The strategy is to bound each term individually using Cauchy-Schwarz (2.2). From this is it easy to see that $|(\mathrm{I})| + |(\mathrm{II})| \lesssim \|v\|_{a_h}\|w\|_{a_h}$. For the terms symmetrical terms (III) and (IV) we apply the inverse inequality (4.10).

$$|(\mathrm{III})| \lesssim \|h^{\frac{1}{2}}\partial_{nn}v\|_{\mathcal{F}_h\cap\Omega}\|h^{-\frac{1}{2}}\,[\![\partial_n w]\!]\,\|_{\mathcal{F}_h\cap\Omega} \lesssim \|v\|_{a_h}\|w\|_{a_h}. \tag{4.29}$$

Here we used that $\|h^{\frac{1}{2}}\partial_{nn}v\|_{\mathcal{F}_h\cap\Omega} \lesssim \|v\|_{A_h}$ thanks to Corollary 4.1. The interior penalty can we easily see that,

$$|(\mathrm{V})| \lesssim \|h^{-\frac{1}{2}}\,[\![\partial_n v]\!]\,\|_{\mathcal{F}_h\cap\Omega}\|h^{-\frac{1}{2}}\,[\![\partial_n w]\!]\,\|_{\mathcal{F}_h\cap\Omega} \lesssim \|v\|_{a_h}\|w\|_{a_h}. \tag{4.30}$$

It remains to handle the symmetry terms (VI) and (VII).

$$|(\mathrm{VI})| \lesssim \|h^{\frac{1}{2}}\partial_{nn}v\|_\Gamma\|h^{-\frac{1}{2}}\partial_n w\|_\Gamma \lesssim \|v\|_{a_h}\|w\|_{a_h} \tag{4.31}$$

Again, here we used the Corollary 4.1. Finally, using the definition of the norm is it easily to see that,

$$|(\mathrm{VIII})| \lesssim \|\partial_n v\|_\Gamma\|\partial_n w\|_\Gamma \lesssim \|v\|_{a_h}\|w\|_{a_h}.$$

Hence, we can conclude

$$|a_h(v,w)| \leqslant \|v\|_{a_h}\|w\|_{a_h} \forall v,w \in V_h. \tag{4.32}$$

Therefore, given that $\|\cdot\|_{a_h} \lesssim \|\cdot\|_{A_h}$, it has been demonstrated that $a_h(\cdot,\cdot)$ is bounded within the $\|\cdot\|_{Ah}$ norm.

Step 2. The goal is to prove (4.25). Let $v \in V_h \oplus V$ and $w \in V_h$. The only difference is that since v is continious we cannot apply to Corollary 4.1 on the estimates (4.29) and (4.31). However, this is not a problem since $\|h^{\frac{1}{2}}\,\{\!\{\partial_{nn}v\}\!\}\,\|_{\mathcal{F}_h\cap\Omega}$ and $\|h^{\frac{1}{2}}\partial_{nn}v\|_\Gamma$ are terms in the norm $\|v\|_{a_h,*}$. Thus, we know that

$$|a_h(v,w)| \leqslant \|v\|_{a_h,*}\|w\|_{A_h}\forall v, w \in V_h \tag{4.33}$$

↳ write Wa

□

4.4 A priori estimates

A priori error analysis (/ estimates)

reviews

In this section, we will only briefly ~~overview the~~ important assumptions and definitions and then prove that the ghost penalty does affect the convergence rate specifically for the energy norm. However, the C^0 discrete solution is to rough for the standard Lagrange interpolation operator. Hence, this motivates us to introduce a method to interpolate a non-smooth function, the so-called Cléments interpolation operator.

↳ this sentence does not make sense.

Not true in our case ($d=2,3$) and we usually assume $u \in H^{\frac{5}{2}+\varepsilon}$

1

~~4.4.1 Energy a priori estimates.~~

If you have only a single subsubsection, you can safely remove it.

Recall that for $v \in H^1(\mathcal{T}_h)$ ~~these~~ the inequalities holds $\forall T \in \mathcal{T}_h$ ~~as~~.

$$\|v\|_{\partial T} \lesssim h_T^{-\frac{1}{2}}\|v\|_T + h^{\frac{1}{2}}\|\nabla v\|_T,$$

$$\|v\|_{\Gamma\cap T} \lesssim h^{-\frac{1}{2}}\|v\|_T + h_T^{\frac{1}{2}}\|\nabla v\|_T.$$

For proof, see [92, Lemma 4.2]. → unclear, a key idea for what?

