Summary of Progress on the Storque 3D state space model

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Abstract

A comprehensive summary on progress to date on the Storque 3D state space simulation include control. This work is largely based on the work by *Mellinger, et. al.*.

1 Nomenclature, Coordinate Systems, and General Setup

We'll use a Z-X-Y Euler angle setup. ϕ is on X', θ is on Y'', ψ is on Z. Note that these X, Y, Z are in the body frame which is described by the unit vectors $\hat{e}_{bx}, \hat{e}_{by}, \hat{e}_{bz}$, respectively. The world frame is described by $\hat{e}_x, \hat{e}_y, \hat{e}_z$, and the rotation matrix from the body frame to the world frame is:

$$^{B}[R]^{W} = \begin{bmatrix} \cos\psi\cos\theta - \sin\phi\sin\psi\sin\theta & \cos\theta\sin\psi + \cos\psi\sin\phi\sin\theta & -\cos\phi\sin\theta \\ -\cos\phi\sin\psi & \cos\phi\cos\psi & \sin\phi \\ \cos\psi\sin\theta + \cos\theta\sin\phi\sin\psi & \sin\psi\sin\theta - \cos\psi\cos\theta\sin\phi & \cos\phi\cos\theta \end{bmatrix} \tag{1.1}$$

$$\bar{\bar{M}} = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \tag{1.2}$$

$$\bar{\bar{I}} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \tag{1.3}$$

$$\vec{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{1.4}$$

$$\vec{\omega} = \begin{bmatrix} p \\ q \\ r \end{bmatrix} \tag{1.5}$$

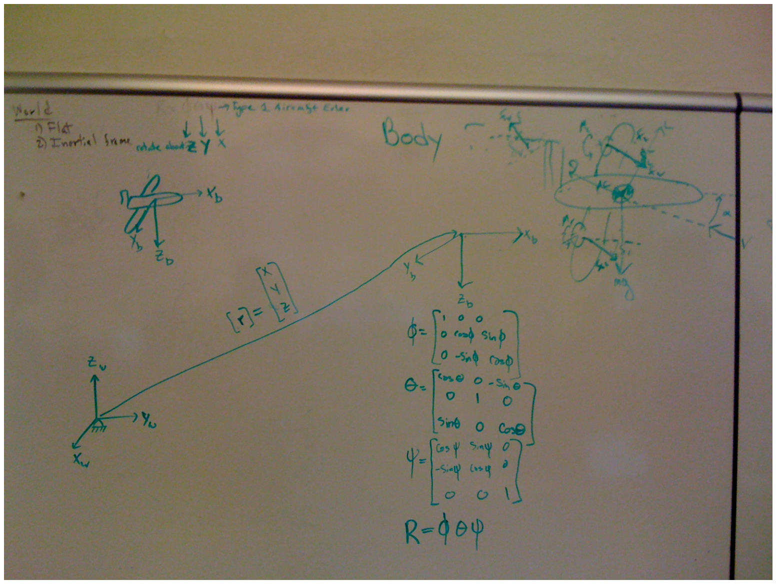

Figure 1: Storque coordinate system

Where p, q, r are angular velocities in the body frame corresponding to the $\hat{e}_{bx}, \hat{e}_{by}, \hat{e}_{bz}$axes, respectively.

$$(T_1, T_2, T_3, T_4)\, and\, (\omega_1, \omega_2, \omega_3, \omega_4)\, and\, (M_1, M_2, M_3, M_4) \tag{1.6}$$

Are the thrust, angular speed, and moment in the e_{bz} direction of each prop (labelled in 1)

2 Equations of Motion

2.1 Linear Acceleration

$$\bar{\bar{M}} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix} + {}^{B}[R]^{W} \begin{bmatrix} 0 \\ 0 \\ -T_1 - T_2 - T_3 - T_4 \end{bmatrix} \tag{2.1}$$

2.2 Angular Acceleration

$$\bar{\bar{I}} \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} L\,(T_4 - T_3) \\ L\,(T_1 - T_2) \\ M_1 + M_2 - M_3 - M_4 \end{bmatrix} - \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times \bar{\bar{I}} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \tag{2.2}$$

3 Analysis

3.1 Motor Characteristics

It has been shown by *Mellinger, et. al.* and suggested by our faculty advisor Dr. Bruce D. Kothmann that the thrust and moment generated by a prop is proportional to angular velocity squared, or $T_i = K_{ti}\omega_i^2$ and $M_i = K_{mi}\omega_i^2$ where K_{ti} and K_{mi} are experimentally determined gains.

