21 Feb

Numerical Computing

Lecture # 03

M T W T F S — Date: 21/02/2022

Section II – Solutions of Non-Linear Equations

→ A non-linear equation is an equation that is not linear.

→ Non-linear equations contain polynomials and trancendental functions.
trigonometric, logarithmic & Exponential

$ax^2 + bx + c = 0$
$x^3 + x^2 = 0$
$x^n = 0$
$n = 2, 3, 4, \ldots$

→ A polynomial is a function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + a_0,$$

where $\boxed{n \geq 0}$ and $a_n \neq 0$.

here n (which is a non-negative integer) is called the degree or the power of the polynomial.

Why n is not negative?

e.g. $y = 2 \cdot x^{-10} = \frac{2}{x^{+10}}$

if $x = 0$

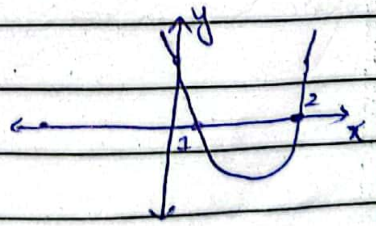
$y = \frac{2}{0^{-10}} = \infty$

[Polynomial Powers]

linear	$ax + b = 0$
quadratic	$ax^2 + bx + c = 0$
cubic	$ax^3 + b = 0$
quatic/ortic	$ax^4 + b = 0$
penta/pentic	$ax^5 + b = 0$

[I learned by hearing and learned more by seeing but i learned the most by doing ~~practical~~]

Non-Linear:



(A) (a)

$x^2 - 3x + 2 = 0$

$x^2 - 2x - x + 2 = 0$

$x(x - 2) - 1(x - 2) = 0$

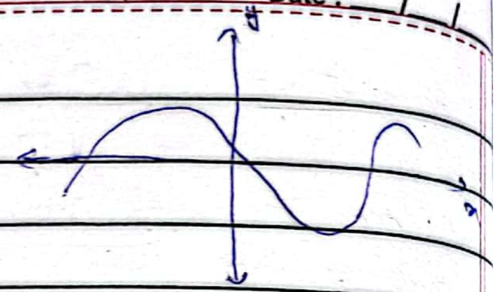
AHMED PAPER PRODUCT

$(x - 1)(x - 2) = 0$

$x = 1 \mid x = 2$

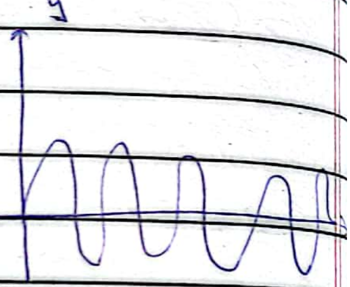
Scanned with CamScanner

(2) $x^3 + 2x^2 + x - 9 = 0.$



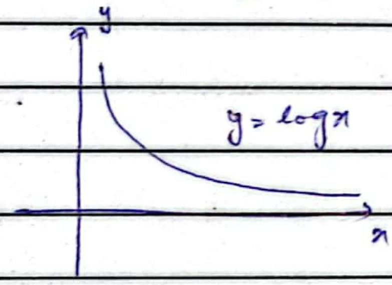
(3)

trigonometric functions are also non-linear function.



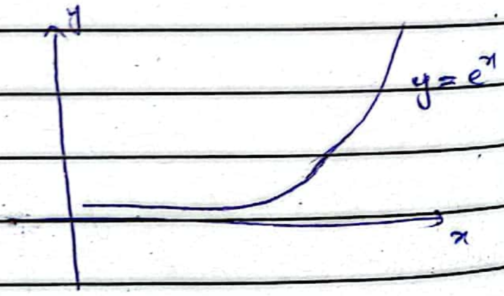
$y = \sin x$

logarithmic



$y = \log x$

Exponential



$y = e^x$

* The **Root** or the **Solution** of an equation is that value of the independent variable (say x) that satisfies the given equation.

i.e. $x^2 - 3x + 2 = 0$

has roots (or solutions) $x = 1$ and $x = 2$.

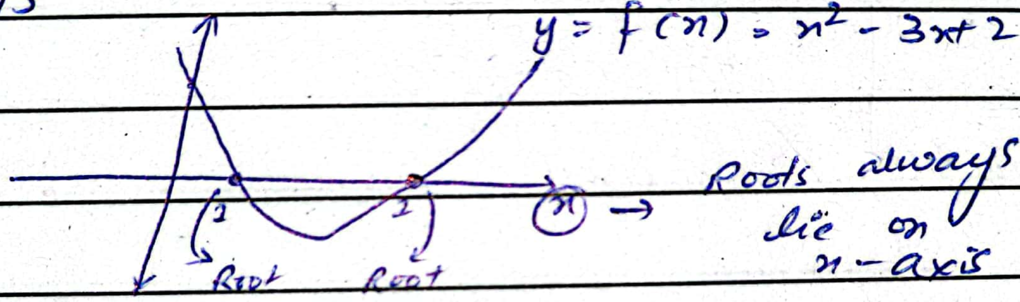
M T W T F S

Date: / /

Graphically, the root or the solution of a non-linear equation

$$y = f(x)$$

is the value of x at which the curve $y = f(x)$ crosses or intersects or touches the x-axis.



→ If a vertical line touches the given graph at more than one point then it is not function (graph).

Chapter #2

Numerical Computing.

Lec # 4

23rd Feb

Bolzano's Theorem:

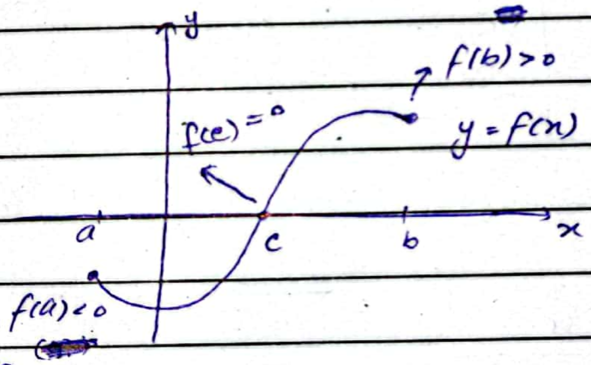
It is a generalization of Intermediate value theorem (I.V.T).

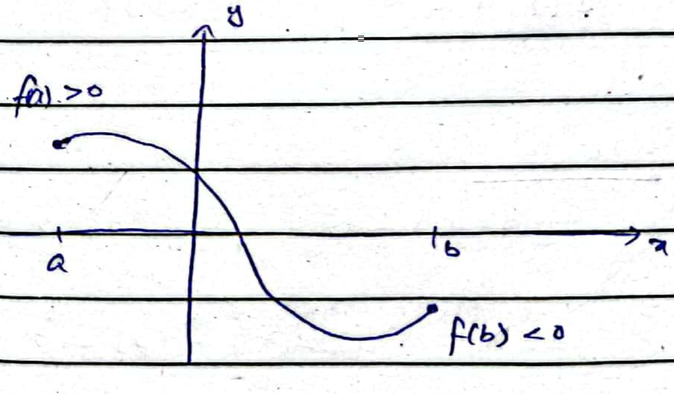
Statement: Suppose $f(x)$ is a continuous function on a closed interval $[a,b]$ and takes opposite sign values at the end-points. $f(a).f(b) < 0$. (that is, either $f(a) < 0$, $f(b) > 0$ or $f(a) > 0$, $f(b) < 0$. Then there exists at least one root c between a and b such that $f(c) = 0$.

Graphical Representation:

Conditions

(1) Continuous Function
(2) Closed Interval
(3) Opposite sign on end-points.





