Very important: Make clear that you review known / standard material here and refer to the main sources from which you collected the presented material. E.g. "We follow the presentation in [...]". Ern Guermond "Finite Elements I-III" is a good reference.

1.3 Outline of the report

This report is a novel stabilized cut continuous interior penalty method (CutCIP) that utilizes the CutFEM framework to handle complex domains and the CIP formulation for its elegant formulation to handle fourth-order spatial derivatives, to solve the CH problem. We will name the cut continuous interior penalty method (CutCIP) method. In the first section prove that the method is stable and has optimal convergence for the BH problem, and then extend the method to handle the CH problem. We will then provide numerical examples.

2 Mathematical Background

2.1 Notation

replace with \leqslant

We will in this report assume Ω to be a compact and open set in $\mathbb{R}^d$. Let $p \in \mathbb{R}$, $1 \leq p \leq \infty$ and ~~We~~ define the space $L^p(\Omega)$ to be the set of all measurable functions $f : \Omega \mapsto \mathbb{R}$ such that $|f|^p$ is Lebesgue ~~measurable~~ integrable, i.e,

$$L^p(\Omega) = \left\{ f : \Omega \mapsto \mathbb{R} \mid \int_\Omega |f|^p \, d\Omega < \infty \right\}.$$

Let $u \in L^p(\Omega)$. We define the integral norm of order p to be

be consistent, stick to either f or u in this chapter. (I would prefer u)

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p \, dx \right)^{\frac{1}{p}}.$$

Try to vary language when you do your final polish round.

Since $p = 2$ is frequently used in this report, we also define for convenience a compact notation $\|u\|_\Omega = \|u\|_{L^2(\Omega)}$. ~~We say~~ Recall that $L^2(\Omega)$ is a Hilbert space if it is equipped with a inner product of two functions $u, v \in L^2(\Omega)$ s.t. $(u, v)_\Omega = (u, v)_{L^2(\Omega)} = \int_\Omega uv dx$. We now use this notation for derivatives,

$$\partial^\alpha f = \frac{\partial^{|\alpha|} f}{\partial^{\alpha_1} x_1 \partial^{\alpha_2} x_2}, \quad \text{for } \alpha = (\alpha_1, \alpha_2) \text{ and } f \in C^{|\alpha|}(\Omega). \tag{6}$$

For d dimensions of order k we define the multi-index $\alpha = (\alpha_1, \ldots, \alpha_d)$ with the absolute value $|\alpha| = \sum_{i=1}^{d} \alpha_i = k$ s.t.

$$\partial^\alpha f = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}} f$$

→ if you introduce/review "known stuff", rather use a passive voice than an active "we" voice. That helps later to distinguish known things from your contributions.

Let $m \geq 0$ be an integer and let $1 \leq p \leq \infty$ be a real number. Then ~~we define~~ the Sobolev space $H^m(\Omega)$ are defined by

$$H^m(\Omega) = \left\{ u \in L^2(\Omega) \mid \partial^\alpha u \in L^2(\Omega) \, \forall \alpha : |\alpha| \leq m \right\}.$$

Equipped with the inner product ~~is $H^m(\Omega)$ denoted as a Hilbert space, that is,~~ for $u, v \in H^m(\Omega)$,

$$(u, v)_{H^m(\Omega)} = \sum_{|\alpha| \leq m} \int_\Omega \partial^\alpha u \partial^\alpha v dx.$$

$H^m(\Omega)$ is a Hilbert space with the corresponding ~~Similarly, the~~ norm ~~is is denoted as,~~ $\|u\|^2_{H^m(\Omega)} = \|u\|^2_{L^2(\Omega)} + \sum_{k=1}^m |u|^2_{H^k(\Omega)}$. ~~where~~ Here, the semi-norm is defined such that, $|u|^2_{H^k(\Omega)} = \sum_{|\alpha|=k} \|\partial^\alpha u\|^2_\Omega$. We will often ~~denote the~~ use shorthand notation $\|u\|_{k,\Omega} = \|u\|_{H^k(\Omega)}$ and $|u|_{k,\Omega} = |u|_{H^k(\Omega)}$.

Operators are usually typed in mathrm. You can use the \declareoperator macro to define a new mathematical operator and then simply write \diam

2.2 Computational Domains

Assume that $\Omega \subset \mathbb{R}^d$ is a compact set with a boundary Γ. In standard FEM methods a key assumption is that the set Ω is a polyhedra. This is useful since a polyhedra can be fully covered by a collection of polyhedra and, hence, motivating us to define a fitted mesh. We define a fitted mesh $\mathcal{T}$ of the domain Ω to be a collection of ~~disjoint~~ closed polyhedra $\{T\}$ with disjoint interior forming a partition of Ω s.t $\overline{\Omega} = \bigcup_{T \in \mathcal{T}} T$, for illustration see Figure 3. Here we say that each $T \in \mathcal{T}$ is a mesh element or an element. The mesh size is defined as the maximum diameter $h := h_{max}$ of any polyhedra in the mesh $\mathcal{T} = \{T\}$, that is, $h_{max} = \max_{T \in \mathcal{T}} h_T$ s.t. $h_T = diam(T) = \max_{x_1, x_2 \in T} dist(x_1, x_2)$ Hence, motivating us to use the notation $\mathcal{T}_h$ for a mesh $\mathcal{T}$ with size h.

