Simply Typed λ-calculus

Function Types

Syntax of Types:

$$T := \text{Bool} \mid T \to T$$

Syntax of Terms

$$t := x \mid \lambda x : T.\, t \mid t\ t \mid b$$
$$\text{if } t \text{ then } t \text{ else } t$$

$b ::= \text{true} \mid \text{false}$

$t : T$

$\underbrace{\lambda x : \text{Bool} . x}_{} : \underbrace{\text{Bool} \to \text{Bool}}_{T}$

$\lambda x : \text{Bool} \; y \; x :$

$T := \mathrm{Bool} \mid T \to T$

$$\boxed{\begin{array}{l} \mathrm{Bool} \\ \mathrm{Bool} \to \mathrm{Bool} \\ (\mathrm{Bool} \to \mathrm{Bool}) \to \mathrm{Bool} \end{array}}$$

Body

Keyword

then

id with Type

else

$$\mathrm{if}\ \ y\ {}^\wedge\ \underbrace{(\lambda x{:}\mathrm{Bool}\ .\ \mathrm{False})}\ {}^\wedge\ \underbrace{\mathrm{false}}$$

$$\frac{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}{\mathrm{true}\ \ \mathrm{true}}$$

Type Environments (Contexts)

$$\Gamma ::= \{\} \mid \Gamma, x : T$$

Γ is a map from identifiers to Types.

$$\underbrace{y : Bool \to Bool} \vdash \lambda x : Bool \; y \, x : Bool \to Bool$$

$$\boxed{\Gamma \vdash t : T}$$

Well Typing Judgements

An inductive defn of well typings for STLC (with if)

Typing Rules:

$$\frac{x : T \in \Gamma}{\Gamma \vdash x : T} \quad \text{(T-VAR)}$$

$$\frac{}{\Gamma \vdash b : Bool} \quad (T\text{-}BOOL)$$

<u>Abstraction:</u>

$$\frac{\Gamma, x : T_1 \vdash t_2 : T_2}{\Gamma \vdash \underbrace{\lambda x : T_1 . t_2}_{\text{function term}} : \boxed{T_1 \rightarrow T_2}} \quad (T\text{-}ABS)$$

$x \notin dom(\Gamma)$

$$\frac{\Gamma \vdash t_1 : \overbrace{T_{11} \to T_{12}} \qquad \Gamma \vdash t_2 : \overbrace{T_{11}}}{\Gamma \vdash \underbrace{t_1\ t_2}_{\text{application}} : T_{12}} \quad (\text{T-APP})$$

$$\frac{\Gamma \vdash t_1 : \mathrm{Bool} \qquad \Gamma \vdash t_2 : T \qquad \Gamma \vdash t_3 : \overline{T}}{\underbrace{\Gamma} \vdash \underbrace{\underline{\mathrm{if}}\ t_1\ \underline{\mathrm{then}}\ t_2\ \mathrm{else}\ t_3}_{\text{expression}} : T} \quad (\text{T-IF})$$

Type

suppose: $\Gamma = x : \text{Bool}, y : \text{Bool} \to \text{Bool}$

① $\Gamma \vdash \text{false} : \text{Bool}$?

1. $\Gamma \vdash \text{false} : \text{BOOL}$ using T-BOOL

② $\Gamma \vdash x : \text{Bool}$

1. $\overbrace{y : \text{Bool}, x : \text{Bool}} \vdash x : \text{Bool}$ using (T-VAR)

alternatively:

$$\left.\begin{array}{l} x : \text{Bool} \in \Gamma \\ \therefore \Gamma \vdash x : \text{Bool} \end{array}\right\} - \text{(T-VAR)}$$

$$\Gamma \vdash_{STLC} y\ x : Bool$$

1. $\Gamma \vdash y : Bool \to Bool$ using (T-VAR) $\because y : Bool \to Bool \in \Gamma$

2. $\Gamma \vdash x : Bool$ using (T-VAR) on x.

3. $\Gamma \vdash y\ x : Bool$ from 1, 2, using (T-APP)

Prove $\Gamma \vdash \lambda z:\mathrm{Bool}.\ x : \mathrm{Bool} \to \mathrm{Bool}$

1. $\overbrace{\underbrace{\Gamma, z:\mathrm{Bool}}_{x:\mathrm{Bool} \in \Gamma, z:\mathrm{Bool}}} \vdash x : \mathrm{Bool}$ using (T-VAR)

2. $\Gamma \vdash \lambda z:\underbrace{\mathrm{Bool}}.\ x : \underbrace{\mathrm{Bool}}_{\text{arg type}} \to \underbrace{\mathrm{Bool}}_{\text{res typ.}}$

Show $\Gamma \vdash \text{if } x \text{ then } (y\ x) \text{ else } x : \text{Bool}$

1. $\Gamma \vdash x : \text{Bool}$ using T-VAR

2. $\Gamma \vdash y : \text{Bool} \to \text{Bool}$ using T-VAR

3. $\Gamma \vdash y\ x : \text{Bool}$

4. $\text{if } x \text{ then } (y\ x) \text{ else } x : \text{Bool}$ from 1, 3, 1 using T-IF