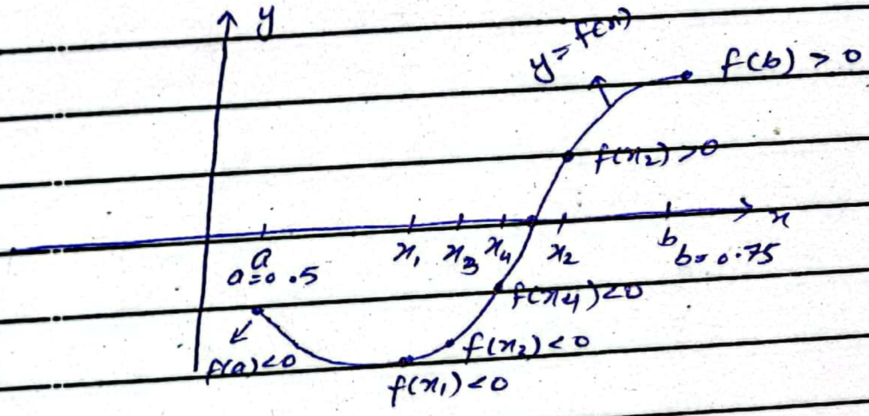
→ $f(a)$ and $f(b)$ should be positive or negative but never zero.

AHMED PAPER PRODUCT

→ Now, we are going to learn the numerical methods to solve non-linear equations of the form $\boxed{f(x) = 0}$.

① **Bisection Method :** It is also called halving the interval method and uses the Bolzano's theorem.
It bisects $[a, b]$ as $x_1 = \frac{a+b}{2}$.

Graphical Representation :



→ Use Bolzano's method (first satisfy all its three condition).

$$\boxed{x_1 = \frac{a+b}{2}}$$

Case I

$f(x_1) = 0$ (root).

Case 2

$f(x_1) < 0$, $f(b) > 0$

$\Rightarrow f(x_1) \cdot f(b) < 0$

$$\boxed{x_2 = \frac{x_1 + b}{2}}$$

M T W T F S Date : __/__/__

$f(x_2) > 0$, $f(x_1) < 0$

$\Rightarrow$ $f(x_2) \cdot f(x_1) < 0$.

$$x_3 = \frac{x_2 + x_1}{2}$$

$f(x_3) < 0$,

$f(x_3) < 0$, $f(x_2) > 0$ $\Rightarrow$ $f(x_3) \cdot f(x_2) < 0$

$$x_4 = \frac{x_3 + x_2}{2}$$

$f(x_4) < 0$,

$f(x_4) < 0$, $f(x_2) > 0$, $\Rightarrow$ $f(x_4) \cdot f(x_2) < 0$

$$x_5 = \frac{x_4 + x_2}{2}$$

$x_5 = \text{root}$

$\boxed{f(x_5) \approx 0}$

$x^3 + x = 1$, $[\underset{a}{0.5} , \underset{b}{0.75}]$.

$f(x) = x^3 + x - 1 = 0$

$x_5 = 0.68 \Rightarrow f(x_5) \approx 0$.

Numerical Computing

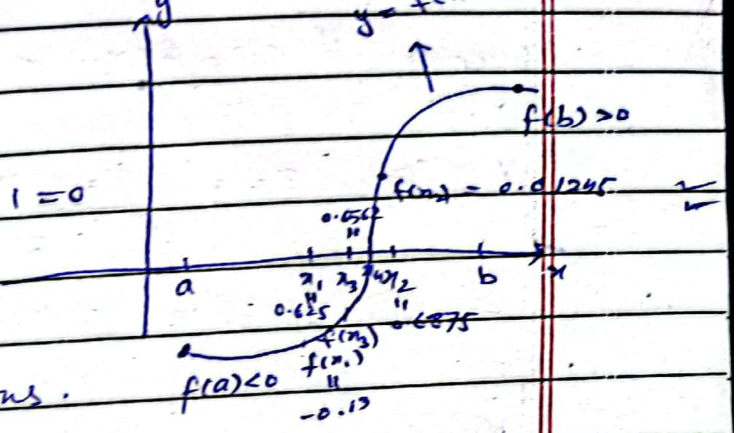
3^{rd} Feb.

M T W T F S — Date: / /

The Bisection method :

Example

Solve $x^3 + x = 2$, in $[0.5, 0.75]$ by the bisection method, correct upto 2 decimal places (2dp).



Solution: Given that

$f(x) = 0$

that is, $f(x) = x^3 + x - 1 = 0$

We know that our given function $f(x)$ is continous.

(because all the Polynomial are continous functions) $\boxed{f(x) = a_n x^n}$ $\boxed{y = f(x)}$

In the given closed intervals $[a, b] = [0.5, 0.75]$.

$f(a) = f(0.5) = -0.375 < 0$

$f(b) = f(0.75) = 0.1718 > 0$.

$\therefore f(a) \cdot f(b) < 0$

$\Rightarrow$ So, by Bolzano's Theorem, there must be at least one root between a and b.

Now, by the bisection method,

$\Rightarrow x_1 = \frac{a+b}{2} = \frac{0.5 + 0.75}{2} = 0.625$.

$\Rightarrow f(x_1) = f(0.625) = -0.130825 < 0$

AHMED PAPER PRODUCT

M T W T F S Date: / /

Since, $f(x_1) \cdot f(b) < 0$, so

$$x_2 = \frac{x_1 + b}{2} = \frac{0.625 + 0.75}{2} = 0.6875$$

$\Rightarrow f(x_2) = 0.01245 > 0$

Since $f(x_2) \cdot f(x_1) < 0$

$$\therefore x_3 = \frac{x_2 + x_1}{2} = 0.6562$$

$\Rightarrow f(x_3) = -0.0612 < 0$

Since $f(x_3) \cdot f(x_2) < 0$, So

$$x_4 = \frac{x_3 + x_2}{2} = 0.6718 \Rightarrow 0.67$$

$\Rightarrow f(x_4) = -0.025 < 0$

Since, $f(x_4) \cdot f(x_2) < 0$

So, $x_5 = \frac{x_4 + x_2}{2} = 0.6796 = 0.68$

$f(x_5) = -0.0065 < 0 \approx 0$

~~x_5 is our root.~~

checking step

Since, $f(x_5) \cdot f(x_2) < 0$, So

$$x_6 = \frac{x_5 + x_2}{2} = 0.6836 = 0.68$$

$\Rightarrow f(x_6) = 0.003 \approx 0$

AHMED PAPER PRODUCT

So, x_5 and x_6 is same at $5^{th}/6^{th}$ operation.
So, x_5 And x_6 are root

M T W T F S   Date: / /

Task: Solve $\sin x = 5x - 2$ in $[0.4, 0.5]$ by the bisection method, in radians, correct up to 2 dp.

Solution: Given that $f(x) = 0$

that is, $f(x) = \sin x - 5x + 2 = 0$

We know that one given function $f(x)$ is continuous as all polynomial trigonometric functions are continuous.

In the given closed interval $[a, b] = [0.4, 0.5]$.

$$f(a) = f(0.4) = 0.3894 - 2 + 2 = 0.3894 > 0$$

$$f(b) = f(0.5) = 0.4794 - 2.5 + 2 = -0.0206 < 0$$

by bisection method

Since, $f(a) \cdot f(b) < 0$, so

$$x_1 = \frac{a+b}{2} = \frac{0.4 + 0.5}{2} = \cancel{\frac{0.3894 + 0.0206}{2}} = \cancel{0.1844 > 0}$$

$$f(x_1) = f(0.1844) = 0.1833 - 0.922 + 2$$

$$f(x_1) = 1.2613 > 0$$

Since, $f(x_1) \cdot f(b) < 0$

$$x_2 = \frac{x_1 + b}{2} = \frac{1.2613 - 0.0206}{2} = 0.62035 > 0$$

$$f(x_2) = f(0.62035) = 0.5813 - 3.10175 + 2$$

$$f(x_2) = -0.51875 < 0$$

AHMED PAPER PRODUCT

Scanned with CamScanner

2nd March

* Numerical Computing

M T W T F S Date: / /

* Bisection Method

Task: Solve $\sin x = 5x - 2$ in $[0.4, 0.5]$ by the bisection method, in radiations (radians) correct up to 2 dp.