1) see next page.

A mesh is conform if $T_1 \neq T_2$ then $T_1 \cap T_2 \neq \emptyset$ for all $T_1, T_2 \in \mathcal{T}_h$. This means that each T share either a vertex or a facet. Let the chunkiness parameter $c_T := h_T / r_T$, where r_T is the largest ball that be inscribed inside a element $T \in \mathcal{T}_h$. A mesh is said to be shape regular if $c_T \leq c$ is independent of T and h. We also say that the mesh is quasi-uniform only if it is shape regular and $h_{max} \leq c h_{min}$.

In this report will we assume that a mesh $\mathcal{T}_h$ is conform, shape regular and quasi-uniform unless specified. The fact that the mesh is conform makes is a useful property since the interface between mesh elements has come into contact in the sense that it is either a vertex or a facet. This with the combination of shape regularity and quasi-uniformity ~~has been~~ is a major key to prove important inequalities in broken Sobolev spaces [81, Chapter 1.4.1]. Hence, the assumptions are very handy when proving convergence.

Let $\mathcal{T}_h = \{T\}$ be a mesh of $\Omega \subset \mathbb{R}^d$ consisting of polygons $T \in \mathbb{R}^d$. The set of all facets is the union of external and internal facets, $\mathcal{F}_h = \mathcal{F}_h^{ext} \cup \mathcal{F}_h^{int}$, where each are defined ~~like this~~ by:

$$\mathcal{F}_h^{int} = \left\{ F = T^+ \cap T^- \mid T^+, T^- \in \mathcal{T}_h \right\} \text{ and } \mathcal{F}_h^{ext} = \left\{ F = \partial T \cap \partial\Omega \mid T \in \mathcal{T}_h \right\}.$$

Assumes $T^+ \neq T^-$

~~Let $\mathcal{T}_h$ be a mesh of Ω equipped with the facets $\mathcal{F}_h$.~~ ~~We will~~ Next, we define the following normal vectors.

too much space. Don write n_T rather $n_{\partial T}$.

1) We define $n = n|_T$ to be unit outward normal on ∂T for each $T \in \mathcal{T}_h$

2) Let $F \in \mathcal{F}_h^{int}$. we define n to be the facet normal $n = n\,|_F = n\,|_{T^+}$ from T^+ to T^-, illustrated in figure 4. → No clear definition. First you choose one of the two normals on F (This is an arbitrary choice). Then you define T^+, T^- w.r.t. that normal choice.

3) Let $F \in \mathcal{F}_h^{ext}$. Then we define the facet normal $n|_F = n|_T$ to the unit outward normal.

Keep in mind that ~~in many that~~ we often will simply use the notation n ~~$= n|_T$ in most cases~~. if it is clear which entity the normal n is associated with.

Let $v \in L^2(\Omega)$ be a scalar function on Ω with a corresponding shape regular and quasi-uniform mesh $\mathcal{T}_h$. We will use the following definitions.

1) Let $F \in \mathcal{F}_h^{int}$ and $v^\pm|_F = \lim_{t \to 0^+} v(x \pm tn)$ ($\mp$) for $x \in F$. We define the mean as $\{\!\{v\}\!\}\,|_F = \frac{1}{2}(v_F^+ + v_F^-)$ and the jump as $[\![v]\!]\,|_F = v_F^+ - v_F^-$.

2) Let $F \in \mathcal{F}_h^{ext}$ and let $v(x) = v(x)|_F$ for $x \in F$. We define the mean as $\{\!\{v\}\!\}|_F = v$ and the jump as $[\![v]\!]\,|_F = v$.

To simplify will we use the notation $\{\!\{v\}\!\} = \{\!\{v\}\!\}|_F$ and $[\![v]\!] = [\![v]\!]|_F$ for all $F \in \mathcal{F}_h$. Remark that if we have to functions $u, v \in L^2(\mathcal{T}_h)$, then the following identity holds $[\![uv]\!] = [\![u]\!]\,\{\!\{v\}\!\} + \{\!\{u\}\!\}\,[\![v]\!]$ along all facets $\mathcal{F}_h$ associated with the triangulation $\mathcal{T}_h$.

↳ for which $u^\pm(x)$ and $v^\pm(x)$ are defined. (to be correct, you would need broken $H^1(\mathcal{T}_h)$ to have traces, but you haven't defined them yet.)

2.3 Broken Sobolev spaces

In this work will we compute norms on discontinuous elements, thus, it will be necessary to define broken Sobolev spaces. Let $\mathcal{T}_h$ be a mesh and some integer $m \leq n$. Then we define the

1) • This is not entirely correct. Since you used polyhedra you made it a bit complicated/cumbersome to define conform meshes :)

See Ern/Guermond Finite elements I, Ch 8, in particular Definition 8.11.

