Part II: $\vec{B}$-fields and all that

20 Magnetic Field

The magnetic field, we shall heavily opt for calling it $\vec{B}$ from now on, is a physical vector field. It is produced by several sources, including **moving charges** (such as current-carrying wires) and **permanent magnets**. When charges move, they generate a magnetic field around them. Additionally, magnetic materials, such as iron or nickel, have their own intrinsic magnetic fields due to the alignment of their atomic or molecular magnetic moments.

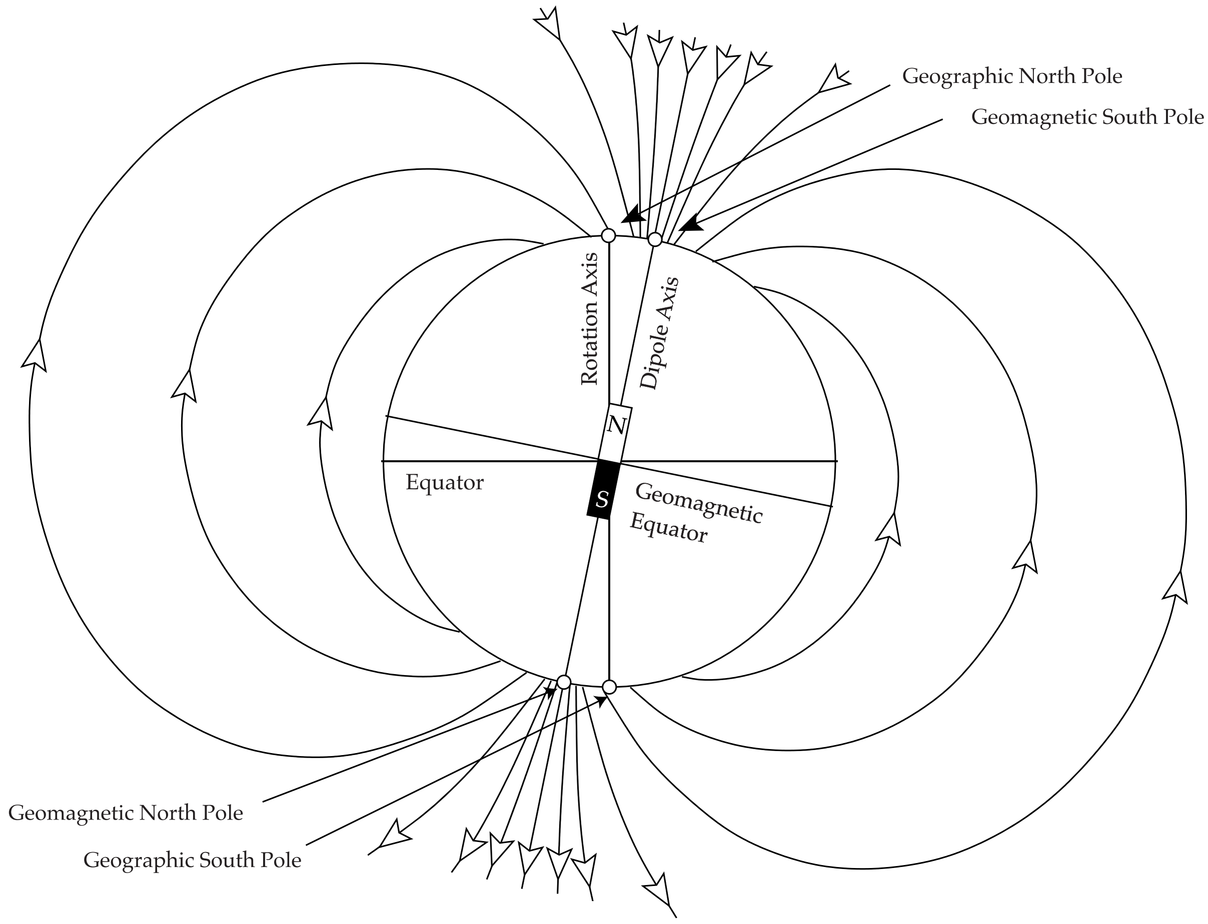


FIGURE 62: Due to the convectional current produced by molten lava inside Earth's core generates a magnetic field (5×10^{-5} T) around the Earth. This field shields us from harsh solar radiation, keeping our atmosphere sustainable for life. The geomagnetic north pole almost coincides with the geographic south pole and vice versa for the geomagnetic south pole.

Mathematically, the magnetic field is defined in terms of **the magnetic force, called Lorentz force, experienced by a moving charge in the presence of a magnetic field**. The magnetic field is represented by the symbol B and is measured in units of Tesla T.

20.1 Permanent Magnet versus Magnetic Substance versus Electromagnet

Permanent magnets are objects that possess their own magnetic fields. They can attract or repel other magnets or magnetic materials. A magnet has two poles, namely the North Pole and the South Pole, and the magnetic field lines form closed loops from the North Pole to the South Pole.

Magnetic substances are materials that can be magnetized in the presence of an external magnetic field, but **do not retain their magnetism** once the external field is removed. Examples include iron, nickel, and cobalt.

An **electromagnet** is a type of magnet that is created by passing an electric current through a coil of wire. The current flowing through the wire produces a magnetic field, and the strength of the magnetic field can be controlled by adjusting the current. Unlike permanent magnets, the magnetic properties of an electromagnet can be turned on and off by controlling the current.

20.2 $\vec{B}$-Field Lines

Magnetic field lines are a visual representation used to depict the direction and strength of the magnetic field at different points in space. They are used to provide an understanding of the magnetic field's behavior and its interaction with other magnetic objects or current-carrying wires.

Magnetic field lines are different from electric field lines in a few ways. While electric field lines start from positive charges and end on negative charges (or go to infinity for isolated charges), **magnetic field lines form closed loops that extend from the North Pole of a magnet to its South Pole**. This is because there are no isolated magnetic charges (i.e., magnetic monopoles) known to exist in nature.

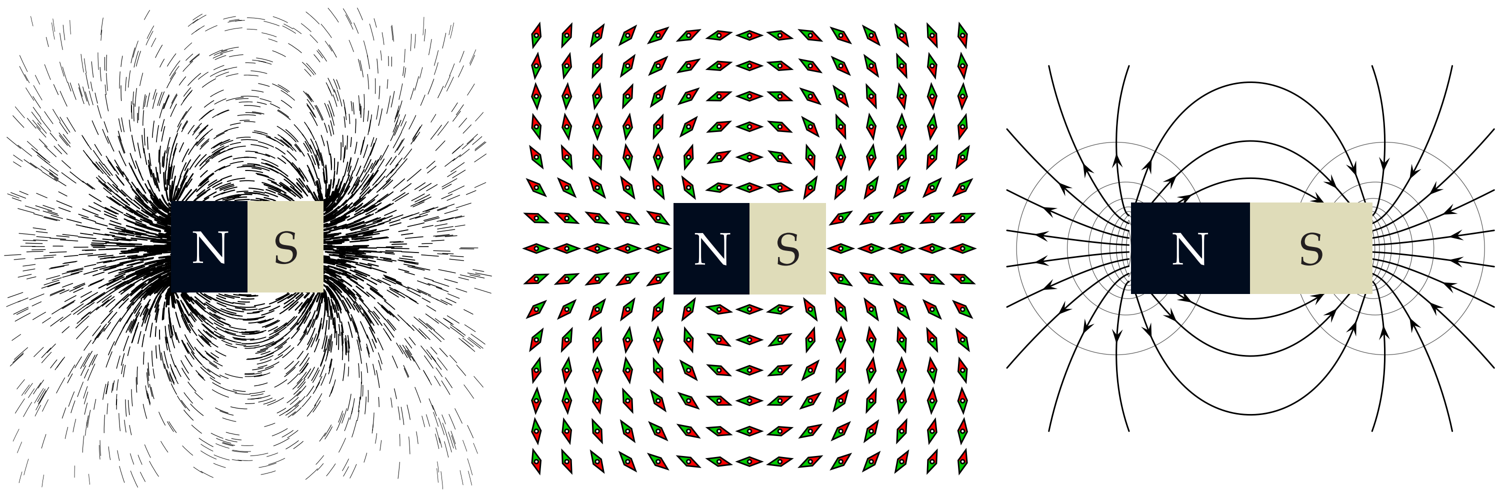


FIGURE 63: Magnetic field lines around a bar magnet. (a) Iron fillings orient themselves in the directions of the magnetic field. (b) Each iron filling can be thought of as tiny bar magnets with their own North and south pole. They orient themselves accordingly to $\vec{B}$. (c) The same field lines are drawn in terms of conventional lines and arrows.

The direction of the magnetic field at any point is tangential to the field line at that point. The density of the magnetic field lines is indicative of the strength of the magnetic field, with a higher field line density representing a stronger field.

21 Gauss's Law for Magnetism

Analogous to the case for $\vec{E}$-field, we may try to associate the $\vec{B}$-field origin with a magnetic charge. Gauss's Law for Magnetism states that **the net magnetic flux through any closed surface is always zero**. Why might that be?

Magnetic field lines always form closed loops, and magnetic field lines always have both a north and south pole. **There are no magnetic monopoles in nature, and magnetic dipoles produce all magnetic fields** (combinations of north and south poles) **or magnetic materials**.

21.1 Magnetic Flux

Magnetic flux is a measure of the total magnetic field passing through a given surface. It quantifies the amount of magnetic field lines penetrating an area. The magnetic flux Φ_B through a closed surface is defined as the surface integral of the magnetic field vector $\vec{B}$ over that closed surface. In mathematical notation, it can be expressed as:

$$\Phi_B = \int \vec{B} \cdot d\vec{A}. \tag{166}$$

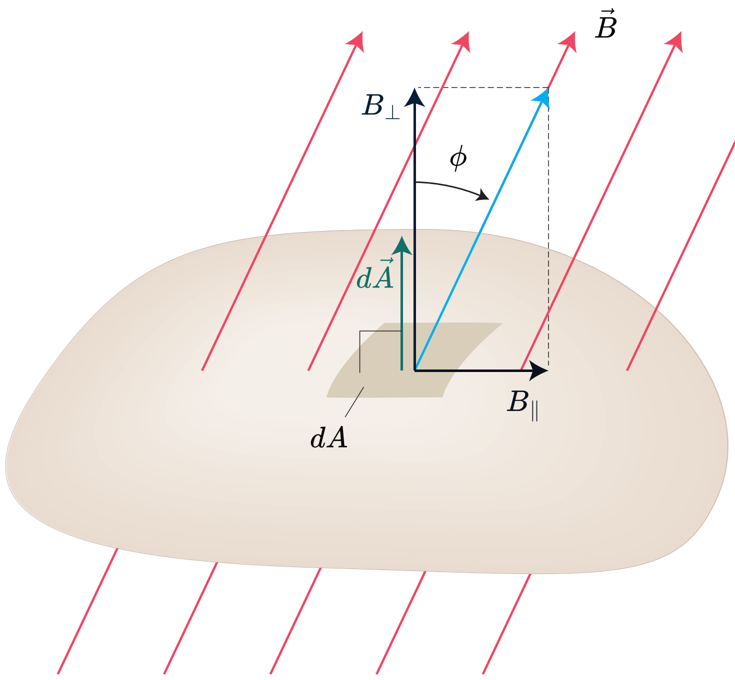


FIGURE 64: The magnetic flux through an area element dA is defined to be $d\Phi_B = B_\perp dA$.

Consider that a magnetic monopole would produce magnetic field lines that radiate outward from the monopole or converge into it, similar to how electric field lines behave around electric charges. In such a case, the net magnetic flux through any closed surface surrounding the magnetic monopole would be non-zero because some magnetic field lines would enter or leave the closed surface.

$$\oint \Phi_B = \oint \vec{B} \cdot d\vec{A} = 0. \tag{167}$$

21.2 Vector Calculus with $\vec{B}$-Field

Outflow of $\vec{B}$-Field

The divergence of $\vec{B}$-field calculates the out(in)flow out of a closed surface.

$$\vec{\nabla} \cdot \vec{B} = 0. \qquad \text{(Differential Form)} \tag{168}$$

NOTE: We have used the Divergence/Gauss's theorem to the integral form of Gauss's law (166) to get the differential form (168).

Circulation of $\vec{B}$-Field

In magnetostatics, where Φ_B is not varying in time, the magnetic fields have zero curl.

$$\vec{\nabla} \times \vec{B} = 0. \tag{169}$$

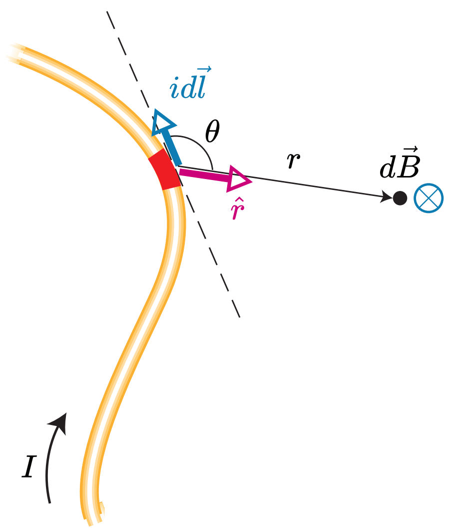
This directly implies, from Stokes's theorem, that in magnetostatics,

$$\oint \vec{E} \cdot d\vec{l} = 0. \tag{170}$$

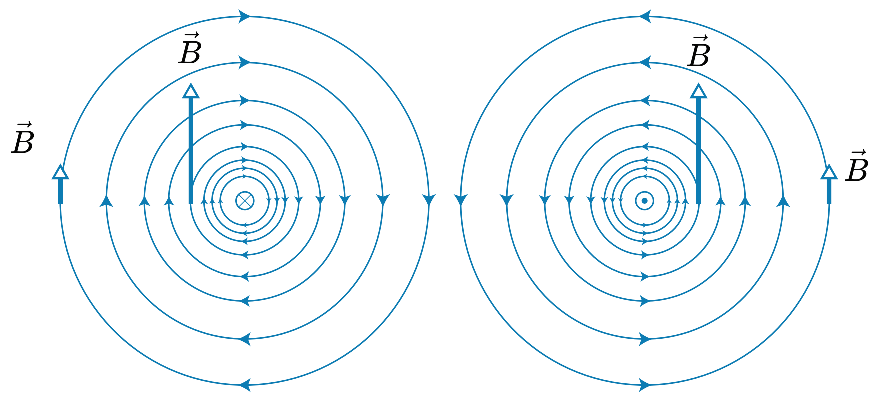
However, when Φ_B varies, some new Physics arises. We shall revisit this part. Put a pin on that.

22 Biot-Savart Law

Consider a straight current-carrying conductor. We'll take a small segment of the conductor with length $d\vec{l}$ at a point P, carrying current I (a small amount of current). We want to find the magnetic field $d\vec{B}$ produced by this small current element at point P.



(a)



(b)

FIGURE 65: (a) A current-length element I produces a differential magnetic field at point P. The blue (the tail of an arrow) at the dot for point P indicates that it is directed to the page there. (b) The magnetic field vector at any point is tangent to a circle for a wire with the current into (out of) the page.

The magnetic field $d\vec{B}$ produced by the small current element $d\vec{l}$ at point P can be given by a mathematical expression:

$$\begin{aligned} |d\vec{B}| &= \frac{\mu_0}{4\pi} \frac{dQ(\vec{v} \times \hat{r})}{r^2} \\ &= \frac{\mu_0}{4\pi} \frac{(nqAv_d)(dl \sin\phi)}{r^2} \\ dB &= \frac{\mu_0}{4\pi} \frac{Idl \sin\phi}{r^2} \end{aligned} \tag{171}$$

$$\therefore \vec{B} = \frac{\mu_0}{4\pi} \int \frac{I d\vec{l} \times \hat{r}}{r^2} \tag{172}$$

This law describes the magnetic field generated by a **steady current**. It mathematically relates the magnetic field at a point to the current flowing through a small wire segment.

The physical significance of the Biot-Savart (*Bee-o Savah*) law lies in its ability to explain how a steady current produces a magnetic field. It shows that the magnetic field at a point is proportional to the current and inversely proportional to the distance from the current element. The law also illustrates that the magnetic field generated by a current-carrying wire circulates the wire in concentric circles, with the strength of the field decreasing as the distance from the wire increases.

The Biot-Savart law is valid for steady currents and in situations where the size of the current-carrying elements is much smaller compared to the distance from the observation point. However, it breaks down under certain conditions, such as changing electric fields or time-varying currents. In those cases, more general equations, such as Ampère's law or Faraday's law, need to be considered.

22.1 $\vec{B}$-Field of a Straight Current Carrying Conductor

Consider a straight current-carrying conductor with current I and length $2L$. Take a small segment of the conductor with length $d\vec{l}$ at a point P, carrying current i (small amount of current). The current I is uniformly distributed along the conductor, so I is proportional to $d\vec{l}$.

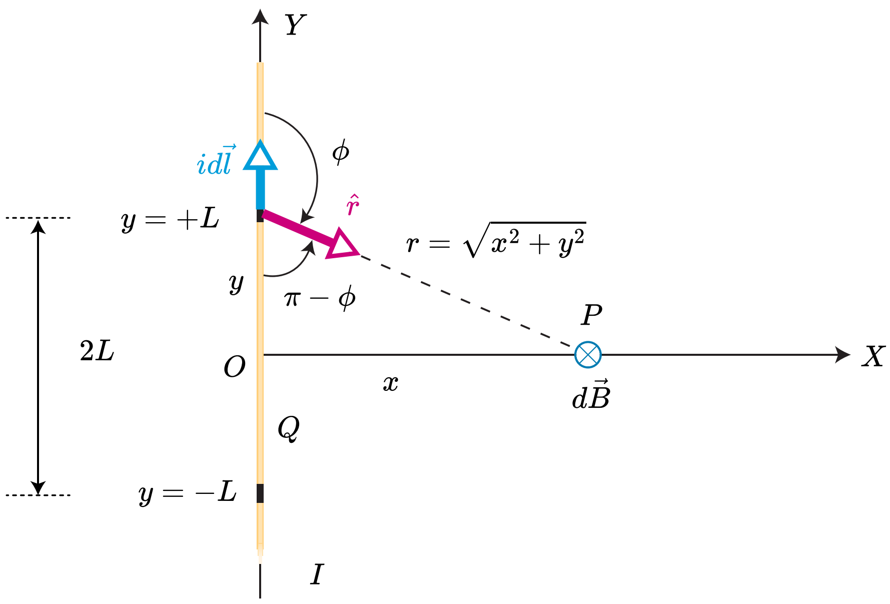


FIGURE 66: Calculating the magnetic field $\vec{B}$ produced by a current I in a long straight wire. The field at P associated with the current-length element $id\vec{l}$ is directed into the page, as shown.

The magnetic field $d\vec{B}$ at point P due to the small current element I is given by Biot-Savart's law:

$$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{I d\vec{l} \times \hat{r}}{r^2} \tag{173}$$

The distance r from the current element $d\vec{l}$ to point P is related to the distance x away, perpendicularly from the conductor: $r = \sqrt{x^2 + y^2}$.

$$B = \frac{\mu_0 I}{4\pi} \int_{-L}^{L} \frac{x dy}{(x^2 + y^2)^{\frac{3}{2}}}$$
$$= \frac{\mu_0 I}{4\pi} \frac{2L}{x\sqrt{x^2 + L^2}}. \tag{174}$$

For a more realistic scenario where $L \gg x$, $L \to \infty$ renders the terms inside the square root to $x^2 + L^2 \sim L^2$. In this limit:

$$B = \frac{\mu_0 I}{2\pi x}. \tag{175}$$

A more general case would be to denote x as r, where r is the distance perpendicularly away from the conductor where the magnetic field is being measured.

$$B = \frac{\mu_0 I}{2\pi r}. \tag{176}$$

22.2 $\vec{B}$-Field of a Curved Current Carrying Conductor (At the Center)

Consider a circular arc of current-carrying wire with a radius R and a current I. We'll take a small segment of the arc with an angular extent $d\phi$ at a point P, carrying current. We want to find the magnetic field $d\vec{B}$ produced by this small current element at P.

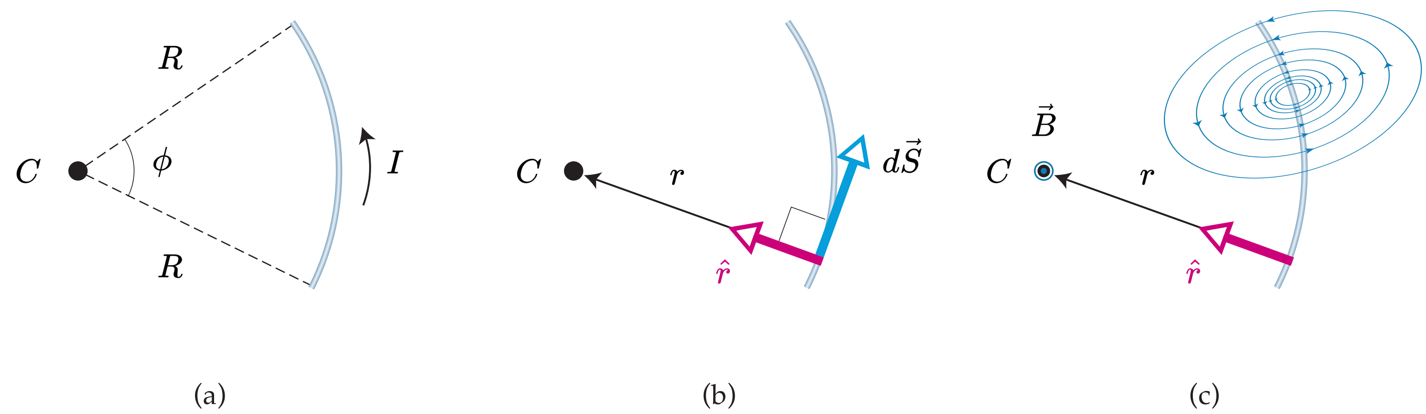


FIGURE 67: (a) A wire in the shape of a circular arc with center C carries current I. (b) For any element of wire along the arc, the angle between the directions of and is 90°. (c) The field is out of the page, as indicated by the colored dot at C.