Solution: Given that $f(x) = 0$

that is, $f(x) = \sin x - 5x + 2 = 0$

→ We know that our given function $f(x)$ is continuous.

→ In the given close interval $[a, b] = [0.4, 0.5]$.)

$f(a) = f(0.4) = \sin(0.4) - 5(0.4) + 2 =$

$= 0.3894 - \cancel{2} + \cancel{2} = 0.39 > 0$

$f(b) = f(0.5) = \sin(0.5) - 5(0.5) + 2$

$= 0.48 - 2.5 + 2 = -0.02 < 0$

Since, $f(a).f(b) < 0$, So by bisection method

$x_1 = \frac{a+b}{2} = 0.45$

$f(x_1) = f(0.45) = \sin(0.45) - 5(0.45) + 2$

$= 0.43 - 2.25 + 2 = 0.18 > 0$

Since, $f(x_1) \cdot f(b) < 0$, So

$x_2 = \frac{x_1 + b}{2} = 0.48$

AHMED PAPER PRODUCT

M T W T F S Date : ___/___/___

$f(x_2) = f(0.48) = \sin(0.48) - 5(0.48) + 2$

$= 0.46 - 2.4 + 2 = 0.06 > 0$

Since, $f(x_2) \cdot f(b) < 0$, So

$$x_3 = \frac{x_2 + b}{2} = 0.49$$

$f(x_3) = f(0.49) = \sin(0.49) - 5(0.49) + 2$

$= 0.47 - 2.45 + 2$

$= 0.02 > 0$

Since, $f(x_3) \cdot f(b) < 0$, So

$$x_4 = \frac{x_3 + b}{2} = 0.50$$

$f(x_4) = f(0.50) = \sin(0.50) - 5(0.50) + 2$

$= 0.48 - 2.5 + 2 = -0.02 \not< 0$

Since, $f(x_4) \cdot f(x_3) < 0$, So

$$x_5 = \frac{x_4 + x_3}{2} = 0.50$$

$f(x_5) = f(0.50) = -0.02 \not< 0$

→ As x_4 and x_5 are same.

→ So, x_5 and x_6 are roots at 4^{th} and 5^{th} iteration respectively.

Home Task: Solve $x^3 = 10 - 4x^2$ in $[1, 2]$, by the bisection method, correct up to 2 dp.

Solution: Given that $f(x) = 0$

that is, $f(x) = x^3 + 4x^2 - 10 = 0$

→ We know that our given function $f(x)$ is continuous.

→ In the given close interval $[a, b] = [1, 2]$

$f(a) = f(1) = (1)^3 + 4(1)^2 - 10 = -5 < 0$

$f(b) = f(2) = (2)^3 + 4(2)^2 - 10 = 14 > 0$

Since, $f(a) \cdot f(b) < 0$, So

$$x_1 = \frac{a+b}{2} = 1.5$$

$f(x_1) = f(1.5) = (1.5)^3 + 4(1.5)^2 - 10$

$= 3.38 + 9 - 10 = 2.38 > 0$

Since, $f(x_1) \cdot f(a) < 0$, So

$$x_2 = \frac{x_1 + a}{2} = 1.25$$

$f(x_2) = f(1.25) = (1.25)^3 + 4(1.25)^2 - 10$

$= 1.95 + 6.25 - 10 = -1.8 < 0$

Since, $f(x_1) \cdot f(x_2) < 0$, So

$$x_3 = \frac{x_1 + x_2}{2} = \frac{1.5 + 1.25}{2} = 1.38$$

$f(x_3) = f(1.38) = (1.38)^3 + 4(1.38)^2 - 10$

$= 2.62 + 7.62 - 10 = 0.23 > 0$

Since $f(x_2).f(x_3) < 0$, so

$x_4 = \frac{x_2 + x_3}{2} = \frac{1.25 + 1.38}{2} = 1.32$

$f(x_4) = f(1.32) = (1.32)^3 + 4(1.32)^2 - 10$

$= 2.30 + 6.97 - 10 = -0.73 < 0$

Since, $f(x_3).f(x_4) < 0$, so

$x_5 = \frac{x_3 + x_4}{2} = \frac{1.38 + 1.32}{2} = 1.35$

$f(x_5) = f(1.35) = (1.35)^3 + 4(1.35)^2 - 10$

$= 2.46 + 7.29 - 10 = -0.25 < 0$

Since, $f(x_3).f(x_5) < 0$, So

$x_6 = \frac{x_3 + x_5}{2} = \frac{1.38 + 1.35}{2} = 1.37$

$f(x_6) = f(1.37) = (1.37)^3 + 4(1.37)^2 - 10$

$= 2.57 + 7.50 - 10 = 0.07 > 0$

Since, $f(x_5).f(x_6) < 0$, So

$x_7 = \frac{x_5 + x_6}{2} = \frac{1.35 + 1.37}{2} = 1.36$

$f(x_7) = f(1.36) = (1.36)^3 + 4(1.36)^2 - 10 = -0.08 \leftarrow$

$= 2.52 + 7.40 - 10$

$= -0.08 \neq 0$

M T W T F S Date :

Since, $f(x_6) \cdot f(x_7) < 0$,

$$x_8 = \frac{x_6 + x_7}{2} = \frac{1.37 + 1.36}{2} = 1.36$$

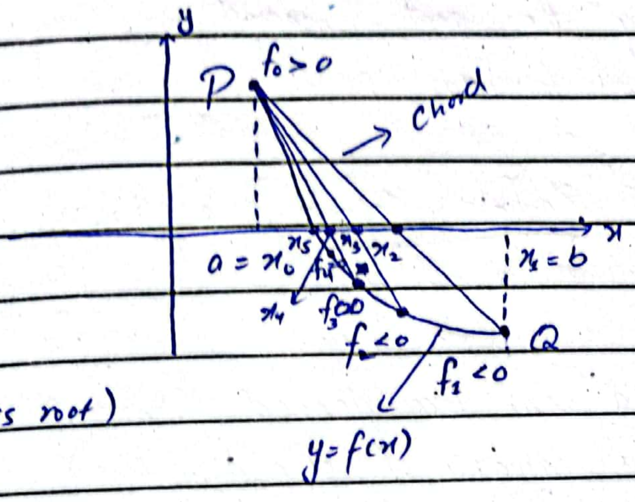
$$f(x_8) = f(1.36) = -0.08 \approx 0$$

→ As x_7 and x_8 are same.

→ So, x_7 and x_8 are roots at 7^{th} and 8^{th} iteration respectively.

(a) The Method of False-Position (or Regula-Falsi Method) :

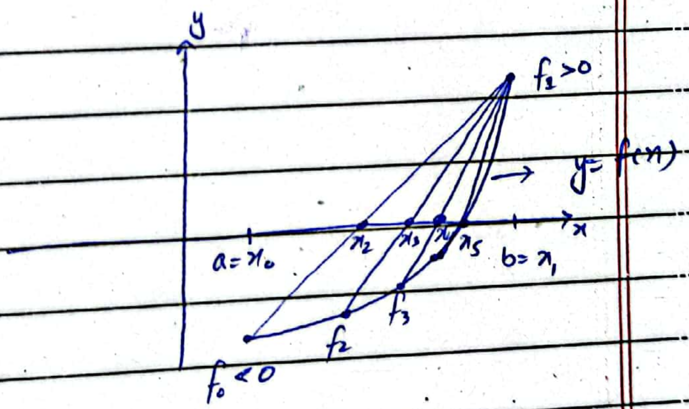
Graphical Illustration : (The method of chord)



(x_5 is root)

(Case 1)

How be at
Peak in your
Field ?
(1) Hard work .
(2) Disscussion
with Seniors
(3) Have your
Relationship
Strength
with Allah



x_5 is root .

