Week 3

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Problem Set 2: Linear Approximations

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Problem 1

Brock Mirman Model

$$u(c) = \sum \beta^t \ln(c_t)$$
$$s.t. \;\; c_t + k_{t+1} = e^{z_t} k_t^{\alpha}$$

FOCs:

$$\begin{aligned}
[k_{t+1}] : -\frac{1}{e^{z_t} k_t^{\alpha} - k_{t+1}} + \beta E[V_{k_{t+1}}(k_{t+1}, \theta_{t+1})] \\
EV : V_{k_t}(k_t, \theta_t) = \frac{\alpha e^{z_t} k_t^{\alpha-1}}{e^{z_t} k_t^{\alpha} - k_{t+1}} \\
\implies \frac{1}{e^{z_t} k_t^{\alpha} - k_{t+1}} = \beta \frac{\alpha e^{z_{t+1}} k_{t+1}^{\alpha-1}}{e^{z_{t+1}} k_{t+1}^{\alpha} - k_{t+2}} \\
\implies 1 = \beta \frac{\alpha e^{z_{t+1}} k_{t+1}^{\alpha-1} (e^{z_t} k_t^{\alpha} - k_{t+1})}{e^{z_{t+1}} k_{t+1}^{\alpha} - k_{t+2}}
\end{aligned}$$

First we take a derivative w.r.t k_{t+2} to find F

$$\begin{aligned}
\frac{d\Gamma}{dX_{t+1}} = \frac{d\Gamma}{dk_{t+2}} &= E\left\{ \beta \frac{\alpha e^{z_{t+1}} k_{t+1}^{\alpha-1} \left(e^{z_t} k_t^{\alpha} - k_{t+1}\right)}{\left(e^{z_{t+1}} k_{t+1}^{\alpha} - k_{t+2}\right)^2} \right\} \\
&= \beta \frac{\alpha k^{\alpha-1} \left(k^{\alpha} - k\right)}{\left(k^{\alpha} - k\right)^2} \\
&= \beta \frac{\alpha k^{\alpha-1}}{\left(k^{\alpha} - k\right)}
\end{aligned}$$

Next w.r.t to k_{t+1} to get G:

$$\begin{aligned}
&\frac{d\Gamma}{dX_t}\left\{\frac{\alpha k_{t+1}^{\alpha-1}k_t^{\alpha}}{\left(k_{t+1}^{\alpha}-k_{t+2}\right)}-\frac{\alpha k_{t+1}^{\alpha}}{\left(k_{t+1}^{\alpha}-k_{t+2}\right)}\right\}\\
&\quad=\frac{\left(k^{\alpha}-k\right)(\alpha)(\alpha-1)k^{\alpha-2}k^{\alpha}-\alpha^2k^{2\alpha-1}}{\left(k^{\alpha}-k\right)^2}\\
&\quad=\frac{\left(x^2-a\right)\left(k^{3a-2}-k^{2a-1}\right)-a^2\left(k^{3a-2}\right)k^{\alpha-1}}{\left(k^{\alpha}-k\right)^2}\\
&\quad=\frac{\left(k^{\alpha}-k\right)\left(\alpha^2\right)k^{\alpha-1}-\alpha k^{\alpha}\alpha k^{\alpha-1}}{\left(k^{\alpha}-k\right)^2}\\
&\quad=\frac{-\alpha k^{3\alpha-2}+\alpha k^{2\alpha-1}-\alpha^2k^{2\alpha-1}+\alpha^2k^{\alpha}}{\left(k^{\alpha}-k\right)^2}\\
&\quad=\frac{\alpha k^{\alpha-1}[-k^{2\alpha-1}+k^{\alpha}-\alpha k^{\alpha}+\alpha k]}{\left(k^{\alpha}-k\right)^2}\\
&\quad=\frac{\alpha k^{\alpha-1}[-k^{\alpha-1}(k^{\alpha}-k)-\alpha(k^{\alpha}-k)]}{\left(k^{\alpha}-k\right)^2}\\
&\quad=-\frac{\alpha k^{\alpha-1}(k^{\alpha-1}+\alpha)}{(k^{\alpha}-k)}
\end{aligned}$$

