With the corresponding energy norms,

$$
\begin{aligned}
\|v\|_{a_h}^2 &= \|v\|_\Omega^2 + \|D^2 v\|_\Omega^2 + \|h^{-\frac{1}{2}} [\![\partial_n v]\!] \|_{\mathcal{F}_h^{int}}^2 + \|h^{-\frac{1}{2}} \partial_n v\|_\Gamma^2, \quad v \in V_h \\
\|v\|_{a_h,*}^2 &= \|v\|_{a_h}^2 + \|h^{\frac{1}{2}} \{\!\{\partial_{nn} v\}\!\} \|_{\mathcal{F}_h^{int}}^2 + \|h^{\frac{1}{2}} \partial_{nn} v\|_\Gamma^2, \quad v \in V \oplus V_h.
\end{aligned}
\tag{10}
$$

2) The Laplace formulation is

$$
\begin{aligned}
a_h^L(u, v) = &(\alpha u, v)_\Omega + (\Delta u, \Delta v)_\Omega \\
&- (\{\!\{\Delta u\}\!\}, [\![\partial_n v]\!])_{\mathcal{F}_h^{int}} - (\{\!\{\Delta v\}\!\}, [\![\partial_n u]\!])_{\mathcal{F}_h^{int}} + \frac{\gamma}{h} ([\![\partial_n u]\!], [\![\partial_n v]\!])_{\mathcal{F}_h^{int}} \\
&- (\Delta u, \partial_n v)_\Gamma - (\Delta v, \partial_n u)_\Gamma + \frac{\gamma}{h} (\partial_n u, \partial_n v)_\Gamma \\
l_h^L(v) = &(f, v)_\Omega - (g_2, v)_\Gamma - (g_1, \Delta v)_\Gamma + \frac{\gamma}{h} (g_1, \partial_n v)_\Gamma
\end{aligned}
\tag{11}
$$

With the corresponding energy norms

$$
\begin{aligned}
\|v\|_{a_h}^2 &= \|v\|_\Omega^2 + \|\Delta v\|_\Omega^2 + \|h^{-\frac{1}{2}} [\![\partial_n v]\!] \|_{\mathcal{F}_h^{int}}^2 + \|h^{-\frac{1}{2}} \partial_n v\|_\Gamma^2, \quad v \in V_h \\
\|v\|_{a_h,*}^2 &= \|v\|_{a_h}^2 + \|h^{\frac{1}{2}} \{\!\{\partial_{nn} v\}\!\} \|_{\mathcal{F}_h^{int}}^2 + \|h^{\frac{1}{2}} \partial_{nn} v\|_\Gamma^2, \quad v \in V \oplus V_h.
\end{aligned}
\tag{12}
$$

Remark. It should be noted that the Hessian formulation has a substantial limitation in that it is only valid for homogeneous Neumann conditions. This constraint arises from the challenges associated with imposing g_1 via the tangential derivative terms in Equation (16) during the proof of Lemma 3.1. From a physical perspective, this is not problematic as it aligns with the boundary conditions of the original CH problem (3). However, from the standpoint of numerical validation, the homogeneous Neumann condition enforces strict rules on the design of manufactured solutions on arbitrary domains. Consequently, the examples illustrated in section 6 are only demonstrated on simple domains. This particular constraint does not apply to the Laplace formulation.

The Hessian formulation is well investigated by Susanne Brenner in several papers for [53, 54, 55] with a corresponding analysis and numerical validation. Similarly, variants of the Laplace formulation can be found here [61, 57]. In these article there is good also evidence that both formulation have the following expected a priori estimates. Let $u \in H^s(\Omega)$, and $u_h \in V_h$ of order k. Then with $r = \min\{s, k+1\}$ the a priori estimates are

Subdivide subsection according to Hessian and Laplacian. $\boxed{3.2.1}$ Construction of the Hessian formulation $\boxed{3.2.2}$ Construction of the ....

$$
\begin{aligned}
\|u - u_h\|_{a_h,*} &\lesssim h^{r-1} \|u\|_{r,\Omega} \\
\|u - u_h\|_\Omega &\lesssim h^r \|u\|_{r,\Omega}
\end{aligned}
$$

Motivate derivation: You want to derive a discrete weak formulation where $V_h \not\subset V$ but which is still consistent for u regular enough. That's why you assume $u \in H^4$ but test with $v_h \in V_h$.

