Investing for retirement

We're planning for retirement and want to know how much to save and invest. We'll use six variables: R for the real annual return on investment, T for years until retirement, D for desired retirement income in today's dollars, W for withdrawal rate, G for inflation-adjusted annual growth of savings, and A for amount of savings already set aside for retirement.

Our task will be easier if we temporarily convert real return into nominal return (B) divided by inflation (C). So $R = B/C$.

Let's start with the flat or time-invariant approach to retirement investing. We have a savings target for the first year (S) determined by the variables above, and the savings targets for later years are increased by anticipated inflation (C). This means that the first year's contribution is S, the second year's contribution is SC, the third year's contribution is SC^2, and so on. In the final year before retirement, the contribution is SC^{T-1}.

The next step is to find the expected value to which each contribution will grow by retirement age. The first year's contribution (S) has T years to appreciate, so its value at retirement is SB^T. The second year's contribution (SC) has $T-1$ years to appreciate so its value is SCB^{T-1}. The third year's contribution (SC^2) has $T-2$ years to appreciate so its value is SC^2B^{T-2}.

So the total value of our nest egg (N) at retirement age will be:

$$N = SB^T + SCB^{T-1} + SC^2B^{T-2} + \ \dots\ + SC^{T-2}B^2 + SC^{T-1}B$$

We can factor out S, and let's call the rest of it Z:

$$N = SZ = S(B^T + CB^{T-1} + C^2B^{T-2} + \ \dots\ + C^{T-2}B^2 + C^{T-1}B)$$

Let's work on simplifying Z for a moment. The first term is already simple: B^T. The second term is CB^{T-1}, but it could be reframed as $B^T(C/B)$. This may seem pointless, but notice that we can define the third term as $B^T(C/B)^2$, the third term as $B^T(C/B)^3$, and so on.

$$Z = B^T + B^T(C/B) + B^T(C/B)^2 + \ \dots\ + B^T(C/B)^{T-2} + B^T(C/B)^{T-1}$$

Let's return to SZ. We can now factor out B^T as well, and we'll call the remainder V:

$$N = SZ = SB^TV = SB^T(1 + C/B + (C/B)^2 + \ \dots\ + (C/B)^{T-2} + (C/B)^{T-1})$$

We know $R = B/C$, so $1/R = C/B$:

$$V = 1 + 1/R + (1/R)^2 + \ \dots\ + (1/R)^{T-2} + (1/R)^{T-1} = \sum_{Y=0}^{T-1} \left(\frac{1}{R}\right)^Y$$

We now have a workable definition of our nest egg from future contributions N. Until now we've neglected the savings already set aside for retirement (A). Similar to the first year's contribution, the anticipated value of A at retirement will be AB^T. The nest egg we're aiming for is actually composed of the nest egg we build from future contributions (N) plus the value our current savings will have at retirement (AB^T). Let's call this total nest egg Q. The desired income in today's dollars D divided by the withdrawal rate W gives us our total nest egg in today's dollars, and multiplying by the inflation factor over the full period C^T gives us Q, the total nest egg in inflated dollars.

$$Q = N + AB^T = DC^T/W$$

Since $N = SB^TV$:

$$SB^TV + AB^T = DC^T/W$$

$$SV + A = \frac{DC^T}{WB^T}$$

Since $C/B = 1/R$, $C^T/B^T = 1/R^T$:

$$SV + A = \frac{D}{WR^T}$$

We can now easily isolate S:

$$S \; = \; \frac{\frac{D}{WR^T} - A}{V} \; = \; \frac{\frac{D}{WR^T} - A}{\sum_{Y=0}^{T-1} \left(\frac{1}{R}\right)^Y}$$

And finally, what if we wish to increase or decrease savings each year after adjusting for inflation? The variable G expresses this growth approach, in contrast to the flat approach.

The first year's contribution will not change: S. The second year's will be SGC, the third year's will be $S(GC)^2$, and so on. So our altered version of V will be:

$$U = 1 + GC/B + (GC/B)^2 + \; ... \; + (GC/B)^{T-2} + (GC/B)^{T-1}$$

As C/B was converted to $1/R$, GC/B will be converted to G/R:

$$U = 1 + G/R + (G/R)^2 + \; ... \; + (G/R)^{T-2} + (G/R)^{T-1} = \sum_{Y=0}^{T-1} \left(\frac{G}{R}\right)^Y$$

We can isolate S with the additional variable G:

$$S \; = \; \frac{\frac{D}{WR^T} - A}{U} \; = \; \frac{\frac{D}{WR^T} - A}{\sum_{Y=0}^{T-1} \left(\frac{G}{R}\right)^Y}$$

What if we wish to reverse this equation and convert a savings pattern into retirement income? We solve for D.

$$SU = \frac{D}{WR^T} - A$$

$$SU + A = \frac{D}{WR^T}$$

$$D = WR^T(A + SU)$$

So we can summarize:

$$S \;=\; \frac{\frac{D}{WR^T} - A}{\sum_{Y=0}^{T-1}\left(\frac{G}{R}\right)^Y}$$

$$D = WR^T\left(A + S\sum_{Y=0}^{T-1}\left(\frac{G}{R}\right)^Y\right)$$

To pursue a flat approach, set G equal to 1. These equations are implemented in WolframAlpha here and here (PDF must be downloaded for hyperlinks to work).