Next we differentiate w.r.t k_t to get H

$$\begin{aligned}
&\frac{d\Gamma}{dX_{t-1}}\left\{\frac{\alpha k_{t+1}^{\alpha-1}k_t^{\alpha}}{\left(k_{t+1}^{\alpha}-k_{t+2}\right)}-\frac{\alpha k_{t+1}^{\alpha}}{\left(k_{t+1}^{\alpha}-k_{t+2}\right)}\right\}\\
&\quad=\frac{d\Gamma}{dX_{t-1}}\left\{\frac{\alpha k_{t+1}^{\alpha-1}k_t^{\alpha}}{\left(k_{t+1}^{\alpha}-k_{t+2}\right)}\right\}\\
&\quad=\left\{\frac{\alpha^2k^{2\alpha-2}}{(k^{\alpha}-k)}\right\}
\end{aligned}$$

Now we differentiate w.r.t Z_{t+1} to get L:

$$
\begin{aligned}
\frac{d\Gamma}{dZ_{t+1}}\beta\frac{\alpha e^{z_{t+1}}k_{t+1}^{\alpha-1}(e^{z_t}k_t^{\alpha}-k_{t+1})}{e^{z_{t+1}}k_{t+1}^{\alpha}-k_{t+2}}& \\
&= \frac{(e^z k^{\alpha}-k)(\alpha e^z k^{\alpha-1}(e^z k^{\alpha}-k))-(\alpha e^z k^{\alpha-1}(e^z k^{\alpha}-k))(e^z k^{\alpha})}{(k^{\alpha}-k)^2} \\
&= \frac{-k(\alpha k^{\alpha-1}(k^{\alpha}-k))}{(k^{\alpha}-k)^2} \\
&= -\frac{k(\alpha k^{\alpha-1})}{(k^{\alpha}-k)} \\
&= -\frac{(\alpha k^{\alpha})}{(k^{\alpha}-k)}
\end{aligned}
$$

Finally we differentiate w.r.t Z_t to get M:

$$
\begin{aligned}
\frac{d\Gamma}{dZ_t}\beta\frac{\alpha e^{z_{t+1}}k_{t+1}^{\alpha-1}(e^{z_t}k_t^{\alpha}-k_{t+1})}{e^{z_{t+1}}k_{t+1}^{\alpha}-k_{t+2}}& \\
&= \frac{d\Gamma}{dZ_t}\beta\frac{\alpha e^{z_{t+1}}k_{t+1}^{\alpha-1}(e^{z_t}k_t^{\alpha})}{e^{z_{t+1}}k_{t+1}^{\alpha}-k_{t+2}} \\
&= \frac{\alpha e^z k^{\alpha-1}e^z k^{\alpha}}{e^z k^{\alpha}-k} \\
&= \frac{\alpha k^{2\alpha-1}}{k^{\alpha}-k}
\end{aligned}
$$

Now we calculate the values of P and Q

$$
P = \frac{-G-\sqrt{G^2-4FH}}{2F}
$$

$$
\begin{aligned}
G &= -\frac{\alpha k^{\alpha-1}(k^{\alpha-1}+\alpha)}{(k^{\alpha}-k)} \\
F &= \frac{\alpha k^{\alpha-1}}{(k^{\alpha}-k)} \\
H &= \left\{\frac{\alpha^2 k^{2\alpha-2}}{(k^{\alpha}-k)}\right\}
\end{aligned}
$$

We plug in the values of G, F, and H into the quadratic formula

$$P = \frac{\left(\frac{\alpha k^{\alpha-1}(k^{\alpha-1}+\alpha)}{(k^{\alpha}-k)}\right) \pm \sqrt{\left(\frac{\alpha k^{\alpha-1}(k^{\alpha-1}+\alpha)}{(k^{\alpha}-k)}\right)^2 - 4\left(\frac{\alpha k^{\alpha-1}}{(k^{\alpha}-k)}\right)\left(\left\{\frac{\alpha^2 k^{2\alpha-2}}{(k^{\alpha}-k)}\right\}\right)}}{2\left(\frac{\alpha k^{\alpha-1}}{(k^{\alpha}-k)}\right)}$$

$$P = \frac{\left(\frac{\alpha k^{\alpha-1}(k^{\alpha-1}+\alpha)}{(k^{\alpha}-k)}\right) \pm \sqrt{\frac{\alpha^4 k^{2(\alpha-1)}+2\alpha^3 k^{3(\alpha-1)}+\alpha^2 k^{4(\alpha-1)}-4\left(\alpha^3 k^{3(\alpha-1)}\right)}{2}}}{2\left(\frac{\alpha k^{\alpha-1}}{(k^{\alpha}-k)}\right)}$$

