Department of Mathematics and Natural Sciences

Pragati Sarani, Merul Badda, Dhaka, 1212

Lecture Notes on PHY-112

Principles of Physics-II

Instructor: Akiful Islam | **FIN**: AZW

E-mail: ext.akiful.islam@bracu.ac.bd | **Consultation Room**: Floor-4, Block-G

This course introduces Electromagnetism, a field of study that explores the interactions between Electric and Magnetic fields. Additionally, it delves into the Electromagnetic Force, recognized as one of the four fundamental forces of nature, and examines the reactions of charged particles when subjected to this force.

The primary topics to be covered in this course encompass Electric Charge, Force, Field, Potential, Capacitance, Resistance, Electric Current, Kirchhoff's laws, Ohm's Law, Electric power, Magnetic Force, Field, Flux, Induction, Faraday's Law, Lenz's Law, Ampère's Law, Gauss's Law, Maxwell's equations, Electromagnetic Oscillations, Impedance, Resonance, Transformers, and Electromagnetic Waves.

The course strongly emphasizes fostering an intuitive understanding of the core principles of Physics. This is accomplished through extensive problem-solving exercises of varying complexity. The course will utilize a practical, calculus-based methodology, making it imperative that students have a solid grasp of calculus before enrollment.

Disclaimer: These notes are based on the PHY-112 course at BracU, taught by the author/instructor during the Summer, Fall 23, and Spring 24 (ongoing) semesters. The author does not assert any originality or copyright ownership over this work. The primary sources for these notes were:

1. Halliday, D., Resnick, R., & Walker, J. (2023). *Principles of Physics*, (Extended, 10th Edition).
2. Young, Hugh D., Roger A. Freedman. *University Physics with Modern Physics*, (15th Edition), 2020.

These notes are not abstract/rigorous for a Physics/Mathematics major. To err is human. Please report any errors found to ext.akiful.islam@bracu.ac.bd.

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Prerequisites I: Vector Algebra

Vector algebra deals with the manipulation and analysis of vectors. It includes the operations of vector **addition**, **subtraction**, scalar **multiplication**, dot (aka scalar) product, and cross (aka vector) product. It is assumed that students have a strong foothold on the prerequisites from having done PHY-111 (Principles of Physics-1) before.

Vector Addition and Subtraction

Vector addition is the process of adding two or more vectors together to **form a new vector**. The sum of two or more vectors is the **resultant vector**.

Vector **subtraction** is subtracting one vector from another. It is the reverse operation of vector addition.

Analytical Approach of Vector Addition

The idea is to resolve all the vectors *to be summed* into their x and y-coordinate axes components. Then, add *like* components ($\hat{\imath}$ with $\hat{\imath}$, $\hat{\jmath}$ with $\hat{\jmath}$, $\hat{k}$ with $\hat{k}$) to construct the resultant vector.

We use the concept of *vector resolution*. It means to **resolve** or break apart the vector into all possible basis directions it can contribute to. These contributions are the *components* of the main vector. 3D parts are not shown in the diagrams below.

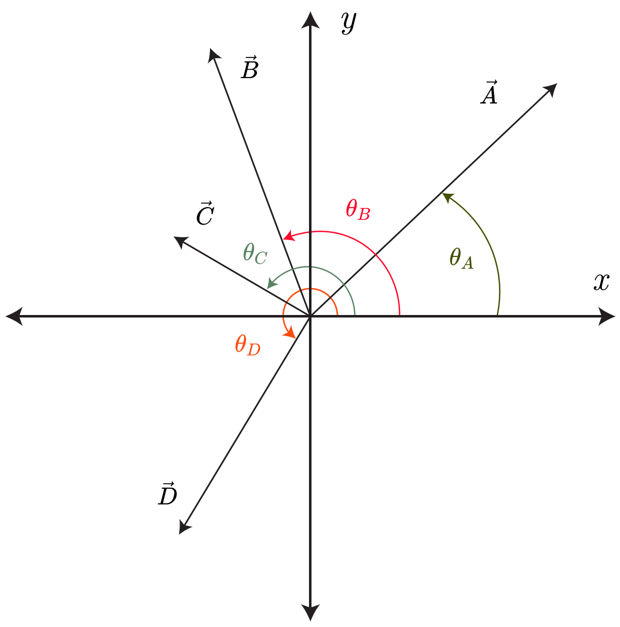


FIGURE 1: A system with four vectors is shown here.

Assume the following setup in Figure 1. We have four vectors $\vec{A}$, $\vec{B}$, $\vec{C}$, $\vec{D}$ making θ_A, θ_B, θ_C, θ_D with the $+x$-axis, according to our counter-clockwise convention of angle measurement.

Vectors	x-components	y-components
$\vec{A}$	$A_x \cos\theta_A$	$A_y \sin\theta_A$
$\vec{B}$	$B_x \cos\theta_B$	$B_y \sin\theta_B$
$\vec{C}$	$C_x \cos\theta_C$	$C_y \sin\theta_C$
$\vec{D}$	$D_x \cos\theta_D$	$D_y \sin\theta_D$

TABLE 1: Resolution of all four vectors into their coordinate components.

The resultant will have its x-component as the sum of all vector contributions in the x-direction and the y-component in the designated y-direction.

$$R_x = A_x \cos\theta_A + B_x \cos\theta_B + C_x \cos\theta_C + D_x \cos\theta_D$$
$$R_y = A_y \sin\theta_A + B_y \sin\theta_B + C_y \sin\theta_C + D_y \sin\theta_D$$

The direction of the resultant will be given by

$$\theta = \begin{cases} \tan^{-1}\left|\dfrac{R_y}{R_x}\right| & ;\ R_x > 0,\ R_y > 0\ (1^{\text{st}}\ \text{quadrant}) \\ \pi - \tan^{-1}\left|\dfrac{R_x}{R_y}\right| & ;\ R_x < 0,\ R_y > 0\ (2^{\text{nd}}\ \text{quadrant}) \\ \pi + \tan^{-1}\left|\dfrac{R_y}{R_x}\right| & ;\ R_x < 0,\ R_y < 0\ (3^{\text{rd}}\ \text{quadrant}) \\ 2\pi - \tan^{-1}\left|\dfrac{R_x}{R_y}\right| & ;\ R_x > 0,\ R_y < 0\ (4^{\text{th}}\ \text{quadrant}) \end{cases}$$

Nothing new here. It's just the same argument we used for the Polar to Cartesian coordinate transformation.

Geometric Approach of Vector Addition

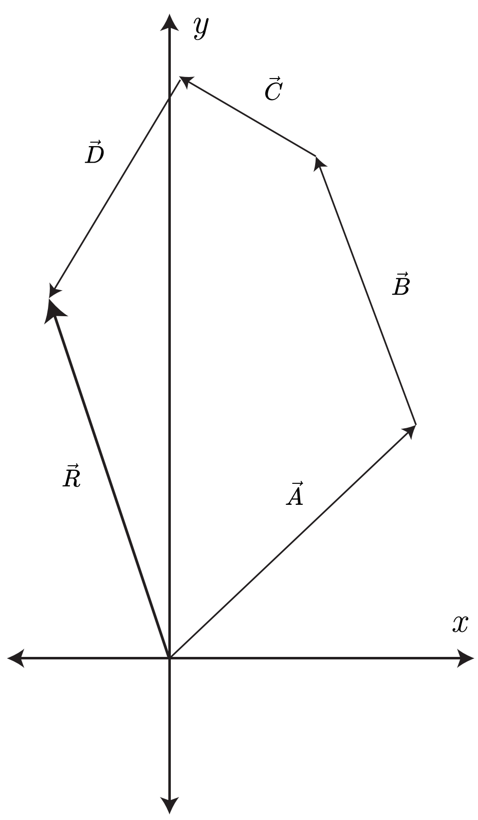


FIGURE 2: A system with four vectors is shown here.

The idea is to draw all the vectors according to the given problem and geometrically add them using the **tip-to-tail** method. The tip of the first vector will be on top of the **tail** of the second vector, followed by the **tip** of the second vector on top of the tail of the third vector, and so on and on until all vectors are accounted for.

Then, using the *geometric* vector sum to find the resultant vector. Assume the same setup above. Keep placing the vectors on top of each other using the tip-to-tail method. Finally, add a vector pointing from the tail of the first vector to the tip of the last vector in the series. That is the resultant vector. Although the analytical method of vector addition may not always place the resultant in its actual quadrant, the geometrical method does.

Vector Multiplication

The last and most crucial part of vector algebra. Unlike scalar multiplication, vectors follow a different concept when it comes to multiplication. We need to understand its physical intuition before doing any math.

Scalar (Dot) Multiplication

The dot product is a scalar quantity obtained by multiplying the magnitudes of the two vectors and the cosine of the angle between them. This multiplication takes the **projection** of one vector in a direction **parallel** to another vector.

For example, take the following vectors $\vec{A}$ and $\vec{B}$. Assume they make an angle θ with each other. There can be two ways we calculate scalar multiplication.

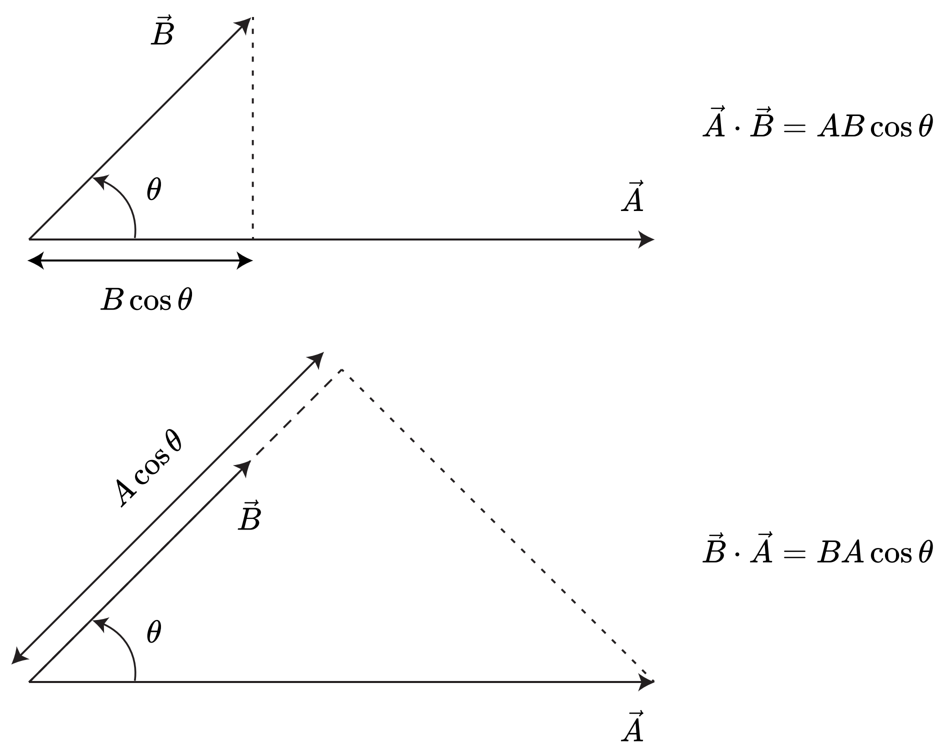


FIGURE 3: Geometrical definition of scalar multiplication.

Geometrical

Measure the magnitude of the vectors and simply plugin their value in the following formula:

$$\vec{A}\cdot\vec{B} = AB\cos\theta = BA\cos\theta = \vec{B}\cdot\vec{A} \tag{1}$$

$B\cos\theta$ is the projection of $\vec{B}$ in a direction parallel to $\vec{A}$.

Analytical

Similar to the vector addition, we first resolve the vectors in the system coordinate axes and then take the products of *like* components and sum all the products.

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z \tag{2}$$

Limiting Cases of Scalar Multiplication

1. $\theta = 0°$: The dot product is maximum.
2. $90° > \theta \geq 0°$: The dot product is **positive** in this range.
3. $\theta = 90°$: The dot product is zero.
4. $180° > \theta > 90°$: The dot product is **negative** in this range.
5. $\theta = 180°$: The dot product is minimum.
6. $270° > \theta > 180°$: The dot product is **negative** in this range.
7. $\theta = 270°$: The dot product is zero.
8. $360° \geq \theta > 270°$: The dot product is **negative** in this range.
9. $\theta = 360°$: The dot product is maximum. Superimposes with the 1^{st} case.

How to use the Dot Product?

1. If vectors are given in *vector* notation, You may use the **analytical** method to measure their scalar product.
2. If vectors are given in *verbal* notation, You may use the **geometrical** method to measure their scalar product.
3. In either notation, the angle between the involved vectors may be found using:

$$\theta = \cos^{-1}\left(\frac{\vec{A} \cdot \vec{B}}{AB}\right) = \cos^{-1}\left\{\frac{A_x B_x + A_y B_y + A_z B_z}{\left(\sqrt{A_x^2 + A_y^2 + A_z^2}\right)\left(\sqrt{B_x^2 + B_y^2 + B_z^2}\right)}\right\} \tag{3}$$

   Remember to place both vectors on top of each other in such a way that their tails overlap.

TAKEAWAY: Dot multiplication is commutative. That is, $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$.

Vector (Cross) Multiplication

The cross product is a vector quantity perpendicular to the plane of the two vectors being multiplied, and its magnitude is equal to the product of the magnitudes of the two vectors and the sine of the angle between them. The direction of the cross product is determined by the **right-hand rule** (Fig. 5). This multiplication takes the **projection** of one vector in a direction **perpendicular** to another vector.

Take the same two vectors $\vec{A}$ and $\vec{B}$ again, making an angle θ with each other. There are two ways we can calculate vector multiplication.

Analytical

Similar to the vector addition, we first resolve the vectors in the system coordinate axes and then take the products of all possible combinations of *unlike* components and follow the following format:

$$\vec{A}\times\vec{B} = \left(A_yB_z - A_zB_y\right)\hat{i} + \left(A_zB_x - A_xB_z\right)\hat{j} + \left(A_xB_y - A_yB_x\right)\hat{k} \tag{4}$$

$$= \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{pmatrix} \tag{5}$$

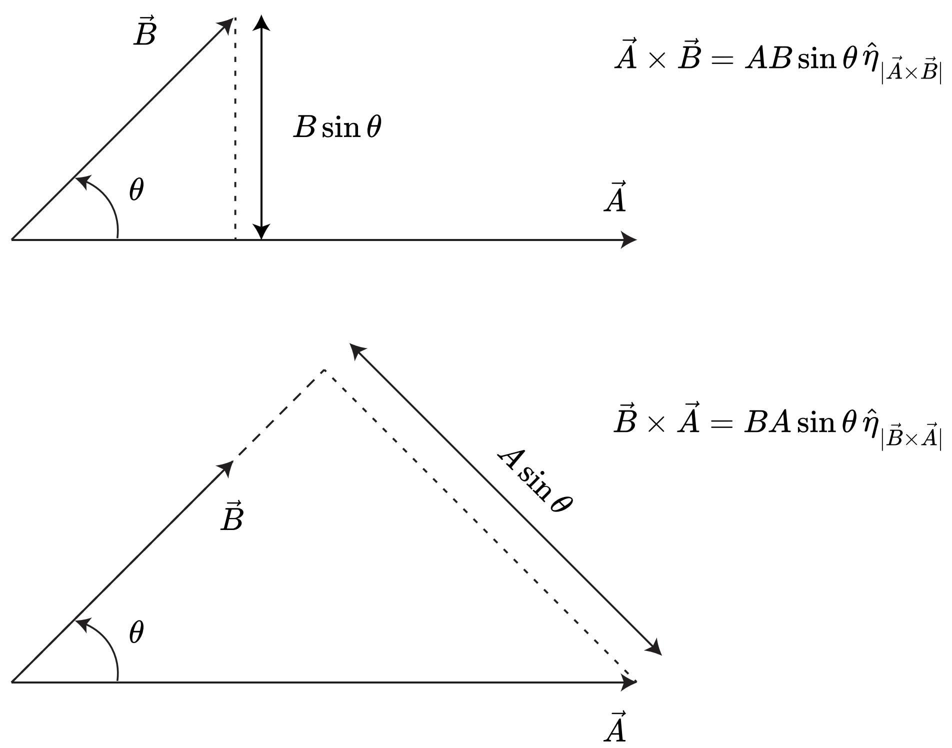


FIGURE 4: Geometrical definition of cross multiplication.

Geometrical

Measure the magnitude of the vectors and simply plugin their value in the following formula:

$$\vec{A}\times\vec{B} = AB\sin\theta\hat{\eta}_{|\vec{A}\times\vec{B}|} \tag{6}$$

$$\vec{B}\times\vec{A} = BA\sin\theta\hat{\eta}_{|\vec{B}\times\vec{A}|} \tag{7}$$

$$\hat{\eta}_{|\vec{A}\times\vec{B}|} = -\hat{\eta}_{|\vec{B}\times\vec{A}|} \tag{8}$$

$B\sin\theta$ is the projection of $\vec{B}$ in a direction perpendicular to $\vec{A}$. The cross product lies in the plane perpendicular to the plane where the parent vector lies in.

Limiting Cases of Vector Multiplication

1. $\theta = 0$: The cross product is zero.

2. $90^\circ \geq \theta > 0^\circ$: The cross product is **positive** in this range.

3. $\theta = 90°$: The cross product is maximum.
4. $180° > \theta > 90°$: The cross product is **positive** in this range.
5. $\theta = 180°$: The cross product is zero.
6. $270° > \theta > 180°$: The cross product is **negative** in this range.
7. $\theta = 270°$: The cross product is maximum in the opposite direction to the 3$^{\text{rd}}$ case.
8. $360° > \theta > 270°$: The cross product is **positive** in this range.
9. $\theta = 360°$: The cross product is zero. Superimposes with the 1$^{\text{st}}$ case.

How to use the Cross Product?

1. If vectors are given in *vector* notation, You may use the **analytical** method to measure their cross product.
2. If vectors are given in *verbal* notation, You may use the **geometrical** method to measure their cross-product.
3. In either notation, the angle between the involved vectors may be found using:

$$\theta = \sin^{-1}\left\{\frac{\left|\vec{A} \times \vec{B}\right|}{AB}\right\} \tag{9}$$

TAKEAWAY: Cross multiplication is anti-commutative. That is, $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$.

Rule of Thumb to Cross Multiplication

Unless explicitly mentioned otherwise, always **follow the right-handed system** Fig. 4 when doing maths with cross products.

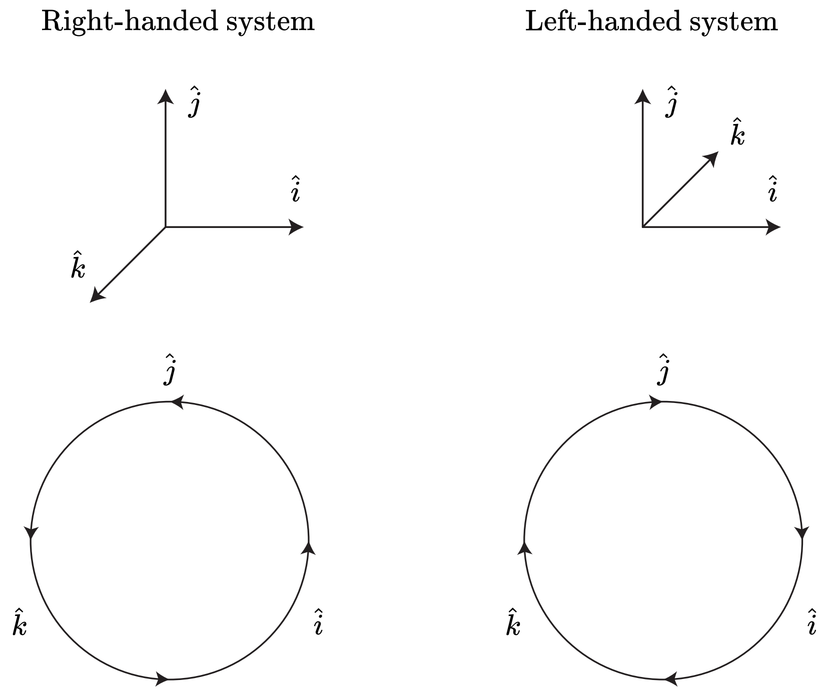


FIGURE 5: The coordinate axes unit vectors follow the cyclic relation in 3D space. The Left-handed system is just the mirror opposite of the Right-handed system. The reflection symmetry lies in the x-y plane.

Vector Division Makes no Sense

Vector division is impractical. Because unlike the mathematical operations of addition and subtraction, which can be applied to vectors with clear geometric interpretations, the operation of division does not have a precise geometric interpretation for vectors.

However, if we divide a vector with a positive scalar, the vector gets down-scaled, more like shrinks, while keeping the same direction. Do the same with a negative scalar; the vector shrinks in the opposite direction.

Prerequisites II: Fields

A field is a physical quantity that fills space and can vary in magnitude and direction at each point. Fields describe how different entities or properties, such as forces or energy, are distributed throughout space.

Fields are typically defined by their value at each point in space and can be represented mathematically as functions or vector quantities. The value of a field at a particular point determines the magnitude and direction of its effect on other objects or particles in that region.

Vector Fields

They are characterized by a vector quantity assigned to each point in space and time. A vector field can be visualized by imagining arrows at each point in space, where the direction and length of the arrow represent the direction and magnitude of the vector at that point. The vector field provides information about the direction and strength of the vector quantity at any given point in the region. Examples of vector fields include electric, magnetic, and velocity fields.

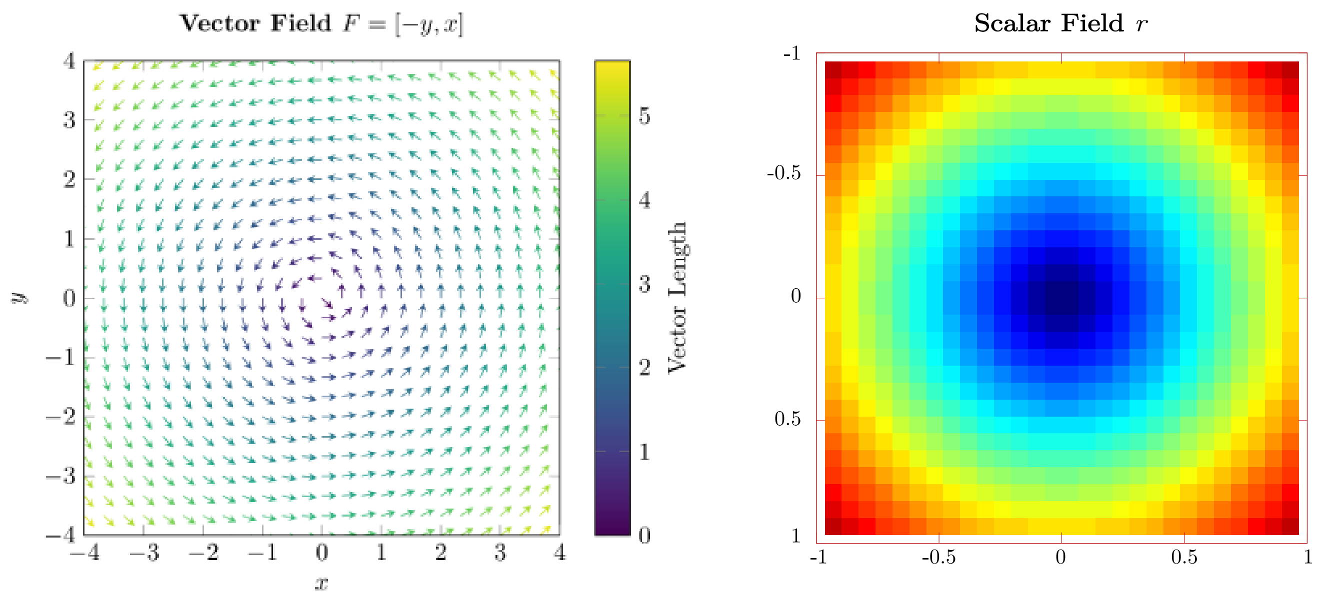


FIGURE 6: Vector fields require the magnitude and the direction at every point in the coordinate grid. The scalar field, however, needs only the magnitude represented by the pseudo color map.

Scalar Fields

They are characterized by a scalar quantity assigned to each point in space and time. Scalar fields can be visualized by imagining a numerical value assigned to each point in space. The value can vary from point to point, representing the magnitude of the scalar quantity at that specific location. Examples of scalar fields include temperature distribution, pressure distribution, and gravitational potential.

Vector Calculus

Vector calculus is a branch of mathematics that deals with vector fields and their derivatives. It provides a robust framework for describing and analyzing various physical phenomena that involve quantities with both magnitude and direction, such as forces, electric and magnetic fields, fluid flow, and more.

There are three main components of vector calculus: gradient, divergence, and curl. Each of these operations captures different aspects of vector fields and offers valuable insights into the behavior and properties of these fields.

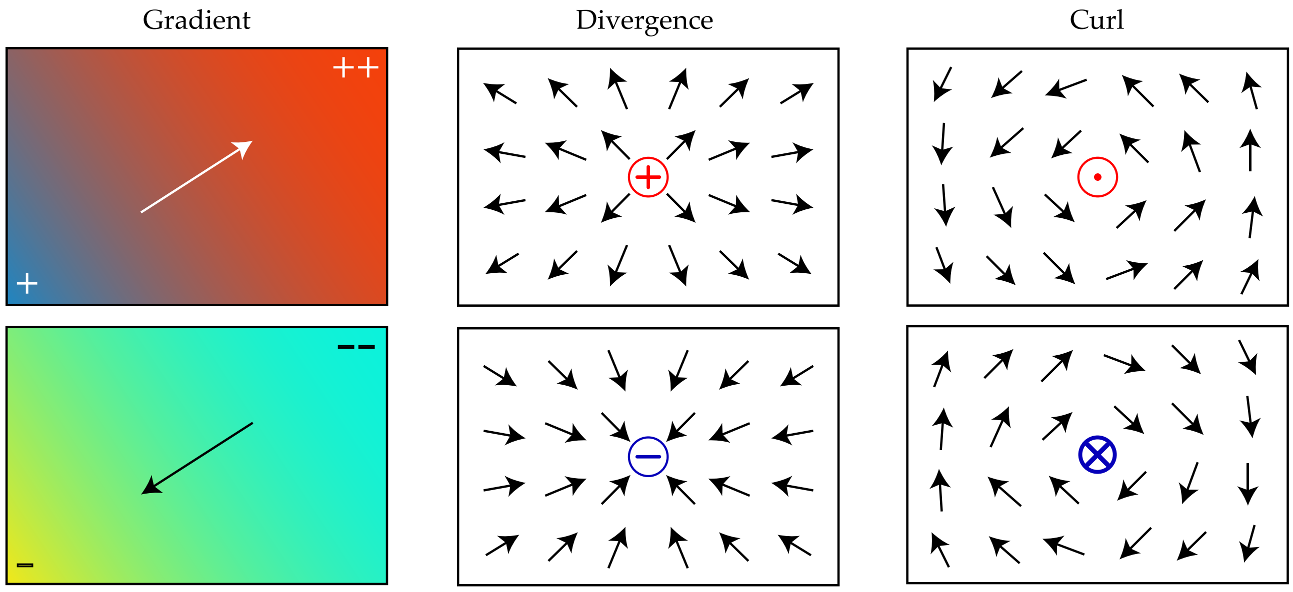


FIGURE 7: (Left) Gradient measures a scalar field's spatial change/rate. (Middle) Divergence measures the divergence and convergence of the vector field from a point source. (Right) Curl measures the vector field's rotation around a point source.

Gradient

The gradient represents the spatial variation or the rate of change of a scalar field. A scalar field assigns a scalar value (e.g., temperature, pressure, electric potential) to every point in space. The gradient of a scalar field yields a vector field that points in the direction of the steepest increase of the scalar field. In simple terms, the gradient tells us how and in which direction a scalar quantity changes rapidly.

$$\vec{\nabla}\,(\text{Scalar Field}) \to \text{Vector}. \tag{10}$$

The gradient helps us understand how a scalar quantity changes in space, providing insights into heat, fluid, or electric potential flow.

Divergence

The divergence measures the outflow of a vector field, i.e., the tendency of a vector field to flow outward or inward from a given point. It quantifies the spreading or convergence of vector field lines. If the divergence at a point is positive, the vector field is **diverging** or spreading out from that point. If the divergence is negative, it suggests that the field is **converging** or coming together at that point. The divergence is associated with the flow of a vector field through a closed surface and is related to the presence of sources or sinks within that region.

$$\begin{aligned} \vec{\nabla}\cdot(\text{Vector Field}) &\to \text{Scalar}, \\ \text{where } \vec{\nabla}\cdot(\text{Vector Field}) &> 0 \Rightarrow \text{Diverging field} \\ \vec{\nabla}\cdot(\text{Vector Field}) &< 0 \Rightarrow \text{Converging field} \\ \vec{\nabla}\cdot(\text{Vector Field}) &= 0 \Rightarrow \text{Solenoidal field} \end{aligned} \tag{11}$$

The divergence informs us about the behavior of a vector field in terms of its sources (positive divergence) or sinks (negative divergence), which is crucial for understanding fluid flow, electric charge distribution, or the behavior of gravitational fields.