(Case 2)

→ In this method, either f_0 or f_1 should be fixed.

→ This method also use Bolzano's theorem

Date : __/__/__

False - Position Method :

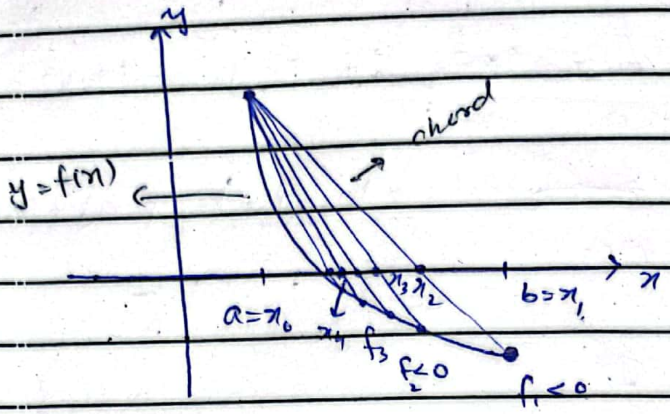
This method also uses the Bolzano's Theorem and hence it is also a bracketing the root method.

→ It's general formula is

$$x_{n+1} = \frac{x_{n-1} f_n - x_n f_{n-1}}{(f_n - f_{n-1})}, \quad n = 1, 2, 3, \ldots \text{ and } f_n \neq f_{n-1}$$

→ This formula can be derived from Calculas (using the equation of straight line).

* The Method of False-Position (Regula-Falsi method):



⇒ The equation of a straight line (or chord $\overline{PQ}$) is

$y - y_0 = m(x - x_0)$, where $\frac{y - y_0}{x_1 - x_0} = \text{slope}$

At the intersection of $\overline{PQ}$ with x-axis, we have

$$x = x_2 \quad \text{and} \quad y = 0$$

$$\Rightarrow \quad 0 - y_0 = m(x_2 - x_0) \Rightarrow \frac{-y_0}{m} = x_2 - x_0 \Rightarrow x_2 = x_0 -$$

$$\Rightarrow \quad x_2 = x_0 - \left(\frac{x_1 - x_0}{y_1 - y_0}\right) y_0 = \frac{x_0 y_1 - \cancel{x_0 y_0} - x_1 y_0 + \cancel{x_0 y_0}}{y_1 - y_0}$$

$$= \frac{x_0 y_1 - x_1 y_0}{y_1 - y_0}$$

$$\Rightarrow \quad x_2 = \frac{x_1 y_0 - x_0 y_1}{y_0 - y_1} = \frac{x_1 f_0 - x_0 f_1}{f_0 - f_1}$$

$$x_3 = \frac{x_2 f_1 - x_1 f_2}{f_1 - f_2}$$

obviously, $f_0 - f_1 \neq 0$, here

Generalization of this result gives

$$x_{n+1} = \frac{x_n f_{n-1} - x_{n-1} f_n}{f_{n-1} - f_n}, \quad \text{for } n = 1, 2, \ldots$$

M T W T F S

Date: __/__/__

Example: Find the root of

$x^3 - 3x = -1$, in $[0, 1]$,

using the false position method, correct up to 2 dp.

Solution: Given that $f(x) = 0$.

$\Rightarrow$ $x^3 - 3x + 1 = 0$

We know that the given function $f(x)$ is continous. (because all the polynomials are continous functions) in the given closed interval.

$[x_0, x_1] = [0, 1]$.

$\Rightarrow$ $f_0 = f(0) = 1 > 0$.

$f_1 = f(1) = -1 < 0$

So, by bolzano's theorem, there must be a root between 0 and 1. $\Rightarrow$ By the false position method,

$f_0 \cdot f_1 < 0$,

$$x_2 = \frac{x_1 f_0 - x_0 f_1}{f_0 - f_1}$$

$$= \frac{(1)(1) - (0)(-1)}{1 - (-1)}$$

$$x_2 = \frac{1 - 0}{1 + 1} = \frac{1}{2}$$

$$\boxed{x_2 = 1/2} = 0.5$$

$f_2 = f(1/2) = (1/2)^3 - 3(1/2) + 1 = \frac{1}{8} - \frac{3}{2} + 1 = \frac{1 - 12 + 8}{8}$

AHMED PAPER PRODUCT

$\boxed{f(1/2) = -3/8}$ $f(1/2) = -0.375 < 0$

M T W T F S Date: / /

Since, $f_2 \cdot f_0 < 0$.

So $x_3 = \dfrac{x_2 f_0 - x_0 f_2}{f_0 - f_2}$

$x_3 = \dfrac{(0.5)(1) - (0)(-0.37)}{1 - (-0.375)}$

$\boxed{x_3 = 0.36}$

$x_3 = \dfrac{0.5}{1.38}$

$\Rightarrow f_3 = f(x_3) = -0.04$

$x_3 = 0.36$

since $f_3 \cdot f_0 < 0$,

So $x_4 = \dfrac{x_3 f_0 - x_0 f_3}{f_0 - f_3} = 0.35$ $\boxed{x_4 = 0.35}$

$\Rightarrow f_4 = f(x_4) = -0.0037 < 0$

checking Since $f_4 \cdot f_0 < 0$, so

$x_5 = \dfrac{x_4 \cdot f_0 - x_0 f_4}{f_0 - f_4}$

$\boxed{x_5 = 0.35}$ $f_5 = -0.0003 \approx 0$

So, x_4 and x_5 are root.

AHMED PAPER PRODUCT

Scanned with CamScanner

Task: Find the root of $x^3 - 3x = -1$ in $[1, 2]$, using the false position method, correct up to 2 dp.

Solution: Given that $f(x) = 0$

$\Rightarrow f(x) = x^3 - 3x + 1 = 0$

we know that function is continuous in the given close interval. $[x_0, x_1] = [1, 2]$.

$\Rightarrow f_0 = f(x_0) = -1 < 0$

$\Rightarrow f_1 = f(x_1) = 3 > 0$

⇒ So, by Bolzano's theorem, there must be a root between 1 and 2.

⇒ By false - position method.

$$x_2 = \frac{x_1 f_0 - x_0 f_1}{f_0 - f_1} = \frac{(2)(-1) - (1)(3)}{(-1) - (3)}$$

$$x_2 = \frac{-2 - 3}{-4} = \frac{5}{4} = 1.25$$

$$f_2 = f(x_2) = (1.25)^3 - 3(1.25) + 1$$

$$f_2 = (1.953125) - 3.75 + 1$$

$$f_2 = -0.796875 < 0$$

Since, $f_1 \cdot f_2 < 0$, so

$$x_3 = \frac{x_2 f_1 - x_1 f_2}{f_1 - f_2} = \frac{(1.25)(3) - (2)(-0.796875)}{3 - (-0.796875)}$$

$$= \frac{3.75 + 1.5375}{3.796875} = \frac{5.2875}{3.796875}$$

$$\boxed{x_3 = 1.3926}$$

M T W T F S   Date: __/__/__

$f_3 = f(x_3) = (1.3926)^3 - 3(1.3926) + 1$

$= 2.7007 - 4.1778 + 1$

$f_3 = -0.4771 < 0$

Since, $f_3 \cdot f_1 < 0$, so

$x_4 = \frac{x_3 f_2 - x_2 f_3}{f_1 - f_3} = \frac{(1.3926)(3) - (2)(-}{3 - (-0.47}$

