Game Theory Notes

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1 Static Games of Complete Information: Strategic-Form

1.1 Introduction to Strategic-Form Games

Overview. General framework:

- One-time Game: strategic-form games & equilibriums
- More than one period: extensive-form games & subgame perfect
- Uncertainty: incomplete information & dynamics

Definition 1.1 (Static Game)**.** A static game $\mathcal{G} = (\mathcal{I}, (S_i)_{i\in\mathcal{I}}, (u_i)_{i\in\mathcal{I}})$ has elements:

(i) Finite set of Players: $\mathcal{I} = \{1, \cdots, n\}$

(ii) A set S_i (pure strategy) available to Player i $\forall i \in \mathcal{I}$

(iii) Payoff function $u_i : S \to \mathbb{R}$ $\forall i \in \mathcal{I}$, where $S = S_1 \times \cdots \times S_n$ is the set of pure strategy profile $s = (s_1, \ldots, s_n)$.

Remark. Generally speaking, Action ($\mathcal{A}_i$) should be different to Strategy (S_i). Only in static games with complete information we can use $\mathcal{A}_i = S_i$.

Mixed Strategy: σ_i

- probability distribution over S_i: $\sigma_i(s_i) \leftarrow$ probability of Player i choosing strategy s_i
- Space of mixed strategy: $\sum_i \Rightarrow \sum = \sum_1 \times \cdots \times \sum_n$

$\Rightarrow$ Player i's payoff to mixed-profile σ: $\sum\limits_{s\in S} \left(\prod\limits_{j=1}^{\mathcal{I}} \sigma_j(s_j) \right) u_i(s)$

Definition 1.2 (Dominated Strategies)**.** Pure strategy S_i is strictly dominated for Player i if there exists $\sigma_i^{'} \in \sum_i$ such that $u_i(\sigma_i^{'}, s_{-i}) \succ u_i(s_i, s_{-i})$.

Remark. S_i is dominated by any pure or mixed profile $\sigma_i^{'}$.

1.2 Nash Equilibrium

Definition 1.3. A mixed-strategy profile σ^* is a NE if $\forall$ player i, we have: $u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(s_i, \sigma_{-i}^*)$ $\forall s_i \in S_i$.

Remark. σ_i^* dominates all profile combination.

- NE is strict $\Leftrightarrow$ $\forall i$ and $s_i \neq s_i^*$ (i.e., s_i not optimal), we have $u_i(s_i^*, s_{-i}^*) > u_i(s_i, s_{-i}^*)$.
- NE is consistent.

1.3 Existence of Nash Equilibria

Theorem 1.1 (Nash 1950b)**.** Every finite strategic-form game has a mixed-strategy equilibrium

Remark. Apply Kakutani's Fixed Point Theorem. Check:

(i) $\sum$ is a non-empty, convex, and compact (finite-dim) subset of Euclidean space.

(ii) Reaction function $r(\sigma)$ is non-empty and convex $\forall \sigma$.

(iii) r has close graph: if $(\sigma^n, \hat{\sigma}^n) \to (\sigma, \hat{\sigma})$ with $\hat{\sigma}^n \in r(\sigma^n)$, then $\hat{\sigma} \in r(\sigma)$.

Motivation: for σ (mixed-NE), players mutually respond best to the others. Say $\sum$ is the mixed strategy profile, and that $r : \sum \rightrightarrows \sum \equiv r_1(\sigma_{-1}) \times \cdots \times r_n(\sigma_{-n})$. Then, a fixed point of r is a σ such that $\sigma \in r(\sigma) \Rightarrow \sigma_i \in r_i(\sigma) \Leftrightarrow$ a fixed point of r is NE. From there, we can use "Kakutani's" to check.

- When payoff function is continuous $\Rightarrow$ response correspondences have close graphs.
- An upper semi-continuous function has upward discontinuities.

Theorem 1.2 (Pure NE- Debreu 1952; Glicksberg 1952; Fan 1952)**.** Consider a strategic-form game whose strategy space S_i are non-empty, compact, and convex subsets of an Euclidean space. If the payoff function u_i are continuous in s and quasi-concave in s_i, there exists a pure-strategy NE.

Remark. Payoff function is required to be continuous and quasi-concave here. If u_i conti, then it is non-empty by EVT (satisfying Cond 1) and has closed graph (satisfying Cond 3) instantaneously. If u_i is quasi-concave, it ensures a convex reaction correspondence.

