Week 3

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Open Source Economics Lab - Math Problem Set

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Exercise 1.3

- $G_1 = \{A : A \subset \mathbb{R}, A \text{ open }\}$
  This is not algebra. Take any subset A_i s.t. $A_i \subseteq A$. A is a strict subset of $\mathbb{R}$. $\exists\ x_0 \in \mathbb{R}$ s.t. $x_0 \notin A$ if $x_0 \notin A, \quad A_i \in A \implies x_o \notin A_i$ but $x \in \mathbb{R} \implies G_1$ is not an algebra

- $\mathcal{G}_2 = \{A : A \text{ is a finite union of intervals of the form (a, b]}, (-\infty, \text{b}], \text{ and } (\text{a}, \infty)\}$
  $\forall x \in \mathbb{R} \| x < b,\ x \in A, \quad \forall x \in \mathbb{R} \| x > a,\ x \in A, \quad a < b \implies \forall x \in \mathbb{R}, x \in A$
  By construction ø ∈ A. Now $\forall A_i \subset A,\ A^c = \mathbb{R} - A. \forall x \in \mathbb{R} \implies A^c \in X$ together $\implies$ A is an algebra

- $\mathcal{G}_3 = \{A : A \text{ is a countable union of (a, b]}, (-\infty, \text{b}], \text{ and } (\text{a}, \infty)\}$
  By the same logic as $\mathcal{G}_2, \mathcal{G}_3$ is an algebra

Exercise 1.7

First we prove that $\{\emptyset, X\}$ is the smallest σ-algebra. Remove any point x_0 from the set. now if $B := \{\emptyset, X\} - x_0, B^c = X - B$. By construction $x_0 \in X, x_0 \notin B \implies B$ is not an algebra. Since the point we removed was arbitrary $\{\emptyset, X\}$ is the smallest σ-algebra

Second we prove that $\mathcal{P}(X)$ is the largest σ-algebra. $\mathcal{P}(X) = \{A : A \subset X\}$ if we add any point x_0 to $\mathcal{P}(X)$, either $x_0 \in X$ in which case we have not changed $\mathcal{P}(X)$ since already $x_0 \in \mathcal{P}(X)$. If $x_0 \notin X$ then $\mathcal{P}(X) + x_0$ is not a σ-algebra

Exercise 1.10

- $\forall \alpha, \emptyset \in S_\alpha \implies \emptyset \in \{\bigcap_\alpha S_\alpha\}$ so (i) of the definition is satisfied.

- Suppose $\exists\ B_0 \subset \{\bigcap_\alpha S_\alpha\}, \implies B_0 \in S_\alpha\ \forall\ \alpha$. If we extend this to an infinite collection of $B_i\ i = \{1, 2, 3, ...\}, \forall\ i,\ B_i \subset S_\alpha \forall\ \alpha$ so $\bigcup B_i \subset \{\bigcap_\alpha S_\alpha\}$. Finally if we take $B \in \{\bigcap_\alpha S_\alpha\},\ B^c = X - B$ assume $\exists\ x_0 \in B^c$ s.t. $x_0 \notin \{\bigcap_\alpha S_\alpha\} \implies \forall \alpha\ \exists\ B_\alpha$ s.t. $B_\alpha \subset S_\alpha$ and $B^c_\alpha \notin S_\alpha$

Exercise 1.22

Let $(X, \mathcal{S}, \mu)$ be a measure space.

- μ is monotone: if $A, B \in \mathcal{S}, A \subset B,$ then $\mu(A) \leq \mu(B)$

$$\begin{aligned}
B &= (A \cap B + A^c \cap B) \\
\mu(B) &= \mu\left(A \cap B \cup A^c \cap B\right) \\
A \subset B \implies \mu(B) &= \mu\left(A \cup A^C \cap B\right) \\
\mu(B) &= \mu(A) + \mu\left(A^c \cup B\right) \\
\text{measure is nonneg} \implies \mu(B) &\geq \mu(A) \quad \square
\end{aligned}$$

- μ is countably subadditive: if $\{A_i\}_{i=1}^\infty \subset \mathcal{A},$ then $\mu\left(\cup_{i=1}^\infty A_i\right) \leq \sum_{i=1}^\infty \mu\left(A_i\right)$
  If $\{A_i\}_{i=1}^\infty$ disjoint then trivially true by the property of measures. Assume that it is not disjoint. Without loss of generality, take A_i, A_j s.t. $A_i \cap A_j = B \neq \emptyset$. By (ii) of definition,

$$\begin{aligned}
\mu(A_i \cup A_j) &= \mu\left((A_i \backslash B) \cup (A_j \backslash B) \cup B\right) \\
&= \mu\left(A_i \backslash B\right) + \mu(A_j \backslash B) + \mu(B) \\
&\leq \mu\left(A_i \backslash B\right) + \mu(A_j \backslash B) + 2\mu(B) \\
&== \mu\left(A_i\right) + \mu(Aj)
\end{aligned}$$

