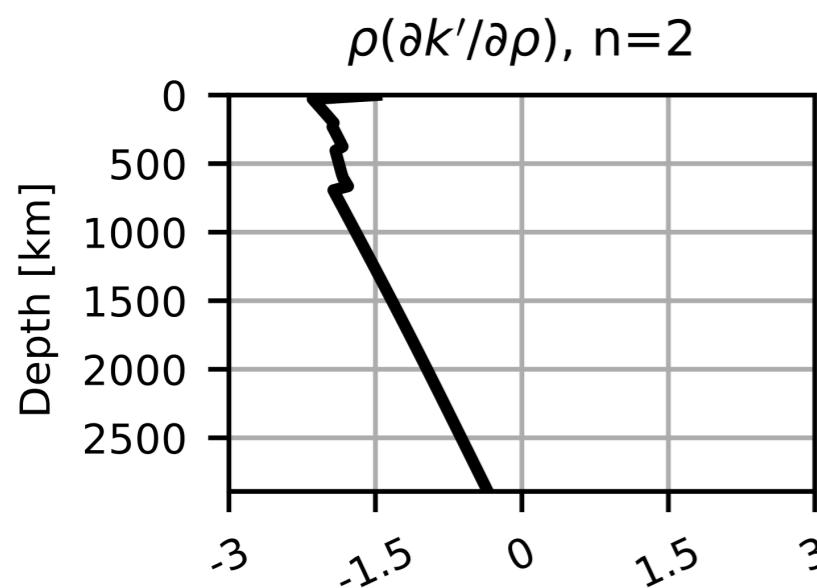


Computing Partial Derivatives of Love Numbers

- **Option A** – From Jupyter Notebook:
 - Launch “working/run_pl.ipynb”
 - Click “run”
- **Option B** – From command line:
 - >> cd ../../working
 - >> mpirun -np 1 python run_pl.py

Plotting Partial Derivatives of Love Numbers

- **Option A** – From Jupyter Notebook:
 - Launch “utility/plots/plot_pl.ipynb”
 - Click “run”
- **Option B** – From command line:
 - >> cd ..utility/plots
 - >> python plot_pl.py



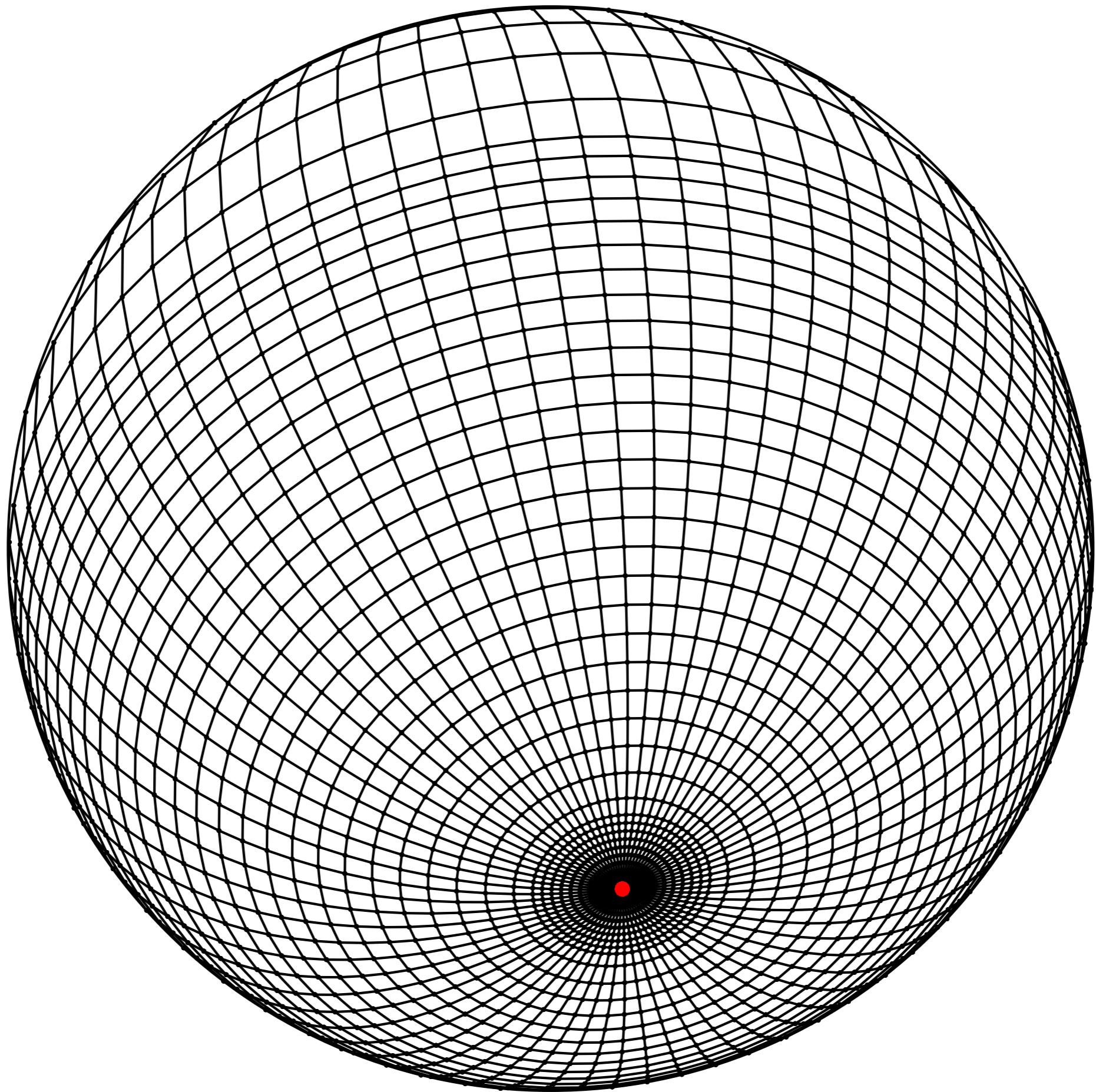
Predicted Load-Induced Displacements

$$\overline{U}(\bar{r}, S, Z, \rho) = \int_{\Omega'} G(|\bar{r}' - \bar{r}|, S) \rho(\bar{r}') Z(\bar{r}') d\Omega'$$