• Also make sure that you at the end clearly state that you in this thesis only consider simplicial elements and quadrilateral/hexahedral elements. That will make your life much easier for the remaining analysis

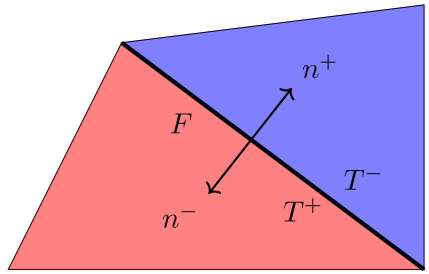


Figure 4: Facet $F \in \mathcal{F}_h^{int}$ shared by the triangles $T^+, T^- \in \mathcal{T}_h$ and the normal unit vector n^+ and n^-. If we pick $T = T^+$ and want to evaluate the normal vector n along a facet F, then we define $n = n\mid_F = n^+$.

broken Sobolev space to be

$$
\begin{aligned}
H^m(\mathcal{T}_h) &:= \left\{ v \in L^2(\Omega) \mid \; v|_T \in H^m(T) \quad \forall T \in \mathcal{T} \right\} \\
L^2(\mathcal{F}_h) &:= \left\{ v \in L^2(\mathcal{T}_h) \mid \; v|_F \in L^2(F) \quad \forall F \in \mathcal{F}_h \right\}.
\end{aligned}
$$

This motivates us to define broken Sobolev norms and inner products using ~~a~~ summation over mesh elements, ~~That is,~~

$$
\|v\|^2_{H^m(\mathcal{T}_h)} = \sum_{T \in \mathcal{T}_h} \|v\|^2_{H^m(T)} \quad \text{and} \quad (v,w)_{H^m(\mathcal{T}_h)} = \sum_{T \in \mathcal{T}_h} (v,w)_{H^m(T)}.
$$

before while shorthand

As ~~expected~~, we use the notation, $\|v\|_{\mathcal{T}_h} = \|v\|_{L^2(\mathcal{T}_h)}$ and $(v,w)_{\mathcal{T}_h} = (v,w)_{L^2(\mathcal{T}_h)}$. That is,

$$
\|v\|^2_{H^m(\mathcal{F}_h)} = \sum_{F \in \mathcal{F}_h} \|v\|^2_{L^2(F)} \quad \text{and} \quad (v,w)_{L^2(\mathcal{F}_h)} = \sum_{T \in \mathcal{F}_h} (v,w)_{L^2(F)}.
$$

mismatch

~~And~~ again, we often use the more compact notation $\|v\|_{\mathcal{F}_h} = \|v\|_{L^2(\mathcal{F}_h)}$ and $(v,w)_{\mathcal{F}_h} = (v,w)_{L^2(\mathcal{F}_h)}$. A very useful lemma when working with ~~on~~ estimates on broken Sobolev spaces is that a if a function is continuous, then the jump between the mesh elements is zero. A function $v \in H^1(\mathcal{T}_h)$ belongs to $H^1(\Omega)$ if and only if $[\![v]\!] = 0$ for $F \in \mathcal{F}_h^{int}$.

2.4 Useful inverse estimates

Some useful inequalities.

(i) Cauchy-Schwarz inequality

Clarify that this holds for every inner product space

$$
\|ab\| \leq \|a\|\|b\|
$$

if $\|\cdot\|$ is induced by the inner product.

(ii) Youngs ε-inequality

$$
2ab \leq \varepsilon a^2 + b^2 \frac{1}{\varepsilon}
$$

See comment next page

(iii) First order inverse inequality

give reference

$$
\|\partial_n v\|_F \leq C_I h^{-\frac{1}{2}} \|\nabla v\|_T
$$

(iv) Second order inverse inequality

give reference

$$
\frac{1}{h}\|\partial_{nn} v_h\|^2_F \leq C_j \|D^2 v_h\|^2_{\mathcal{T}_h}
$$

Very important: iii) - v) only holds for discrete functions $v_h \in V_h$. So state those one first after you introduce FEM spaces. !

(v) General inverse inequality. Let $\alpha = (\alpha_1, \dots, \alpha_N)$ and $\beta = (\beta_1, \dots, \beta_N)$. Assume $u \in H^{|\beta|}(T)$. The following inverse inequalities hold.

$$\|\partial^\alpha u\|_T \lesssim h^{-|\beta|} \|\partial^{\alpha-\beta} u\|_T$$
$$\|\partial_n^\alpha u\|_F \lesssim h^{-\frac{1}{2}} \|\partial^\alpha u\|_T$$

assumes that $\beta \leq \alpha$ (component-wise), state this clearly.

here v is as in iii) - iv)

For a full triangle we have

$$\|v\|_{\partial T} \lesssim h_T^{-\frac{1}{2}} \|v\|_T + h_T^{\frac{1}{2}} \|\nabla v\|_T$$

