Prompt for Vector Autoregression (VAR) Model Mathematical Expressions

Please provide rigorous mathematical derivations of the following components for a Vector Autoregression (VAR) model of order p, represented as:

$$X_{t} = B(1)X_{t-1} + B(2)X_{t-2} + \ldots + B(p)X_{t-p} + v_{t}$$

where:

- X_t is an $(n \times 1)$ vector of variables
- $B(i)$ are $(n \times n)$ coefficient matrices for $i = 1, 2, \ldots, p$
- v_t is an $(n \times 1)$ vector of error terms, with $v_t \sim N(0, \Sigma_v)$, where Σ_v is the $(n \times n)$ variance-covariance matrix of the errors

1. Canonical Form (VAR(p) in VAR(1) Form)

Derive the canonical form (companion form) of the VAR(p) model, expressing it as a VAR(1) system. Specifically:

a) Define the stacked vector $Y_t = [X_t', X_{t-1}', \ldots, X_{t-p+1}']'$

b) Express the system in the form $Y_t = AY_{t-1} + U_t$, where:

- Derive the exact form of the $(np \times np)$ coefficient matrix A
- Specify the structure of the error vector U_t

c) Show that this representation is mathematically equivalent to the original VAR(p) specification

2. Impulse Response Functions (IRFs)

Derive the impulse response functions that track how a shock to one variable affects all variables in the system over time:

a) Express the VAR(p) model in moving average (MA) representation (Wold decomposition): $$X_t = \mu + \sum_{i=0}^{\infty} \Psi_i v_{t-i}$$

b) Derive the formula for the impulse response matrices Ψ_i in terms of the original VAR coefficients $B(1), B(2), \ldots, B(p)$

c) Derive both:

- The cumulative impulse response functions
- The orthogonalized impulse response functions using Cholesky decomposition of Σ_v

3. Variance Decomposition

Derive the mathematical expressions for forecast error variance decomposition (FEVD):

a) Express the h-step ahead forecast error as: $$X_{t+h} - E_t[X_{t+h}] = \sum_{i=0}^{h-1} \Psi_i v_{t+h-i}$$

b) Calculate the mean squared error (MSE) matrix of the h-step forecast: $$MSE(h) = \sum_{i=0}^{h-1} \Psi_i \Sigma_v \Psi_i'$$

c) Derive the formula for the contribution of innovations in variable j to the h-step forecast error variance of variable k

d) Express the proportion of the forecast error variance of variable k attributable to shocks in variable j at horizon h

4. Likelihood Function

Derive the log-likelihood function for the VAR(p) model:

a) Assuming multivariate normal errors, write out the likelihood function in full matrix notation

b) Express the concentrated log-likelihood function for the model

c) Show how to estimate the coefficient matrices $B(1), B(2), \ldots, B(p)$ and the variance-covariance matrix Σ_v using maximum likelihood estimation

d) Derive the information criteria (AIC, BIC, HQ) for model selection based on the likelihood function

Additional Requirements

1. Use consistent matrix notation throughout all derivations
2. Clearly distinguish between vectors and matrices, using bold font or appropriate notation
3. Provide step-by-step derivations with sufficient intermediate steps to follow the logic
4. Include any necessary assumptions or conditions for each derivation to be valid
5. For each major result, provide a brief interpretation of its economic or statistical significance