Displacement Displacement/Force Force

- Response at observation point r
- Load Green's function
- Spherically symmetric structure
- Load density at load point r'
- Load height at load point r'
- Integrate over Earth's surface

To generate the forward model, both the load model ($Z\rho$) and the load Green's functions (G) must be given



Copy external load files into appropriate directories

- **IMPORTANT:** The GRACE and tide models cannot be shared beyond this class. If you want to use them in your research, you must register for access to each data product.
- Place the GRACE data file into:
 - `/input/Load_Models/GRACE-Tellus-RL06/`
- Place the got410c data files into:
 - `/input/Load_Models/GOT410c/`

Four Example Applications

- Hydrological loading from GRACE mascons
- Disk loading
- Ocean tidal loading
- Creating a design matrix to use for inversions

Example 1: GRACE

- Example 1: Compute time series of displacements caused by global hydrological loading at the locations of GPS stations
 - Use data from GRACE mascon solutions
 - Convolve with load Green's functions for PREM
- Step 1: Compute Love numbers and Green's functions for PREM.
- Step 2: Create a load grid for GRACE mascons.
- Step 3: Run the convolution.

Generating the Load Grids

- **Option A** – From Jupyter Notebook:
 - Launch “GRDGEN/load_files/gen_grace_tellus_rl06.ipynb”
 - Click “run”
- **Option B** – From command line:
 - >> cd ../../GRDGEN/load_files
 - >> python gen_grace_tellus_rl06.py

Run the Convolution

- Option A – From Jupyter Notebook:
 - Launch “working/run_cn_grace.ipynb”
 - Click “run”
- Option B – From command line:
 - >> cd ../../working
 - >> mpirun -np 1 python run_cn_grace.py

Plot the Results

- **Option A – From Jupyter Notebook:**
 - Launch “utility/plots/plot_ts.ipynb”
 - Click “run”
- **Option B – From command line:**
 - >> cd ..utility/plots
 - >> python plot_ts.py

Where are the Stations?

- [https://www.unavco.org/instrumentation/networks/
map/map.html#!/](https://www.unavco.org/instrumentation/networks/map/map.html#!/)
- [http://geodesy.unr.edu/NGLStationPages/
GlobalStationList](http://geodesy.unr.edu/NGLStationPages/
GlobalStationList)
- [http://geodesy.unr.edu/NGLStationPages/gpsnetmap/
GPSNetMap.html](http://geodesy.unr.edu/NGLStationPages/gpsnetmap/
GPSNetMap.html)

Example 2: Disks

- Example 2: Compute peak displacements cause by disk loads of various sizes
 - Create disk loads
 - Convolve with load Green's functions for PREM
- Step 1: Compute Love numbers and Green's functions for PREM.
- Step 2: Create a load grid for disks with various radii.
- Step 3: Run the convolution.

Generating the Load Grids

- **Option A – From Jupyter Notebook:**
 - Launch “GRDGEN/load_files/gen_disk.ipynb”
 - Click “run”
- **Option B – From command line:**
 - >> cd ../../GRDGEN/load_files
 - >> python gen_disk.py
- **Repeat for a disk of radius 25 km**

Running the Convolution

- **Option A** – From Jupyter Notebook:
 - Launch “working/run_cn_disk.ipynb”
 - Click “run”
- **Option B** – From command line:
 - >> cd ../../working/
 - >> mpirun -np 2 python run_cn_disk.py

