Best of Put option pricing in C++ using Monte Carlo simulation with variance reduction

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April 29, 2017

Abstract

Best of Put is a put option on the best performing stock from a basket of correlated stocks. In order to simulate the stock movement a vector of independent normal random variables is drawn. Correlation of the normal random variables is introduced using Choleski decomposition. Stock movement is simulated using Black-Scholes dynamics. Monte Carlo method is used for pricing with two methods of variance reduction: antithetic and control variables. The pricing is implented in C++.

1 Simulating independent standard normal random variables

Function Marsagliapolar generates 2 standard normal random variables using Marsaglia polar method. Variables v_1 and v_2 are drawn from uniform distribution on [-1,1] until

$$s = v_1^2 + v_2^2 < 1 \tag{1}$$

Then random variables calculated as:

$$z_i = v_i \sqrt{\frac{-2ln(s)}{s}}, \quad i = 1, 2 \tag{2}$$

are independent standard normal. Each loop iteration in simulation function is using both of them and n=4 random variables are simulated with n/2 iterations.

2 Correlation

Choleski decomposition of correlation matrix is used to generate n correlated standard normal random variables from n independent ones. Any correlation matrix C can be decomposed: $C = LL^T$ where L is lower triangular matrix. If Z is a vector of n independent standard normal random variables then $X = LZ$ is a vector of n correlated standard normal random variables with correlation matrix C:

$$EXX^T = ELZ(LZ)^T = ELZZ^T L^T = LE(ZZ^T)L^T = LIL^T = LL^T = C \tag{3}$$

Choleski function uses Choleski algorithm to decompose the correlation matrix:

$$L[j,j] = \Big(C[j,j] - \sum_{k=1}^{j-1} L[j,k]^2\Big)^{1/2} \tag{4}$$

$$L[i,j] = \frac{C[i,j] - \sum_{k=1}^{j-1} L[i,k]L[j,k]}{L[j,j]}, \quad i > j \tag{5}$$

The constructor of BestOfPut ensures that entries to correlation matrix are between -1 and 1 and that C is symmetric. However, for C to be a valid correlation matrix it has also to be positive-semidefinite, otherwise it cannot be decomposed in the way described above.

3 Efficiency and variance reduction methods

Given the vector of correlated standard normal random variables X, the stock prices are generated using the Black-Scholes result:

$$S_i(T) = S_i(0)exp((r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}X_i), \quad i = 1, ...n \tag{6}$$

This method of simulation is more precise than Euler or Milstein scheme because there is no discretisation error. The first method of variance reduction is antithetic variate method. -X which is also a vector of standard normal random variables with correlation matrix C is used again to generate a new set of stock prices:

$$S_i^A(T) = S_i(0)exp((r - \frac{1}{2}\sigma^2)T - \sigma\sqrt{T}X_i), \quad i = 1, ...n \tag{7}$$

The payoff of the best-of-put option at maturity is then calculated as:

$$BoPPayoff = ((K - \max_i S_i)^+ + (K - \max_i S_i^A)^+)/2 \tag{8}$$

Averaging the two payoffs reduces the variance of the Monte Carlo estimator because X and -X and as a consequence two payoffs are negatively correlated. The other method of variance reduction uses the average price of plain vanilla put options as control variate. Based on the simulated stock prices (including antithetic variates) the average payoff of put options is calculated:

$$PutPayoff = \frac{1}{2n}\sum_{i=1}^{n}((K - S_i)^+ + (K - S_i^A)^+) \tag{9}$$

Discounted to time 0 it consist an unbiased estimator of average price of put options, which under Black-Scholes setting is known to be:

$$PutPrice = \frac{1}{n}\sum_{i=1}^{n}(Ke^{-rT}\Phi(-d_2^i) - S_i(0)\Phi(-d_1^i)) \tag{10}$$

$$d_1^i = \frac{ln(\frac{S_i(0)}{K}) + (r + \frac{1}{2}\sigma_i^2)T}{\sigma_i\sqrt{T}}, \quad d_2^i = d_1^i - \sigma_i\sqrt{T} \tag{11}$$

Finally, the price of best-of-put option is calculated in each simulation as:

$$BoPPrice = e^{-rT}BoPPayoff + (PutPrice - e^{-rT}PutPayoff) \tag{12}$$

and then it is averaged across all simulations. Including the put adjustment in the final formula is the essence of the control variable method. The average price of put options is positively correlated with the price of the best-of-put option. Therefore whenever Monte Carlo estimator of best-of-put option deviates from its true value, it will be brought to it by the put adjustment. In this framework the Monte Carlo estimator of vanilla put price is itself subject to antithetic variate variance reduction method.