Curl

The curl measures the circulation of a vector field, i.e., the tendency of a vector field to circulate or form vortices around a given point. It characterizes the rotational behavior of a vector field. If the curl at a point is non-zero, it signifies the presence of swirling motion or rotation in the vector field. The curl is associated with the circulation of a vector field around a closed loop and is often encountered in the study of fluid dynamics and electromagnetism.

$$
\begin{aligned}
\vec{\nabla} \times (\text{Vector Field}) &\rightarrow \text{Vector}, \qquad (12)\\
\text{where } \vec{\nabla} \times (\text{Vector Field}) > 0 &\Rightarrow \text{Counterclockwise rotation}\\
\vec{\nabla} \times (\text{Vector Field}) < 0 &\Rightarrow \text{Clockwise rotation}\\
\vec{\nabla} \times (\text{Vector Field}) = 0 &\Rightarrow \text{No rotation}
\end{aligned}
$$

The curl captures the rotational aspects of a vector field, shedding light on phenomena such as fluid vortices or the behavior of magnetic fields around current-carrying wires.

Types of Integrals and Vector Calculus

Line, surface, and volume integrals are mathematical tools for calculating physical quantities associated with one-dimensional lines, two-dimensional surfaces, and three-dimensional volumes.

Line Integrals

They are also known as **path integrals**. They are used to calculate quantities along curves or paths in space. They are typically used to determine quantities associated with a line, such as work, circulation, or flux.

A line integral can be visualized as **the addition of** the contributions of a quantity along a curve. Think of walking along a curved path and measuring some physical quantity as you move. The line integral accumulates the contributions of the quantity along the entire path, taking into account the direction and magnitude at each point.

Surface Integrals

They are used to calculate quantities that are distributed over two-dimensional surfaces. They are commonly employed to determine flux, representing the flow of a vector quantity through a surface.

Imagine a fluid flowing through a surface to grasp the physical intuition behind surface integrals. The surface integral calculates the total amount of fluid passing through the surface, considering the direction and magnitude of the flow at each point on the surface. It measures how much vector quantity passes through the surface per unit area.

Volume Integrals

They are used to calculate quantities associated with three-dimensional volumes. They are frequently used to determine quantities like total mass, charge, energy, or electric current.

Physically, volume integrals can be understood as **the summation of** the contributions of a quantity throughout a three-dimensional region. Picture a quantity distributed within a solid object. The volume integral integrates the contributions of the quantity over the entire volume, considering the variation of the quantity within the region.

Stoke's Theorem

This theorem relates **the circulation of a vector field around a closed curve to the flux of the curl of the vector field through the surface bounded by the curve**. It establishes **a connection between a line integral around a closed curve and a surface integral over the region bounded by that curve**.

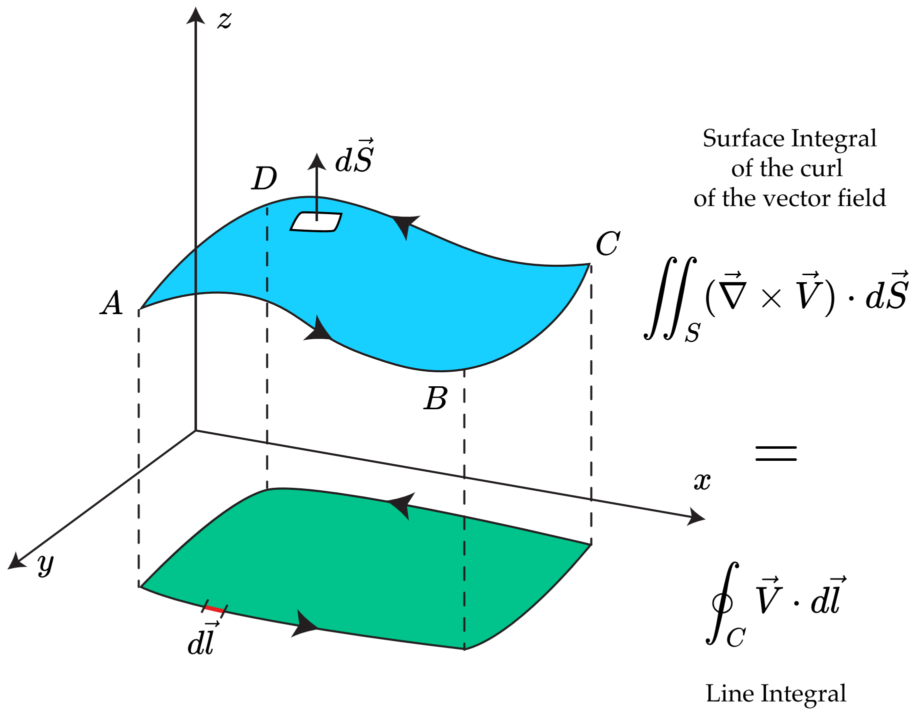


FIGURE 8: Stoke's theorem states the contribution of a vector field through a surface measured across the boundary line of that surface is the same if it were measured using the surface integral of the curl of the vector field.

Mathematically, this theorem is stated as follows:

$$\oint_C \vec{V} \cdot d\vec{l} = \iint_S (\vec{\nabla} \times \vec{V}) \cdot d\vec{S}, \tag{13}$$

where $\oint_C$ represents the line integral around the closed curve C, V is the vector field, dr is the differential vector along the curve, $\iint_S$ represents the surface integral over the surface S bounded by the curve, curl $\vec{\nabla} \times \vec{V}$ is the curl of the vector field V, and dS is the differential surface area vector.

Divergence Theorem

Also known as **Gauss' theorem**, it relates **the flux of a vector field through a closed surface to the divergence of the vector field within the volume enclosed by that surface**. It establishes a **connection between a surface integral over a closed surface and a volume integral over the region bounded by that surface**.

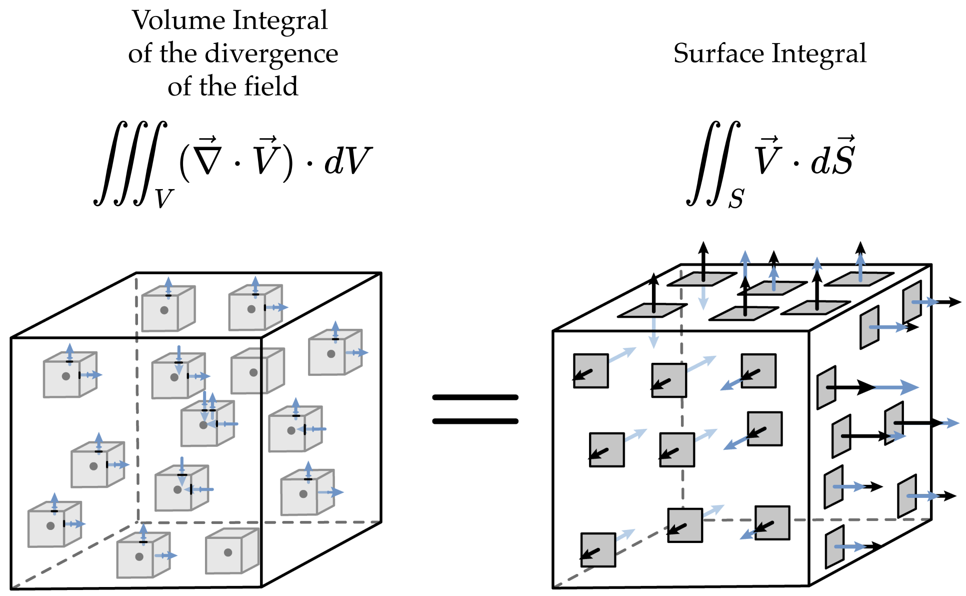


FIGURE 9: Divergence theorem states the contribution of a vector field through a surface measured using the surface integral is the same if it were measured using the volume integral of the divergence of the vector field.

Mathematically, this theorem is stated as follows:

$$\iint_S \vec{V} \cdot d\vec{S} = \iiint_V (\vec{\nabla} \cdot \vec{V}) \cdot dV, \tag{14}$$

Where $\iint_S$ represents the surface integral over the closed surface S, V is the vector field, dS is the differential surface area vector, $\iiint_V$ represents the volume integral over the region V enclosed by the surface, $\vec{\nabla} \cdot \vec{V}$ is the divergence of the vector field V, and dV is the differential volume element.

Part I: $\vec{E}$-fields and all that

1 Electrostatics

Electrostatics is a branch of physics that studies electric charges at rest and their interactions. This also includes the study of charge distribution that stays static in time. It focuses on the behavior of stationary electric charges and the electric fields they create.

1.1 Electric Charge

Electric charge is a **fundamental property of matter** that determines how objects interact through electromagnetic forces. It is a property carried by particles such as protons and electrons, which are the building blocks of atoms.

At its core, electric charge measures the imbalance of positive and negative particles within an object. It is denoted by the symbol q and is typically measured in coulombs (C) units. There are two types of electric charges: positive and negative.

NOTE: **The magnitude of the charge of the electron or proton is a natural unit of charge**.

What Makes Positive Charges *positive* and Negative Charges *negative*?

Electric charges determine the ability of elementary particles to experience electromagnetic forces. Objects with like charges (either positive or negative) repel each other, while objects with opposite charges attract each other. This fundamental force of attraction and repulsion is responsible for various phenomena we encounter in everyday life, from the spark produced by static electricity to the operation of electronic devices. **That's it!!! The signs of electric charges are just a convention**. No divine intervention here.

1.2 Properties of Electric Charge

- Conservation of Charge: Charge is conserved, meaning it cannot be created or destroyed. The total amount (algebraic sum) of charge in an isolated system remains constant. This principle allows us to analyze charge interactions and predict the outcomes of various electrical processes.
- Quantization of Charge: Electric charge comes in discrete units. An electron carries the smallest unit of charge and is equal to approximately -1.602×10^{-19} coulombs. This quantization explains why we observe whole number multiples of the electron's charge in most charge interactions.
- Additivity of Charge: The total charge of an object is the algebraic sum of the individual charges within it. Charges can be added or subtracted, leading to a net charge for the object. For example, if we have two objects with charges +2C and -1C, the net charge of the system is +1C.
- Charge Interactions: Charged objects exert forces on each other through electric fields. Like charges repel each other, while opposite charges attract. These interactions are fundamental to understanding the behavior of charged particles and the formation of electric fields.
- Coulomb's Law: Coulomb's Law describes the force between two charged objects. It states that the force between two charges is directly proportional to the product of their magnitudes and inversely proportional to the square of the distance between them. This mathematical relationship helps quantify the strength of the electrostatic force.

2 Electric Charge Distribution

A *point charge*, denoted q, is an idealized charged object with negligible physical size. Any system larger than a single point charge is called a *charge distribution*. These are usually classified as either *discrete* and/or *continuous* distributions.

2.1 Discrete Charge Distribution

A discrete charge distribution is modeled as a collection of individual point charges. The total charge of such a distribution is the algebraic sum of individual point charges.

$$Q = \sum_{i}^{N} q_i. \tag{15}$$

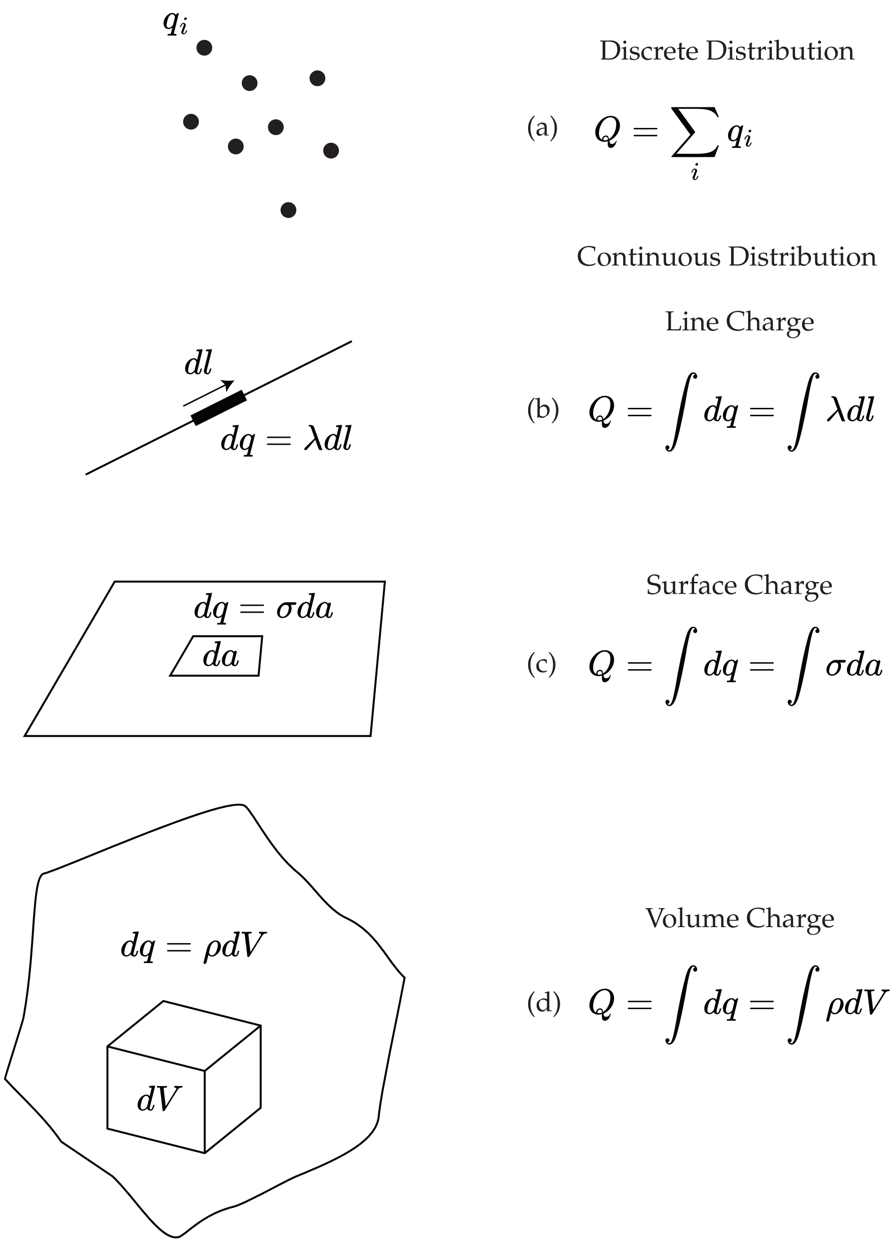


FIGURE 10: (a) Discrete charge distribution. The total charge is the sum of individual point charges. (b) Continuous charge distributions. The total charge is found by integrating over the charge elements, dq.

2.2 Continuous Charge Distribution

A continuous charge distribution is modeled as charge spread *continuously* over a spatial region. We may split the system into infinitesimal elements of charge, dq, and treat these elements as point charges. The total charge is then found by integrating over the elements:

$$Q = \int dq. \tag{16}$$

Charge Densities

While dealing with a continuous charge distribution, we can have the distribution in all three possible spatial dimensions. It is convenient to express the element sq in terms of *charge density*, then.

A line charge λ [$\mathrm{C\,m^{-1}}$], is a charge per unit length:

$$\lambda = \frac{dq}{dl}. \tag{17}$$

A surface charge σ [$\mathrm{C\,m^{-2}}$], is a charge per unit area:

$$\sigma = \frac{dq}{da}. \tag{18}$$

A volume charge ρ [$\mathrm{C\,m^{-3}}$], is a charge per unit volume:

$$\rho = \frac{dq}{dV}. \tag{19}$$

Elements of charge and the total charge of a continuous distribution are thus written as:

$$\text{distribution} \equiv \begin{cases} \dfrac{dq}{dl} & \text{in one dimension (1D)} \\ \dfrac{dq}{da} & \text{in two dimension (2D)} \\ \dfrac{dq}{dV} & \text{in three dimension (3D)} \end{cases} \tag{20}$$

$$Q = \int dq = \begin{cases} \lambda dl & \text{in one dimension (1D)} \\ \sigma da & \text{in two dimension (2D)} \\ \rho dV & \text{in three dimension (3D)} \end{cases} \tag{21}$$

3 Coulomb's Law of Electrostatics

Coulomb's Law is a fundamental principle in physics that describes the force between two electrically charged particles. It states that **the magnitude of the electrostatic force between two charged particles is directly proportional to the product of their charges and inversely proportional to the square of the distance between them**.

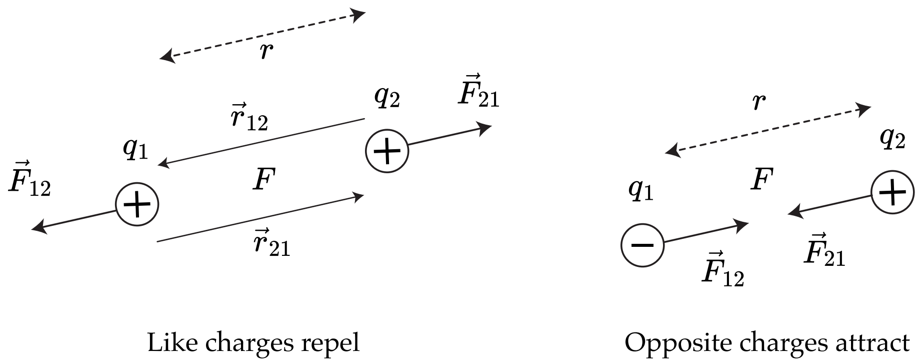


FIGURE 11: Electrostatic force $\vec{F}_E$ between two point charges q_1 and q_2. This force can be attractive or repulsive depending on the signs of the point charges.

Intuitively, Coulomb's Law can be understood as follows: If you have two charged objects, whether they are positively or negatively charged, they will either attract or repel each other. The strength of this attraction or repulsion depends on the magnitude of the charges and the distance between them. Like charges (positive-positive or negative-negative) repel each other, while opposite charges (positive-negative) attract each other.

Mathematically, Coulomb's Law (Only magnitude) can be expressed as:

$$F_E = \frac{1}{4\pi\epsilon_0}\frac{|q_1 q_2|}{r^2}, \tag{22}$$

where F is the magnitude of the electrostatic force between the charges q_1 and q_2, r is the distance between them, and $k = \dfrac{1}{4\pi\epsilon_0} = 8.998 \times 10^9\,\mathrm{N\,m^2C^{-2}}$ is the electrostatic constant.

In vector form, the equation reduces to

$$\vec{F}_E = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^3}\vec{r}. \tag{23}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\hat{r} \tag{24}$$

Comparing Electrostatic Force with Gravitational Force

Coulomb's Law has a similar form to Newton's Law of gravitation, which gives the gravitational force, $\vec{F}_G$, exerted on a mass m_2 by a mass m_1 a distance r apart:

$$\vec{F}_G = \frac{G m_1 m_2}{r^2}\hat{r}. \tag{25}$$

Since mass is always positive, the force of gravity is always attractive. Electric charges can be positive or negative, resulting in repulsive or attractive electrostatic forces.

If we calculate the ratio of the magnitudes of the gravitational force to the electrostatic force, we get the following:

$$\left|\frac{\vec{F}_G}{\vec{F}_E}\right| = 4\pi\epsilon_0 G\left(\frac{m_q}{q}\right)^2. \tag{26}$$

We can get an approximation for this by setting the mass and charge of elementary particles like the proton or electron we get:

$$\left|\frac{\vec{F}_G}{\vec{F}_E}\right| \sim 10^{-36}. \tag{27}$$

This means it is safe to ignore gravitational effects in many electrostatic setups.

TAKEAWAY: Each and every one of us possesses such a strong force on an elementary level, keeping our existence together that trumps the force that binds the galaxy and stars together.

The Principle of Superposition for Coulomb Forces

The total electric force on a charged particle with charge Q is the vector sum of the individual forces exerted on that particle by other n amounts of charged particles in the distribution.

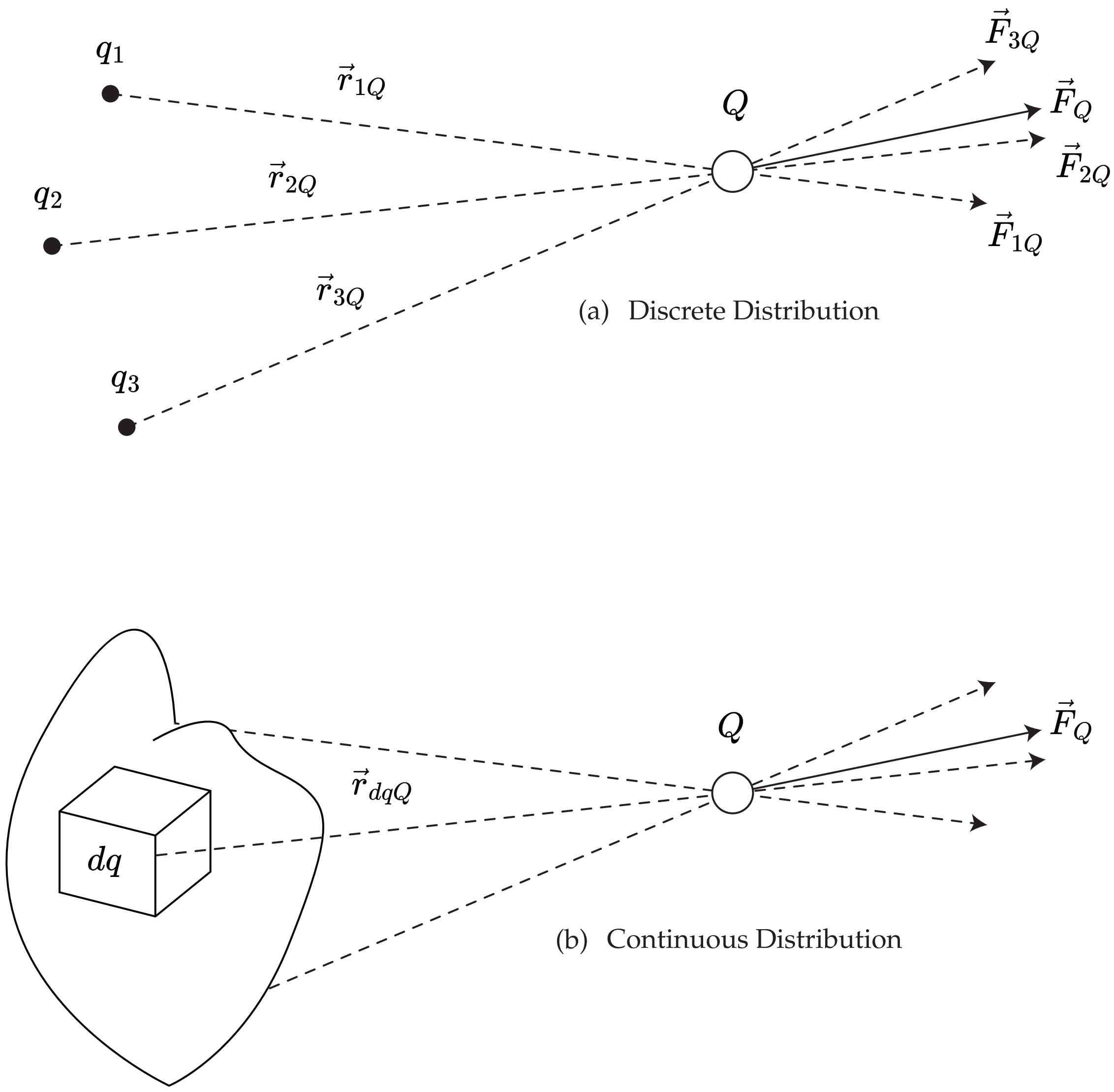


FIGURE 12: (a) Discrete charge distribution. The total force is the vector sum of individual forces. (b) Continuous charge distributions. The total force is found by integrating over the force elements, $d\vec{F}$.

The total force on q_1 is:

$$\vec{F}_Q = \vec{F}_{1Q} + \vec{F}_{2Q} + \vec{F}_{3Q} + \ldots \quad \ldots \vec{F}_{nQ} \tag{28}$$

This type of linear addition of individual components to determine the total applies to many so-called *linear systems* and is referred to as the *principle of superposition*. The total force on Q is given by the sum of the n individual Coulomb interactions over charges q_i:

$$\begin{aligned} \vec{F}_Q &= \frac{Qq_1}{4\pi\epsilon_0 r_{1Q}^2}\hat{r}_{1Q} + \frac{Qq_2}{4\pi\epsilon_0 r_{2Q}^2}\hat{r}_{2Q} + \frac{Qq_3}{4\pi\epsilon_0 r_{3Q}^2}\hat{r}_{3Q} + \dots \dots \frac{Qq_n}{4\pi\epsilon_0 r_{nQ}^2}\hat{r}_{nQ} \\ &= \sum_i^N \frac{Qq}{4\pi\epsilon_0 r_{iQ}^2}\hat{r}_{iQ}, \end{aligned} \tag{29}$$

where $\vec{r}_{iQ} = \vec{r}_Q - \vec{r}_i$. $\vec{r}_Q$ is the position vector of the observer charge. $\vec{r}_i$ is the position vector of the i^{th} charge in the distribution, measured in the Cartesian coordinate system.

For a continuous distribution, the sum becomes an integral over elements of charge dq:

$$\vec{F}_Q = \int \frac{Qq}{4\pi\epsilon_0 r_{dqQ}^2}\hat{r}_{dqQ}, \tag{30}$$

where $\vec{r}_{dqQ} = \vec{r}_Q - \vec{r}_{dq}$. $\vec{r}_Q$ is the position vector of the observer charge. $\vec{r}_{dq}$ is the position vector of the charge element dq in the distribution, measured in the Cartesian coordinate system.

3.1 Electric Field Intensity/Strength

Electric field intensity, also known as **electric field strength** or simply **electric field**, is a fundamental concept in physics that describes the force experienced by a positive test charge placed in an electric field. It is defined as the force per unit positive charge exerted on the test charge.

Mathematically, electric field intensity E at a point in space is given by the equation:

$$\vec{E} = \lim_{q_0 \to 0} \frac{\vec{F}}{q_0} = \frac{1}{4\pi\epsilon_0}\frac{|q|}{r^2}, \tag{31}$$

where E is the electric field intensity, F is the electrostatic force experienced by the test charge, and q is the magnitude of the test charge. The limit is there to ensure the existence of the field even in the absence of Q,

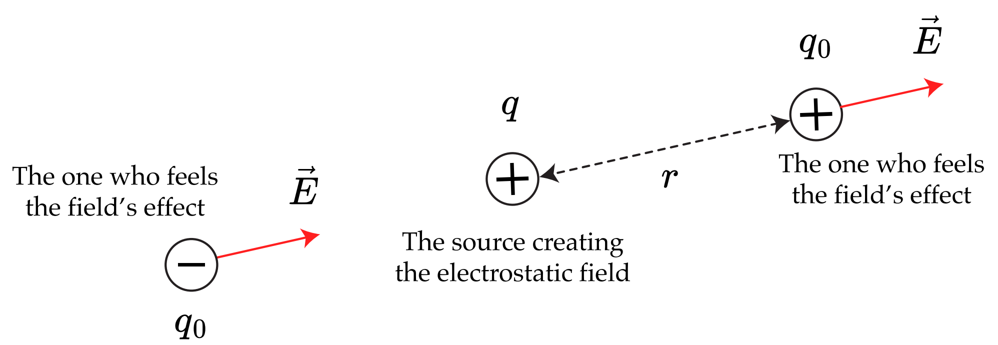


FIGURE 13: The electric field $\vec{E}$ felt by a point charge q_0 by an isolated point charge q.

In vector form, the equation reduces to

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^3}\vec{r}. \tag{32}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r} \tag{33}$$

NOTE: **The electric field at a point does not depend on the incoming charge at *that* point**. The electric field at a certain point is present whether or not there is a second point charge to experience the field.

Electric field intensity is a vector quantity with both magnitude and direction. The direction of the electric field at a given point is **the direction in which a positive test charge would experience a force if placed at that point**. Another crucial relation is as follows:

$$\vec{F}_E = q\vec{E}, \tag{34}$$

where $\vec{F}_E$ is the Coulomb force felt by the charge q.