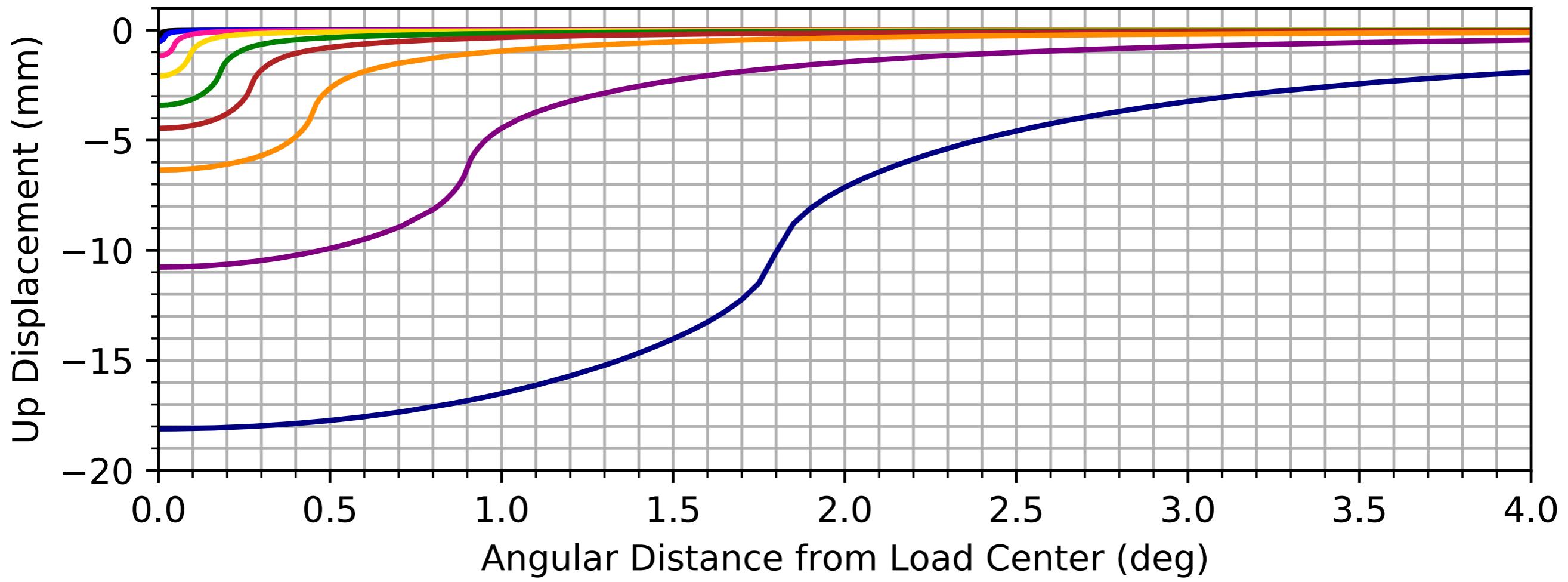
Combining the Stations

- Option A – From Jupyter Notebook:
 - Launch “utility/pmes/run_combine_stations.ipynb”
 - Click “run”
- Option B – From command line:
 - >> cd ..utility/pmes/
 - >> python run_combine_stations.py

Plotting the Results

- **Option A** – From Jupyter Notebook:
 - Launch “utility/plots/plot_dk.ipynb”
 - Click “run”
- **Option B** – From command line:
 - >> cd ../plots/
 - >> python plot_dk.py

Disks | PREM | 1 m Fresh Water



Example 3: Load Tides

- Example 3: Compute surface displacements caused by ocean tidal loading
 - Generate a load grid for an ocean-tide model
 - Convolve with load Green's functions for PREM
- Step 1: Compute Love numbers and Green's functions for PREM.
- Step 2: Create a load grid for an ocean-tide model.
- Step 3: Run the convolution.

Generating the Load Grids

- **Option A – From Jupyter Notebook:**
 - Launch “GRDGEN/load_files/gen_otl.ipynb”
 - Click “run”
- **Option B – From command line:**
 - >> cd ../../GRDGEN/load_files
 - >> python gen_otl.py
- **Optionally repeat for O1 tide**

Running the Convolution

- **Option A** – From Jupyter Notebook:
 - Launch “working/run_cn_olt.ipynb”
 - Click “run”
- **Option B** – From command line:
 - >> cd ../../working/
 - >> mpirun -np 1 python run_cn_olt.py

Combining the Harmonics

- **Option A** – From Jupyter Notebook:
 - Launch “utility/pmes/run_combine_harmonics.ipynb”
 - Click “run”
- **Option B** – From command line:
 - >> cd ..utility/pmes/
 - >> python run_combine_harmonics.py

Check out the Results

- **Option A** – From Jupyter Notebook:
 - Navigate into the “utility/pmes/output” folder
 - Open the file called:
`cn_OceanOnly_M2_cm_convgf_GOT410c_PREM.txt`
- **Option B** – From command line:
 - `>> cd ./output`
 - `>> more cn_OceanOnly_M2_cm_convgf_GOT410c_PREM.txt`

Example 4: Design Matrix

- **Example 3:** Compute a design matrix to solve for load distribution given a set of surface-displacement observations
 - Generate the grid cells
 - Convolve with load Green's functions for PREM
- **Step 1:** Compute Love numbers and Green's functions for PREM.
- **Step 2:** Create a grid of cells.
- **Step 3:** Run the convolution.

Generating the Load Grids

- **Option A** – From Jupyter Notebook:
 - Launch “GRDGEN/design_matrix/gen_cells.ipynb”
 - Click “run”
- **Option B** – From command line:
 - >> cd ../../GRDGEN/design_matrix
 - >> python gen_cells.py

Running the Convolution

- **Option A** – From Jupyter Notebook:
 - Launch “working/run_dm.ipynb”
 - Click “run”
- **Option B** – From command line:
 - >> cd ../../working/
 - >> mpirun -np 1 python run_dm.py