  So without loss of generality the measure of the union is less than or equal to the sum of the measures $\square$

Exercise 1.23

Let $(X, \mathcal{S}, \mu))$ be a measure space. Let $B \in \mathcal{S}$. Show that $\lambda : \mathcal{S} \to [0, \infty]$
defined by $\lambda(A) = \mu(A \cap B)$ is also a measure $(X, \mathcal{S})$

- (i) $\quad \lambda(\emptyset) = \mu(B \cap \emptyset) = \mu(\emptyset) = 0 \quad \checkmark$

- (ii) Take a set of sets $\{A_i\}_{i=1}^\infty$

$$\begin{aligned}
\lambda(\bigcup_{i=1}^\infty A_i) &= \mu(\bigcup_{i=1}^\infty A_i \cap B) \\
&= \sum_{i=1}^\infty \mu(A_i \cap B) \\
&= \sum_{i=1}^\infty \lambda(A_i)
\end{aligned}$$

Exercise 1.26

Given $A_i \forall x \in A_i,\ x \in A_j \forall j < i.$ Given any point $y \notin A_N, y \notin \bigcap_{i=1}^{N} A_n \forall N$

$$\implies \bigcap_{i=1}^{i} A_n = A_i$$

$$\implies \mu(\bigcap_{i=1}^{i} A_n) = \mu(A_i) \quad \forall i$$

$$\implies \lim_{i \to \infty} \mu(\bigcap_{i=1}^{i} A_n) = \lim_{i \to \infty} \mu(A_i)$$

Exercise 2.10

Explain why $(*)$ in the preceding theorem could be replaced by
$\mu^*(B) = \mu^*(B \cap E) + \mu^* (B \cap E^c)$

$$\begin{aligned}
\text{s.t.s} \quad \mu^*(B) &\leq \mu^*(B \cap E) + \mu^* (B \cap E^c) \\
B &= (B \cap E) \cup (B \cap E^c) \quad \text{Basic Set Property} \\
\mu^*(B) &= \mu^*((B \cap E) \cup (B \cap E^c)) \\
(B \cap E) \cap (B \cap E^c) &= \emptyset \quad \text{so by prop (ii) of outer measure} \\
\implies \mu^*((B \cap E) \cup (B \cap E^c)) &\leq \mu^*(B \cap E) + \mu^* (B \cap E^c) \\
\mu^*(B) &\leq \mu^*(B \cap E) + \mu^* (B \cap E^c)
\end{aligned}$$

Exercise 2.14

By Thm 2.12 if $\mathcal{M}$ is the σ algebra from Thm 2.8 If $\sigma(\mathcal{A})$ be the σ -algebra generated by $\mathcal{A}$ then $\sigma(\mathcal{A}) \subset \mathcal{M}$

If the Borel Sigma algebra is generated by $\sigma(\mathcal{O})$ where $\mathcal{O}$ is the set of all open sets in R. Clearly this is an algebra so $B(\mathbb{R}) = \sigma(\mathcal{O})$ and by Thm 2.12 $B(\mathbb{R}) \subset \mathcal{M}$

Exercise 3.1

Given a subset $A \subset \mathbb{R}$ s.t. A is countable.

The lebesgue measure $\mu^*(B) := \inf \{\sum_{n=1}^{\infty} (b_n - a_n) : B \subset \bigcup_{i=1}^{\infty} (a_i, b_i]\}$ If we take $A : \{x_1, x_2, ...\}$ we can construct intervals of the form

$$\left[x_i - \frac{\epsilon}{2^{i+2}}, x_i + \frac{\epsilon}{2^{i+2}}\right]$$

Then

$$\sum_{i=1}^{\infty}\left[x_i + \frac{\epsilon}{2^{i+2}} - \left(x_i - \frac{\epsilon}{2^{i+2}}\right)\right]$$
$$= \sum_{i=1}^{\infty}\frac{\epsilon}{2^i+1} = \frac{\epsilon}{2} < \epsilon \quad \square$$

Exercise 3.7

By Thm 3.3:

Let $(\mathbb{X}, \mathcal{M}_X, \mu_X), (\mathbb{Y}, \mathcal{M}_Y, \mu_Y)$ be a measure spaces. Let $\mathcal{G}_Y$ be a set that generates the σ -algebra $\mathcal{M}_Y$, i.e. $\sigma(\mathcal{G}_Y) = \mathcal{M}_Y$.

Then $f : \mathbb{X} \to \mathbb{Y}$ is $\mathcal{M}_X, \mathcal{M}_Y$ -measurable if and only if $\forall B \in \mathcal{G}_Y, f^{-1}(B) \in \mathcal{M}_X$

Since the thm requires that any set work the idea set B in $\mathcal{G}_\dagger$ must be sufficiently general. The sets given in the examples below are equally general as they all allow for any value of a and thus follow identical logic.