3.2 Detailed Construction of Hessian and Laplacian Formulations

Out goal is to derive the Hessian formulation.

Ikke formuler det som separat lemma, det er for spesifikt. Bare utled identitetene du trenger. Start fra motivasjonen beskrevet i

~~**Lemma 3.1.**~~ *Assume the homogeneous Neumann conditions, $g_1(x) = 0$. Let $u \in H^4(\Omega)$ be the solution to (7). And let $V = \{v \in H^1(\Omega) \mid v|_T \in H^m(T)\ \forall T \in \mathcal{T}_h\}$. Then does the following identity hold.* For the entire derivation write v_h instead of v, and assume $v_h \in V_h$

$$
(\Delta^2 u, v)_\Omega = (D^2 u, D^2 v)_\Omega + (g_2, v)_\Gamma - (\{\!\{\partial_{nn} u\}\!\}, [\![\partial_n v]\!])_{\mathcal{F}_h^{int}} - (\partial_{nn} u, \partial_n v)_\Gamma \tag{13}
$$

~~*Proof.*~~ We will start constructing a local theory for a triangle K and then extend it to the full mesh $\mathcal{T}_h$. Using Greens Theorem is it obvious that $(\Delta^2 u, v)_T = (\partial_n \Delta u, v)_{\partial T} - (\nabla(\Delta u), \nabla v)_T$. We can expand the second term in the following way.

Eq 1

display on separate line, with the eq. number.

$$
\begin{aligned}
(\nabla(\Delta u), \nabla v)_T &= \sum_{i=1}^d (\Delta \partial_{x_i} u, \partial_{x_i} v)_T = \sum_{i=1}^d (\nabla \cdot (\nabla \partial_{x_i} u), \partial_{x_i} v)_T \\
&= \sum_{i=1}^d (\partial_n \partial_{x_i} u, \nabla \partial_{x_i} v)_{\partial T} - (\nabla \partial_{x_i} u, \nabla \partial_{x_i} v)_T = (\partial_n \nabla u, \nabla v)_{\partial T} - (D^2 u, D^2 v)_T
\end{aligned}
$$

Eq 2

See comment next page.

Hence, the boundary condition of Δu is integrated into the formulation. It can be denoted that D^2 is the Hessian matrix operator. Also remark that we apply the notation $(D^2u, D^2v)_\Omega = \int_\Omega D^2u : D^2v dx$ for the inner product $D^2u : D^2v$.

We want to decompose the evaluation of ∇u on the boundary ∂T in the tangential and normal direction. Pick a facet $F \in \partial T$, then we define the following decomposition of linear transformation $\nabla u = P_F\nabla u + Q_F\nabla u$ s.t. the orthogonality, $P_F\nabla u \cdot Q_F\nabla u = 0$, holds. The normal projection matrix is defined as $Q_F = n \otimes n$ and the tangential decomposition follows from $P_F = I - Q_F = I - n\otimes n = \sum_{i=1}^{d-1} t_i \otimes t_i$, which is a orthonormal basis t_i, $i = 1, \dots, d-1$ for the space orthogonal to the outer normal vector n on a facet F. Let $a_1, a_2, a_3 \in R^d$ be any vectors, then it is well known that the following identity holds $(a_1 \otimes a_2)a_3 = (a_2^T a_3)a_1$. Hence, we have

$$
\begin{aligned}
Q_F\nabla u &= (n\otimes n)\nabla u = (n^T\nabla u)n \\
P_F\nabla u &= (I - n\otimes n)\nabla u = \nabla u - (n^T\nabla u)n = \sum_{i=1}^{d-1}(t_i^T\nabla u)t_i
\end{aligned} \tag{14}
$$

Given that u is evaluated only on ∂T can we write $\nabla u = \left(n^T\nabla u\right) n + \sum_i^{d-1}\left(t_i^T\nabla u\right) t_i$ s.t.

$$
\begin{aligned}
(\partial_n\nabla u, \nabla v)_{\partial T} &= (\partial_n(\partial_n u\cdot n), \partial_n v\cdot n)_{\partial T} + \sum_{i=1}^{d-1}(\partial_n(\partial_{t_i}u\cdot t_i), \partial_{t_i}v\cdot t_i)_{\partial T} \\
&= (\partial_{nn}u, \partial_n v)_{\partial T} + \sum_{i=1}^{d-1}(\partial_{nt_i}u, \partial_{t_i}v)_{\partial T}
\end{aligned}
$$

Eq 3.