The magnetic field $d\vec{B}$ produced by the small current element $d\vec{l}$ at point P can be given by a modified version of the Biot-Savart law, considering the circular geometry:

$$\vec{B} = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{s} \times \hat{R}}{R^2}$$
$$= \frac{\mu_0 I}{4\pi} \int \frac{R d\phi \sin 90^\circ}{R^2}$$

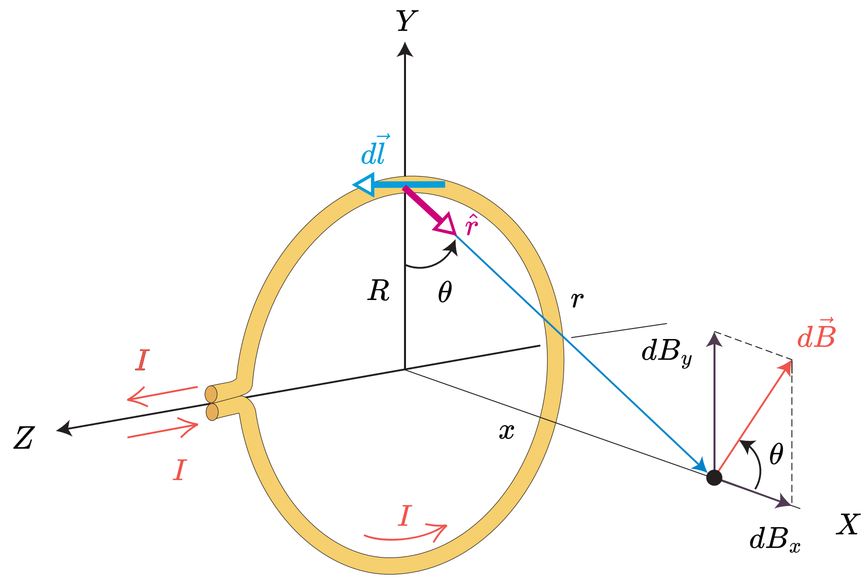


FIGURE 68: (Magnetic field on the axis of a circular loop. The current in the segment $d\vec{S}$ causes the field $d\vec{B}$ at P, which lies in the xy-plane. The currents in other $d\vec{S}$'s cause $d\vec{B}$'s with different components perpendicular to the x-axis, adding to zero. The x-components of the $d\vec{B}$'s combine to give the total $\vec{B}$ field at point P.

$$= \frac{\mu_0 I R}{4\pi} \int_0^{\phi} \frac{d\phi}{R^2}$$

$$= \frac{\mu_0 I \phi}{4\pi R}. \tag{177}$$

For a complete circular wire, this becomes:

$$B = \frac{\mu_0 I \times 2\phi}{4\pi R} = \frac{\mu_0 I}{2R}. \tag{178}$$

This $\vec{B}$ is measured in the center of the circular arc wire or the circular wire.

22.3 $\vec{B}$-Field of a Looped Current Carrying Conductor (Axial)

Consider a looped current-carrying conductor with a radius R and a constant current I. We want to find the magnetic field $\vec{B}$ at a point P located at a distance r away from the center of the conductor (perpendicularly). Since the conductor is circular, we can express the differential magnetic field at a point x distance away from the center up in the following form of Biot-Savart Law:

$$B = \frac{\mu_0 I}{4\pi} \int \frac{dS}{(x^2 + R^2)^{\frac{3}{2}}} \tag{179}$$

This dB will then have to be resolved into its two components:

$$B_x = \frac{\mu_0 I}{4\pi} \int \frac{dS}{(x^2 + R^2)^{\frac{3}{2}}} \cos\theta \tag{180}$$

$$B_y = \frac{\mu_0 I}{4\pi} \int \frac{dS}{(x^2 + R^2)^{\frac{3}{2}}} \sin\theta. \tag{181}$$

We can neglect the vertical component since they cancel out during the integration due to mirror symmetry. Take the orientation of the coil's axis in an arbitrary direction $\hat{r}$.

$$
\begin{aligned}
B = \frac{\mu_0 I}{4\pi}\int \frac{dS}{(r^2+R^2)}\cos\theta &= \frac{\mu_0 I}{4\pi}\int \frac{dS}{(r^2+R^2)}\frac{R}{\sqrt{r^2+R^2}} \\
&= \frac{\mu_0 IR}{4\pi}\int \frac{dS}{(r^2+R^2)^{\frac{3}{2}}} \\
&= \frac{\mu_0 IR}{4\pi}\frac{1}{(r^2+R^2)^{\frac{3}{2}}}\int_0^{2\pi R} dS \\
&= \frac{\mu_0 IR^2}{2}\frac{1}{(r^2+R^2)^{\frac{3}{2}}}
\end{aligned} \tag{182}
$$

In the limit $R \gg r$:

$$
B = \frac{\mu_0 I}{2R}. \tag{183}
$$

This is identical to Eq. (178).

23 Magnetic Force Caused by $\vec{B}$-Fields

The force on a moving charge q in a magnetic field B can be described by the equation:

$$
\vec{F}_B = qvB\sin\phi = q\vec{v}\times\vec{B}, \tag{184}
$$

where F_B is the Magnetic force, v is the velocity of the charge, and ϕ is the angle between the velocity vector and the magnetic field vector.

An external electric charge will also experience an electrostatic force $\vec{F}_E = q\vec{E}$.

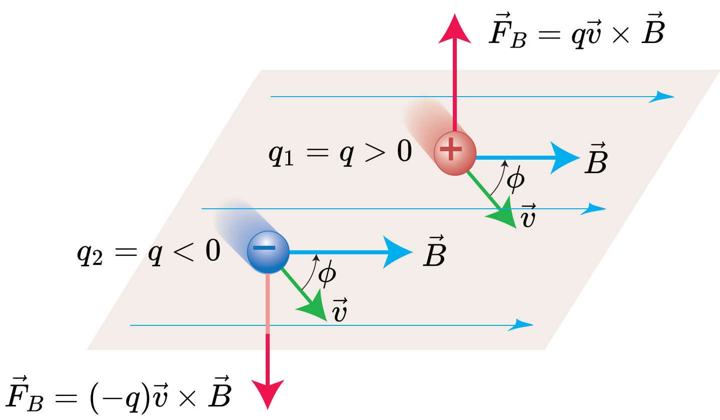


FIGURE 69: The magnetic force $\vec{F}_B$ acting on a positive charge q moving with velocity $\vec{v}$ is perpendicular to both $\vec{v}$ and the magnetic field $\vec{B}$. For given values of speed v and magnetic field strength B, the force is greatest when $\vec{v}$ and $\vec{B}$ are perpendicular.

The total force on the external charge is then

$$
\vec{F} = \vec{F}_B + \vec{F}_E
$$

$$
\begin{aligned}
&= q\vec{v} \times \vec{B} + q\vec{E} \\
&= q\left(\vec{v} \times \vec{B} + \vec{E}\right). \qquad (185)
\end{aligned}
$$

This is called the Lorentz force.

23.1 Magnetic Force on a Current-Carrying Conductor

Consider a current-carrying conductor with a current I (Figure 70). We'll take a small segment of the conductor with a length $d\vec{l}$ at a point P. The current segment $d\vec{l}$ experiences a magnetic force $\vec{F}_B$ due to the magnetic field $\vec{B}$ at that point.

$$
\begin{aligned}
\vec{F}_B &= qv_d B \sin\phi \\
&= (nAl)(qv_d B) \\
&= (nqv_d A)(lB_\perp) \\
&= IlB_\perp \\
&= IlB \sin\phi \\
&= I\vec{l} \times \vec{B}. \qquad (186)
\end{aligned}
$$

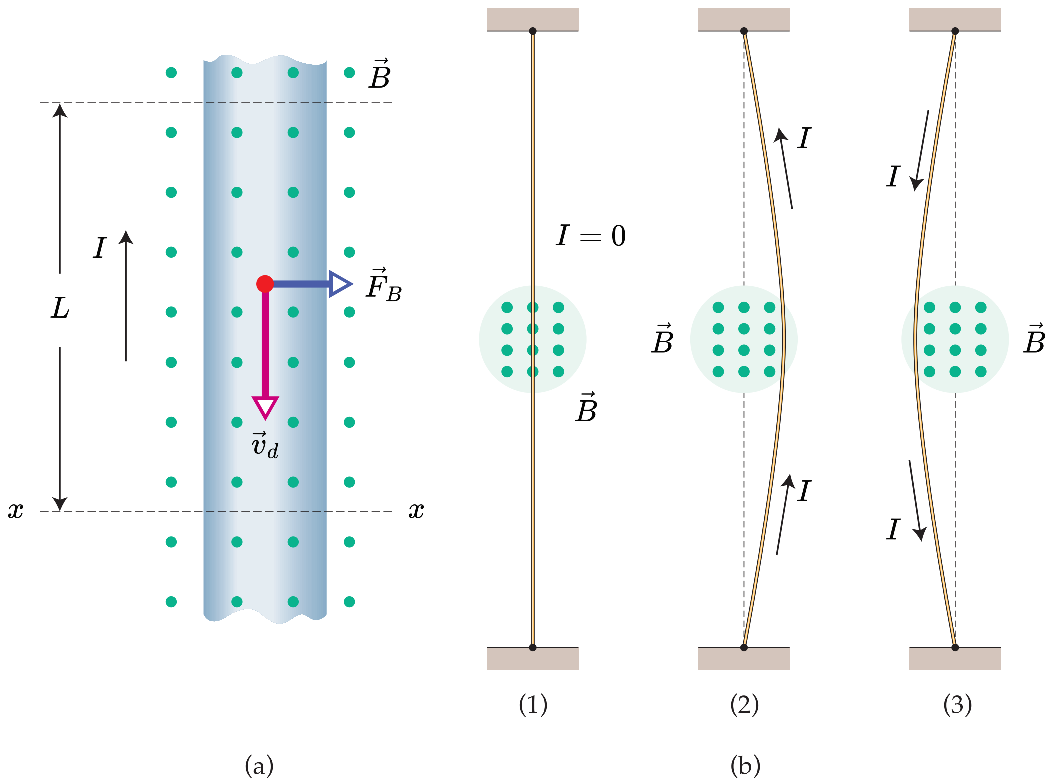


FIGURE 70: (a) A close-up view of a section of the wire. The current direction is upward, which means that electrons drift downward. A magnetic field that emerges from the plane of the page causes the electrons and the wire to be deflected to the right. (b) A flexible wire passes between the pole faces of a magnet (only the farther pole face is shown). (b1) Without current in the wire, the wire is straight. (b2) With an upward current, the wire is deflected rightward. (b3) With a downward current, the deflection is leftward. The connections for getting the current into the wire at one end and out of it at the other end are not shown.

Magnetic force on an infinitesimal wire segment:

$$d\vec{F}_B = Id\vec{l} \times \vec{B}. \tag{187}$$

Remember, the current I is not a vector. The direction of the current flow is described by $d\vec{l}$, not I. If the conductor is curved, I is the same at all points along its length, but $d\vec{l}$ changes direction—. It is always tangent to the conductor.

23.2 Magnetic Force on a Current-Carrying Loop

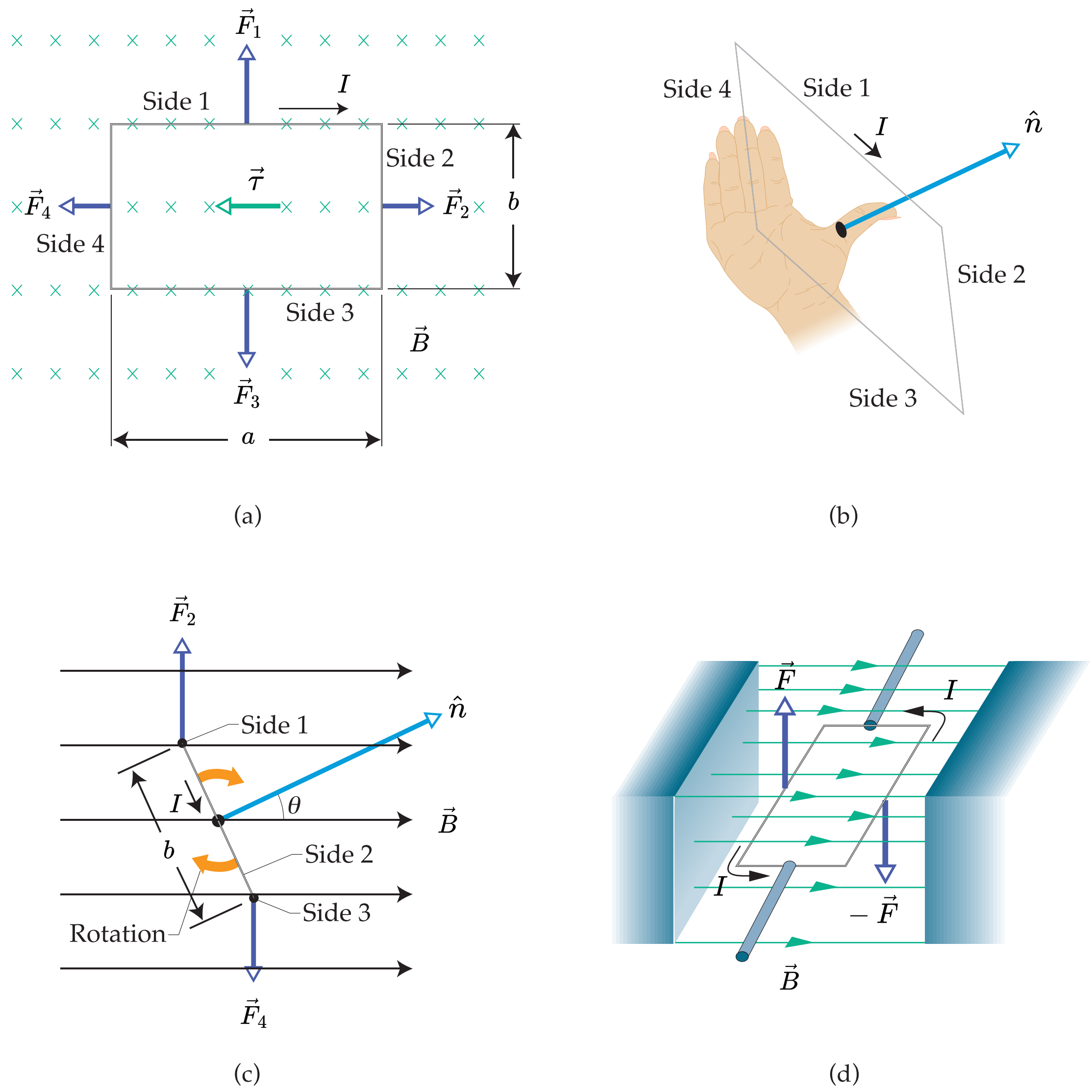


FIGURE 71: A rectangular loop, of length a and width b and carrying a current I, is located in a uniform magnetic field. A torque $\vec{\tau}$ acts to align the normal vector with the direction of the field. (a) The loop as seen by looking in the direction of the magnetic field. (b) A perspective of the loop showing how the right-hand rule gives the direction of $\vec{\tau}$, perpendicular to the loop's plane. (c) A side view of the loop from side 2. The loop rotates as indicated. (d) The elements of an electric motor. A rectangular loop of wire, carrying a current and free to rotate about a fixed axis, is placed in a magnetic field. Magnetic forces on the wire produce a torque that rotates it. A commutator (not shown) reverses the direction of the current every half-revolution so that the torque always acts in the same direction.

Consider a rectangular loop of wire with side lengths a and b. A line perpendicular to the plane of the loop (i.e., a normal to the plane) makes an angle θ with the direction of the magnetic field $\vec{B}$, and the loop carries a current I. The wires leading the current into and out of the loop and the source of EMF are omitted to keep the diagram simple.

The force $\vec{F}$ on the right side of the loop (length a) is to the right, in the $+x$ direction, as shown. On this side, $\vec{B}$ is perpendicular to the current direction, and the force on this side has magnitude:

$$F_1 = IaB. \tag{188}$$

A force $\vec{F}_3 = -\vec{F}_1$ with the same magnitude but in opposite directions acts on the opposite side of the loop. The sides with length b make an angle $90° - \theta$ with the direction of $\vec{B}$. The forces on these sides are the vectors $\vec{F}_3$ and $\vec{F}_4 = -\vec{F}_2$; their magnitude $\vec{F}_2$ and $\vec{F}_4$ are given by

$$F_3 = IbB\sin(90° - \theta) = IdB\cos\theta \tag{189}$$

The lines of action of both forces lie along the y axis. The total force on the loop is *zero* because the forces on opposite sides cancel out in pairs.

TAKEAWAY: **The net force on a current loop in a uniform magnetic field is zero. However, the net torque is not, in general, equal to zero.**

23.3 Magnetic Torque on a Current-Carrying Loop and Electromagnet

The two forces $\vec{F}_1$ and $\vec{F}_3$ in Figure 71(c) lie along the same line and so give rise to zero net torque with respect to any point. The two forces $\vec{F}_2$ and $\vec{F}_4$ lie along different lines, and each gives rise to a torque about the y-axis.

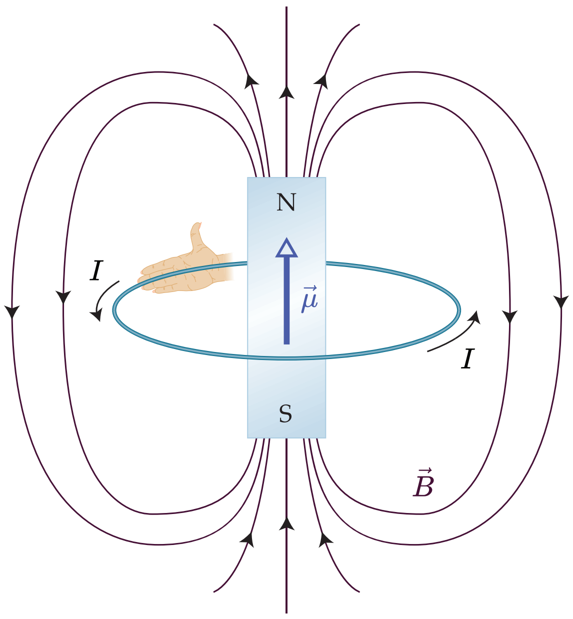


FIGURE 72: The magnetic moment represents the strength and orientation of a magnetic source, such as a magnet or a current loop. It helps us understand how *magnetically strong* and in which direction a magnetic object or system influences its surroundings. We use the right-hand palm rule to find the direction of $\vec{\mu}$. Curl the four fingers in the current direction, and the thumb will point along $\vec{\mu}$. The magnetic field lines work as if the moment was a bar magnet from the North Pole to the South. $\vec{\mu}$ points from the South Pole to the North.

According to the right-hand rule for determining the direction of torques, the vector torques due to $\vec{F}_2$ and $\vec{F}_4$ are both in the $+y$ direction; hence the net vector torque $\vec{\tau}$ is in the $+y$ direction as well. The moment arm for each of these forces (equal to the perpendicular distance from the rotation axis to the line of action of the force) is $\frac{b}{2}\sin\theta$, so the torque due to each force has magnitude $F\frac{b}{2}\sin\theta$.

The net torque is then given by the following:

$$\begin{aligned} |\vec{\tau}| &= 2F\frac{b}{2}\sin\theta \\ &= (IBa)(b\sin\theta) \\ \tau &= IBA\sin\theta \\ \vec{\tau} &= I\vec{B}\times\vec{A}. \end{aligned} \tag{190}$$

The product IA is defined as the **magnetic dipole moment** or **magnetic moment** $\vec{\mu}$ of the loop. It is analogous to the electric dipole moment.

In terms of μ, the net torque can be expressed as

$$|\vec{\tau}| = \mu B\sin\theta \tag{191}$$

$$\vec{\tau} = \vec{\mu}\times\vec{B}. \tag{192}$$

where θ is the angle between the normal to the loop (the direction of the vector area $\vec{A}$) and $\vec{B}$. A current loop, or any other object that experiences a magnetic torque, is also called a magnetic dipole. $\vec{\mu}$ points in the direction of the normal vector to the plane of the current loop.

24 Charged Particles in $\vec{B}$-Field

24.1 Circular Deflection of Charged Particles

24.2 The Thompson Experiment: Discovery of Electron

The cross-field cathode ray tube experiment, also known as the Thomson experiment, was conducted by J.J. Thomson in 1897. This experiment played a crucial role in the discovery of the electron and provided strong evidence for the existence of subatomic particles.

Thomson used a cathode ray tube, a sealed glass tube containing a vacuum with two metal electrodes. The cathode, a negatively charged electrode, emitted a stream of electrons, known as cathode rays, towards the anode, a positively charged electrode. Additionally, perpendicular to the path of the cathode rays, Thomson introduced a magnetic field and an electric field.

