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INTRODUCTION TO CUT FINITE ELEMENT METHOD

And why this is potentially a great technology for solving complex PDE problems.

Isak Hammer

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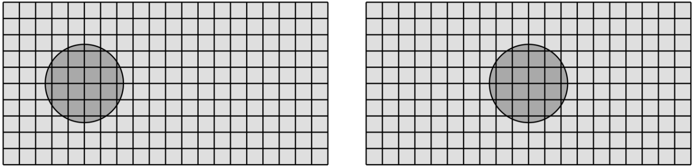
I will present a method to solve PDE's on an unfitted mesh



- Fairly new class of methods! 5-10 years old.
- Can handle smooth boundaries and complex geometries!
- Possible to apply moving domains without re-meshing!

Maybe I should introduce myself

- Isak Hammer, 27 year old, Lofoten
- Graduate student in Industrial Mathematics
- Department of Mathematical Sciences (IMF)
- Specialization in
  1. Finite element methods
     - 1.1 Writing master thesis on unfitted methods for Cahn-Hilliard.
  2. Optimal control problems for PDEs
     - 2.1 Side projects on biomembrane dynamics

Master student for one the CFD groups at IMF

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- Research aligned towards biophysics problems
- 1 Postdoc
- 2 PhD candidates
- 1 Master's student

Introduction

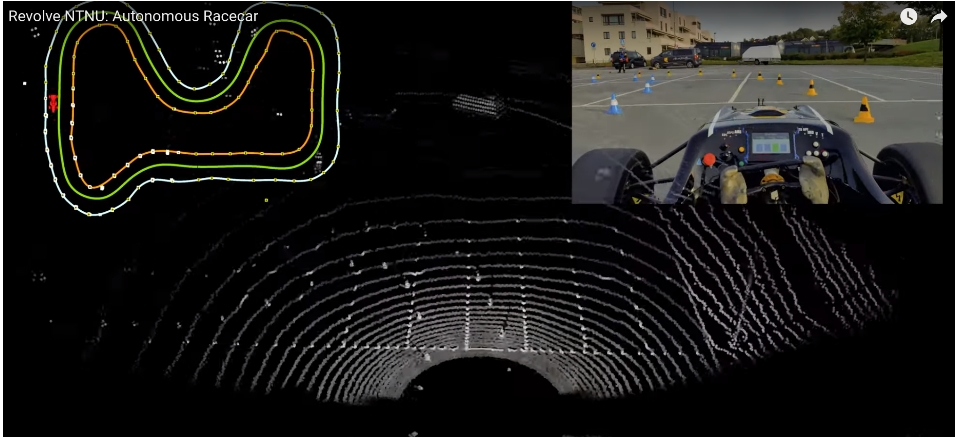
- Offshore apprentice for 2 years
- Worked with downhole electric instruments find reservoir properties
  1. Reservoir permeability and pressure
  2. Geological characteristics as rock dating, porosity etc.

Introduction

- Robotics Engineer for 2 years
- Revolve NTNU
- Autonomous Systems
  1. Vehicle Dynamics
  2. Nonlinear MPC and trajectory optimization
  3. Unknown path planning
  4. Linux, C++, and Python (Casadi, acados, ROS, Docker etc)

Introduction

Introduction



Live demonstration of the system: here

Contents

Plan for today

1. Crash course in basic FEM - 10 min
2. Introduction to CutFEM - 10 min

My goal is to give you a taste of modern finite element methods!

Finite Element Methods!

- To simplify will I only consider Poisson equation $\Delta u = f$
  - However, same ideas applies to Navier-Stokes and Cahn-Hilliard
- But to tell the story we need to do math
  - Notation, inverse inequalities, variational forms, domains etc
  - **I will skip a lot of details,** but still promote the main ideas.

We start with finite difference

- Then we gradually introduce flaws and problems!

Poisson Problem

Let us define the physical domain $\Omega \subset \mathbb{R}^2$ and the scalar functions $f : \Omega \to \mathbb{R}$ and $g : \Omega \to \mathbb{R}$.

Poisson Problem

We define the strong formulation of the Possion problem to find a scalar solution $u : \Omega \to \mathbb{R}$ s.t.