2.5 Lax-Milgram lemma

Definition 2.1 (Linear bounded functional)**.** *Let X be a Hilbert space. Furthermore, we define the dual space ~~the~~ to be the space of linear and bounded functionals $F : X \mapsto \mathbb{R}$, i.e.,*

$$X' = \begin{Bmatrix} F : X \mapsto \mathbb{R} \text{ s.t. } \forall v, w \in X, \forall a, b \in \mathbb{R} \text{ and } C > 0 \text{ is} \\ F(\lambda v + \mu w) = \lambda F(v) + \mu F(w) \text{ and } |F(v)| \leq C\|v\|_X \end{Bmatrix}$$

incorrect notation, X must be a set. Just define first what a linear bounded functional is, then simply write $X' = \{ F : X \to \mathbb{R} \mid F \text{ is linear and bounded} \}$.

Problem 2.1 (Abstract linear problem)**.** *Assume X and Y to be two Hilbert spaces. Let the vector space $\mathcal{L}(X, Y)$ be all linear bounded operators spanned from X to Y. We define the abstract linear problem as follows; find $u \in X$ s.t.*

This notation is not correct (1)

$$a(u, v) = l(v) := \langle f, v \rangle_{X', X} \, \forall v \in X$$

Where $a \in \mathcal{L}(X \times X, \mathbb{R})$ is a bounded ~~linear~~ bilinear form and $f \in X' := \mathcal{L}(X, \mathbb{R})$ is a bounded linear form. Here we denote by $\langle \cdot, \cdot \rangle_{X', X}$ ~~as~~ the duality pairing between X' and X.

Definition 2.2 (Coercivity)**.** *Let X be a Hilbert space and let $a \in \mathcal{L}(X \times X, \mathbb{R})$. We say that the bilinear form a is coercive on X if there exists an constant $C > 0$ s.t.*

$$a(v, v) \geq C\|v\|_X^2 \; \forall v \in X$$

(2)

always write $a(\cdot,\cdot)$ or $l(\cdot)$, single letters a, l can easily get lost in a sentence.

Lemma 2.1 (Lax-Milgram)**.** ~~*We say that*~~ *the abstract linear problem 2.1 is well-posed if a is coercive. Moreover, the following a priori estimate holds true.*

$$\|v\|_X \leq \frac{1}{C} \|f\|_{X'}$$

Proof. The problem can easily be proved using a special case of the Banach–Nečas–Babuška theorem. See [81, Lemma 1.4] □

2.6 Finite element method

The finite element method (FEM) is a numerical method to solve partial differential equation by finding an approximation of the Problem 2.1. Let X_h be a finite-dimensional (polynomial) approximation space on the mesh $\mathcal{T}_h$. We say that a method is conform if $X_h \subset X$ and non-conform if $X_h \not\subset X$. We define the approximate problem as follows.

(2)

Problem 2.2 (The approximate problem)**.** *Find $u \in X_h$ s.t.*

$$a_h(u, v) = \langle f, v \rangle \, \forall v \in X_h$$

We denote the functional $a_h : X_h \times X_h \to \mathbb{R}$ as an consistent approximation of $a : X \times X \to \mathbb{R}$, and similarly for the right-hand side.

Definition 2.3 (Broken polynomial spaces)**.** *Let $\mathcal{T}_h$ be a mesh of $\Omega \in \mathbb{R}^d$. Let $\mathcal{P}^k(T)$ be the space of all polynomials of degree k in the mesh element T. We define the broken polynomial space as*

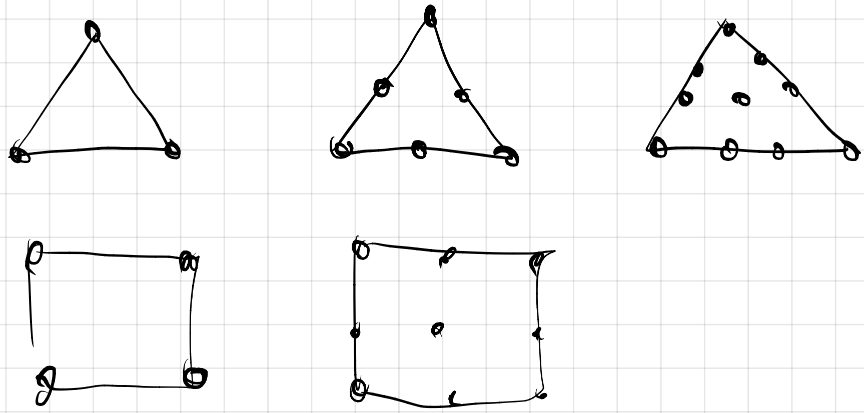
$$\mathcal{P}^k(\mathcal{T}_h) := \left\{ v \in L^2(\Omega) \mid v|_T \in \mathcal{P}^k(T) \quad \forall T \in \mathcal{T}_h \right\}.$$

① $a \in \mathcal{L}(X \times X, \mathbb{R})$ means in your notation that
a is linear on the product space $X \times X$
which is not the same as being <u>bilinear</u> on X ?
Just state what it means for $a(\cdot,\cdot)$ to be bilinear.
Also define "boundedness of a bilinear form"
Then you can use the more standard notation

$$\mathcal{L}(X, X; \mathbb{R}) \text{ or } \mathcal{L}^2(X, \mathbb{R})$$

② The section on FEM needs to be extended a bit.