The Superposition Principle for Electric Fields

Like the Coulomb force, the electric field also obeys the principle of superposition. The electric field experienced by a point charge Q due to a charge distribution q_i or dq is given by:

$$\begin{aligned}\vec{E} = \frac{\vec{F}}{Q} &= \frac{q_1}{4\pi\epsilon_0 r_{1r}^2}\hat{r}_{1r} + \frac{q_2}{4\pi\epsilon_0 r_{2r}^2}\hat{r}_{2r} + \frac{q_3}{4\pi\epsilon_0 r_{3r}^2}\hat{r}_{3r} + \dots \dots \frac{q_n}{4\pi\epsilon_0 r_{nr}^2}\hat{r}_{nr} \\ &= \sum_i^N \frac{q_n}{4\pi\epsilon_0 r_{ir}^2}\hat{r}_{ir},\end{aligned} \tag{35}$$

where $\vec{r}_{ir} = \vec{r} - \vec{r}_i$ are the separation coordinates. $\vec{r}$ is the position vector of the point where the observation is being made. $\vec{r}_i$ is the position vector of the i^{th} source charge, measured in the Cartesian coordinate system.

We can translate this to a continuous source charge distribution:

$$\vec{E} = \int \frac{dq}{4\pi\epsilon_0 r_{dqr}^2}\hat{r}_{dqr}, \tag{36}$$

where $\vec{r}_{dqr} = \vec{r} - \vec{r}_{dq}$ are the separation coordinates. $\vec{r}$ is the position vector of the point where the observation is being made. $\vec{r}_{dq}$ is the position vector of the dq source charge element, measured in the Cartesian coordinate system.

3.2 Electric Field Lines

They are imaginary lines used to visualize and represent the direction and strength of an electric field. They are drawn **tangent to the electric field vector** at each point and point away from positive charges and towards negative charges.

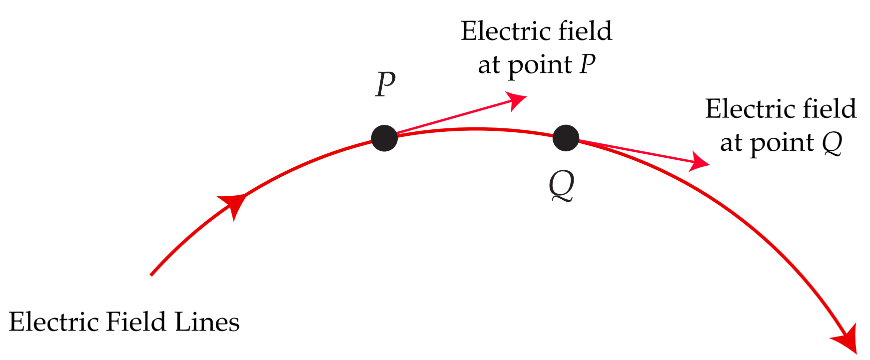


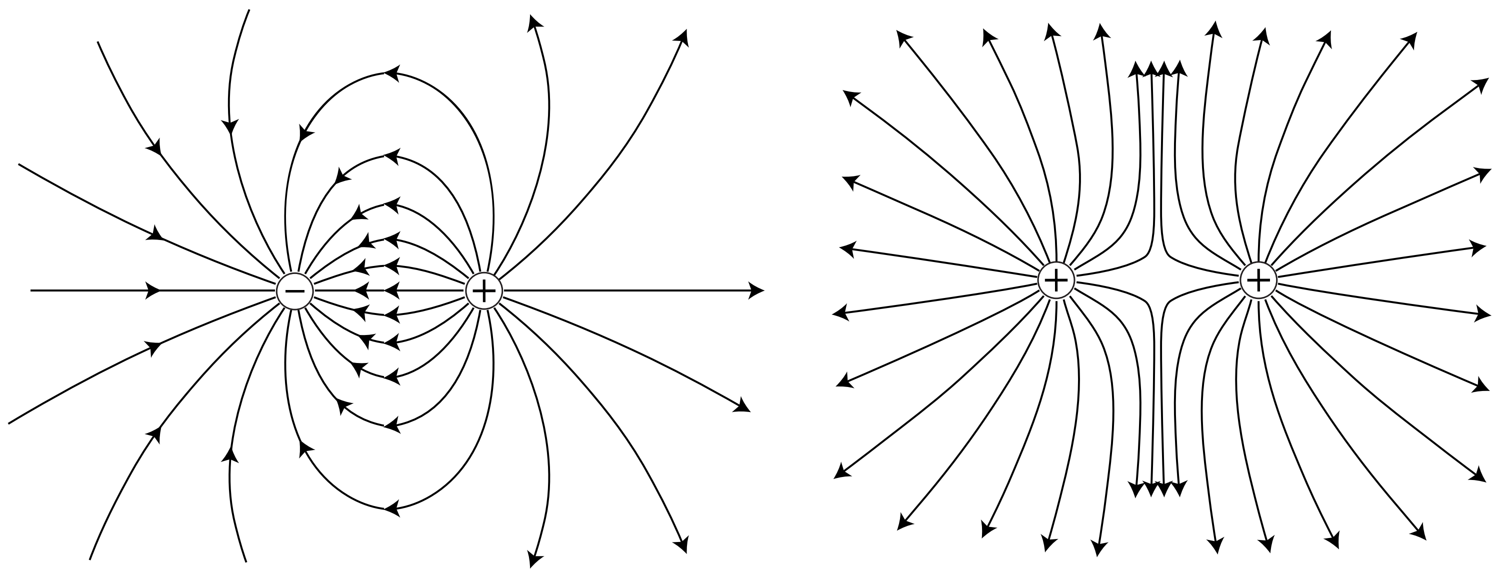
FIGURE 14: The direction of the electric field at any point is tangent to the field line through that point.

NOTE: **Electric field lines are not trajectories. No two electric field lines will intersect.**

Electric field lines provide a qualitative representation of the electric field's behavior and direction. The density of field lines indicates the strength of the electric field, with closer lines representing a stronger field. The direction of the field lines shows the direction in which a positive test charge would move if placed in the field.

Why can't two electric field lines intersect each other?

If two different field lines were to intersect, it would mean that a test charge would feel two different fields pointing in two different directions, which is forbidden.



(a) (b)

FIGURE 15: Electric field lines for two different charge distributions. The magnitude of $\vec{E}$ generally differs at different points along a given field line.

3.3 Electric Field due to a Charged Particle

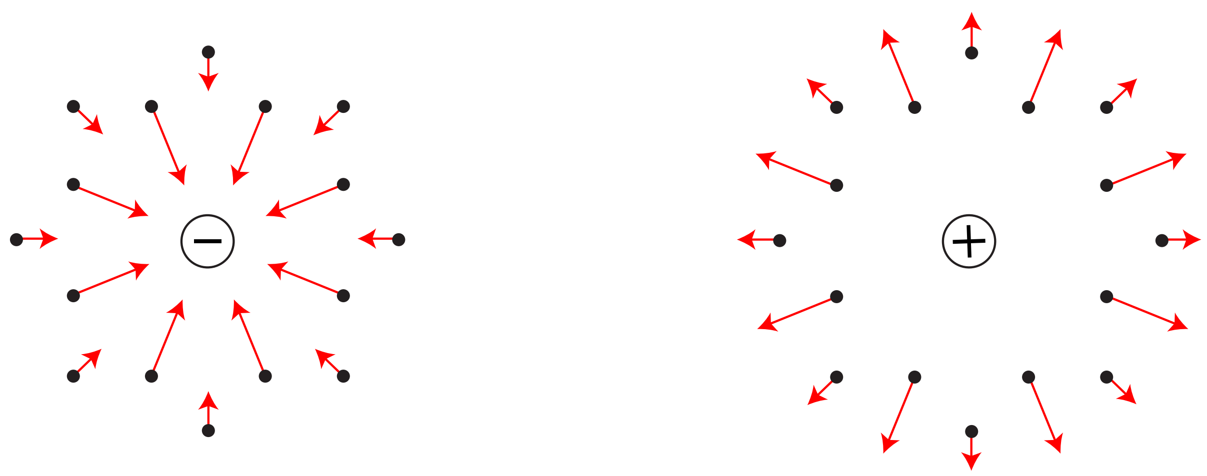


FIGURE 16: A point charge q produces an electric field $\vec{E}$ at all points in space. The field strength decreases with increasing distance.

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{|q|}{r^2}\hat{r}. \tag{37}$$

We write $|q|$ to avoid the danger of getting a negative E when q is negative and then think the negative sign has something to do with direction.

4 Electric Dipole

An electric dipole is a system consisting of two equal and opposite point charges separated by a small distance. The charges in an electric dipole have opposite signs, typically denoted as $q = +ne^-$ and $q = -ne^-$, and are located close to each other. The separation between the charges is denoted as $d = 2a$, where a represents half the distance between the charges, which is quite essential in the calculation.

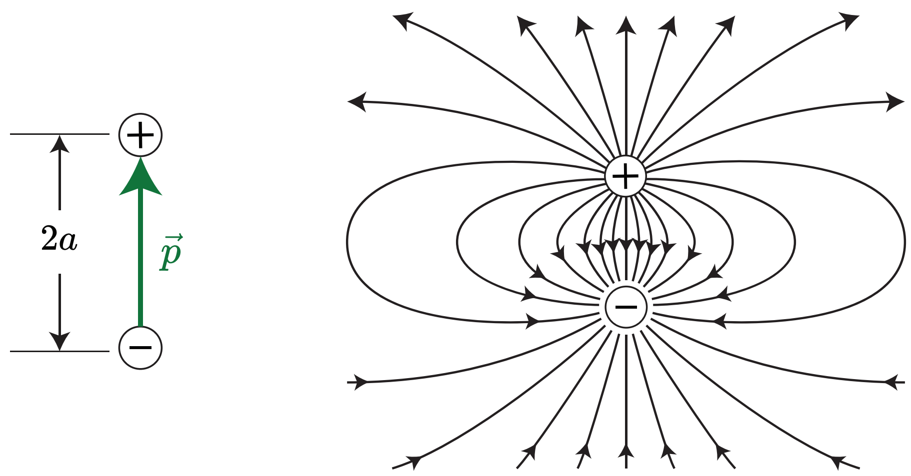


FIGURE 17: (Left) The dipole moment $\vec{p}$ of the dipole points from the negative charge to the positive charge. (Right) The pattern of electric field lines around an electric dipole.

NOTE: **The reason to take the distance between the point charges to be $2a$ is to avoid a non-fractional mess in the calculation**.

Electric dipoles also exhibit an electric dipole moment vector $\vec{p}$, which points from the negative to the positive charge. The direction of the dipole moment indicates the orientation of the dipole. The magnitude of the dipole moment measures the strength of the dipole. Mathematically, it can be expressed as:

$$\vec{p} = q \cdot 2\vec{a}. \tag{38}$$

NOTE: **While point charges have a net charge and produce an electric field around them, electric dipoles have a zero net charge since the charges cancel each other out**.

There are no theoretical limits on the separation distances of the two charges in an electric dipole. The smallest meaningful separation between charges in a physical system will depend on the scale of the particles involved and the conditions under which the dipole operates. We can set a limit $r \gg a$ for practical purposes, where r is the distance from the common center of the dipole charges.

4.1 Force on an Electric Dipole

When placed in an electric field $\vec{E}$, each charge experiences a force. The torque on the dipole is the sum of the torques on the two charges.

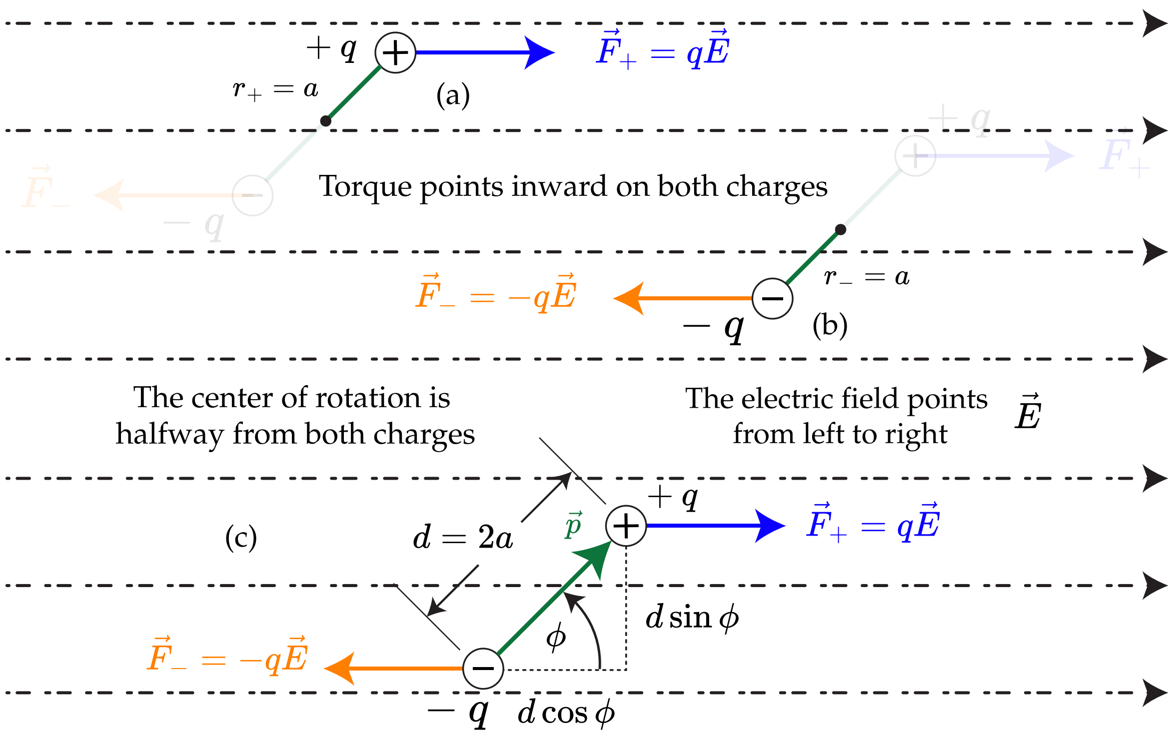


FIGURE 18: (a) The positive charge experiences the force F_+, along $\vec{E}$. The torque points inward with r_+ being the moment arm. (b) The negative charge experiences the force F_-, opposite to $\vec{E}$. The torque still points inward, with r_- being the moment arm. (c) The net force on this electric dipole is zero, but a torque directed into the page tends to rotate the dipole clockwise.

The positive charge experiences a Coulomb force $\vec{F}_+ = +Q\vec{E}$ that points in the field direction. The negative charge of the dipole also feels an equal but opposite force $\vec{F}_- = -Q\vec{E}$ to that of Q_+. The net force on the dipole thus $\vec{F}_{\text{net}} = \vec{F}_+ + \vec{F}_- = 0$. This does not mean the dipole is motionless.

4.2 Torque on an Electric Dipole

The moment arm vector $\vec{r}_+ = a\hat{d}$ for the positive charge points from the axis of rotation (the midpoint of the dipole) to the positive charge, perpendicular to the separation vector $d = 2a$. The torque for the positive charge would then be:

$$\tau_+ = \vec{r}_+ \times \vec{F}_+ = a\hat{d} \times Q\vec{E} = (Qa)\hat{d} \times \vec{E} = \frac{1}{2}\left(\vec{p} \times \vec{E}\right). \tag{39}$$

Similarly, the negative charge will feel a torque of the form, $\tau_- = \vec{r}_- \times \vec{F}_- = -L\hat{d} \times -Q\vec{E} = (Qa)\hat{d} \times \vec{E} = \frac{1}{2}\left(\vec{p} \times \vec{E}\right)$. The total torque of the dipole system is the vector sum of the two torque.

$$\vec{\tau} = \vec{\tau}_+ + \vec{\tau}_- = \frac{1}{2}\left(\vec{p} \times \vec{E}\right) + \frac{1}{2}\left(\vec{p} \times \vec{E}\right) = \vec{p} \times \vec{E}. \tag{40}$$

When placed in an external electric field, an electric dipole experiences a torque that tends to align it with the electric field. The torque exerted on the dipole is proportional to the product of the electric field $\vec{E}$ and the dipole moment $\vec{p}$. This torque tends to rotate the dipole such that its dipole moment aligns with the electric field.

4.3 Potential Energy in an Electric Dipole

Since the dipole system experiences a net torque $\vec{\tau}$ [N m] that rotates the dipole in an electric field, it gains some rotational potential energy.

$$
\begin{aligned}
U_{\text{dipole}} &= -\int \tau d\theta \\
&= -\int_{\frac{\pi}{2}}^{\theta} pE \sin\theta \, d\theta \\
&= \Big[pE \cos\theta \Big]_{\frac{\pi}{2}}^{\theta} \\
&= \vec{p} \cdot \vec{E}
\end{aligned} \tag{41}
$$

5 Electric Field Intensity Measurements for Charge Distribuions

5.1 Discrete I: Electric Field due to an Electric Dipole (Parallel on the Dipole Axis)

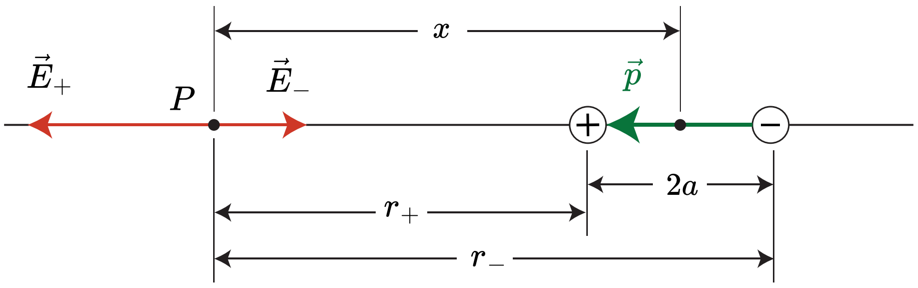


FIGURE 19: An electric dipole. The dipole's two charges result from the electric field vectors at point P on the dipole axis. Point P is at distances $r_+ = x - a$ and $r_- = x + a$ from the individual charges that make up the dipole. Here, $x \gg a$.

Measuring the field x distance away from the bisecting point of the dipole:

$$
\begin{aligned}
\vec{E}_{\text{dipole}} &= \vec{E}_+ + \vec{E}_- \\
&= \frac{1}{4\pi\epsilon_0}\left(-\frac{q}{r_+^2} + \frac{q}{r_-^2}\right)\hat{i} \\
&= \frac{q}{4\pi\epsilon_0}\left\{\frac{1}{(x-a)^2} - \frac{1}{(x+a)^2}\right\}\hat{i} \\
&= \frac{q}{4\pi\epsilon_0}\left[\frac{(x-a)^2 - (x+a)^2}{\{(x+a)(x-a)\}^2}\right]\hat{i} \\
&= \frac{q}{4\pi\epsilon_0}\left[\frac{4xa}{(x^2-a^2)^2}\right]\hat{i} \\
&= \frac{q}{4\pi\epsilon_0}\left[\frac{2\times 2xa}{x^4}\right]\hat{i}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{4\pi\epsilon_0} \times 2 \times \frac{q \times 2a}{x^3}\hat{i} \\
&= \frac{1}{4\pi\epsilon_0}\frac{2p}{x^3}\hat{i} \\
&= \frac{1}{2\pi\epsilon_0}\frac{p}{x^3}\hat{i}. \qquad (42)
\end{aligned}
$$

The field points from the direction of the dipole moment. i.e., toward the positive charge.

To account for the arbitrary direction of the observation point P along the dipole axis. We replace x with r, giving us:

$$\vec{E}_{\text{dipole}} = \frac{1}{2\pi\epsilon_0}\frac{\vec{p}}{r^3}\hat{r}_{(-)\to(+)} = \frac{1}{2\pi\epsilon_0}\frac{\vec{p}}{r^3}\hat{p} \qquad (43)$$

The electric field of a dipole thus points in the direction of the electric dipole moment, regardless of whatever orientation we try. It further establishes the *why* behind the convention of $\vec{p}$'s direction.

5.2 Discrete II: Electric Field due to an Electric Dipole (Perpendicular to the Dipole Axis)

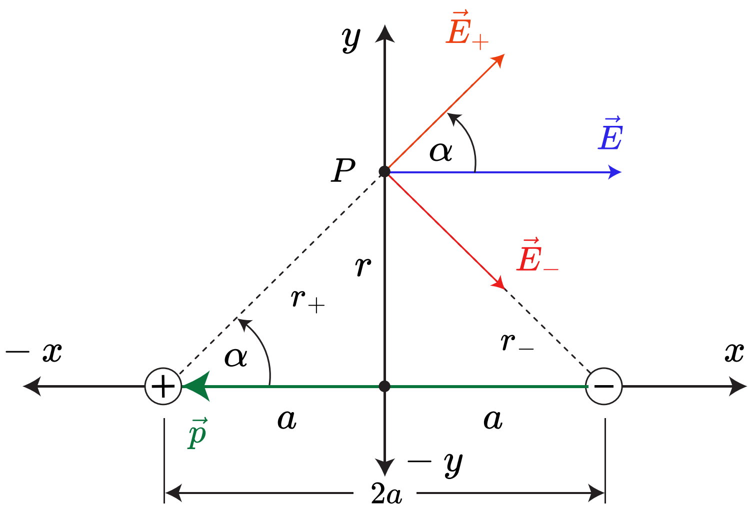


FIGURE 20: An electric dipole produces an electric field at point P off the dipole axis. Point P is at a distance r perpendicularly off the dipole axis. Here, $r \gg a$.

Measuring the field r distance, perpendicularly away from the bisecting point of the dipole:

$$
\begin{aligned}
\vec{E}_{\text{dipole}} &= \vec{E}_+ + \vec{E}_- \\
&= \{E_+(x) + E_-(x)\}\,\hat{i} + \{E_+(y) + E_-(y)\}\,\hat{j} \\
&= \{E_+(x) + E_-(x)\}\cos\alpha\hat{i} + 0\hat{j} \\
&= \frac{2}{4\pi\epsilon_0} \times \frac{q}{y^2+a^2} \times \frac{a}{\sqrt{y^2+a^2}}\hat{i} \\
&= \frac{q \times 2a}{4\pi\epsilon_0}\frac{1}{(y^2+a^2)^{\frac{3}{2}}}\hat{i} \\
&= \frac{1}{4\pi\epsilon_0}\frac{p}{(y^2+a^2)^{\frac{3}{2}}}\{-\hat{p}\}. \qquad (44)
\end{aligned}
$$

The field points from the direction of the dipole moment. i.e., toward the positive charge.

To account for the arbitrary direction of the observation point P along the dipole axis. We replace y with r, giving us:

$$\vec{E}_{\text{dipole}} = \frac{1}{4\pi\epsilon_0} \frac{p}{(r^2+a^2)^{\frac{3}{2}}} \{-\hat{p}\} \tag{45}$$

In the limit $r \gg a$, we get the following:

$$\vec{E}_{\text{dipole}} = \frac{1}{4\pi\epsilon_0} \frac{p}{r^3} \{-\hat{p}\} . \tag{46}$$

We can see that Eqs. (46) and (43) are identical except for a factor of two, which shows up only when the field is measured along the dipole axis.

In both cases, it is evident that the electric field due to an electric dipole falls off as $\frac{1}{r^3}$. This is called the *Inverse Cube Law*, unlike the well-known *Inverse Square Law* we encountered for a single-point charge.

5.3 Continuous I: Electric Field due to a Charged Line (Finite) Segment

A line segment, of length $2L$, is uniformly charged with a charge density $\lambda = \frac{Q}{2L}\,\mathrm{C\,m^{-1}}$. We divide the charged line segment into infinitesimal point charge elements λdy.

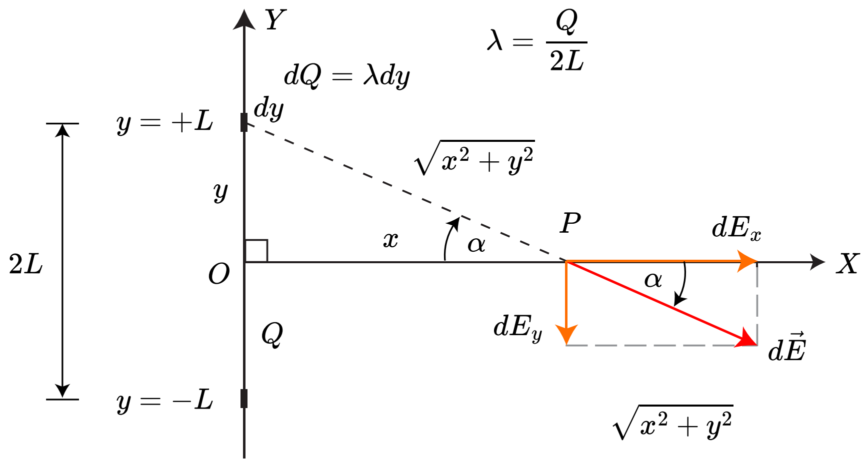


FIGURE 21: Measuring the field x distance, *perpendicularly* away from the center point of a line segment of length $2a$, uniformly charged with $+Q$ charge.

At $y = +L$, we can use one of these elements and measure the infinitesimal field $d\vec{E}$ produced at point P. We then resolve this $d\vec{E}$ into its spatial components dE_x and dE_y. This line element has an identical twin on the opposite end of the segment at $y = -L$. This element also produces a $d\vec{E}$ of its own, having dE_x and dE_y. This twin point charge element cancels their y-components while adding x-components. We get a similar outcome if we exhaust all possible charge elements on this line. Each symmetric charge element gives a summed x-components and canceled y-components. We now integrate these infinitesimal field components between the limits of $y = -L$ to $y = +L$ and calculate the net E at P.

$$\begin{aligned} \vec{E} &= \int d\vec{E} = \int dE_x \hat{\imath} \\ &= \frac{1}{4\pi\epsilon_0} \int \frac{\lambda dy}{\sqrt{x^2+y^2}} \cos\alpha\, \hat{\imath} \end{aligned} \tag{47}$$

$$
\begin{aligned}
&= \frac{1}{4\pi\epsilon_0} \int \frac{\lambda dy}{x^2+y^2} \times \frac{x}{\sqrt{x^2+y^2}} \hat{i} \\
&= \frac{\lambda x}{4\pi\epsilon_0} \int \frac{dy}{\left(x^2+y^2\right)^{3/2}} \hat{i}. \qquad (48)
\end{aligned}
$$

NOTE: We are taking the charge Q to be positive in the following calculation. For the same case where Q is taken to be negative, we flip the direction of the electric field at each point, even the final net direction, for that matter.

In Eq. (48), we substitute $y = x \tan u$ with $u = \tan^{-1}\left(\frac{y}{x}\right)$.

$$
\begin{aligned}
y &= x \tan u \\
dy &= x \sec^2 u du. \\
x^2 + y^2 &= x^2 + (x \tan u)^2 \\
&= x^2 \left(1 + \tan^2 u\right) \\
&= x^2 \sec^2 u.
\end{aligned}
$$

With these substitutions, Eq. (48) now becomes:

$$
\begin{aligned}
\vec{E} &= \frac{\lambda x}{4\pi\epsilon_0} \int \frac{x \sec^2 u}{\left(x^2 \sec^2 u\right)^{3/2}} du \\
&= \frac{\lambda x}{4\pi\epsilon_0} \int \frac{x \sec^2 u}{x^3 \sec^3 u} du \\
&= \frac{\lambda x}{4\pi\epsilon_0} \int \frac{du}{x^2 \sec u} \\
&= \frac{\lambda x}{4\pi\epsilon_0 x^2} \int \cos u du
\end{aligned}
$$

Here, we must apply the limits of integration for u and not y because we swapped the integrating variable y.

y	$u = \tan^{-1}\left(\frac{y}{x}\right)$
$+L$	$\tan^{-1}\left(\frac{+L}{x}\right)$
$-L$	$\tan^{-1}\left(\frac{-L}{x}\right)$

We now evaluate the integral with the properly set limits:

$$
\vec{E} = \frac{\lambda x}{4\pi\epsilon_0 x^2} \int_{\tan^{-1}\left(\frac{-L}{x}\right)}^{\tan^{-1}\left(\frac{+L}{x}\right)} \cos u du \, \hat{i}
$$

$$
\begin{aligned}
&= \frac{\lambda x}{4\pi\epsilon_0 x^2} \Big[\sin u\Big]_{\tan^{-1}\left(\frac{-L}{x}\right)}^{\tan^{-1}\left(\frac{+L}{x}\right)} \hat{\imath} \\
&= \frac{\lambda x}{4\pi\epsilon_0 x^2} \left[\sin\left\{\tan^{-1}\left(\frac{L}{x}\right)\right\} - \sin\left\{\tan^{-1}\left(\frac{-L}{x}\right)\right\}\right] \hat{\imath} \\
&= \frac{\lambda x}{4\pi\epsilon_0 x^2} \left[\sin\left\{\sin^{-1}\left(\frac{L}{x\sqrt{\frac{L^2}{x^2}+1}}\right)\right\} - \sin\left\{\sin^{-1}\left(\frac{-L}{x\sqrt{\frac{L^2}{x^2}+1}}\right)\right\}\right] \hat{\imath} \\
&= \frac{\lambda x}{4\pi\epsilon_0 x^2} \left(\frac{L}{x\sqrt{\frac{L^2}{x^2}+1}} - \frac{-L}{x\sqrt{\frac{L^2}{x^2}+1}}\right) \hat{\imath} \\
&= \frac{\lambda x}{4\pi\epsilon_0 x^2} \left(\frac{2L}{x\sqrt{\frac{L^2}{x^2}+1}}\right) \hat{\imath} \\
&= \frac{\lambda}{4\pi\epsilon_0 x^2} \left(\frac{2L}{\sqrt{\frac{L^2+x^2}{x^2}}}\right) \\
&= \frac{\lambda \times 2L}{4\pi\epsilon_0 x\sqrt{L^2+x^2}} \hat{\imath} && (49) \\
&= \frac{1}{4\pi\epsilon_0} \frac{Q}{x\sqrt{x^2+L^2}} \hat{\imath}. && (50)
\end{aligned}
$$

For an observation made in any arbitrary orientation of the line charge segment:

$$
\begin{aligned}
\vec{E} &= \frac{\lambda \times 2L}{4\pi\epsilon_0 r\sqrt{r^2+L^2}} \hat{r} && (51) \\
&= \frac{1}{4\pi\epsilon_0} \frac{Q}{r\sqrt{r^2+L^2}} \hat{r}. && (52)
\end{aligned}
$$

$\hat{r}$ denotes the direction (radially outward) of the field (due to $+Q$) *away* from the source *toward* the observation point.