A key idea is to utilize between the relationship between the physical space Ω and the active mesh $\mathcal{T}_h$. Assume that Ω has a boundary Γ in C^1, then does it exist an bounded extension operator,

$$(\cdot)^e : W^{m,q}(\Omega) \to W^{m,q}(\Omega^e)$$

→ what is Ω^e? Just make it $\mathbb{R}^d$.

for all $v \in W^{m,q}(\Omega)$ where $0 < m \leqslant \infty$ and $1 \leqslant q \leqslant \infty$ which satisfies

$$v^e|_\Omega = v,$$

$$\|v^e\|_{m,q,\Omega^e} \lesssim \|v\|_{m,q,\Omega}.$$

This extension theorem primarily utilizes [93, Theorem 9.7] by ~~Brezis~~ with valuable context provided by [94, p.181, p. 185].

word choice. Just refer to [93], [94] for proofs.

① Rewrite the introducing outline of this section.

Outline should include:

- Describe goal: for proposed method, we want to derive a priori error estimates with respect to both the energy and L^2-norm

- Briefly outline how to reach goal
  - Construction of a suitable (quasi-) interpolation operator, here we use the Clément quasi-interpolation operator, which in contrast to the std. Lagrange nodal interpolation operator is also defined for low regularity functions $u \in L^2$.
  - In combination with discrete coercivity this allows you to derive an a priori error estimate in the energy norm
  - Finally you use a std. duality argument (Aubin-Nitsche trick) to derive the L^2 error estimate.

Why? We don't need that... We can simply stick to $\Omega^e = \mathbb{R}^d$.

Lipschitz boundary is enough.

Remark. Be aware that this theorem requires that we have a sufficiently smooth domain, hence, emphasizing the assumption of a sufficiently smooth boundary. Also keep in mind that the original theorem states the mapping $(\cdot)^e : W^{m,q}(\Omega) \to W^{m,q}(\mathbb{R}^d)$, ~~but we restrict ourself to a slightly bigger extension $\Omega^e \supset \Omega$. This is useful because we are now able to extend the function to the active set, while preserving the Sobolev regularity.~~

~~TODO: Not sure if this argument holds. At least from my point of view is this generalization is not trivial.~~

refer back to local estimates you covered in Ch. 2.

Set / construct

~~Now construct the extended Sobolev space s.t.~~ that $\Omega_h = \mathcal{T}_h \subset \Omega^e$. We ~~define~~ an unfitted Clément interpolator $C_h^e : H^m(\Omega_h^e) \to V_h$ s.t. $C_h^e v := C_h v^e$. We can immediately observe that the interpolation satisfies the global error estimates, that is,

You cannot write it like this. Note that Ω_h is suppose to be a set of points and $\mathcal{T}_h$ is a set / collection of triangles. So you have to write $\Omega_h = \bigcup_{T \in \mathcal{T}_h} T$.

Always write out! Please replace in all files.

don't we have extra 2

$$\|v - C_h^e v\|_{m,2,\mathcal{T}_h} \lesssim h^{l-m} \sum_{T \in \mathcal{T}_h} \|v\|_{l,2,\omega(T)}, \quad m \leqslant l \leqslant k+1$$

need to swap m and l?

$$\|v - C_h^e v\|_{m,2,\mathcal{F}_h} \lesssim h^{l-m-\frac{1}{2}} \sum_{T \in \mathcal{T}_h} \|v\|_{l,2,\omega(F)}, \quad m + \frac{1}{2} \leqslant l \leqslant k+1$$

$$\|v - C_h^e v\|_{m,2,\Gamma} \lesssim h^{l-m-\frac{1}{2}} \sum_{T \in \mathcal{T}_h} \|v\|_{l,2,\omega(T)}, \quad m + \frac{1}{2} \leqslant l \leqslant k+1$$

Don't forget extension symbol, otherwise v on $\mathcal{T}_h$ does not make sense. Or you state that it is implicitly understood that $(\,)^e$ is applied whenever v is considered outside its original domain.

Typically you would square all expression since $\|v\|^2_{\mathcal{T}_h} = \sum_{T \in \mathcal{T}_h} \|v\|^2_T$ but $\|v\|_{\mathcal{T}_h} \neq \sum_{T \in \mathcal{T}_h} \|v\|_T$.

and argue that $\sum_T \|v\|_{s,\omega(T)} \leqslant C\|v\|_{s,\mathcal{T}_h}$ where C is some constant decided by the maximum number of elements in a patch $\omega(T)$ for all $T \in \mathcal{T}_h$. This also holds for the inequality $\sum_T \|v\|_{s,\omega(F)} \leqslant C\|v\|_{s,\mathcal{T}_h}$. Hence, we have perhaps an even more useful set of inequalities.