Theory

Equations of motion for spheroidal oscillations

$$\rho \bar{a} = \bar{\nabla} \cdot \bar{\bar{\sigma}} + \rho \bar{F}$$

$$\nabla^2 \psi_1 = -4\pi G \rho_1$$

$$\frac{dy}{dr} = Ay$$

System of six first-order differential equations

radial displacement and stress

tangential displacement and stress

gravitational potential

$$\begin{aligned}
 \dot{y}_1 &= \frac{-2\lambda}{\lambda+2\mu} \frac{y_1}{r} + \frac{y_2}{\lambda+2\mu} + \frac{\lambda n(n+1)}{\lambda+2\mu} \frac{y_3}{r}, \\
 \dot{y}_2 &= \left[-\omega^2 \rho r^2 - 4\rho g r + \frac{4\mu(3\lambda+2\mu)}{\lambda+2\mu} \right] \frac{y_1}{r^2} - \frac{4\mu}{\lambda+2\mu} \frac{y_2}{r} \\
 &\quad + \left[n(n+1)\rho g r - \frac{2\mu(3\lambda+2\mu)n(n+1)}{\lambda+2\mu} \right] \frac{y_3}{r^2} \\
 &\quad + n(n+1) \frac{y_4}{r} - \rho y_6, \\
 \dot{y}_3 &= -\frac{y_1}{r} + \frac{y_3}{r} + \frac{y_4}{\mu}, \\
 \dot{y}_4 &= \left[g \rho r - \frac{2\mu(3\lambda+2\mu)}{\lambda+2\mu} \right] \frac{y_1}{r^2} - \frac{\lambda}{\lambda+2\mu} \frac{y_2}{r} \\
 &\quad + \left[-\omega^2 \rho r^2 + \frac{2\mu}{\lambda+2\mu} [\lambda(2n^2+2n-1) + 2\mu(n^2+n-1)] \right] \frac{y_3}{r^2} \\
 &\quad - \frac{3y_4}{r} - \rho \frac{y_5}{r}, \\
 \dot{y}_5 &= 4\pi G \rho y_1 + y_6, \\
 \dot{y}_6 &= -4\pi G \rho n(n+1) \frac{y_3}{r} + n(n+1) \frac{y_5}{r^2} - \frac{2y_6}{r},
 \end{aligned}$$

- Function of elastic and density structure
- Function of spherical harmonic degree

Equations of motion for spheroidal oscillations

$$\frac{dy}{dr} = Ay$$

radial displacement
and stress

tangential
displacement and
stress

gravitational
potential

$$\begin{aligned}
 \dot{y}_1 &= \frac{-2\lambda}{\lambda + 2\mu} \frac{y_1}{r} + \frac{y_2}{\lambda + 2\mu} + \frac{\lambda n(n+1)}{\lambda + 2\mu} \frac{y_3}{r}, \\
 \dot{y}_2 &= \left[-\omega^2 \rho r^2 - 4\rho g r + \frac{4\mu(3\lambda + 2\mu)}{\lambda + 2\mu} \right] \frac{y_1}{r^2} - \frac{4\mu}{\lambda + 2\mu} \frac{y_2}{r} \\
 &\quad + \left[n(n+1)\rho g r - \frac{2\mu(3\lambda + 2\mu)n(n+1)}{\lambda + 2\mu} \right] \frac{y_3}{r^2} \\
 &\quad + n(n+1) \frac{y_4}{r} - \rho y_6, \\
 \dot{y}_3 &= -\frac{y_1}{r} + \frac{y_3}{r} + \frac{y_4}{\mu}, \\
 \dot{y}_4 &= \left[g \rho r - \frac{2\mu(3\lambda + 2\mu)}{\lambda + 2\mu} \right] \frac{y_1}{r^2} - \frac{\lambda}{\lambda + 2\mu} \frac{y_2}{r} \\
 &\quad + \left[-\omega^2 \rho r^2 + \frac{2\mu}{\lambda + 2\mu} [\lambda(2n^2 + 2n - 1) + 2\mu(n^2 + n - 1)] \right] \frac{y_3}{r^2} \\
 &\quad - \frac{3y_4}{r} - \rho \frac{y_5}{r}, \\
 \dot{y}_5 &= 4\pi G \rho y_1 + y_6, \\
 \dot{y}_6 &= -4\pi G \rho n(n+1) \frac{y_3}{r} + n(n+1) \frac{y_5}{r^2} - \frac{2y_6}{r},
 \end{aligned}$$

Same equations used
to compute normal-
mode oscillations and
solid Earth body tides

$$\rho \bar{a} = \bar{\nabla} \cdot \bar{\sigma} + \rho \bar{F},$$

$$u_n = U(r) P_n(\cos \theta) e^{i\omega t}$$

$$v_n = V(r) \frac{\partial P_n(\cos \theta)}{\partial \theta} e^{i\omega t}$$

$$\psi_n = P(r) P_n(\cos \theta) e^{i\omega t},$$

$$y_1 = U$$

$$y_2 = \lambda X + 2\mu \dot{U}$$

$$y_3 = V$$

$$y_4 = \mu \left(\dot{V} - \frac{V}{r} + \frac{U}{r} \right)$$

$$y_5 = P$$

$$y_6 = \dot{P} - 4\pi G \rho_0 U.$$

Surface Boundary Conditions

	Free Oscillations	External Potential	Surface Mass Loading	Surface Shear Forcing	Surface Stress (n=1)
y_2	0	0	$-g_S^2 \frac{2n+1}{4\pi G}$	0	$-\frac{3g_S^2}{4\pi G}$
y_4	0	0	0	$\frac{(2n+1)g_S^2}{4\pi G n(n+1)}$	$\frac{3g_S^2}{8\pi G}$
$y_6 + \frac{n+1}{a} y_5$	0	$(2n+1)g_S$	$(2n+1)g_S$	0	0

Love Number Development

Same as free-oscillation calculation

- We need to solve the **equations of motion** (i.e. *conservation of linear momentum*) for the spheroidal deformation of an elastic, self-gravitating and spherically symmetric Earth
- To eliminate derivatives in the elastic parameters, we reduce the equations of motion to a system of six **first-order ODEs**
- Compute three linearly independent **starting solutions** based on the analytical solutions for a homogeneous sphere
- **Integrate** the three solutions through the spherically symmetric Earth model, applying boundary conditions at all internal interfaces
- Apply stress and gravitational-potential boundary conditions at the surface to derive the remaining solutions to the equations of motion, or Love numbers, for each spherical harmonic degree

Love Number Development

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Green's Function Development

