2 Biharmonic Equation

2.1 Strong form of the Biharmonic Equation

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain and $\partial\Omega$ be its corresponding boundary. Let the fourth order biharmonic equation have the form,

$$\begin{aligned} \Delta^2 u + \alpha u &= f \quad \text{in } \Omega, \\ \partial_n u &= g_1(x) \quad \text{on } \partial\Omega, \\ \partial_n \nabla^2 u &= g_2(x) \quad \text{on } \partial\Omega. \end{aligned} \tag{1}$$

should be Δ

$= \Delta(\Delta)$

Here is Δ^2 the biharmonic operator, also known as the bilaplacian. We will assume for now that $u \in H^4(\Omega)$, α is a nonnegative constant and $f \in L_2(\Omega)$. We may consider the functions g_1 and g_2 as ~~time independent~~ boundary conditions. Such problems as (1) are often associated with the Cahn-Hilliard model [1] for phase seperation. ~~As a matter of fact, the major difference is that (1) has no time dependencie.~~ However, depending on how Cahn-Hilliard model is time discretized numerically can (1) naturally arise. I refer to [2] for more informastion on this.

We don't consider time-dependent problems, as no need to explicitly mention this. instead say "inhomogeneous"

Before you submitt make sure you have used a proper spell checker program.

2.2 Computational Domains

first we need to

~~We may want~~ to define the computational domain. Recall that $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain. Let $\mathcal{T}_h$ be a triangular mesh of Ω where every triangle $T \in \mathcal{T}_h$, which is illustrated in figure 1. We define h as the max diameter of the triangle T such that $h = \max_{T \in \mathcal{T}_h} h_T$. ① See next page.

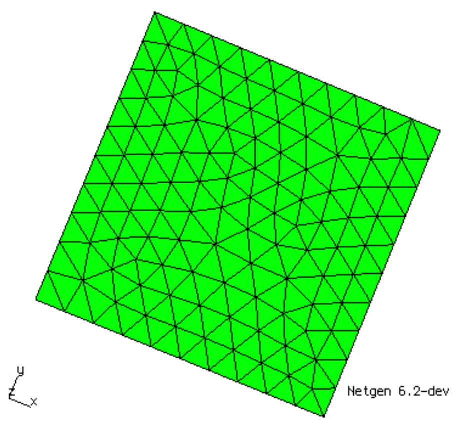

Figure 1: Example of a mesh of $\Omega \subset \mathbb{R}^2$ with triangulation $\mathcal{T}_h$.

facets facet F

We may also define the set of all ~~edges~~ $\mathcal{F}_h$ where every ~~edge~~ is denoted by $\cancel{E} \in \mathcal{F}_h$. However, we will distinguish between the set of external edges $\mathcal{F}_h^{ext}$, which is all edges along $\partial\Omega$, and the interior edges $\mathcal{F}_h^{int}$. Let the edges be denotes as $E \in \mathcal{F}_h$, then the normal vector n is across the edge from T^+ to T^-, illustraded in figure 2.

↳ Replace edges (works on in 2d) with facets (works in all dim.)

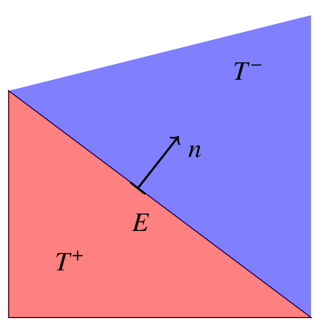

Figure 2: Edge $E \in \mathcal{F}_h$ shared by the triangles $T^+, T^- \in \mathcal{T}_h$ and the normal unit vector n.

① Be more specific. E.g.

"Let $\mathcal{T}_h = \{T\}$ be a tessellation of Ω consisting of triangles T."

Then introduce

$$h_T = \operatorname{diam} T$$

$$h_{min} = \min_{T \in \mathcal{T}_h} h_T$$

$$h_{max} = \max_{T \in \mathcal{T}_h} h_T =: h$$

- Assume mesh is conform i.e. if $T_1 \neq T_2$ and $T_1 \cap T_2 \neq \emptyset$ then they share either a vertex, an edge (in 2D) or a facet.
- Introduce chunkiness parameter: $c_T := \frac{h_T}{r_T}$ where r_T is the radius of the largest ball that be be inscribed the element T.
- Assume that mesh is shape-regular; i.e. that $c_T \leq c$ independent of T and h.
- mesh is quasi-uniform: shape regular + $h_{max} \leq c\, h_{min}$

Note: Everything about meshes, computational domains etc (that means Section 2.2) should be moved into Section 3. You don't need meshes to talk about the derivation of the weak (continuous) formulation of the biharmonic equation

2.3 Weak Form of Biharmonic Operator in $H^4(\cancel{\mathcal{T}}\ \Omega)$

Let $w, v \in H^4(T)$ and $\mathcal{T}_h$ the simplicial triangulation of Ω. Using the same method as in [2, 3] can we deduce that for every triangle $T \in \mathcal{T}_h$ it holds that

Don't write ∇^2, write Δ instead!

