Master Thesis

Mathematical Modelling of Cell Membrane Dynamics

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1 Introduction

The goal of the thesis implement modern FEM methods to solve the Cahn-Hilliard equation. The plan is to first understand the methods we want to use for the Poisson equation and then implement and do a thoughtful analysis for the method for the fourth order Biharmonic equation. Once this framework is established will we aim for solving the Chan-Hilliard equation using well known techniques for time-discretization and nonlinear problems.

The thesis is planned to have the following structure:

(a) Understand and implement a discontinuous Galerkin method for the Poisson equation.

(b) Derive an unfitted continuous interior penalty method for the biharmonic equation and show that the method is well-posed.

(c) Implement a time discretization scheme of an linear Cahn-Hilliard equation

(d) Implement nonlinear iteration methods for the (nonlinear) Cahn-Hilliard Equation

2 Unfitted cut discontinuous Galerkin method for the Poisson equation

2.1 Introduction

2.2 Notation

2.3 Hilbert Spaces

This subsection is copied from the project thesis

We will in this report assume Ω to be a compact and open set in $\mathbb{R}^2$. Now let the parameter $p \in \mathbb{R}$, $p \geq 1$. We then define the space $L^p(\Omega)$ to be the set of all measurable functions $f : \Omega \mapsto \mathbb{R}$ such that $|f|^p$ is Lebesgue measurable, i.e,

$$L^p(\Omega) = \left\{ f : \Omega \mapsto \mathbb{R} \mid \int_\Omega |f|^p \, d\Omega < \infty \right\}.$$

A useful extension, which we will use later, are the set of locally integrable functions for any compact subset $K \subseteq \text{Interior}(\Omega)$ [1], that is,

$$L^1_{loc}(\Omega) = \left\{ f : f \in L^1(\Omega) \quad \forall K \right\}.$$

Let $u \in L^p(\Omega)$. We define the integral norm of order p to be

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p \, dx \right)^{\frac{1}{p}}.$$

Since $p = 2$ is frequently used in this report, we also define for convenience a compact notation $\|u\|_\Omega = \|u\|_{L^2(\Omega)}$. We say that $L^2(\Omega)$ is a Hilbert space if it is equipped with a inner product of two functions $u, v \in L^2(\Omega)$ such that

$$(u, v)_\Omega = (u, v)_{L^2(\Omega)} = \int_\Omega uv dx.$$

To generalize, we denote the notation $\mathcal{V}$ for a arbitrary Hilbert space. Furthermore, we define the dual space the be the space of linear and bounded functionals $F : \mathcal{V} \mapsto \mathbb{R}$[2], i.e.,

$$\mathcal{V}^* = \left\{ \begin{array}{l} F : \mathcal{V} \mapsto \mathbb{R} \text{ such that } \forall v, w \in \mathcal{V}, \forall a, b \in \mathbb{R} \text{ and } C > 0 \text{ is} \\ F(\lambda v + \mu w) = \lambda F(v) + \mu F(w) \text{ and } |F(v)| \leq C \|v\|_\mathcal{V} \end{array} \right\}$$

and we equip it with the functional norm,

$$\|F\|_{\mathcal{V}^*} = \sup_{v\in\mathcal{V}} \frac{|F(v)|}{\|v\|_{\mathcal{V}}}.$$

We will now establish a notion of the weak derivative, but first are we going to characterize some useful definitions of continuity. The space $C^k(\Omega)$ for $k \geq 0$ denotes the set of functions whose derivatives, up to order of k , is continuous in Ω. Note that we often use the shorthand notation $C^0 = C(\Omega) = C^0(\Omega)$. From this, let $C^\infty(\Omega)$ be the set of infinitely differentiable functions in Ω. Furthermore, we then denote the space $C_0^\infty(\Omega)$ as the space of all functions, $u \in C^\infty(\Omega)$, vanishing outside of any compact subset of Ω. Let $u, v \in C^1(\Omega)$ and the define boundary $\Gamma = \partial\Omega$ with a corresponding outer normal vector n. It is well known that this partial integration formula holds [3],

$$\int_\Omega \nabla u \cdot v dx = \int_\Gamma u \cdot v n ds - \int_\Omega u \cdot \nabla v dx.$$