Make sure that regularity assumption and polynomial order fit together. (!)

$$\|v - C_h^e v\|_{m,2,\mathcal{T}_h} \lesssim h^{l-m} \|v\|_{l,2,\mathcal{T}_h}, \quad m \leqslant l \leqslant k+1 \tag{4.34}$$

$$\|v - C_h^e v\|_{m,2,\mathcal{F}_h} \lesssim h^{l-m-\frac{1}{2}} \|v\|_{l,2,\mathcal{T}_h}, \quad m + \frac{1}{2} \leqslant l \leqslant k+1 \tag{4.35}$$

$$\|v - C_h^e v\|_{m,2,\Gamma} \lesssim h^{l-m-\frac{1}{2}} \|v\|_{l,2,\mathcal{T}_h}, \quad m + \frac{1}{2} \leqslant l \leqslant k+1 \tag{4.36}$$

Maybe hard to argue (4.36) to hold on Γ, but may be related to some generalization of (4.11) and (4.9). Anyhow, (4.35) and (4.36) was never used in the proof of Lemma 4.4 since we used inverse estimates and ended up with (4.34) instead on all of them.

Naturally can we see this is the tools we need to construct an estimate for the energy norm.

Lemma 4.4. *Let $u \in H^s(\Omega)$ for $s \geq 3$ be a exact solution to (??). Then we have*

$$\|u - C_h u\|_{a_h,*} \lesssim h^{s-2} \|u\|_{H^s(\Omega)}$$

make sure that you state order k of FEM space here explicitly and introduce again $r = \min\{s, k+1\}$, so that the notation is consistent with Theorem 4.7.

Proof. By definition is

$$\begin{aligned}
\|u - C_h^e u\|^2_{a_h,*} &= \||\alpha|^{\frac{1}{2}}(u - C_h^e u)\|^2_{\mathcal{T}_h \cap \Omega} + \|D^2(u - C_h^e u)\|^2_{\mathcal{T}_h \cap \Omega} \\
&\quad + \gamma \|h^{-\frac{1}{2}} [\![\partial_n (u - C_h^e u)]\!]\|^2_{\mathcal{F}_h \cap \Omega} + \gamma \|h^{-\frac{1}{2}} \partial_n (u - C_h^e u)\|^2_\Gamma \\
&\quad + \|h^{\frac{1}{2}} \{\!\{\partial_{nn}(u - C_h^e u)\}\!\}\|^2_{\mathcal{F}_h \cap \Omega} + \|h^{\frac{1}{2}} \partial_{nn}(u - C_h^e u)\|^2_\Gamma.
\end{aligned}$$

The strategy is to bound each term individually. Starting with the first two terms we get

$$\begin{aligned}
\||\alpha|^{\frac{1}{2}}(u - C_h^e u)\|^2_{\mathcal{T}_h \cap \Omega} &\lesssim \|(u - C_h^e u)\|^2_{0,2,\mathcal{T}_h} \\
&\lesssim h^{2(s-0)} \|u\|^2_{s,\mathcal{T}_h} \lesssim h^{2(s-2)} \|u\|^2_{s,\mathcal{T}_h} \\
\|D^2(u - C_h^e u)\|^2_{\mathcal{T}_h \cap \Omega} &\lesssim \|u - C_h^e u\|^2_{2,\mathcal{T}_h} = \|u - C_h^e u\|^2_{2,2,\mathcal{T}_h} \\
&\lesssim h^{2(s-2)} \|u\|^2_{s,\mathcal{T}_h}.
\end{aligned}$$

① Your notation / index use in Clément operator is quite messed up ☺

Also, for later use, already start using approximation index.

Fix Clément estimates both here and in Section 2.

For $v \in H^s(\Omega_h)$ and $C_{\hat{h}} : L^2(\Omega_h) \to \mathbb{P}^k_c(\mathcal{T}_h)$

and $r = \min\{s, k+1\}$ we have that

- $\| v - C_{\hat{h}} v \|_{\ell, T} \lesssim h_T^{r-\ell} \, |v|_{r, \omega(T)} \qquad 0 \le \ell \le r$
- $\| v - C_{\hat{h}} v \|^2_{\ell, F} \lesssim h_F^{r-\ell-\frac{1}{2}} \, |v|_{r, \omega(T)} \qquad 0 \le \ell \le r - \frac{1}{2}$.

Also adapt / check remaining section / Lemma 4.4, Thm. 4.7.