Same as free-oscillation calculation

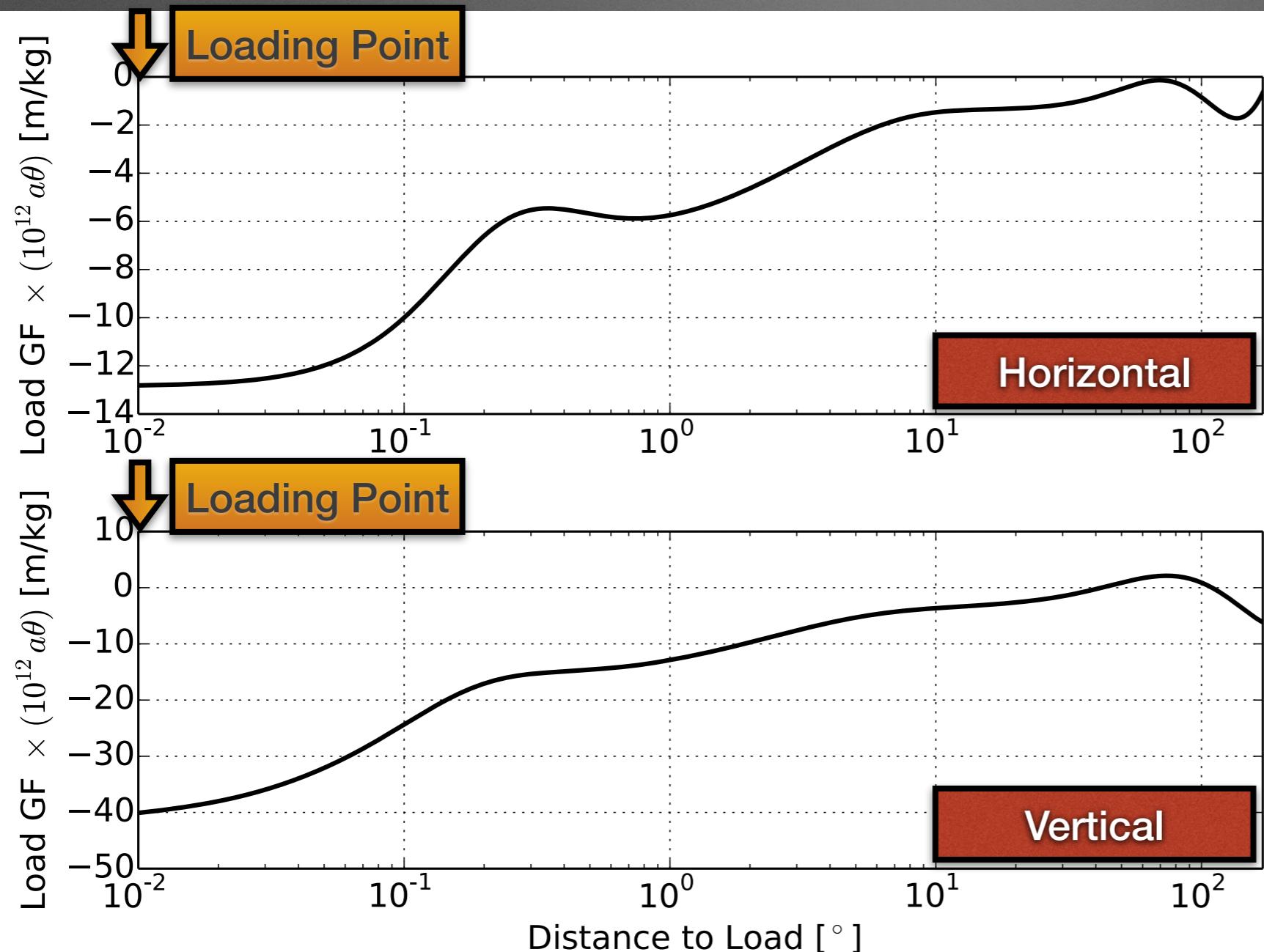
- Develop the **equations of motion** (*conservation of momentum*) for the spheroidal deformation of an elastic, self-gravitating and hydrostatically pre-stressed body $\rho \bar{a} = \bar{\nabla} \cdot \bar{\sigma} + \rho \bar{F}$
- Reduce to a system of six **first-order ODEs** that depend on the elastic moduli, density, spherical harmonic degree, and radius
- Compute analytical **starting solutions** for a homogeneous sphere
- Integrate through a spherically symmetric Earth model (including fluid layers) to the surface
- Apply **boundary conditions** at the surface to derive solutions to the equations of motion, and **Love numbers** for each spherical harmonic degree

$$u_n = U_n P_n(\cos \theta)$$

$$v_n = V_n \frac{\partial P_n(\cos \theta)}{\partial \theta}$$

Load Green's Functions

Characterize the deformation response to a delta-function unit normal force



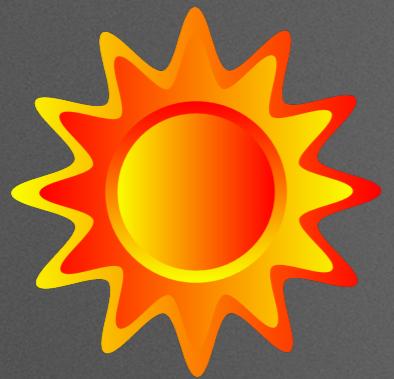
Horizontal Displacement

$$v = \frac{a}{m_E} \sum_{n=1}^{\infty} l'_n \frac{\partial P_n(\cos \theta)}{\partial \theta}$$