As promised, for the case of $-Q$, the field will point radially inward toward the source. The field strength will remain the same; only the direction will earn a negative sign before it to that of the positive case.

LIMITING CASES:

1. $r \gg L$: It reduces to the field due to a single-point charged particle.

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \hat{r}. \tag{53}$$

2. $r \to \infty$:

$$\vec{E} = 0. \tag{54}$$

3. $r = L$:

$$\vec{E} = \frac{1}{2\pi\epsilon_0}\frac{Q}{r^2}\hat{r}. \tag{55}$$

4. $r \ll L$: This case gives us the Electric Field due to an Infinite Line of Charge.

$$\vec{E} = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{r}\hat{r}. \tag{56}$$

5.4 Continuous II: Electric Field due to an Infinite Line of Charge

NOTE: We are taking the charge Q to be positive in the following calculation. For the same case where Q is taken to be negative, we simply flip the direction of the electric field at each point, even the final net direction, for that matter.

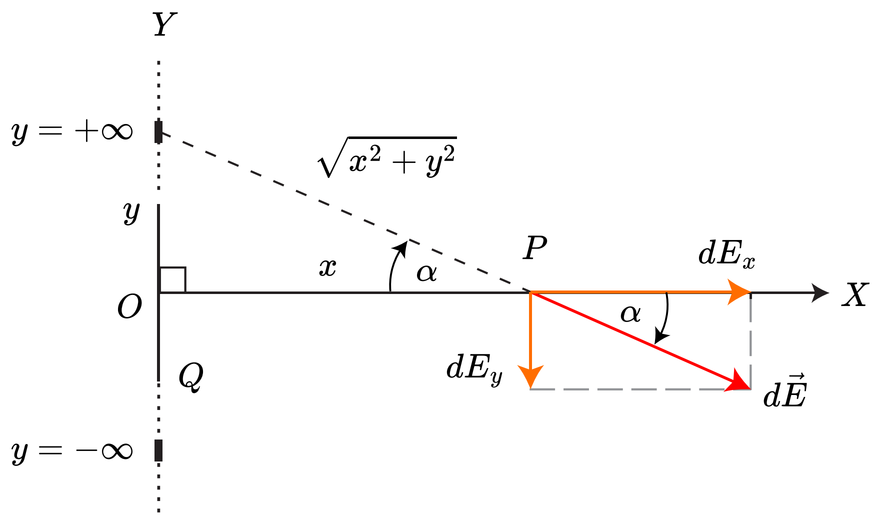


FIGURE 22: Measuring the field x distance away from an infinite line uniformly charged with $+Q$ charge.

The calculations for this are the same for the charged line segment. The integration limit should be $y = -\infty$ to $y = +\infty$ instead of $y = -L$ to $y = +L$. Here, we must apply the limits of integration for u and not y because we swapped the integrating variable y. This type of integration usually requires a splitting up of the limits.

y	$u = \tan^{-1}\left(\frac{y}{x}\right)$
$+\infty$	$+\frac{\pi}{2}$
$-\infty$	$-\frac{\pi}{2}$
0	0

We directly import the steps from the line segment field derivation (48) and evaluate the integral with the properly set limits:

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\int_{y=-\infty}^{y=0}\frac{\lambda x dy}{(x^2+R^2)^{3/2}}\hat{i} + \frac{1}{4\pi\epsilon_0}\int_{y=0}^{y=+\infty}\frac{\lambda x dy}{(x^2+R^2)^{3/2}}\hat{i}$$

$$= \frac{\lambda x}{4\pi\epsilon_0 x^2} \int_{-\frac{\pi}{2}}^{0} \cos u du\, \hat{\imath} + \frac{\lambda x}{4\pi\epsilon_0 x^2} \int_{0}^{+\frac{\pi}{2}} \cos u du\, \hat{\imath}$$

$$= \frac{\lambda}{4\pi\epsilon_0 x} \left(\int_{-\frac{\pi}{2}}^{0} \cos u du + \int_{0}^{+\frac{\pi}{2}} \cos u du \right) \hat{\imath}$$

$$= \frac{1}{4\pi\epsilon_0} \frac{2\lambda}{x} \hat{\imath}.$$

For an observation made in any arbitrary orientation of the line charge segment:

$$\vec{E} = \frac{\lambda}{2\pi\epsilon_0 r}\hat{r}. \tag{57}$$

$\hat{r}$ denotes the direction (perpendicularly outward) of the field (due to $+Q$) *away* from the source *toward* the observation point.

As promised, for the case of $-Q$, the field will point radially inward toward the source. The field strength will remain the same; only the direction will earn a negative sign in front of it to that of the positive case.

NOTE: We will have a *given* line charge density λ in this case. Unlike the previous cases where we could define $\lambda = \dfrac{\text{Total Charge}}{\text{Total Length}}$, we can't do the same here due to the infinite length.

5.5 Continuous III: Electric Field due to a Ring of Charge

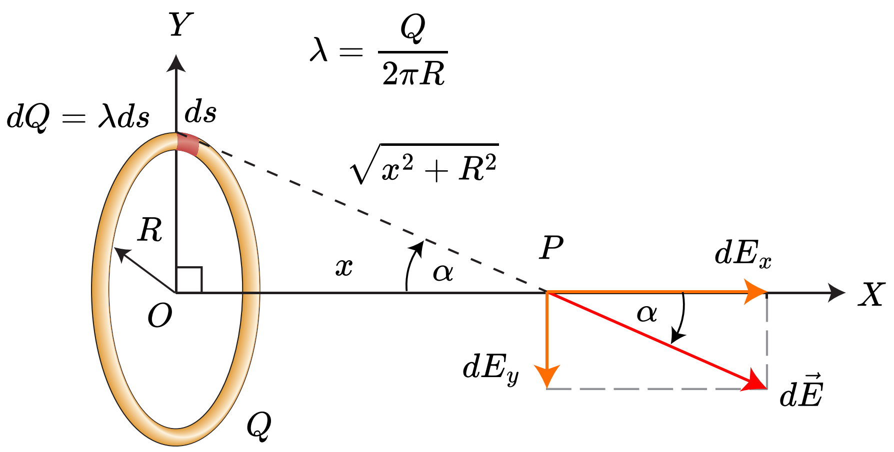


FIGURE 23: Measuring the field x distance away from the center of a ring of radius a, uniformly charged with $+Q$ charge.

The ring, having radius R, is uniformly charged with a density $\lambda = \dfrac{Q}{2\pi R}$ [$\mathrm{C\,m^{-2}}$]. We divide the charged ring into infinitesimal point charge elements λds across the ring's circumference.

NOTE: We are taking the charge Q to be positive in the following calculation. For the same case where Q is taken to be negative, we simply flip the direction of the electric field at each point, even the final net direction, for that matter.

At $s = 0$ (suppose it denotes $y = +R$ in the diagram), we can use one of these elements and measure the infinitesimal field $d\vec{E}$ produced at point P. We then resolve this $d\vec{E}$ into its spatial components dE_x and

dE_y. This line element has an identical twin on the opposite end of the segment at $y = -R$ across the circumference. This element also produces a $d\vec{E}$ of its own, having dE_x and dE_y. This twin point charge element cancels their y-components while adding x-components. We get a similar outcome if we exhaust all possible charge elements on this line. Each symmetric charge element gives a summed x-components and canceled y-components. We now integrate these infinitesimal field components between the limits of $s = 0$ to $s = 2\pi R$ and calculate the net E at P.

$$\vec{E} = \int d\vec{E} = \int dE_x \hat{i} \tag{58}$$

$$= \frac{1}{4\pi\epsilon_0} \int_0^{2\pi R} \frac{\lambda ds}{\sqrt{x^2 + R^2}} \cos\alpha\, \hat{i}$$

$$= \frac{1}{4\pi\epsilon_0} \int_0^{2\pi R} \frac{\lambda ds}{x^2 + R^2} \times \frac{x}{\sqrt{x^2 + R^2}}\, \hat{i}$$

$$= \frac{\lambda x}{4\pi\epsilon_0 \left(x^2 + R^2\right)^{3/2}} \int_0^{2\pi R} dS\, \hat{i}$$

$$= \frac{2\pi\lambda R x}{4\pi\epsilon_0 \left(x^2 + R^2\right)^{3/2}}\, \hat{i}$$

$$= \frac{\lambda R x}{2\epsilon_0 \left(x^2 + R^2\right)^{3/2}}\, \hat{i}. \tag{59}$$

For an observation made in any arbitrary orientation of the line charge segment:

$$\vec{E} = \frac{\lambda R r}{2\epsilon_0 \left(r^2 + R^2\right)^{3/2}}\, \hat{r}.$$

$$= \frac{Q r}{4\pi\epsilon_0 \left(r^2 + R^2\right)^{3/2}}\, \hat{r}. \tag{60}$$

LIMITING CASES:

1. $r \gg R$:

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2}\, \hat{r}. \tag{61}$$

   This reduces to a field due to a single-point charge.

2. $r = 0$:

$$\vec{E} = 0. \tag{62}$$

   The field is zero at the center of the ring.

3. $r = R$:

$$\vec{E} = \frac{1}{(2)^{3/2}} \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2}\, \hat{r}. \tag{63}$$

4. $r \ll R$:

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Qr}{R^2}\hat{r}. \tag{64}$$

This reduces to a field due to a single-point charge.

$\hat{r}$ denotes the direction (axially outward) of the field (due to $+Q$) *away* from the source *toward* the observation point.

As promised, for the case of $-Q$, the field will point radially inward toward the source. The field strength will remain the same; only the direction will earn a negative sign in front of it compared to that of the positive case.

5.6 Continuous IV: Electric Field due to a Uniformly Charged Disk

NOTE: We are taking the charge Q to be positive in the following calculation. For the same case where Q is taken to be negative, we simply flip the direction of the electric field at each point, even the final net direction for that matter.

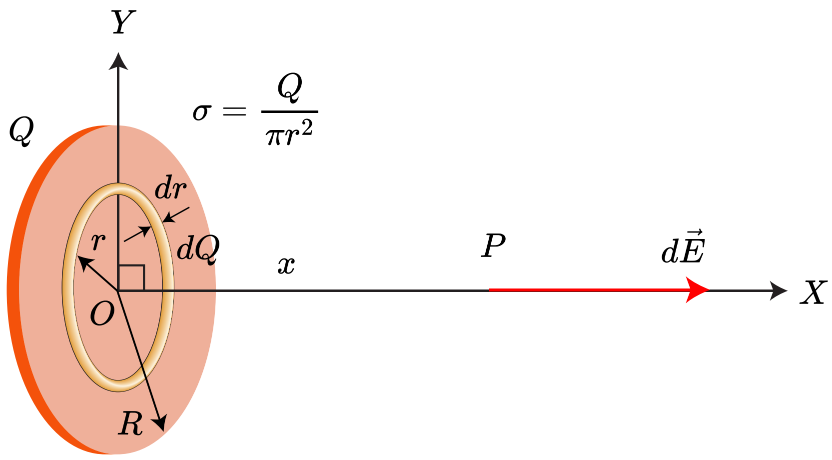


FIGURE 24: Measuring the field x distance away from the center point of a disk of radius R, uniformly charged with $+Q$ charge.

We divide the charged disk into a collection of infinitesimal ring charge elements. We already know the electric field produced by a ring charge distribution from Eq. (60). We start by taking one ring charge element of radius r' and integrate it from $r' = 0$ to $r' = R$ to measure the net $\vec{E}$ produced at P. A ring charge produces a field along the central axis as per Eq. (60). In order to integrate the ring charge element, we need to vary it infinitesimally. Take $dQ = \sigma dA = 2\pi\sigma r' dr'$, where σ is the surface charge density of the disk, $2\pi r'$ is the circumference of the initial ring charge, and dr' is the variation.

$$\begin{aligned}
\vec{E} &= \frac{1}{4\pi\epsilon_0}\int \frac{dQ}{x^2 + r'^2} \times \frac{x}{\sqrt{x^2 + r'^2}}\hat{i} \\
&= \frac{1}{4\pi\epsilon_0}\int \frac{2\pi\sigma r' dr'}{(x^2 + r'^2)^{\frac{3}{2}}}\hat{i} \\
&= \frac{x\sigma\pi}{4\pi\epsilon_0}\int_{r'=0}^{r'=R} \frac{2r' dr'}{(x^2 + r'^2)^{\frac{3}{2}}}\hat{i}
\end{aligned}$$

$$= \frac{\sigma x}{4\epsilon_0}\left[\frac{\left(x^2+r'^2\right)^{-\frac{1}{2}}}{-\frac{1}{2}}\right]_{r'=0}^{r'=R}\hat{i}$$

$$\because \int X^M dX = \frac{X^{M+1}}{M+1} \text{ with } X = x^2 + r'^2,\ m = -\frac{3}{2},\ dX = 2r' dr'$$

$$= \frac{2\sigma x}{4\epsilon_0}\left[\frac{1}{x} - \frac{1}{\sqrt{x^2+R^2}}\right]\hat{i}$$

$$= \frac{\sigma}{2\epsilon_0}\left(1 - \frac{x}{\sqrt{x^2+R^2}}\right)\hat{r}. \tag{65}$$

For an observation made in any arbitrary orientation of the line charge segment:

$$\vec{E} = \frac{\sigma}{2\epsilon_0}\left(1 - \frac{r}{\sqrt{r^2+R^2}}\right)\hat{r}. \tag{66}$$

$$= \frac{Q}{2\epsilon_0 \pi R^2}\left(1 - \frac{r}{\sqrt{r^2+R^2}}\right)\hat{r}. \tag{67}$$

$\hat{r}$ denotes the direction (axially outward) of the field (due to $+Q$) *away* from the source *toward* the observation point.

As promised, for the case of $-Q$, the field will point radially inward toward the source. The field strength will remain the same; only the direction will earn a negative sign in front of it compared to that of the positive case.

LIMITING CASES:

1. $R \gg r$: This case reduces to a field due to an infinite sheet of charge. We will return to this important result later in the course.

$$\vec{E} = \frac{\sigma}{2\epsilon_0}. \tag{68}$$

2. $R \ll r$: Observing too far from the source essentially will result in zero field intensity.

$$\vec{E} = 0. \tag{69}$$

5.7 Continuous V: Electric Field due to an Infinite Sheet of Charge

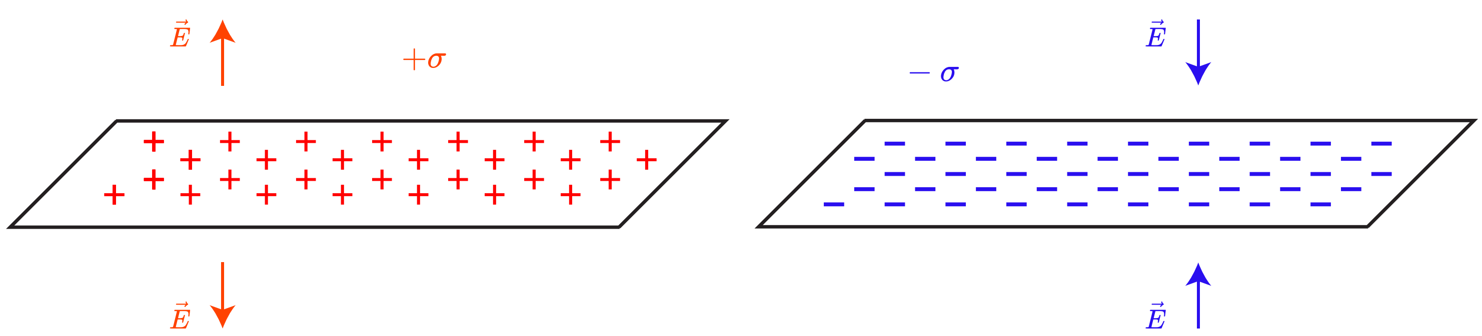


FIGURE 25: Measuring the field from a random distance away from an infinitely long sheet of charge.

$$\vec{E} = \frac{\sigma}{2\epsilon_0}. \tag{70}$$

This field does not depend on the distance from the sheet. It is uniform everywhere perpendicular to the sheet, away from it.

NOTE: There is no such thing as an infinite sheet of charge, but if the dimensions of the sheet are much larger than the observation distance r of the field from the sheet, the field is very nearly given by Eq. (70).

5.8 Continuous VI: Electric Field due to Two Oppositely Charged Infinite Sheets

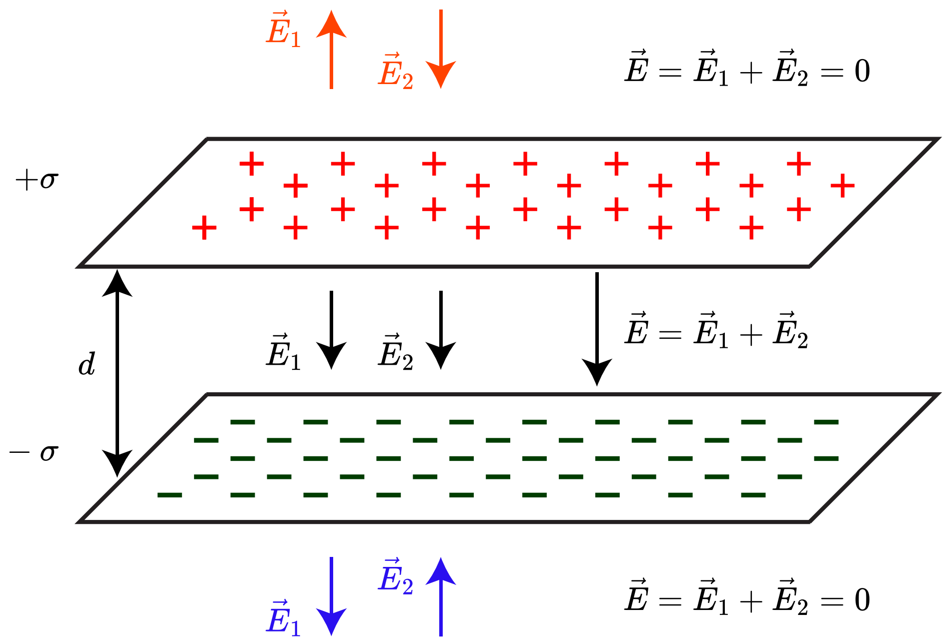


FIGURE 26: Measuring the field in between and away from two infinite plane sheets with uniform surface charge densities $+\sigma$ and $-\sigma$ are placed parallel to each other with separation d.

$$\vec{E} = \vec{E}_1 + \vec{E}_2 = \begin{cases} 0 & \text{above the upper (positive) sheet} \\ \frac{\sigma}{\epsilon_0}\hat{n}_{(+)\to(-)} & \text{between the sheet} \\ 0 & \text{below the lower (negative) sheet} \end{cases} \tag{71}$$

Where both the sheets have the same magnitude at all points, independent of the distance from either sheet:

$$E_1 = E_2 = \frac{\sigma}{2\epsilon_0}.$$

6 Gauss's Law of Electrostatics

Gauss's Law of Electrostatics is an alternative form of Coulomb's Law. The latter provides information about the electric field once the source electric charge distribution is given. Gauss's Law does the opposite. It provides information on the amount of charge, given the field.

It states that **the total electric flux through any closed surface is equal to the net electric charge enclosed by that surface divided by the permittivity of free space**.

6.1 Electric Flux

It measures **the number of electric field lines passing through a given surface**. It is defined as the dot product of the electric field vector E and the area vector A of the surface.

Mathematically, electric flux Φ_E is given by the equation:

$$\Phi_E = \sum \text{Electric field lines crossing through a chosen area,} \tag{72}$$

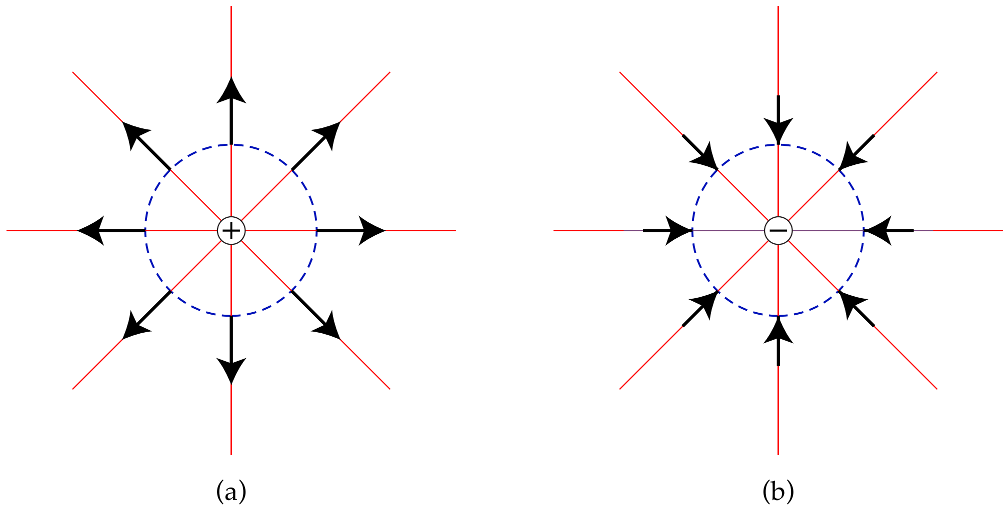


FIGURE 27: Electric field vectors and field lines pierce an imaginary, spherical Gaussian surface that encloses a particle with charge $+Q$ in (a) and $-Q$ in (b).

Electric flux provides information about the *strength* and *distribution* of the electric field through a surface. It is **proportional to the number of electric field lines** passing through the surface, indicating the intensity of the electric field.

The flux is zero if the number of electric field lines piercing the surface equals the number of field lines piercing out of the surface. We can quickly get a head start on the flux problem just by drawing a relevant diagram. Depending on the angle between $\vec{E}$ and $\vec{A}$ or $d\vec{A}$, electric flux can be positive, zero, or negative.

6.2 Flux of a Uniform Electric Field

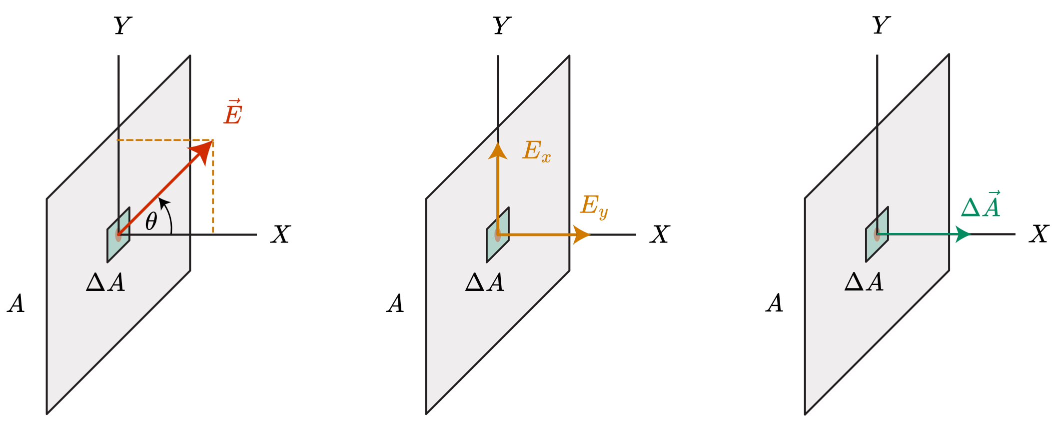


FIGURE 28: (a) An electric field vector pierces a small square patch on a flat surface. (b) Only the x component pierces the patch; the y component skims across it. (c) The area vector of the patch is perpendicular to the patch, with a magnitude equal to the patch's area.

The flux is mathematically calculated as the total contribution of the electric field across the Gaussian surface area, precisely given by the following dot product.

$$\begin{aligned}\Phi_E &= \vec{E}\cdot\vec{A} = E_x A_x + E_y A_y + E_z A_z \\ &= E(A\cos\theta) \\ &= A(E\cos\theta).\end{aligned} \tag{73}$$

Just like the notion that we use in dot product, the parallel projection of $\vec{E}$ onto $\vec{A}$ gives the same flux as the projection of $\vec{A}$ onto $\vec{E}$. Here, $\vec{E}$ is uniform across the Gaussian surface area.

6.3 Flux of a Non-Uniform Electric Field

In this case, we cannot take the same field $\vec{E}$ across a common cross-sectional area. We must measure the field flux across all possible differential area elements and then integrate all contributions.

$$\Phi_E = \int E\cos\theta dA \tag{74}$$

$$= \int E_{\parallel}\, dA$$

$$= \int \vec{E}\cdot d\vec{A} \tag{75}$$

The differential element is defined arbitrarily across the Gaussian surface. We take all the possible combinations of $\vec{E}\cdot d\vec{A}$ and then integrate them through the whole area.

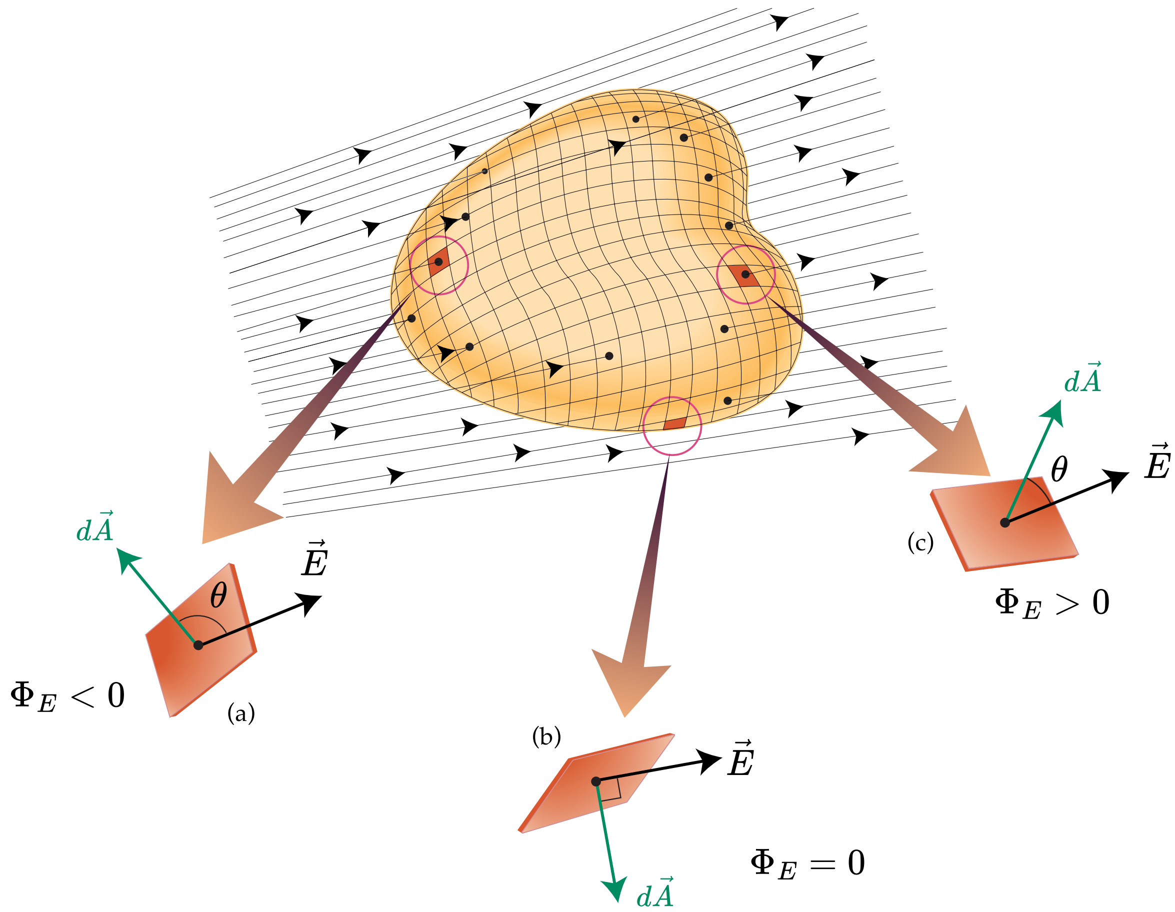


FIGURE 29: A Gaussian surface of arbitrary shape immersed in an electric field. The surface is divided into small squares of area A. The electric field vectors and the area vectors for three representative squares, marked (a), (b), and (c), are shown.