Theorem 1.3 (Mixed NE- Glicksberg 1952)**.** Consider a strategic-form game whose strategy space S_i are non-empty compact subsets of a metric space. If the payoff function u_i are continuous then there exists a mixed-strategy NE.

Remark. No compactness here ($\Rightarrow$ not Euclidean space). No quasi-concave condition here. If u_i is disconti, then there is no Pure NE and may not have Mixed NE.

2 Iterated Strict Dominance, Rationalizablity, and Correlated Equilibrium

2.1 IESDS, Rationalizablity

Motivation. What strategies *could* a rational player play? Recall that in IESDS, we assume rational players will NEVER play those dominated (so we delete such strategies). Here, a rational agent play a best response to some beliefs about the opponents' strategies $\Rightarrow$ **conjectures**

Definition 2.1 (The Process of IESDS)**.** We can follow the steps:

(i) Set $S_i^0 \equiv S_i$ and $\sum_i^0 \equiv \sum_i$.

(ii) Define S_i^n recursively by:
$S_i^n = \{s_i \in S_i^{n-1} |$ exists no $\sigma_i \in \sum_i^{n-1}$ s.t. $u_i(\sigma_i, s_{-i} > u_i(s_i, s_{-i}) \forall s_{-i} \in S_{-i}^{n-1}\}$

(iii) And, define $\sum_i^n = \{\sigma_i \in \sum_i \,|\, \sigma_i(s_i) > 0$ only if $s_i \in S_i^n\}$

(iv) Set $S_i^\infty = \bigcap\limits_{n=0}^{\infty} S_i^n$ is the set of Player i's Pure strategies that survive IESDS.

(v) Set $\sum_i^\infty = \{\sigma_i \in \sum_i \,|$ no σ_i' s.t. $u_i(\sigma_i', s_{-i}) > u_i(\sigma_i, s_{-i}) \forall s_{-i} \in S_{-i}^\infty\}$ is the set of Player i's Mixed strategies that survive IESDS.

Definition 2.2. A game is solvable by IESDS if, for each player i, S_i^∞ is a singleton (i.e., a one-element set).

Remark. When the IESDS yields a qnique strategy profile, this strategy profile is necessarily a NE (think the prisoner's dilemma).

Definition 2.3 (Rationalizable Strategies)**.** Set $\tilde{\sum}_i^0 \equiv \sum_i$, and for each i recursively define:

$$\tilde{\sum}_i^n = \left\{ \sigma_i \in \tilde{\sum}_i^{n-1} |\ \exists \sigma_{-i} \in \bigtimes_{j \neq i} \text{convex hull } (\tilde{\sum}_j^{n-1}) \text{ s.t. } u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i}) \forall \sigma_i' \in \tilde{\sum}_i^{n-1} \right\}.$$

Then, the rationalizable strategies for player i are $R_i = \bigcap\limits_{n=0}^{\infty} \tilde{\sum}_i^n$.

Remark. $\tilde{\sum}_i^{n-1}$ is the set of i's surviving strategies that are best response to some strategies in $\tilde{\sum}_{-i}^{n-1}$, i.e., opponents' surviving strategies through (IESDS) round $n-1$.

- The convex hull of a set $\tilde{\sum}$ is the smallest convex set that contains it.
- In general, rationalizability will yield sharper predictions than IESDS.
- Every NE is rationalizable.

Theorem 2.1 (Rationalizability and IESDS- Pearce 1984)**.** Rationalizability and IESDS coincide in two-player game. Specifically, if $\mathcal{I} = 1, 2$ (two players), then $R_i = \sum_i$ for $i = 1, 2$ (set of rationalizability = set of mixed strategies surviving IESDS).

Remark. This theorem hints that:

- $R_i \subseteq \sum_i$ for all i (so that when $i = 2,\ R_i = \sum_i$)
- Set of rationalizable strategies survive IESDS (so is a subset to it).

2.2 Correlated Equilibrium

Motivation. Aumann (1974) motivates such an idea of a correlated equilibrium achieved by some signaling devices/models. Suppose a correlating device $(\Omega, (\mathcal{T}_{i\in\mathcal{I}}), p)$:

Definition 2.4 (Correlated Equilibrium A)**.** A correlated equilibrium ζ relative to information structure $(\Omega, (\mathcal{T}_{i\in\mathcal{I}}), p)$ is a NE in strategies that are adapted to this info structure.