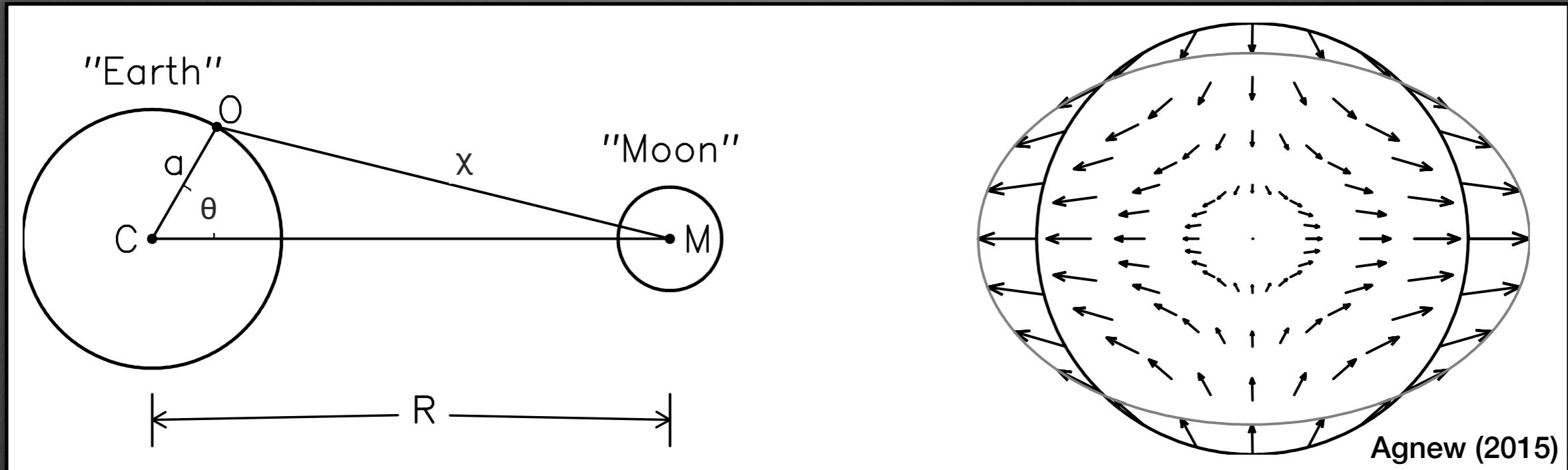
Vertical Displacement

$$u = \frac{a}{m_E} \sum_{n=0}^{\infty} h'_n P_n(\cos \theta)$$

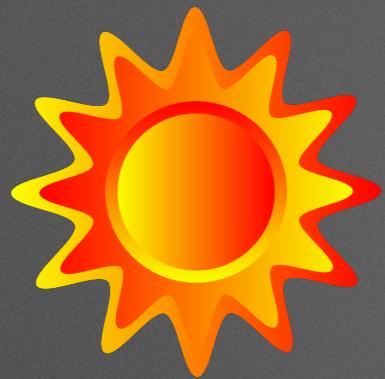
Tides



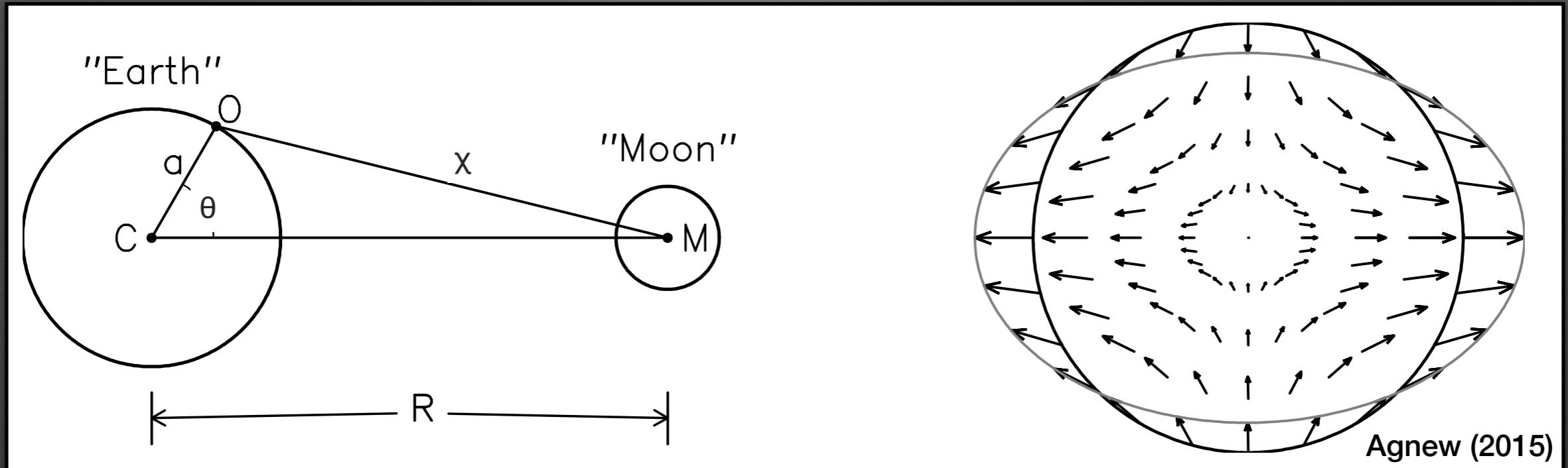
Tidal Potential



Tides arise due to differential, or unbalanced, gravitational forces within the oceans and the solid Earth