Here we used that $n^Tn = 1$ and $t_i^Tt_i = 1$. We applied the simple relation,

$$
\begin{aligned}
\partial_n(\partial_n u) &= n^T\nabla(\partial_n u) = n^T(D^2u\ n) = n^TD^2u\ n = \partial_{nn}u, \\
\partial_n(\partial_{t_i}u) &= t_i^T\nabla(\partial_n u) = t_i^T(D^2u\ n) = n^TD^2u\ t_i = \partial_{nt_i}u.
\end{aligned}
$$

We may also deduce the relationship $\partial_{nt_i}u = \partial_{t_in}u$ which arise from the fact that $n^TD^2u\ t_i = (t_i^TD^2u\ n)^T = t_i^TD^2u\ n$, where we utilized the symmetry $D^2u = (D^2u)^T$ and that the product is a scalar. Adding all these calculations together we have the following local identity,

Refer to relevant equations, e.g. "Combining the identities Eq 1, Eq 2 and Eq 3 we see that"

$$
(\Delta^2u, v)_T = (D^2u, D^2v)_T + (\partial_n\Delta u, v)_{\partial T} - (\partial_n(\partial_n u), \partial_n v)_{\partial T} - \sum_{i=1}^{d-1}(\partial_n(\partial_{t_i}u), \partial_{t_i}v)_{\partial T}
$$

be consistent and write it as $\partial_{nn}u$.

same here, write $\partial_{nt_i}u$ instead.

sum over all

For global continuity we ~~add all~~ ~~the~~ triangles in the mesh $\mathcal{T}_h$,

? What do you mean with that ?

$$
\left(\Delta^2u, v\right)_\Omega = \sum_{T\in\mathcal{T}_h}(D^2u, D^2v)_T + (\partial_n\Delta u, v)_{\partial T} - (\partial_n(\partial_n u), \partial_n v)_{\partial T} - \sum_{i=1}^{d-1}(\partial_{nt_i}u, \partial_{t_i}v)_{\partial T} \tag{15}
$$

$\partial_{nn}u$

Our goal is to simplify the equation above so we can take account for discontinuities of the derivatives. By integrating over exterior facets $\mathcal{F}_h^{ext}$ and interior facets $\mathcal{F}_h^{int}$ we will get e more suitable formulation which makes it easier to control the jumps between the elements, hence makes it possible to penalize discontinuities.

$$
\begin{aligned}
(\Delta^2 u, v)_\Omega &= \sum_{T\in\mathcal{T}_h} (D^2u, D^2v)_T + (\partial_n \Delta u, v)_{\partial T} - (\partial_{nn}u, \partial_n v)_{\partial T} - \sum_{i=1}^{d-1} (\partial_{t_i n} u, \partial_{t_i} v)_{\partial T} \\
&= \sum_{T\in\mathcal{T}_h} (D^2u, D^2v)_T + \sum_{F\in\mathcal{F}_h^{ext}} (\partial_n \Delta u, v)_F - (\partial_{nn}u, \partial_n v)_F - \sum_{i=1}^{d-1} (\partial_{t_i n} u, \partial_{t_i} v)_F \\
&\quad + \sum_{F\in\mathcal{F}_h^{int}} \underbrace{\left((\partial_{n^+}\Delta u^+, v^+)_F + \left(\partial_{n^-}\Delta u^+, v^-\right)_F\right)}_{(I)} \\
&\qquad - \underbrace{\left(\left(\partial_{n^+n^+}u^+, \partial_{n^+}v^+\right)_F + \left(\partial_{n^-n^-}u^-, \partial_{n^-}v^-\right)_F\right)}_{(II)} \\
&\qquad - \sum_{i=1}^{d-1} \underbrace{\left((\partial_{n^+t_i}u^+, \partial_{t_i}v^+)_F + \left(\partial_{n^-t_i}u^-, \partial_{t_i}v^-\right)_F\right)}_{(III)} .
\end{aligned}
$$