Remark. That is, the vector $(\zeta_1, \cdots, \zeta_n)$ is a correlated equilibrium if, for every player i and adapted strategy $\tilde{\zeta}_i$,

$$\sum_{\omega\in\Omega} p(\omega)u_i(\zeta_i(\omega), \zeta_{-i}(\omega)) \geq \sum_{\omega\in\Omega} p(\omega)u_i(\tilde{\zeta}_i(\omega), \zeta_{-i}(\omega))$$

We require that ζ_i maximize player i's ex-ante payoffs.

Definition 2.5 (Correlated Equilibrium B)**.** A correlated equilibrium is any probability distribution $p(\cdot)$ over the pure strategies $S_1 \times \cdots S_{\mathcal{I}}$ such that, for every player i and every function $d_i(\cdot)$ that maps S_i to S_i, we have:

$$\sum_{s\in S} p(s)u_i(s_i, s_{-i}) \geq \sum_{s\in S} p(s)u_i(d_i(s_i), s_{-i})$$

Remark. It is equivalent to consider: $\sum_{s_{-i}\in S_{-i}} p(s_{-i}|s_i)u_i(s_i, s_{-i}) \geq \sum_{s_{-i}\in S_{-i}} p(s_{-i}|s_i)u_i(s_i', s_{-i})$

Summary. Both Pure and Mixed NEs are correlated equilibrium. Pure: $p(\cdot)$ degenerate. Mixed: $p(\cdot)$ can be joint distribution. The set of correlated equilibrium is convex and is at least as large as the convex hull of NE. For the existence problem, also use fixed point theorems to deal with.

2.3 Rationalizability and Subjective Correlated Equilibria

Summary. In matching pennies, players are allowed to have inconsistent strategic beliefs, by rationalizability. Meanwhile, subjective correlated equilibrium allows the belief to be *completely* arbitrary. It does not capture the restrictions implied by common knowledge and thus is less restrictive than rationalizability (check adapted strategy definition on Fudenberg and Tirole (1991) p.60).

3 Dynamic Games of Complete Information: Extensive-Form

Overview. Use extensive-form games to capture the dynamic structure. For instance, entry and entry deterrence in IO and time-consistency problem in Macro. Very essential example "Stackelberg leader" duopoly: one producer chooses action (i.e., output level) first and the other follows up. In this case, NE is defined as the strategy profile that neither players would gain from switching strategies. *Backward induction* is the key to solve these kinds of scenarios, alongside the idea of *subgame-perfect equilibrium*.

3.1 Commitment and Perfection in Multi-Stage Games: Observable Action

Summary. Multi-stage game: (1) all players knew the action chosen in previous stage $0, \cdots, k-1$, and (2) all players move simultaneously in each stage k (including the singleton of "do nothing"). Some notations:

- Players: $i \in \mathcal{I}$
- Stages: $0, \cdots, k, k+1$
- Action set (choice): players simultaneously choose actions from $\mathcal{A}_i(h^0)$
- Observable action profile (stage 0): $a^0 \equiv (a_1^0, \cdots, a_{\mathcal{I}}^0)$
- History: let $h^0 = \emptyset$, then history h^1 at the beginning of stage 1 is just $h^1 = a^0$
- Next stage: $\mathcal{A}_i(h^1)$ then will be the action set for possible second-stage. Iteratively, h^{k+1} will be the history at the end of stage k/by start of stage $k+1$: $h^{k+1} = (a^0, \cdots, a^k)$.
- Pure strategy: let H^k be the all stage-k history and let $\mathcal{A}_i(H^k) = \bigcup_{h^k \in H^k} \mathcal{A}_i(h^k)$. A pure strategy for player i is a sequence $\{s_i^k\}_{k=0}^K$

Summary. Backward induction can be applied to **any** finite game of perfect information (finite: finite stages & finite actions each stage).

Definition 3.1 (Subgame-Perfect Equilibrium: SPE)**.** A strategy profile s of a multi-stage game with observed actions is a subgame-perfect equilibrium **if** for every h^k, the restriction $s|h^k$ to $\mathcal{G}(h^k)$ is a NE of $\mathcal{G}(h^k)$.

Example 3.1 (Two-stage game with Cournot)**.** Firm 1 and Firm 2 both have a const average cost of \$2 per unit. Firm 1 decides whether to invest in new technology with \$0 per unit but the technology itself costs f. Firm 2 observes Firm 1's investment and then propose q_2 to compete in cournot format (against q_1 by Firm 1). Suppose demand curve is given: $p(q) = 14 - q$. Firm 1's payoff will be: $((14-2) - q_1 - q_2)q_1$ if not invest and $((14-0) - q_1 - q_2)q_1 - f$ if invest. Firm 2's will be: $((14-2) - q_1 - q_2)q_2$.