The basic idea behind the experiment was to observe the deflection of the cathode rays in the presence of both electric and magnetic fields. The magnetic force $\vec{F}_B$ on a charged particle moving through a magnetic field $\vec{B}$ is given by Eq. (184):

$$F_B = qvB. \tag{193}$$

The electric force $\vec{F}_E$ on a charged particle moving through an electric field $\vec{E}$ is given by:

$$F_E = qE. \tag{194}$$

Thompson first applied the $\vec{B}$ field and recorded a deflection by the charged particle. He then applied an $\vec{E}$

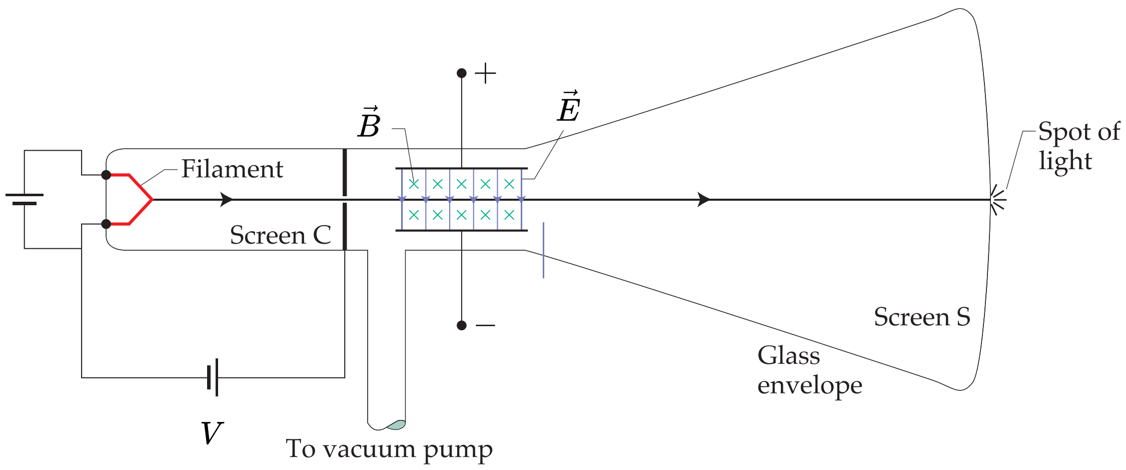


FIGURE 73: A modern version of J. J. Thomson's apparatus for measuring the ratio of mass to charge for the electron. An electric field is established by connecting a battery across the deflecting plate terminals. The magnetic field is set up by means of a current in a system of coils (not shown).

field perpendicularly to $\vec{B}$ and tuned it till the deflection was nullified. Such effect claims:

$$\begin{aligned} F_B &= F_E \\ qvB &= qE \\ v &= \frac{E}{B}. \end{aligned} \tag{195}$$

The accelerating voltage V that was applied to the particle stored the gained electric potential energy in the form of kinetic energy due to acceleration.

$$\begin{aligned} qV &= \frac{1}{2}mv^2 \\ \frac{q}{m} &= \frac{1}{2}\frac{v^2}{V} \\ &= \frac{1}{2}\frac{E^2}{B^2V}. \end{aligned} \tag{196}$$

$$= \frac{1}{2}\frac{V^2}{B^2VL^2} = \frac{1}{2}\frac{V}{B^2L^2} \tag{197}$$

24.3 Particle Accelerators: The Cyclotron

A particle accelerator is a device that accelerates charged particles, such as protons or electrons, to high speeds and energies. The cyclotron is a type of circular accelerator. It consists of two hollow, D-shaped metal electrodes called dees, which are placed in a strong magnetic field. The dees are connected to a high-frequency alternating voltage source, creating an electric field that keeps the particle accelerating. The charged particles, say, a proton, are injected at the center of the cyclotron, and the electric field between the dees accelerates them. As the particles gain energy, they move in a circular path due to the magnetic field. **The magnetic field acts as a constant force perpendicular to the particle's velocity, causing it to undergo circular motion according to the Lorentz force equation**:

$$\vec{F}_B = q\vec{v} \times \vec{B}. \tag{198}$$

As the particles complete each circular orbit in the dees, the electric field polarity switches at precisely the right moment to accelerate the particles further. This process repeats with each half-cycle of the alternating

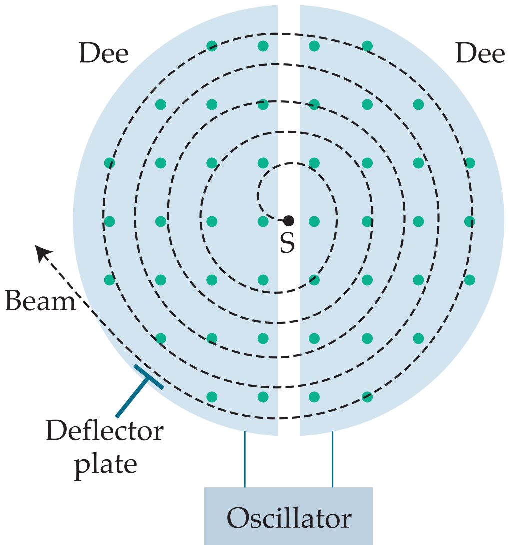


FIGURE 74: The elements of a cyclotron, showing the particle source S and the dees. A uniform magnetic field is directed up from the plane of the page. Circulating protons spiral outward within the hollow dees, gaining energy every time they cross the gap between the dees.

voltage, allowing the particles to gain energy with each pass through the dees.

The particle has a centripetal force given by:

$$F_c = \frac{mv^2}{R}. \tag{199}$$

Since the centripetal force F_c is provided by the magnetic force F_B, we can equate these forces:

$$\begin{aligned} q|v|B &= \frac{mv^2}{R} \\ R &= \frac{mv}{|q|B}. \end{aligned} \tag{200}$$

The period T (the time for one full revolution) is equal to the circumference divided by the speed:

$$T = \frac{2\pi R}{v} = \frac{2\pi}{v}\frac{mv}{|q|B} = \frac{2\pi m}{|q|B}. \tag{201}$$

The frequency f (the number of revolutions per unit time) is

$$f = \frac{1}{T} = \frac{|q|B}{2\pi m}. \tag{202}$$

It is also known as the *Cyclotron Frequency*.

The angular frequency ω of the motion is then

$$\omega = 2\pi f = \frac{|q|B}{m}. \tag{203}$$

NOTE: The quantities T, f, and v do not depend on the speed of the particle (provided the speed is much less than the speed of light). Fast particles move in large circles and slow ones in small circles, but all particles with the same charge-to-mass ratio $\frac{q}{m}$ take the same time T (the period) to complete one round trip.

24.4 The Hall Effect

The Hall Effect occurs when a current-carrying conductor (usually a metal or semiconductor) is placed in a magnetic field perpendicular to the direction of the current flow. When this happens, an electric field is induced across the conductor, perpendicular to both the current and magnetic field directions. This leads to a voltage difference, known as the Hall voltage, across the width of the conductor.

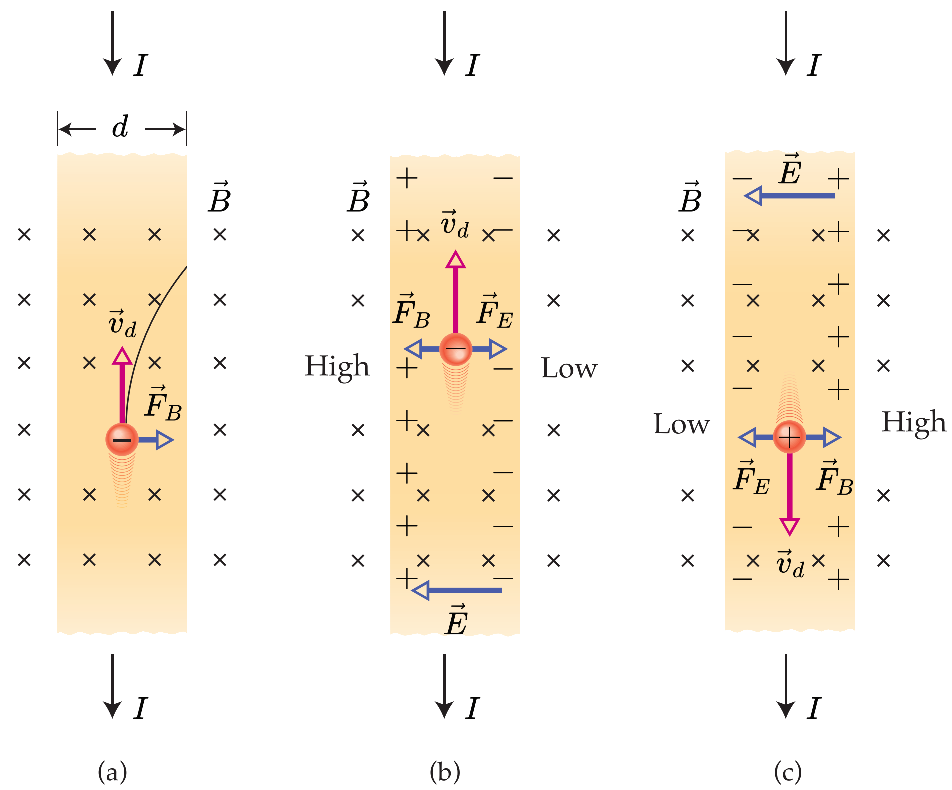


FIGURE 75: A strip of copper carrying a current i is immersed in a magnetic field. (a) The situation immediately after the magnetic field is turned on. The curved path that an electron will then take is shown. (b) The situation is at equilibrium, which quickly follows. Note that negative charges pile up on the right side of the strip, leaving uncompensated positive charges on the left. Thus, the left side has a higher potential than the right one. (c) For the same current direction, if the charge carriers were positively charged, they would pile up on the right side, and the right side would be at a higher potential.

Imagine a flat metal (Figure 75) (charge carriers are electrons) plate with length l, width d, and thickness t through which an electric current I flows from one edge to the other. Suppose we apply a magnetic field B perpendicular to the plate. In that case, the moving charge carriers (electrons in metals) experience a magnetic force $\vec{F}_{\text{magnetic}} = q\vec{v} \times \vec{B}$ that acts perpendicular to both the current direction and the magnetic field direction.

Due to this force, the electrons are pushed to one side of the plate, creating an excess of negative charges on one edge and a deficit of negative charges on the other edge. This separation of charges establishes an electric field $\vec{E}$ within the plate, pointing from the negative charge side to the positive charge side. It exerts an electric force $\vec{F}_{\text{electric}} = q\vec{E}$ on the charge carriers opposite to the magnetic force $\vec{F}_{\text{magnetic}}$: The induced electric field (Hall Field) will eventually balance the magnetic force on the electrons, resulting in a steady-state condition with a constant Hall voltage V_{Hall} across the plate. This voltage difference can be measured and is proportional to the magnetic field strength and the current in the conductor.

Since the system reaches a steady state, the net force on the charge carriers is zero:

$$\begin{aligned}\vec{F}_{\text{magnetic}} + \vec{F}_{\text{electric}} &= 0 \\ q\vec{v} \times \vec{B} &= q\vec{E}_{\text{Hall}} \\ \vec{E}_{\text{Hall}} &= -\vec{v}_d \vec{B}_\perp \\ &= -\frac{\vec{J}\vec{B}_\perp}{nq} \quad \because \vec{J} = \frac{I}{A} = \frac{nq\vec{v}_d A}{A} = nq\vec{v}_d. \end{aligned} \tag{204}$$

When q is positive, $\vec{E}$ is negative. When q is negative, $\vec{E}$ is positive (depending on the dimension).

When positive charge carriers are present, they accumulate at the upper edge, creating a potential difference opposite to negative charges. Shortly after the Hall effect was discovered in 1879, it was noted that some materials, particularly semiconductors, exhibited a Hall EMF that was opposite to that of metals, as if their charge carriers were positively charged. It is now understood that these materials are conducted through a process called *hole conduction*. Within these materials, there are locations known as holes that would usually be occupied by electrons but are actually vacant. The absence of a negative charge is equivalent to the presence of a positive charge. When an electron moves to fill a hole, it leaves another hole behind it, causing the hole to migrate in the opposite direction of the electron.

The Hall voltage V_{Hall} across the width of the plate is given by the electric field multiplied by the width d:

$$V_{\text{Hall}} = E_{\text{Hall}} d = -\left(\frac{1}{nq}\right)(\vec{J}\vec{B}_\perp)d = -R_H \vec{J}\vec{B}_\perp d, \tag{205}$$

where R_H is the Hall coefficient for the specific material.

Note: The $\perp$ sign with $\vec{B}$ implies that $\vec{B}$ is always perpendicular to the vector ($\vec{E}_{\text{Hall}}$ or $\vec{J}$) in front of it.

25 Ampère's Law

Ampère's Law is a fundamental principle in electromagnetism that relates the magnetic field around a closed loop to the electric current passing through the loop.

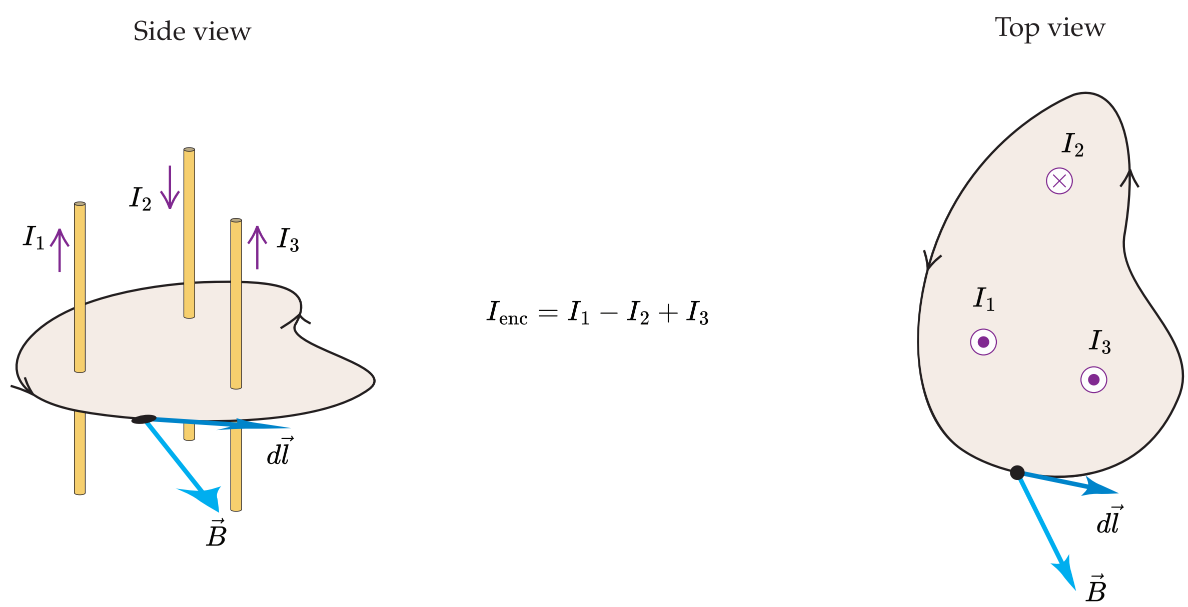


FIGURE 76: Using Ampère's Law to find the magnetic field that a current i produces outside a long straight wire of circular cross-section. The Amperian loop is a concentric circle that lies outside the wire.

It states that the line integral of the magnetic field along a closed loop is equal to the product of the permeability of free space and the total current enclosed by the loop.

Mathematically, Ampère's Law is expressed as:

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{enclosed}}. \tag{206}$$

Ampère's Law is a powerful tool for calculating the magnetic field in cases with a high degree of symmetry, such as when dealing with cylindrical or spherical symmetry. It allows us to determine the magnetic field produced by a current-carrying wire, a solenoid, or other geometries with symmetry.

This demonstrates the intimate relationship between electricity and magnetism, indicating that an electric current produces a magnetic field, and the strength of the field is directly proportional to the current.

Once again, just like Gauss' Law, Ampère's Law is only valid when the electric current is steady and the magnetic field is constant over time. It breaks down when there are time-varying electric fields or changing magnetic fields, as in cases involving rapidly changing currents or electromagnetic induction. In such scenarios, Maxwell's extension of Ampère's Law, known as Maxwell's equations, must accurately describe the electromagnetic phenomena. We will get to it shortly.

Convention for Current while using Ampère's Law

Curl your right hand around the Amperian loop, with the fingers pointing in the direction of integration. A current through the loop in the general direction of your outstretched thumb is assigned a plus sign, and a current generally in the opposite direction is assigned a minus sign.

$$\oint B_{\|} dl = \mu_0 (i_1 + i_2 - i_3).$$

Condition to Choose Appropriate Amperian Loop

Choosing the appropriate Amperian loop is crucial when applying Ampère's Law.

To ensure accurate and meaningful results when using Ampère's Law, selecting an Amperian loop that satisfies certain conditions is important. Here are the key considerations:

- Symmetry
- Uniform $\vec{B}$-Field at all points on the loop
- Avoid Singularities

25.1 $\vec{B}$-Field Outside a Long Straight Current-Carrying Wire

Consider a long straight wire carrying a current I. We want to find the magnetic field $\vec{B}$ at a distance $|\vec{r}|$ from the wire, where r is greater than the radius of the wire. Choose an Amperian loop of radius r concentric with the wire's center. Due to symmetry, the field $\vec{B}$ will remain constant across the Amperian loop.

Apply Ampère's Law:

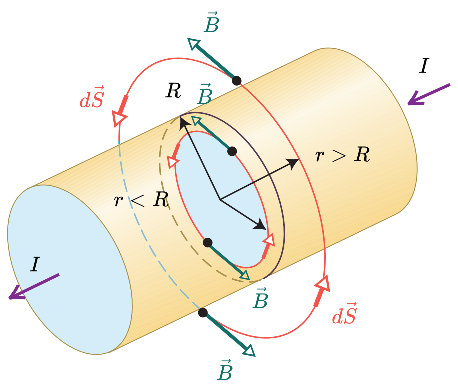
$$\begin{aligned}
\oint B_{\|} dA &= \mu_0 I_{\text{enc}} = B \oint dl = B \times 2\pi r \\
B \times 2\pi r &= \mu_0 I_{\text{enc}} \\
B &= \frac{\mu_0 I_{\text{enc}}}{2\pi r}
\end{aligned} \tag{207}$$

25.2 $\vec{B}$-Field Inside a Long Straight Current-Carrying Wire

Consider a long straight wire carrying a current I. We want to find the magnetic field $\vec{B}$ at a distance $|\vec{r}|$ from the wire, where r is smaller than the radius of the wire. Choose an Amperian loop of radius r concentric with the wire's center. Due to symmetry, the field $\vec{B}$ will remain constant across the Amperian loop.

Here,

$$I_{\text{enc}} = I_{\text{actual}} \times \frac{\pi r^2}{\pi R^2}.$$



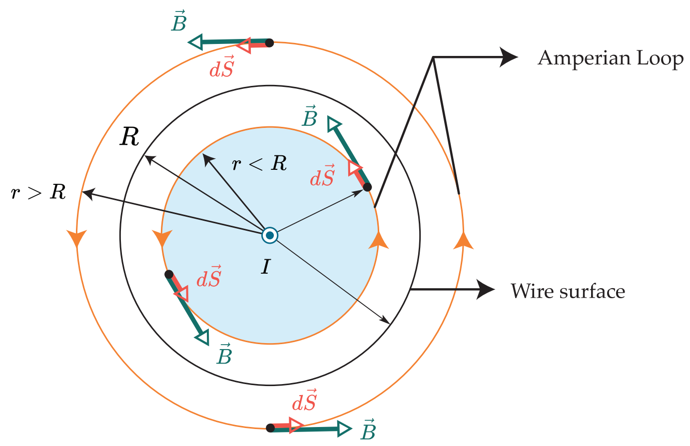


FIGURE 77: To find the magnetic field at radius $r < R$, we apply Ampère's Law to the circle enclosing the gray area. The current through the gray area is $I_{\text{actual}} \times \dfrac{\pi r^2}{\pi R^2}$. To find the magnetic field at radius $r > R$, we apply Ampère's Law to the circle enclosing the entire conductor.

Apply Ampère's Law:

$$\begin{aligned} \oint B_{\parallel} dA = \mu_0 I_{\text{enc}} &= B \oint dl = B \times 2\pi r \\ B \times 2\pi r &= \mu_0 I_{\text{actual}} \times \frac{\pi r^2}{\pi R^2} \\ B &= \left(\frac{\mu_0 I_{\text{enc}}}{2\pi R^2}\right) r \end{aligned} \tag{208}$$

Region	**$\vec{B}$-Field**
$r < R$	$B = \left(\dfrac{\mu_0 I_{\text{enc}}}{2\pi R^2}\right) r$
$r = R$	$B = \dfrac{\mu_0 I_{\text{enc}}}{2\pi R}$
$r > R$	$B = \dfrac{\mu_0 I_{\text{enc}}}{2\pi r}$

25.3 $\vec{B}$-Field of a Solenoid

Consider a solenoid, which is a long cylindrical coil of wire with N turns per unit length and carrying a current I. We want to find the magnetic field $\vec{B}$ inside the solenoid.

Choose a rectangular loop that passes through the center of the solenoid and is parallel to the axis of the solenoid. The magnetic field $\vec{B}$ inside the solenoid is uniform and directed along the axis of the solenoid. The length of the closed path l of the rectangular loop equals the length of the solenoid L.

