Balancing Efficiency and Interpretability: A New Approach to Multi-Objective Optimization with High Computation Costs in Lipschitz Functions

Ильгам Магданович Латыпов

Московский физико-технический институт

18 мая 2024 г.

Научный руководитель:

к.т.н. Ю. В. Дорн

The problem

- Practical tasks are often given as a problem of multiobjective optimization. However, the high cost of computations and even impossibility of repeatedly computing the functions makes iterative optimizers impractical. Example of such task is a graph optimization problems.
- In this paper, we present a method for finding compromise solution without iterative computations in case of Lipschitz functions. The found solution has a good practical interpretability.

Problem statement

Multiobjective optimization problem is formulated as follows:

$$\min_x f \triangleq (f_1(x), ..., f_m(x))^T \quad \text{s.t. } x \in K \qquad (T_0)$$

- $x \in \mathbb{R}^n$
- functions $f_i : \mathbb{R}^n \to \mathbb{R} \quad i = \overline{1, m}$
- Decision set K is convex, nontrivial, compact and equipped with a norm $\|\cdot\|$

▶ It remains to determine what means optimality.

Optimization problem

For all $i = \overline{1, m}$ for $f_i(x) : \mathbb{R}^n \to \mathbb{R}$:

- It computed at x_i : $f_i(x_i) = v_i$
- It satisfy
$$1 - \gamma \leq \frac{f(x)}{f_i(x_i)} \leq 1 + \gamma$$

Can be rewritten as an optimization problem:

$$\begin{aligned} &\min_{x,\gamma} \gamma \\ &\text{s.t. } x \in K \\ &\qquad |f_i(x) - v_i| \leq \gamma v_i \ \ \text{for} i \in \overline{1, m} \end{aligned} \tag{T_1}$$

Examples

Example

Function f_i is a loss function, computed on a model with parameters x. x_i is an optimal parameter for this loss function. Then solution of task T_1 is a parameter, that spoils metrics less than with factor $1 + \gamma$. Note that in this case x_i is hardly computable point.

Example

The delivery service have been working with different strategies – points x_i and compute metrics – $f_i(x_i)$. Now they want to set some parameters, that will guarantee the values of the metrics if the situation will be similar to the ones a already known.

Method

Definition

Function $f : \mathbb{R}^n \to \mathbb{R}+$ is satisfy Lipschitz condition on K with norm $\|\cdot\|$:

$$\exists L > 0 : \ \forall x, y \in K : \ |f(x) - f(y)| \leq L\|x - y\|$$

- For all $i = \overline{1, m}$ for $f_i(x) : \mathbb{R}^n \to \mathbb{R}$:
  - It satisfies Lipschitz condition 3 with constant L_i.
  - It computed at $x_i : f_i(x_i) = v_i$.
  - Function is to be minimized.

We search $x \in K$ that is feasible for T_1.
Method should not compute functions while solution search.

Method

- Consider some $x \in K$ and γ that $\forall i = \overline{1, m}$ satisfy

$$\|x_i - x\| \le \frac{\gamma v_i}{L_i} \Rightarrow f_i(x) - v_i \le |f_i(x) - v_i| \le \gamma v_i \tag{1}$$

- Which means, that x, γ is feasible for T_1. We can define new optimization problem:

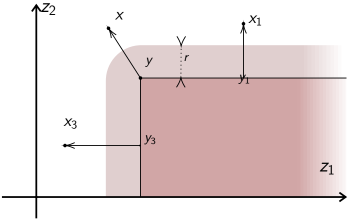
$$\begin{aligned} &\min_{x,\gamma} \gamma \\ &\text{s.t. } x \in K \\ &\qquad \|x - x_i\| \le \frac{1}{L_i}(\gamma v_i) \quad \forall i \in \overline{1, m} \end{aligned} \tag{T_2}$$

Method

We can use monotonicity of functions on parameters to ease constraints. For this we introduce $\texttt{clip} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ operator:

$$\texttt{clip}(x, y, f)_i = \begin{cases} \max(x_i - y_i, 0) & f \text{ increase on } x_i \\ \min(y_i - x_i, 0) & f \text{ decrease on } x_i \\ x_i - y_i & \text{none of the above} \end{cases} \tag{2}$$

$$\min c^T x$$
$$Ax \leq b$$



Method

$$\min_{x,\gamma} \gamma \qquad (T_3)$$

$$\text{s.t. } x \in K$$

$$\|clip(x, x_i, f_i)\| \leq \frac{1}{L_i}(\gamma v_i) \quad \forall i \in \overline{1, m}$$