Derive these formulas for Ω not for T!

$$
\begin{aligned}
\left(\Delta^2 w, v\right)_{\mathcal{T}} &= \langle \partial_n \nabla^2 w, v\rangle_{\partial T} - \left(\nabla\left(\nabla^2 w\right), \nabla v\right)_{\mathcal{T}} \\
&= \left(D^2 w, D^2 v\right)_{\mathcal{T}} + \langle \partial_n \nabla^2 w, v\rangle_{\partial T} - \langle \partial_n \nabla w, \nabla v\rangle_{\partial T} && (2)\\
&= \left(D^2 w, D^2 v\right)_{\mathcal{T}} - \langle \partial_{nt} w, \partial_t v\rangle_{\partial T} - \langle \partial_{nn} w, \partial_n v\rangle_{\partial T} + \langle \partial_n \nabla^2 w, v\rangle_{\partial T}. && (3)
\end{aligned}
$$

Note that we in the second step we used that

$\partial_t(\partial_n u) = 0$ på Ω hvis $\partial_n u = 0$ på $\partial\Omega$ $\Rightarrow \partial_n v = 0$ on $\partial\Omega$ for test function.

$$
\begin{aligned}
\left(\nabla\left(\nabla^2 w\right), \nabla v\right)_{T} &= \sum_{i=1}^{2} \int_{T} (\nabla\nabla w_{x_i}) \cdot v_{x_i}\, dx \\
&= \int_{T} D^2 w : D^2 v\, dx - \int_{\partial T} (\partial_n \nabla w) \cdot \nabla v\, ds \\
&= \left(D^2 w : D^2 v\right)_{T} + \langle \partial_n \nabla w, \nabla v\rangle_{\partial T}
\end{aligned}
$$

We denote D^2 as the Hessian matrix operator such that

$$
(D^2 u, D^2 v)_\Omega = \int_\Omega D^2 u : D^2 v dx,
$$

where $D^2 u : D^2 v$ is the inner product. Also keep in mind that the last result naturally arise when defining $\nabla = (\partial_n, \partial_t)$ such that

$$
\langle \partial_n \nabla w, \nabla v\rangle_{\partial T} = \langle \partial_{nt} w, \partial_t v\rangle_{\partial T} + \langle \partial_{nn} w, \partial_n v\rangle_{\partial T}.
$$

Remark that we have two formulations (2) and (3)

2.4 Weak Form Biharmonic Equation in $H^4(\Omega)$

We might want to introduce the full basic weak formulation of (1). Now, let the solution space be ~~on the form,~~

$$
V = \left\{ v \in H^2(\Omega) : \partial_n v = g_1 \text{ on } \partial\Omega \right\}..
$$

Since $\partial_n v \neq 0$ you need to distinguish between trial and test function space!

Consider the weak formulation to solve for a $u \in V$ such that

$$
a(u, v) = F(v). \quad \forall v \in V. \tag{4}
$$

Formulate weak formulation at end of section when you motivated derivation of $a(\cdot,\cdot)$.

By using the identity (2) and utilizing the solution space can we easily do a global summation over the triangulation. Hence, the global weak formulation can be expressed as,

No need to involve triangles T. Just do the integration by parts on the entire domain Ω.

$$
\begin{aligned}
\left(\Delta^2 u, v\right)_\Omega &= \sum_{T\in\mathcal{T}_h} -(\nabla\Delta u, \nabla v)_T + \langle \partial_n(\Delta u), v\rangle_{\partial T} \\
&= \sum_{T\in\mathcal{T}_h} -(\nabla\Delta u, \nabla v)_T + \langle \partial_n(\Delta u), v\rangle_{\partial T} \\
&= \sum_{T\in\mathcal{T}_h} (\Delta u, \Delta v)_T - \underbrace{\boxed{\langle \partial_n \nabla u, v\rangle_{\partial T}}}_{\langle \nabla g_1, v\rangle_{\partial T}} + \underbrace{\langle \partial_n(\Delta u), v\rangle_{\partial T}}_{\langle g_2, v\rangle_{\partial T}} \\
&= (\Delta u, \Delta v)_\Omega - \langle \nabla g_1, v\rangle_{\partial\Omega} + \langle g_2, v\rangle_{\partial\Omega}
\end{aligned}
$$

where did this come from?

