→ since you present several (a^H, a^L).

4 Cut continuous interior penalty methods for the biharmonic problem

→ not a good reason... Curved elements can be used. Don't provide motivation for using unfitted methods here, that's "high-level" motivation and should be done in the introduction.

Questions arise when we want to allow for complex geometries where some physical domain Ω has a smooth boundary Γ and, thus, cannot be fully covered of a fitted mesh. ~~This motivated us to define a unfitted finite element method~~. The concept is to generate a structured background mesh which fully covers Ω , but does not necessarily fit ~~it perfectly~~ to the boundary. Conceptionally, the bilinear form $a_h(\cdot,\cdot)$ and linear form $l_h(\cdot)$ still associated with the physical domain with the discrete space V_h defined in the unfitted mesh. However, we will run into geometrical problems when so-called cut finite elements has a very small intersection of the physical domain, i.e, $|T \cap \Omega|_d \ll |T|_d \lesssim h^d$ and $|\Gamma \cap T|_{d-1} \ll h^{d-1}$, where $|\cdot|_d$ is the measure of the volume in dimension d. The cut elements are identified as a crucial geometric issue since the inverse inequalities outlined in Subsection 2.7 do not generalize well to these elements. This highlights? challenges in establishing stability and a priori estimates for any geometric arrangement. word choice.

be more specific: Integrations of the contributions in a_h and l_h still performed on the physical domain (to keep method consistent) but the involved discrete test/trial functions are defined on the background mesh.

One way do handle this issue is to introduce the so-called cut finite element method (CutFEM). The method is adding ~~an~~ certain stabilization term, also known as the so-called ghost penalty term, which extends the finite element space from the interior elements outside the physical domain (1) stable way which also guarantees stabilization and geometrical robustness, optimal approximation properties. For more information, see [73, 89, 90, 91]. ~~Utilizing~~ Inspired by a corresponding CutFEM DG elliptic framework [81] and starting from the CIP methods introduced in Section 3, is the goal in this section to engineer suitable stabilized ghost penalty terms for the biharmonic problem. We will show what assumptions ~~that is~~ are needed for the ghost-penalty method for the discrete problem to be stable in Subsection 4.3 and ~~obtain~~ derive optimal convergence in Subsection 4.4. Once this is fulfilled will we propose a g_h ghost penalty which fulfills these assumptions in Subsection 4.5. grammar!

4.1 Computational domain

you can safely merge these two figures. No need to have 2.

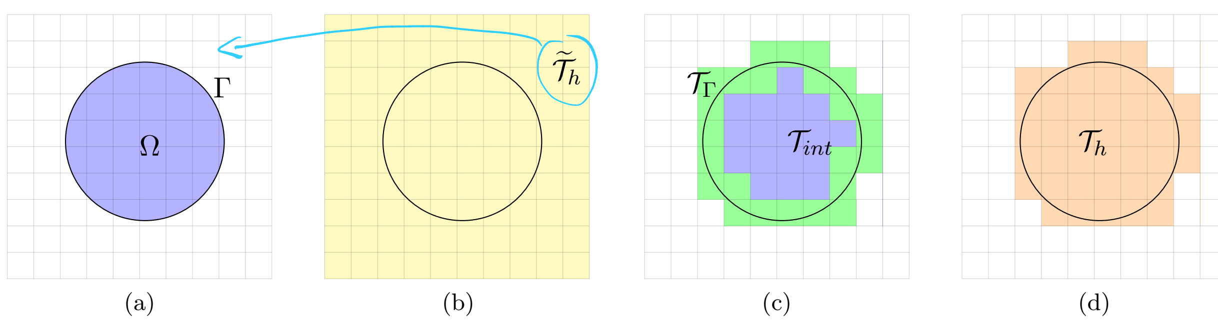

Figure 12: Illustration of the domain Ω with the corresponding boundary Γ, the background mesh $\widetilde{\mathcal{T}}_h$, the cut cells $\mathcal{T}_\Gamma$, the interior cells $\mathcal{T}_{int}$ and the active set $\mathcal{T}_h = \mathcal{T}_{int} \cup \mathcal{T}_\Gamma$.

We want to devise a CutFEM based on the CIP formulation for the biharmonic problem. Assume that the physical domain $\Omega \subseteq \mathbb{R}^d$ to be open and bounded with a corresponding a sufficiently smooth boundary Γ. Let $\widetilde{\mathcal{T}}_h$ be a shape-regular and quasi-uniform mesh which covers Ω, but does not need to fit the domain. Let us denote the active set $\mathcal{T}_h \subseteq \widetilde{\mathcal{T}}_h$ which intersects the interior of the active domain Ω, that is

$$\mathcal{T}_h = \left\{ T \in \widetilde{\mathcal{T}}_h \mid T \cap \Omega \neq \emptyset \right\}. \tag{4.1}$$

We define the corresponding set of interior facets,

mathrm

$$\mathcal{F}_h^{int} = \left\{ F = T^+ \cap T^- \mid T^+, T^- \in \mathcal{T}_h \text{ and } T^+ \neq T^- \right\},$$

and the set of elements cut by the boundary

$$\mathcal{T}_h^{\Gamma} \leftarrow \mathcal{T}_\Gamma = \{ T \in \mathcal{T}_h \mid T \cap \Gamma \neq \emptyset \}.$$

① Not correct. It does not extend the discrete function space, the function space stays the same. Rather the natural energy norm associated with the augmented bilinear form provides control over $v_h \in V_h$ on the <u>entire</u> background mesh, and not only the physical domain Ω.