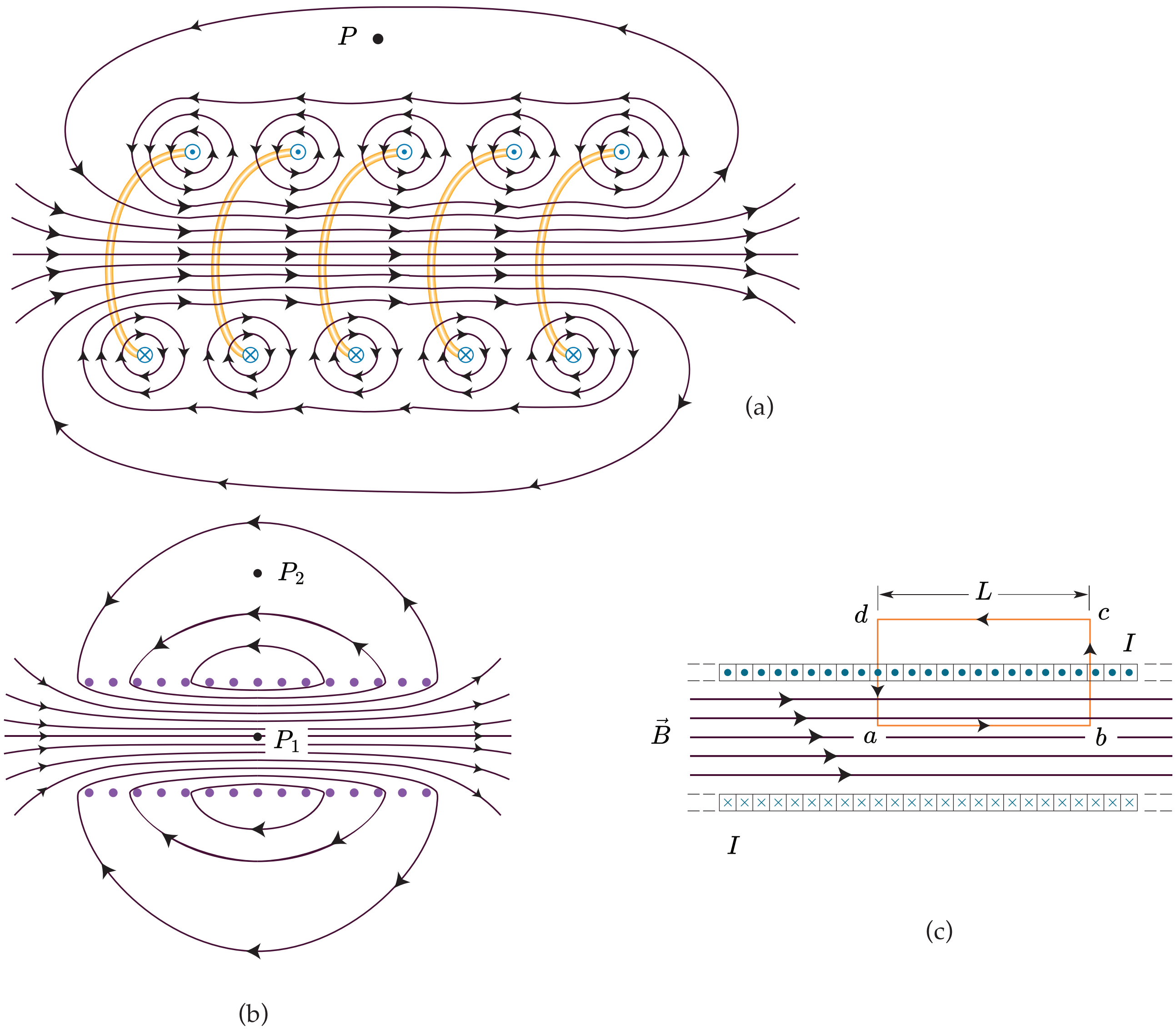


FIGURE 78: (a) A vertical cross-section through the central axis of a "stretched-out" solenoid. The back portions of five turns are shown, as are the magnetic field lines due to a current through the solenoid. Each turn produces circular magnetic field lines near itself. Near the solenoid's axis, the field lines combine into a net magnetic field that is directed along the axis. The closely spaced field lines there indicate a strong magnetic field. Outside the solenoid the field lines are widely spaced; the field there is very weak. (b) Magnetic field lines for a real solenoid of finite length. The field is strong and uniform at interior points such as P_1 but relatively weak at external points such as P_2. (c) Application of Ampère's Law to a section of a long ideal solenoid carrying a current I.The Amperian loop is the rectangle *abcda*.

Here,

$$I_{\text{enc}} = N I_{\text{actual}} L.$$

Apply Ampère's Law:

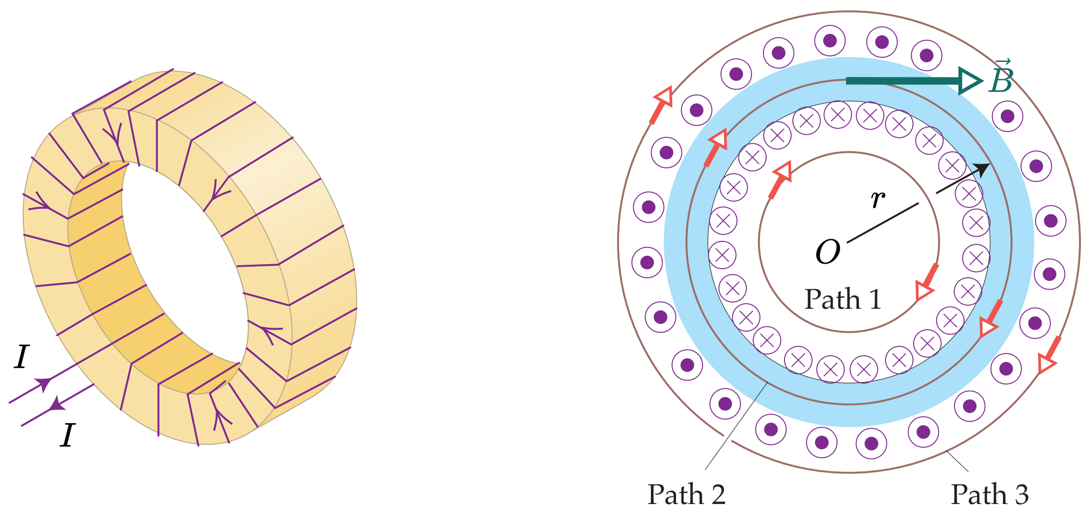
$$\oint B_{\|} dA = \mu_0 I_{\text{enc}} = B \oint dl = BL$$

$$BL = \mu_0 \times N I_{\text{actual}} L$$
$$B = \mu_0 N I_{\text{actual}}. \tag{209}$$

25.4 $\vec{B}$-Field of a Toroid

Consider a toroid, which is a hollow, circular, ring-shaped coil of wire with multiple turns. The toroid has a mean radius R and a cross-sectional area A. The current flowing through the toroid is I. We want to find the magnetic field $\vec{B}$ inside the toroid.

Choose a circular loop that lies in the same plane as the toroid and passes through the center of the toroid. The magnetic field $\vec{B}$ inside the toroid is circular and concentric with the toroid's circular cross-section. A toroid is clearly may describe as a (hollow) solenoid that has been curved until its two ends meet, forming a sort of hollow bracelet.



(a) (b)

FIGURE 79: (a) A toroidal solenoid. For clarity, only a few turns of the winding are shown. (b) Integration paths (black circles) are used to compute the magnetic field $\vec{B}$ set up by the current (shown as dots and crosses).

Apply Ampère's Law:

$$\oint B_{\|} dA = \mu_0 I_{\text{enc}} = B \oint dl = B \times 2\pi r$$
$$B \times 2\pi r = \mu_0 N I_{\text{actual}}$$
$$B = \frac{\mu_0 N I_{\text{actual}}}{2\pi r}. \tag{210}$$

Part III: Uniting $\vec{E}$ and $\vec{B}$-fields

26 Electromagnetic Induction

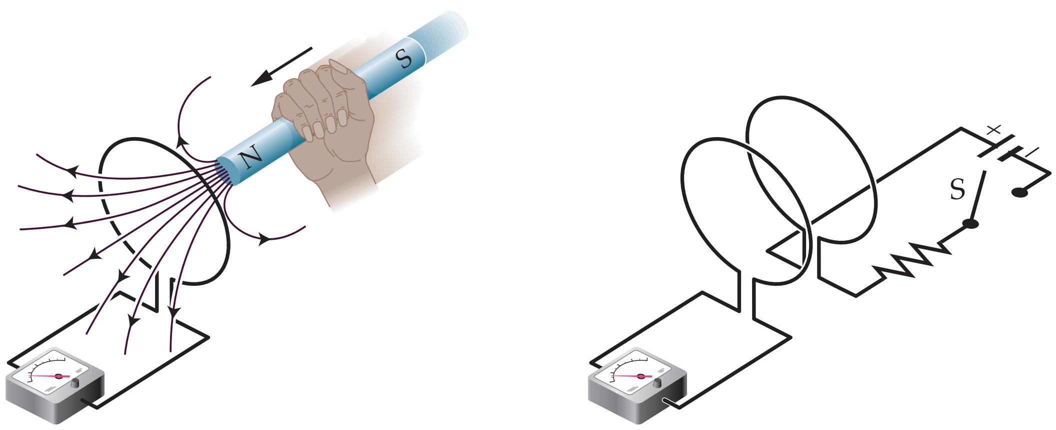
Electromagnetic induction refers to the process by which an electric current is induced in a conductor when exposed to a changing magnetic field. This phenomenon was first discovered by Michael Faraday in the early 19$^{\text{th}}$ century and is governed by Faraday's Law of electromagnetic induction and Lenz's Law.

A British scientist, Michael Faraday, conducted a famous experiment in the early 19$^{\text{th}}$ century to demonstrate electromagnetic induction. He observed that when he moved a magnet (Experiment #1) through a coil of wire or changed the magnetic field strength near the coil, a current was induced in the wire. The same thing happens when a current-carrying loop is brought near a coil (Experiment #2). There appears a current only when the incoming circuit opens and/or closes. These groundbreaking experiments showed that a changing magnetic field could create an electric current in a conductor, revealing the relationship between electricity and magnetism.

26.1 Induced EMF and Faraday's Law

Induced EMF refers to the electromotive force or voltage generated in a conductor due to electromagnetic induction. When a conductor is exposed to a changing magnetic field or experiences a relative motion with respect to a magnetic field, the magnetic flux linking the conductor changes, leading to the generation of an induced EMF. This phenomenon is the basis for generating electrical power in generators and is essential in operating transformers. The induced EMF $\mathcal{E}$ in the conductor is given by:

$$\mathcal{E} = -\frac{d\Phi_B}{dt}, \tag{211}$$



(a) (b)

FIGURE 80: (a) An ammeter registers a current in the wire loop when the magnet is moving with respect to the loop. (b) An ammeter registers a current in the left-hand wire loop just as switch S is closed (to turn on the current in the righthand wire loop) or opened (to turn off the current in the right-hand loop). No motion of the coils is involved.

Eq. (211) is known as **Faraday's Law** of electromagnetic induction. It states that **the magnitude of the induced electromotive force (EMF) in a circuit is directly proportional to the rate of change of magnetic flux passing through the circuit**.

Faraday's Law tells us that **a changing magnetic field near a conductor induces an electric field within the conductor, which in turn causes an electric current to flow**. This principle forms the basis for operating devices such as electric generators and transformers.

26.2 Lenz's Law

Lenz's Law, which is a consequence of Faraday's Law, states that the direction of the induced current in a conductor is such that it opposes the change in magnetic flux that produced it. In other words, the induced current creates a magnetic field that opposes the original change in a magnetic field. **This Law can be understood as a consequence of the conservation of energy**. Lenz's Law ensures that the induced current and magnetic field always acts in a way that opposes the original cause of the change in flux.

How to Find the Direction of the Induced EMF Using Lenz's Law

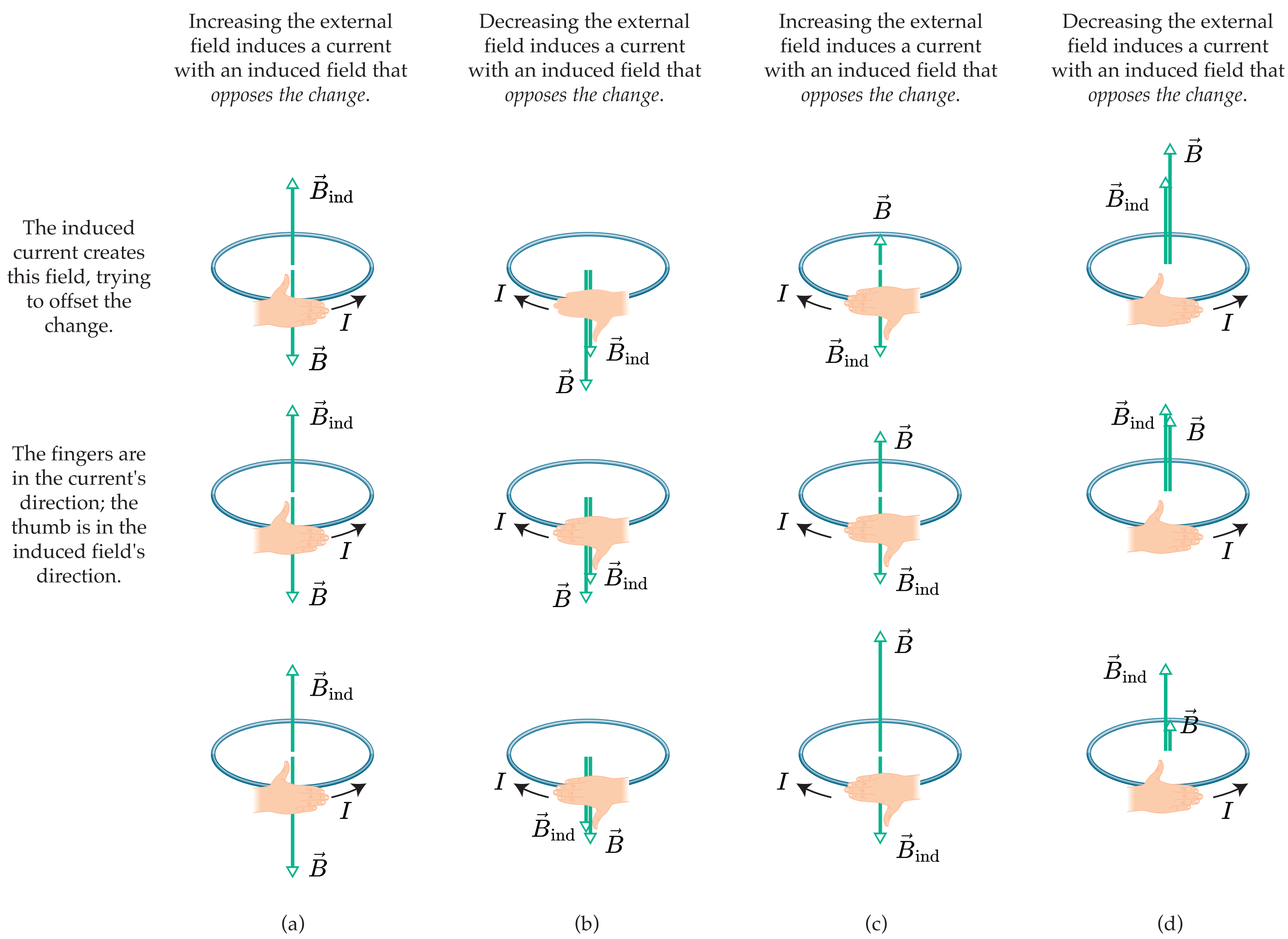


FIGURE 81: The direction of the current I induced in a loop is such that the current's magnetic field $\vec{B}_{\text{ind}}$ opposes the change in the magnetic field $\vec{B}$ inducing I. The field $\vec{B}_{\text{ind}}$ is always directed opposite an increasing field $\vec{B}$ (a, c) and in the same direction as a decreasing field B (b, d). The curled–straight right-hand rule gives the direction of the induced current based on the direction of the induced field.

1. Substitute the *incoming* current loop (Experiment #2) or bar magnet (Experiment #1) with a magnetic moment $\vec{\mu}$ aligned in the direction from the south pole of a bar magnet towards its north pole (similar to the orientation of the bar magnet).
2. In accordance with Lenz's Law, which necessitates the induced magnetic field to *oppose* the change introduced by the incoming components, introduce a current in the induced loop such that the poles

of $\vec{\mu}_{\text{ind}}$ are oriented in a direction *opposite* to the main magnetic moment $\vec{\mu}$ (in contrast to the direction of the main magnetic field $\vec{B}$).

26.3 A Reformulation of Faraday's Law

Consider a charge carrier of charge q_0 moving around the looped conducting wire. The work done W done on it in one complete revolution by the induced electric field is $W = \mathcal{E}q$.

$$\begin{aligned}
W &= (q_0 E)(2\pi r) \\
&= q_0 \int (2\pi r E) = \oint \vec{F} \cdot d\vec{s} \\
&= q_0 \oint \vec{E} \cdot d\vec{l} \\
\frac{W}{q_0} &= \mathcal{E} = \oint \vec{E} \cdot d\vec{l} \\
\therefore \oint \vec{E} \cdot d\vec{l} &= -\frac{d\Phi_B}{dt}. \qquad (212)
\end{aligned}$$

Apply Stoke's theorem to (212) to get the differential form of Faraday's Law.

$$\begin{aligned}
\oint_S (\vec{\nabla} \times \vec{E}) \cdot d\vec{A} &= -\oint_S \cdot \frac{\partial}{\partial t}(\vec{B} \cdot d\vec{A}) \\
\vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t}. \qquad (213)
\end{aligned}$$

Eq. (213) is the 3$^{\text{rd}}$ Maxwell's Equation that says there should exist a divergence of the electric field if the magnetic field is varying. No variations in $\vec{B}$ give zero divergence.

26.4 Magnetic Properties of Matter

Magnetic properties of matter arise from the interactions between the magnetic fields generated by the atomic or molecular structure of materials and external magnetic fields. There are three main types of magnetic behavior exhibited by materials: diamagnetism, paramagnetism, and ferromagnetism.

- Diamagnetism: Diamagnetic materials exhibit a weak repulsion when placed in an external magnetic field. This behavior arises due to the induced magnetic fields that oppose the applied field. In diamagnetic materials, the electron orbital motion within atoms or molecules creates small, individual magnetic fields that align in the opposite direction to the external field. As a result, the net magnetic moment of the material is reduced, leading to repulsion. Examples of diamagnetic substances include water, wood, and most organic compounds.
- Paramagnetism: Paramagnetic materials are weakly attracted to an external magnetic field. Unlike diamagnetic materials, paramagnetic substances possess unpaired electrons in their atomic or molecular orbitals. When exposed to a magnetic field, these unpaired electrons align their magnetic moments in the direction of the field, resulting in a net attraction. However, the alignment is weak and disappears when the external field is removed. Paramagnetic behavior is observed in aluminum, platinum, and oxygen materials.
- Ferromagnetism: Ferromagnetic materials are strongly attracted to an external magnetic field and can retain magnetization even after removing the field. In ferromagnetic substances, groups of atoms or ions called magnetic domains have aligned magnetic moments that reinforce each other, creating a macroscopic magnetization. The alignment is a result of interactions between neighboring atomic

magnetic moments. This spontaneous alignment gives rise to the characteristic properties of ferromagnetic materials, such as permanent magnetism and the ability to attract other magnetic materials. Iron, Nickel, and Cobalt are examples of ferromagnetic materials.

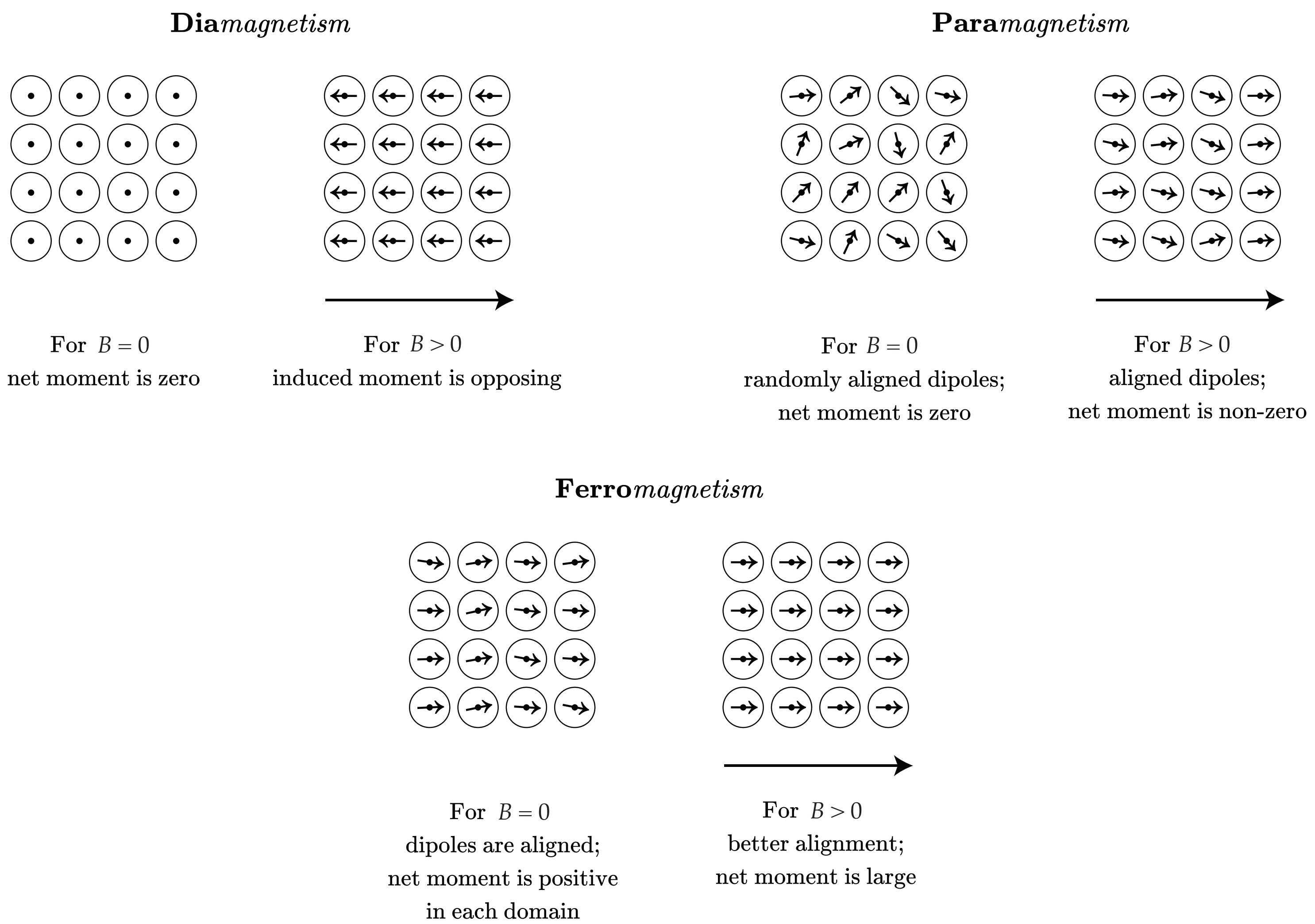


FIGURE 82: A magnified diagram of the inner workings of magnetic dipoles, the arrows show the directions of magnetization in the domains of a single crystal of Nickel. Domains that are magnetized in the direction of an applied magnetic field grow larger.

