TYPE Inference / Reconstruction

The type checking problem:

$$\boxed{\Gamma \vdash_C e : T}$$

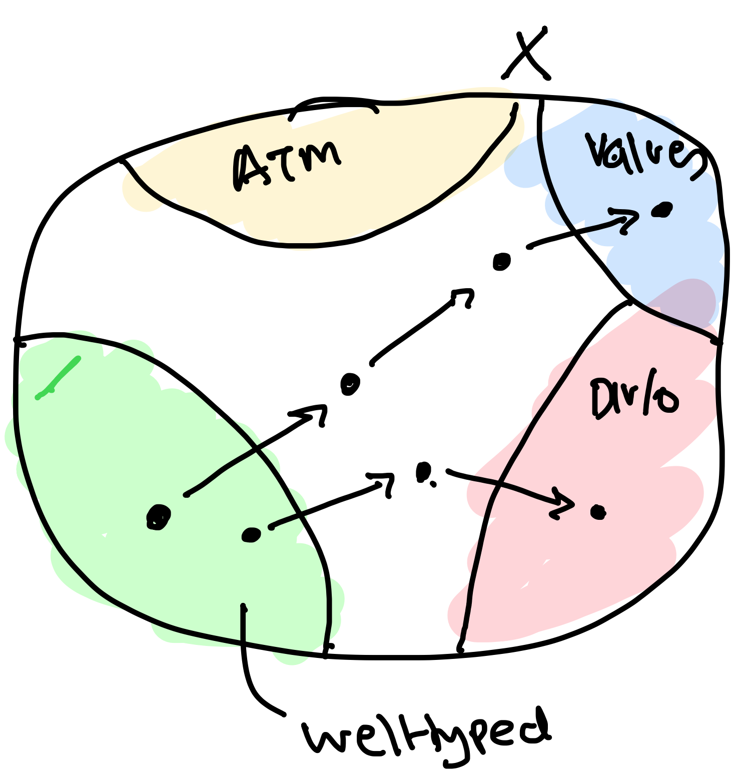
TyENV

Type System

Exp

Type

IF+/



if $\vdash e : T$ then

either $e \xrightarrow{*} v$ or

$e \xrightarrow{*} e'$ where

e' DIV/0

but $e \not\xrightarrow{*} e'$ where

e' ATM

Let $\vdash_T e : T$

Safety = Preservation + Progress

Preservation: if $\vdash_T e : \tau$ and

$$e \longrightarrow e'$$

then $\vdash_T e' : \tau$

Progress: if $\vdash_T e : \tau$ then

either e is a VALUE $v : \tau$

or e DIV/0

or $\exists\, e'$:

$$e \longrightarrow e'$$

$$\text{TYEXP} \quad T ::= t \mid \text{num} \mid \text{bool} \mid T \to T$$

$$\text{TYVAR} \quad t$$

$$\text{ID} \quad x$$

$$\text{EXP} \quad e ::= \bar{n} \mid \bar{b} \mid x \mid \text{if } e\ e\ e \mid$$
$$\lambda \underbrace{x : T}_{\text{Type declaration}} .\ e \mid @e\ e$$

$$\Gamma : \text{TYENV} = \text{ID} \xrightarrow{\text{fin}} \text{TYEXP}$$

$$\Gamma = [x : \text{num} \to \text{bool},\ y : \text{num}]$$

$$\text{dod}(\Gamma) = \{x, y\}$$
$$\text{dom}(\Gamma) = \text{ID}$$
$$\text{rng}(\Gamma) = \{\text{num} \to \text{bool}, \text{num}\}$$

Type System $\vdash_c$

$$\boxed{\Gamma \vdash_c e : \tau}$$

$$\frac{}{\Gamma \vdash \bar{n} : \text{num}} \text{ NUM}$$

$$\frac{}{\Gamma \vdash \bar{b} : \text{bool}} \text{ BOOL}$$

$$\frac{}{\underbrace{\Gamma, x : \tau}_{} \vdash x : \tau} \text{ VAR}$$

$$\frac{\Gamma \vdash e_1 : \text{bool} \quad \Gamma \vdash e_2 : \tau \quad \Gamma \vdash e_3 : \tau}{\Gamma \vdash \text{if } e_1\, e_2\, e_3 : \tau} \text{ IF}$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \to \tau_2 \quad \Gamma \vdash e_2 : \tau_1}{\Gamma \vdash @ e_1\, e_2 : \tau_2} \text{ APP}$$

$$\frac{\Gamma, x : \tau_1 \vdash e : \tau_2}{\Gamma \vdash \lambda x : \tau_1 . e : \tau_1 \to \tau_2} \text{ ABS}$$

Example:

Show:

$$\boxed{\underbrace{\phi}_{\Gamma} \vdash_c \underbrace{\lambda x : t \,.\, x}_{e} : \underbrace{t \to t}_{\tau}}$$

1. $x : t \vdash x : t$ VAR
2. $\phi \vdash \lambda x : t \,.\, x : t \to t$ from 1, using ABS

Verify:

$$\boxed{\phi \vdash_c \lambda x : \text{bool} \,.\, x : \text{bool} \to \text{bool}}$$