We now use this notation for derivatives [1] so

$$\partial^\alpha f = \frac{\partial^{|\alpha|} f}{\partial^{\alpha_1} x_1 \partial^{\alpha_2} x_2}, \quad \text{where } \alpha = (\alpha_1, \alpha_2) \text{ and } f \in C^{|\alpha|}(\Omega). \tag{1}$$

Finally, let $u \in L^1_{loc}(\Omega)$. We call the function $w \in L^1_{loc}(\Omega)$ the α-th weak derivative of u if

$$\int_\Omega w\varphi dx = (-1)^{|\alpha|} \int_\Omega u \cdot \partial^\alpha \varphi dx, \quad \forall \varphi \in C_0^\infty(\Omega).$$

We are now able to construct the Sobolev space [3],

$$H^m(\Omega) = \left\{ u \in L^2(\Omega) \mid \partial^\alpha u \in L^2(\Omega) \, \forall \alpha : |\alpha| \leq m \right\} \text{ for } m > 1$$

Equipped with the inner product is $H^m(\Omega)$ denoted as a Hilbert space, that is, for $u, v \in H^m(\Omega)$,

$$(u, v)_{H^m(\Omega)} = \sum_{|\alpha|\leq m} \int_\Omega \partial^\alpha u \partial^\alpha v dx.$$

Similarly, the integral norm is denoted as,

$$\|u\|_{H^m(\Omega)} = \left(\|u\|_{L^2(\Omega)} + \sum_{k=1}^m |u|^2_{H^k(\Omega)} \right)^{\frac{1}{2}},$$

[1] In literature is often $D^\alpha f$ commonly used, but later in the report is this notation reserved for the Hessian operator. Therefore, we then the notation $\partial^\alpha f$ in this report.

where the seminorm is defined such that,

$$|u|_{H^k(\Omega)} = \left(\sum_{|\alpha|=k} \|\partial^\alpha u\|_\Omega^2 \right).$$

For convenience, we also entitle the notation,

$$H_0^m(\Omega) = \left\{ \text{completion of } C_0^\infty(\Omega) \text{ w.r.t. } \|\cdot\|_{H^m(\Omega)} \right\}.$$

Write definitions considering $H^{\frac{1}{2}}(\Gamma)$

2.4 Possion problem

Let $f \in H^1(\Omega)$ and $g \in H^{\frac{1}{2}}(\Gamma)$ and $\Omega \in \mathbb{R}^d$. We then define the strong formulation of the Possion problem to be

$$\begin{aligned} -\Delta u &= f \in \Omega \\ u &= g \in \Omega \end{aligned}.$$

Let us define the Hilbert spaces $V = H^1(\Omega)$, $V_g = \{v \in H^1(\Omega) : v\,|_\Gamma = g\}$, the bilinear form $a : V \times V \to \mathbb{R}$ and the linear form $l : V' \to \mathbb{R}$ s.t.

$$a(u, v) = (\nabla u, \nabla v)_\Omega, \quad l(v) = (f, v)_\Omega.$$

We say the weak formulation is to find a $u \in V_g$ so this equation holds

$$a(u, v) = l(v), \quad \forall v \in V$$

References

[1] S. Brenner and R. Scott. *The Mathematical Theory of Finite Element Methods*. Texts in Applied Mathematics. Springer New York, 2007, pp. 271, 76–77. ISBN: 9780387759340. DOI: 10.1007/978-0-387-75934-0. URL: https://link.springer.com/content/pdf/10.1007/978-0-387-75934-0.pdf.

[2] A. Quarteroni. *Numerical Models for Differential Problems, Third Edition*. DOI: 10.1007/978-3-319-49316-9. URL: https://doi.org/10.1007/978-3-319-49316-9.

[3] A. Manzoni, A. Quarteroni, and S. Salsa. *Optimal Control of Partial Differential Equations: Analysis, Approximation, and Applications*. Applied Mathematical Sciences. Springer International Publishing, 2021. ISBN: 9783030772253. URL: https://books.google.no/books?id=V3NpzgEACAAJ.