Week 3

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Open Source Economics Lab - Kirk Phillips
Problem Set 1: DSGE

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1 Problem 1

For the Brock and Mirman model, find the value of A in the policy function. Show that your value is correct.

$$u(c) = \sum \beta^t \ln(c_t)$$
$$s.t. \;\; c_t + k_{t+1} = e^{z_t} k_t^\alpha$$

FOCs:

$$[c_1] : \frac{\beta^t}{C_+} = \lambda_+$$
$$[k_{t+1}] : \left(e^{z_{t+1}} k_{t+1}^{\alpha-1}\right) \lambda_{t+1} \alpha = \lambda_t$$

$$\implies \alpha \left(e^{z_{t+1}} k_{t+1}^{a-1}\right) \left(\frac{\beta^{t+1}}{C_{t+1}}\right) == \frac{\beta^t}{C^t}$$
$$\implies \frac{\beta \alpha e^{z_{t+1}} k_{t+1}^\alpha}{e^{z_{t+1}} k_{t+1}^\alpha - k_{t+2}} = \frac{1}{e^{z_t} k_t^\alpha - k_{t+1}}$$
$$\implies \frac{\beta \alpha e^{z_{t+1}} k_{t+1}^\alpha}{e^{z_{t+1}} k_{t+1}^\alpha - A(e^{z_{t+1}} k_{t+1}^\alpha)} = \frac{1}{e^{z_t} k_t^\alpha - A(e^{z_t} k_t^\alpha)}$$
$$\implies \beta\alpha \frac{e^{z_{t+1}} k_{t+1}^\alpha}{(1-A) e^{z_{t+1}} k_{t+1}^\alpha} = \frac{1}{(1-A)(e^{z_t} k_t^\alpha)}$$
$$\implies \beta\alpha \frac{1}{(1-A) k_{t+1}} = \frac{1}{(1-A)(e^{z_t} k_t^\alpha)}$$
$$\implies \frac{\beta\alpha}{k_{t+1}} = \frac{1}{(e^{z_t} k_t^\alpha)}$$
$$\implies k_{t+1} = \beta\alpha(e^{z_t} k_t^\alpha)$$

2 Exercise 2

$$\max \ln (a_t) + \alpha \ln (1 - l_t) + BE\left[V\left(k_{+1}, \theta_{t+1}\right)\right]$$
$$F\left(K_t, L_t, z_t\right) = e^{z_t} K_t^{\alpha} L_t^{1-\alpha}$$
$$\text{s.t. } (1-\tau)\left[w_t \ell_t + (r_t - \delta) k_t\right] + k_t + T_t = c_t + k_{t+1}$$

Write out all the characterizing equations for the model using these functional forms:

$$(1)\ (1-\tau)\left[w_t \ell_t + (r_t - \delta) k_t\right] + k_t + T_t = c_t + k_{t+1}$$
$$(2)\ \frac{1}{c_t} = \beta E_t\left[\frac{1}{c_{t+1}}\left[(r_{t+1} - \delta)(1-\tau) + 1\right]\right]$$
$$(3)\ \frac{\omega_t(1-\tau)}{c_t} = \frac{\alpha}{1-\ell_t}$$
$$(4)\ r_t = \alpha e^{z_t} k_t^{\alpha-1} \ell_t^{1-\alpha}$$
$$(5)\ \omega_t = (1-\alpha) e^{z_t} k_t^{\alpha} \ell_t^{-\alpha}$$
$$(6)\ \tau\left[w_t \ell_t + (r_t - \delta) k_t\right] = T_t$$
$$(7)\ z_t = (1-\rho_z)\,\overline{z} + \rho_z z_{t-1} + \epsilon_t^z; \quad \epsilon_t^z \sim \text{ i.i.d. } N\left(0, \sigma_z^2\right)$$

We can't use the same trick because of the fact that there are too many variables and simply isolating one will not give a closed form solution.