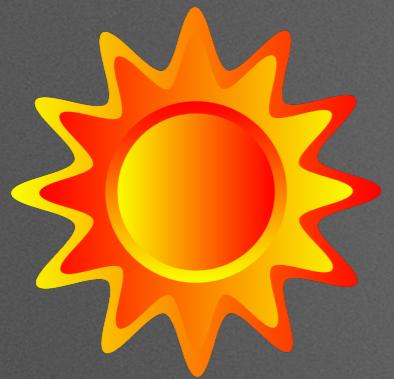


Tidal Potential

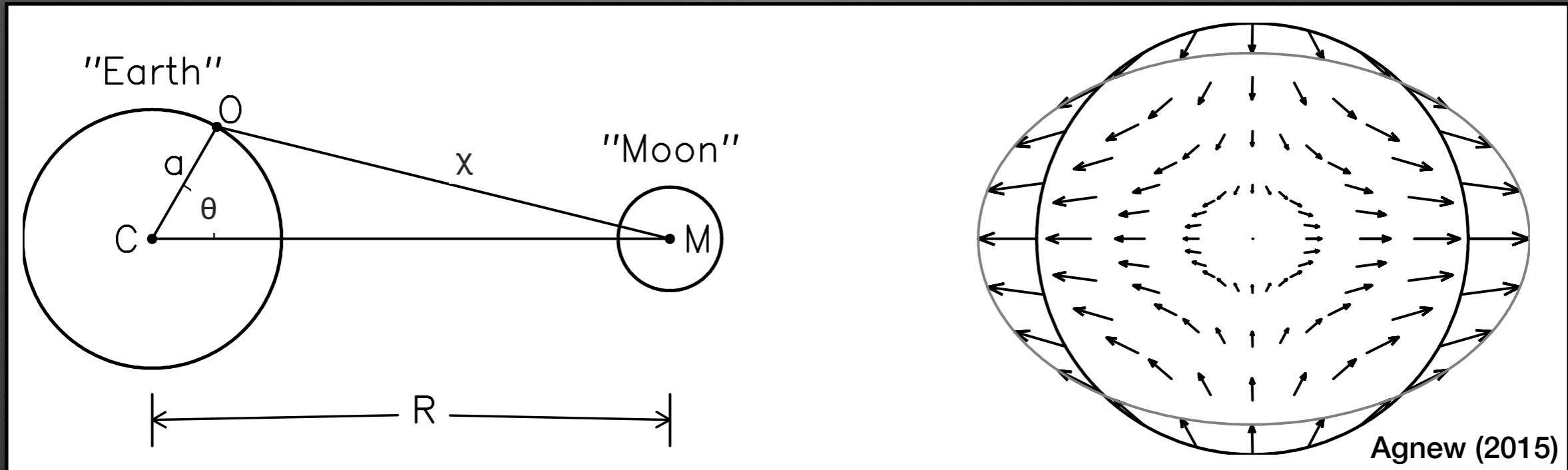


Tides arise due to differential, or unbalanced, gravitational forces within the oceans and the solid Earth

$$V_O = \frac{GM_{\text{ext}}}{\chi} = \frac{GM_{\text{ext}}}{R} \sum_{n=0}^{\infty} \left(\frac{a}{R}\right)^n P_n(\cos \theta)$$



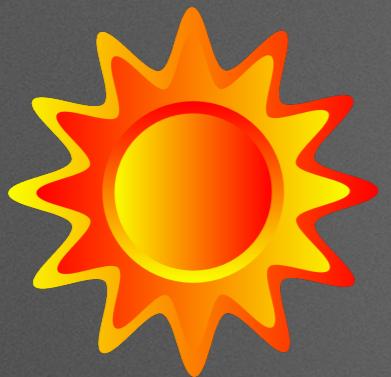
Tidal Potential



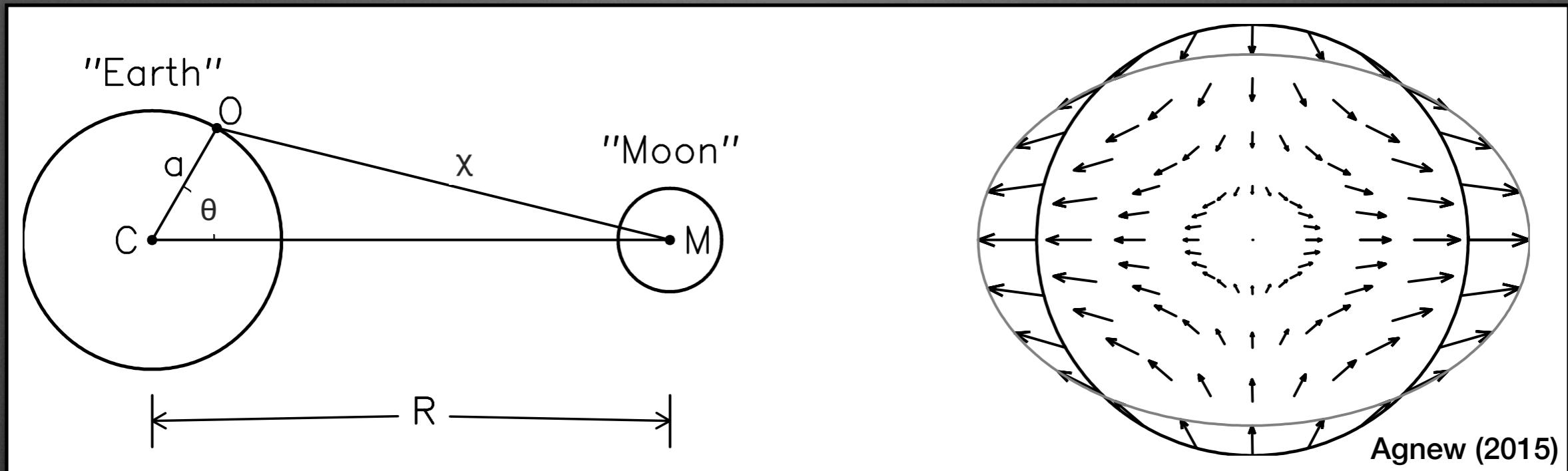
Tides arise due to differential, or unbalanced, gravitational forces within the oceans and the solid Earth

$$V_O = \frac{GM_{\text{ext}}}{\chi} = \frac{GM_{\text{ext}}}{R} \sum_{n=0}^{\infty} \left(\frac{a}{R}\right)^n P_n(\cos \theta)$$