Mathematically, Gauss's Law can be expressed as:

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{\text{enclosed}}}{\varepsilon_0}. \qquad \text{(Integral Form)} \tag{76}$$

This Law demands that **the total electric flux passing through a closed surface is proportional to the total electric charge enclosed within that surface**. In other words, it relates the behavior of the electric field to the distribution of electric charge.

INTRIGUE: The prime reason for choosing a specific Gaussian Surface comes once we define a term called *Equipotential Surface*. It is to be introduced soon with proper context. Stand by!

Gauss's Law's ability to provide a simplified method for calculating electric fields in highly symmetrical situations is critical. The electric flux and the enclosed charge can be determined by choosing an *appropriate* Gaussian surface that *matches the symmetry of the problem*. This allows for calculating the electric field at any point of interest without considering the details of the charge distribution.

6.4 Applications of Gauss's Law

Spherical Symmetry

We want to measure $\vec{E}$ in three regions divided by the limits: $r < R, r = R, r > R$, where r is the radius of the chosen Gaussian surface, R is the radius of the spherical charge distribution or the spherical shell charge distribution, both uniformly charged with an amount Q.

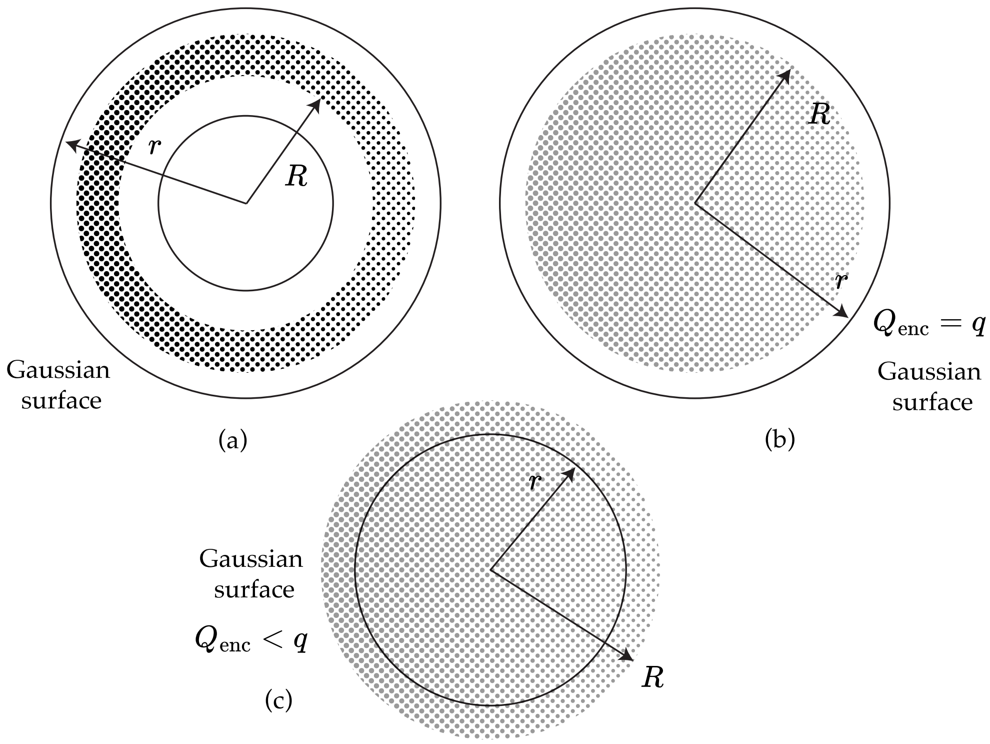


FIGURE 30: (a) Charge Q is uniformly distributed across a shell with a radius of R. (b) The dots represent a spherically symmetric charge distribution with a radius of R, whose volume charge density ρ is only a function of the distance from the center. The charged object is not a conductor, so the charge is assumed to be fixed in position. A concentric spherical Gaussian surface with $r > R$ is shown. A similar Gaussian surface with $r > R$ is shown in (c).

CASE 1: **Spherical Shell Charge Distribution**:

Region	**Electric Field**
$r < R$	$E = 0$
$r = R$	$E = \dfrac{Q}{4\pi\epsilon_0 R^2}$
$r > R$	$E = \dfrac{Q}{4\pi\epsilon_0 r^2} \sim E \propto \dfrac{1}{r^2}$

CASE 2: **Spherical Charge Distribution**:

Region	**Electric Field**
$r < R$	$E = \dfrac{Qr}{4\pi\epsilon_0 R^3}$
$r = R$	$E = \dfrac{Q}{4\pi\epsilon_0 R^2}$
$r > R$	$E = \dfrac{Qr}{4\pi\epsilon_0 R^3} \sim E \propto \dfrac{1}{r^2}$

We used the uniform behavior of the charge distribution, meaning the volume charge density is constant within the distribution. Our chosen Gaussian surface encloses a fraction of the total charge $Q_{\text{enc}} = \dfrac{Qr^3}{R^3}$.

CASE 3: **Excess Charge Distribution on a Solid Spherical Conductor**:

Inside a conductor, the charges are free to move due to the presence of mobile electrons or ions. When a conductive sphere is charged, the excess charge distributes itself on the sphere's outer surface. This occurs because like charges repel each other, and the charges on the surface of the sphere naturally move as far away from each other as possible.

If there were any excess charges inside the sphere, it would create an electric field within the conductor. The charges inside the sphere would experience a force due to this electric field, causing them to redistribute until the electric field inside the conductor becomes zero. This redistribution of charges continues until the system reaches electrostatic equilibrium.

Cylindrical Symmetry

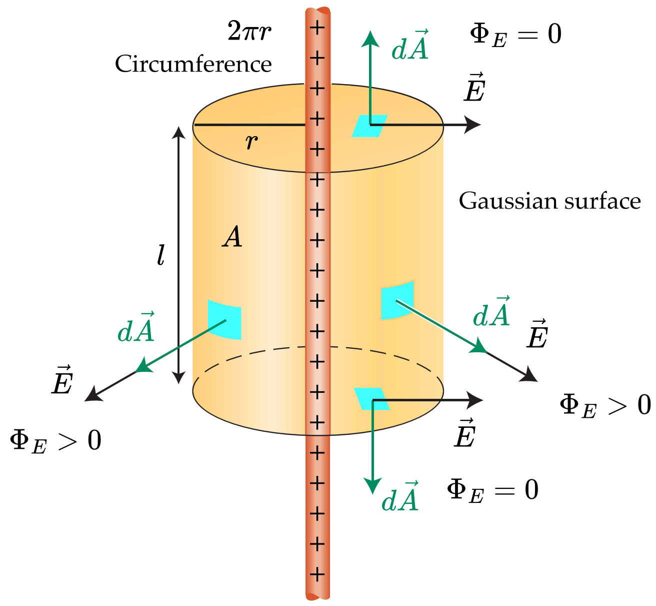


FIGURE 31: A Gaussian surface in the form of a closed cylinder surrounds a section of a very long, uniformly charged, cylindrical plastic rod. Only the boundary wall across the cylinder's length gives a non-zero flux. Both ends of the cylinder contribute nothing to the flux. That is why we ignored the $2\pi r^2$ term from the area.

An infinitely long charged wire with a charge density λ exhibits cylindrical symmetry in its vicinity. We choose a cylinder as our Gaussian surface to preserve symmetry in our calculations. The cylinder has a radius r at both ends and a length of l units, with an area of $2\pi rl + 2\pi r^2$.

Gauss's law states that

$$\begin{aligned} EA &= \frac{Q_{\text{enc}}}{\epsilon_0} \\ \Rightarrow E \times 2\pi rl &= \frac{\lambda l}{\epsilon_0} \\ \Rightarrow E &= \frac{\lambda}{2\pi\epsilon_0 r}\hat{r}. \end{aligned} \tag{77}$$

Here r represents the distance from which the field is being measured. We may arbitrarily choose this in any direction. It predominantly doesn't have to be the x or y axis.

Planar Symmetry

An infinitely long charged sheet with a charge density σ shows a head-on cylindrical symmetry around its vicinity. To maintain symmetry in our calculation, we consider a cylinder as our Gaussian surface. The cylinder has a radius r on both ends and is l units long with an area of $2\pi rl + 2\pi r^2$.

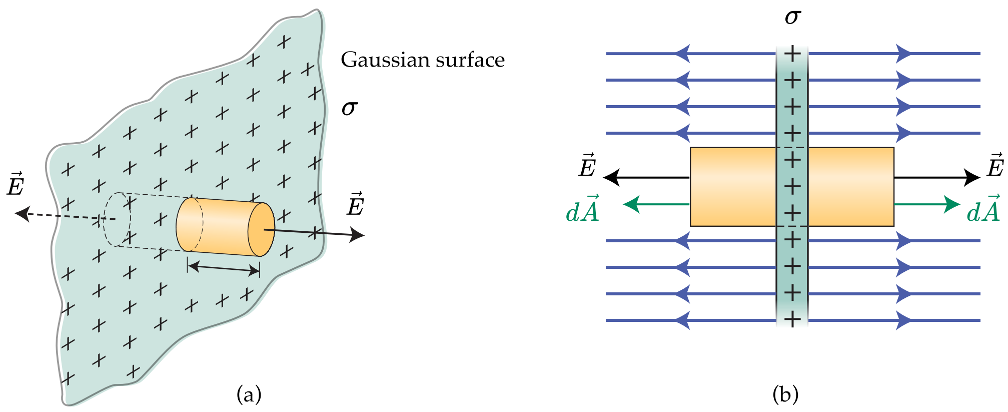


FIGURE 32: (a) Perspective view and (b) side view of a portion of a very large, thin plastic sheet, uniformly charged on one side to surface charge density σ. A closed cylindrical Gaussian surface passes through the sheet perpendicular to it. This time, only both ends of the cylinder contribute to the flux. That is why we shall ignore the $2\pi rl$ term from the area.

Gauss's Law demands

$$\begin{aligned} EA &= \frac{Q_{\text{enc}}}{\epsilon_0} \\ \Rightarrow 2EA &= \frac{\sigma A}{\epsilon_0} \\ \Rightarrow E &= \frac{\sigma}{2\epsilon_0}. \end{aligned} \tag{78}$$

6.5 Vector Calculus with $\vec{E}$-Field

Outflow of $\vec{E}$-Field

The divergence of $\vec{E}$-field calculates the out(in)flow out of a closed surface.

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho_{\text{enclosed}}}{\epsilon_0}. \qquad \text{(Differential Form)} \tag{79}$$

NOTE: We have used the Divergence/Gauss's theorem to the integral form of Gauss's Law (76) to get the differential form (79).

Circulation of $\vec{E}$-Field

In electrostatics, the electric fields have zero curl.

$$\vec{\nabla} \times \vec{E} = 0. \tag{80}$$

This directly implies, from Stokes's theorem, that in electrostatics,

$$\oint \vec{E} \cdot d\vec{l} = 0. \tag{81}$$

The rest is another story. We shall deal with that in the latter half of the course.

7 Work Done due to Electrostatic Force

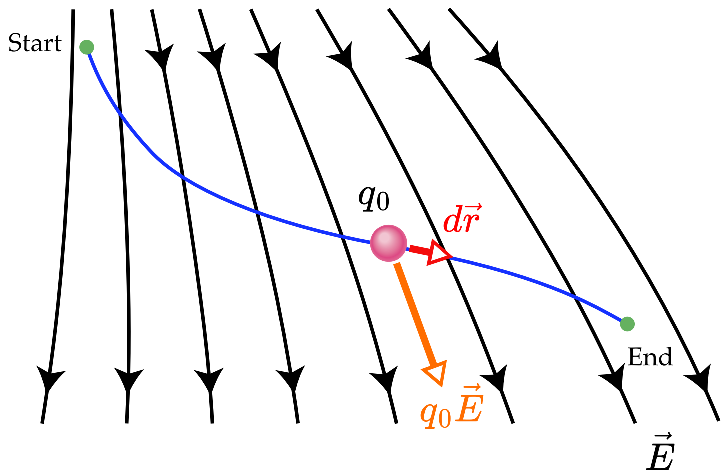


FIGURE 33: A test charge q_0 moves from one point to another along the path shown in a non-uniform electric field. During a displacement, an electric force acts on the test charge. This force points in the direction of the field line at the location of the test charge.

When a charge is moved against the electric field, work is done by an external force to overcome the repulsive forces between the moving charge and the other charges in the field. This work done is stored as potential energy in the system.

$$W = \int dW = \int \vec{F} \cdot d\vec{r}$$

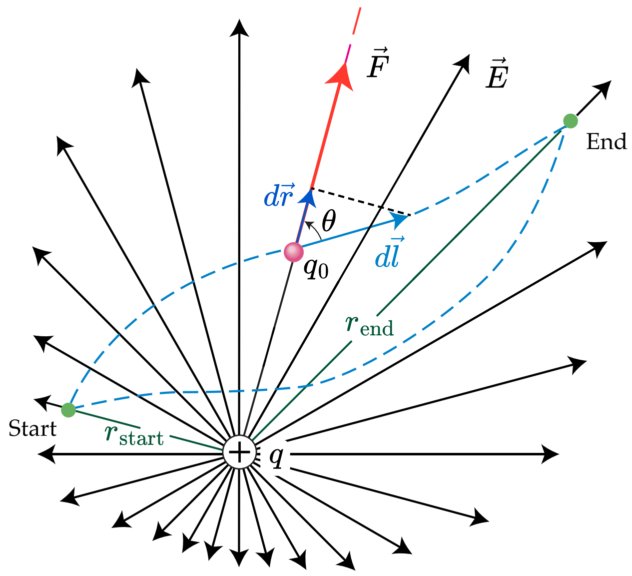


FIGURE 34: A test charge q_0 moves from one point to another along the path shown in a uniform electric field produced by a point charge q.

$$= q_0 \int_{\text{start}}^{\text{final}} \vec{E} \cdot d\vec{r}. \tag{82}$$

The above formula gives you the work done to move one charge from one point to another within an electric field. There needs not be a target charge at the final point. You only need one charge to displace in the field.

$$\begin{aligned} W &= \int dW \\ &= \int \vec{F} \cdot d\vec{r} \\ &= \frac{1}{4\pi\epsilon_0} \int_{\text{start}}^{\text{final}} \frac{q_0 q}{r^2} \cdot dr \\ &= \frac{q_0 q}{4\pi\epsilon_0} \int_{\text{start}}^{\text{final}} \frac{dr}{r^2} \\ &= \frac{q_0 q}{4\pi\epsilon_0} \left(\frac{1}{r_{\text{initial}}} - \frac{1}{r_{\text{final}}} \right) \end{aligned} \tag{83}$$

The above formula gives you the work done to move one charge from one point to another within the vicinity of a charge, producing an electric field. There needs to be a target charge at the final point. You need two or more charges to displace in the field.

If the general displacement in which the start and end points do not lie in the same radial line, the work done is given by

$$W = \int_{\text{start}}^{\text{final}} F \cos\theta dl = \int_{\text{start}}^{\text{final}} \frac{q_0 q}{r^2} \cos\theta dl, \tag{84}$$

where $dr = \cos\theta dl$ is the radial component of the line element.

What do signs of work done mean here?

Imagine you are moving a test charge (positive charge) within a field produced by a positive charge. The test charge will feel a repulsive force during this displacement. If the displacement is toward the target positive charge, the work done comes out negative. In a different case where the same test charge was to be moved toward a negative charge, the force is attractive, making the work done positive.

7.1 To (Dis)Assemble Many Point Charges in the presence of One Point Charge

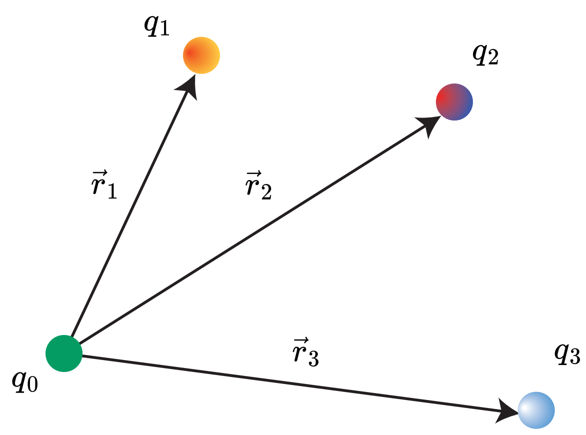


FIGURE 35: The potential energy associated with a charge q_0 at point a depends on the other charges q_1, q_2, and q_3 and on their distances r_1, r_2, and r_3 from point q_0.

Suppose a charge q_0 moves in an electric field $\vec{E}$ made or caused by several point charges $q_1, q_2, q_3, \ldots\ldots$ at distances $r_1, r_2, r_3, \ldots\ldots$ from q_0. The superposition principle will find the total work done.

$$W = \frac{q_0}{4\pi\epsilon_0}\left(\frac{q_1}{r_1} + \frac{q_2}{r_2} + \frac{q_3}{r_3} + \ldots\ldots\right) = \frac{q_0}{4\pi\epsilon_0}\sum_i \frac{q_i}{r_i}. \tag{85}$$

NOTE: The r_i in the above equation are distances between the i^{th} charge and q_0. The source charge q_0 must be located from the origin of a chosen coordinate system first.

7.2 To (Dis)Assemble a Discrete Point Charge Distribution from Scratch

Imagine instead of the motion of the charge q_0; we now try to calculate work done to form the collection of point charges $q_1, q_2, q_3, \ldots\ldots$ all separated from each other by infinite distances and then bringing them together. The distance between q_i and q_j is r_{ij}. The work done now is given by

$$W = \frac{1}{4\pi\epsilon_0}\sum_{i<j, i\neq j} \frac{q_i q_j}{r_{ij}}. \tag{86}$$

NOTE: The r_{ij} in the above equation are distances between the i^{th} and j^{th} charges. Those charges must be located from the origin of a chosen coordinate system. Thus, $\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$.

From the conservation of mechanical energy and the work-energy theorem, we get $\Delta U = -W$.

8 Electric Potential Energy

Physically, **Electric potential energy is the work done to bring a charge from infinity (where the electric field is considered to be zero) to a particular location in an electric field**. Electrical potential energy U is required to move within an electric field from one point to another. It is the negative of the work done to execute the same displacement.

$$\begin{aligned}\Delta U &= U_{\text{final}} - U_{\text{initial}} \\ &= -\frac{q_0 q}{4\pi\epsilon_0}\left(\frac{1}{r_{\text{initial}}} - \frac{1}{r_{\text{final}}}\right) \\ &= \frac{q_0 q}{4\pi\epsilon_0}\left(\frac{1}{r_{\text{final}}} - \frac{1}{r_{\text{initial}}}\right) \qquad (87) \\ &= -q_0 \int_{\text{start}}^{\text{final}} \vec{E}\cdot d\vec{r} \\ \Rightarrow \Delta U &= q_0 \int_{\text{final}}^{\text{start}} \vec{E}\cdot d\vec{r}. \qquad (88)\end{aligned}$$

9 Electric Potential

Electric potential describes **the amount of electric potential energy per unit charge at a specific point in an electric field**. It represents the *electrical potential energy* that a positive test charge would possess if placed at that point in the field.

The electric potential at a specific point in an electric field **is defined as the electric potential energy per unit charge**. It is denoted by the symbol V and is measured in volts V.

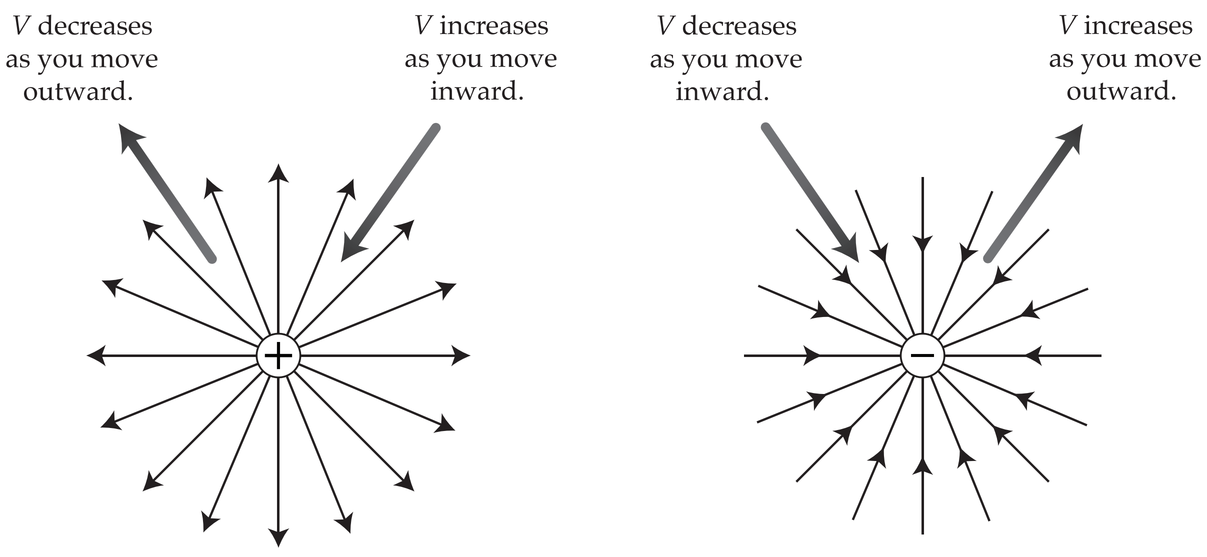


FIGURE 36: If you move in the direction of $\vec{E}$, electric potential V decreases; if you move in the direction opposite , V increases.

The electric potential at a point is a *scalar quantity* and can be positive or negative, depending on the relative positions and magnitudes of the charges involved.

$$\Delta V = \frac{\Delta U}{q_0} = \frac{-W}{q_0}. \tag{89}$$

9.1 Electric Potential Difference

Also known as voltage difference, it measures the difference in electric potential between two points in an electric field. It represents the work done per unit charge moving a charge between those two points.

It is defined as the amount of work required to move a charge from one point to another in an electric field. It is directly related to the electric field, which determines the force experienced by a charge. The electric field pushes or pulls the charge, and the potential difference measures the energy transfer as the charge moves through the field.

Consider a region where there is a uniform electric field. If a positive test charge is moved against the direction of the electric field, from point A to point B, work is done against the electric field, and the charge gains potential energy. The electric potential at point A is higher than at point B, and the potential difference V between the two points is given by:

$$\begin{aligned}
\Delta V &= V_B - V_A \\
&= \frac{U_B - U_A}{q_0} \\
&= \frac{-W_{AB}}{q_0} \qquad (90) \\
&= \frac{-\displaystyle\int_A^B \vec{F} \cdot d\vec{r}}{q_0} \\
&= \frac{\displaystyle\int_B^A q_0 \vec{E} \cdot d\vec{r}}{q_0} \\
&= \int_B^A \vec{E} \cdot d\vec{r}. \qquad (91)
\end{aligned}$$

where ΔV represents the change in electric potential, V_B is the potential at point B (higher), and V_A is the potential at point A (lower).

The relationship between electric potential difference and the electric field (uniform) is given by:

$$\Delta V = Ed, \tag{92}$$

where d is the separation distance between the two points. This equation shows that the potential difference is directly proportional to the electric field strength and the separation distance.

9.2 Equipotential Surfaces and Gauss's Law (Again!)

These are imaginary surfaces in an electric field where all points on the surface have the same electric potential. In other words, they are surfaces that connect points with the same electric potential value.

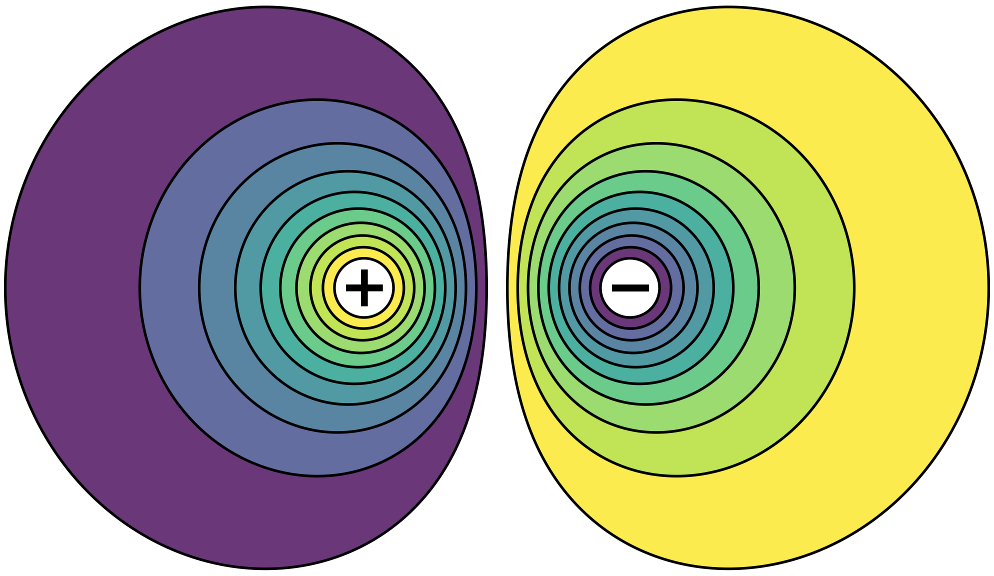


FIGURE 37: Contour map of the range of equipotential surfaces of an electric dipole. These surfaces get dense because the electric field is strong between the two component charges. This is so because a strong field means the places where you would get the same field reading change drastically, giving closely spaced equipotential surfaces.

A few important properties of equipotential surfaces are:

- **Gaussian surfaces**: All the Gaussian surfaces that we considered during field calculation were chosen to make up equipotential surfaces. That way, we can integrate within the surface without changing the field component anywhere.
- **Perpendicular to Electric Field Lines**: Equipotential surfaces are always perpendicular to the electric field lines at every point. This means the electric field lines intersect the equipotential surfaces at right angles.
- **No Potential Difference**: The electric potential remains constant within an equipotential surface. This implies that no work is required to move a charge along the surface because the potential difference is zero.
- **Uniform Potential**: All points on an equipotential surface have the same electric potential. The potential values may vary between different equipotential surfaces, but the potential remains constant within each surface.
- **Closer Spacing Indicates Stronger Field**: The spacing between equipotential surfaces indicates the strength of the electric field. Closer spacing between the surfaces indicates a stronger electric field, while broader spacing indicates a weaker field.

They represent regions in an electric field where the work done in moving a charge between any two points on the surface is zero. Since the electric potential is constant along an equipotential surface, no work is required to move a charge within that surface because the potential difference is zero. This implies that the electric field is perpendicular to the equipotential surface at every point.

10 Electric Potential Measurements for Charge Distributions

10.1 Discrete I: Electric Potential due to a Charged Particle

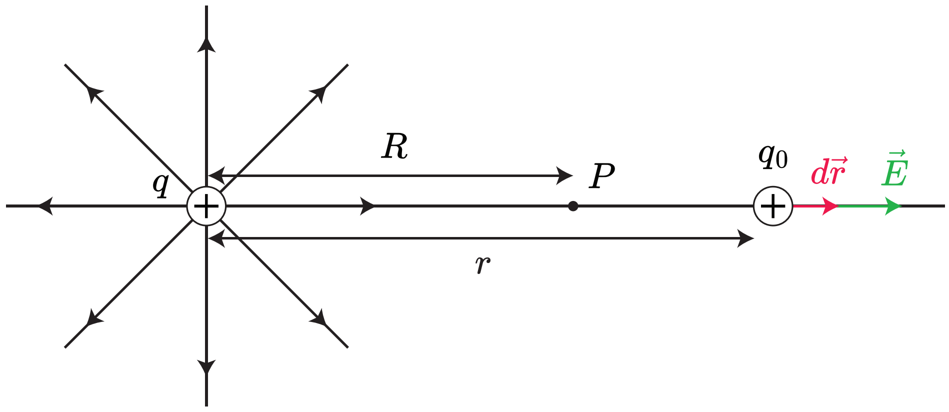


FIGURE 38: The particle with positive charge q produces an electric field and an electric potential V at point P. Moving a test charge q_0 from P to infinity, we find the potential. The test charge is shown at a distance r from the particle during differential displacement.