- Define X_h properly:
  - Assume simplicial / quadrilateral meshes
  - Define local polynomial space $\mathbb{P}^\ell(T)$ and $\mathbb{Q}^\ell(T)$
  - Add figures to illustrate $\mathbb{P}^1, \mathbb{P}^2, \mathbb{P}^3$ elements and $\mathbb{Q}^1, \mathbb{Q}^2, \mathbb{Q}^3$ "



  - Define corresponding global spaces X_h

- Also introduce $\mathbb{P}^\ell_c(\mathcal{T}_h)$ (globally continuous versions) of $\mathbb{P}^\ell(\mathcal{T}_h)$.

<u>Bonus/extra</u> (if time permits):

- Give a short presentation of general error theory for conform methods as we did in Opt 2 or DA8502 (Galerkin Orthog + Cea's lemma)

See e.g. my lecture notes for DA8502 (the latex ones)

As a general writing rule: At the beginning of each chapter (sometimes also Section) write a short "Methods for the" paragraph summarizing what the main objectives of this chapter. E.g. the main points would be: - Recall 4th order biharmonic problem and its weak formulation - Derivation of 2 weak discrete forms - Mention your sources!

3 Continuous Interior Penalty Biharmonic Problem ~~on a Polygonal Domain~~ with Cahn-Hilliard type boundary conditions.

Add subsection title 3.1: The biharmonic problem with C-H type boundary conditions

Recall the biharmonic problem. Let $\Omega \subseteq \mathbb{R}^d$ be a bounded polygonal domain and Γ be its corresponding boundary. Also let $\mathcal{T}_h = \{T\}$ be a shape-regular fitted mesh s.t. $\mathcal{T}_h = \Omega$. Let the BH problem have the form,

Add equation numbers everywhere

$$\begin{aligned} \Delta^2 u + \alpha u &= f(x) \quad \text{in } \Omega, && 3.1 && \text{or } 3.1a \\ \partial_n u &= g_1(x) \quad \text{on } \Gamma, && 3.2 && 3.1b \\ \partial_n \Delta u &= g_2(x) \quad \text{on } \Gamma. && 3.3 && 3.1c \end{aligned} \tag{7}$$

Here is $\Delta^2 = \Delta(\Delta)$ the biharmonic operator, also known as the bilaplacian. We will assume for the strong form that $u \in H^4(\Omega)$, $\alpha > 0$ and $f \in L^2(\Omega)$. The functions $g_1, g_2 : \Omega \to \mathbb{R}$ are denoted as boundary conditions similar to the CH problem.

Add discussion on regularity?

always write out

Remark. It ~~is worth noting that the~~ BH problem is closely related to the Kirchhoff's plate problem by changing the boundary conditions s.t. $u = \partial_n u = 0$ on Γ, which is in the literature known as so-called clamped boundary conditions. Many of the papers we refer to may consider clamped boundary condition and not the CH boundary conditions. C-H The main difference relies on if the problem is treated with homogeneous or non-homogeneous boundary conditions and if the discrete space is imposing the Dirichlet and Neumann conditions strongly in the discrete solution space or weakly using the Nitsche's method [82].

We want to construct a weak form for the strong BH problem (7). Let $v \in H_2(\Omega)$ $H^2(\Omega)$ Using Greens Theorem is it obvious that $(\Delta^2 u, v)_\Omega = (\partial_n \Delta u, v)_\Gamma - (\nabla(\Delta u), \nabla v)_\Omega$. Now doing a new iteration of the Greens theorem we get $-(\nabla(\Delta u), \nabla v)_\Omega = (\Delta u, \Delta v)_\Omega - (\partial_n v, \Delta v)_\Gamma$. Hence, we ~~have~~ obtain the identity ~~s~~.

avoid "doing" in general, try to find more precise formulations, e.g. "Applying Green's theorem once more"

$(\Delta u, \partial_n v)$

$$(\Delta^2 u, v)_\Omega = (\Delta u, \Delta v)_\Omega + (\partial_n \Delta u, v)_\Gamma - (\partial_n v, \Delta u)_\Gamma \tag{8}$$

into

don't overuse abbreviations!

1

Taking account ~~for~~ the boundary conditions, ~~do~~ we end up with the following corresponding weak ~~problem to the BH~~ formulation of the biharmonic problem (7). Find a $u \in H^2(\Omega)$ s.t. $a(u,v) = l(v) \forall v \in H^2(\Omega)$ where

$$\begin{aligned} a(u,v) &= (\alpha u, v)_\Omega + (\Delta u, \Delta v)_\Omega + (\partial_n \Delta u, v)_\Gamma - (\partial_n v, \Delta u)_\Gamma - (\partial_n u, \Delta v)_\Gamma \\ l(v) &= (f,v)_\Omega - (g_2, v)_\Omega + (g_1, \Delta v)_\Gamma \end{aligned}$$

need to be replaced with b.c., so they don't appear in lhs/$a(\cdot,\cdot)$!