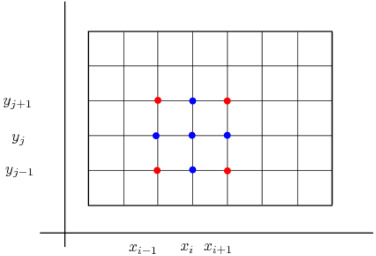
$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= g \quad \text{on } \Gamma \end{aligned}.$$

Finite difference approach

- Taylor expansion in y and x direction.

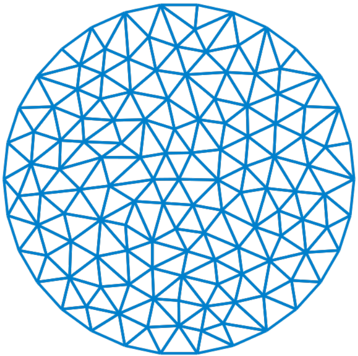
$$\Delta u = u_{xx} + u_{yy}$$
$$\approx \frac{u(x-h,y) - 2u(x,y) + u(x+h,y)}{h^2} + \frac{u(x,y-h) - 2u(x,y) + u(x,y+h)}{h^2}$$

- Works very well for perfectly square domains



Finite difference approach

- Can be very messy for unstructured mesh!



- It does exists methods for handling nodes on these domains.
  - The implementation can be very messy
  - Does not scale that well as far as I know.

Finite element method

Finite element approach for triangulations

1. We want to write the problem on a equivalent integral form.
2. Introduce so-called test functions space
3. Introduce a discrete polynomial space as an approximation space.

Finite element method

Let us introduce some notation

- Function spaces

$$L^2(\Omega) = \left\{ f : \Omega \mapsto \mathbb{R} \mid \int_\Omega |f|^2 \, d\Omega < \infty \right\} .$$
$$H^1(\Omega) = \left\{ u \in L^2\left(\Omega\right) \text{ and } \nabla u \in L^2\left(\Omega\right) \right\} .$$

- Essentially is $L^2(\Omega)$ the space of all integrable functions.
  - **Example:** $\int_{-1}^{1} (\frac{1}{x}) dx$ is not $L^2([-1, 1])$ integrable.
  - **Example:** $\int_{2}^{3} (\frac{1}{x}) dx$ is $L^2([2, 3])$ integrable.
- $H^1(\Omega)$ is the space off all functions where both the function and its first derivative is integrable.

Finite element method

Norms and inner products

- For $u, v \in L^2(\Omega)$ we define

$$\|u\|_\Omega = \|u\|_{L^2(\Omega)} = \left(\int_\Omega |u|^2\, dx\right)^{\frac{1}{2}}$$
$$(u, v)_\Omega = (u, v)_{L^2(\Omega)} = \int_\Omega uv dx.$$

- For $u, v \in H^1(\Omega)$ we define

$$\|u\|_{H^1(\Omega)} = \|u\|_\Omega + \|\nabla u\|_\Omega,$$
$$(u, v)_{H^1(\Omega)} = (u, v)_\Omega + (\nabla u, \nabla v)_\Omega.$$

Finite element method

Writing the Poission problem on an integral form

1. We extent the definitions for the boundary conditions s.t.
   - $H_0^1 = \{v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma\}$
   - $V_g = \{v \in H^1(\Omega) \mid u = g \text{ on } \Gamma\}$

   Notice that the boundary conditions is built in to the space!
2. Let $u \in V_g$. If we multiply with a so-called test function $v \in H_0^1(\Omega)$ and apply Greens theorem we get

$$-\int_\Omega \Delta uvdx = \int_\Omega \nabla u \nabla vdx - \int_\Gamma \partial_n uvdx = \int_\Omega \nabla u \nabla vdx$$

   We will use the following compact notation:

$$-(\Delta u, v)_\Omega = (\nabla u, \nabla v)_\Omega - (\partial_n u, v)_\Gamma = (\nabla u, \nabla v)_\Omega$$

Finite element method

Writing the Poisson problem on an integral form

Let us denote a bilinear form and a linear form

$$a(u,v) := (\nabla u, \nabla v)_\Omega \text{ and } \quad l(v) := (f,v)_\Omega$$

The weak formulation of the Poisson problem is to find a $u \in V_g$ s.t.

$$a(u,v) = l(v) \quad \forall v \in H_0^1(\Omega)$$

How can we discretize the following function spaces?