To find the potential of a point charge Q, we need to bring the test charge q_0 from P toward infinity. We need to remember that $V = 0$ at $r \to \infty$.

$$
\begin{aligned}
V_f - V_i &= -\int_{r=R}^{r=\infty} E dr \\
\Rightarrow 0 - V &= -\frac{Q}{4\pi\epsilon_0}\int_{r=R}^{r=\infty} \frac{dr}{r^2} \\
&= \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r}\right]_{r=R}^{r=\infty} \\
&= -\frac{Q}{4\pi\epsilon_0 R} \\
\therefore V &= \frac{Q}{4\pi\epsilon_0 R}.
\end{aligned}
\tag{93}
$$

10.2 Discrete II: Electric Potential due to a Collection of Charged Particle

Using the superposition principle, we need to find the net electric potential at a point due to a collection of point charges.

$$
V = \sum_i^n V_i = \frac{1}{4\pi\epsilon_0}\sum_i^n \frac{q_i}{r_i}. \tag{94}
$$

10.3 Continuous I: Electric Potential due to a Continuous Charge Distribution

When the charge distribution is continuous, we cannot sum the potential contribution but integrate it.

Let us take the $V = 0$ at $r \to \infty$. If we treat the element of charge dq as a particle, then we can express the potential dV due to dq:

$$dV = \frac{1}{4\pi\epsilon_0}\frac{dq}{r}, \tag{95}$$

dq can be either positive or negative.

The total electric potential would then be

$$V = \int dV = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{r}. \tag{96}$$

The integral must be taken over the entire charge distribution. Note that because the electric potential is a scalar, there are no vector components to consider.

10.4 Continuous II: Electric Potential due to a Charged Conducting Sphere

We can easily infer this using our ideas on the electric field of a spherical shell charge distribution or an excess charge distributed on a conducting sphere case. We found $\vec{E} = 0$ inside the shell(or sphere) in both cases. We found a maximum at the shell(or sphere) radius and points away from the source, $\vec{E}$ dies off as $\frac{1}{r^2}$. We use these values to calculate the work done per unit test charges to move it within this $\vec{E}$ in the three regions.

Region	**Electric Field**	**Electric Potential**
$r < R$	$E = 0$	$V = \frac{Q}{4\pi\epsilon_0 R}$
$r = R$	$E = \frac{Q}{4\pi\epsilon_0 R^2}$	$V = \frac{Q}{4\pi\epsilon_0 R}$
$r > R$	$E = \frac{Q}{4\pi\epsilon_0 r^2}$	$V = \frac{Q}{4\pi\epsilon_0 r}$

With no field inside the shell(or sphere), no work is done on a test charge that moves from any point to any other point inside the sphere, giving us a constant V. V becomes the maximum where E is the maximum and later dies off as you move farther outward.

10.5 Continuous III: Electric Potential due to a Finitely Long Line of Charge

Consider a thin nonconducting rod of length $2L$ with a positive charge of uniform linear density λ.

$$\begin{aligned} V = \int dV &= \frac{1}{4\pi\epsilon_0}\int \frac{dq}{r} \\ &= \frac{1}{4\pi\epsilon_0}\int_{y=-L}^{y=L} \frac{\lambda dy}{\sqrt{x^2+y^2}} \end{aligned} \tag{97}$$

In Eq. (97), we substitute $y = x\tan u$ with $u = \tan^{-1}\left(\frac{y}{x}\right)$.

$$\begin{aligned} y &= x\tan u \\ dy &= x\sec^2 u du. \\ x^2+y^2 &= x^2+(x\tan u)^2 \end{aligned}$$

$$
\begin{aligned}
&= x^2\left(1+\tan^2 u\right)\\
&= x^2 \sec^2 u.
\end{aligned}
$$

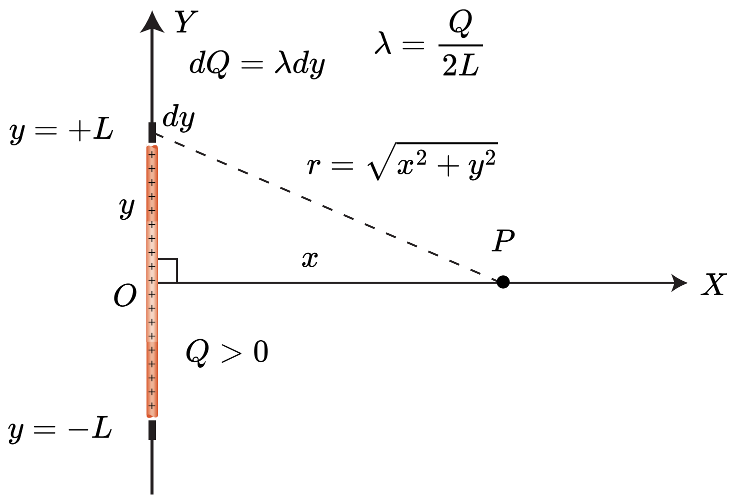


FIGURE 39: A thin, uniformly charged rod of length $2L$, produces an electric potential V at point P. An element is treated as a particle with a small charge $dQ = \lambda dy$.

With these substitutions, Eq. (97) now becomes:

$$V = \frac{\lambda}{4\pi\epsilon_0}\int \sec u du$$

Here, we must apply the limits of integration for u and not y because we swapped the integrating variable y.

y	$u = \tan^{-1}\left(\frac{y}{x}\right)$
$+L$	$\tan^{-1}\left(\frac{+L}{x}\right)$
$-L$	$\tan^{-1}\left(\frac{-L}{x}\right)$

With these substitutions, Eq. (97) now becomes:

$$
\begin{aligned}
V &= \frac{\lambda}{4\pi\epsilon_0}\int_{\tan^{-1}\left(\frac{-L}{x}\right)}^{\tan^{-1}\left(\frac{+L}{x}\right)} \sec u du\\
&= \frac{\lambda}{4\pi\epsilon_0}\left(\left[\ln|\sec u + \tan u|\right]_{\tan^{-1}\left(\frac{-L}{x}\right)}^{\tan^{-1}\left(\frac{+L}{x}\right)}\right)\\
&= \frac{\lambda}{4\pi\epsilon_0}\left(\ln\left|\sec\left[\tan^{-1}\left(\frac{L}{x}\right)\right] + \tan\left[\tan^{-1}\left(\frac{L}{x}\right)\right]\right| - \ln\left|\sec\left[\tan^{-1}\left(\frac{-L}{x}\right)\right] + \tan\left[\tan^{-1}\left(\frac{-L}{x}\right)\right]\right|\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\lambda}{4\pi\epsilon_0}\left(\ln\left|\sec\left[\sec^{-1}\left(\frac{\sqrt{x^2+L^2}}{x}\right)\right]+\tan\left[\tan^{-1}\left(\frac{L}{x}\right)\right]\right|\right.\\
&\qquad\qquad\left.-\ln\left|\sec\left[\sec^{-1}\left(\frac{\sqrt{x^2+L^2}}{x}\right)\right]+\tan\left[\tan^{-1}\left(\frac{-L}{x}\right)\right]\right|\right)\\
&= \frac{\lambda}{4\pi\epsilon_0}\left(\ln\left|\frac{\sqrt{x^2+L^2}}{x}+\frac{L}{x}\right|-\ln\left|\frac{\sqrt{x^2+L^2}}{x}+\left(\frac{-L}{x}\right)\right|\right)\\
&= \frac{\lambda}{4\pi\epsilon_0}\left(\ln\left|\frac{\sqrt{x^2+L^2}+L}{x}\right|-\ln\left|\frac{\sqrt{x^2+L^2}-L}{x}\right|\right)\\
&= \frac{\lambda}{4\pi\epsilon_0}\left(\ln\left|\sqrt{x^2+L^2}+L\right|-\ln x-\ln\left|\sqrt{x^2+L^2}-L\right|+\ln x\right)\\
&= \frac{\lambda}{4\pi\epsilon_0}\left(\ln\left|\frac{\sqrt{x^2+L^2}+L}{\sqrt{x^2+L^2}-L}\right|\right).
\end{aligned}
\tag{98}
$$

10.6 Continuous IV: Electric Potential due to an Infinitely Long Line of Charge

We may take this from an electric field point of view. But unlike the finite charged line segment, defining a potential bringing a charge from infinity and placing it near the source is impractical. Instead, we bring in the charge from a finite distance where $V = 0$ and move to another finite distance.

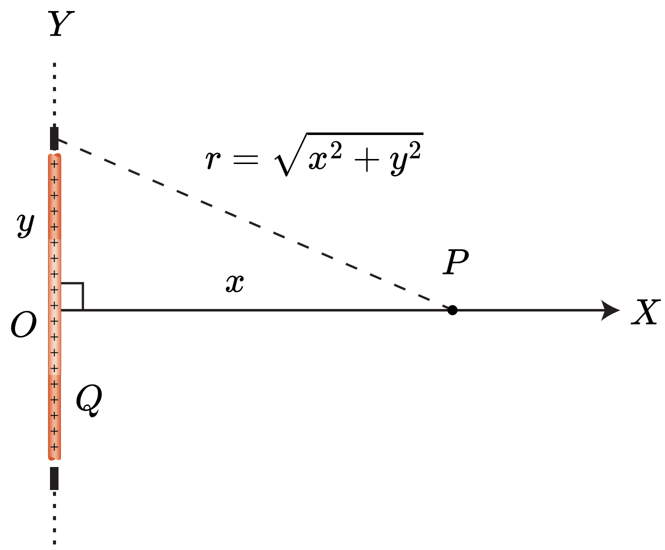


FIGURE 40: A thin, uniformly charged rod of infinite length produces an electric potential V at point P. The charged line has a uniform charge density λ.

Consider a positively charged infinitely long line of charge with positive line charge density λ. The line has field lines emanating radially outward from it. The potential of any point a with respect to b, radial distances r_a, and r_b from the line charge is

$$
\begin{aligned}
V_a - V_b &= \int_a^b E dr\\
&= \frac{\lambda}{2\pi\epsilon_0}\int_a^b \frac{dr}{r}
\end{aligned}
$$

$$= \frac{\lambda}{2\pi\epsilon_0} \ln \left| \frac{r_b}{r_a} \right|. \tag{99}$$

Consider a finite starting distance where $V = 0$. Then, we would get the potential for the point a with a radial distance $r_a = r$ from the source. If we take the starting point $r_b = r_0$.

$$V_b - 0 = V = \frac{\lambda}{2\pi\epsilon_0} \ln \frac{r_0}{r}. \tag{100}$$

10.7 Continuous V: Electric Potential due to an Infinitely Long Charged Cylinder

This is the extended case of potential due to a charged conducting sphere, except the electric field resembles that of an infinitely long charged line. Similar to the finite charged line segment, defining a potential bringing a charge from infinity and placing it near the source is impractical. Instead, we bring in the charge from a finite distance where $V = 0$ and move to another finite distance.

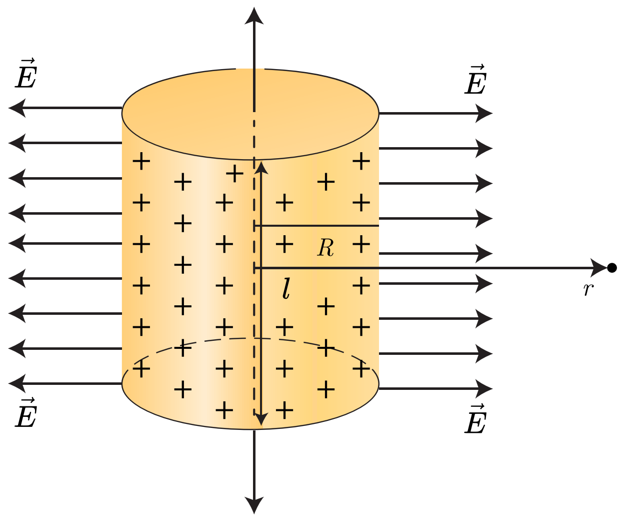


FIGURE 41: A cylinder of radius R, positively charged on its outer surfaces to a uniform surface charge density σ. Find the potential V at point P on the central axis of the cylinder.

Region	**Electric Field**	**Electric Potential**	
$r < R$	$E = 0$	$V = \frac{\lambda}{2\pi\epsilon_0} \ln \frac{R}{r}$	
$r = R$	$E = \frac{\lambda}{2\pi\epsilon_0 R}$	$V = 0$	
$r > R$	$E = \frac{\lambda}{2\pi\epsilon_0 r}$	$V = \ln \left(\frac{r_a}{r_b} \right)$	$r_a > r > r_b$

10.8 Continuous VI: Electric Potential due to a Charged Ring

Consider a ring of radius R with a positive charge Q of uniform line density λ (Fig. 42).

$$\begin{aligned} V = \int dV = &= \frac{1}{4\pi\epsilon_0} \int \frac{dq}{r} \\ &= \frac{1}{4\pi\epsilon_0} \frac{1}{\sqrt{x^2 + R^2}} \int dq \\ &= \frac{1}{4\pi\epsilon_0} \frac{Q}{\sqrt{x^2 + R^2}}. \end{aligned} \tag{101}$$

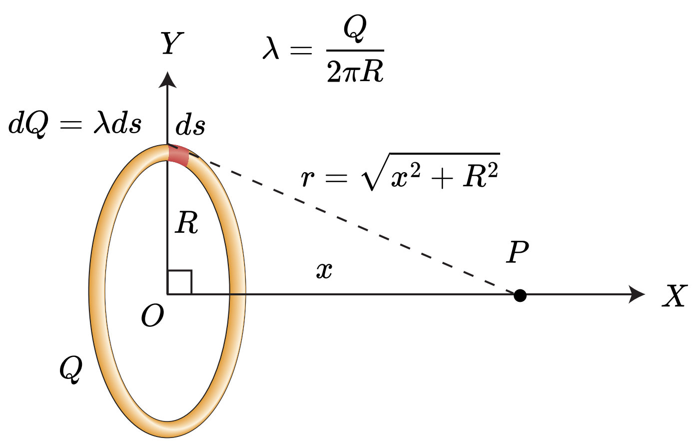


FIGURE 42: A ring of radius R, charged on its circumference to a uniform line charge density λ. Find the potential V at point P on the central axis of the disk.

10.9 Continuous VII: Electric Potential due to a Charged Flat Disk

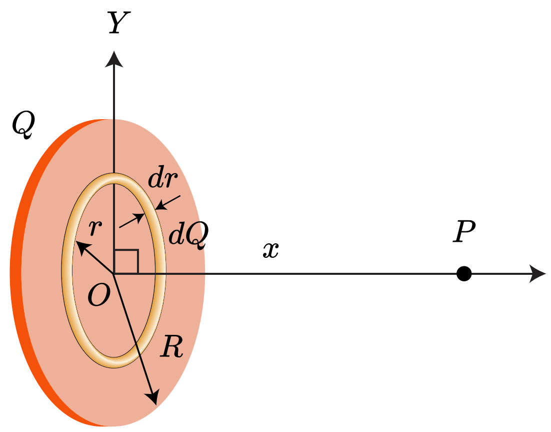


FIGURE 43: A flat disk of radius R, charged on its top surface to a uniform surface charge density σ. Find the potential V at point P on the central axis of the disk.

Consider a flat disk of radius R with a positive charge of uniform surface density σ. In this case $dq = \sigma(2\pi r)(dr)$.

$$\begin{aligned} V = \int dV &= \frac{1}{4\pi\epsilon_0} \int \frac{dq}{r} \\ &= \frac{1}{4\pi\epsilon_0} \int_{r=0}^{r=R} \frac{2\pi\sigma r dr}{\sqrt{x^2 + r^2}} \\ &= \frac{\sigma}{2\epsilon_0} \int_{r=0}^{r=R} \frac{r dr}{\sqrt{x^2 + r^2}} \end{aligned} \tag{102}$$

In Eq. (102), we substitute $u = x^2 + r^2$ such that $du = 2rdr$.

With these substitutions, Eq. (102) now becomes:

$$V = \frac{\sigma}{2\epsilon_0} \int \frac{du}{2\sqrt{u}}$$

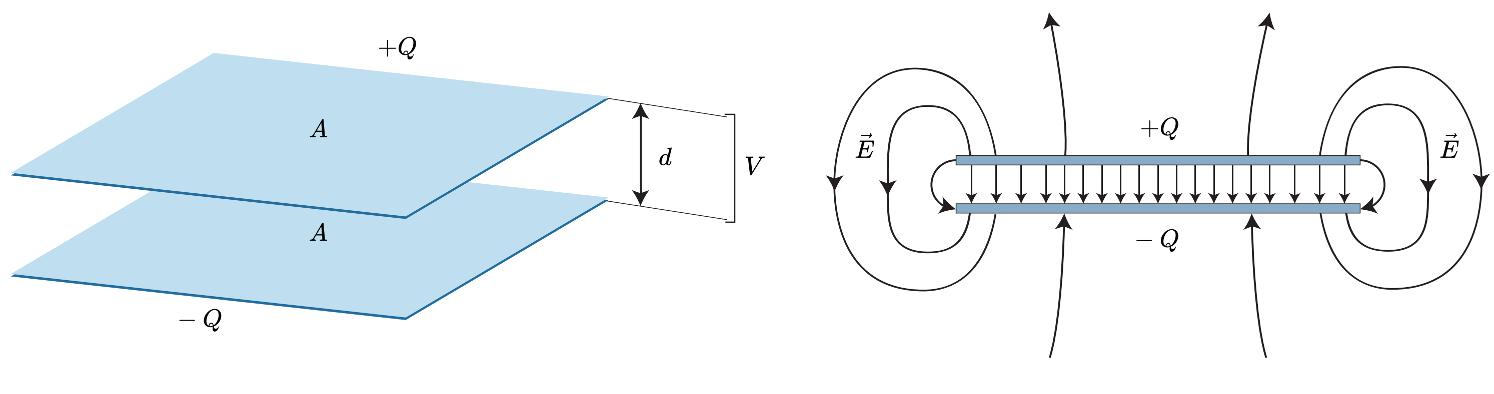
Here, we must apply the limits of integration for u and not r because we swapped the integrating variable r.

y	$u = x^2 + r^2$
0	x^2
R	$x^2 + R^2$

With these substitutions, Eq. (102) now becomes:

$$\begin{aligned} V &= \frac{\sigma}{2\epsilon_0}\left[\sqrt{u}\right]_{u=x^2}^{u=x^2+R^2} \\ &= \frac{\sigma}{2\epsilon_0}\left[\sqrt{x^2+R^2} - x\right]. \end{aligned} \tag{103}$$

11 Capacitance and Capacitors



(a) (b)

FIGURE 44: (a) A parallel-plate capacitor comprises two plates of area A separated by a distance d.The charges on the facing plate surfaces have the same magnitude, q, but opposite signs. (b) As the field lines show, the electric field due to the charged plates is uniform in the central region between the plates. The field is not uniform at the edges of the plates, as indicated by the **fringing** of the field lines.

Capacitors are electronic components designed to store and release electrical energy. They consist of two conductive plates separated by a dielectric material. When a voltage difference is applied across the plates, an electric field is established, and the capacitor stores electric charge, which can be later released.

When a capacitor charges, its plates have charges of equal magnitudes but opposite signs: $+Q$ and $-Q$.

NOTE: We call the charge of a capacitor Q. Remember, Q is not the net charge on the capacitor, which is zero.

Because the plates are conductors, they are equipotential surfaces; both share the same electric potential. There is a potential difference between the two plates applied using a battery source.

NOTE: Although we will calculate the potential difference across the two plates, we shall use V and not ΔV. Why? Historical reasons and conventions.

The capacitance of the two capacitors is then defined by

$$C = \frac{Q}{V}. \tag{104}$$

The capacitance measures how much charge must be put on the plates to produce a specific potential difference between them: *The greater the capacitance, the more charge is stored*. The SI unit for capacitance is the *farad* (F) 1 farad $= 1F = 1$ coulomb per volt $= \frac{\text{C}}{\text{V}}$.

11.1 Calculating Capacitance

We follow the following two steps:

1. **Calculate the Charge of the capacitor using Gauss's Law**

$$\begin{aligned} \epsilon_0 \oint \vec{E} \cdot d\vec{A} = Q_{\text{enc}} = Q. \\ Q = \epsilon_0 E \oint dA \\ \therefore Q = \epsilon_0 EA. \end{aligned} \tag{105}$$

2. **Calculate the Voltage of the capacitor**

$$\begin{aligned} V = V_f - V_i = -\int_i^f \vec{E} \cdot d\vec{r} \\ \therefore V = \int_{r=0}^{r=d} Edr = Ed. \end{aligned} \tag{106}$$

Thus, the primary formula for capacitance (of a parallel-plate capacitor) is

$$C = \frac{\epsilon_0 A}{d}. \tag{107}$$

ϵ_0 is the permittivity of free space in this equation. It measures the tendency of resistance to an electric field when a dielectric material is placed in the field. For any dielectric material, ϵ is the permittivity. For vacuum/air/free space, $\epsilon = \epsilon_0$.

NOTE: The Capacitance of a capacitor only depends on the geometry of the conductor plates.

The dielectric material, an insulator, prevents the direct flow of current between the plates. Instead, it allows the build-up of an electric field, causing the capacitor to store electrical energy in the form of accumulated charge.

11.2 How do Capacitors store Energy?

Capacitors retain electrical energy due to the separation of charge on the plates and the presence of the dielectric material. An external agent must do work to charge a capacitor. We can imagine doing the work ourselves by transferring electrons from one plate to the other, one by one. As the charges pile up, the electric field between the plates increases, which opposes the continued transfer. So, a greater amount of work is required. A battery does all this for us at the expense of its stored chemical energy. This spent energy is stored in the capacitor plate.

The stored energy can be released rapidly When the capacitor is connected to a circuit. The charge flows from one plate to another, discharging the capacitor and releasing the stored energy. This discharge can occur in a controlled manner, allowing capacitors to provide bursts of energy when needed.

Suppose $q^{'}$ is transferred to the plates in a short period. The potential difference between the plates at that same time is $V^{'} = \frac{q^{'}}{C}$. If an extra small amount of charge dq is transferred, the work increases as follows:

$$dW = V^{'}dq^{'} = \frac{q^{'}}{C}dq^{'}. \tag{108}$$

The total work done is evaluated by integrating both sides

$$W = \int_0^W dW = \frac{1}{C}\int_0^Q q^{'}dq^{'} = \frac{Q^2}{2C}. \tag{109}$$

This work is stored as electric potential energy in the field.

$$U = \frac{1}{2C}Q^2 = \frac{1}{2}CV^2. \tag{110}$$

In a parallel-plate capacitor, neglecting fringing, the electric field has the same value at all points between the plates. Thus, the energy density u—the potential energy per unit volume between the plates—should also be uniform.

$$u = \frac{U}{V} = \frac{CV^2}{2Ad} = \frac{1}{2}\epsilon_0\left(\frac{V}{d}\right)^2 = \frac{1}{2}\epsilon_0 E^2 \tag{111}$$

12 Capacitors in Circuit

12.1 Series Connection

In a series connection, the voltage across each capacitor is the same, and the total voltage across the series combination is equal to the sum of the voltages across each individual capacitor.

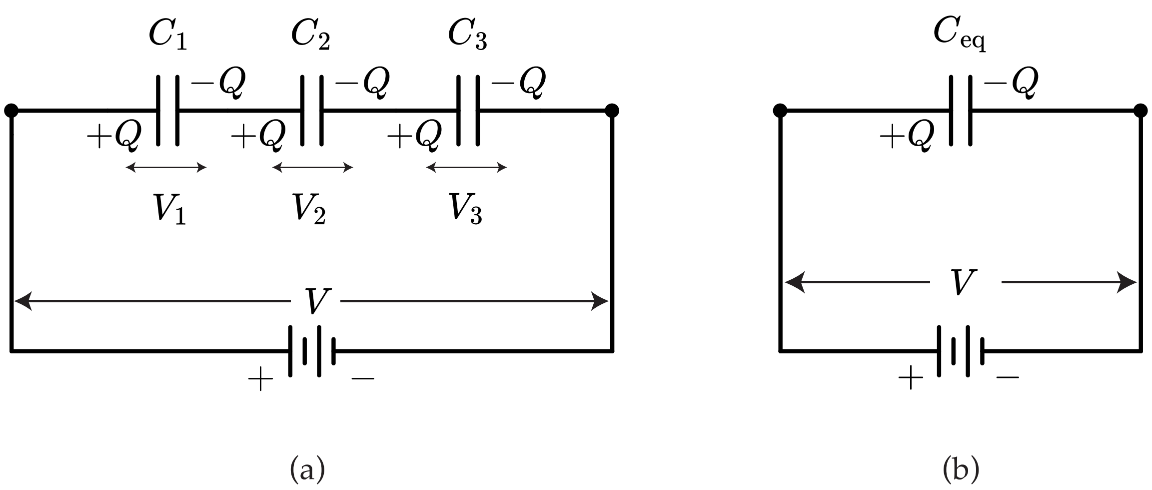


FIGURE 45: (a) Three capacitors connected in series to a battery. The battery maintains a potential difference V between the top and bottom plates of the series combination. (b) The equivalent capacitor, with capacitance C_{eq}, replaces the series combination.

When a potential difference V is applied across several capacitors connected in series, the capacitors have identical charge Q. The sum of the potential differences across all the capacitors is equal to the applied potential difference V.

From the circuit,

$$\begin{aligned} V_1 &= \frac{Q}{C_1}, \qquad V_2 = \frac{Q}{C_2}, \qquad V_3 = \frac{Q}{C_3} \qquad (112) \\ V &= V_1 + V_2 + V_3 \\ &= \frac{Q}{C_1} + \frac{Q}{C_2} + \frac{Q}{C_3} \\ V &= Q\left(\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}\right) \\ \frac{V}{Q} &= \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} \\ \frac{1}{C_{\text{eq}}} &= \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} = \sum_i^n \frac{1}{C_i}. \qquad (113) \end{aligned}$$

12.2 Parallel Connection

In a parallel connection, the voltage across each capacitor is the same, and the total charge stored in the combination is the sum of the charges stored in each individual capacitor.

From the circuit (Fig. 46),

$$\begin{aligned} Q_1 &= C_1 V, \qquad Q_2 = C_2 V, \qquad Q_3 = C_3 V \qquad (114) \\ Q &= Q_1 + Q_2 + Q_3 \\ &= C_1 V + C_2 V + C_3 V \\ Q &= V\left(C_1 + C_2 + C_3\right) \\ \frac{Q}{V} &= C_1 + C_2 + C_3 \\ C_{\text{eq}} &= C_1 + C_2 + C_3 = \sum_i^n C_i. \qquad (115) \end{aligned}$$

When a potential difference V is applied across several capacitors connected in parallel, that potential difference V is applied across each capacitor. The total charge q stored on the capacitors is the sum of the charges stored on all the capacitors.

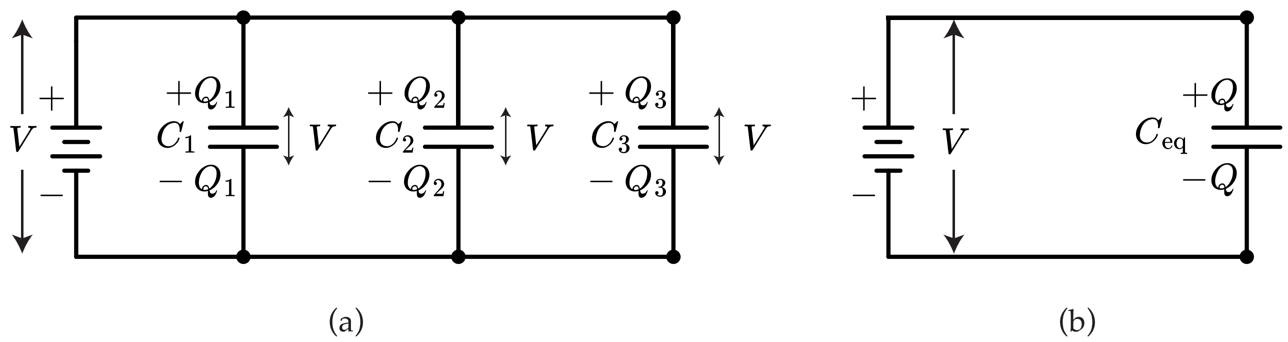
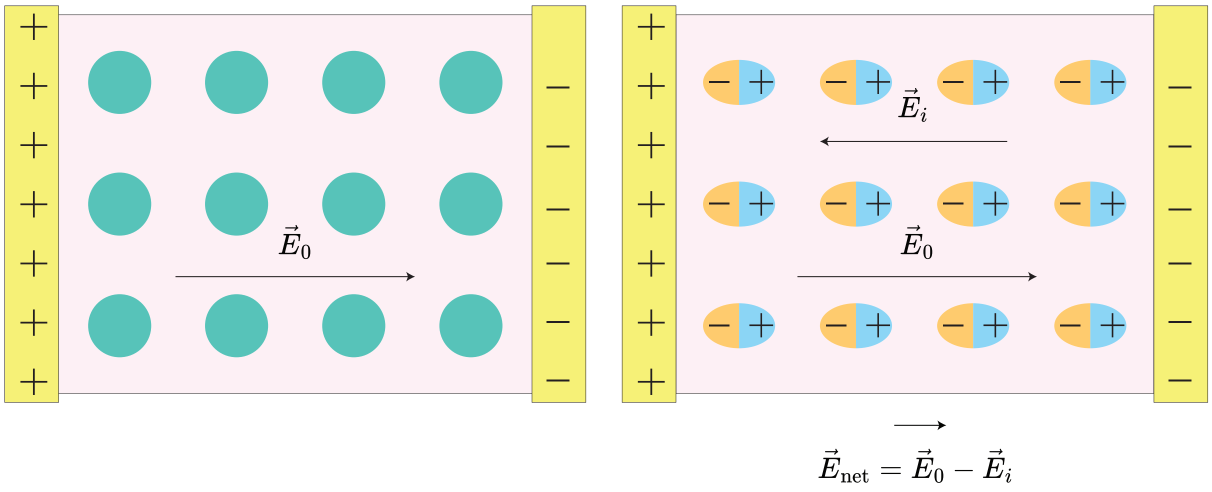


FIGURE 46: (a) Three capacitors connected in parallel to a battery. The battery maintains potential difference V across its terminals and, thus, across each capacitor. (b) The equivalent capacitor, with capacitance C_{eq}, replaces the parallel combination.