$x_4 = \frac{4.1778 + 0.9542}{3.4771} = 1.4759$

$f_4 = f(x_4) = (1.4759)^3 - 3(1.4759) + 1$

$= 3.2152 - 4.4277 + 1$

$f_4 = -0.2125 < 0$

Since, $f_4 \cdot f_1 < 0$, so

$x_5 = \frac{x_4 f_2 - x_1 f_4}{f_1 - f_4} = \frac{(1.4759)(3) - (2)(}{3 - (-0.2125)}$

$x_5 = \frac{4.4277 + \cancel{0.2125}\ 0.425}{3.2125} = 1.5106$

$f_5 = f(x_5) = (1.5106)^3 - 3(1.5106) + 1$

$= 3.4468 - 4.5318 + 1$

$f_5 = -0.085 < 0$

Since, $f_5 \cdot f_1 < 0$, so

$x_6 = \frac{x_5 f_1 - x_1 f_5}{f_1 - f_5}$

M T W T F S   Date: / /

$$x_6 = \frac{(1.5106)(3) - (2)(-0.085)}{(3) - (-0.085)}$$

$$x_6 = \frac{4.5318 + 0.17}{3.085} = 1.5240$$

$$f_6 = f(x_6) = (1.5240)^3 - 3(1.5240) + 1$$
$$= 3.5401 - 4.572 + 1$$
$$= -0.0319 < 0$$

Since, $f_6 \cdot f_1 < 0$, so

$$x_7 = \frac{x_6 f_1 - x_1 f_6}{f_1 - f_6} = \frac{(1.5240)(3) - (2)(-0.0319)}{(3) - (-0.0319)}$$

$$= \frac{4.572 + 0.0638}{3.0319} = 1.5290$$

$$f(x_7) = f_7 = (1.5290)^3 - 3(1.5290) + 1$$
$$= 3.5745 - 4.587 + 1$$
$$f_7 = -0.0125$$

Since, $f_7 \cdot f_1 < 0$, so

$$x_8 = \frac{x_7 f_1 - x_1 f_7}{f_1 - f_7} = \frac{(1.5290)(3) - (2)(-0.0125)}{(3) - (-0.0125)}$$

$$x_8 = \frac{4.587 + 0.025}{3.0125} = 1.5309.$$

$$f_8 = f(x_8) = (1.5309)^3 - 3(1.5309) + 1$$
$$= 3.5879 - 4.5927 + 1$$
$$= -\cancel{0.0169}\ 0.0048$$

AHMED PAPER PRODUCT

→ So, x_8 is our root.

M T W T F S   Date: / /

* Home Task

Find a root of $xe^x = \cos x$, in $[0, 1]$, using the false position method, correct up to 2 dp (in radius).

Solution: Given that $f(x) = 0$

$\Rightarrow \quad f(x) = xe^x - \cos x = 0$

$\Rightarrow$ As we know that function is continuous in the given closed interval $[x_0, x_1] = [0, 1]$

So,

$f_0 = f(x_0) = (0)e^0 - \cos(0) = -1 < 0$

$f_1 = f(x_1) = (1)e^1 - \cos(1) = 2.7182 - 0.5403$

$f_1 = 2.~~68~~1779 > 0$

Since, $f_0 \cdot f_1 < 0$, So

$$x_2 = \frac{x_1 f_0 - x_0 f_1}{f_0 - f_1} = \frac{(1)(-1) - (0)(2.1779)}{-1 - 2.1779}$$

$$x_2 = \frac{-1}{-3.1779} = 0.3146$$

$f_2 = f(x_2) = (0.3146)e^{(0.3146)} - \cos(0.3146)$

$= 0.4309 - 0.9509$

$f_2 = -0.52 < 0$

Since, $f_1 \cdot f_2 < 0$, So

$$x_3 = \frac{x_2 f_1 - x_1 f_2}{f_1 - f_2} = \frac{(0.3146)(2.1779) - (1)(-0.52)}{2.1779 - (-0.52)}$$

=

AHMED PAPER PRODUCT

Scanned with CamScanner

M T W T F S   Date: / /

$$x_3 = \frac{0.6851 + 0.52}{2.6979} = 0.4466$$

$$f_3 = f(x_3) = (0.4466)e^{(0.4466)} - \cos(0.4466)$$

$$= 0.6980 - 0.9019$$

$$f_3 = -0.2039 < 0$$

$\Rightarrow$ Since, $f_1 \cdot f_3 < 0$, so

$$x_4 = \frac{x_3 f_1 - x_1 f_3}{f_1 - f_3} = \frac{(0.4466)(2.1779) - (1)(-0.2039)}{(2.1779) - (-0.2039)}$$

$$x_4 = \frac{0.9726 + 0.2039}{2.3818} = 0.4939$$

$$f_4 = f(x_4) = (0.4939)e^{(0.4939)} - \cos(0.4939)$$

$$= 0.8093 - 0.8804$$

$$f_4 = -0.0711 < 0$$

$\Rightarrow$ Since, $f_1 \cdot f_4 < 0$, so

$$x_5 = \frac{x_4 f_1 - x_1 f_4}{f_1 - f_4} = \frac{(0.4939)(2.1779) - (1)(-0.0711)}{(2.1779) - (-0.0711)}$$

$$x_5 = \frac{1.0756 + 0.0711}{2.249} = 0.5098$$

$$f_5 = f(x_5) = (0.5098)e^{(0.5098)} - \cos(0.5098)$$

$$= 0.8487 - 0.8728$$

$$f_5 = -0.0241$$

$\Rightarrow$ Since $f_1 \cdot f_5 < 0$, so

$$x_6 = \frac{x_5 f_1 - x_1 f_5}{f_1 - f_5} = \frac{(0.5098)(2.1779) - (1)(-0.0241)}{(2.1779) - (-0.0241)} = \frac{1.1343}{2.202}$$

$$x_6 = 0.5151$$

$$f_6 = f(x_6) = (0.5151)e^{(0.5151)} - \cos(0.5151)$$

$$= 0.8621 - 0.8702$$

$$f_6 = -0.0081 \approx 0$$

So, the root is x_6.

9th March

* Numerical Computing.

M T W T F S Date: / /

(03) Fixed Point Iteration Method:

we usually find the root at which the graph of $y = f(x)$ crosses the x-axis, or equivalently, $f(x) = 0$. Similarly, we can find a fixed point at which the graph of $y = g(x)$ and $y = h(x)$ intersect each other, by solving $g(x) = h(x)$. we can easily reduce the second case to the first by setting

$$f(x) = g(x) - h(x) = 0$$

and solving $f(x) = 0$, instead.

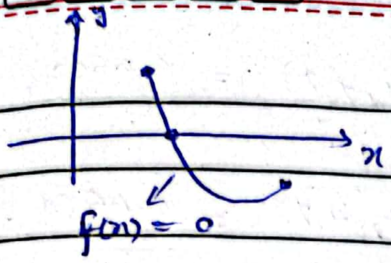
⇒ Iteration means the repitition of a process. This method is applicable to $f(x) = 0$ in the modified from as $f(x) + x = x$.

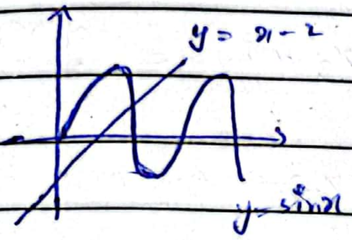
⇒ $x = g(x)$, with the condition of convergence that $|g'(x)| < 1$.

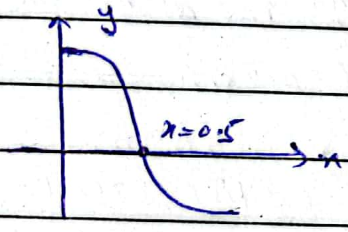
In terms of iterations, we have

$$x_{n+1} = g(x_n) \quad , \quad n = 0, 1, 2, 3, \ldots$$

and a known x_0 (the initial guess).