Answer. Consider each cases and take FOC for each Firm for reaction functions:

- Firm 1 not invest: $\frac{\partial((14-2)-q_1-q_2)q_1}{\partial q_1} = 12 - 2q_1 - q_2 = 0 \Rightarrow r_1(q_2) = 6 - \frac{q_2}{2}$. By symmetry, $r_2(q_1) = 6 - \frac{q_1}{2}$. Solve this and get $(4, 4)$ with both payoffs of 16.
- Firm 1 invest: $r_1(q_2) = 7 - \frac{q_2}{2}$ and $r_2(q_1) = 6 - \frac{q_1}{2}$. Solve this and get $(\frac{16}{3}, \frac{10}{3})$. Firm 1's payoff will be $\frac{256}{9} - f$.

Thus, as long as the fixed cost f satisfies $\frac{256}{9} - f > 16 \Rightarrow f < \frac{112}{9} \approx 12.44$, Firm 1 will choose to invest.

Motivation. Players can sometimes gain benefit from making a binding commitment. With more than one players, commitments can be of value since such commitment would alter a sequence of actions the opponents may play. Typically, the optimal reaction function will be decreasing function of opponent's action/output. Commitment can also be achieved by "moving earlier" as it appears in Stackelberg.

Example 3.2 (Odyssey-Commitment)**.** A general can burn the bridges as a commitment not to retreat (i.e., no way back) and Odysseus has himself lashed to the mast and order his sailors to plug their ears with wax as a commitment not to go to the Siren's island. Once they make these commitments, the cost of turning back is often modeled as ∞.

3.2 Extensive Form

Definition 3.2 (Elements for Extensive Form)**.** The extensive-form game should contain:

(i) A finite set of player $i \in \mathcal{I}$

(ii) The order of moves and active players: "game tree" $i : X \to \mathcal{I}$

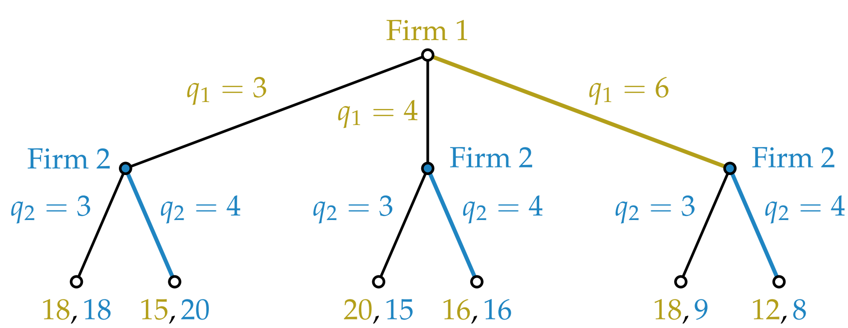
(iii) Payoff function for each player of the move: $u_i : \mathbb{Z} \to \mathbb{R}$

(iv) A set of actions: $\mathcal{A}(h)$ at information set h

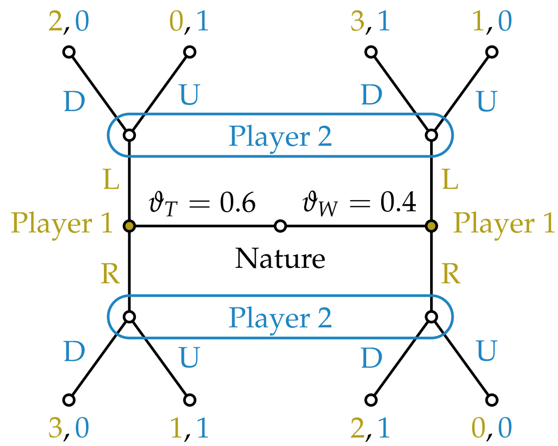
(v) A partition of history H: what each player knows when making moves

(vi) Probability distribution over exogenous events: "Nature" $\mathcal{N}$

Definition 3.3 (Perfect Information)**.** Information is *perfect* if nature has no moves AND each information set is a *singleton*. Else, imperfect information.