The underlying principles that govern these magnetic behaviors involve the interaction of magnetic moments at the atomic or molecular level. In diamagnetism, the magnetic moments of individual atoms or molecules oppose the applied field due to the electron orbital motion. In paramagnetism, the unpaired electrons align with the external field, resulting in a net attraction. In ferromagnetism, the alignment of magnetic domains leads to strong and persistent magnetization.

26.5 Motional EMF

The motional EMF is the electromotive force induced in a circuit due to the motion of a conductor through a magnetic field. When the conductor moves through the field, the motion creates a change in magnetic flux, which induces a voltage across the circuit. This induced voltage can drive a current in the circuit, even if the circuit is initially open.

Consider a conducting rod of length L moving with a constant velocity $\vec{v}$ perpendicular to a magnetic field $\vec{B}$. The rod is connected to a circuit that forms a complete loop with a total resistance R. As the rod moves through the magnetic field, an EMF is induced in the circuit due to the motion of the rod cutting across the magnetic field lines.

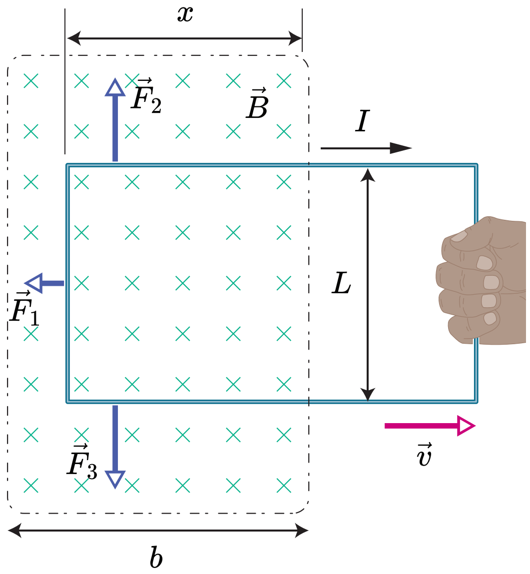


FIGURE 83: You pull a closed conducting loop out of a magnetic field at constant velocity $\vec{V}$. While the loop is moving, a clockwise current i is induced in the loop, and the loop segments still within the magnetic field experience forces $\vec{F}_1$, $\vec{F}_2$ and $\vec{F}_3$.

The EMF induced in the circuit is given by Faraday's Law of electromagnetic induction:

$$\begin{aligned} \mathcal{E} &= -\frac{d\Phi_B}{dt} = -\frac{d(BA)}{dt} \\ &= -B\frac{dA}{dt} = -BL\frac{dx}{dt} \\ \therefore \mathcal{E} &= -BLv. \end{aligned} \tag{214}$$

The induced current in the circuit is:

$$i = \frac{|\mathcal{E}|}{R} = \frac{BLv}{R} \tag{215}$$

The unbalanced force in the current loop with side length L is the magnetic force:

$$|\vec{F}| = i\vec{L} \times \vec{B} = iLB \tag{216}$$

Mixing Eq. (215) and (216) we get the following:

$$F = \frac{B^2L^2v}{R}. \tag{217}$$

The rate of work, i.e., Power on the loop as you move it in the magnetic field:

$$P = Fv = \frac{B^2L^2v^2}{R}. \tag{218}$$

This will also be the thermal energy appearance in the loop when moving at a constant speed v.

$$P = i^2R = \frac{B^2L^2v^2}{R^2}R = \frac{B^2L^2v^2}{R}. \tag{219}$$

27 Inductance

Inductance L is a fundamental property of an electrical circuit or component that describes its ability to resist changes in current. It quantifies how much magnetic flux is produced per unit of current change. Inductance is typically measured in henries (H), and its value depends on the geometry and arrangement of the circuit elements.

27.1 Self Inductance

Self-inductance refers to the property of a circuit or component to induce an electromotive force (EMF) in itself when the current through it changes. It occurs due to the changing magnetic field generated by the current.

Consider a solenoid (a coil of wire) with N turns and length L, carrying a current I. When a current flows through the solenoid, it generates a magnetic field $\vec{B}$ inside the coil. The magnetic flux Φ_B is linked with each turn of the coil.

The induced magnetic flux Φ_B (One unit of it per turn of the coil) is proportional to the current i in the coil.

$$\begin{aligned} N\Phi_B &\propto i \\ N\Phi_B &= Li \\ \therefore L &= \frac{N\Phi_B}{i}. \end{aligned} \tag{220}$$

Eq. (220) defines the self-inductance or simply inductance of a current-carrying coil.

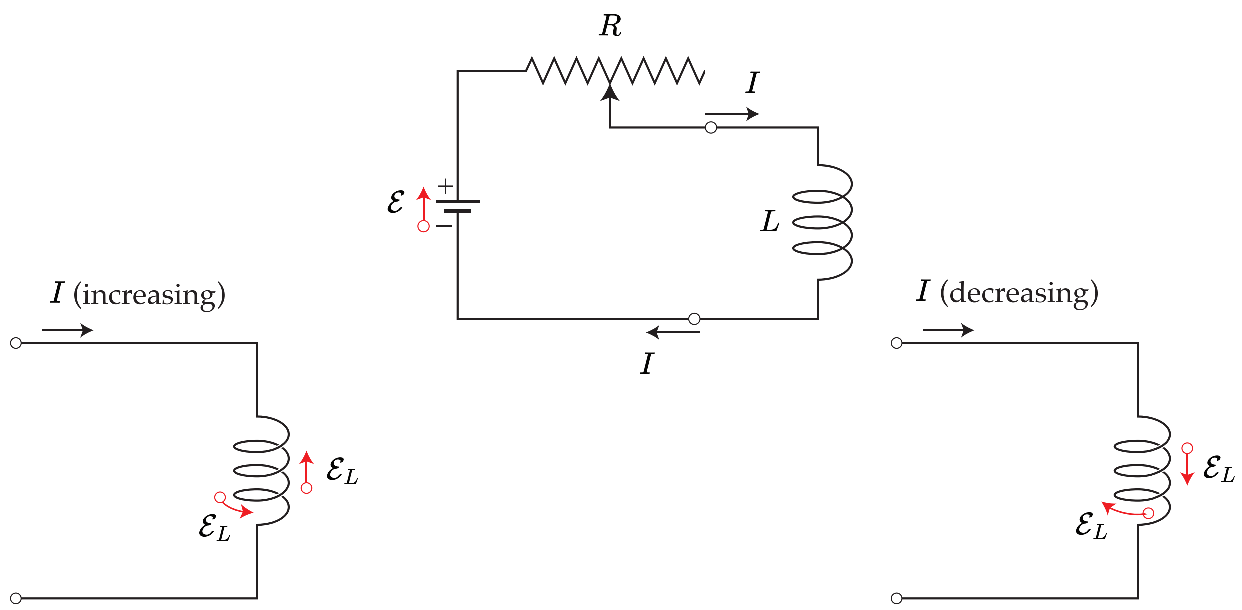


FIGURE 84: If the current in a coil is changed by varying the contact position on a variable resistor, a self-induced EMF L will appear in the coil while the current is changing. The current i is increasing, and the self-induced EMF L appears along the coil in a direction such that it opposes the increase. The arrow representing L can be drawn along a turn of the coil or alongside the coil. Both are shown. The current i is decreasing, and the self-induced EMF appears in a direction such that it opposes the decrease.

Induced EMF and Current Due to an Inductor

The induced EMF $\mathcal{E}$ can now be defined using Faraday's Law of induction.

$$\mathcal{E}_L = -\frac{d(N\Phi_B)}{dt} = -L\frac{di}{dt}. \tag{221}$$

When you increase the current i in a coil, the rate of change of current $\frac{di}{dt}$ represents the change that self-inductance resists. Consequently, a self-induced EMF appears in the coil, opposing this increase in current, striving to maintain the initial state. Conversely, if the current decreases over time, the self-induced EMF resists this decrease, attempting to sustain the initial state. This behavior aligns with Lenz's Law.

Furthermore, a self-induced potential difference V_L can be defined across an inductor assumed to be between its terminals outside the changing flux region. In the case of an ideal inductor (with negligible wire resistance), the magnitude of V_L equals the magnitude of the self-induced EMF $\mathcal{E}_L$. However, if the inductor's wire has resistance r, we mentally separate it into a resistance r (assumed outside the changing flux region) and an ideal inductor with self-induced EMF L. Similar to a real battery with both EMF and internal resistance r, the potential difference across a real inductor's terminals differs from its EMF. It's important to note that unless stated otherwise, we consider inductors to be ideal.

27.2 Mutual Inductance

Mutual inductance M is a fundamental property of two nearby electrical circuits or components that describes how changes in the current in one circuit induce an electromotive force (EMF) in the other circuit.

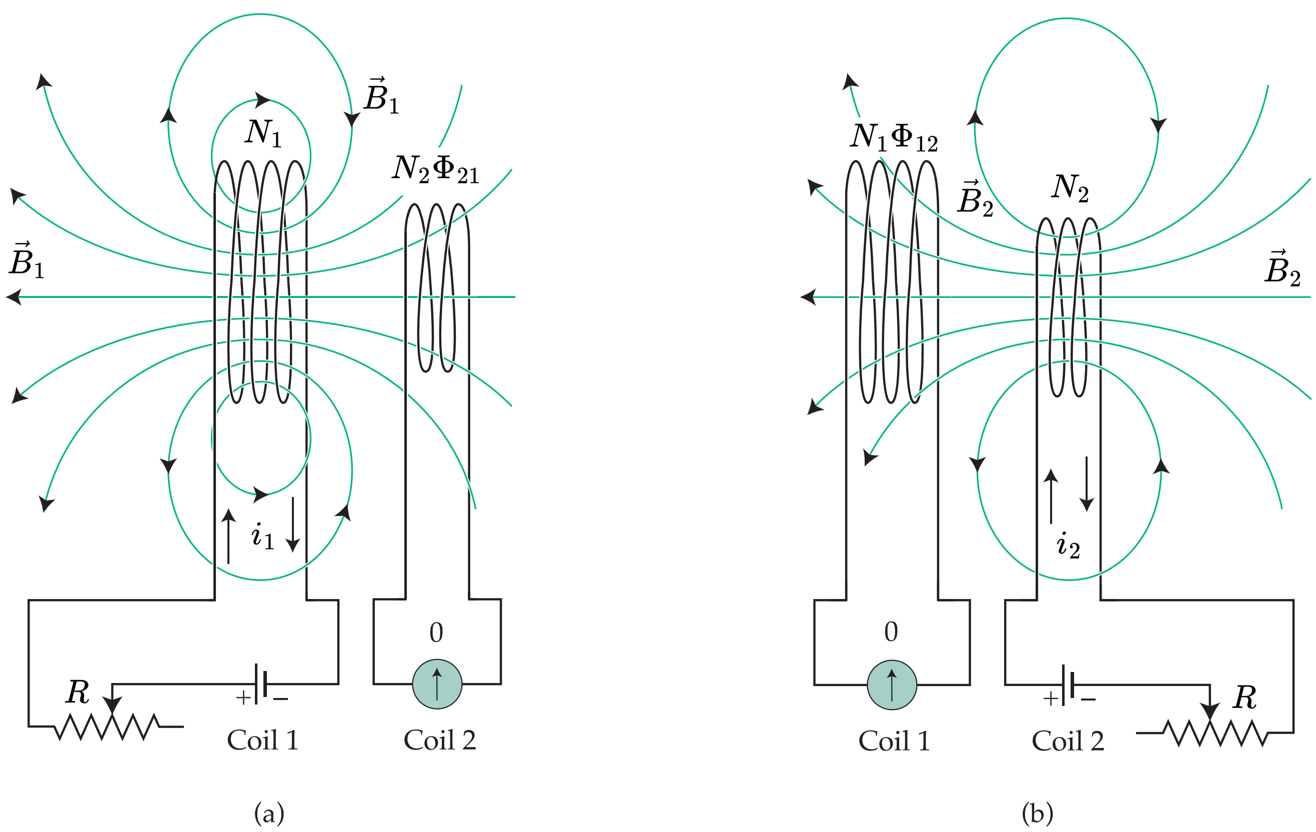


FIGURE 85: Mutual induction. (a) The magnetic field produced by current i_1 in Coil 1 extends through Coil 2. If i_1 is varied (by varying resistance R), an EMF is induced in Coil 2, and current registers on the meter connected to Coil 2. (b) The roles of the coils are interchanged.

Consider two separate coils, Coil 1 and Coil 2, placed close to each other. When the current in Coil 1 changes, it generates a changing magnetic field $\vec{B}_1$ that links with Coil 2. The magnetic flux Φ_{21} due to Coil 1's changing current linked with Coil 2 is given by the following equation, mirroring Eq. (220):

$$M_{21} = \frac{N_2 \Phi_{12}}{i_1}. \tag{222}$$

Similarly, when the current in Coil 2 changes, it generates a changing magnetic field $\vec{B}_2$ that links with Coil 1. The magnetic flux Φ_{12} due to Coil 2's changing current linked with Coil 1 is given by:

$$M_{12} = \frac{N_1 \Phi_{21}}{i_2}. \tag{223}$$

Combining Eqs. (222) and (222) gives us:

$$M = M_{21} = M_{12} = \frac{N_1 \Phi_{21}}{i_2} = \frac{N_2 \Phi_{12}}{i_1}. \tag{224}$$

28 Energy Stored in $\vec{B}$-Field

The energy stored in a magnetic field can be derived using the formula for the energy stored in an inductor. When a current flows through an inductor, it generates a magnetic field, and energy is stored in this field. We begin with the power stored in the circuit.

The power in the RL circuit P_L can be calculated using the following steps:

$$P_L = \mathcal{E} i = L i \frac{di}{dt} + i^2 R, \tag{225}$$

where the first term is the power stored in the inductor and the second term is the power dissipated across the wire resistance. We can consider the wire resistance R out of the jurisdiction of the inductor.

Differentiating Eq. (225), we get the energy in the circuit:

$$\begin{aligned} \frac{dU_B}{dt} &= P_L = L i \frac{di}{dt} \\ U_B &= \int_0^{U_B} dU_B = L \int_0^i i di \\ U_B &= \frac{1}{2} L i^2. \end{aligned} \tag{226}$$

This is the magnetic field energy stored in the inductor.

28.1 Energy Density of $\vec{B}$-Field

Consider a length l near the middle of a long solenoid of cross-sectional area A carrying current i; the volume associated with this length is Al. The energy U_B stored by the length l of the solenoid must lie entirely within this volume because the magnetic field outside such a solenoid is approximately zero.

Moreover, the stored energy must be uniformly distributed within the solenoid because the magnetic field is (approximately) uniform everywhere inside. Thus, the energy stored per unit volume of the field is

$$u_B = \frac{U}{V} = \frac{\frac{1}{2} L i^2}{Al} = \frac{L i^2}{2Al} = \frac{L}{l} \frac{i^2}{2A}$$

$$
\begin{aligned}
&= \frac{i^2}{2Al}\frac{N\Phi_B}{i} \\
&= \frac{i}{2Al}(nl)(BA) \\
&= \frac{i}{2Al}(nl)(\mu_0 in)A \\
&= \frac{n^2i^2\mu_0}{2} = \frac{(\mu_0 ni)^2}{2\mu_0} \\
\therefore u_B &= \frac{1}{2\mu_0}B^2.
\end{aligned} \tag{227}
$$

29 Transient Series RL Circuits

Similar to RC circuits, we define RL circuits where the capacitor is replaced with an inductor with inductance L.

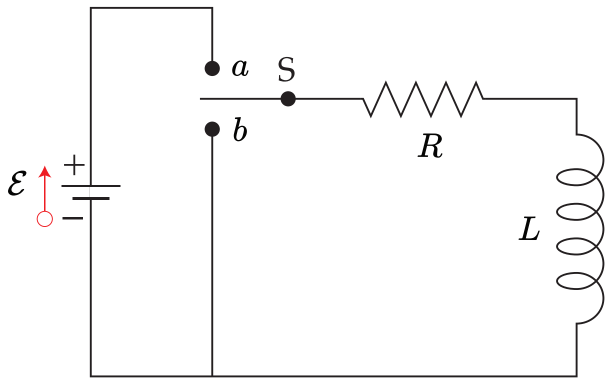


FIGURE 86: When switch S is closed on a, the inductor is activated through the resistor when the current changes only. When the switch is afterward closed on b, the inductor does not let the current drop sharply through the resistor.

29.1 Turn on the Circuit: Growth of Current

Consider the circuit with the battery, resistor, and inductor connected in series. The voltage across the capacitor at any time t is denoted as $V_L(t)$. The transient current in this circuit is $i(t)$.

Use the KVL for the circuit. The instantaneous potential difference V_{ab} and V_{bc} are: $v_{ab} = iR$ and $v_{bc} = -L\frac{di}{dt}$.

$$
\begin{aligned}
\mathcal{E} - iR - L\frac{di}{dt} &= 0 \\
\frac{di}{dt} &= \frac{\mathcal{E} - iR}{L} \\
\frac{di}{dt} &= \frac{\mathcal{E}}{L} - \frac{iR}{L} \\
\frac{di}{i - \frac{\mathcal{E}}{R}} &= -\frac{R}{L}dt.
\end{aligned} \tag{228}
$$

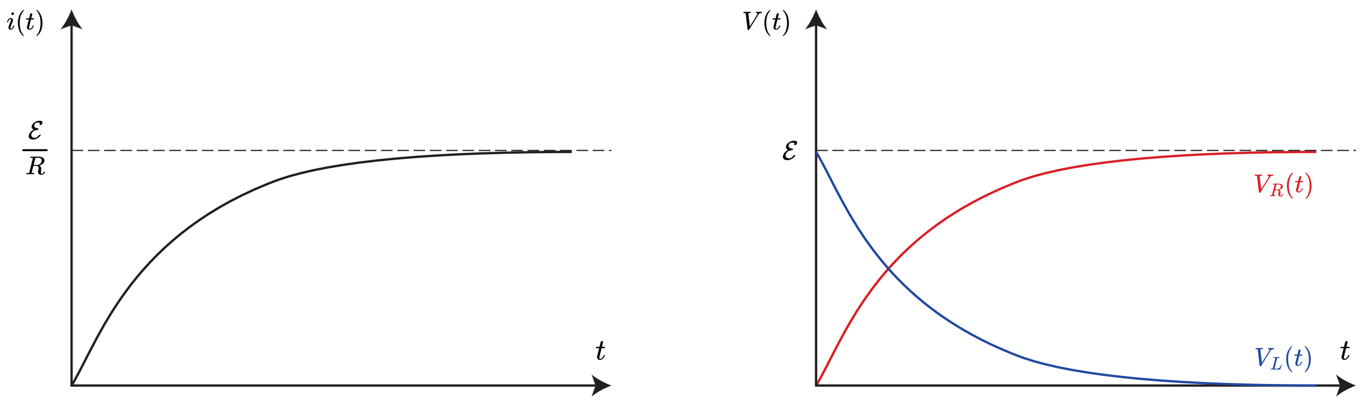


FIGURE 87: Both currents i (following the Eqs. (229)) and voltage V_L (following the Eqs. (230)) increases exponentially with time t.

The integration limits are $i = 0$ at $t = 0$ to $i = i$ at t,

$$\begin{aligned}
\int_0^i \frac{di}{i - \frac{\mathcal{E}}{R}} &= -\int_0^t \frac{dt}{\frac{L}{R}} \\
\ln\left(\frac{i - \frac{\mathcal{E}}{R}}{-\frac{\mathcal{E}}{R}}\right) &= -\frac{t}{\frac{L}{R}} \\
i - \frac{\mathcal{E}}{R} &= e^{-\frac{t}{\frac{L}{R}}} \\
i &= \frac{\mathcal{E}}{R}\left(1 - e^{-\frac{t}{\frac{L}{R}}}\right) \\
i &= \frac{\mathcal{E}}{R}\left(1 - e^{-\frac{t}{\tau_L}}\right).
\end{aligned} \tag{229}$$

This is the current equation for an inductor during a current surge in a series RL circuit. It shows how the current in the inductor increases with time as it charges up to the maximum charge allowed by the circuit.

