Summary of Progress on the Storque 3D state space model

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Abstract

A comprehensive summary on progress to date on the Storque 3D state space simulation include control. This work is largely based on the work by *Mellinger, et. al.*.

1 Nomenclature, Coordinate Systems, and General Setup

We'll use a Z-X-Y Euler angle setup. ϕ is on X', θ is on Y'', ψ is on Z. Note that these X, Y, Z are in the body frame which is described by the unit vectors $\hat{e}_{bx}, \hat{e}_{by}, \hat{e}_{bz}$, respectively. The world frame is described by $\hat{e}_x, \hat{e}_y, \hat{e}_z$, and the rotation matrix from the body frame to the world frame is:

$$^{B}[R]^{W} = \begin{bmatrix} \cos\psi\cos\theta - \sin\phi\sin\psi\sin\theta & \cos\theta\sin\psi + \cos\psi\sin\phi\sin\theta & -\cos\phi\sin\theta \\ -\cos\phi\sin\psi & \cos\phi\cos\psi & \sin\phi \\ \cos\psi\sin\theta + \cos\theta\sin\phi\sin\psi & \sin\psi\sin\theta - \cos\psi\cos\theta\sin\phi & \cos\phi\cos\theta \end{bmatrix} \tag{1.1}$$

$$\bar{\bar{M}} = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \tag{1.2}$$

$$\bar{\bar{I}} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \tag{1.3}$$

$$\vec{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{1.4}$$

$$\vec{\omega} = \begin{bmatrix} p \\ q \\ r \end{bmatrix} \tag{1.5}$$

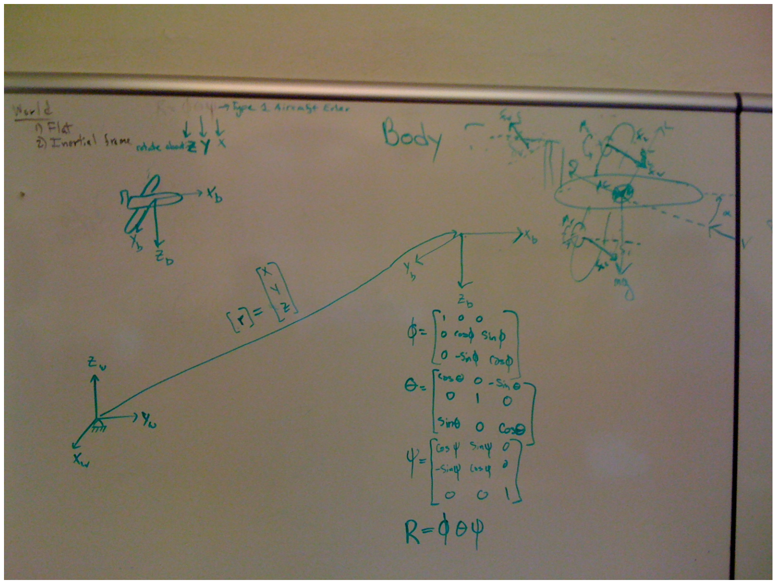

Figure 1: Storque coordinate system

Where p, q, r are angular velocities in the body frame corresponding to the $\hat{e}_{bx}, \hat{e}_{by}, \hat{e}_{bz}$axes, respectively.

$$(T_1, T_2, T_3, T_4)\, and\, (\omega_1, \omega_2, \omega_3, \omega_4)\, and\, (M_1, M_2, M_3, M_4) \tag{1.6}$$

Are the thrust, angular speed, and moment in the e_{bz} direction of each prop (labelled in **??**)

2 Equations of Motion

2.1 Linear Acceleration

$$\bar{\bar{M}} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix} + {}^{B}[R]^{W} \begin{bmatrix} 0 \\ 0 \\ -T_1 - T_2 - T_3 - T_4 \end{bmatrix} \tag{2.1}$$

2.2 Angular Acceleration

$$\bar{\bar{I}} \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} L\,(T_4 - T_3) \\ L\,(T_1 - T_2) \\ M_1 + M_2 - M_3 - M_4 \end{bmatrix} - \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times \bar{\bar{I}} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \tag{2.2}$$

Where:

$$\begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = nastyvectorIneedtolookupagain \tag{2.3}$$

3 Analysis

3.1 Motor Characteristics

It has been shown by *Mellinger, et. al.* and suggested by our faculty advisor Dr. Bruce D. Kothmann that the thrust and moment generated by a prop is proportional to angular velocity squared, or $T_i = K_{ti}\omega_i^2$ and $M_i = K_{mi}\omega_i^2$ where K_{ti} and K_{mi} are experimentally determined gains.