Abstract definition of a finite element

We define a element as the triple $(T, \mathcal{P}, \Sigma)$ where,

- T is an triangle
- $\mathcal{P}^k(T)$ is a finite polynomial basis $\{\phi\}_i^n$ of dimension k, also known as shape functions.
- Σ is the dual of $\mathcal{P}^k(T)$, that is, the set of linear forms $\{\sigma_i\}_i^n$ with the mapping,

  $$\sigma_i : \mathcal{P}^k(T) \to \mathbb{R} \quad \text{such that} \quad \sigma_i(\phi_j) = \delta_{ij}$$

  Also contains the DOFs or the coefficients in the polynom!

Example: For $k = 1$ the DOF in a triangle is one coefficient per mesh node, i.e., number of DOFs is $n = 3$.

Finite element method

The absolute key idea of the finite element methods is the following;

- **The goal is to approximate the (infinite dimensional) function space $H^1(\Omega)$ with a finite dimensional polynomial space $\mathcal{P}^k(\Omega) = span\{\phi_1, \ldots, \phi_N\}$ of order k**

Finite element method

To solve the Poisson problem using FEM we have the following discrete problem;

Discrete Poisson problem

We want to find an $u_h \in V_h := \mathcal{P}^k(\Omega)$ s.t.

$$a(u_h, v_h) = l(v_h) \quad \forall v_h \in V_h$$

Finite element method

Constructing the linear system

- Since $v_h, u_h \in V_h := \mathcal{P}^k(\Omega)$ can we write $u_h = \sum_{i=0}^{N} U_i \phi_i$ and $v_h = \sum_{i=0}^{N} U_i \phi_i$ with coefficients $\{U_i\}_{i=0}^{N}$ and $\{V_i\}_{i=0}^{N}$.
- Let us define a matrix $[\mathcal{A}]_{ji} = a(\phi_i, \phi_j)$ and $[F]_j = l(\phi_j)$.

Thus, we have

$$\sum_{i,j}^{N} V_j U_i \; a(\phi_j, \phi_i) = \sum_{j}^{N} V_j \; l(\phi_j).$$

Hence, we have the following equivalent linear system, that is

$$\mathcal{A} U = F.$$

Energy norm

We will now show requirements for uniqueness of numerical solution, but we need the so-called energy norm.

The energy norm is denoted by

$$\|v\|_a^2 = a(v, v)$$

Requirements for a well-posed problem

Theorem (The (discrete) Lax-Milgram Theorem)

Assume we have a general the bilinear form $a_h : V_h \times V_h \to \mathbb{R}$ *and a bounded linear form* $l_h : V_h \to \mathbb{R}$*. If there exists some constants* $C_1 > 0$ *and* $C_2 > 0$ *such that;*

1. *The bilinear form is bounded,*

$$|a_h(v, w)| \leq C_1 \|v\|_{a_h} \|w\|_{a_h} \quad \forall v, w \in V_h.$$

2. *The bilinear form is coercive (one-to-one),*

$$a_h(v, v) \geq C_2 \|v\|_{a_h}^2 \quad \forall v \in V_h.$$

then it exists a unique discrete solution $u \in V_h$ *s.t.*

$$a_h(u, v) = l_h(v) \quad \forall v \in V_h$$

Why is a well-posed problem so interesting?

Consequences of Lax Milgram

1. The solution is accurate and reliable.
2. The discrete problem converges when $h \to 0$
3. Does not exists infinite solutions
4. The problem is stable respect to small perturbation.

Conclusion

- Lax-Milgram very useful and rigour tool when developing new FEM schemes.
- Still hard to apply directly nonlinear problems, but we use it as a basis on simplified problems before we introduce nonlinear terms.
    1. Stokes equation vs Navier Stokes equation
    2. Biharmonic equation vs Chan Hilliard equation

Great we have a solution

What is the problem?

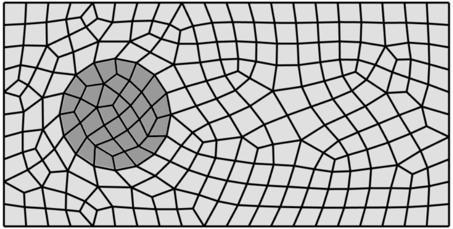
- It is suboptimal on moving domains $\Omega(t)$.
- And only works if Ω can be fully covered by by the mesh

$$\Omega = \bigcup_i T_i$$

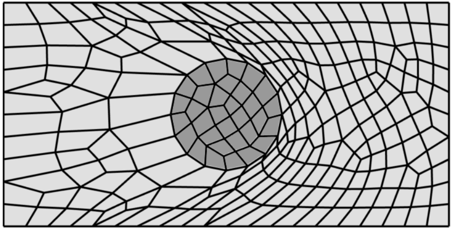
Thus, cannot handle smooth boundaries.