The inductive time constant is given by $\tau_L = \frac{L}{R}$. This has the same dimensions of time similar to the capacitive time constant $\tau_C = RC$. At $t = 0$, Eq. (229) leads to:

$$i = \frac{\mathcal{E}}{R}\left(1 - e^{-\frac{0}{\tau_L}}\right) = 0.$$

At $t \to \infty$, Eq. (229) leads to:

$$i = \frac{\mathcal{E}}{R}\left(1 - e^{-\frac{\infty}{\tau_L}}\right) = \frac{\mathcal{E}}{R}.$$

29.2 Turn on the Circuit: Growth of Resistive Voltage

Multiplying Eq. (229) with the resistor R gives the voltage equation during the current surge period of the resistor.

$$V_R^{\text{circuit-on}} = iR = \mathcal{E}\left(1 - e^{-\dfrac{t}{\tau_L}}\right). \tag{230}$$

At $t = 0$, Eq. (230) leads to:

$$V_R^{\text{circuit-on}} = \mathcal{E}\left(1 - e^{-\dfrac{0}{\tau_L}}\right) = 0.$$

At $t \to \infty$, Eq. (230) leads to:

$$V_R^{\text{circuit-on}} = \mathcal{E}\left(1 - e^{-\dfrac{\infty}{\tau_L}}\right) = \mathcal{E}.$$

29.3 Turn on the Circuit: Decay of Inductive Voltage

Differentiating Eq. (229) with time once and multiplying it by L gives the voltage equation during the current surge period of the inductor.

$$V_L^{\text{circuit-on}} = L\frac{di}{dt} = \frac{L\mathcal{E}}{R} \times \frac{1}{\tau_L}\left(e^{-\dfrac{t}{\tau_L}}\right) = \mathcal{E}e^{-\dfrac{t}{\tau_L}}. \tag{231}$$

At $t = 0$, Eq. (235) leads to:

$$V_L^{\text{circuit-on}} = \mathcal{E}e^{-\dfrac{0}{\tau_L}} = \mathcal{E}.$$

At $t \to \infty$, Eq. (235) leads to:

$$V_L^{\text{circuit-on}} = \mathcal{E}e^{-\dfrac{\infty}{\tau_L}} = 0.$$

29.4 Turn off the Circuit: Decay of Current

Consider the same RL circuit above but have the battery turn off. The current then decays slowly instead of a sharp, instantaneous drop.

$$\begin{aligned} -iR - L\frac{di}{dt} &= 0 \\ i &= -\frac{L}{R}\frac{di}{dt} \\ \frac{di}{i} &= -\frac{dt}{\dfrac{L}{R}} \end{aligned} \tag{232}$$

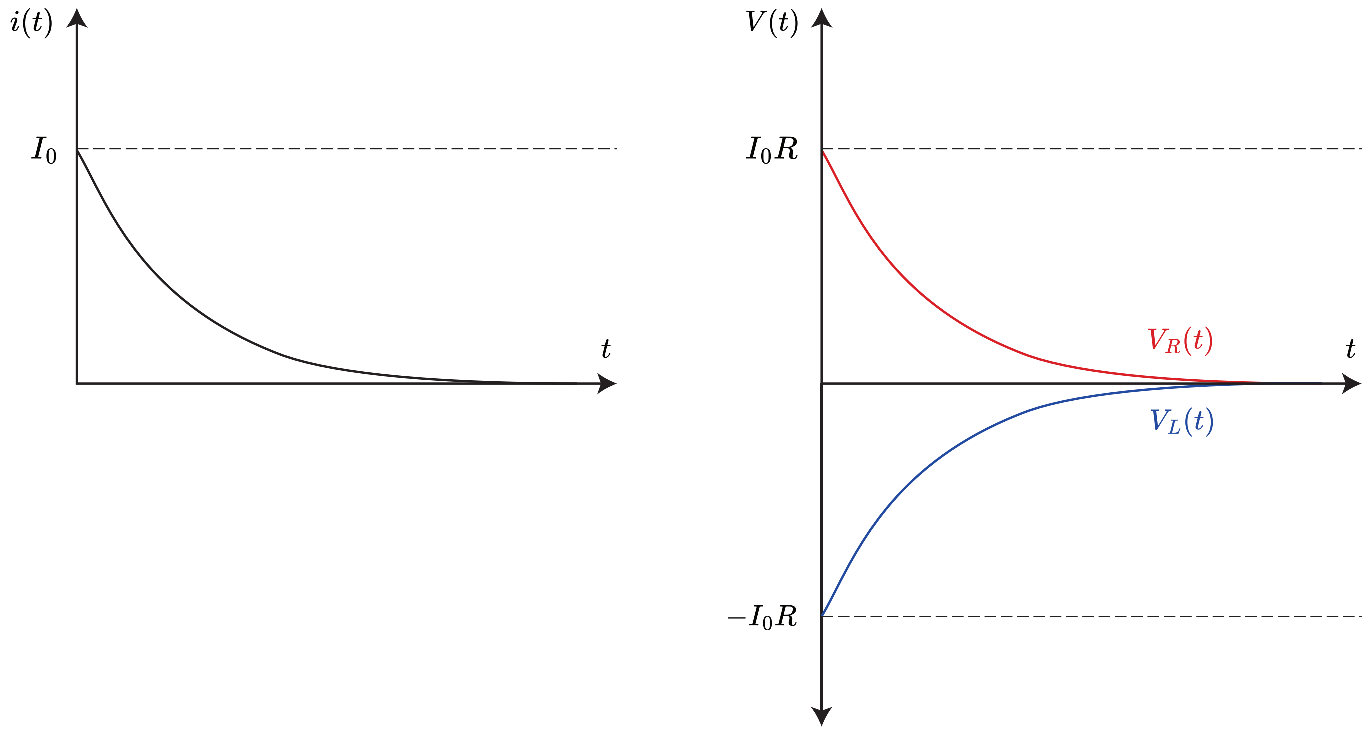


FIGURE 88: Both currents i (following the Eqs. (233)) and voltage V_L (following the Eqs. (234)) decreases exponentially with time t.

The integration limits are $i = I_0$ at $t = 0$ to $i = i$ at t,

$$\begin{aligned}
\int_{I_0}^{i} \frac{di}{i} &= -\int_0^t \frac{dt}{\frac{L}{R}} \\
\ln\left(\frac{i}{I_0}\right) &= -\frac{t}{\tau_L} \\
\frac{i}{I_0} &= e^{-\frac{t}{\tau_L}} \\
i &= I_0 e^{-\frac{t}{\tau_L}}.
\end{aligned} \tag{233}$$

At $t = 0$, Eq. (234) leads to:

$$i = I_0 e^{-\frac{0}{\tau_L}} = I_0.$$

At $t \to \infty$, Eq. (234) leads to:

$$i = I_0 e^{-\frac{t}{\tau_L}} = 0.$$

This is the decay current equation for an inductor in a series RL circuit. It shows how the current in the inductor decreases with time as it decays to the minimum charge allowed by the circuit.

29.5 Turn off the Circuit: Decay of Resistive Voltage

Multiplying Eq. (233) with the resistor R gives the voltage equation during the current surge period of the inductor.

$$V_R^{\text{circuit-off}} = iR = (I_0 R)e^{-\frac{t}{\tau_L}}. \tag{234}$$

At $t = 0$, Eq. (234) leads to:

$$V_R^{\text{circuit-off}} = iR = (I_0 R)e^{-\frac{0}{\tau_L}} = I_0 R.$$

At $t \to \infty$, Eq. (234) leads to:

$$V_R^{\text{circuit-off}} = iR = (I_0 R)e^{-\frac{\infty}{\tau_L}} = 0.$$

29.6 Turn off the Circuit: Growth of Inductive Voltage

Differentiating Eq. (233) with time once and multiplying it by L gives the voltage equation during the current surge period of the inductor.

$$V_L^{\text{circuit-off}} = L\frac{di}{dt} = -L\frac{I_0}{\tau_L}\left(e^{-\frac{t}{\tau_L}}\right) = -\frac{LI_0}{\frac{L}{R}}\left(e^{-\frac{t}{\tau_L}}\right) = -V_0 e^{-\frac{t}{\tau_L}}. \tag{235}$$

At $t = 0$, Eq. (235) leads to:

$$V_L^{\text{circuit-off}} = -V_0 e^{-\frac{0}{\tau_L}} = -V_0.$$

At $t \to \infty$, Eq. (235) leads to:

$$V_L^{\text{circuit-off}} = -V_0 e^{-\frac{\infty}{\tau_L}} = 0.$$

30 Electromagnetic Oscillations I: Series LC Circuits

LC oscillations refer to the periodic exchange of energy between an inductor L and a capacitor C in an electrical circuit. This phenomenon occurs in circuits known as *LC circuit*s. LC oscillations are a form of simple harmonic motion where the electric and magnetic fields in the circuit store and exchange energy at a specific resonant frequency. Consider a series LC circuit consisting of an inductor (L) and a capacitor (C) connected in series. The current I flowing through the circuit at any time t charges the capacitor and creates a magnetic field in the inductor. The voltage across the capacitor V_C and the voltage across the inductor V_L contribute to the total voltage V across the circuit. Use KVL to find:

$$\begin{aligned} V = V_C + V_L &= \frac{q}{C} + L\frac{di}{dt} \\ L\frac{d^2q}{dt^2} + \frac{q}{C} &= V. \end{aligned} \tag{236}$$

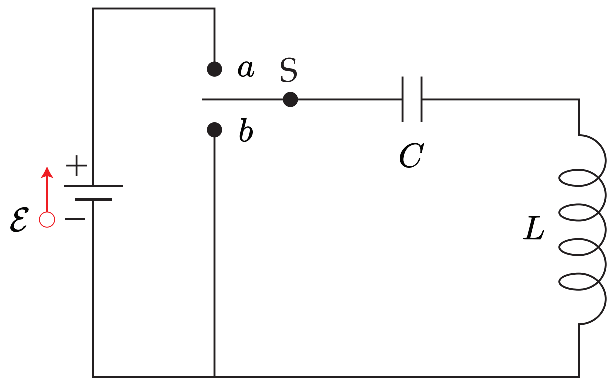


FIGURE 89: When switch S is closed on a, the capacitor charges only. When the switch is afterward closed on b, the inductor exchanges the stored energy back and forth, creating an *electromagnetic oscillation*.

This is a 2nd order linear differential equation that describes the behavior of an LC circuit. Its solution gives the charge q as a function of time t and relates to the LC oscillations in the circuit.

$$q(t) = Q\cos(\omega t + \phi), \tag{237}$$

where Q is the maximum charge, ω is the angular frequency of the oscillation, and ϕ is the phase angle.

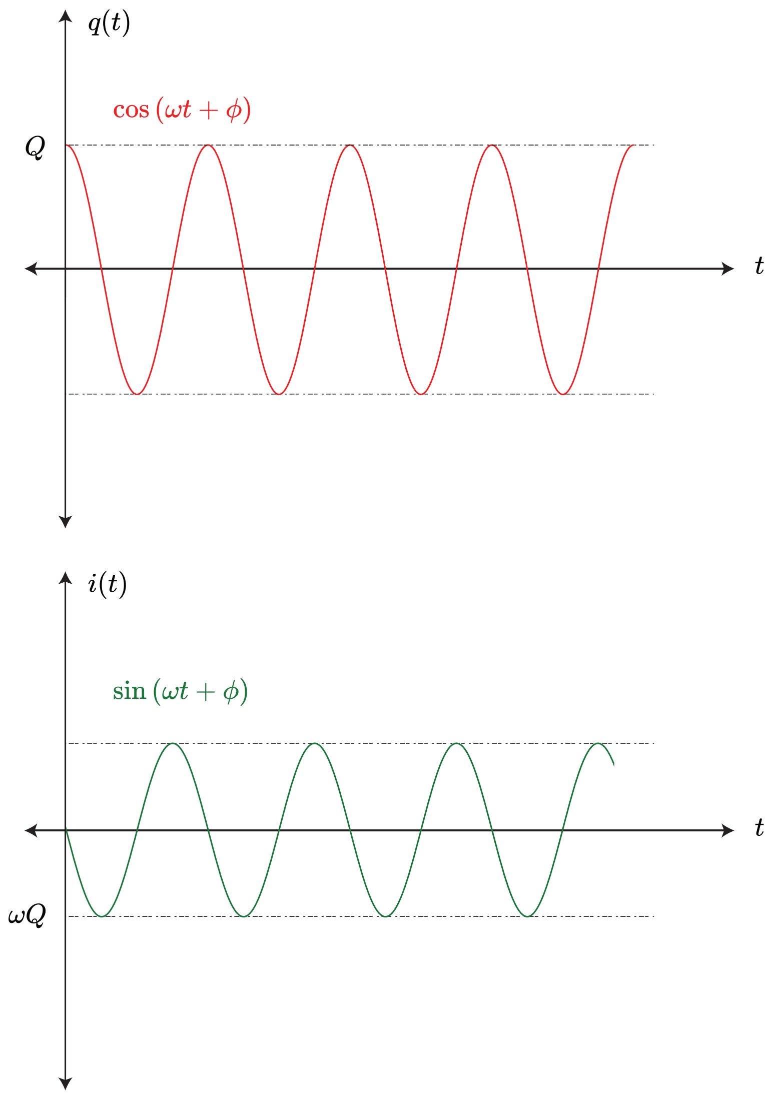


FIGURE 90: When switch S is closed on a, the capacitor charges only. When the switch is afterward closed on b, the inductor exchanges the stored energy back and forth, creating an *electromagnetic oscillation*; while the resistor dissipates the energy with an exponentially decreasing amplitude.

If we turn on the voltage and allow the capacitor to store some charges, thus introducing electrical energy into the circuit. Turning off the source will lead to the following:

$$L\frac{d^2q}{dt^2} + \frac{q}{C} = 0. \tag{238}$$

This leads to a harmonic *LC* oscillations of energy transfer with essentially no one to damp the oscillation. The charge oscillation (and thus, by extension, the energy oscillates) following Eq. (237).

This sinusoidal variation in charge corresponds to oscillations in the current and voltage across the inductor and capacitor, leading to LC oscillations in the circuit. The resonant frequency ω_0 of these oscillations is determined by:

$$\omega_0 = \frac{1}{\sqrt{LC}}. \tag{239}$$

We can measure the sinusoidal current in the circuit by differentiating Eq. (237) once:

$$i = \frac{dq}{dt} = -\omega Q \sin(\omega t + \phi) = -I \sin(\omega t + \phi). \tag{240}$$

The maximum amplitude of this oscillating current is $I = \omega Q$.

Theoretically, this oscillation is undamped, just like a simple pendulum, because no one is present in the circuit to dissipate any energy.

30.1 Electrical and Magnetic Energy Oscillations in *LC* Circuits

The electrical energy stored in the *LC* circuit at time t is,

$$U_E(t) = \frac{q^2}{2C} = \frac{Q^2}{2C}\cos^2(\omega t + \phi). \tag{241}$$

The magnetic energy stored in the *LC* circuit at time t is,

$$U_B(t) = \frac{1}{2}Li^2 = \frac{1}{2}L\omega^2 Q^2 \sin^2(\omega t + \phi) = \frac{Q^2}{2C}\sin^2(\omega t + \phi). \tag{242}$$

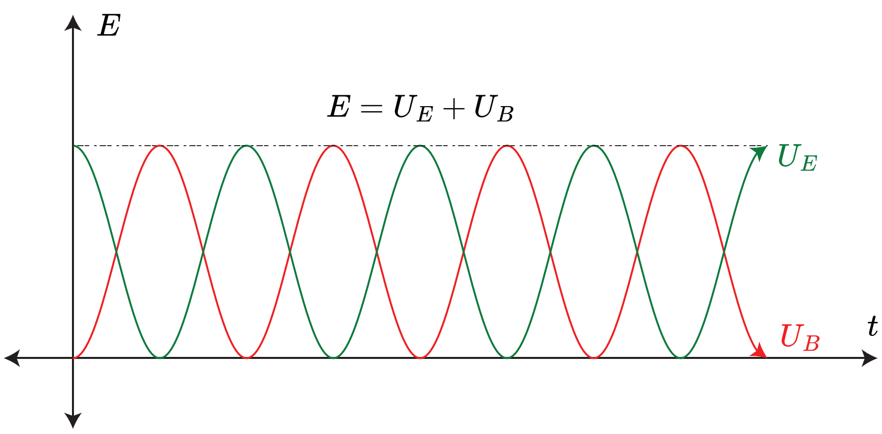


FIGURE 91: Energy oscillation graph for an LC circuit. At any instant, the total energy is constant (dashed line).

The total circuit energy (undamped) would then be

$$E_{\text{no-damping}} = U_E(t) + U_B(t) = \frac{Q^2}{2C}\left[\sin^2(\omega t + \phi) + \cos^2(\omega t + \phi)\right] = \frac{Q^2}{2C}. \tag{243}$$

Figure 91 plots of $U_E(t)$ and $U_B(t)$ for the case of $\phi = 0$. Note that

- The maximum values of U_E and U_B are both $\frac{Q^2}{2C}$
- At any instant, the total energy $\frac{Q^2}{2C}$ is constant
- When U_E is maximum, and U_B is minimum, and vice versa.

31 Electromagnetic Oscillations II: Series RLC Circuits

RLC oscillations refer to the periodic oscillatory behavior that occurs in circuits with a resistor R, an inductor L, and a capacitor C in an electrical circuit. This phenomenon occurs in circuits known as *RLC circuit*s. RLC oscillations are a form of damped simple harmonic motion where the electric and magnetic fields in the circuit store and exchange energy at a specific resonant frequency while the energy is dissipated with an exponential envelope.

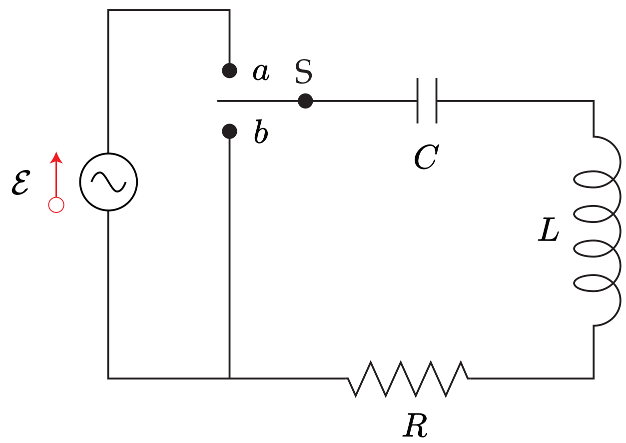


FIGURE 92: When switch S is closed on a, the capacitor charges only. When the switch is afterward closed on b, the inductor exchanges the stored energy back and forth, creating an *electromagnetic oscillation*; while the resistor dissipates the energy with an exponentially decreasing amplitude.

Consider a series RLC circuit consisting of a resistor R, an inductor L, and a capacitor C connected in series. The current I flowing through the circuit at any time t charges the capacitor and creates a magnetic field in the inductor. The voltage across the resistor V_R, voltage across the capacitor V_C, and the voltage across the inductor V_L contribute to the total voltage V across the circuit. Use KVL to find:

$$\begin{aligned} V = V_R + V_C + V_L = iR + \frac{q}{C} + L\frac{di}{dt} \\ \frac{dq}{dt} + L\frac{d^2q}{dt^2} + \frac{q}{C} = V. \end{aligned} \tag{244}$$

This is a 2^{nd} order linear differential equation that describes the behavior of an RLC circuit. Its solution gives the charge q as a function of time t identical to Eq. (237) with a damping exponential amplitude.

$$q(t) = Qe^{-\alpha t}\cos(\omega_{\text{damping}}\, t + \phi), \tag{245}$$

where Q is the maximum charge, $\omega_{\text{damping}} = \sqrt{\omega_0^2 - \left(\frac{R}{2L}\right)^2}$ is the damping angular frequency of the oscillation, α is the damping parameter and ϕ is the phase angle.

If we turn on the voltage and allow the capacitor to store some charges, thus introducing electrical energy into the circuit. Turning off the source will lead to the following:

$$\frac{dq}{dt} + L\frac{d^2q}{dt^2} + \frac{q}{C} = 0. \tag{246}$$

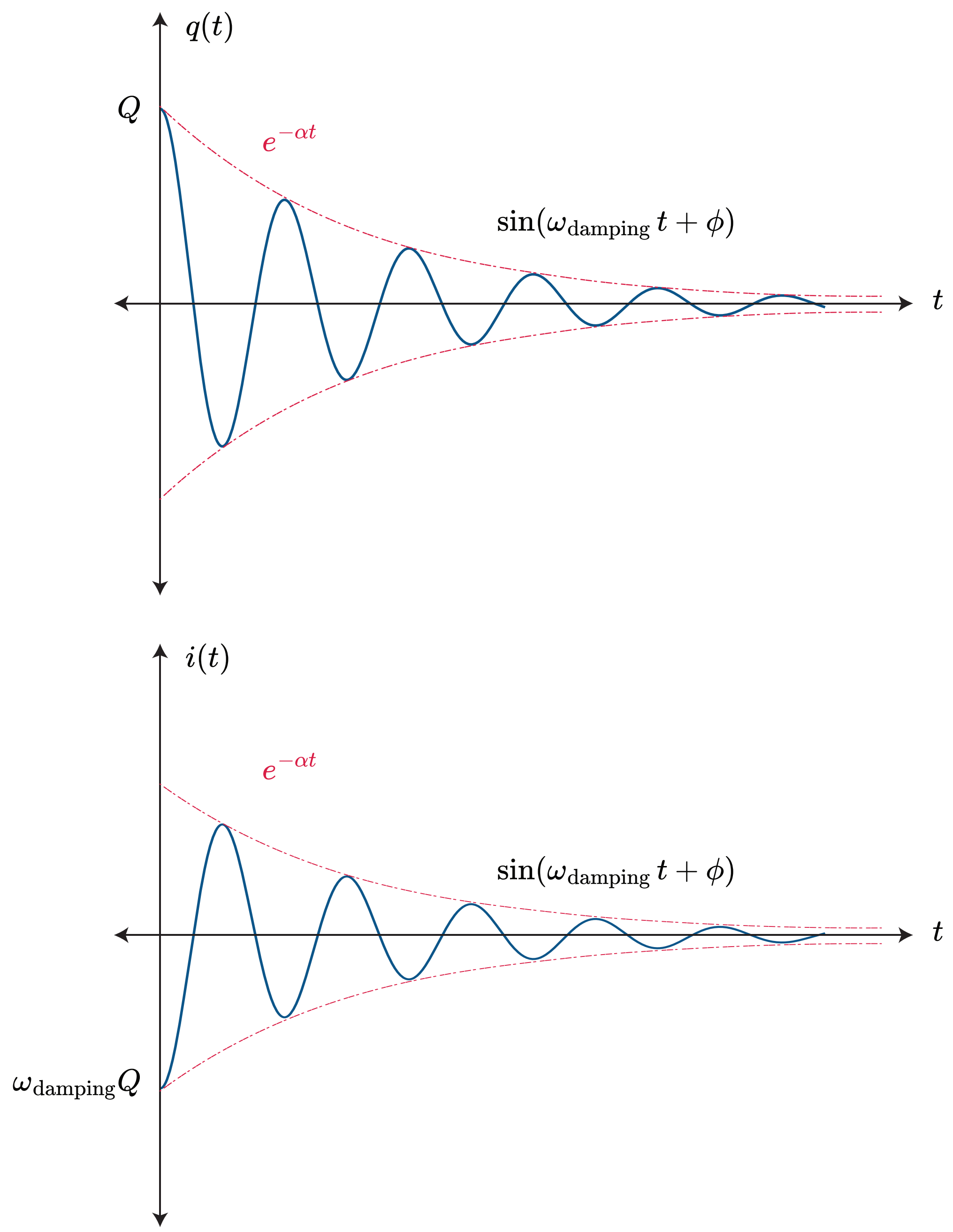


FIGURE 93: When switch S is closed on a, the capacitor charges only. When the switch is afterward closed on b, the inductor exchanges the stored energy back and forth, creating an *electromagnetic oscillation*; while the resistor dissipates the energy with an exponentially decreasing amplitude.