Replication. Figure 3.4 in Fudenberg and Tirole (1991)



Replication. Figure 3.6 in Fudenberg and Tirole (1991)

Remark (Signaling game)**.** Note that the Nature moves first and choose "type" for player 1 (i.e., private information). For instance, in Fudenberg and Tirole (1991) Figure 3.6, Nature determines that Player 1 will be "Tough" type with probability $\vartheta_T = 0.6$ and be "Weak" type with probability $\vartheta_W = 0.4$. Player 2 observes the action but not knowing Player 1's type, which forms this *signaling game* (imperfect information).

3.3 Strategies and Equilibria in Extensive-Form Games

Definition 3.4 (Reduced Strategic Form)**.** The *reduced strategic form* of an extensive-form game is obtained by identifying equivalent pure strategies. And, two pure strategies s_i and s_i' are *equivalent* if they lead to the same probability distribution over outcomes for all pure strategies of opponents.

Definition 3.5 (Behavior Strategy)**.** Let $\Delta(\mathcal{A}(h_i))$ be the probability distribution on $\mathcal{A}(h_i)$. A *behavior strategy for player* i, denoted σ_i, is an element of the Cartesian product $\bigtimes_{h_i \in H_i} \Delta(\mathcal{A}(h_i))$.

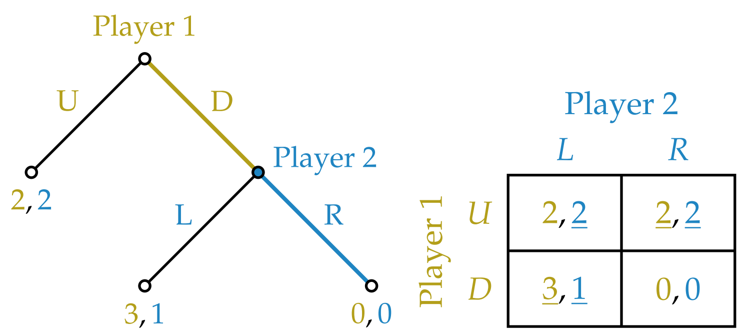
Remark. A behavior strategy specifies a probability distribution over actions at each h_i, and probability distribution at different information sets are independent.

Theorem 3.1 (Kuhn 1953)**.** In a game of perfect recall, mixed strategies are equivalent to the unique behavior strategies they generates.

Theorem 3.2 (Zermelo 1913; Kuhn 1953)**.** A finite (extensive-form) game of perfect information has a pure-strategy NE.

3.4 Backward Induction and Subgame Perfection

Motivation. Though complex extensive-form game can be represented by strategic form one (and thus the concept of NE can be used), Selton (1965) argued that some NE are *more reasonable* than others.



Replication. Figure 3.14 in Fudenberg and Tirole (1991)

For instance, in extensive for game, the equilibrium is (D, L), but, in strategic form, (D, L) and (U, R) are both NE. We say (U, R) is *not credible* as it relies on empty threat that Player 2 will only play R. We see Backward Induction yields the "correct" answer.

Definition 3.6 (Subgame)**.** A proper subgame $\mathcal{G}$ of an extensive-form game T consists of a *single* node and all its successors in T, with the property that if $x' \in \mathcal{G}$ and $x'' \in h(x')$, then $x'' \in \mathcal{G}$. the information sets and payoffs of the subgame are inherited from the original game.

Remark. The requirements that the subgame begins with a single node and heritage info sets imply x in original game must be a singleton information set, i.e., $h(x) = \{x\}$.

Definition 3.7 (Subgame-Perfect Equilibrium; SPE)**.** A behavior-strategy profile σ of an extensive-form game is a *subgame-perfect equilibrium* if the restriction of $\sigma|x$ to $\mathcal{G}(x)$ is a NE for every proper subgame $\mathcal{G}$.

Remark. Every new stage can begin a proper subgame. Subgame perfection coincides with backward induction in finite games of perfect information.

Summary. Subgame perfection supposes not only the players expect NE in all subgames but also that ALL players expect the SAME equilibria.

4 Applications of Multi-Stage Games with Observed Actions

Overview. Include "open-loop" and "close-loop" equilibria, finite-horizon and infinite-horizon games, and some dynamic optimization. Consider mostly games with an infinite horizon as they are better models to many (real) situations. For instance, the bargaining game of Rubinstein (1982). Though those bargaining models are of infinite horizon, there exist some actions– accept the offer or exiting from the market– to end the game.

4.1 The Principle of Optimality and Subgame Perfection

Motivation. Check whether there are any history h^t where Player i can gain by *deviating* from the actions prescribed by s_i at h^t to verify a strategy profile of a multi-stage game with observed actions is subgame perfect.