For convenience, will we define also the interior of the active set as $\mathcal{T}_{int}$.

To be consistent $\mathcal{T}_h^{int}$ with notation $\mathcal{F}_h^{int}$.

$$\boxed{\mathcal{T}_{int}} = \{T \in \mathcal{T}_h \mid T \cap \mathrm{Int}(\Omega) \neq \emptyset\}\,.$$

Hence, we have that the active set is the union of the interior and cut elements, $\mathcal{T}_h = \mathcal{T}_{int} \cup \mathcal{T}_\Gamma$. For an illustration, see Figure 12.

$\mathcal{T}_h^{int} \cup \mathcal{T}_h^{\Gamma}$

4.2 Cut continuous interior penalty methods

As Ω is static is it easy to observe that having a polynomial basis on the full mesh, $\widetilde{\mathcal{T}}_h$, is not necessary. Hence, we define the polynomial space ~~using~~ only on the active set $\mathcal{T}_h$ from (4.1),

$$V_h = \left\{ v \in C^0\left(\mathcal{T}_h\right) : v_T = v|_T \in \mathcal{P}^k\left(T\right), \forall T \in \mathcal{T}_h \right\}.$$

not defined set. Rather write $C(\Omega_h)$ and define $\Omega_h = \bigcup_{T \in \mathcal{T}_h} T$.

Furthermore, drawing on the principles outlined in Section 3, we can indeed recall two CIP formulations for the biharmonic equation: the Hessian formulation (3.15) and the Laplace formulation (3.18).

Function space on meshes ($C^0(\mathcal{T}_h)$) are often understood as broken spaces, only defined element wise.

that will be required later...

To make sure the problem is stabilized will we add a hypothetical consistent symmetric positive semi-definite bilinear ghost-penalty term $g_h : V_h \times V_h \to \mathbb{R}$ ~~in addition~~ to our bilinear. That is, we define the discrete problem to ~~s.t.~~ be: Remove all such abbreviations!

$$\text{Find } u_h \in V_h \text{ such that } A_h(u_h, v) := a_h(u_h, v) + g_h(u_h, v) = l_h(v) \quad \forall v \in V_h. \tag{4.2}$$

don't do "do" → "..section, we provide.."

Here ~~is~~ $a_h(\cdot,\cdot)$ ~~denoted as~~ stands for either $a_h^L(\cdot,\cdot)$ or $a_h^H(\cdot,\cdot)$.

~~We will~~ in this section do a full proof for the Hessian formulation, however, the proofs does ~~should~~ not differ to much ~~for the Laplace formulation.~~ For simplification will we use the notation $a_h(u, v) = a_h^H(u, v)$ and $l_h(v) = l_h^H(v)$ for the rest of the stability and convergence analysis. "for the Laplace formulation_"

Keep in mind that in contrast to the standard CIP methods, ~~this~~ method is defined on an unfitted mesh. As we will see in the analysis, the ghost penalty is a method to ensure numerical stability on cut cell $\mathcal{T}_\Gamma$. The main reason why this numerical instability is happening for a unfitted mesh is that when a cell is badly cut, see examples in Figure 14. In other words, when a cell is "badly cut," it means that it is intersected by the boundary Γ in such a way that only a very small part of the interior of an element T intersects with the physical domain Ω , i.e. $|\Omega \cap T|_d \ll h^d$. This can lead to "both stability issues and" a very poor condition number of the ~~local~~ system matrix ~~corresponding to such a cell, causing numerical instability~~.

"our proposed"

The ghost penalty stabilization technique is designed to tackle this issue. Essentially, this approach introduces additional terms into the finite element method that penalize jumps in the discrete solution and its gradients across cell interfaces, typically in the cut cells. This penalty not only improves the conditioning of the system matrix but also enhances the robustness of the method with respect to the location of the boundary inside each cell. However, to make this possible, we assume a so-called fat-intersection property, which will be relevant in Section 4.5.

Our first assumption as as follows; For a $T \in \mathcal{T}_\Gamma$ there always exists a patch $\omega(T)$ which contains T and an element T' with a so-called fat intersection $|T' \cap \Omega|_d \gtrsim |T'|_d$, where $|\cdot|_d$ is the measure of an element of dimensions $d = 2, 3$. For an illustration, see Figure 13.

We define the underlying norms for $v \in V_h$ as

$$\|v\|_{a_h}^2 = \alpha \|v\|_{\mathcal{T}_h \cap \Omega}^2 + \|D^2 v\|_{\mathcal{T}_h \cap \Omega}^2 + \|h^{-\frac{1}{2}} [\![\partial_n v]\!] \|_{\mathcal{F}_h \cap \Omega}^2 + \|h^{-\frac{1}{2}} \partial_n v\|_\Gamma^2, \tag{4.3}$$

$$|v|_{g_h}^2 = g(v, v), \tag{4.4}$$

$$\|v\|_{A_h}^2 = \|v\|_{a_h}^2 + |v|_{g_h}^2, \tag{4.5}$$

$$\tag{4.6}$$

and for $v \in V \oplus V_h$ we get, also introduce

$$\|v\|_{a_h,*}^2 = \|v\|_{a_h}^2 + \|h^{\frac{1}{2}} \{\!\{\partial_{nn} v\}\!\} \|_{\mathcal{F}_h \cap \Omega}^2 + \|h^{\frac{1}{2}} \partial_{nn} v\|_\Gamma^2 . \tag{4.7}$$