3 Exercise 3

$$u\left(c_t, \ell_t\right) = \frac{c_t^{1-\gamma} - 1}{1-\gamma} + a \ln (1 - \ell_t)$$
$$F\left(K_t, L_t, z_t\right) = e^{z_t} K_t^{\alpha} L_t^{1-\alpha}$$

$$\text{s.t. } (1-\tau)\left[w_t \ell_t + (r_t - \delta) k_t\right] + k_t + T_t = c_t + k_{t+1}$$

Write out all the characterizing equations for the model using these functional forms.

$$(1)\ (1-\tau)\left[w_t \ell_t + (r_t - \delta) k_t\right] + k_t + T_t = c_t + k_{t+1}$$
$$(2)\ (c_t)^{-\gamma} = \beta E_t\left[(c_{t+1})^{-\gamma}\left[(r_{t+1} - \delta)(1-\tau) + 1\right]\right]$$
$$(3)\ \frac{\omega_t(1-\tau)}{c_t} = \frac{\alpha}{1-\ell_t}$$
$$(4)\ r_t = \alpha e^{z_t} k_t^{\alpha-1} \ell_t^{1-\alpha}$$
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4 Exercise 4

$$u\left(c_t, \ell_t\right) = \frac{c_t^{1-\gamma} - 1}{1-\gamma} + a\frac{\left(1-\ell_t\right)^{1-\xi} - 1}{1-\xi}$$
$$F\left(K_t, L_t, z_t\right) = e^{z_t}\left[\alpha K_t^{\eta} + (1-\alpha)L_t^{\eta}\right]^{\frac{1}{\eta}}$$

$$\text{s.t. } (1-\tau)\left[w_t\ell_t + \left(r_t - \delta\right)k_t\right] + k_t + T_t = c_t + k_{t+1}$$

Write out all the characterizing equations for the model using these functional forms.

$$
\begin{aligned}
&(1)\ (1-\tau)\left[w_t\ell_t + \left(r_t - \delta\right)k_t\right] + k_t + T_t = c_t + k_{t+1}\\
&(2)\ (c_t)^{-\gamma} = \beta E_t\left[(c_{t+1})^{-\gamma}\left[\left(r_{t+1}-\delta\right)(1-\tau)+1\right]\right]\\
&(3)\ \omega_t(1-\tau)(c_t)^{-\gamma} = a(1-\ell_t)^{-\xi}\\
&(4)\ r_t = e^{z_t}\alpha K_t^{\eta-1}\left[\alpha K_t^{\eta} + (1-\alpha)\ell_t^{\eta}\right]^{\frac{1-\eta}{\eta}}\\
&(5)\ \omega_t = e^{z_t}(1-\alpha)\ell_t^{\eta-1}\left[\alpha K_t^{\eta} + (1-\alpha)L_t^{\eta}\right]^{\frac{1-\eta}{\eta}}\\
&(6)\ \tau\left[w_t\ell_t + \left(r_t-\delta\right)k_t\right] = T_t\\
&(7)\ z_t = \left(1-\rho_z\right)\overline{z} + \rho_z z_{t-1} + \epsilon_t^z; \quad \epsilon_t^z \sim \text{ i.i.d. } N\left(0, \sigma_z^2\right)
\end{aligned}
$$

5 Exercise 5

$$u\left(c_t\right) = \frac{c_t^{1-\gamma} - 1}{1-\gamma}$$
$$F\left(K_t, L_t, z_t\right) = K_t^{\alpha}\left(L_t e^{z_t}\right)^{1-\alpha}$$

$$\text{s.t. } (1-\tau)\left[w_t\ell_t + \left(r_t - \delta\right)k_t\right] + k_t + T_t = c_t + k_{t+1}$$

Write out all the characterizing equations for the model using these functional forms.

$$
\begin{aligned}
&(1)\ (1-\tau)\left[w_t\ell_t + \left(r_t - \delta\right)k_t\right] + k_t + T_t = c_t + k_{t+1}\\
&(2)\ (c_t)^{-\gamma} = \beta E_t\left[(c_{t+1})^{-\gamma}\left[\left(r_{t+1}-\delta\right)(1-\tau)+1\right]\right]\\
&(3)\ r_t = \alpha k_t^{\alpha-1}(\ell_t e^{z_t})^{1-\alpha}\\
&(4)\ \omega_t = (1-\alpha)(e^{z_t})^{1-\alpha}k_t^{\alpha}(\ell_t)^{-\alpha}\\
&(5)\ \tau\left[w_t\ell_t + \left(r_t-\delta\right)k_t\right] = T_t\\
&(6)\ z_t = \left(1-\rho_z\right)\overline{z} + \rho_z z_{t-1} + \epsilon_t^z; \quad \epsilon_t^z \sim \text{ i.i.d. } N\left(0, \sigma_z^2\right)
\end{aligned}
$$