Where integration over all interior facets $\forall F \in \mathcal{F}_h^{int}$ is computed in this way. ? Not sure what this sentence mean

$$
\begin{aligned}
(I) &= \left(\partial_{n^+}\Delta u^+, v^+\right)_F + \left(\partial_{n^-}\Delta u^-, v^-\right)_F \\
&= \int_F [\![\partial_n \Delta u \cdot v]\!] = \int_F \{\!\{\partial_n \Delta u\}\!\} \underbrace{[\![v]\!]}_{=0} + \underbrace{[\![\partial_n \Delta u]\!]}_{=0} \{\!\{v\}\!\} = 0 \\
(II) &= \left(\partial_{n^+n^+}u^+, \partial_{n^+}v^+\right)_F + \left(\partial_{n^-n^-}u^-, \partial_{n^-}v^-\right)_F \\
&= \int_F [\![\partial_{nn} u \cdot \partial_n v]\!] = \int_F \{\!\{\partial_{nn} u\}\!\} \underbrace{[\![\partial_n v]\!]}_{\neq 0} + \underbrace{[\![\partial_{nn} u]\!]}_{=0} \{\!\{\partial_n v\}\!\} . \\
(III) &= \left(\partial_{n^+t_i}u^+, \partial_{t_i}v^+\right)_F + \left(\partial_{n^-t_i}u^-, \partial_{t_i}v^-\right)_F \\
&= \int_F [\![\partial_{nt_i} u \cdot \partial_{t_i} v]\!] = \int_F \{\!\{\partial_{nt_i} u\}\!\} \underbrace{[\![\partial_{t_i} v]\!]}_{=0} + \underbrace{[\![\partial_{nt_i} u]\!]}_{=0} \{\!\{\partial_{t_i} v\}\!\} = 0
\end{aligned}
$$

Observe that the cancellations in the term (I) and term (III) appears of the continuity of $v \in V$ and $u \in H^4(\Omega)$ which makes the jumps and derivative jumps zero. On the other hand, the second term (II) is does not vanish since the ~~discontinuity is~~ "derivative of" normal ~~vector~~ for $v \in V_h$ is "has" a jump. It can also be raised that $\{\!\{\partial_{nn} u\}\!\} = \partial_{nn} u$ holds ~~by the continuity~~ since $u \in$ of $H^4(\Omega)$. Hence, we have the following identity,

Also mention that you replaced $\partial_n \Delta u$ with g_2 ?.

Eq 1

$$
\begin{aligned}
(\Delta^2 u, v)_\Omega = \left(D^2u, D^2v\right)_{\mathcal{T}_h} + \boxed{(g_2}, v)_{\mathcal{F}_h^{ext}} - \left(\{\!\{\partial_{nn} u\}\!\}, [\![\partial_n v]\!]\right)_{\mathcal{F}_h^{int}} \\
- (\partial_{nn}u, \partial_n v)_{\mathcal{F}_h^{ext}} - \sum_{i=1}^{d-1} (\partial_{t_i n} u, \partial_{t_i} v)_{\mathcal{F}_h^{ext}}
\end{aligned}
\tag{16}
$$

Explain that these naturally appears during partial integration, and therefore they are called consistency terms. They appear since $v_h \notin V$?

Under the assumption that $g_1 = 0$ on Γ, and given that the tangent vector is orthogonal to n, we can assert that $\partial_{t_i n} u = \partial_{t_i}(\partial_n u) = 0$ holds for any $i = 1, \ldots, d$. This implies that the last term of the equation vanish, thus providing the validity of the stated identity. □

$\hookrightarrow = \partial_{t_i} g = 0$

We will now assemble the Hessian CIP formulation. ~~Assume $u, v \in V_h$ and~~ $g_1 = 0$, then we have the linear form

Use same assumptions as suggested above $u \in H^4(\Omega)$, $v_h \in V_h$.