This leads to a damped harmonic *RLC* oscillations of circuit energy with R essentially being the damping component of the oscillation. The charge oscillation (and thus, by extension, the energy oscillates) following Eq. (245).

We can measure the sinusoidal current in the circuit by differentiating Eq. (237) once:

$$
\begin{aligned}
i = \frac{dq}{dt} &= -\omega_{\text{damping}} Q e^{-\alpha t} \sin\left(\omega_{\text{damping}}\, t + \phi\right) - \alpha Q e^{-\alpha t} \cos\left(\omega_{\text{damping}}\, t + \phi\right) \\
&= -\omega_{\text{damping}} Q e^{-\alpha t} \sin\left(\omega_{\text{damping}}\, t + \phi\right) - \alpha q(t) \\
&= -\omega_{\text{damping}} Q e^{-\alpha t} \sin\left(\omega_{\text{damping}}\, t + \phi\right) - \alpha \int i dt. \\
i - \alpha \int i dt &= -\omega_{\text{damping}} Q e^{-\alpha t} \sin\left(\omega_{\text{damping}}\, t + \phi\right) \\
\therefore i &= -\left(\frac{1}{1-\alpha t}\right) \omega_{\text{damping}} Q e^{-\alpha t} \sin\left(\omega_{\text{damping}}\, t + \phi\right) && (247) \\
&= -I e^{-\alpha t} \sin\left(\omega_{\text{damping}}\, t + \phi\right). && (248)
\end{aligned}
$$

The decaying (maximum) amplitude of this oscillating current is $I = \left(\frac{1}{1-\alpha t}\right) \omega_{\text{damping}} Q$.

This oscillation is damped, just like a practical simple pendulum.

31.1 Electrical and Magnetic Energy Oscillations in *RLC* Circuits

The electrical energy stored in the LC circuit at time t is,

$$
U_E(t) = \frac{q^2}{2C} = \frac{Q^2}{2C} e^{-2\alpha t} \cos^2\left(\omega_{\text{damping}}\, t + \phi\right). \tag{249}
$$

The magnetic energy stored in the LC circuit at time t is,

$$
\begin{aligned}
U_B(t) &= \frac{1}{2} L i^2 = \frac{1}{2} L I^2 \sin^2\left(\omega_{\text{damping}}\, t + \phi\right) \\
&= \frac{Q^2}{2C} \left[\frac{\omega_{\text{damping}}}{\omega_0 (1-\alpha t)}\right]^2 e^{-2\alpha t} \sin^2\left(\omega_{\text{damping}}\, t + \phi\right). && (250)
\end{aligned}
$$

The total circuit energy (damped) would then be

$$
\begin{aligned}
E_{\text{damping}} = U_E(t) + U_B(t) &= \frac{Q^2}{2C} e^{-2\alpha t} \left[\cos^2\left(\omega_{\text{damping}} t + \phi\right) - \left\{\frac{\omega_{\text{damping}}}{\omega_0 (1-\alpha t)}\right\}^2 \sin^2\left(\omega_{\text{damping}} t + \phi\right)\right] \\
&= \frac{Q^2}{2C} e^{-2\alpha t} \cos^2(\omega_{\text{damping}} t + \phi) \left[1 - \left\{\frac{\omega_{\text{damping}}}{\omega_0 (1-\alpha t)}\right\}^2 \tan^2\left(\omega_{\text{damping}} t + \phi\right)\right] \\
&= U_E \left[1 - \left\{\frac{\omega_{\text{damping}}}{\omega_0 (1-\alpha t)}\right\}^2 \tan^2\left(\omega_{\text{damping}} t + \phi\right)\right]. && (251)
\end{aligned}
$$

32 Alternating Current & Forced Electromagnetic Oscillation

The oscillations in an RLC circuit will not damp out if an external EMF device supplies enough energy to *compensate* for the energy dissipated as thermal energy in the resistance R. This is called *Forced Oscillation*. Forced oscillation is the phenomenon of an external alternating voltage source being applied to the circuit, driving it to oscillate at a frequency determined by the source. This external voltage source is often sinusoidal and leads to generating an alternating current (AC) in the circuit. The circuit's natural frequency of oscillation $\omega_0 = \frac{1}{\sqrt{LC}}$ and the frequency of the external driving source ω_d play a significant role in determining the response.

Alternating current periodically changes its direction and magnitude. It is the type of current commonly found in power grids and household electrical systems. AC is generated by power stations and transmitted over long distances. It can easily be converted to different voltages using transformers.

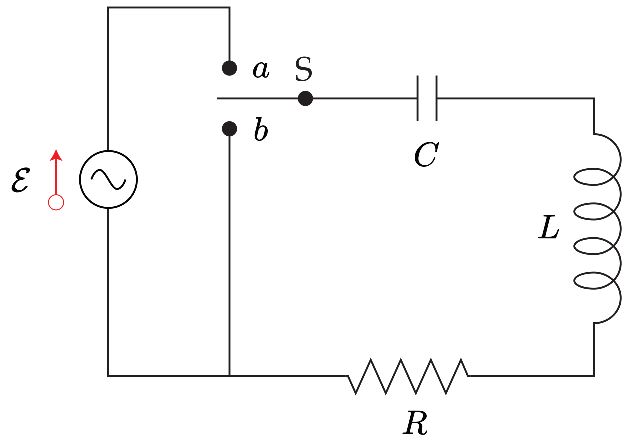


FIGURE 94: When switch S is closed on a, the capacitor charges only. When the switch is afterward closed on b, the inductor exchanges the stored energy back and forth, creating an *electromagnetic oscillation*, while the resistor dissipates this energy as thermal energy.

Consider a series RLC circuit driven by an external sinusoidal EMF source $\mathcal{E}_{\text{source}}(t) = \mathcal{E}_m \sin(\omega_d t + \phi)$, where $\mathcal{E}_m$ is the amplitude of the EMF, ω_d is the driving angular frequency of the source, and $\omega_d t$ is the phase of the EMF. An alternator can produce this type of alternating EMF to the damping RLC circuit.

The current would then be of the following form:

$$i(t) = I_m \sin(\omega_d t + \phi), \tag{252}$$

where I_m is the magnitude of the driven alternating current.

NOTE: Whatever the natural angular frequency ω_0 of a circuit may be, forced oscillations of charge, current, and potential difference in the circuit always occur at the driving angular frequency ω_d.

32.1 A Quick Detour through Phasors

Phasors are a geometrical representation that simplifies the analysis of sinusoidal oscillations in alternating current (AC) circuits and wave phenomena. They allow us to analyze complex oscillatory behaviors using simpler mathematical operations and concepts. Phasors are especially useful when dealing with the amplitude, phase, and frequency relationships of oscillating quantities. We will better understand them once we see them in action.

Phasors have the following properties:

- **Angular speed**: Phasors rotate counterclockwise (positive) about the origin with an angular speed equal to the driving angular frequency ω_d
- **Length**: The length of each phasor represents the amplitude of the alternating quantities, v, I.
- **Projection**: The projection of each phasor on the vertical axis represents the value of the alternating quantity at time t
- **Rotation angle**: The rotation angle of each phasor is equal to the phase of the alternating quantity at time t

NOTE: Phasors are not exactly vectors in the traditional sense, but they are often treated as vectors in mathematical analysis due to their similarity in behavior and algebraic operations. Phasors are complex numbers that represent the magnitude and phase of a sinusoidal oscillation. While complex numbers have both real and imaginary components, they can also be represented as vectors in a complex plane. **Whatever You do, do not draw them in a Cartesian coordinate system with XY axis**.

32.2 Foced Oscillations in Three Simple Circuits

Before we get to the whole RLC circuit being operated under the influence of an alternating source, it is good practice to see how this EMF affects each circuit element independently.

A Resistive Load

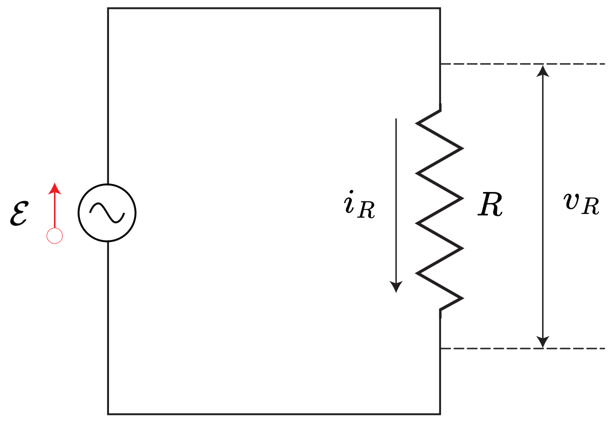


FIGURE 95: A resistor is connected across an alternating-current generator.

Consider the following circuit where an alternating EMF $\mathcal{E}$ is connected in series with a resistor R. Use KVL to find:

$$\mathcal{E} - v_R = 0, \tag{253}$$

v_R is the resistive voltage drop.

This gives us

$$v_R = \mathcal{E}_m \sin(\omega_d t + \phi) = V_R \sin(\omega_d t + \phi), \tag{254}$$

V_R is the maximum amplitude of the resistive voltage drop.

From Ohm's Law,

$$\begin{aligned} i_R &= \frac{v_R}{R} = \frac{V_R}{R}\sin(\omega_d t + \phi) \\ i_R &= I_R \sin(\omega_d t + \phi), \end{aligned} \tag{255}$$

where I_R is the maximum amplitude of the alternating current in the resistive circuit, giving us:

$$V_R = I_R R. \tag{256}$$

Comparing Eqs. (254) and (255), we see that voltage and current in a resistive circuit are in phase.

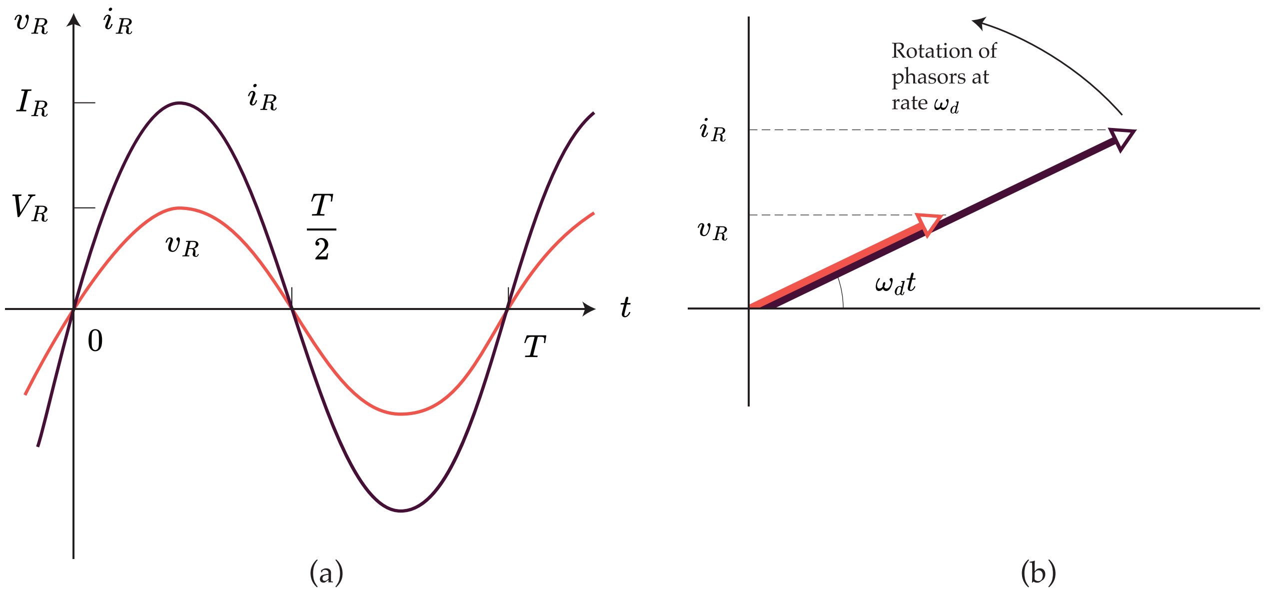


FIGURE 96: (a) The current i_R and the potential difference v_R across the resistor are plotted on the same graph, both versus time t. They are in phase and complete one cycle in one period T. (b) A phasor diagram shows the same thing as (a).

An Inductive Load

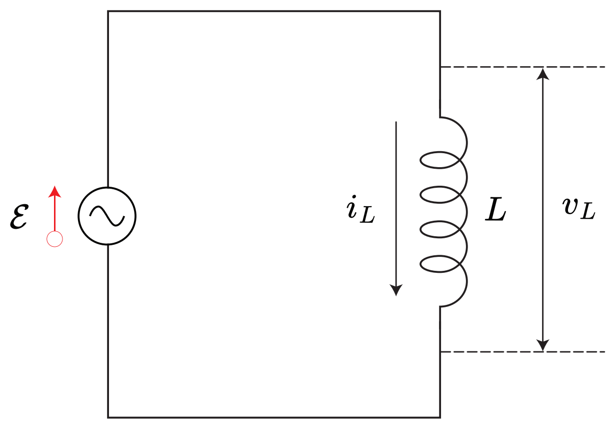


FIGURE 97: An inductor is connected across an alternating-current generator.

Consider the following circuit where an alternating EMF $\mathcal{E}$ is connected in series with an inductor L. Use KVL to find:

$$\mathcal{E} - v_L = 0, \tag{257}$$

v_L is the inductive voltage drop.

This gives us

$$v_L = \mathcal{E}_m \sin(\omega_d t + \phi) = V_L \sin(\omega_d t + \phi), \tag{258}$$

V_L is the maximum amplitude of the inductive voltage drop.

The voltage drop across an inductor,

$$v_L = L\frac{di}{dt}$$

$$\begin{aligned}
\frac{di}{dt} &= \frac{V_L}{L}\sin(\omega_d t+\phi) \\
i_L = \int di_L &= \frac{V_L}{L}\int \sin(\omega_d t+\phi) \\
&= -\left(\frac{V_L}{\omega_d L}\right)\cos(\omega_d t+\phi) \\
&= -\left(\frac{V_L}{X_L}\right)\cos(\omega_d t+\phi) \\
\therefore i_L &= \left(\frac{V_L}{X_L}\right)\sin[(\omega_d t+\phi)-90^\circ],
\end{aligned} \tag{259}$$

where $X_L = \omega_d L$ is the *inductive reactance*, giving us:

$$V_L = I_L X_L. \tag{260}$$

Comparing Eqs. (258) and (259) we see that voltage and current in an inductive circuit are out of phase. The inductive current is 90° behind in phase with the inductive voltage. The current I_L *lags* the voltage V_L.

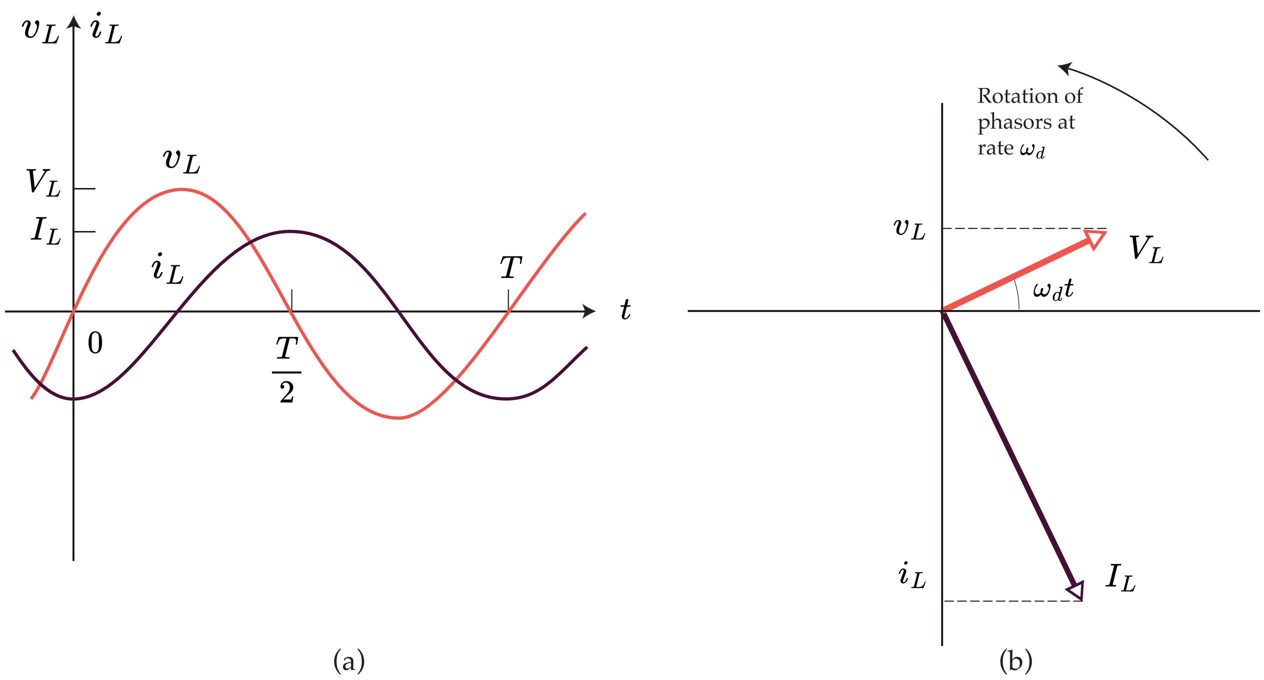


FIGURE 98: (a) The current i_L and the potential difference v_L across the inductor are plotted on the same graph, both versus time t. The current in the inductor lags the voltage by 90° in one period T. (b) A phasor diagram shows the same thing as (a).

A Capacitive Load

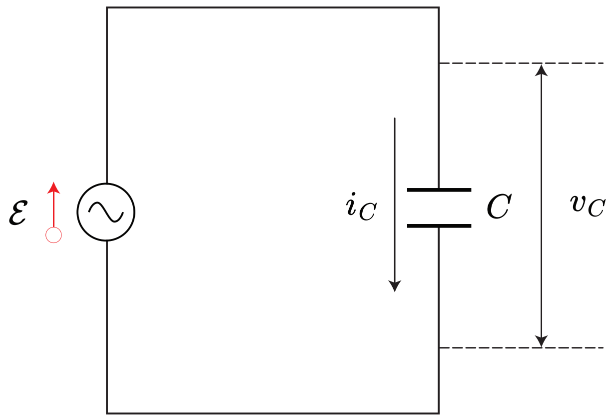


FIGURE 99: A capacitor is connected across an alternating-current generator.

Consider the following circuit where an alternating EMF $\mathcal{E}$ is connected in series with a capacitor C. Use KVL to find:

$$\mathcal{E} - v_C = 0, \tag{261}$$

v_C is the capacitive voltage drop.

This gives us

$$v_C = \mathcal{E}_m \sin(\omega_d t + \phi) = V_C \sin(\omega_d t + \phi), \tag{262}$$

V_C is the maximum amplitude of the capacitive voltage drop.

From the Capacitance definition,

$$\begin{aligned}
q_C &= C v_C \\
&= C V_C \sin(\omega_d t + \phi) \\
i_C = \frac{dq}{dt} &= \omega_d C V_C \cos(\omega_d t + \phi) \\
i_C &= \frac{V_C}{\left(\dfrac{1}{\omega_d C}\right)} \cos(\omega_d t + \phi) \\
i_C &= \frac{V_C}{X_C} \cos(\omega_d t + \phi) \\
i_C &= I_C \cos[(\omega_d t + \phi) + 90^\circ],
\end{aligned} \tag{263}$$

where I_C is the maximum amplitude of the alternating current in the capacitive circuit, giving us:

$$V_C = I_C X_C. \tag{264}$$

Comparing Eqs. (262) and (263), we see that voltage and current in an inductive circuit are out of phase. The capacitive current is 90° ahead in phase with the capacitive voltage. The current I_C *leads* the voltage V_C.

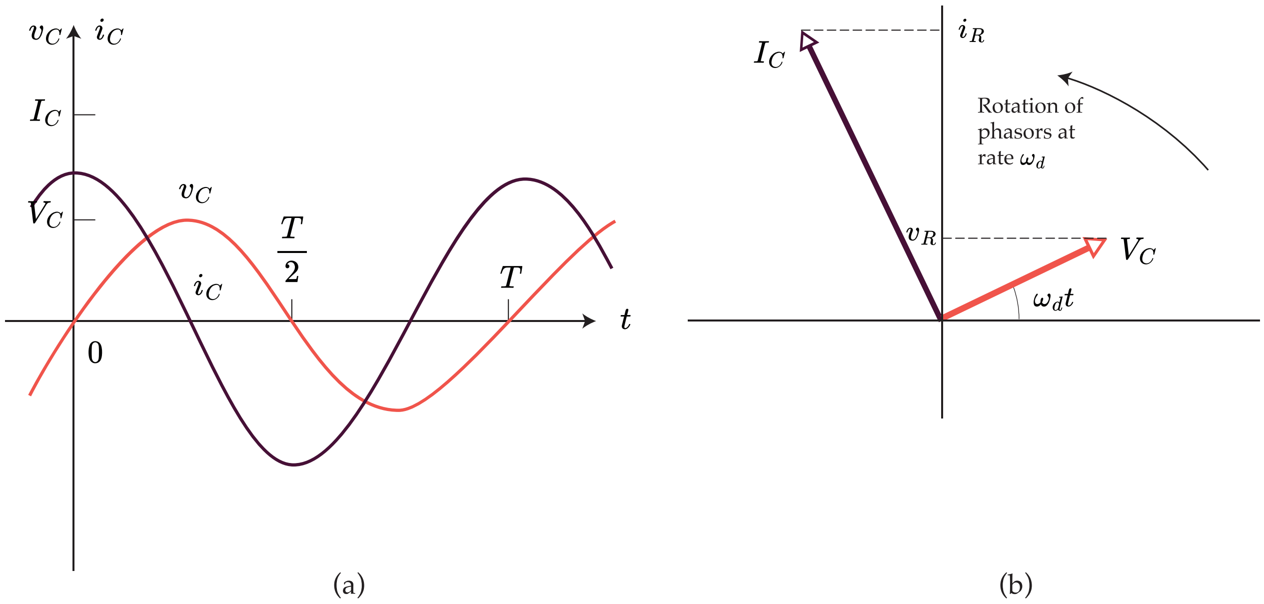


FIGURE 100: (a) The current i_C and the potential difference v_C across the capacitor are plotted on the same graph, both versus time t. The current in the capacitor leads the voltage by 90° in one period T. (b) A phasor diagram shows the same thing as (a).