Exercise 3.10

If f, g are continuous then the property is immediate from 1. If f, g are not continuous then the property follows immediately for $f * g, f + g, |f|$ since the functions are continuous from the domain $\mathbb{R}^2 \to \mathbb{R}$. Now given that f, g are measurable then $\forall B \in \mathcal{M}, f^{-1}(B) \in \mathcal{M}_X, g^{-1}(B) \in \mathcal{M}_X$. Since the max/min of two functions is the true value of one them, the pre-image of any point in max or min function lies in the pre-image of f or g and thus the function is measurable.

3.17

If f is bounded then $max(f) - min(x) = b - a = c$. Assume w/out loss of generality that $a = 0$, if not then the same effect is accomplished by redefining $f' = f - a$ and we assert that the function a is integrable so by Thm 3.10 s.t.s f' is integrable. For a given S_n The range of f is now partitioned into a set of $2^n * n$ intervals of height $\frac{1}{2^n}$ and by construction of if we take S_b then $max(f - S_n) = \frac{1}{2^n} \quad \forall n > b$. A sequence of functions f_n uniformly converges to a function f if $\forall \epsilon > 0, \exists N_\epsilon$ s.t $.\forall x$ if $n > N_\epsilon f(x) - f(n) < \epsilon$. Take $\epsilon > 0$ and since we can take n arbitarily high, take N_ϵ s.t. $\frac{1}{2^{N_\epsilon}} < \epsilon \implies N_\epsilon > -\frac{\ln(\epsilon)}{\ln(2)}$ and thus for N sufficiently large S_n converges uniformly to f

4.13

$$
\begin{aligned}
\int_E f^d \mu &= \sup \left\{ \int_E s d\mu : 0 \leq s \leq f, \text{ s simple, measurable } \right\} \\
&= \sup \sum_{i=1}^{N} c_i \mu \left(E \cap E_i \right) \\
&\forall i, c_i = f(x) \text{ for some value of x and since} |f| < x, |c_i| < M \\
\sup \sum_{i=1}^{N} c_i \mu \left(E \cap E_i \right) &\leq \sup \sum_{i=1}^{N} |c_i| \mu \left(E \cap E_i \right) \\
&\leq \sup \sum_{i=1}^{N} M \mu \left(E \cap E_i \right) \\
&= M \sup \sum_{i=1}^{N} \mu \left(E \cap E_i \right) \\
&\leq M\mu(E) < \infty \implies \text{ f is integrable}
\end{aligned}
$$

4.14

Assume $\exists E_i \subset E$ some subset with $\mu(E_i) > 0$ s.t f attains a value that is not finite for $x \in E_i$

$$
\int_E f = \sup \sum_{j=1}^{N} c_i \mu \left(E_j \cap E \right)
$$

$$
\text{By applying the sup operator we can assume } \exists N \text{ s.t. } , c_i = \infty
$$

$$
= \sup \sum_{j=1}^{N} c_i \mu \left(E_j \cap E \right) = \infty
$$

$f \notin L^1(\mu, E) \quad \square$

4.15

$$\int_E f = \sup \sum_{i=1}^{N} c_i \mu \left(E_i \cap E\right)$$

$$\int_E g = \sup \sum_{i=1}^{N} d_i \mu \left(E_i \cap E\right)$$

$$\forall i, c_i \leq d_i, \mu \left(E_i \cap E\right) identical for both.$$

$$\implies \sup \sum_{i=1}^{N} c_i \mu \left(E_i \cap E\right) \leq \sup \sum_{i=1}^{N} d_i \mu \left(E_i \cap E\right)$$

$$\implies \int_E f \leq \int_E g$$

4.16

$$\int_E f = \sup \sum_{i=1}^{N} c_i \mu \left(E_i \cap E\right)$$

$$\int_A f = \sup \sum_{i=1}^{N} c_i \mu \left(E_i \cap A\right)$$

$$\begin{aligned} A \subset E &\implies A \cap E = A \\ \implies \mu(E_i \cap A) &= \mu(E_i \cap (A \cap E) \\ &\leq \mu(E_i \cap E) \\ \implies \int_A f &\leq \int_E f \end{aligned}$$

4.21

$$C = A - B, \quad \mu(C) = 0, \quad A = B \cup C, \quad B \cap C = \emptyset$$

By prop (ii) of measure $\mu(\bigcup A_i) = \sum \mu(A_i)$ if $A_i \cap A_j = \emptyset \forall i \neq j$. If $\{A\} = \{B, C\}$ then

$$\begin{aligned} \mu(B \cup C) &= \mu(B) + \mu(C) \\ \mu(A) &= \mu(B) + 0 \quad \square \end{aligned}$$