Mellinger, et. al. and Dr. Kothmann also suggested that:

$$\dot{\omega_i} = K_{mot}\left(\omega_{i,com} - \omega_i\right)$$

Where $\omega_{i,com}$ is the commanded motor speed.

Solving explicitly for ω_i with $\omega_i(t = 0) = 0$ we have:

$$\omega_i\left(t\right) = \omega_{i,com}\left(1 - e^{-K_{mot}t}\right)$$

Which shows that $K_{mot} = 1/\tau_{mot}$ which can be determined experimentally. We have limited useful experimental data for transient motor response because our initial results exhibited signs of rate limiting, and so the data are not entirely useful. However, using what we have $\tau_{mot} \approx 0.2$ [s] so:

$$K_{mot} \approx 5 \quad [1/s] \tag{3.1}$$

3.2 K_t and K_m From Experimental Results

Figure 2 shows raw data from the thrust tests of the Turnigy 35-42D motor with the Master Airscrew 12"x6" Tri Blade prop. Each point represents the average of 2 [s] of data at a certain RPM. The last 3 and 4 points in the thrust and moment plots, respectively, were ignored in the 2nd order polynomial fit because they were deemed to be the result of current limiting in our test system, and so were not consistent with the full prototype in situ environment. These tests will be redone, eventually, to verify this hypothesis.

3.3 Equations of Motion with Motor Characteristics

3.3.1 Linear Acceleration

$$\bar{\bar{M}}\begin{bmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{bmatrix} = \begin{bmatrix}0\\0\\mg\end{bmatrix} + {}^{B}[R]^{W}\begin{bmatrix}0\\0\\K_t\left(\omega_1^2+\omega_2^2+\omega_3^2+\omega_4^2\right)\end{bmatrix} \tag{3.2}$$

3.3.2 Angular Acceleration

$$\bar{\bar{I}}\begin{bmatrix}\dot{p}\\ \dot{q}\\ \dot{r}\end{bmatrix} = \begin{bmatrix}K_tL\left(\omega_4^2-\omega_3^2\right)\\ K_tL\left(\omega_1^2-\omega_2^2\right)\\ K_m\left(\omega_1^2+\omega_2^2\right) - K_m\left(\omega_3^2+\omega_4^2\right)\end{bmatrix} - \begin{bmatrix}p\\q\\r\end{bmatrix} \times \bar{\bar{I}}\begin{bmatrix}p\\q\\r\end{bmatrix} \tag{3.3}$$

Where we assume $K_{t1} = K_{t2} = ... = K_t$ and $K_{m1} = K_{m2} = ... = K_m$ because all four motor/prop combinations are identical.