Moving Domains

- Potentially very costly re-meshing procedures.
- Ill-conditioned if mesh is too bad



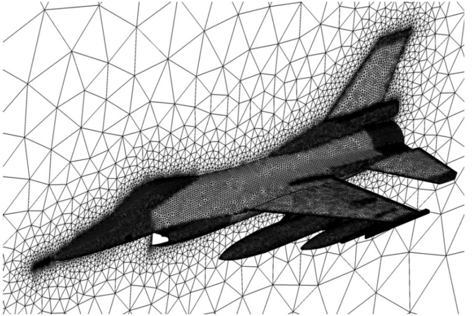
(a) Initial mesh.



(b) Transformed mesh.

Other problems of unstructured mesh

- Unstructured mesh is difficult to parallelize
- Cannot handle smooth boundaries. Some application may actually require smooth boundaries (shape optimization etc).



Ways to solve this problem

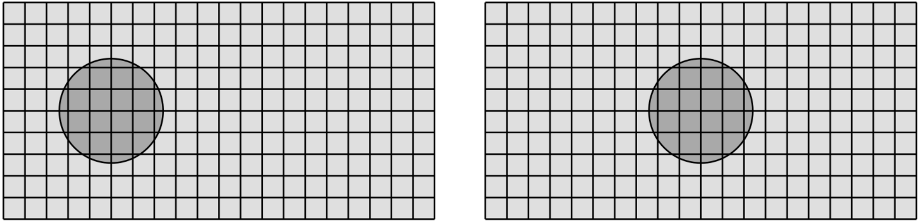
- Change the geometry.
  1. Time-dependent mesh elements on moving domains.
  2. Delete and add nodes when necessary.
  3. Full re-mesh generation.
  4. Probably many more clever methods . . .
- Utilize the geometry instead of modifying it
  1. Method that can handle smooth boundaries.
  2. Do some smart transformations

Question

How do you approach this problem?

Cut finite element method

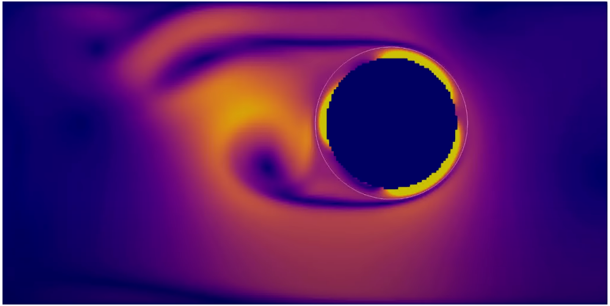
Method to solve PDE's with moving domains on a unfitted mesh!



- Here we considering an smooth boundary Γ in C^2
- No re-meshing on moving domains, only new configuration of cut cells.
- Potential to handle very complex geometries

Example of complex domains

Navier-Stokes on moving domains



Link to video

Computational Domains

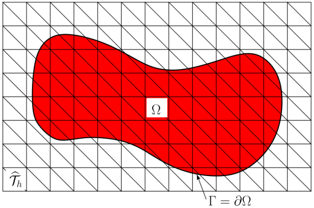


Figure: Physical domain

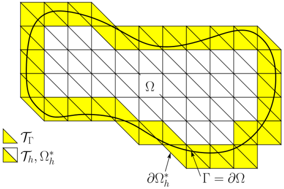
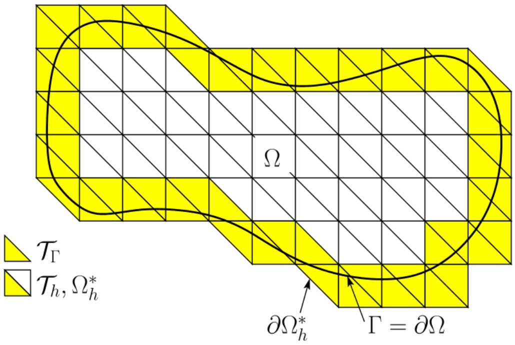


Figure: Cut cells

- We define a background mesh $\widetilde{\mathcal{T}}_h$
- An active submesh $\mathcal{T}_h \subset \widetilde{\mathcal{T}_h}$ containing physical domain Ω.
- Cut cells $\mathcal{T}_\Gamma \subset \mathcal{T}_h$ is the mesh elements that intersects with the boundary Γ.