Theorem 4.1 (One-stage deviation Principle for finite-horizon)**.** In a finite multi-stage game with observed actions, strategy profile s is subgame perfect **if and only if** it satisfies the one-stage-deviation condition, meaning that NO Player i can gain by deviating from s in a single stage and still confronting to s thereafter.

Remark. It is equivalent to say: profile s is SP **if and only if** there exists NO Player i and NO strategy $\hat{s}_i$ that agrees with s_i except at a single stage t and history h^t, and such that $\hat{s}_i$ is a better response to s_{-i} than s_i conditional on h^t being reached.

Definition 4.1 (Continuity at Infinity)**.** A game is continuous at infinity if, for each Player i, the utility function u_i satisfies:

$$\sup_{h,\tilde{h} \text{ s.t. } h^t - \tilde{h}^t} |u_i(h) - u_i(\tilde{h})| \to 0 \text{ as } t \to \infty.$$

Remark. Events in distant future are relatively unimportant. Continuity at infinity is satisfied if the overall payoff is a discounted present-value sum $g_i^t(a^t)$ and per-period payoffs are uniformly bounded, i.e., there is a $\mathcal{M}$ such that $\max_{t,a^t} |g_i^t(a^t)| < \mathcal{M}$.

Definition 4.2 (One-stage deviation Principle for infinite-horizon)**.** In an infinite-horizon multi-stage game with observed actions that is continuous at infinity, profile s is SP **if and only if** there exists NO Player i and NO strategy $\hat{s}_i$ that agrees with s_i except at a single stage t and history h^t, and such that $\hat{s}_i$ is a better response to s_{-i} than s_i conditional on h^t being reached.

Remark. This Theorem shows if s satisfies one-stage-deviation condition, then it cannot be improved by any finite sequence of deviations in any subgame. It is also the essential idea of the Principle of Optimality for discounted dynamic programming (also see Blackwell (1965) and Appendix I for basics of Linear Programming).

4.2 The Rubinstein-Stahl Bargaining Model

Motivation. Two players must agree on how to share a pie of size 1. In period $0, 2, 4 \cdots$:

- Player 1 proposes a sharing rule $(x, 1-x)$, and Player 2 can accept or reject.
  - If Player 2 accepts, then game **ends**.
  - If Player 2 rejects, then proposes a sharing rule $(x, 1-x)$ in next period.
    - If Player 1 accepts, then game **ends**.
    - If Player 1 rejects, then proposes a sharing rule $(x, 1-x)$ in next period. $\ddots$

Consider discount factors $\delta_1^t, \delta_2^t \in (0,1)$, then payoffs are $(\delta_1^t x, \delta_2^t(1-x))$ if sharing rule $(x, 1-x)$ is accepted at round t.

Example 4.1 (Profile 1: A NE but not SP)**.** One NE can be: *Player 1 always propose $x = 1$ and refuse all smaller shares; Player 2 also always offers $x = 1$ and accept any offers*. Yet, this will not be SP. Why? If Player 2 rejects Player 1's first offer and offer Player 1 a share $x > \delta_1$, then Player 1 should accepts since the best possible outcome if Player 1 rejects is to receive full pie (recall: Player 2 also proposes $x = 1$) but only worth δ_1 next round.

Example 4.2 (Profile 2: SPE)**.** Consider the following: *Player i always demands a share $\frac{1-\delta_j}{1-\delta_i\delta_j}$ when it is their turn. They accept any share $\geq \frac{\delta_i(1-\delta_j)}{1-\delta_i\delta_j}$ and refuse any smaller share*. Why? Note that Player i's demand of share is:

$$\frac{(1-\delta_j)}{1-\delta_i\delta_j} = \frac{(1-\delta_i\delta_j) + \delta_i\delta_j - \delta_j}{1-\delta_i\delta_j} = 1 - \underbrace{\delta_j \frac{(1-\delta_j)}{1-\delta_i\delta_j}}_{\text{Player 2 will accept}},$$

which is the highest share for Player i that is accepted by Player j (Player i cannot make any lower offer as it will be accepted by j). Player i would also hurt if proposing a higher share, getting reject, and waiting till next round when Player j demands $\frac{1-\delta_i}{1-\delta_i\delta_j}$, i.e., Player i gets $\left(1 - \frac{1-\delta_i}{1-\delta_i\delta_j}\right)$:

$$\delta_i \left(1 - \frac{1-\delta_i}{1-\delta_i\delta_j}\right) = \delta_i^2 \frac{1-\delta_j}{1-\delta_i\delta_j} < \frac{1-\delta_j}{1-\delta_i\delta_j}$$

It is thus optimal for Player i to accept any offer that at least receives $\frac{\delta_i(1-\delta_j)}{1-\delta_i\delta_j}$.