Laplace operator Δu^2. Not ∂_u^2.

From?

For our problem $\partial_n u = g_1$ is an essential b.c. (i.e built into the function space) and not a natural one.

such that

$$
\begin{aligned}
a(u, v)_\Omega &= \left(D^2 u, D^2 v\right)_\Omega + \alpha(u, v)_\Omega, \\
F(v)_\Omega &= (f, v)_\Omega - \langle g_2, v\rangle_{\partial\Omega} \cancel{+ \langle \nabla g_1, \nabla v\rangle_{\partial\Omega}}.
\end{aligned}
$$

In fact, the solution is unique for $\alpha > 0$. However, for $\alpha = 0$ must we assume the solvability condtion,

$$
\int_\Omega f dx = \int_{\partial\Omega} g_2 ds.
$$

2.1. Strong formulation

2.2. Weak formulation

- Integration by parts rule for Ω

$$(\Delta^2 u, v)_\Omega = (D^2 u, D^2 v)_\Omega + \dots$$

- Introduce $V = \{ v \in H^2(\Omega) \mid \partial_n v = 0 \}$

Derivation of the weak form where $\partial_n v$ is built in weakly.

$$(\Delta^2 u, v)_\Omega = (\partial_n \Delta u, v)_{\partial\Omega} - (\nabla(\Delta u), \nabla v)_\Omega \quad \text{integration by parts}$$

$$= (g_2, v)_{\partial\Omega} - (\nabla(\Delta u), \nabla v)_\Omega$$

Simplest integration by part would be
$$-(\nabla(\Delta u), \nabla v)_\Omega = -(\Delta u, \partial_n v)_{\partial\Omega} + (\Delta u, \Delta v)_\Omega$$

but we don't have b.c. for Δu.

Instead we write:

$$(\nabla(\Delta u), \nabla v)_\Omega = \sum_{i=1}^{d} (\Delta \partial_{x_i} u, \partial_{x_i} v)_\Omega$$

$$= \sum_{i=1}^{d} (\nabla \cdot (\nabla \partial_{x_i} u), \partial_{x_i} v)_\Omega$$

$$= \sum_{i=1}^{d} \left(\frac{\partial}{\partial n} \partial_{x_i} u, \partial_{x_i} v\right)_{\partial\Omega} - \sum_{i=1}^{d} (\nabla \partial_{x_i} u, \nabla \partial_{x_i} v)_\Omega$$

$$= \left(\frac{\partial}{\partial n} \nabla u, \nabla v\right)_{\partial\Omega} - (D^2 u, D^2 v)_\Omega$$

$$= (\partial_{nn} u, \overbrace{\partial_n v}^{=0})_{\partial\Omega} + (\partial_{nt} u, \partial_t v)_{\partial\Omega} - (D^2 u, D^2 v)_\Omega$$

This condition easily arise when using the substitution $v = 1$ in (4). To handle this, can we extended the solution space

replace with

$$V^* = \begin{cases} V & \alpha > 0 \\ \{v \in V : v(p_*) = 0\}, & \alpha = 0 \end{cases} \qquad \int_\Omega v = 0$$

where p_* is a corner of the polygonal domain Ω. Thus, the unique solution in $v \in V^*$ belongs to $H^3(\Omega)$ and we get the follwing elliptic regularity estimate [3],

$$|u|_{H^3(\Omega)} \le C_\Omega \left(\|f\|_{L_2(\Omega)} + (1 + \alpha^{\frac{1}{2}}) \cdot \|w\|_{H^4(\Omega)} \right), \quad w \in H^4(\Omega). \tag{5}$$

This regularity estimate may be important for further usecases in terms of error analysis.

3 Continious Interior Penalty Method

3.1 Introduction

To solve (1) numerically do we want to introduce the Continious Interior Penalty Method (CP), which is a Discontinious Galerkin method (DG) using C^0 finite elements. There is several reasons why we want to apply C^0 instead of the often used C^1 finite elements for fourth order problems. First and foremost is the C^0 finite elements simpler than obtaining C^1 finite elements. Also, compared to other methods similar to the mixed finite element method for the problem (1), CP has in fact preserved the symmetric positive definiteness, which means the stability analysis is more straight forward. Finally and most importantly according to [2] can naive use mixed methods of splitting the boundary conditions of the problem (1) produce wrong solutions if Ω is nonconvex.

Write about this:

Conformal methods $V_h \subset V$ requires C^1. Exists in a good manner in 2D, but does not exist generalization in 3D. Need reference.

Use Bspline as alternative basis. . Less flexible when generating meshes for complicated domains. Need reference.