Constructing the method



Observation

1. The interior is pretty nice to deal with.
2. The boundary must be parametrized somehow.
   - Level-set functions $\varphi(x) = 0$ is one way.
   - Splines is also possible
3. How do we deal with Dirichlet conditions?
4. How do we deal with elements with "bad" cuts?
5. Can we show that the problem is still well-posed?

Recall Poisson problem

Recall the formulation $(\nabla u, \nabla v)_\Omega - (\partial_n u, v)_\Gamma = (f, v)_\Omega$

Problem

- Dirichlet conditions is embedded in the function space, $V_g = \{v \in H^1(\Omega) \mid u = g \text{ on } \Gamma\}$
- But is difficult to handle when Γ is smooth.

Can we impose the Dirichlet conditions naturally?

Yes! We add a penalty on the boundary $\mu(u - g, v)_\Gamma$,

$$(\nabla u, \nabla v)_\Omega - (\partial_n u, v)_\Gamma + \mu(u, v)_\Gamma = (f, v)_\Omega + \mu(g, v)_\Gamma.$$

For symmetry we can also add $(u - g, \partial_n v)_\Gamma$,

$$(\nabla u, \nabla v)_\Omega - (\partial_n u, v)_\Gamma + (-u, \partial_n v)_\Gamma + \mu(u, v)_\Gamma = (f, v)_\Omega + \mu(g, v)_\Gamma + (-g, \partial_n v)_\Gamma.$$

Poisson formulation on a smooth boundary

Recall that $\mathcal{T}_h$ is the active mesh, that is, all trianges intersection with the interior of the domain Ω.

Definitions

Let $V_h := \mathcal{P}^k(\mathcal{T}_h) \cap C^0(\Omega)$.We denote the bilinear form $a_h : V_h \times V_h \to \mathbb{R}$ and the linear form $l_h : V_h \to \mathbb{R}$ to be,

$$
\begin{aligned}
a_h(u, v) &:= (\nabla u, \nabla v)_\Omega - (\partial_n u, v)_\Gamma - (u, \partial_n v)_\Gamma + \mu(u, v)_\Gamma \\
l_h(v) &:= (f, v)_\Omega + \mu(g, v)_\Gamma - (g, \partial_n v)_\Gamma
\end{aligned}
$$

Problem Statement

We want to find a $u \in V_h$ s.t. $a_h(u, v) = l_h(v) \quad \forall v \in V_h$.

Recall Lax Milgram

Theorem

$a_h(u,v) = l_h(v)$ *well-posed if both of these statements holds;*

- *The bilinear form is bounded,*

$$|a_h(v,w)| \leq C_1 \|v\|_{a_h} \|w\|_{a_h} \quad \forall v, w \in V_h.$$

- *The bilinear form is coercive (one-to-one),*

$$a_h(v,v) \geq C_2 \|v\|_{a_h}^2 \quad \forall v \in V_h.$$

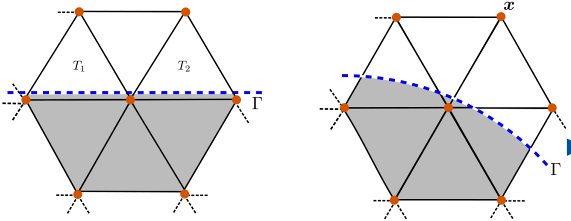
Is the new system well-posed?

- The Dirchlet conditions problem is in good shape for smooth domains!
- But from basic FEM theory it is now necessarry to apply

$$h^{\frac{1}{2}}\|\partial_n v\|_{\Gamma\cap T} \leq C\|v\|_{\Omega\cap T}$$

  to obtain well-posedness!
- But what about integration on very bad cut elements?
  - The relation between length $|F\cap\Omega|$ and volume $|T\cap\Omega|$ is very different on cut elements, thus, the norm is unbounded if the cut is bad.

The problem with the bad cuts



Observation

- Bad cuts makes it hard to justify length $F \cap \Omega$ vs area $|T \cap \Omega|$. Thus, the necessarry

$$h^{\frac{1}{2}} \|\partial_n v\|_{\Gamma \cap T} \leq C \|v\|_{\Omega \cap T}$$

become unbounded. Hence, the system is ill-conditoned.