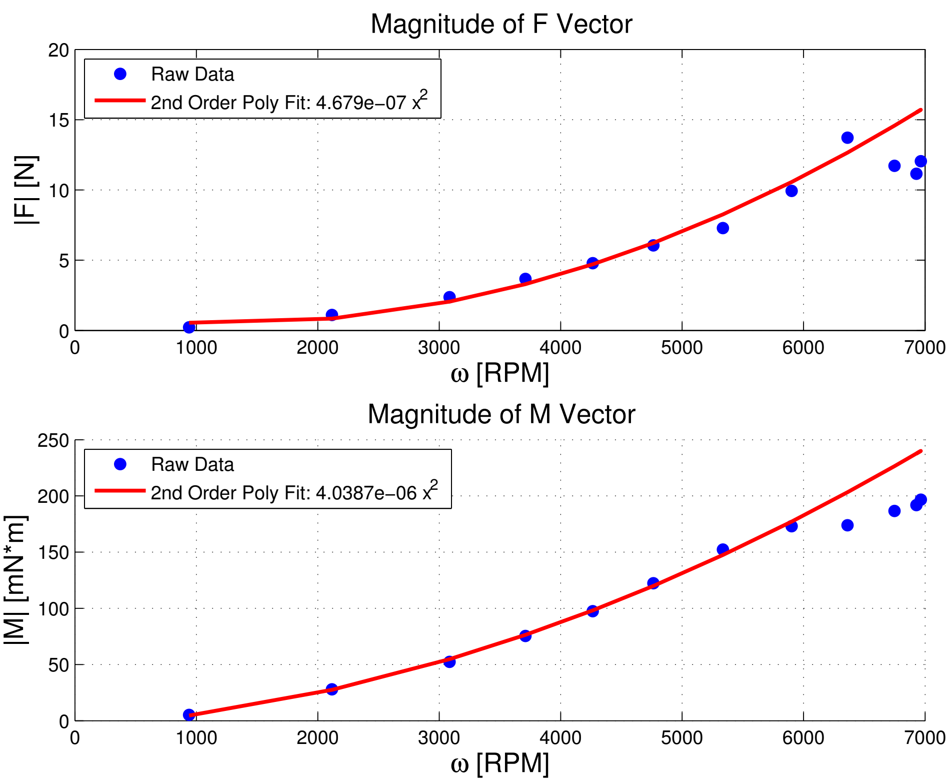

Figure 2: Thrust test data with 2nd order polynomial fit for thrust and moment coefficients with erroneous data points ignored in fit

3.4 Hover Trim

At hover, $\dot{\vec{\omega}} = \vec{\omega} = \vec{0}$ and $\phi = \theta = \psi = 0$ and $\ddot{r} = \dot{r} = \vec{0}$. This says:

$$^B[R]^W = [I]_{3x3} = identity$$

And so:

$$\begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix} = {}^B[R]^W \begin{bmatrix} 0 \\ 0 \\ K_t\left(\omega_1^2 + \omega_2^2 + \omega_3^2 + \omega_4^2\right) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ K_t\left(\omega_1^2 + \omega_2^2 + \omega_3^2 + \omega_4^2\right) \end{bmatrix} \tag{3.4}$$

Also:

$$\begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} K_t L\left(\omega_4^2 - \omega_3^2\right) \\ K_t L\left(\omega_1^2 - \omega_2^2\right) \\ K_m\left(\omega_1^2 + \omega_2^2\right) - K_m\left(\omega_3^2 + \omega_4^2\right) \end{bmatrix} \tag{3.5}$$

Which implies:

$$\omega_1^2 = \omega_2^2 = \omega_3^2 = \omega_4^2 = \omega_{trim}^2$$

Finally we have:

$$4K_t\omega_{trim}^2 = mg$$

Or:

$$\omega_{trim} = \sqrt{\frac{mg}{4K_t}} \tag{3.6}$$

3.5 Control

3.6 Non-Linear State Space Representation

Our state vector is defined as:

$$\vec{x} = \begin{pmatrix} x & y & z & \dot{x} & \dot{y} & \dot{z} & \phi & \theta & \psi & p & w & r & \omega_1 & \omega_2 & \omega_3 & \omega_4 \end{pmatrix}^T \tag{3.7}$$

Our input vector is:

$$\vec{u} = \begin{pmatrix} \dot{x}_{com} & \dot{y}_{com} & \dot{z}_{com} & \phi_{com} & \theta_{com} & \psi_{com} \end{pmatrix}^T \tag{3.8}$$

The rate of change of state without control is defined, in pieces, below:

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \left(\bar{\bar{M}}\right)^{-1} \left\{ \begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix} + {}^{B}[R]^{W} \begin{bmatrix} 0 \\ 0 \\ K_t\left(\omega_1^2+\omega_2^2+\omega_3^2+\omega_4^2\right) \end{bmatrix} \right\} \tag{3.9}$$

$$\begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \left(\bar{\bar{I}}\right)^{-1} \left\{ \begin{bmatrix} K_tL\left(\omega_4^2-\omega_3^2\right) \\ K_tL\left(\omega_1^2-\omega_2^2\right) \\ K_m\left(\omega_1^2+\omega_2^2\right) - K_m\left(\omega_3^2+\omega_4^2\right) \end{bmatrix} - \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times \bar{\bar{I}} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \right\} \tag{3.10}$$

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & -\cos\phi\sin\theta \\ 0 & 1 & \sin\phi \\ \sin\theta & 0 & \cos\phi\cos\theta \end{bmatrix}^{-1} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \tag{3.11}$$

$$\begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \\ \dot{\omega}_4 \end{bmatrix} = K_{mot} \begin{bmatrix} \omega_{1,des} - \omega_1 \\ \omega_{2,des} - \omega_2 \\ \omega_{3,des} - \omega_3 \\ \omega_{4,des} - \omega_4 \end{bmatrix} \tag{3.12}$$

Where we have not defined $\omega_{i,des}$ yet.

3.7 Linearization

If we assume $\bar{\bar{I}}_{ij} \ll \bar{\bar{I}}_{ii}$ because of the symmetry of our design, then the $\vec{\omega} \times \bar{\bar{I}}\vec{\omega}$ term in (3.10) is $\approx \vec{0}$.