Capacitance with a Dielectric



(a) (b)

FIGURE 47: (a) A nonpolar dielectric slab is placed between the capacitor plates. The circles represent the electrically neutral atoms within the dielectric. (b) The field slightly stretches the atoms, separating the centers of a positive and negative charge. The separation produces surface charges on the slab faces. These charges set up a field in the opposite direction, generating $\vec{E}_i$. $\vec{E}_{\text{net}}$ is the new, post-dielectric field that works for the capacitor.

If we fill the space between the plates of a capacitor with a dielectric, the induced $\vec{E}_i$ reduces the original $\vec{E}$ field. The net field $\vec{E}_{\text{net}}$ reduces the potential difference between the capacitor plates. The capacitance must increase since the battery source must still provide Q to the plates. Assume without the dielectric, the capacitance is C_0. The increased capacitance is C. They are related to a constant called *dielectric constant*.

$$\frac{C}{C_0} = K = \frac{\frac{\epsilon A}{d}}{\frac{\epsilon_0 A}{d}} = \frac{\epsilon}{\epsilon_0}. \tag{116}$$

We can extend the above equation to other important parameters related to capacitance.

$$K = \frac{C}{C_0} = \frac{\frac{Q}{V}}{\frac{Q}{V_0}} = \frac{V_0}{V} = \frac{E_0 d}{E_{\text{net}} d} = \frac{E_0}{E_{\text{net}}}. \tag{117}$$

In the limit, the plate separation is very small in practice; we can assume the following for the electric fields at work for both cases: capacitors with and without a dielectric.

$$E_0 = \frac{\sigma}{\epsilon_0}, \qquad E_i = \frac{\sigma_i}{\epsilon_0}. \tag{118}$$

$$E_{\text{net}} = E_0 - E_i = \frac{1}{\epsilon_0}(\sigma - \sigma_i). \tag{119}$$

$$K = \frac{E_0}{E_{\text{net}}} = \frac{\sigma}{\sigma - \sigma_0}. \tag{120}$$

Using (116), we can generate all the equations for a capacitor with a dielectric inside.

$$C = \frac{\epsilon A}{d} = \frac{K \epsilon_0 A}{d}. \tag{121}$$

$$u = \frac{1}{2}\epsilon E^2 = \frac{1}{2}K\epsilon_0 E^2. \tag{122}$$

13 Electric Current versus Steady Electric Current

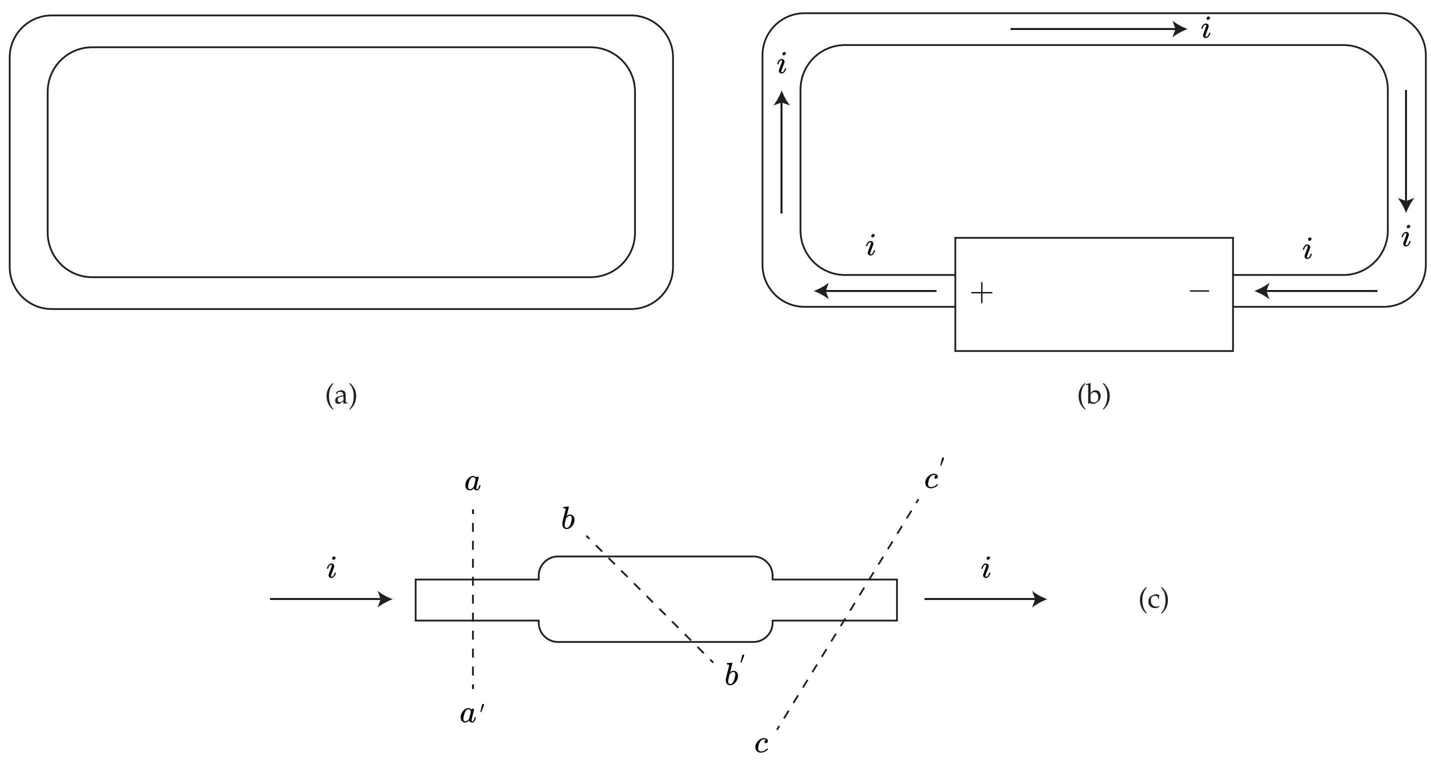


FIGURE 48: (a) A loop of copper in electrostatic equilibrium. The entire loop is at a single potential, and the electric field is zero at all points inside the copper. (b) Adding a battery imposes an electric potential difference between the ends of the loop that are connected to the terminals of the battery. The battery thus produces an electric field within the loop, from terminal to terminal, and the field causes charges to move around the loop. This movement of charges is a current i.

Current is defined as the rate of flow of electric charge. In the context of steady current, although the individual charges within the conductor are stationary, there is a continuous movement of charge carriers. This movement occurs at a constant rate, resulting in a steady flow of charges in a particular direction.

Mathematically, electric current is defined as:

$$I = \frac{dq}{dt}. \tag{123}$$

According to this equation, the current is directly proportional to the amount of charge passing through a point and inversely proportional to the time taken for that charge to pass.

We can find the charge that passes through a cross-section in a time interval:

$$Q = \int dq = \int idt \tag{124}$$

The ampere (A) is the SI unit for the current. One ampere is defined as the flow of one coulomb of charge per second:

$$1\text{ A} = \frac{1\text{ C}}{1\text{ s}}. \tag{125}$$

Steady current refers to the flow of electric charges in a conductor that remains constant over time.

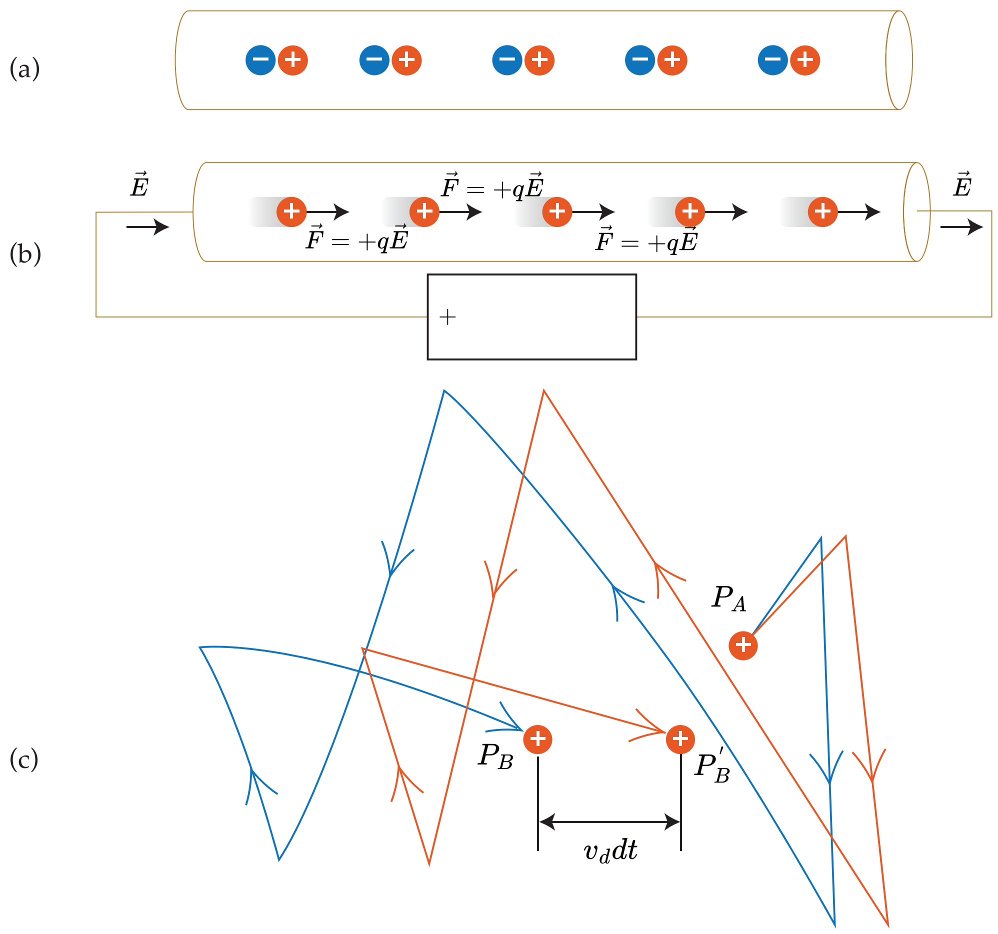


FIGURE 49: (a) If there is no electric field inside a conductor, a positive charge carrier moves randomly from point P_A to point P_B in a time dt. (b) If an electric field $\vec{E}$ is present, the electric force $\vec{F} = +q\vec{E}$ imposes a small drift (greatly exaggerated here). (c) This drift velocity takes the positive charge to point $P_B^{'}$, a distance $v_d dt$ from P_B in the direction of the force.

The charge carriers are typically electrons in a conductor, such as a wire. Under the influence of an electric field generated by a voltage source, electrons experience a force that propels them to move in a specific direction. As electrons move, they transfer their charge to neighboring electrons, creating a *domino effect* of charge transfer throughout the conductor.

Even though the individual charges (electrons) may not have a net motion, the transfer of charge from one electron to another results in a net flow of charge in the conductor. This net flow constitutes the steady current.

13.1 A Deeper Look at Current: Drift Velocity and Model of Metallic Conduction

Drift velocity refers to the *average* velocity of charged particles, typically electrons, in a conductor due to an applied electric field. It measures the charge carriers' net motion in the direction of the electric current.

In a conductor (Figure 49), electrons cannot move at high speeds due to collisions with atoms or ions in the conductor's lattice structure. These collisions cause the charged particles to scatter and change direction randomly. An electric field exerts a force on the electrons, causing them to move in the direction opposite to the field (from negative to positive potential). As the electrons move, they experience frequent collisions with the lattice, which randomize their direction. However, an overall **drift motion** emerges under the influence of the applied electric field.

Suppose there are n moving charged particles per unit volume. Assume that all the particles move with the same drift velocity with magnitude v_d. In a time interval dt, each particle moves a distance $v_d dt$. The particles that flow out of the right end of the shaded cylinder with length $v_d dt$ during dt are the particles within this cylinder at the beginning of the interval dt. This interval gets to have a name of its own, *mean time between collisions*, τ. It measures the average time elapsed between two successive collisions.

The volume of the cylinder is $V = Av_d dt$, and the number of particles within it is $nAv_d dt$. If each particle has a charge q, the charge dQ that flows out of the end of the cylinder during time dt is $dQ = q(nAv_d dt) = nqv_d Adt$.

The drift velocity v_d is related to the current I and the number of charge carriers n per unit volume, passing through a cross-section A by the equation:

$$I = \frac{dQ}{dt} = nqv_d A. \tag{126}$$

This equation shows that the current is directly proportional to the drift velocity. The higher the drift velocity of charge carriers, the greater the current flow.

However, drift velocity is relatively low, typically on the order of millimeters per second, due to the frequent collisions experienced by the charge carriers. Using 1st laws of motion from mechanics:

$$v_d = \left(\frac{qE}{m}\right)\tau. \tag{127}$$

NOTE: It is important to note that the drift velocity is not the speed at which electrical signals or energy propagate through a circuit. Signals propagate nearly at the speed of light.

13.2 A Deeper Look at Current: Current Density and its Vector Property

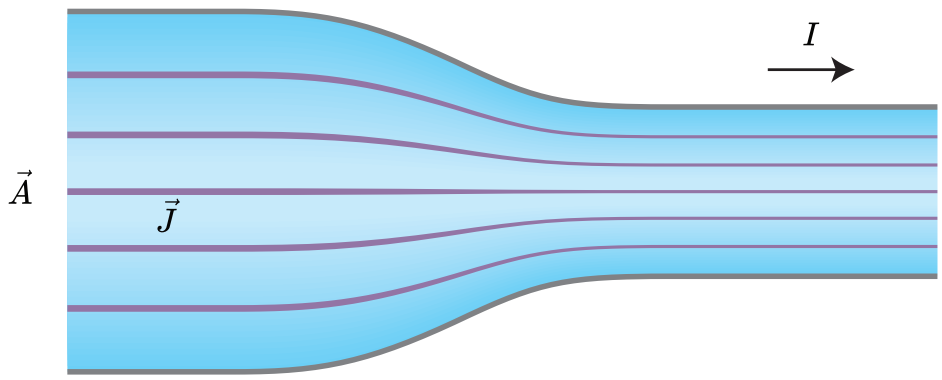


FIGURE 50: Streamlines representing current density in the flow of charge through a constricted conductor.

Imagine a localized view and study the flow of charge through a cross-section A of the conductor at a particular point, having the current per unit density J. This density is a vector quantity and points in the direction of the drift velocity of the charge carrier producing the current. The current through a small cross-sectional element is

$$dI = \vec{J} \cdot d\vec{A} \tag{128}$$
$$I = \int dI = \int JdA = J \int dA = JA$$

$$J = \frac{I}{A} = nqv_d. \tag{129}$$

14 Ohm's Law

From the previous section, we found from Eqs. (126) and (127):

$$I = nqA\left(\frac{qE}{m}\right)\tau \tag{130}$$

$$J = \frac{I}{A} = nq\left(\frac{qE}{m}\right)\tau$$

$$J = \left(\frac{nq^2\tau}{m}\right)E \tag{131}$$

The current density $\vec{J}$ in a conductor depends on the electric field $\vec{E}$ and on the properties of the material (everything inside the parentheses). In general, this dependence can be quite complex. But for some materials, especially metals, at a given temperature, $\vec{J}$ is nearly directly proportional to $\vec{E}$. The proportionality constant is called Conductivity (measured in Siemens $\sigma = \Omega^{-1}\,\mathrm{m}^{-1}$).

Taking the reciprocal of Eq. (131), gives us a different look at the same relation:

$$E = \left(\frac{m}{nq^2\tau}\right)J \tag{132}$$

The terms in the parentheses also form a new proportionality constant called *Resistivity* ρ, measured in $\Omega\,\mathrm{m}$, which is simply the reciprocal of σ.

This relationship (131 and 132) is known as Ohm's Law. The more famous form can be easily derived from them.

$$E = \rho J \tag{133}$$

$$\frac{\Delta V}{L} = \rho\frac{I}{A} \tag{134}$$

$$V = \left(\frac{\rho L}{A}\right)I \tag{135}$$

The term in parenthesis is called *Resistance*:

$$R = \rho\frac{L}{A}, \tag{136}$$

where ρ is the specific resistance of the material making up the conductor. Resistance is not necessarily constant for each wire or conductor. It can vary depending on temperature, length, cross-sectional area, and the material used.

Resistance refers to the property of a material or component to resist the flow of electric current. It measures how effectively a material hinders the movement of charged particles (usually electrons) when an electric potential difference is applied across it. Resistance is denoted by the R symbol and measured in ohms Ω.

Mathematically, Ohm's Law (Eq. (135)) can be expressed as:

$$\Delta V = IR. \tag{137}$$

Ohm's Law can be understood using an analogy of water flowing through a pipe. Imagine the flow of water as the flow of electric current, the pressure difference in the pipe as the voltage, and the narrowness of the pipe as the resistance. If you increase the pressure difference (voltage) in the pipe, more water will flow through it. Similarly, if you increase the voltage across a conductor, more current will flow through it. On the other hand, if the pipe becomes narrower (increased resistance), the water flow will decrease. Likewise, if the resistance of a conductor increases, the current flow will decrease.

A more realistic form of Ohm's Law is:

$$I = \frac{\Delta V}{R} \tag{138}$$

Since current is produced in the first place due to potential differences, it should convince You of this form. Remember, ΔV is the cause, and I is the effect, in practice. Materials that follow Ohm's law are called Ohmic materials.

14.1 When Ohm's Law no Longer Works

While Ohm's Law holds true for many materials and circuits under certain conditions, it is not universally valid in all situations. Here are some scenarios where Ohm's Law may break down:

1. **Non-Ohmic Materials**: Ohm's Law specifically applies to ohmic materials, which have a linear relationship between voltage and current. Ohmic materials obey Ohm's Law within a specific range of voltages and currents. However, some materials, like diodes or transistors, exhibit non-linear behavior and do not follow Ohm's Law.

2. **Temperature Dependency**: Ohm's Law assumes a constant temperature. However, in some cases, temperature changes can lead to variations in resistance, which can affect the accuracy of Ohm's Law. For example, as the temperature increases, the resistance of some materials may also increase, causing deviations from Ohm's Law.

3. **Nonlinear Circuit Elements**: Circuits that include nonlinear elements, such as transistors or diodes, cannot accurately be described by Ohm's Law since these elements exhibit nonlinear voltage-current relationships.

4. **High Frequencies**: At high frequencies, especially in circuits involving inductors and capacitors, the reactance (frequency-dependent resistance) may come into play and alter the behavior of the circuit. Ohm's Law may not fully describe the circuit behavior in such cases.

15 Electric Circuits

Electric circuits are interconnected paths through which electric current can flow. They consist of electrical components such as resistors, capacitors, inductors, and switches connected by conductive wires. From previous chapters, we know that moving a charge q within an electric field $\vec{E}$ causes it to gain/lose electric potential energy ΔU_E. Let this current pass through a circuit element, and we can receive this electric potential energy in a more usable, practical form, say a light bulb, cell phone, or computer. Electric circuits are the foundation of modern electrical and electronic devices, providing a means to control and utilize electrical energy for various applications.

15.1 Electromotive Force of a Power Source

Electromotive force (EMF) is the electrical potential difference across the terminals of a power source, like a battery or a generator, **when it is not connected to any external circuit**. It represents the ability of the source to drive or push charges through a circuit. EMF is measured in volts V.

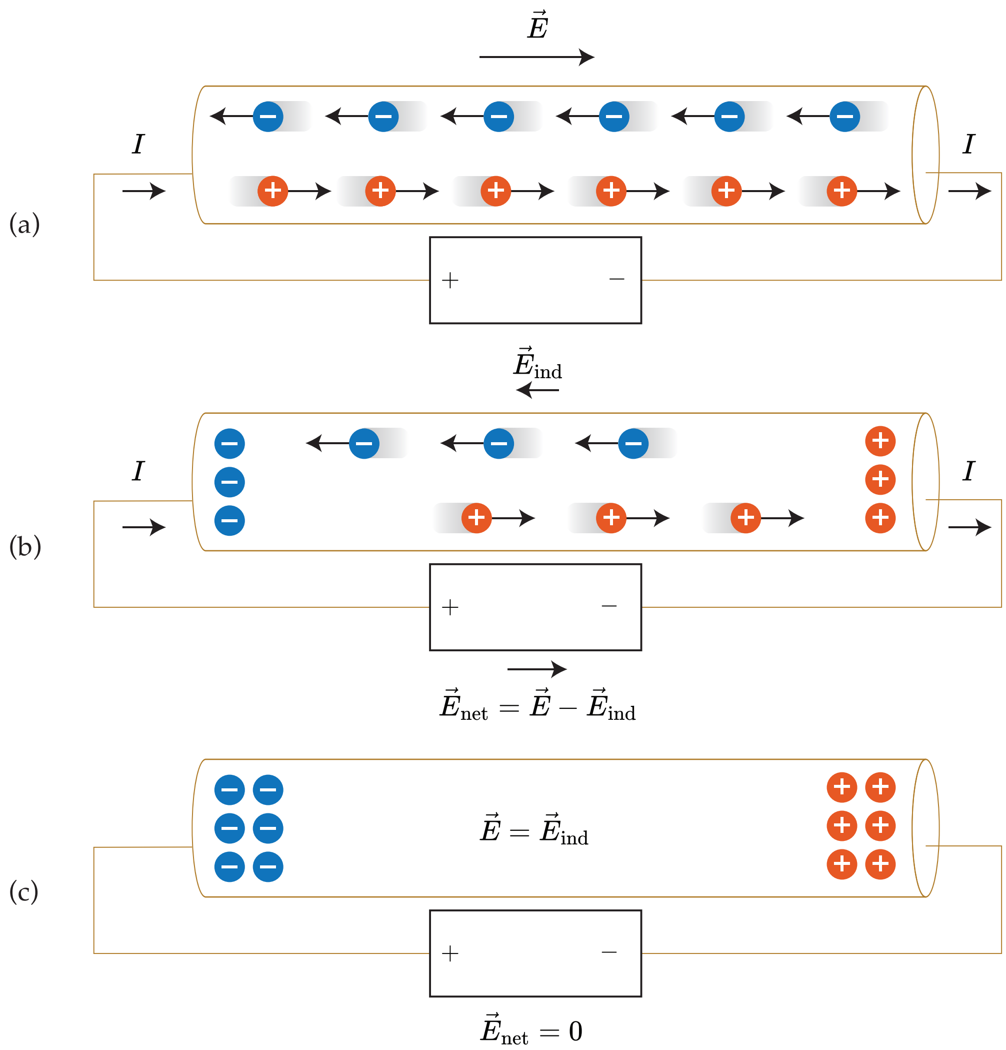


FIGURE 51: (a) An electric field $\vec{E}$ produced inside an isolated conductor causes a current. (b) The current causes charge to build up at the ends. The charge build-up produces an opposing field $\vec{E}_{\text{ind}}$, thus reducing the current. (c) After a very short time, $\vec{E}_{\text{ind}}$ has the same magnitude as $\vec{E}$; then the net field is $\vec{E}_{net} = \vec{E} - \vec{E}_{\text{ind}}$=0, and the current stops completely.

When a source has an EMF, it creates an electric field within the source that pushes charges from one terminal to another, creating a persistent potential difference between the terminals. To understand the necessity of an EMF, we must learn the scenario without it.

Consider a small segment of the current-carrying conductor, and an electric field is applied across it. Due to this field, the charge carriers will gain a net drift motion and produce a flow of current. But inside the conductor segment, soon, charge carriers of opposite signs pile up in the opposite end, creating an induced electric field $\vec{E}_{\text{ind}}$ that opposes the original field. The current I now owes its strength to the net surviving electric field $\vec{E}_{net} = \vec{E} - \vec{E}_{\text{ind}}$. Sooner or later, this induced field matches the strength of the original field, and the current stops flowing, beating the whole purpose of a continuous current flow by means of an electric circuit. This is where EMF comes in.

The presence of EMF establishes an electric potential difference that can make charges move in a circuit against the battery's polarity. A persistent EMF hinders the build-up of an induced field and keeps the circuit *on*.

Define the EMF $\mathcal{E}$ as the voltage difference between the two terminals of an ideal battery source:

$$\mathcal{E} = \frac{dW}{dq} = V_{ab}. \tag{139}$$

15.2 A Tale of Real Batteries

In practice, however, batteries have internal resistance r due to the inherent characteristics of their chemical composition and construction. The physical intuition behind the internal resistance can be understood in terms of the obstacles the charge carriers face as they travel within the battery. When an external load (e.g., a light bulb or a motor) is connected to the battery, a current is drawn from the battery to power the load. However, as the current flows through the battery, it encounters resistance within the battery itself. This internal resistance acts like a **bottleneck** that hinders the smooth flow of current.

The internal resistance effectively reduces the voltage available at the battery terminals compared to its open-circuit voltage $\mathcal{E}$ (voltage without any load connected). The net available potential difference a battery can then supply is:

$$V_{ab} = \mathcal{E} - Ir. \tag{140}$$

Comparing this to Ohm's Law, the net current provided by the battery when an external load R is connected would become:

$$V_{ab} = IR = \mathcal{E} - Ir$$
$$I = \frac{\mathcal{E}}{r+R}. \tag{141}$$
$$r = \frac{\mathcal{E} - IR}{I} \tag{142}$$

16 Resistors in Electric Circuits

16.1 Series Connection

In a series connection, the voltage across each resistor is not the same, and the total voltage across the series combination is equal to the sum of the voltages across each individual resistor. **When a potential difference V is applied across several capacitors connected in series, the resistors allow the identical current flow I. The sum of the potential difference drops across all the resistors is equal to the applied potential difference V.**

From the circuit in Figure (52),

$$V_1 = IR_1, \qquad V_2 = IR_2, \qquad V_3 = IR_3 \tag{143}$$
$$\begin{aligned} V &= V_1 + V_2 + V_3 \\ &= IR_1 + IR_2 + IR_3 \\ V &= I(R_1 + R_2 + R_3) \\ \frac{V}{I} &= R_1 + R_2 + R_3 \end{aligned}$$
$$R_{\text{eq}} = R_1 + R_2 + R_3 = \sum_i^n R_i. \tag{144}$$

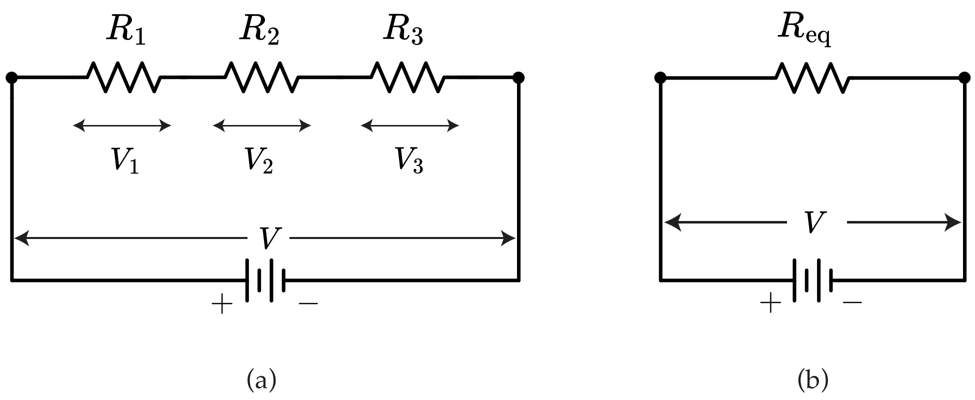


FIGURE 52: (a) Three resistors connected in series to a battery. The battery maintains a potential difference V given by Eq. (140) between the top and bottom plates of the series combination. (b) The equivalent resistor, with resistance R_{eq}, replaces the series combination.

16.2 Parallel Connection

In a parallel connection, the voltage across each resistor is the same, and the current, however, splits up in the junction to a different value for each individual resistor.