(1)

$$
\begin{aligned}
a_h^H(u, v) &= (\alpha u, v)_\Omega + \left(D^2u, D^2v\right)_\Omega \\
&\quad - \left(\{\!\{\partial_{nn} u\}\!\}, [\![\partial_n v]\!]\right)_{\mathcal{F}_h^{int}} - \left(\{\!\{\partial_{nn} v\}\!\}, [\![\partial_n u]\!]\right)_{\mathcal{F}_h^{int}} + \frac{\gamma}{h}\left([\![\partial_n u]\!], [\![\partial_n v]\!]\right)_{\mathcal{F}_h^{int}} \\
&\quad - (\partial_{nn}u, \partial_n u)_\Gamma - (\partial_{nn}v, \partial_n u)_\Gamma + \frac{\gamma}{h}(\partial_n u, \partial_n v)_\Gamma \\
l_h^H(v) &= (f, v)_\Omega - (g_2, v)_\Gamma
\end{aligned}
\tag{17}
$$

Not motivated

① Provide more motivation for the 4 new terms you added, also do this stepwise starting from Eq 1.

1) State that a smooth enough solution $u \in H^4$ of the biharmonic problem satisfy the identity

$$(D^2 u, D^2 v_h)_{\mathcal{T}_h} - (\{\partial_{nn} u\}, [\partial_n v_h])_{\mathcal{F}_h^{int}} - (\partial_{nn} u, \partial_n v_h)_{\mathcal{F}_h^{ext}} \qquad (\text{Eq } 2)$$

$$= (f, v)_{\mathcal{T}_h} - (g_2, v)_{\Gamma}.$$

Want to use this weak problem to find a discrete solution $u_h \in V_h$.

2) Next point out that this candidate (Eq 2) for a discrete weak formulation is not symmetric due to the consistency terms appearing on $\mathcal{F}_h^{int}$ and Γ. Thus one adds symmetrization terms where one exploits that $[\partial_n u] = 0$ and $\partial_n u = 0$ on $\mathcal{F}_h^{int}$ and Γ respectively.

3) Finally, point out that in order to prove discrete coercivity, one adds a "stability term" $\frac{\gamma}{h}([\partial_n u], [\partial_n v])_{\mathcal{F}_h^{int}} + \frac{\gamma}{h}(\partial_n u, \partial_n v)_{\Gamma}$, which are consistent modifications again since $[\partial_n u] = 0$ and $\partial_n u = 0$ on Γ.

State final discrete weak problem: Find $u_h \in V_h$ s.t. $\forall v_h \in V_h$
$a_h^H(u_h, v_h) = l_h^H(v_h)$ where
$a_h^H = \dots$ and $l_h^H = \dots$ Also identify/state the natural norms associated with H.

Hence, the Hessian formulation is derived.

Subsection 3.2.2

~~Lemma 3.2.~~ *Assume $u \in H^4(\Omega)$ and $v \in V = \{v \in H^1(\Omega) \mid v|_T \in H^m(T)\ \forall T \in \mathcal{T}_h\}$. Then we have the following identity.*

$$(\Delta^2 u, v)_\Omega = (\Delta u, \Delta v)_{\mathcal{T}_h} + (\llbracket \partial_n v \rrbracket, \{\!\{\Delta u\}\!\})_{\mathcal{F}_h} + (g_2, v)_\Gamma - (\partial_n v, \Delta u)_\Gamma.$$

Please follow same recipe as outlined for 3.2.1 Hessian formulation.

~~Proof.~~ Similarly, we start by constructing ~~a local theory~~ integration by parts identities for a triangle T and then extend it to the full mesh $\mathcal{T}_h$. Utilizing (8) can we ~~construct a local identity s.t.~~ See that

$$(\Delta^2 u, v)_T = (\Delta u, \Delta v) + (\partial_n \Delta u, v)_{\partial T} - (\partial_n v, \Delta u)_{\partial T}$$

Now, doing a summation over all elements we get don't do "dos" :) just say: "summing over all"...

$$(\Delta^2 u, v)_\Omega = \sum_{T \in \mathcal{T}_h} \Big((\Delta u, \Delta v)_T + (\partial_n \Delta u, v)_{\partial T} - (\partial_n v, \Delta u)_{\partial T} \Big)$$

use parenthesis if you have several summands

$$= (\Delta u, \Delta v)_{\mathcal{T}_h} + \sum_{F \in \mathcal{F}_h^{ext}} \Big(\overbrace{(\partial_n \Delta u, v)_F}^{=(g_2, v)_F} - (\partial_n v, \Delta u)_F \Big)$$

$$+ \sum_{F \in \mathcal{F}_h^{int}} \Big(\underbrace{((\partial_{n^+} \Delta u, v)_F + (\partial_{n^-} \Delta u, v)_F)}_{(I)} - \underbrace{((\partial_{n^+} v, \Delta u)_F + (\partial_{n^-} v, \Delta u)_F)}_{(II)} \Big)$$

larger parenthesis.

be more precise. Also as before ensure that $u \in H^4(\Omega)$ but $v = v_h \in V_h$.