The first two terms in the expansion ($n=0, 1$) represent a constant and uniform force along the CM direction, respectively

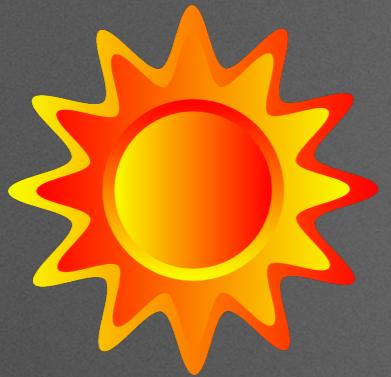


Tidal Potential

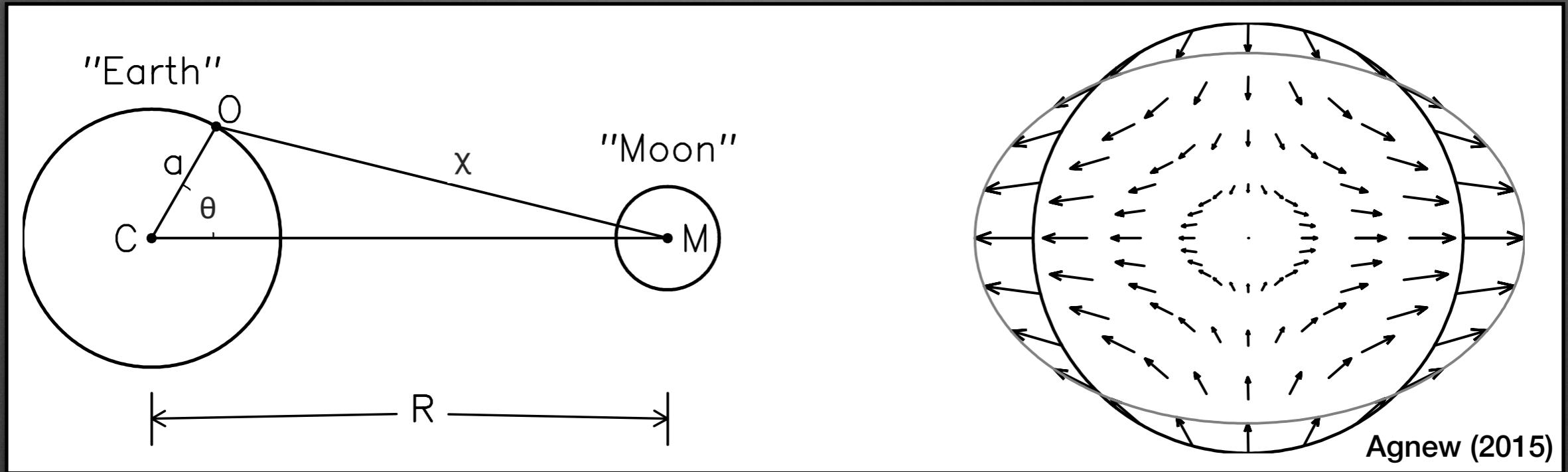


The degree-2 term generates the largest tidal response

$$V_O = GM_{\text{ext}} \frac{a^2}{R^3} \left(\frac{3}{2} \cos^2 \theta - \frac{1}{2} \right)$$



Tidal Potential



Agnew (2015)

The degree-2 term generates the largest tidal response

$$V_O = GM_{\text{ext}} \frac{a^2}{R^3} \left(\frac{3}{2} \cos^2 \theta - \frac{1}{2} \right)$$

θ and R : Complicated functions of the astronomical ephemeris

Astronomical Parameters

T = mean lunar time	1.035 days
s = mean longitude of moon	27.322 days
h = mean longitude of sun	365.242 days
p = mean longitude of lunar perigee	8.85 years
N' = negative mean longitude of lunar ascending node	18.61 years
p' = mean longitude of perihelion	20942 years

Astronomical Parameters

T = mean lunar time	1.035 days
s = mean longitude of moon	27.322 days
h = mean longitude of sun	365.242 days
p = mean longitude of lunar perigee	8.85 years
N' = negative mean longitude of lunar ascending node	18.61 years
p' = mean longitude of perihelion	20942 years

Doodson Number

$$[\eta_{\tau} \quad \eta_s \quad \eta_h \quad \eta_p \quad \eta_{N'} \quad \eta_{p'}]$$

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τ = mean lunar time	1.035 days
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Doodson Number

$$[\eta_{\tau} \quad \eta_s \quad \eta_h \quad \eta_p \quad \eta_{N'} \quad \eta_{p'}]$$

Astronomical argument (reference phase)

$$\Phi_{\eta}(t) = \eta_{\tau} \tau(t) + \eta_s s(t) + \eta_h h(t) + \eta_p p(t) + \eta_{N'} N'(t) + \eta_{p'} p'(t)$$

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Harmonics = Unique sums/differences of astronomical parameters

Total Tide = Superposition of tidal harmonics