- We are forced to extend the norm s.t.

$$h^{\frac{1}{2}} \|\partial_n v\|_{\Gamma \cap T} \leq C \|v\|_{T}$$

But we are integrating outside of our domain :(

Ghost penalty

The solution of the inverse estimate problem is simple! We add a **ghost penalty** $g_h(u, v)$ to handle the bad cuts as an regularization!

$$h^{\frac{1}{2}} \|\partial_n v\|_{\Gamma \cap T} \leq C\|v\|_T + g_h(u, v)$$

The goal is to regulate the ill-conditioned problem!

- We give it the necessarry assumptions for Lax-Milgram to hold!
- Same strategy is used to to obtain optimal convergence!
- We then engineer the g_h given the assumptions!

Hence, we end up with this stabilized problem formulation

Stabilized Poission Problem

Let $A_h(u, v) := a_h(u, v) + g_h(u, v)$. We want to find a $u \in V_h$ s.t.

$$A_h(u, v) = l_h(v) \quad \forall v \in V_h$$

The reality is more nasty

Proposition 4.2 (Discrete coercivity)
The discrete form A_h is coercive, that is,

$$A_h(v,v) \gtrsim \|v\|^2_{A_h} \quad \forall v \in V_h. \tag{4.17}$$

Proof. The proof of this statement follows the presentation in [6,16], but was first given for $\mathbb{P}_1$ elements in [9].

First take $v \in V_h$ and set $u = v$, and then using the Cauchy-Schwartz inequality in addition to the inequality $2ab \leqslant a^2\epsilon + b^2/\epsilon$ for real numbers a, b and ϵ. This yields,

$$\begin{aligned} A_h(v,v) &= (\nabla v, \nabla v)_{\mathcal{T}_h \cap \Omega} - (\partial_n v, v)_\Gamma - (v, \partial_n v)_\Gamma + \gamma/h(v,v)_\Gamma + g_h(v,v) \\ &= (\nabla v, \nabla v)_\Omega - 2(h^{1/2}\partial_n v, h^{-1/2}v)_\Gamma + \gamma/h(v,v)_\Gamma + g_h(v,v) \\ &\geqslant \|\nabla v\|^2_\Omega - 2\|h^{1/2}\partial_n v\|_\Gamma \|h^{-1/2}v\|_\Gamma + \gamma/h\|v\|^2_\Gamma + |v|^2_{g_h} \\ &\geqslant \|\nabla v\|^2_\Omega - \epsilon\|h^{1/2}\partial_n v\|^2_\Gamma - \frac{1}{\epsilon}\|h^{-1/2}v\|^2_\Gamma + \gamma\|h^{-1/2}v\|^2_\Gamma + |v|^2_{g_h}. \end{aligned} \tag{4.18}$$

Then using the inverse estimate from Proposition 4.1 enables bounding the flux over the boundary by the gradient over the whole active mesh. This yields

$$A_h(v,v) \geqslant \|\nabla v\|^2_\Omega - \epsilon C_\Gamma \|\nabla v\|^2_{\mathcal{T}_h} + (\gamma - \epsilon^{-1})\|h^{-1/2}v\|^2_\Gamma + |v|^2_{g_h}. \tag{4.19}$$

Further, we use the requirement put on g_h in Assumption 1 and collect the terms

$$\begin{aligned} A_h(v,v) &\geqslant \|\nabla v\|^2_\Omega - \epsilon C_\Gamma C_g \left(\|\nabla v\|^2_\Omega + |v|^2_{g_h}\right) + (\gamma - \epsilon^{-1})\|h^{-1/2}v\|^2_\Gamma + |v|^2_{g_h} \\ &= (1 - \epsilon C_\Gamma C_g)\|\nabla v\|^2_\Omega + (\gamma - \epsilon^{-1})\|h^{-1/2}v\|^2_\Gamma + (1 - \epsilon C_\Gamma C_g)|v|^2_{g_h}. \end{aligned} \tag{4.20}$$

If we let $\epsilon = 1/(C_\Gamma C_g)$ and $\gamma = 4C_\Gamma C_g$, we can finally assert that

$$A_h(v,v) \geqslant C\|v\|^2_{A_h}, \tag{4.21}$$

from some constant $C > 0$ for all $v \in V_h$. □

But I hope at least you have learned something new :)

Questions?