$\sin x = x - 2$. $\sin x - x + 2 = 0$
$\downarrow$
$f(x) = 0$

* <u>Graphical Illustration using an Example :</u>

⇒ Solve $\sin x = 5x - 2$, using the fixed point iteration method, near $x_0 = 0.5$, correct up to 3 dp in radians.

⇒ <u>Solution :</u> For fixed point iteration method, we have

$$5x = \sin x + 2 \Rightarrow x = \frac{1}{5}(\sin x + 2) = g(x)$$

Condition ⇒ $|g'(x)| = \frac{2}{5}|\cos x| < 1$, as $\cos x \le 1$

Now, we set the iteration as

$$x_{n+1} = g(x_n) = \frac{1}{5}(\sin x_n + 2), \text{ with } x_0 = 0.5$$

$n = 0 \Rightarrow x_1 = \frac{1}{5}(\sin x_0 + 2) = 0.4959$

M T W T F S Date: / /

$n = 1 \Rightarrow x_2 = \frac{1}{5}(\sin x_1 + 2) = 0.4952$

$n = 2 \Rightarrow x_3 = \frac{1}{5}(\sin x_2 + 2) = 0.4950$

$n = 3 \Rightarrow x_4 = \frac{1}{5}(\sin x_3 + 2) = 0.4950$

Checking step: $f(x_4) = 0.00003 \approx 0$

where $f(x) = \sin x - 5x + 2$

———x———

Task: Find a positive real root of

$$x^3 + x^2 - 1 = 0,$$

by fixed point iteration method, with $x_0 = 0.$ correct up to 2 dp.

Solution: For fixed point iteration method, we have

$$x^3 + x^2 - 1 = 0$$

$$x^3 + x^2 = 1$$

$$x^2(x + 1) = 1$$

$$x^2 = \frac{1}{x+1}$$

$$\sqrt{x^2} = \sqrt{\frac{1}{x+1}}$$

$$x = \frac{1}{\sqrt{x+1}} = g(x)$$

~~Now, we set the iteration as~~

$$|g'(x)| = \left|\frac{-1}{2(x+1)^{3/2}}\right| = \left|\frac{-1}{2}(x+1)^{3/2}\right| = \frac{1}{2(x+1)^{3/2}}$$

AHMED PAPER PRODUCT

M T W T F S Date: / /

$$|g'(x)| = \frac{1}{2(x+1)^{3/2}} = \frac{1}{2(0.5+1)^{3/2}} = \frac{1}{3.6742} = 0.2721 < 1$$

Condition True.

Now, we set the iterations as

$$x_{n+1} = g(x_n) = \frac{1}{\sqrt{x_n+1}}, \text{ with } x_0 = 0.5$$

$n=0 \Rightarrow x_1 = \frac{1}{\sqrt{x_0+1}} = \frac{1}{\sqrt{0.5+1}} = 0.8164$

$n=1 \Rightarrow x_2 = \frac{1}{\sqrt{x_1+1}} = \frac{1}{\sqrt{0.8164+1}} = 0.7419$

$n=2 \Rightarrow x_3 = \frac{1}{\sqrt{x_2+1}} = \frac{1}{\sqrt{0.7419+1}} = 0.7576$

$n=3 \Rightarrow x_4 = \frac{1}{\sqrt{x_3+1}} = \frac{1}{\sqrt{0.7576+1}} = 0.7542$

Checking step:

$$f(x_4) = (0.7542)^3 + (0.7542)^2 - 1$$
$$= 0.4291 + 0.5688 - 1$$
$$= -0.0021 \approx 0.$$

where

$$f(x) = x^3 + x^2 - 1$$

AHMED PAPER PRODUCT

14th March

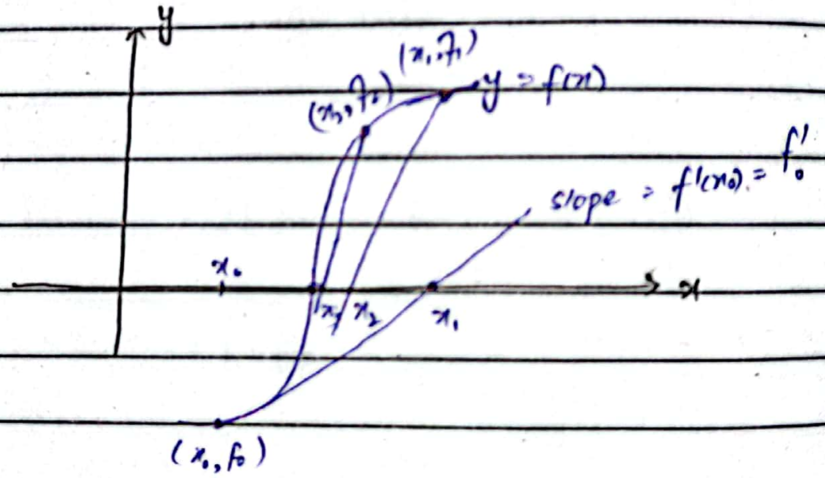
* Numerical Computing.

M T W T F S Date: / /

(4) Newton's Method:

It is also known as Newton-Raphson method and is the most powerful numerical method to find the roots of a non-linear equation $f(x)=0$, using an initial guess (or approximation) x_0. It is also fastly convergent, provided that x_0 is chosen sufficiently close to the actual root.

Graphical Illustration



Statement: If x_n is an approximation to a root of $f(x)=0$, then a better approximation is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

$f'(x_n) \neq 0$ and $n = 0, 1, 2, 3, \ldots$

if $n=0$,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

M T W T F S

Date: / /

Proof: The equation of tangent at (x_0, f_0) is $y - y_0 = m(x - x_0)$,

$\Rightarrow y - f_0 = f_0'(x - x_0)$, where $m = f'(x_0) = f_0'$ = slope of the tangent line

At the x-intercept, $x = x_1$ and $y = 0$.

$\therefore 0 - f_0 = f_0'(x_1 - x_0) \Rightarrow x_1 = x_0 - \frac{f_0}{f_0'} = x_0 - \frac{f(x_0)}{f'(x_0)}$

Similarly, $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad f'(x_n) \neq 0 \text{ and } n = 0, 1, 2, 3, \ldots$$

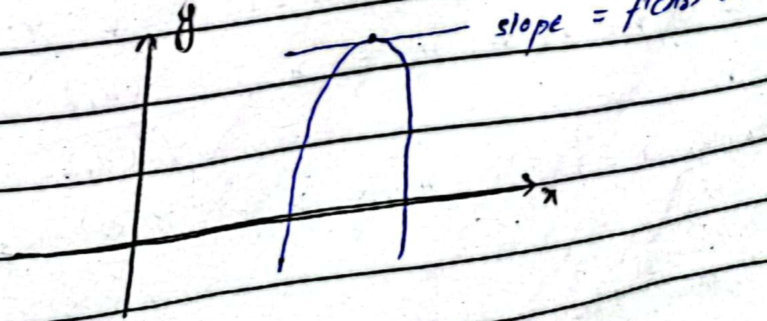
[~~when convergence fails~~ divergence arrives].

Limitations:

Newton's method fails to convergence if

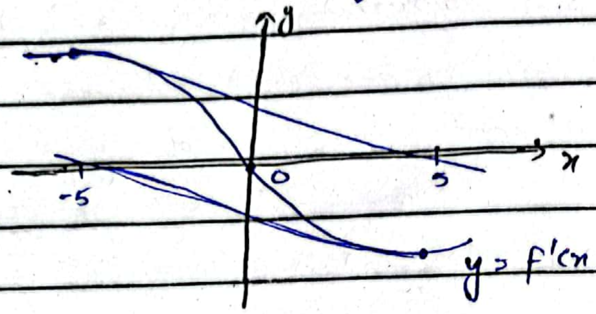
(i) The root is very close to the maximum and minimum value of the function because at such points,

$f'(x) = 0$



M T W T F S — Date: / /

(02) The initial guess x_0 lies away from the actual root and the function has a special shape that makes the gusses in a cyclic loop.