Remark that the g_1 boundary condition is weakly imposed by adding a symmetry term, $-(\partial_n u, \Delta v)_\Gamma - (g_1, \Delta v)_\Gamma = 0$.

be consistent with capitalization

3.1 Continuous Interior Penalty formulations

We define the two relevant CIP formulations for the (7) as follows. Let

we generally have

$$V_h = \left\{ v \in C^0(\Omega) : \cancel{v_T} = v|_T \in \mathcal{P}_2(T), \forall T \in \mathcal{T}_h \right\}$$

P^2 not P_2, see Def 2.3

The general idea is to construct a bilinear problem $a_h : V_h \times V_h \to \mathbb{R}$ and $l_h : V_h \to \mathbb{R}$ ~~s.t.~~ "and to solve the problem: find $u \in V_h$ ..." ~~there exists an~~ $u \in V_h$ s.t. $a_h(u_h, v_h) = l_h(v)$ for all $v \in V_h$. For the CIP formulation we will investigate two formulations of the forms a_h and l_h.

don't use "we" and "will investigate". That sounds like you present something new. Either use passive voice or use words like "we require".

1) Assuming that $\boxed{g_1 = 0}$. Then Hessian problem formulation is

Again, cite sources!

$$\begin{aligned} a_h^H(u,v) &= (\alpha u, v)_\Omega + \left(D^2 u, D^2 v\right)_\Omega \\ &\quad - (\{\!\{\partial_{nn} u\}\!\}, [\![\partial_n v]\!])_{\mathcal{F}_h^{int}} - (\{\!\{\partial_{nn} v\}\!\}, [\![\partial_n u]\!])_{\mathcal{F}_h^{int}} + \frac{\gamma}{h}([\![\partial_n u]\!], [\![\partial_n v]\!])_{\mathcal{F}_h^{int}} \\ &\quad - (\partial_{nn} u, \partial_n v)_\Gamma - (\partial_{nn} v, \partial_n u)_\Gamma + \frac{\gamma}{h}(\partial_n u, \partial_n v)_\Gamma \\ l_h^H(v) &= (f,v)_\Omega - (g_2, v)_\Gamma - (g_1, \partial_{nn} v)_\Gamma + \frac{\gamma}{h}(g_1, \partial_n v)_\Gamma \end{aligned} \tag{9}$$

? You just assumed $g_1 = 0$?

1. It does not make sense to weakly impose $\partial_{n_h} u = g_n$ for the continuous weak problem, as we cannot prove coercivity for the resulting bilinear form ! (Recall that for the coercivity of a_h, inverse estimates are crucial, those holds only for discrete function spaces !)

- Formulate the continuous weak formulation by building $\partial_n u = g_n$ strongly into function space (as Brenner did).
- Then when you derive the discrete weak formulation make a point that
  - C^1 continuity is enforced only weakly
  - and in a similar way $\partial_n u = g_n$ is enforced weakly.

2. Move the entire section 3.1 on the C1P formulations past section 3.2 "Detailed Construction of ... "

- Also discuss that (12) actually defines a norm (see. Fang/Karakashian)
- Present a slightly more detailed version of the a priori error estimates in particular suboptimal estimates for L^2 (only order 2) if $p=2$ and general estimates for $k \geq 2$ (not only $k=2$)
- Also clarify assumptions on boundary

<u>A priori</u> error estimates:

$u \in H^s(\Omega)$, $s \geq \frac{5}{2} + \varepsilon$ $\quad u_h \in P_k(\mathcal{T}_h)$ $\quad k \geq 2$. Set $r = \min\{s, k+1\}$

Then

- $\|u - u_h\|_{a_h, *} \lesssim h^{r-2} \|u\|_{r,\Omega}$
- $\|u - u_h\|_{\Omega} \lesssim h^{r - \max\{0, 3-k\}} \|u\|_{r,\Omega}$

number equations w.r.t. chapters, so (3.x)

With the corresponding energy norms,

$$\|v\|_{a_h}^2 = \|v\|_\Omega^2 + \|D^2 v\|_\Omega^2 + \|h^{-\frac{1}{2}} [\![\partial_n v]\!] \|_{\mathcal{F}_h^{int}}^2 + \|h^{-\frac{1}{2}} \partial_n v\|_\Gamma^2, \quad v \in V_h$$
$$\|v\|_{a_h,*}^2 = \|v\|_{a_h}^2 + \|h^{\frac{1}{2}} \{\!\{\partial_{nn} v\}\!\} \|_{\mathcal{F}_h^{int}}^2 + \|h^{\frac{1}{2}} \partial_{nn} v\|_\Gamma^2, \quad v \in V \oplus V_h. \tag{10}$$

define this at first occurrence.