Decomposing the terms and utilizing the regularity of u and v is it easy to see that,

$$(I) = (\partial_{n^+} \Delta u, v)_F + (\partial_{n^-} \Delta u, v)_F = \int_F [\partial_n \Delta u \cdot v] = (\{\!\{\partial_{n^+} \Delta u\}\!\}, \underbrace{\llbracket v \rrbracket}_{=0})_F + (\underbrace{\llbracket \partial_n \Delta u \rrbracket}_{=0}, \{\!\{v\}\!\})_F$$

Write v_h not v.

$$(II) = (\partial_{n^+} v, \Delta u)_F + (\partial_{n^-} v, \Delta u)_F = \int_F \llbracket \partial_n v \cdot \Delta u \rrbracket = (\underbrace{\llbracket \partial_n v \rrbracket}_{\neq 0}, \{\!\{\Delta u\}\!\})_F + (\{\!\{\partial_n v\}\!\}, \underbrace{\llbracket \Delta u \rrbracket}_{=0})_F$$

Hence, we end up with the identity,

$$(\Delta^2 u, v)_\Omega = (\Delta u, \Delta v)_{\mathcal{T}_h} + (\llbracket \partial_n v \rrbracket, \{\!\{\Delta u\}\!\})_{\mathcal{F}_h} + (g_2, v)_\Gamma - (\partial_n v, \Delta u)_\Gamma.$$

$\square$

Finally, can we construct the Laplace CIP formulation. Let $u, v \in V_h$, then can finally write the following respective forms.

$$\begin{aligned} a_h^L(u, v) &= (\alpha u, v)_\Omega + (\Delta u, \Delta v)_\Omega \\ &\quad - (\{\!\{\Delta u\}\!\}, \llbracket \partial_n v \rrbracket)_{\mathcal{F}_h} - (\{\!\{\Delta v\}\!\}, \llbracket \partial_n u \rrbracket)_{\mathcal{F}_h^{int}} + \frac{\gamma}{h} (\llbracket \partial_n u \rrbracket, \llbracket \partial_n v \rrbracket)_{\mathcal{F}_h^{int}} \\ &\quad - (\Delta u, \partial_n v)_\Gamma - (\Delta v, \partial_n u)_\Gamma + \frac{\gamma}{h} (\partial_n u, \partial_n v)_\Gamma \\ l_h^L(v_h) &= (f, v)_\Omega - (g_2, v)_\Gamma - (g_1, \Delta v)_\Gamma + \frac{\gamma}{h} (g_1, \partial_n v)_\Gamma \end{aligned} \tag{18}$$

~~In contrast to the Hessian formulation, is the boundary Neumann booundary condition~~ g_1 ~~is imposed weakly using the Nitsche's method~~, hence, the regularization term. This consists of adding a symmetry term and adding an additional stabilization regularisation term [84], that is,

Not true, also for the Hessian $\partial_n u = 0$ is imposed weakly with a Nitsche-type method, but there we assume that $g_1 = 0$.

$$(\Delta v, \partial_n u)_\Gamma + (\Delta v, g_1)_\Gamma + \frac{\gamma}{h} (\partial_n u - g_1, \partial_n v)_\Gamma = 0$$

It is worth noting that technically is the interior regularisation equivalent to do a Nitsche's method in all interior boundaries of the elements with boundary conditions of each element weakly imposed to zero. Thus, we expect the penalty parameter γ to be the same interior and exterior elements. Let where $k \geq 2$ is the polynomial order, then for the Hessian formulation is it theoretically proven that $\gamma = 2k(k-1)$ [54, 53]. However, we still prefer to experimentally verify the best parameter.