Theorem

Solution of T_3 feasible for T_1

Theorem

Let γ^* be a solution of T_3. If we have approximations of the Lipschitz constants $\widetilde{L_i} = \kappa_i L_i$, then by solving T_3, in which the Lipschitz constants are changed to $\widetilde{L_i}$ we obtain x, for which $f(x) - f(x_i) \leq \frac{\kappa_{\max}}{\kappa_i}\gamma^* w_i$. Here, $\kappa_{\max} = \max_{i=\overline{1,m}} \kappa_i$.

Experiment: notations

Legend:

- The company produces k types of products with amounts $x \in \mathbb{R}^k$ and realize it with cost c.
- For this it needs n types of resources with amounts $b \in \mathbb{R}^n$, which are purchased at a cost c_b at the beginning of the period.
- Some of the resources may run out during the period, so the company buys additional resources y at the cost $c_a \geq c_b$ componentwise.
- For the product i it uses a_{ij} amount of resource j. Denote it $A = \|a_{i,j}\|_{i,j=1} \in \mathbb{R}^{n \times k}$.

$$
\begin{aligned}
f(b, x, c, c_b, c_a) &= c^T x - c_b^T b - c_a^T y \\
y &= \max(0, Ax - b)
\end{aligned}
$$

Experiment: statement

- The company has worked for m periods and observed for $i \in \overline{1, m} : c_i, x_i, c_b^i, c_a^i$ for resource amounts b_i.

It is necessary to find a strategy for purchasing resources that will give good enough income if situation will look as one of observed situations. Besides that there is a constraint for budget:
$K = \{b : c_b^T b \leq B\}$
Explanation for the graphs:

- Functions to be maximized, so plot $f^*/f(x)$.
- We search $b, \gamma : 1 - \gamma \leq \frac{f(x)}{f^*} \longrightarrow \frac{f^*}{f(x)} \leq \frac{1}{1-\gamma}$
- In our method we use $\|\cdot\|_2$ instead of $\|\cdot\|_1$, so we do not know exact Lipschitz constant and use $\gamma \cdot C$.

Experiment: results

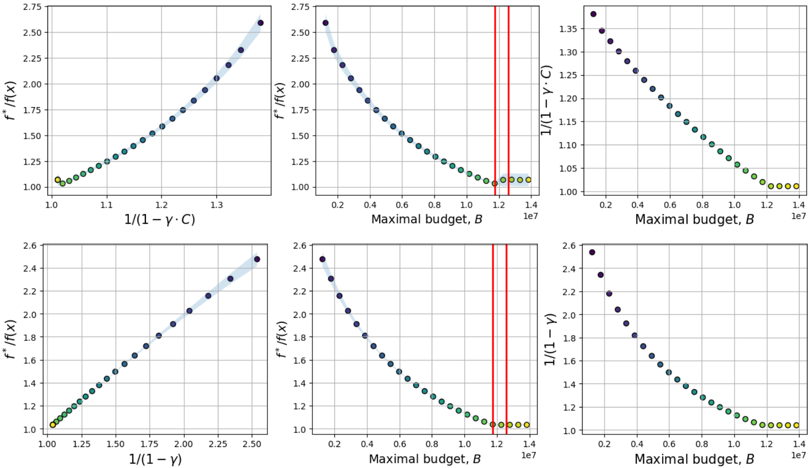


Рис.: The dependence of the function relative increments on the budget and the resulting γ for scenarios with differet budget. Upper is our solution, lower is exact solution.

Experiment: results

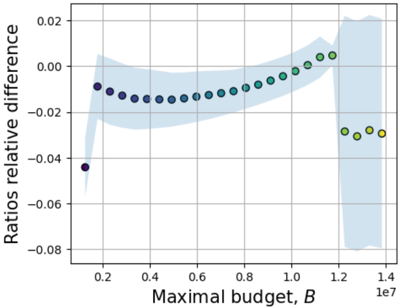


Рис.: Relative relations and computation times dependence on budget.

Summary

- Future work: сделать алгоритм, который выдает одно решение, которое будет "хорошим"на все сценарии.