Remark. This Rubinstein (1982) extends Stahl (1971) by considering infinite-horizon game. Two drawbacks for the finite-horizon version: ① solution for finite version depends on the length of game and who makes offer in last round, and ② it is more natural to continue the bargain if the offer at the last round is rejected. The uniqueness of the Infinite-Horizon Equilibrium matters here.

5 Repeated Games (skip)

6 Static Games of Incomplete Information: Bayesian Equilibrium

(under construction)

Appendix I. Introduction of Linear Programming

Illustrative problem: Let's consider:

$$\min_{x_1,x_2} x_1 + 3x_2 \text{ subject to } \quad \begin{aligned} &x_1 + x_2 \geq 2, \\ &x_2 \geq 1, \\ &x_1 - x_2 \geq 3 \end{aligned}$$

Let some helper coeffs $p_1, p_2, p_3 \geq 0$, we can rewrite our desired lower bound B:

$$p_1 \cdot (\underbrace{x_1 + x_2 \geq 2}_{\text{CSTR. 1}}) + p_2 \cdot (\underbrace{x_2 \geq 1}_{\text{CSTR. 2}}) + p_3 \cdot (\underbrace{x_1 - x_2 \geq 3}_{\text{CSTR. 3}}) = x_1 + 3x_2 \geq B$$

But by comparing the coefficients to objective function ($x_1 + 3x_2$), we require:

$$\begin{aligned} x_1 : &\quad p_1 + p_3 = 1 \\ x_2 : &\quad p_1 + p_2 - p_3 = 3 \\ &\quad p_1, p_2, p_3 \geq 0 \\ \rightarrow B &\quad = 2p_1 + 1p_2 + 3p_3 \end{aligned}$$

Now we notice that our original minimization problem (primal) can be transformed into the following maximization problem (dual):

$$\max_{p_1,p_2,p_3} 2p_1 + p_2 + 3p_3 \text{ subject to } \quad \begin{aligned} &p_1 + p_3 = 1, \\ &p_1 + p_2 - p_3 = 3, \\ &p_1, p_2, p_3 \geq 0 \end{aligned}$$

We call this derived optimization problem the **dual LP problem**.

Theorem (Dual Problem). If we transform the dual into an equivalent minimization problem and then form its dual, we obtain a problem equivalent to the original problem. In mathematical words,

$$\min c'\mathbf{x} \text{ subject to } A\mathbf{x} \geq b \Leftrightarrow \max p'b \text{ subject to } p'A = c',\ p \geq 0.$$

These two problems have the same values.

Remark. For non-negative vector $p \geq 0$, we have: $A\mathbf{x} \geq b \Rightarrow (p'A)\,\mathbf{x} \geq p'b$. Therefore, to minimize (get lower bound) of objective function, it is equivalent to think maximizing $p'b$ by the derived constraint $p'A = c'$ matching the coefficient of x.

- Min problem of budget constraint given optimal utility $\Leftrightarrow$ Max problem of utility function given budget constraint

Appendix II. Supermodular Games

Overview. Developed by Topkis (1979), applied to economic problem by Vives (1990) and then Milgrom and Roberts (1990). The idea is that in some games best response correspondences are *increasing* so that players' strategies are "strategic complements." Supermodular games are well-behaved- have pure strategy NE. The upper and lower bounds of the NE sets and the rationalozble sets coincide. A supermodular game does not require convexity and differentiability assumptions, and it needs an **order structure on strategy space**, **weak continuity on payoff functions**, and that the marginal utility of each player's strategy is **monotonic**.