* EX: Solve $\sin x + 2 = 5x$, near $x_0 = 0.5$, correct up to 3 dp in radians.

Solution : Let

$$f(x) = \sin x - 5x + 2 = 0$$

$$\Rightarrow f'(x) = \cos x - 5$$

using Newton's Method, we write

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad f'(x_n) \neq 0 \text{ and } n = 0,1,2,3\ldots$$

$$n = 0 \Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 0.5 - \left[\frac{-0.0257}{-4.1224}\right] = 0.495009$$

$$n = 1 \Rightarrow x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.495007$$

$$n = 2 \Rightarrow x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.495007$$

Checking step

$f(x_3) = 0.0000002 \approx 0$.

Task : Solve $x^3 + 1 = 5x$, near $x_0 = 1$, using Newton's method, correct up to 3 dp.

Solution : Let $f(x) = x^3 - 5x + 1 = 0$

$\Rightarrow f'(x) = 3x^2 - 5$

using Newton's method, we write

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} , \quad f'(x_n) \neq 0 , \text{ and } n = 0, 1, 2, 3 \ldots$$

$n = 0 \Rightarrow$ $$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{(1)^3 - 5(1) + 1}{(3(1)^2 - 5)}$$

$$x_1 = 1 - \frac{(-3)}{(-2)} = \frac{2 - 3}{2} = \frac{-1}{2} = -0.5$$

when $n = 1 \Rightarrow$ $$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = -0.5 - \frac{((-0.5)^3 - 5(-0.5) + 1)}{(3(-0.5)^2 - 5)}$$

$$= -0.5 - \frac{(-0.125 + 2.5 + 1)}{(+0.75 - 5)}$$

$$x_2 = -0.5 + \frac{3.375}{4.25} = 0.2941$$

~~when $n = 2 \Rightarrow x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.019 - \frac{((0.019)^3 - 5(0.019) + 1)}{3(0.}$~~

$\Rightarrow$ Next Page.

15th March

* Numerical Computing.

M T W T F S   Date: __/__/__

Task: Solve $x^3 + 1 = 5x$, near $x_0 = 1$, using Newton's method, correct up to 3 dp.

Solution: Let $f(x) = x^3 - 5x + 1 = 0$

$\Rightarrow f'(x) = 3x^2 - 5$

using Newton's Method, we write

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad f'(x_n) \neq 0 \text{ and } n = 0, 1, 2, 3 \ldots$$

when $n = 0$, $\Rightarrow$ $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$$= 1 - \frac{((1)^3 - 5(1) + 1)}{(3(1)^2 - 5)}$$

$$= 1 - \frac{(1 - 5 + 1)}{3 - 5} = 1 - \frac{3}{2}.$$

$$\boxed{x_1 = -0.5}$$

when $n = 1$ $\Rightarrow$ $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$$= -0.5 - \frac{((-0.5)^3 - 5(-0.5) + 1)}{(3(-0.5)^2 - 5)}$$

$$= -0.5 - \frac{(-0.125 + 2.5 + 1)}{(0.75 - 5)} = -0.5 + \frac{3.375}{4.25}$$

$$\boxed{x_2 = 0.2941}$$

when $n = 2$ $\Rightarrow$ $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

$$= 0.2941 - \frac{((0.2941)^3 - 5(0.2941) + 1)}{(3(0.2941)^2 - 5)}$$

AHMED PAPER PRODUCT

$$= 0.2941 - \frac{0.4450}{4.7405}$$

$$\boxed{x_3 = 0.2003}$$

when $n = 3 \Rightarrow x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 0.2003 - \frac{((0.2003)^3 - 5(0.2003)}{(3(0.2003)^2 - 5}$

$= 0.2003 + \frac{0.0065}{4.9598}$

$$\boxed{x_4 = 0.2016}$$

when $n = 4 \Rightarrow x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} = 0.2016 - \frac{(0.2016)^3 - 5(0.2016)}{(3(0.2016)^2 - 5}$

$= 0.2016 + \frac{0.0001}{4.8781}$

$$\boxed{x_5 = 0.2016}$$

checking step

$f(x_5) = 0.0007 \approx 0$

* Solve with

$\Rightarrow x_0 = 3$

when $n = 0 \Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 3 - \frac{((3)^3 - 5(3) + 1)}{(3(3)^2 - 5)}$

$= 3 - \frac{27 - 15 + 1}{27 - 5}$

$$\boxed{x_1 = 2.4091}$$

when $n = 1 \Rightarrow x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.4091 - \frac{((2.4091)^3 - 5(2.4091) + 1)}{(3(2.4091)^2 - 5)}$

$= 2.4091 - \frac{(13.9818 - 12.0455 + 1)}{(17.4112 - 5)}$

$$\boxed{x_2 = 2.1726}$$

M T W T F S Date: / /

when $n=2 \Rightarrow x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.1726 - \frac{((2.1726)^3 - 5(2.1726)+1)}{(3(2.1726)^2 - 5)}$

$= 2.1726 - \frac{(10.2550 - 10.863 + 1)}{(14.1605 - 5)}$

$\boxed{x_3 = 2.1299}$

when $n=3 \Rightarrow x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 2.1299 - \frac{((2.1299)^3 - 5(2.1299)+1)}{(3(2.1299)^2 - 5)}$

$= 2.1299 - \frac{(9.6622 - 10.6495 + 1)}{(13.6094 - 5)}$

$x_4 = 2.1285$

when $n=4 \Rightarrow x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} = 2.1285 - \frac{((2.1285)^3 - 5(2.1285)+1)}{(3(2.1285)^2 - 5)}$

$= 2.1285 - \frac{(9.6431 - 10.6425 + 1)}{(13.5915 - 5)}$

$\boxed{x_5 = 2.1285}$

checking step. $f(x_5) = 0.0006 \approx 0$

—x—

M T W T F S Date: / /

when $n=2 \Rightarrow x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.1726 - \frac{((2.1726)^3 - 5(2.1726)+1)}{(3(2.1726)^2 - 5)}$

$= 2.1726 - \frac{(10.2550 - 10.863 + 1)}{(14.1605 - 5)}$

$\boxed{x_3 = 2.1299}$

when $n=3 \Rightarrow x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 2.1299 - \frac{((2.1299)^3 - 5(2.1299)+1)}{(3(2.1299)^2 - 5)}$

$= 2.2299 - \frac{(9.6622 - 10.6495 + 1)}{(13.6099 - 5)}$

$x_4 = 2.1285$

when $n=4 \Rightarrow x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} = 2.1285 - \frac{((2.1285)^3 - 5(2.1285)+1)}{(3(2.1285)^2 - 5)}$

$= 2.1285 - \frac{(9.6431 - 10.6425 + 1)}{(13.5915 - 5)}$

$\boxed{x_5 = 2.1285}$

checking step. $f(x_5) = 0.0006 \approx 0$

—x—

March

* Numerical Computing ...