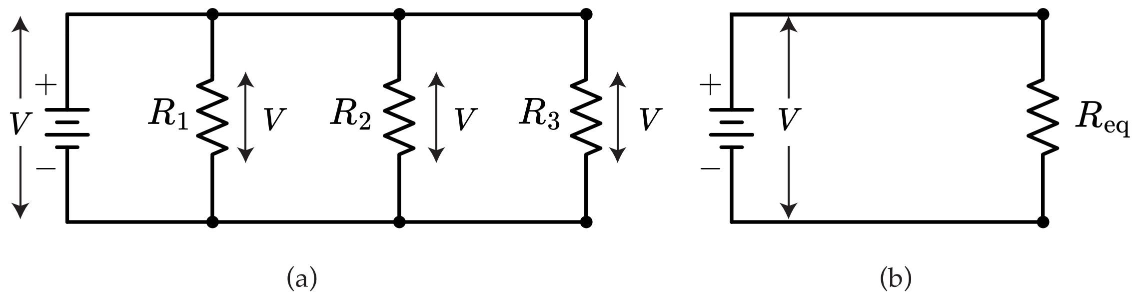


FIGURE 53: (a) Three resistors connected parallel to a battery. The battery maintains a potential difference V given by Eq. (140) between the top and bottom plates of the parallel combination. (b) The equivalent resistor, with resistance R_{eq}, replaces the parallel combination.

From the circuit in Figure (53),

$$I_1 = \frac{V}{R_1}, \qquad I_2 = \frac{V}{R_2}, \qquad I_3 = \frac{V}{R_3} \tag{145}$$

$$\begin{aligned} I &= I_1 + I_2 + I_3 \\ &= \frac{V}{R_1} + \frac{V}{R_2} + \frac{V}{R_3} \\ I &= V\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right) \\ \frac{I}{V} &= \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \\ \frac{1}{R_{eq}} &= \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} = \sum_i^n \frac{1}{R_i}. \end{aligned} \tag{146}$$

When a potential difference V is applied across several resistors connected in parallel, the total current I breaks into different parts for each resistor, and the sum of all the currents recombines at the junction at the other end of the connection.

17 Power in Electric Circuits

In an electric circuit, power is the rate at which electrical energy U is transferred or consumed. It represents how quickly electrical energy is converted into other forms of energy (such as heat, light, or mechanical work) within the circuit.

$$P = \frac{dW_{ab}}{dt} = \frac{-dU_{ba}}{dt} = \frac{-dqV_{ba}}{dt} = V_{ab}\frac{dq}{dt} = V_{ab}I. \tag{147}$$

If the potential at a is lower than at b, then V_{ab} is negative, and there is a net transfer of energy (and power) out of the circuit element. The element then acts as a source, delivering electrical energy (and power) into the circuit to which it is attached. This is the usual situation for a battery, which converts chemical energy into electrical energy and delivers it to the external circuit. Thus qV_{ab} can denote a quantity of energy that is either delivered to a circuit element or extracted from that element, say R.

$$P = I^2R = \frac{V_{ab}^2}{R}. \tag{148}$$

The unit of electrical power is Watt (W).

$$1\text{W} = \frac{1\text{J}/1\text{C}}{1\text{C}/1\text{s}} = \frac{1\text{J}}{1\text{s}}.$$

What Happens to this Power?

The moving charges collide with atoms in the resistor and transfer some of their energy to these atoms, increasing the internal energy of the material. Either the temperature of the resistor increases, or there is a flow of heat out of it, or both. Every resistor has a power rating, the maximum power the device can dissipate without becoming overheated and damaged.

17.1 Power Output of a Circuit

The power output of a battery source with an electromotive force (EMF) and internal resistance is the electrical power the battery delivers to an external circuit. The power output is the rate at which the battery transfers energy to the load connected to it.

Consider a battery with an EMF $\mathcal{E}$ (voltage across its terminals when no current is flowing) and an internal resistance r. The battery is connected to an external load with resistance R, forming a complete circuit.

When a current I flows through the circuit, there is a voltage drop across the internal resistance r, and the voltage across the external load V_{ab} (terminal voltage) is given by Eq. (140). Merging that with Eq. (147) gives us:

$$P_{\text{out}} = V_{ab}I = \mathcal{E}I - I^2r = P_B - P_R. \tag{149}$$

$$P_B = \mathcal{E}I. \tag{150}$$

$$P_R = I_2R. \tag{151}$$

P_B is the rate at which work is done on the circulating charges by whatever agency causes the nonelectrostatic force in the source, i.e., the battery. This term represents the rate of conversion of nonelectrical energy to electrical energy within the source.

P_R is the rate at which electrical energy is dissipated in the internal resistance of the source.

The difference $P_{\text{out}} = \mathcal{E}I - I^2 r$ is the net electrical power output of the source—that is, the rate at which the source delivers electrical energy to the remainder of the circuit.

17.2 Power Input of a Circuit

To calculate the power input to a circuit, we only need to flip the direction of the current I by applying a battery with higher EMF to the lower-EMF battery present in the circuit.

Because of this reversal of current, instead of Eq. (140), we have for the power receiving source:

$$V_{ab} = \mathcal{E} + Ir. \tag{152}$$

Merging that with Eq. (147) gives us:

$$P_{\text{in}} = V_{ab}I = \mathcal{E}I + I^2 r = P_B + P_R. \tag{153}$$
$$P_B = \mathcal{E}I. \tag{154}$$
$$P_R = I_2 R. \tag{155}$$

This is what happens when a rechargeable battery (a storage battery) is connected to a charger. The charger supplies electrical energy to the battery; part of it is converted to chemical energy, to be reconverted later, and the remainder is dissipated (wasted) in the battery's internal resistance, warming the battery and causing a heat flow out of it. If you have a power tool or laptop computer with a rechargeable battery, you may have noticed that it gets warm while it is charging.

18 Solving DC Circuits: Kirchoff's Laws

Named after German physicist Gustav Kirchhoff (*Kir-koff*), critical in electrical circuit analysis and governs the current and voltage behavior in a circuit. These laws, known as Kirchhoff's current law (KCL) and Kirchhoff's voltage law (KVL), are essential tools for understanding and analyzing complex electrical circuits.

18.1 Kirchhoff's Current Law (KCL)

KCL states that the algebraic sum of currents at any junction (or node) in a circuit is zero. In other words, the total current flowing into a node equals the total current flowing out. This law is based on the principle of charge conservation.

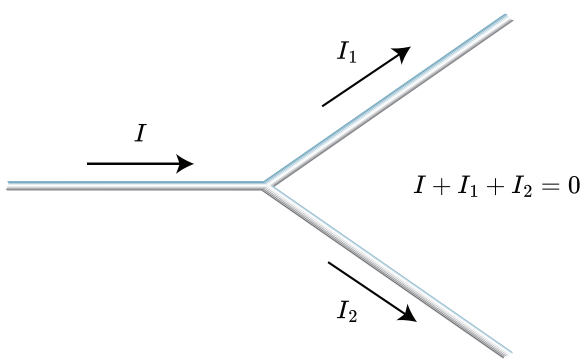


FIGURE 54: Kirchhoff's Current law or junction rule states that as much current flows into a junction as flows out of it.

Imagine a junction in a circuit as a meeting point where different paths for current flow converge, much like a traffic intersection. The sum of all the incoming currents I at that junction must equal the sum of all the outgoing currents $I_1 + I_2$. It's like cars entering an intersection from different directions and then dispersing along different roads. The total number of cars entering the intersection must equal the number of cars leaving it.

We will use **Nodal analysis** to implement KCL in our circuit-solving problems. Nodal analysis, also known as the node-voltage method, is a technique that uses Kirchhoff's Current Law (KCL) to determine the voltage at each node in a circuit.

TAKEAWAY: KCL establishes the Conservation of Electric Charge principle for electric circuits.

18.2 Kirchhoff's Voltage Law (KVL)

KVL states that the sum of the voltage drops (or rises) around any closed loop in a circuit equals the sum of the voltage sources (such as batteries) in that loop. This law is based on the principle of energy conservation.

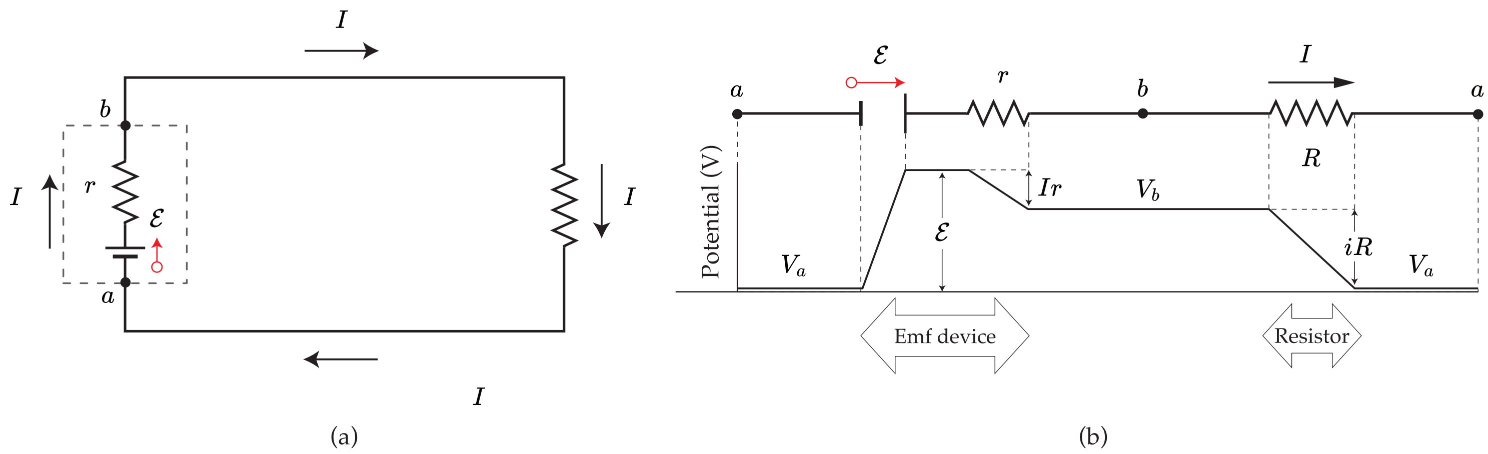


FIGURE 55: Kirchhoff's Voltage law or loop rule states that the algebraic sum of the changes in potential encountered in a complete traversal of any loop of a circuit must be zero.

$$\mathcal{E} - iR - IR = \sum V = 0$$

Think of a closed loop in a circuit as a path you can travel, much like a loop in a roller coaster ride. You may encounter uphill climbs and downhill descents as you move along the loop. The sum of all the changes in height (potential) you experience during the loop must be zero. The energy you gain during uphill climbs must equal the energy you lose during downhill descents.

We will use **Loop analysis** to implement KVL. Loop analysis, also known as the mesh current method, uses Kirchhoff's Voltage Law (KVL) to determine the currents flowing in each circuit loop.

TAKEAWAY: KCL establishes the Conservation of Energy principle for electric circuits and the fact that **Electric Force is conservative**.

Sign Convention for the KVL

Resistance Rule: For a move through resistance in the direction of the current, the change in potential is $-iR$; in the opposite direction, it is $+iR$.

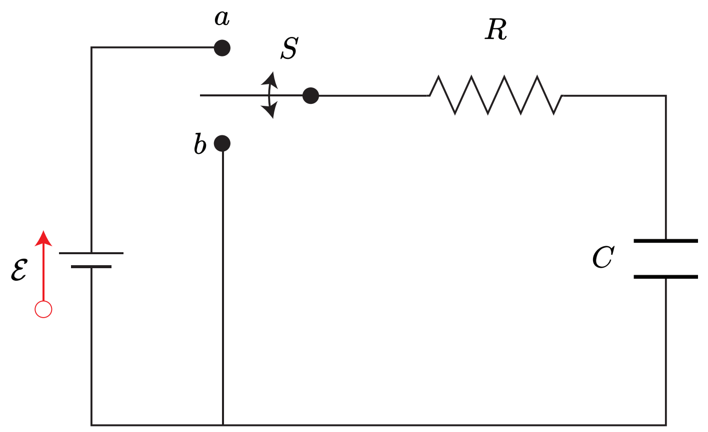


FIGURE 56: When switch S is closed on a, the capacitor is charged through the resistor. When the switch is afterward closed on b, the capacitor discharges through the resistor.

EMF Rule: For a move through an ideal EMF device in the direction of the EMF arrow ($-$ to $+$), the change in potential is $+\mathcal{E}$; in the opposite direction, it is $-\mathcal{E}$.

19 Transient Series RC Circuits

A transient current, also known as a *transient response* or transient behavior, refers to **the temporary, short-lived flow of electric current in an electrical circuit that occurs in response to a sudden change or disturbance in the circuit**. Transient currents are characterized by their relatively brief duration, and they occur during the transient period when the circuit transitions from one steady-state condition to another. We will look at two different transient response systems in this course. The first one is the RC circuit discussed below. The second one, the RL circuit, will be discussed after we are introduced to *Inductors*.

19.1 Charging of a Capacitor: Growth of Charge

Consider the circuit with the battery, resistor, and capacitor connected in series. The voltage across the capacitor at any time t is denoted as V, and the charge stored in the capacitor at time t is $q(t)$.

Use the KVL for the circuit. The instantaneous potential difference V_{ab} and V_{bc} are: $v_{ab} = iR$ and $v_{bc} = \dfrac{q}{C}$.

$$\begin{aligned}
\mathcal{E} - iR - \frac{q}{C} &= 0 \\
i &= \frac{\mathcal{E}}{R} - \frac{q}{RC} \\
\frac{dq}{dt} &= \frac{\mathcal{E}}{R} - \frac{q}{RC} \\
\frac{dq}{dt} &= -\frac{1}{RC}\left(q - C\mathcal{E}\right) \\
\frac{dq}{q - C\mathcal{E}} &= -\frac{dt}{RC}.
\end{aligned} \tag{156}$$

The integration limits are $q = 0$ at $t = 0$ to $q = q$ at t,

$$\int_0^q \frac{dq}{q - C\mathcal{E}} = -\int_0^t \frac{dt}{RC}$$

$$\begin{aligned}
\ln\left(\frac{q - C\mathcal{E}}{-C\mathcal{E}}\right) &= -\frac{t}{RC} \\
\frac{q - C\mathcal{E}}{-C\mathcal{E}} &= e^{-\dfrac{t}{RC}} \\
q(t) &= C\mathcal{E}\left(1 - e^{-\dfrac{t}{RC}}\right). \qquad (157)
\end{aligned}$$

This is the charge equation for a capacitor while charging in a series RC circuit. It shows how the charge in the capacitor increases with time as it charges up to the maximum charge allowed by the circuit and the capacitor.

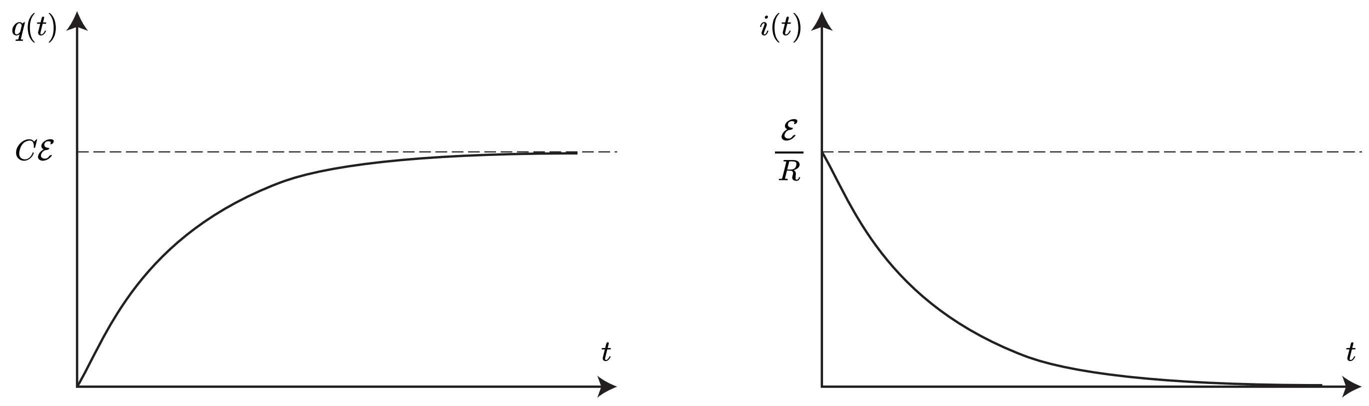


FIGURE 57: Charge q increases exponentially with time t. At the same time, Current i decreases exponentially.

At $t = 0$, Eq. (157) leads to:

$$q(t) = C\mathcal{E}\left(1 - e^{-\dfrac{0}{RC}}\right) = 0.$$

At $t \to \infty$, Eq. (157) leads to:

$$q(t) = C\mathcal{E}\left(1 - e^{-\dfrac{\infty}{RC}}\right) = C\mathcal{E} = Q_f.$$

19.2 Charging of a Capacitor: Decay of Current

Differentiating Eq. (157) with respect to t once gives the transient current equation during the charging period of the capacitor.

$$i(t) = \frac{dq}{dt} = \frac{\mathcal{E}}{R}e^{-\dfrac{t}{RC}}. \qquad (158)$$

At $t = 0$, Eq. (158) leads to:

$$i(t) = \frac{\mathcal{E}}{R}e^{-\dfrac{0}{RC}} = \frac{\mathcal{E}}{R} = I_0.$$

At $t \to \infty$, Eq. (158) leads to:

$$i(t) = \frac{\mathcal{E}}{R} e^{-\frac{\infty}{RC}} = 0.$$

19.3 Charging of a Capacitor: Decay of Resistive Voltages

During the charging phase, the resistive voltage V_R can be found by manipulating Eq. (157):

$$V_R^{\text{charging}}(t) = i(t)R = \mathcal{E} e^{-\frac{t}{RC}}. \tag{159}$$

At $t = 0$, Eq. (159) leads to:

$$V_R^{\text{charging}}(t) = \mathcal{E}\left(1 - e^{-\frac{0}{RC}}\right) = \mathcal{E}.$$

At $t \to \infty$, Eq. (159) leads to:

$$V_R^{\text{charging}}(t) = \mathcal{E}\left(1 - e^{-\frac{\infty}{RC}}\right) = 0.$$

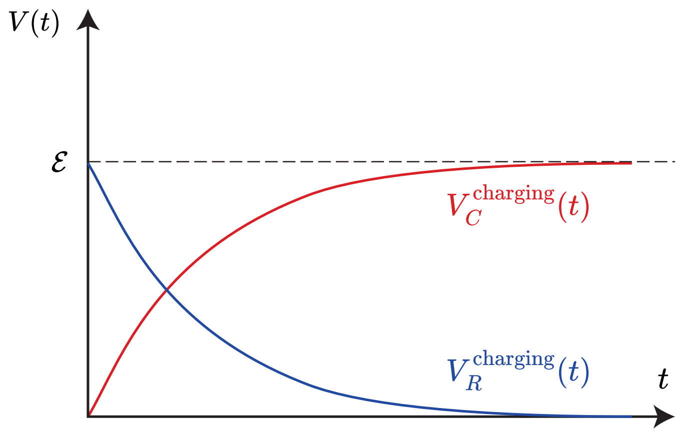


FIGURE 58: Resistive voltage V_R decreases exponentially with time t. At the same time, the Capacitive voltage V_C increases exponentially during charging.

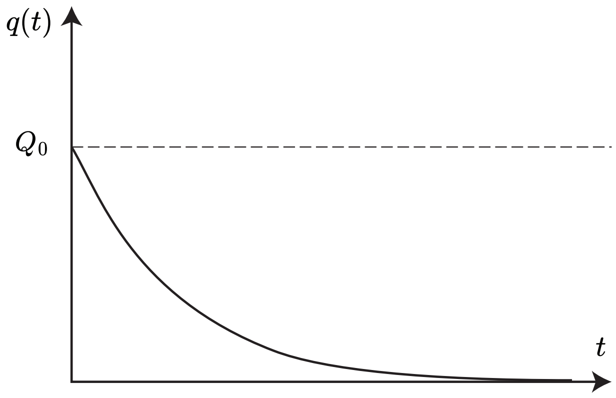
19.4 Charging of a Capacitor: Growth of Capacitive Voltages

During the charging phase, the voltage can be found by manipulating Eq. (157):

$$V_C^{\text{charging}}(t) = \frac{q(t)}{C} = \mathcal{E}\left(1 - e^{-\frac{t}{RC}}\right). \tag{160}$$

At $t = 0$, Eq. (160) leads to:

$$V_C^{\text{charging}}(t) = \mathcal{E}\left(1 - e^{-\frac{0}{RC}}\right) = 0.$$



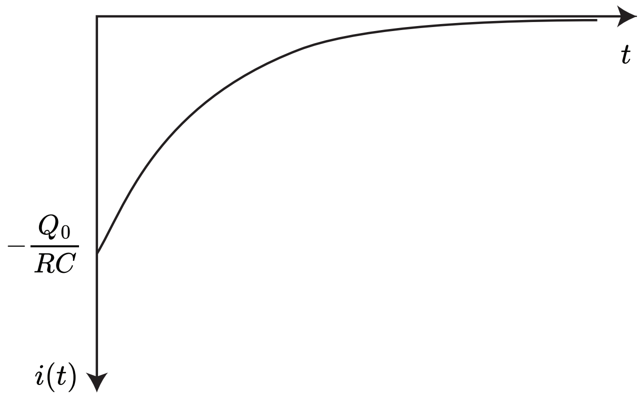


FIGURE 59: Charge q decreases exponentially with time t. At the same time, the Current i increases exponentially.

At $t \to \infty$, Eq. (160) leads to:

$$V_C^{\text{charging}}(t) = \mathcal{E}\left(1 - e^{-\frac{\infty}{RC}}\right) = \mathcal{E}.$$

19.5 Discharging of a Capacitor: Decay of Charge

Consider the same RC circuit above but have the capacitor discharge now from an arbitrary maximum charge Q_0 across the load R. Only the EMF is absent now in the circuit.

$$\begin{aligned} iR - \frac{q}{C} &= 0 \\ i &= -\frac{q}{RC} \\ \frac{dq}{dt} &= -\frac{q}{RC} \\ \frac{dq}{q} &= -\frac{dt}{RC} \end{aligned} \tag{161}$$

The integration limits are $q = Q_0$ at $t = 0$ to $q = q$ at t,

$$\begin{aligned} \int_{Q_0}^{q} \frac{dq}{q} &= -\int_0^t \frac{dt}{RC} \\ \ln\left(\frac{q}{Q_0}\right) &= -\frac{t}{RC} \\ \frac{q}{Q_0} &= e^{-\frac{t}{RC}} \\ q(t) &= Q_0 e^{-\frac{t}{RC}}. \end{aligned} \tag{162}$$

At $t = 0$, Eq. (162) leads to:

$$q(t) = Q_0 e^{-\frac{t}{RC}} = Q_0.$$

At $t \to \infty$, Eq. (162) leads to:

$$q(t) = Q_0 e^{-\frac{\infty}{RC}} = 0.$$

This is the discharge equation for a capacitor in a series RC circuit. It shows how the charge in the capacitor decreases with time as it discharges to the minimum charge allowed by the circuit and the capacitor.

19.6 Discharging of a Capacitor: Growth of Current

Differentiating Eq. (162) with respect to t once gives the transient current equation during the discharging period of the capacitor.

$$i = \frac{dq}{dt} = -\frac{Q_0}{RC} e^{-\frac{t}{RC}}. \tag{163}$$

At $t = 0$, Eq. (163) leads to:

$$i = -\frac{Q_0}{RC} e^{-\frac{0}{RC}} = -\frac{Q_0}{RC} = I_0.$$

At $t \to \infty$, Eq. (163) leads to:

$$i = -\frac{Q_0}{RC} e^{-\frac{\infty}{RC}} = 0.$$

19.7 Discharging of a Capacitor: Growth of Resistive Voltages

During the discharging phase, the voltage can be found by manipulating Eq. (162):

$$V_R^{\text{discharging}}(t) = \frac{q(t)}{C} = \frac{Q_0}{C} e^{-\frac{t}{RC}}. \tag{164}$$

At $t = 0$, Eq. (164) leads to:

$$V_R^{\text{discharging}}(t) = -\frac{Q_0}{C} e^{-\frac{0}{RC}} = -\frac{Q_0}{C}.$$

At $t \to \infty$, Eq. (164) leads to:

$$V_R^{\text{discharging}}(t) = -\frac{Q_0}{C} e^{-\frac{\infty}{RC}} = 0.$$

19.8 Discharging of a Capacitor: Decay of Capacitive Voltages

During the discharging phase, the voltage can be found by manipulating Eq. (162):

$$V_C^{\text{discharging}}(t) = \frac{q(t)}{C} = \frac{Q_0}{C} e^{-\frac{t}{RC}}. \tag{165}$$

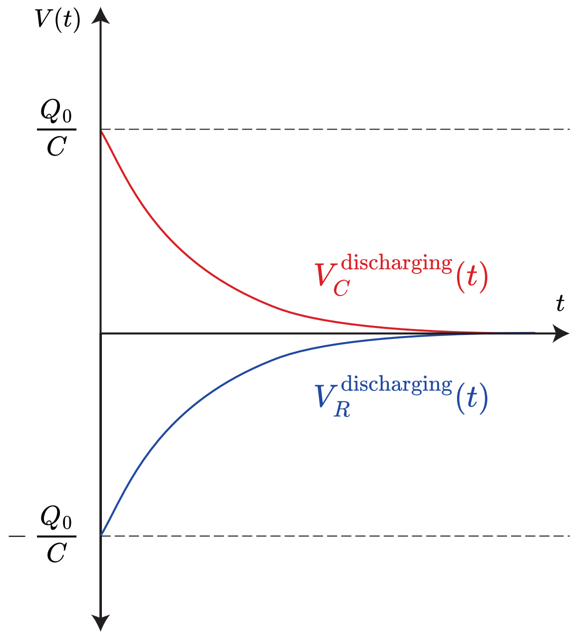


FIGURE 60: Resistive voltage V_R increases exponentially with time t to zero. At the same time, the Capacitive voltage V_C decreases exponentially during discharging.

At $t = 0$, Eq. (165) leads to:

$$V_C^{\text{discharging}}(t) = \frac{Q_0}{C} e^{-\frac{0}{RC}} = \frac{Q_0}{C} = V_0 = \mathcal{E}.$$

At $t \to \infty$, Eq. (165) leads to:

$$V_C^{\text{discharging}}(t) = \frac{Q_0}{C} e^{-\frac{\infty}{RC}} = 0.$$

In practice, both these cycles occur one after the other. We may attach each characteristic curve with others in sequence to get the whole picture of charging and discharging.

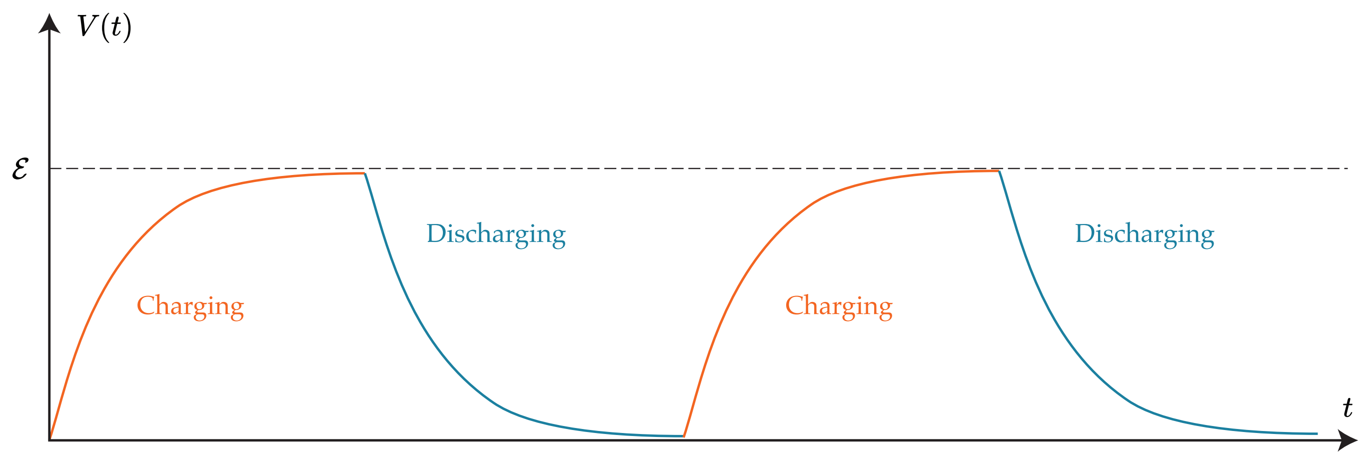


FIGURE 61: Voltage across the capacitor plates rises and decays following the Eqs. (160) and (165) with time.