Notations. Suppose each player i's strategy set S_i is a subset of a finite-dim Euclidean space $\mathbb{R}^m$. Then we define $S \equiv \bigtimes_{i=1}^{\mathcal{I}} S_i$ is a subset of $\mathbb{R}^m$, where $m \equiv \sum_{i=1}^{\mathcal{I}} m_i$. Let x and y denote two vectors in some Euclidean Space $\mathbb{R}^k$. Also, let $x \geq y$ if $x_k \geq y_k$ for all $k = 1, \cdots, K$, and let $x > y$ if $x \geq y$ and there exists k such that $x_k > y_k$. The order $\geq$ is only a partial order: if a vector dominates another in one component but is dominated in another component, the vectors cannot be compared. Next, we consider some lattice idea and define the *"meet"* as $x \wedge y$ and *"join"* as $x \vee y$.

$$x \wedge y \equiv (\min(x_1, y_1), \cdots, \min(x_k, y_k))$$
$$x \vee y \equiv (\max(x_1, y_1), \cdots, \max(x_k, y_k))$$

S is a sublattice of $\mathbb{R}^m$ if $s \in S$ and $\tilde{s} \in S$ implies that $s \wedge \tilde{s} \in S$ and $s \vee \tilde{s} \in S$

Definition (Increasing difference)**.** $u_i(s_i, s_{-i})$ has *increasing difference* in (s_i, s_{-i}) **if**, for all $(s_i, \tilde{s}_i) \in S_i^2$ and $(s_{-i}, \tilde{s}_{-i}) \in S_{-i}^2$ s.t. $s_i \geq \tilde{s}_i$ and $s_{-i} \geq \tilde{s}_{-i}$, we have:

$$u_i(s_i, s_{-i}) - u_i(\tilde{s}_i, s_{-i}) > u_i(s_i, \tilde{s}_{-i}) - u_i(\tilde{s}_i, \tilde{s}_{-i})$$

Remark. Change the $\geq$ to $>$ in conditions for *strictly increasing diff*. This definition says that an increase in the strategies of player i's rivals (i.e., from playing $\tilde{s}_{-i}$ to s_{-i}) raises the desirability of playing high strategy for player i (i.e., from playing $\tilde{s}_i$ to s_i).

Definition (Supermodular)**.** $u_i(s_i, s_{-i})$ is *supermodular* in s_i **if** for each s_{-i} we have:

$$u_i(s_i \wedge \tilde{s}_i, s_{-i}) + u_i(s_i \vee \tilde{s}_i, s_{-i}) \geq u_i(s_i, s_{-i}) + u_i(\tilde{s}_i, s_{-i})$$

for all $(s_i, s_{-i}) \in s_i^2$. Strictly supermodular if changing the sign from $\geq$ to $>$.

Definition (Supermodular Game)**.** A *supermodular game* is such that, for each i, S_i is a sublattice of $\mathbb{R}^m$, u_i shows increasing differences in (s_i, s_{-i}), and u_i is supermodular in s_i.

Remark. Suppose u_i is twice continuously differentiable, then u_i is supermodular **if and only if** $\partial^2 u_i / \partial s_l \partial s_k \geq 0$.

Example. Consider à la Diamond (1982) search model with the following payoff function:

$$u_i(s) = \alpha s_i \sum_{j \neq i} s_j - c(s_i) \begin{cases} s_i : \text{ player } i\text{'s search intensity} \\ c(s_i) : \text{ cost of search} \\ s_i \sum_{j \neq i} s_j : \text{ probability of finding a trade partner} \\ \alpha : \text{ standalone gain if a partner is found} \end{cases}$$

Since $\partial^2 u_i / \partial s_i \partial s_j = \alpha > 0$, this search (game) is supermodular.

Theorem (Topkis 1979 & Vives 1990). (a) (Topkis 1979) If, for each i, S_i is compact and u_i is upper semi-continuous in s_i for each s_{-i}. Suppose the game is supermodular, then the Pure NE strategy is non-empty (i.e., $\neq \emptyset$) and possesses greatest/least equilibrium $\bar{s}$ and $\underline{s}$. (b) (Vives 1990) if furthermore the game is strictly supermodular, the set of NE is a non-empty "complete" sublattice ("complete" means sup and inf of any subset $\in$ that set).

Theorem (Topkis 1979). Consider a supermodular game with strictly increasing differences. If $s_i \in r_i^*(s_{-i})$, $\tilde{s}_i \in r_i^*(\tilde{s}_{-i})$ and that $s_{-i} \geq \tilde{s}_{-i}$, then $s_i \geq \tilde{s}_i$.

Theorem (Milgrom and Roberts 1990). Consider a supermodular game such that, for each i, S_i is a complete sublattice and is bounded, and such that u_i is continuous and is bounded above. Then, the IESDS strategies yield a set od strategies in which the greatest and the least elements are NE: $\bar{s}$ and $\underline{s}$.

(For more detailed walk-through and examples, check Fudenberg and Tirole (1991), p.496.)

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