M T W T F S   Date: / /

(05) The Secant Method :

This method uses a succession of roots of secant lines to find the desired root of a given non-linear function $f(x)$. It can be thought of a finite-difference approximation of Newton's method by replacing the derivative $f'(x_n)$ by

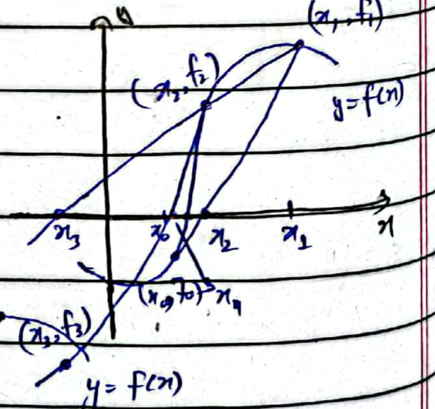
$$\text{gradient (or slope) of the Secant Line} = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

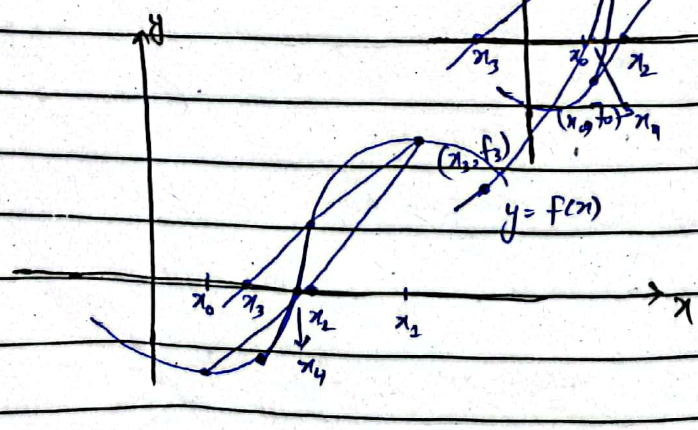
Statement : This method states that if x_{n-1} - x_n are two approximation to a root of a non-linear equation $f(x)=0$, then a better approximation is

$$x_{n+1} = \frac{x_n f(x_{n-1}) - x_{n-1} f(x_n)}{f(x_{n-1}) - f(x_n)}$$

where $f(x_{n-1}) \neq f(x_n)$ and $n = 1, 2, 3, \ldots$

Graphical Illustration :





Scanned with CamScanner

M T W T F S

Date: __/__/__

Derivation: Newton's Method is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

we replace $f'(x_n)$ by the slope of a secant line and so

$$x_{n+1} = x_n - \frac{f(x_n)}{\left[\frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}\right]}$$

$$= x_n - \left[\frac{f(x_n)}{f(x_n) - f(x_{n-1})} \times (x_n - x_{n-1})\right]$$

$$= \frac{\cancel{x_n f(x_n)} - x_n f(x_{n-1}) - \cancel{f(x_n) x_n} + f(x_n) x_{n-1}}{f(x_n) - f(x_{n-1})}$$

$$= \frac{x_n f(x_{n-1}) - x_{n-1} f(x-1)}{f(x_{n-1}) + f(x_n)}$$

Example: Slove $\sin x + 2 = 5x$, near $x_0 = 0.4$ and $x_1 = 0.5$, by the secant method, in radians correct up to 3 dp.

Solution: Let $f(x) = \sin x - 5x + 2 = 0$

using the secant method we write

$$x_{n+1} = \frac{x_n f(x_{n-1}) - x_{n-1} f(x_n)}{f(x_{n-1}) - f(x_n)}$$

$n = 1 \Rightarrow$ $x_2 = \dfrac{x_1 f(x_0) - x_0 f(x_1)}{f(x_0) - f(x_1)} = 0.49498$.

AHMED PAPER PRODUCT

$$n=2 \Rightarrow x_3 = \frac{x_2 f(x_1) - x_1 f(x_2)}{f(x_1) - f(x_2)} = 0.0711$$

$$n=3 \Rightarrow x_4 = \frac{x_3 \overset{\text{Positive}}{\boxed{f(x_2)}} - x_2 \overset{\text{Positive}}{\boxed{f(x_3)}}}{f(x_2) - f(x_3)} = 0.495007$$

$$n=4 \Rightarrow x_5 = \frac{x_4 f(x_3) - x_3 f(x_4)}{f(x_3) - f(x_4)} = 0.495007$$

checking step.

$$f(x_5) = 0.000003 \approx 0 .$$

— x —

Task : Using Secant method to find a positive, real root of $x^3 - x = 10$, near $x_0 = 2$ and $x_1 = 2.5$, Correct up to 3 dp.

Solution : Let $f(x) = x^3 - x - 10 = 0$

using the secant method, we write

$$x_{n+1} = \frac{x_n f(x_{n-1}) - x_{n-1} f(x_n)}{f(x_{n-1}) - f(x_n)}$$

$$n=1 \Rightarrow x_2 = \frac{x_1 f(x_0) - x_0 f(x_1)}{f(x_0) - f(x_1)} = \frac{(2.5)(-4) - (2)(3.125)}{-4 - 3.125}$$

$$x_2 = \frac{-10 - 6.25}{-7.125} = \frac{16.25}{7.125} = 2.2807$$

$$n=2 \Rightarrow x_3 = \frac{x_2 f(x_1) - x_1 f(x_2)}{f(x_1) - f(x_2)} = \frac{(2.2807)(3.125) - (2.5)(\ldots}{3.125 - (-0.4\ldots}$$

M T W T F S Date: ___/___/___

$$x_3 = \frac{7.1271 + 1.0428}{3.5421} = 2.3065$$

$n=3 \Rightarrow$
$$x_4 = \frac{x_3 f(x_2) - x_2 f(x_3)}{f(x_2) - f(x_3)} = \frac{(2.3065)(-0.4171) - (2.2807)(-0.0361)}{-0.4171 - (-0.0361)}$$

$$x_4 = \frac{-0.9620 + 0.0823}{-0.4171 + 0.0361} = \frac{-0.8797}{-0.381} = 2.3089$$

$n=4 \Rightarrow$
$$x_5 = \frac{x_4 f(x_3) - x_3 f(x_4)}{f(x_3) - f(x_4)} = \frac{(2.3089)(-0.0361) - (2.3065)(-0.0001)}{-0.0361 + 0.0001}$$

$$x_5 = \frac{-0.0834 + 0.0002}{-0.036} = \frac{-0.0832}{-0.036} = 2.3111$$

As $x_3 \approx x_4$

So, $f(x_4) = -0.0001 \approx 0$

(Root)

x

25th March

* Numerical Computing ..

M T W T F S Date: / /

* Rate of Convergence

→ Let x_0, x_1, x_2 —— is a sequence which converges to x^* (root/approximate) and error $e_k = x_k - x^*$ (actual root - approximate root).
If there exist number p and constant $C \neq 0$. such that

Pages 19, 20, 21, 25, 31, 32, 41, 42

$$\lim_{k \to \infty} \frac{|e_{k+1}|}{|e_k|^p} = C,$$

then p is called the order of convergence and C is error constant.

⇒ We say that, Convergence is

linear, $p = 1$ and $C < 1$.
super linear, if $p > 1$,
quadratic, $p = 2$.

⇒ We say that the method converges with order p, if all convergent sequences obtained by this method, have the order of convergence greater or equal to p and at least one of them has order of convergence exactly equal to p.

AHMED PAPER PRODUCT

Scanned with CamScanner

[order of convergence of Bisection Method]

Date :

Bisection Method :

$p = 1$, linear

$c = \frac{1}{2}$

Midpoint x_{k+1} of interval (a_k, b_k) is an approximation of x^* with error

$$|x_{k+1} - x^*| \le \frac{1}{2}(b_k - a_k)$$
$$= 2^{-k-1}(b_0 - a_0)$$

$$\lim_{k \to \infty} \frac{|x_{k+1} - x^*|}{|x_k - x^*|^p} = \frac{2^{-k-1}(b_0 - a_0)}{[2^{-k}(b_0 - a_0)]^p}$$
$$= \frac{1}{2}\left[\frac{2^k}{b_0 - a_0}\right]^{p-1}$$

Regula Falsi (False-Position) Method :

$p = 1$, linear

Secant Method :

$p = 1.618$, super linear

→ It converge, if initial points x_1 and x_2 are close enough to root x^* .

Newton's Method :

$p = 2$, quadratic

→ Newton's Method is always convergent if the initial approximation is sufficiently close to the root.