2) The Laplace formulation is

$$\begin{aligned} a_h^L(u,v) = &(\alpha u, v)_\Omega + (\Delta u, \Delta v)_\Omega \\ &- (\{\!\{\Delta u\}\!\}, [\![\partial_n v]\!])_{\mathcal{F}_h^{int}} - (\{\!\{\Delta v\}\!\}, [\![\partial_n u]\!])_{\mathcal{F}_h^{int}} + \frac{\gamma}{h}([\![\partial_n u]\!], [\![\partial_n v]\!])_{\mathcal{F}_h^{int}} \\ &- (\Delta u, \partial_n v)_\Gamma - (\Delta v, \partial_n u)_\Gamma + \frac{\gamma}{h}(\partial_n u, \partial_n v)_\Gamma \\ l_h^L(v) = &(f, v)_\Omega - (g_2, v)_\Gamma - (g_1, \Delta v)_\Gamma + \frac{\gamma}{h}(g_1, \partial_n v)_\Gamma \end{aligned} \tag{11}$$

With the corresponding energy norms

$$\begin{aligned} \|v\|_{a_h}^2 &= \|v\|_\Omega^2 + \|\Delta v\|_\Omega^2 + \|h^{-\frac{1}{2}} [\![\partial_n v]\!] \|_{\mathcal{F}_h^{int}}^2 + \|h^{-\frac{1}{2}} \partial_n v\|_\Gamma^2, \quad v \in V_h \\ \|v\|_{a_h,*}^2 &= \|v\|_{a_h}^2 + \|h^{\frac{1}{2}} \{\!\{\partial_{nn} v\}\!\} \|_{\mathcal{F}_h^{int}}^2 + \|h^{\frac{1}{2}} \partial_{nn} v\|_\Gamma^2, \quad v \in V \oplus V_h. \end{aligned} \tag{12}$$

Remark. It should be noted that the Hessian formulation has a substantial limitation in that it is only valid for homogeneous Neumann conditions. This constraint arises from the challenges associated with imposing g_1 via the tangential derivative terms in Equation (16) during the proof of Lemma 3.1. From a physical perspective, this is not problematic as it aligns with the boundary conditions of the original CH problem (3). However, from the standpoint of numerical validation, the homogeneous Neumann condition enforces strict rules on the design of manufactured solutions on arbitrary domains. Consequently, the examples illustrated in section 6 are only demonstrated on simple domains. This particular constraint does not apply to the Laplace formulation.

The Hessian formulation is well investigated by Susanne Brenner in several papers for [53, 54, 55] with a corresponding analysis and numerical validation. Similarly, variants of the Laplace formulation can be found here [61, 57]. In these article there is good also evidence that both formulation have the following expected a priori estimates. Let $u \in H^s(\Omega)$, and $u_h \in V_h$ of order k. Then with $r = \min\{s, k+1\}$ the a priori estimates are

Also mention Feng/Karakashian 2005/6 and explain that their DG method reduces to CIP methods if DG spaces are swapped with C^0 spaces.

$$\|u - u_h\|_{a_h,*} \lesssim h^{r-1} \|u\|_{r,\Omega}$$
$$\|u - u_h\|_\Omega \lesssim h^r \|u\|_{r,\Omega}$$

3.2 Detailed Construction of Hessian and Laplacian Formulations

Out goal is to derive the Hessian formulation.

Lemma 3.1. *Assume the homogeneous Neumann conditions, $g_1\cancel{(x)} = 0$. Let $u \in H^4(\Omega)$ be the solution to (7). And let $V = \{v \in H^1(\Omega) \mid v|_T \in H^m(T)\ \forall T \in \mathcal{T}_h\}$. Then does the following identity hold.*

just use the finite element space here.

$$(\Delta^2 u, v)_\Omega = (D^2 u, D^2 v)_\Omega + (g_2, v)_\Gamma - (\{\!\{\partial_{nn} u\}\!\}, [\![\partial_n v]\!])_{\mathcal{F}_h^{int}} - (\partial_{nn} u, \partial_n v)_\Gamma \tag{13}$$

Proof. We will start constructing a local theory for a triangle ~~K~~ T and then extend it to the full mesh $\mathcal{T}_h$. Using Greens Theorem is it obvious that $(\Delta^2 u, v)_T = (\partial_n \Delta u, v)_{\partial T} - (\nabla(\Delta u), \nabla v)_T$. We can expand the second term in the following way.

$$\begin{aligned} (\nabla(\Delta u), \nabla v)_T &= \sum_{i=1}^d (\Delta \partial_{x_i} u, \partial_{x_i} v)_T = \sum_{i=1}^d (\nabla \cdot (\nabla \partial_{x_i} u), \partial_{x_i} v)_T \\ &= \sum_{i=1}^d \left((\partial_n \partial_{x_i} u, \cancel{\nabla} \partial_{x_i} v)_{\partial T} - (\nabla \partial_{x_i} u, \nabla \partial_{x_i} v)_T\right) = (\partial_n \nabla u, \nabla v)_{\partial T} - (D^2 u, D^2 v)_T \end{aligned}.$$

add parenthesis

typo

(you are still on a single T so it does not make sense to talk about b.c.