Mellinger, et. al. and Dr. Kothmann also suggested that:

$$\dot{\omega_i} = K_{mot}\left(\omega_{i,com} - \omega_i\right)$$

3.2 Equations of Motion with Motor Characteristics

3.2.1 Linear Acceleration

$$\bar{\bar{M}} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix} + {}^{B}[R]^{W} \begin{bmatrix} 0 \\ 0 \\ K_t\left(\omega_1^2 + \omega_2^2 + \omega_3^2 + \omega_4^2\right) \end{bmatrix} \tag{3.1}$$

3.2.2 Angular Acceleration

$$\bar{\bar{I}} \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} K_tL\left(\omega_4^2 - \omega_3^2\right) \\ K_tL\left(\omega_1^2 - \omega_2^2\right) \\ K_m\left(\omega_1^2 + \omega_2^2\right) - K_m\left(\omega_3^2 + \omega_4^2\right) \end{bmatrix} - \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times \bar{\bar{I}} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \tag{3.2}$$

Where we assume $K_{t1} = K_{t2} = ... = K_t$ and $K_{m1} = K_{m2} = ... = K_m$ because all four motor/prop combinations are identical.

3.3 Hover Trim

At hover, $\dot{\vec{\omega}} = \vec{\omega} = \vec{0}$ and $\phi = \theta = \psi = 0$ and $\ddot{r} = \dot{r} = \vec{0}$. This says:

$${}^{B}[R]^{W} = [I]_{3x3} = identity$$

And so:

$$\begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix} = {}^{B}[R]^{W} \begin{bmatrix} 0 \\ 0 \\ K_t\left(\omega_1^2 + \omega_2^2 + \omega_3^2 + \omega_4^2\right) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ K_t\left(\omega_1^2 + \omega_2^2 + \omega_3^2 + \omega_4^2\right) \end{bmatrix} \tag{3.3}$$

Also:

$$\begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} K_tL\left(\omega_4^2 - \omega_3^2\right) \\ K_tL\left(\omega_1^2 - \omega_2^2\right) \\ K_m\left(\omega_1^2 + \omega_2^2\right) - K_m\left(\omega_3^2 + \omega_4^2\right) \end{bmatrix} \tag{3.4}$$

Which implies:

$$\omega_1^2 = \omega_2^2 = \omega_3^2 = \omega_4^2 = \omega_{trim}^2$$

Finally we have:

$$4K_t\omega_{trim}^2 = mg$$

Or:

$$\omega_{trim} = \sqrt{\frac{mg}{4K_t}} \tag{3.5}$$

3.4 Control

3.5 Non-Linear State Space Representation

Our state vector is defined as:

$$\vec{x} = \begin{pmatrix} x & y & z & \dot{x} & \dot{y} & \dot{z} & \phi & \theta & \psi & p & w & r & \omega_1 & \omega_2 & \omega_3 & \omega_4 \end{pmatrix}^T \tag{3.6}$$

Our input vector is:

$$\vec{u} = (something) \tag{3.7}$$

The rate of change of state without control is defined, in pieces, below:

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \left(\bar{\bar{M}}\right)^{-1} \left\{ \begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix} + {}^{B}[R]^{W} \begin{bmatrix} 0 \\ 0 \\ K_t\left(\omega_1^2 + \omega_2^2 + \omega_3^2 + \omega_4^2\right) \end{bmatrix} \right\} \tag{3.8}$$

$$\begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \left(\bar{\bar{I}}\right)^{-1} \left\{ \begin{bmatrix} K_tL\left(\omega_4^2 - \omega_3^2\right) \\ K_tL\left(\omega_1^2 - \omega_2^2\right) \\ K_m\left(\omega_1^2 + \omega_2^2\right) - K_m\left(\omega_3^2 + \omega_4^2\right) \end{bmatrix} - \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times \bar{\bar{I}} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \right\} \tag{3.9}$$

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & -\cos\phi\sin\theta \\ 0 & 1 & \sin\phi \\ \sin\theta & 0 & \cos\phi\cos\theta \end{bmatrix}^{-1} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \tag{3.10}$$

$$\begin{bmatrix} \dot{\omega_1} \\ \dot{\omega_2} \\ \dot{\omega_3} \\ \dot{\omega_4} \end{bmatrix} = K_{mot} \begin{bmatrix} \omega_{1,des} - \omega_1 \\ \omega_{2,des} - \omega_2 \\ \omega_{3,des} - \omega_3 \\ \omega_{4,des} - \omega_4 \end{bmatrix} \tag{3.11}$$

Where we have not defined $\omega_{i,des}$ yet.

3.6 Linearization

If we assume $\bar{\bar{I}}_{ij} \ll \bar{\bar{I}}_{ii}$ because of the symmetry of our design, then the $\vec{\omega} \times \bar{\bar{I}}\vec{\omega}$ term in (**??**) is $\approx \vec{0}$.