$$P = \frac{\left(\frac{\alpha k^{\alpha-1}(k^{\alpha-1}+\alpha)}{(k^{\alpha}-k)}\right) \pm \sqrt{\frac{\alpha^4 k^{2(\alpha-1)}-2\alpha^3 k^{3(\alpha-1)}+\alpha^2 k^{4(\alpha-1)}}{(k^{\alpha}-k)^2}}}{2\left(\frac{\alpha k^{\alpha-1}}{(k^{\alpha}-k)}\right)}$$

$$P = \frac{\left(\frac{\alpha k^{\alpha-1}(k^{\alpha-1}+\alpha)}{(k^{\alpha}-k)}\right) \pm \sqrt{\frac{\alpha^4 k^{2(\alpha-1)}-2\alpha^3 k^{3(\alpha-1)}+\alpha^2 k^{4(\alpha-1)}}{(k^{\alpha}-k)^2}}}{2\left(\frac{\alpha k^{\alpha-1}}{(k^{\alpha}-k)}\right)}$$

$$P = \frac{\left(\frac{\alpha k^{\alpha-1}(k^{\alpha-1}+\alpha)}{(k^{\alpha}-k)}\right) \pm \frac{\alpha^2 k^{\alpha-1}-\alpha k^{2(\alpha-1)}}{(k^{\alpha}-k)}}{2\left(\frac{\alpha k^{\alpha-1}}{(k^{\alpha}-k)}\right)}$$

$$P = \frac{\alpha k^{\alpha-1}(k^{\alpha-1}+\alpha) \pm (\alpha^2 k^{\alpha-1} - \alpha k^{2(\alpha-1)})}{2\alpha k^{\alpha-1}}$$

$$P = \frac{\alpha k^{2\alpha-2} + \alpha^2 k^{\alpha-1} \pm (\alpha^2 k^{\alpha-1} - \alpha k^{2(\alpha-1)})}{2\alpha k^{\alpha-1}}$$

$$(+)P = \frac{\alpha k^{2\alpha-2} + \alpha^2 k^{\alpha-1} + (\alpha^2 k^{\alpha-1} - \alpha k^{2(\alpha-1)})}{2\alpha k^{\alpha-1}} = \frac{2\alpha^2 k^{\alpha-1}}{2\alpha k^{\alpha-1}} = \alpha = P_1$$

$$(-)P = \frac{\alpha k^{2\alpha-2} + \alpha^2 k^{\alpha-1} - (\alpha^2 k^{\alpha-1} - \alpha k^{2(\alpha-1)})}{2\alpha k^{\alpha-1}} = \frac{2\alpha k^{2(\alpha-1)}}{2\alpha k^{\alpha-1}} = k^{\alpha-1} = P_2$$

Now we must find Q

$$Q = -\frac{LN+M}{FN+FP+G}$$

We set N = 0 to simplify the math.

$$M = \frac{\alpha k^{2\alpha-1}}{k^{\alpha}-k}$$

$$F = \frac{\alpha k^{\alpha-1}}{(k^{\alpha}-k)}$$

$$P = k^{\alpha-1}, \alpha$$

$$G = -\frac{\alpha k^{\alpha-1}(k^{\alpha-1}+\alpha)}{(k^{\alpha}-k)}$$

Now we plug and chug

$$
\begin{aligned}
Q &= -\frac{\left[\frac{\alpha k^{2\alpha-1}}{k^\alpha - k}\right]}{\left[\frac{\alpha k^{\alpha-1}}{(k^\alpha - k)}\right][P] + \left[-\frac{\alpha k^{\alpha-1}(k^{\alpha-1}+\alpha)}{(k^\alpha - k)}\right]} \\
&= -\frac{\alpha k^{2\alpha-1}}{\alpha k^{\alpha-1}P + \alpha k^{\alpha-1}(k^{\alpha-1}+\alpha)} \\
&= -\frac{\alpha k^{2\alpha-1}}{\alpha k^{\alpha-1}P - \alpha^2 k^{\alpha-1} - \alpha k^{2\alpha-2}} \\
(P=\alpha) \implies & -\frac{\alpha k^{2\alpha-1}}{\alpha^2 k^{\alpha-1} - \alpha^2 k^{\alpha-1} - \alpha k^{2\alpha-2}} = \frac{-\alpha k^{2\alpha-1}}{-\alpha k^{2\alpha-2}} = \frac{1}{k} \\
(P=k^{\alpha-1}) \implies & -\frac{\alpha k^{2\alpha-1}}{\alpha k^{2\alpha-2} - \alpha^2 k^{\alpha-1} - \alpha k^{2\alpha-2}} = \frac{-\alpha k^{2\alpha-1}}{-\alpha^2 k^{\alpha-1}} = \frac{k^\alpha}{\alpha}
\end{aligned}
$$