1. $x : \text{bool} \vdash x : \text{bool}$ VAR

2. $\underbrace{\phi}_{\Gamma} \vdash \underbrace{\lambda x : \text{bool} \,.\, x}_{e} : \underbrace{\text{bool} \to \text{bool}}_{\tau}$ from 1, using ABS

$\lambda x \,.\, x$

$\Gamma \vdash_C \lambda x : \overbrace{\qquad}^{\tau_1} \quad x : \overbrace{\qquad}^{\tau_2}$

TYPE SUBSTITUTION

σ : TYSUBST = TYVAR $\xrightarrow{\text{fin}}$ TYEXP

$$\sigma = [\underbrace{t_1}_{} : \underbrace{num \to bool}_{\tau},\ \underbrace{t_2}_{} : \underbrace{num}_{\tau},\ \underbrace{t_3}_{} : \underbrace{t_4}_{\tau}]$$

$apply_{TYEXP}$: TYSUBST , TYEXP $\longrightarrow$ TYEXP

$$apply(\sigma, \underbrace{t_2}_{\tau_1} \to \underbrace{t_3}_{\tau_2}) = \underbrace{num}_{apply(\sigma;\ \tau_1)} \to \underbrace{t_4}_{apply(\sigma,\ \tau_2)}$$

$apply_{TYEXP}(\sigma, \tau) =df=$

$$apply(\sigma, t) = t \quad \text{if } t \notin dod(\sigma)$$
$$= \sigma(t) \quad \text{if } t \in dod(\sigma)$$

$$apply(\sigma, num) = num$$
$$apply(\sigma, bool) = bool$$
$$apply(\sigma, \tau_1 \to \tau_2) = apply(\sigma; \tau_1) \to apply(\sigma; \tau_2)$$

$apply(\sigma, \tau)$ is written as $\sigma(\tau)$

Application of a type substitution on Type Env

$$\text{apply}_{\text{TYENV}} : \text{TYSUBST}, \text{TYENV} \longrightarrow \text{TYENV}$$

$\text{apply}_{\text{TYENV}}(\sigma, \Gamma)$ where

$\sigma = [t_1 : \text{num} \to \text{bool}, t_2 : \text{num}, t_3 : t_4]$

and $\Gamma = [x : t_1, \; y : t_2 \to t_3]$

$\text{apply}_{\text{TYENV}}(\sigma, \Gamma) = \sigma(\Gamma)$

$= [x : \text{num} \to \text{bool}, y : \text{num} \to t_4]$

$\sigma([x_1 : T_1, \; x_2 : T_2, \; \ldots \; x_n : T_n])$

$\stackrel{\text{def}}{=} [x_1 : \sigma(T_1), \; x_2 : \sigma(T_2) \ldots \; x_n : \sigma(T_n)]$

$\text{apply}_{\text{EXP}}(\sigma, e)$

$\sigma(e) =$ df.

$\sigma(\bar{n}) = \bar{n}$

$\sigma(\bar{b}) = \bar{b}$

$$\sigma(@e_1\ e_2) = @\ \sigma(e_1)\ \sigma(e_2)$$

$$\sigma(\text{if } e_1\ e_2\ e_3) = \text{if } \sigma(e_1)\ \sigma(e_2)\ \sigma(e_3)$$

$$\sigma(\lambda x : T.\ e) = \lambda x : \sigma(T).\ \sigma(e)$$

Example:

$$\sigma = [\,t_1 : num \to bool,\ t_2 : num\ \ t_3 : t_4\,]$$

$$\sigma(t_1 \to t_2) = (num \to bool) \to num$$

TYEQN $\boxed{T_1 \stackrel{?}{=} T_2}$

Solve $t_1 \to num \stackrel{?}{=} bool \to t_3$

Solutions are in terms of substitutions

let $\sigma = [t_1 : bool,\ t_3 : num]$ σ is a solution

$$\therefore\ \sigma(t_1 \to num) = \sigma(bool \to t_3)$$

$$= bool \to num \qquad\qquad bool \to num$$

Solve

$$\overbrace{t_1 \rightarrow num}^{T_1} \stackrel{?}{=} \overbrace{t_2 \rightarrow t_3}^{T_2}$$

$\sigma_1 = [t_1 : num,\ t_2 : num,\ t_3 : num]$

$\sigma_1(T_1) = num \rightarrow num$

$\sigma_2(T_2) = num \rightarrow num$

$\therefore\ \sigma_1$ solves $T_1 = T_2$

$\sigma_2 = [t_1 : bool,\ t_2 : bool,\ t_3 : num]$

$\sigma_2(T_1) = bool \rightarrow num$

$= \sigma_2(T_2)$

$\sigma_3 : [t_1 : t,\ t_2 : t,\ t_3 : num]$

$\sigma_3(T_1) = t \rightarrow num$

$\sigma_3(T_2) = t \rightarrow num$

Example 3:

$$t_1 \rightarrow bool = t_2 \rightarrow num$$

No solution!

Suppose Γ, e are given,

can I find a σ & τ s.t.

$\underbrace{\sigma\Gamma} \vdash_C \underbrace{\sigma e} : \tau$

TYPE INFERENCE

$$\phi \vdash \underbrace{\lambda x . x} : t \to t$$