Write in mixed formulation $\overline{w} = \Delta w$

None-conform discretization of 4th order problem using C0 Elements. Hence CP Method

Move section 2.2 here

3.2 Constructing Continious Interior Penalty Method

Let us ~~again~~ ~~define~~ $u, v \in H^4(T)$ and ~~recall~~ ~~the~~ results (3) so then applied to T yields

assume that

$$\left(\Delta^2 u, v\right)_T = \left(D^2 u, D^2 v\right)_T - \langle \partial_{nt} u, \partial_t v\rangle_{\partial T} - \langle \partial_{nn} u, \partial_n v\rangle_{\partial T} + \left\langle \partial_n \nabla^2 u, v\right\rangle_{\partial T}. \tag{6}$$

For global continuity, let $v \in V = \left\{v \in H^1(\Omega) : v_T \in H^4(T),\ \forall T \in \mathcal{T}_h\right\} \cap C^0(\overline{\Omega})$ and $u \in H^4(\Omega)$ such that,

$$\left(\Delta^2 u, v\right)_\Omega = \sum_{T\in\mathcal{T}_h} \left(D^2 u, D^2 v\right)_T - \langle \partial_{nt} u, \partial_t v\rangle_{\partial T} - \langle \partial_{nn} u, \partial_n v\rangle_{\partial T} + \left\langle \partial_n \nabla^2 u, v\right\rangle_{\partial T}. \tag{7}$$

However, this can be simplified to

$$\begin{aligned}
\left(\Delta^2 u, v\right)_\Omega &= \sum_{T\in\mathcal{T}_h} \left(D^2 u, D^2 v\right)_T + \sum_{E\in\mathcal{F}^{ext}} \left\langle \partial_n \nabla^2 u, v\right\rangle_E - \langle \partial_{nt} u, \partial_t v\rangle_E + \langle \partial_{nn} u, \partial_n v\rangle_E + \sum_{E\in\mathcal{F}^{int}} \langle \partial_{nn} u, [\![\partial_n v]\!]\rangle_E \\
&= \sum_{T\in\mathcal{T}_h} \left(D^2 u, D^2 v\right)_T + \sum_{E\in\mathcal{F}^{ext}} \langle g_2, v\rangle_E + \langle n g_2, \nabla_n v\rangle_E + \langle \partial_t g_1, \partial_t v\rangle_E + \sum_{E\in\mathcal{F}^{int}} \langle \partial_{nn} u, [\![\partial_n v]\!]\rangle_E
\end{aligned} \tag{8}$$

Where $\mathcal{F}_h^{int}, \mathcal{F}_h^{ext} \subset \mathcal{F}_h$ be the set of interior and exterior facets of the triangulation $\mathcal{T}_h$. Keep in mind that the jump over and edge E, visualized in figure 2, is defined as $[\![a]\!] = a^+ - a^-$ and similarly will the mean be defined as $\{\!\{a\}\!\} = \frac{1}{2}(a^+ + a^-)$. The equivalence of (7) and (8) comes from the following argumentation.

$$\begin{aligned}
\left(\Delta^2 u, v\right)_\Omega &= \sum_{T\in\mathcal{T}_h} \left(D^2 u, D^2 v\right)_T - \langle \partial_{nt} u, \partial_t v\rangle_{\partial T} - \langle \partial_{nn} u, \partial_n v\rangle_{\partial T} + \left\langle \partial_n \nabla^2 u, v\right\rangle_{\partial T} \\
&= \sum_{T\in\mathcal{T}_h} \left(D^2 u, D^2 v\right)_T \\
&\quad + \sum_{E\in\mathcal{F}_h^{ext}} \underbrace{\left\langle \partial_n \nabla^2 u, v\right\rangle_E}_{=\langle g_2, v\rangle_E} - \underbrace{\langle \partial_{nt} u, \partial_t v\rangle_E}_{=\langle \partial_t g_1, \partial_t v\rangle} - \underbrace{\langle \partial_{nn} u, \partial_n v\rangle}_{=\langle n g_2, \partial_n v\rangle_E} \\
&\quad + \sum_{E\in\mathcal{F}_h^{int}} \underbrace{\left(\left\langle \partial_{n^+} \nabla^2 u^+, v^+\right\rangle_E + \left\langle \partial_{n^-} \nabla^2 u^+, v^-\right\rangle_E\right)}_{(I)} + \underbrace{\left(\left\langle \partial_{n^+t} u^+, \partial_t v^+\right\rangle_E + \langle \partial_{n^-t} u^-, \partial_t v^-\rangle_E\right)}_{(II)} + \underbrace{\left(\left\langle \partial_{n^+n^+} u^+, v^+\right\rangle_E + \langle \partial_{n^-n^-} u^-, v^-\rangle_E\right)}_{(III)}
\end{aligned}$$