"the normal flux of" "appears naturally in"

Hence, ~~the boundary condition of~~ Δu ~~is integrated into~~ the formulation. It can be denoted that D^2 is the Hessian matrix operator. Also remark that we apply the notation $(D^2u, D^2v)_\Omega = \int_\Omega D^2u : D^2v dx$ for the inner product $D^2u : D^2v$. $\partial_n \nabla u$

Next, We want to decompose the evaluation of ∇u on the boundary ∂T in the tangential and normal direction. Pick a facet $F \in \partial T$, then we define the following decomposition of linear transformation $\nabla u = P_F \nabla u + Q_F \nabla u$ s.t. the orthogonality, $P_F \nabla u \cdot Q_F \nabla u = 0$, holds. ~~The~~ "Here, the" normal projection matrix is defined as $Q_F = n \otimes n$, and the tangential decomposition follows from $P_F = I - Q_F = I - n \otimes n = \sum_{i=1}^{d-1} t_i \otimes t_i$, ~~which~~ "where" is a orthonormal basis t_i, $i = 1, \dots, d-1$ for the space orthogonal to the outer normal vector n on a facet F. Let $a_1, a_2, a_3 \in R^d$ be any vectors, then it is well known that the following identity holds $(a_1 \otimes a_2)a_3 = (a_2^T a_3)a_1$. Hence, we have

$$
\begin{aligned}
Q_F \nabla u &= (n \otimes n)\nabla u = (n^T \nabla u)n \\
P_F \nabla u &= (I - n \otimes n)\nabla u = \nabla u - (n^T \nabla u)n = \sum_{i=1}^{d-1} (t_i^T \nabla u)t_i
\end{aligned}
\tag{14}
$$

Given that u is evaluated only on ∂T can we write $\nabla u = \left(n^T \nabla u\right) n + \sum_i^{d-1} \left(t_i^T \nabla u\right) t_i$ s.t.

$\partial_n u\, n$, no $(\cdot)$? $\partial_n u\, n$

$$
\begin{aligned}
(\partial_n \nabla u, \nabla v)_{\partial T} &= (\partial_n(\partial_n u \cdot n), \partial_n v \cdot n)_{\partial T} + \sum_{i=1}^{d-1} (\partial_n(\partial_{t_i} u \cdot t_i), \partial_{t_j} v \cdot t_j)_{\partial T} \\
&= (\partial_{nn} u, \partial_n v)_{\partial T} + \sum_{i=1}^{d-1} (\partial_{nt_i} u, \partial_{t_i} v)_{\partial T}
\end{aligned}
$$

$i,j=1$ need to have 2 separate summation indices? But $t_i \cdot t_j = \delta_{ij}$.

Here we used that $n^T n = 1$ and $t_i^T t_j = 1$ δ_{ij}. We applied the simple relation,

$$
\begin{aligned}
\partial_n(\partial_n u) &= n^T \nabla(\partial_n u) = n^T(D^2 u\ n) = n^T D^2 u\ n = \partial_{nn} u, \\
\partial_n(\partial_{t_i} u) &= t_i^T \nabla(\partial_n u) = t_i^T(D^2 u\ n) = n^T D^2 u\ t_i = \partial_{nt_i} u.
\end{aligned}
$$

We may also deduce the relationship $\partial_{nt_i} u = \partial_{t_i n} u$ which arise from the fact that $n^T D^2 u\ t_i = (t_i^T D^2 u\ n)^T = t_i^T D^2 u\ n$, where we utilized the symmetry $D^2 u = (D^2 u)^T$ and that the product is a scalar. Adding all these calculations together we have the following local identity,

$$
(\Delta^2 u, v)_T = (D^2 u, D^2 v)_T + (\partial_n \Delta u, v)_{\partial T} - (\partial_n(\partial_n u), \partial_n v)_{\partial T} - \sum_{i=1}^{d-1} (\partial_n(\partial_{t_i} u), \partial_{t_i} v)_{\partial T}
$$

For global continuity we ~~add~~ sum over all the triangles in the mesh $\mathcal{T}_h$.

$$
\left(\Delta^2 u, v\right)_\Omega = \sum_{T \in \mathcal{T}_h} (D^2 u, D^2 v)_T + (\partial_n \Delta u, v)_{\partial T} - (\partial_n(\partial_n u), \partial_n v)_{\partial T} - \sum_{i=1}^{d-1} (\partial_{nt_i} u, \partial_{t_i} v)_{\partial T} \tag{15}
$$

Our goal is to simplify the equation above so we can take account for discontinuities of the derivatives. By integrating over exterior facets $\mathcal{F}_h^{ext}$ and interior facets $\mathcal{F}_h^{int}$ we will get e more suitable formulation which makes it easier to control the jumps between the elements, hence makes it possible to penalize discontinuities.