32.3 Phasor Diagrams of the Three Simple Circuits

We try to represent the following table full of information in their respective phasor diagrams.

Components	Voltage	Current	Reactance	Phase Relations	Phase Angle (I with V)
Resistor	V_R	I_R	R	I_R in phase with v_R	0°
Inductor	V_L	I_L	$X_L = \omega_d L$	I_L *lags* V_L	+90°
Capacitor	V_C	I_C	$X_C = \dfrac{1}{\omega_d C}$	I_C *leads* V_C	−90°

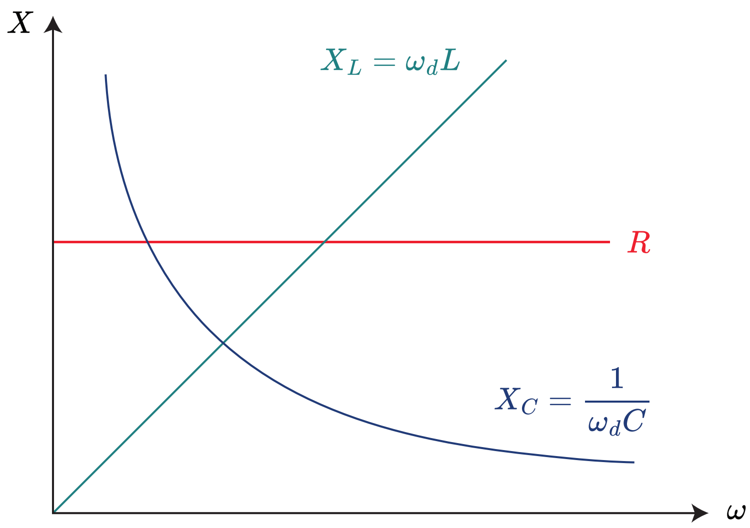


FIGURE 101: Circuit element reactances X variations with time t in an alternating circuit.

32.4 The Series RLC Circuits in Alternating Source

We are now finally ready to see what happens when all three circuit elements come together in a circuit where the source has an alternating EMF.

Consider the very first circuit of this chapter (Figure 94), and use KVL to find:

$$\begin{aligned} \mathcal{E}(t) &= V_R + V_L + V_C \\ \mathcal{E}_m \sin(\omega_d t) &= IR + L\frac{di}{dt} + \frac{q}{C} \end{aligned} \tag{265}$$

Differentiating Eq. (265) once with t we get the following:

$$-\omega_d \mathcal{E}_m \sin(\omega_d t) = R\frac{dI}{dt} + L\frac{d^2 i}{dt^2} + \frac{1}{C}\int I dt \tag{266}$$

This is a 2nd order linear ordinary differential equation with variable coefficients. To solve it, we assume a solution of the form $I = I_m \cos(\omega_d t + \phi)$, where I_0 is the amplitude of the alternating current and ϕ is the phase angle.

Substituting this into Eq. (266) and simplifying:

$$-\omega_d \mathcal{E}_m \sin(\omega_d t) = -\omega_d R I_m \sin(\omega_d t + \phi) - \omega_d^2 L I_m \cos(\omega_d t - \phi) + \frac{I_m}{C} \tag{267}$$

A simple matching of coefficients gives us the following relationship:

$$\omega_d R I_m = \omega_d \mathcal{E} \tag{268}$$

$$\omega_d^2 L I_m = \omega^2 C I_m. \tag{269}$$

Solving for I gets us:

$$I_m = I = \frac{\mathcal{E}}{\sqrt{R^2 + \left(\omega_d L - \frac{1}{\omega_d C}\right)^2}} \tag{270}$$

This equation describes the amplitude of the current through the circuit in response to the external sinusoidal voltage source. The current amplitude depends on the impedance Z of the circuit, given by:

$$Z = \sqrt{R^2 + \left(\omega_d L - \frac{1}{\omega_d C}\right)^2} = \sqrt{R^2 + (X_L - X_C)^2}. \tag{271}$$

Special Cases

- $X_L = X_C$: $Z = R \rightarrow$ circuit is resistive
- $X_L > X_C$: $R \leq (X_L - X_C) \rightarrow$ circuit is inductive
- $X_L < X_C$: $R \geq (X_L - X_C) \rightarrow$ circuit is capacitive

We can overlap the (b) diagrams (96, 98, 100) to get a single phasor diagram representing the series RLC circuit altogether.

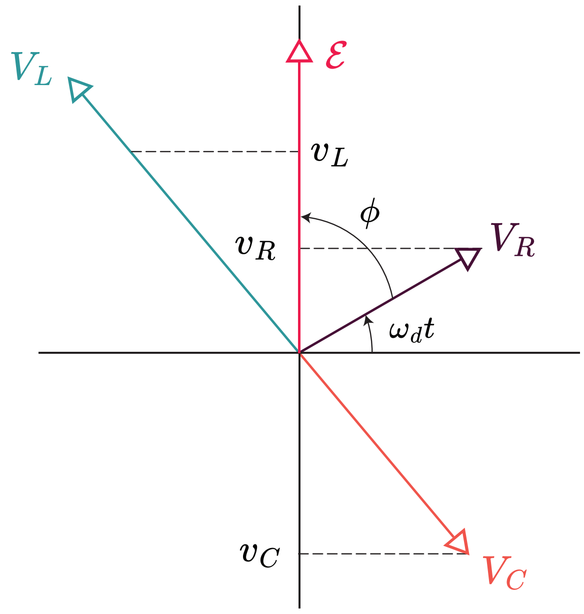


FIGURE 102: A capacitor is connected across an alternating-current generator.

The Phase Constant

For an RLC circuit or any oscillatory system, the phase constant in the phasor diagram is crucial in determining the behavior of the circuit. It determines the phase relations between the source EMF and the resistive voltage. It influences how the current and voltage across different circuit elements are related to each other, affecting the circuit's overall response.

$$\tan\phi = \frac{V_L - V_C}{V_R} = \frac{IX_L - IX_C}{IR} = \frac{X_L - X_C}{R}. \tag{272}$$

Special Cases

- $X_L = X_C$: $\phi = 0^\circ \rightarrow V_R$ is in phase with I; Resonance; $\omega_d = \omega_0$
- $X_L > X_C$: $\phi \equiv +\text{ve} \rightarrow V_L$ leads I
- $X_L < X_C$: $\phi \equiv -\text{ve} \rightarrow V_L$ lags I

32.5 Power in Alternating Circuits

Recall the equation relating instantaneous power dissipated across a resistor by a current:

$$\begin{aligned} P &= i^2 R \\ &= [I\sin(\omega_d t - \phi)]^2 R \\ P &= I^2 R \sin^2(\omega_d t - \phi). \end{aligned} \tag{273}$$

The average rate at which energy is dissipated in the resistor, however, is the average of Eq. (273) over time. Over one complete cycle, the average value of $\sin\theta$, where θ is any variable, is zero. However, the average value of $\sin^2\theta$ is $\frac{1}{2}$.

$$P_{\text{avg}} = (I^2 R) \times \frac{1}{2} = \left(\frac{I}{\sqrt{2}}\right)^2 R. \tag{274}$$

The term in the bracket is labelled the **root-mean-square**, or **RMS**, value of the current I. The root-mean-square (RMS) value is a way of expressing the effective or average value of an alternating current (AC) voltage, current, or power in a way that's comparable to the direct current (DC) values.

Thus, Eq. (274) can now be written as

$$P_{\text{avg}} = I_{\text{RMS}}^2 R. \tag{275}$$

We can also define RMS values of voltages and EMFs for alternating-current circuits:

$$V_{\text{RMS}} = \frac{V}{\sqrt{2}}. \tag{276}$$

$$\mathcal{E}_{\text{RMS}} = \frac{\mathcal{E}_m}{\sqrt{2}}. \tag{277}$$

Because the proportionality factor in Eqs. (274) and (276-277) is the same for all three variables, we can write

$$I_{\text{RMS}} = \frac{\mathcal{E}_{\text{RMS}}}{\sqrt{R^2 + \left(\omega_d L - \dfrac{1}{\omega_d C}\right)^2}}. \tag{278}$$

Part IV: A teensy Glimpse into Electrodynamics

33 Maxwell's Equations and Where to Find Them

Maxwell's four fundamental equations describe the behaviour of electric and magnetic fields and their interactions. These equations, formulated by the Scottish physicist James Clerk Maxwell in the 19$^{\text{th}}$ century, played a pivotal role in unifying the study of electricity and magnetism and laid the foundation for the field of electromagnetism.

In fact, we have been building them throughout this course. They come in the form of Gauss's Law [79, 168], Faraday's Law [213], and Ampère' Law [206, 286]. Four equations there are, two for $\vec{E}$-fields and two for $\vec{B}$-fields. Two measure the outflow of the two fields out of a closed surface, and the other two measure the circulation of the fields. Literally, everything we derived and *suffered* through this course can be boiled down to these four elegant equations.

But the story is not over yet. These equations hold keys to yet another paradigm-shifting outcome. You would be surprised. I know I was when I had first found out.

33.1 Displacement Current and Ampère-Maxwell Law

Displacement current is a concept in electromagnetism that was introduced by James Clerk Maxwell to explain certain behaviors in electric circuits. It's not an actual flow of charge like current in a wire, but rather, it's a mathematical term that accounts for changing electric fields in the context of Maxwell's equations.

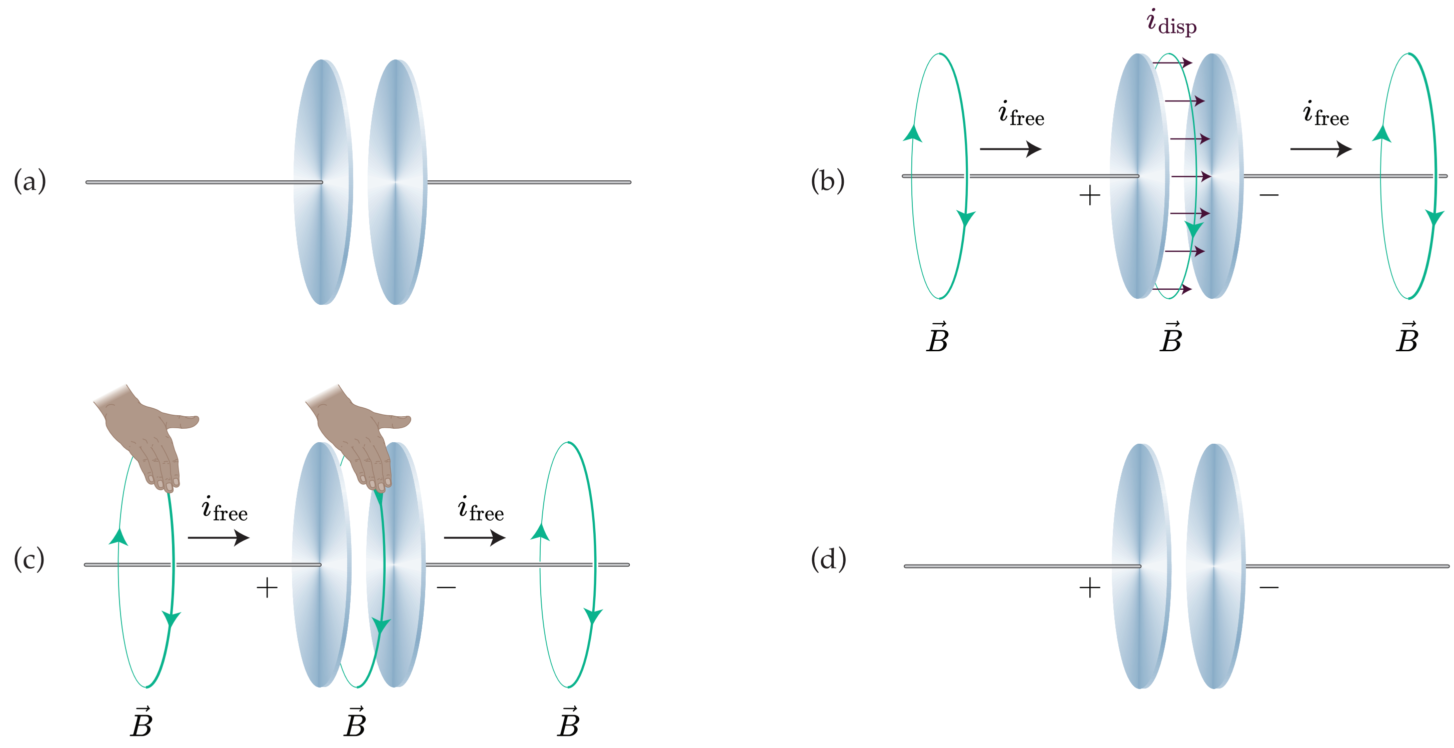


FIGURE 103: (a) Before and (d) after the plates are charged, there is no magnetic field. (b) During the charging, a magnetic field is created by both the real and the (fictional) displacement currents. (c) The same right-hand rule works for both currents to give the direction of the magnetic field.

Consider a circular parallel plate capacitor setup in a circuit. When the capacitor charges, the field in between the plates $\vec{E}$ also increases. The charge q on the plates at any time is related to the magnitude E of the field between the plates at that time, and the plate area A by

$$q = \epsilon_0 A E. \tag{279}$$

Differentiating this equation once with time gives the variation of $\vec{E}$ and a fictitious current between the plates. This current is fictitious because there is no actual flow of charge carriers in between the plates, unlike the conventional current. This fictitious current is called the *displacement current*.

$$i_{\text{disp}} = \frac{dq}{dt} = \epsilon_0 \frac{d(EA)}{dt} = \epsilon_0 \frac{d\Phi_E}{dt}. \tag{280}$$

This current is generated when there's a variation in $\vec{E}$ without the involvement of moving charge carriers. Maxwell added this current contribution to *fix* the Ampère's law.

With this Maxwell correction, Ampère's Law states:

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \left(i_{\text{free}} + i_{\text{disp}}\right) \tag{281}$$

33.2 Then God said the Beautiful Maxwell's Equations

- **Gauss's Law for Electric Fields**: Electric fields are generated by isolated electric charges. Electric field lines originate from positive charges and terminate on negative charges. The differential form claims that the total electric flux (flow of electric field lines) through a closed surface is proportional to the enclosed electric charge. The integral form claims that field lines diverge from a positive charge source and converge for a negative charge.

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho_{\text{enc}}}{\varepsilon_0} \qquad \text{(Differential Form)} \tag{282}$$

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{\text{enc}}}{\varepsilon_0} \qquad \text{(Integral Form)} \tag{283}$$

- **Gauss's Law for $\vec{B}$-Fields**: Unlike electric charges, there are no magnetic monopoles (sinks or sources) in nature. Magnetic field lines always form closed loops, as they cannot start or end on a single magnetic pole. The total magnetic flux through any closed surface is always zero, emphasizing that magnetic field lines always form closed loops.

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \text{(Differential Form)} \tag{284}$$

$$\oint \vec{B} \cdot d\vec{A} = 0 \qquad \text{(Integral Form)} \tag{285}$$

- **Faraday's Law of Electromagnetic Induction**: A changing magnetic field creates an electric field. When the magnetic field through a closed loop changes, it induces an electric field that drives current in the loop. When a moving magnet nears a coil of wire, the changing magnetic field induces a flow of electrons (current) in the wire.

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \text{(Differential Form)} \tag{286}$$

$$\oint \vec{E} \cdot d\vec{l} = -\frac{\partial}{\partial t} \oint \vec{B} \cdot d\vec{A} \qquad \text{(Integral Form)} \tag{287}$$

- **Ampère's Law with Maxwell's Correction**: Both electric currents and changing electric fields contribute to the creation of magnetic fields.

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \qquad \text{(Differential Form)} \tag{288}$$

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{free}} + \mu_0 \epsilon_0 \oint \vec{E} \cdot d\vec{A} \qquad \text{(Integral Form)} \tag{289}$$

33.3 And There Was Light

Electromagnetic waves are self-propagating waves that consist of electric and magnetic fields oscillating perpendicular to each other and perpendicular to the direction of wave propagation. These waves are produced by the acceleration of charged particles or by varying electric and magnetic fields. **They are Light** as we know them to be. The light that we can see and the light we can't see.

Electromagnetic waves span a wide range of frequencies and wavelengths, forming the electromagnetic spectrum. This spectrum includes various types of waves, such as radio waves, microwaves, infrared radiation, visible light, ultraviolet radiation, X-rays, and gamma rays. Each type of wave has a different frequency and wavelength range, and they all exhibit the same basic properties of electromagnetic waves.

By combining Maxwell's equations, we can obtain a wave equation that describes the propagation of electromagnetic waves. For the electric field, we get the following when we take the Laplacian of the field $\nabla^2\vec{F} = \vec{\nabla}\left(\vec{\nabla}\cdot\vec{F}\right) - \vec{\nabla}\times\left(\vec{\nabla}\times\vec{F}\right)$:

$$\nabla^2\vec{E} - \varepsilon_0\mu_0\frac{\partial^2\vec{E}}{\partial t^2} = 0. \tag{290}$$

For the magnetic field, we get the following:

$$\nabla^2\vec{B} - \varepsilon_0\mu_0\frac{\partial^2\vec{B}}{\partial t^2} = 0. \tag{291}$$

These equations have plane wave solutions of the form:

$$\vec{E}(\vec{r},t) = \vec{E}_{\text{max}}\sin(\vec{k}\cdot\vec{r} - \omega t) \tag{292}$$

$$\vec{B}(\vec{r},t) = \vec{B}_{\text{max}}\sin(\vec{k}\cdot\vec{r} - \omega t) \tag{293}$$

Substituting it into the wave equation, we get:

$$(-k^2 + \varepsilon_0\mu_0\omega^2)\vec{E}(\vec{r},t) = 0 \tag{294}$$

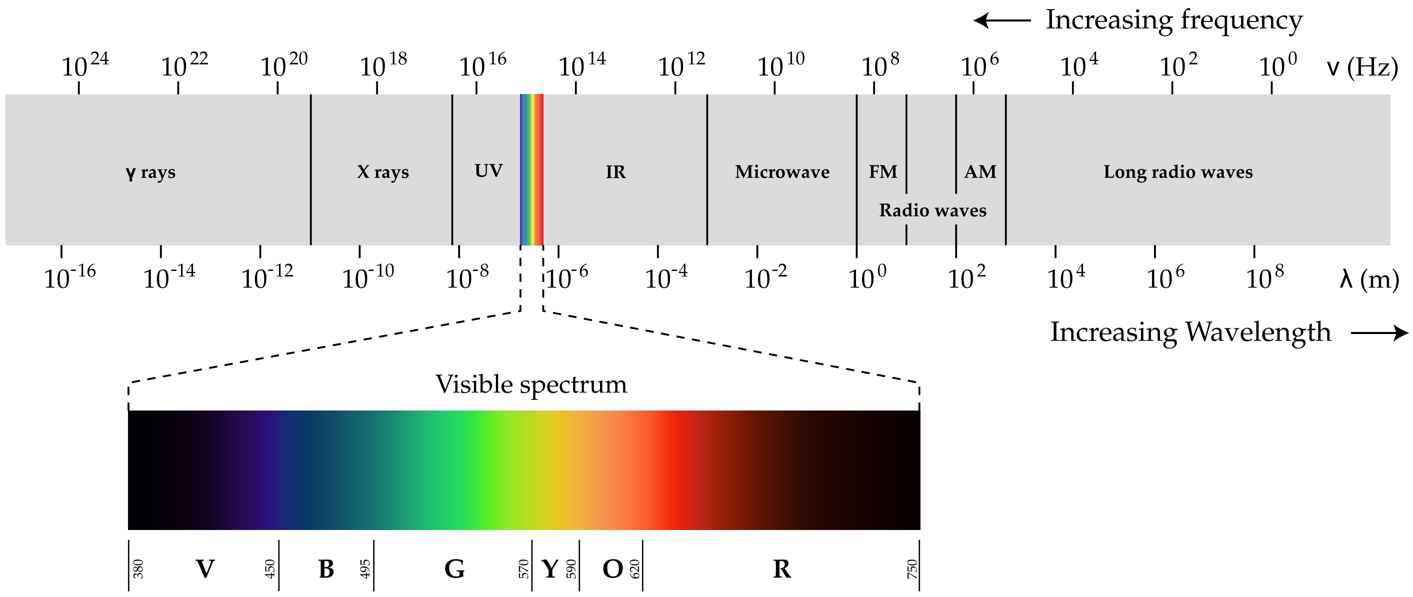
For this equation to hold true for all $\vec{r}$ and t, the term inside the parentheses must be equal to zero:

$$\begin{aligned} k^2 &= \varepsilon_0\mu_0\omega^2 \\ k &= \pm\omega\sqrt{\varepsilon_0\mu_0} \end{aligned} \tag{295}$$

Now, the speed of a wave is given by the ratio of the angular frequency ω to the wave number k:

$$\begin{aligned} v &= \frac{\omega}{k} = \frac{\omega}{\pm\omega\sqrt{\varepsilon_0\mu_0}} \\ v &= \frac{1}{\sqrt{\varepsilon_0\mu_0}} = c. \end{aligned} \tag{296}$$

This is the speed of electromagnetic waves in a vacuum, equal to light speed. Electromagnetic waves travel at the speed of light because light itself is an electromagnetic wave.



The journey is far from over here. If You ask a Physics major, actually this is where the story begins. But, I now must pull the curtain. We really ran out of time. Namárië.

The End!