Problem 2

$$
\ln\left(E_t\left\{\beta\frac{\alpha e^{z_{t+1}+(\alpha-1)k_{t+1}}\left(e^{z_t+\alpha k_t} - e^{k_{t+1}}\right)}{e^{z_{t+1}+\alpha k_{t+1}} - e^{k_{t+2}}}\right\}\right) = 0
$$

$$
\ln\beta + \ln\left(\alpha e^{z_{t+1}+(\alpha-1)k_{t+1}+z_t+\alpha k_t} - \alpha e^{z_{t+1}+\alpha k_{t+1}}\right) - \ln\left(e^{z_{t+1}+\alpha k_{t+1}} - e^{k_{t+2}}\right) = 0
$$

Now we must take FOCs w.r.t $\{X_{t+1}, X_t, X_{t-1}, Z_{t+1}, Z_t\}$

$$
\begin{aligned}
F &= \frac{e^{k_{t+2}}}{e^{z_{t+1}+\alpha k_{t+1}} - e^{k_{t+2}}} \\
G &= \frac{(\alpha-1)e^{z_{t+1}+(\alpha-1)k_{t+1}+z_t+\alpha k_t} - \alpha e^{z_{t+1}+\alpha k_{t+1}}}{e^{z_{t+1}+(\alpha-1)k_{t+1}+z_t+\alpha k_t} - e^{z_{t+1}+\alpha k_{t+1}}} - \frac{\alpha e^{z_{t+1}+\alpha k_{t+1}}}{e^{z_{t+1}+\alpha k_{t+1}} - e^{k_{t+2}}} \\
H &= \frac{\alpha e^{z_{t+1}+(\alpha-1)k_{t+1}+z_t+\alpha k_t}}{e^{z_{t+1}+(\alpha-1)k_{t+1}+z_t+\alpha k_t} - e^{z_{t+1}+\alpha k_{t+1}}} \\
L &= 1 - \frac{e^{z_{t+1}+\alpha k_{t+1}}}{e^{z_{t+1}+\alpha k_{t+1}} - e^{k_{t+2}}} \\
M &= \frac{e^{z_{t+1}+(\alpha-1)k_{t+1}+z_t+\alpha k_t}}{e^{z_{t+1}+(\alpha-1)k_{t+1}+z_t+\alpha k_t} - e^{z_{t+1}+\alpha k_{t+1}}}
\end{aligned}
$$

Problem 3

$$
E_t\left\{F\tilde{X}_{t+1} + G\tilde{X}_t + H\tilde{X}_{t-1} + L\tilde{Z}_{t+1} + M\tilde{Z}_t\right\} = 0
$$

We Will transform using the following identities:

- $\tilde{Z}_t = N\tilde{Z}_{t-1} + \varepsilon_t$
- $\tilde{X}_t = P\tilde{X}_{t-1} + Q\tilde{Z}_t$

$$
\begin{aligned}
&\left\{F\tilde{X}_{t+1} + G\tilde{X}_t + H\tilde{X}_{t-1} + L\tilde{Z}_{t+1} + M\tilde{Z}_t\right\} \\
&= F\left(P\left(\tilde{X}_t\right) + Q\left(\tilde{Z}_{t+1}\right)\right) + G\left(P\left(\tilde{X}_{t+1}\right) + Q\left(\tilde{Z}_t\right)\right) + H\tilde{X}_{t-1} + L\left(N\tilde{Z}_{t-1}\right) + M\tilde{Z}_t \\
&= FP\left(P\left(\tilde{X}_{+-1}\right) + Q\left(\tilde{Z}_{+}\right)\right) + FQ\left(N\tilde{Z}_{t-1}\right) + GP\tilde{X}_{t-1} + GQ\tilde{Z}_t + H\tilde{X}_{t-1} + L\left(N\tilde{Z}_{t-1}\right) + M\tilde{Z}_t \\
&= (FPP + GP + H)\tilde{X}_{t-1} + [FPQ + FQN + GQ + LN + M]\tilde{Z}_t \\
&= [(FP + G)P + H]\tilde{X}_{t-1} + [(FQ + L)N + (FP + G)Q + M]\tilde{Z}_t = 0
\end{aligned}
$$

The rest are in jupyter notebook