If we impose SS:

$$
\begin{aligned}
&(1)\ (1-\tau)\left[w+(r-\delta)\,k\right]+T=c\\
&(2)\ (c)^{-\gamma}=\beta\left[(c)^{-\gamma}\left[(r-\delta)\,(1-\tau)+1\right]\right]\\
&(3)\ r=\alpha(e^{z})^{1-\alpha}k^{\alpha-1}\\
&(4)\ \omega=(1-\alpha)(e^{z})^{1-\alpha}k^{\alpha}\\
&(5)\ \tau\left[w+(r-\delta)\,k\right]=T\\
&(6)\ z=\overline{z}=0
\end{aligned}
$$

From Equation (3)

$$
\begin{aligned}
r&=\alpha(e^{z})^{1-\alpha}k^{\alpha-1}\\
\implies K^{\alpha-1}&=\left(\frac{r}{\alpha(e^{z})^{1-\alpha}}\right)\\
K&=\left(\frac{r}{\alpha}\right)^{\frac{1}{\alpha-1}}
\end{aligned}
$$

From Equation (2)

$$
\begin{aligned}
(c)^{-\gamma}&=\beta\left[(c)^{-\gamma}\left[(r-\delta)\,(1-\tau)+1\right]\right]\\
1&=\beta\left[(r-\delta)\,(1-\tau)+1\right]\\
r&=\frac{1-\beta}{\beta(1-\tau)}+\delta
\end{aligned}
$$

Together Implies

$$
K=\left(\frac{\frac{1-\beta}{\beta(1-\tau)}+\delta}{\alpha}\right)^{\frac{1}{\alpha-1}}
$$

See Attached DSGE.tex for numerical Results

6 Exercise 6

$$
\begin{aligned}
u\left(c_t,\ell_t\right)&=\frac{c_t^{1-\gamma}-1}{1-\gamma}+a\frac{\left(1-\ell_t\right)^{1-\xi}-1}{1-\xi}\\
F\left(K_t,L_t,z_t\right)&=K_t^{\alpha}\left(L_te^{z_t}\right)^{1-\alpha}
\end{aligned}
$$

$$
\text{s.t. } (1-\tau)\left[w_t\ell_t+\left(r_t-\delta\right)k_t\right]+k_t+T_t=c_t+k_{t+1}
$$

Write out all the characterizing equations for the model using these functional forms.

$$(1)\ (1-\tau)\left[w_t\ell_t + (r_t-\delta)\,k_t\right] + k_t + T_t = c_t + k_{t+1}$$
$$(2)\ (c_t)^{-\gamma} = \beta E_t\left[(c_{t+1})^{-\gamma}\left[(r_{t+1}-\delta)(1-\tau)+1\right]\right]$$
$$(3)\ \omega_t(1-\tau)(c_t)^{-\gamma} = a(1-\ell_t)^{-\xi}$$
$$(4)\ r_t = \alpha k_t^{\alpha-1}(\ell_t e^{z_t})^{1-\alpha}$$
$$(5)\ \omega_t = (1-\alpha)(e^{z_t})^{1-\alpha}k_t^{\alpha}(\ell_t)^{-\alpha}$$
$$(6)\ \tau\left[w_t\ell_t + (r_t-\delta)\,k_t\right] = T_t$$
$$(7)\ z_t = (1-\rho_z)\,\overline{z} + \rho_z z_{t-1} + \epsilon_t^z; \quad \epsilon_t^z \sim \text{ i.i.d. } N\left(0,\sigma_z^2\right)$$

When we impose steady state, equilibrium is characterized by

$$(1)\ (1-\tau)\left[w\ell + (r-\delta)\,k\right] + T = c$$
$$(2)\ (c)^{-\gamma} = \beta\left[(c)^{-\gamma}\left[(r-\delta)(1-\tau)+1\right]\right]$$
$$(3)\ \omega(1-\tau)(c)^{-\gamma} = a(1-\ell)^{-\xi}$$
$$(4)\ r = \alpha(e^z\ell)^{1-\alpha}k^{\alpha-1}$$
$$(5)\ \omega = (1-\alpha)(e^z)^{1-\alpha}k^{\alpha}\ell^{-\alpha}$$
$$(6)\ \tau\left[w\ell + (r-\delta)\,k\right] = T$$
$$(7)\ z = \overline{z} = 0$$

See Attached DSGE.tex for numerical Results