4.4 A priori error estimate

For the proposed method, we want to derive a priori error estimate with respect to both the $\|\cdot\|_{a_h,*}$-norm and the $\|\cdot\|_\Omega$-norm. These estimates are geometrically robust in that they remain unaffected by specific cut configurations, thanks to the ghost penalty they incorporate. First, we construct a suitable (quasi-)interpolation operator, here we use the Clement quasi interpolation operator which in contrast to the standard Lagrange nodal interpolation operator is also defined for low regularity function $u \in L^2(\Omega)$. In combination with discrete coercivity this allows us to derive an a priori error estimate in the energy norm. Finally, we use a standard duality argument, also known as Aubin-Nitsche trick, to derive the $L^2(\Omega)$-error estimate.

Recall that for $v \in H^3(\mathcal{T}_h)$ the following inequalities.

$$\|\nabla v\|_{\partial T} \lesssim h_T^{-\frac{1}{2}}\|\nabla v\|_T + h^{\frac{1}{2}}\|D^2 v\|_T, \tag{4.36}$$

$$\|\nabla v\|_{\Gamma\cap T} \lesssim h^{-\frac{1}{2}}\|\nabla v\|_T + h_T^{\frac{1}{2}}\|D^2 v\|_T, \tag{4.37}$$

$$\|D^2 v\|_{\Gamma\cap T} \lesssim h^{-\frac{1}{2}}\|D^2 v\|_T + h_T^{\frac{1}{2}}\|D^3 v\|_T, \tag{4.38}$$

holds $\forall T \in \mathcal{T}_h$, for proof see [93, Lemma 4.2]. In this context is $D^3 v$ a tensor of third partial derivates, $\left[D^3 v\right]_{ijk} = \frac{\partial^3 v}{\partial x_i \partial x_j \partial x_k} \forall i,j,k \in \{1,\ldots,d\}$ where the norm $\|D^3 v\|_\Omega^2 = \int_\Omega D^3 v : D^3 v \; dx$ is defined via the standard Frobenius inner product.

Assume that Ω has a boundary Γ in C^1, then there exists a bounded extension operator,

$$(\cdot)^e : H^m(\Omega) \to H^m(\mathbb{R}^d), \tag{4.39}$$

for all $v \in H^m(\Omega)$ which satisfies

$$\begin{aligned} v^e|_\Omega &= v, \\ \|v^e\|_{m,\mathbb{R}^d} &\lesssim \|v\|_{m,\Omega}. \end{aligned} \tag{4.40}$$

For more information, see [94, Theorem 9.7] and [95, p.181, p.185]. For the notation we simply write $v := v^e$ for $v \in \mathbb{R}^d \setminus \Omega$.

Starting from Lemma 2.8, assume $v \in H^s(\Omega)$ and let $r = \min(s, k+1)$. Revisit the definition of V_h from (4.2), which is a polynomial of degree k. We can then employ the combination of the Clément interpolator with the extension operator to create $C_h^e : H^m(\mathbb{R}^d) \to V_h$, such that $C_h^e v := C_h v^e$. Next, recall that $\sum_T \|v\|_{s,\omega(T)} \leqslant C\|v\|_{s,\mathcal{T}_h}$ where C is some constant decided by shape regularity of the mesh and the maximal number of different patches a single element can belong to. This also holds for the inequality $\sum_T \|v\|_{s,\omega(F)} \leqslant C\|v\|_{s,\mathcal{T}_h}$. The following estimates are thereby established.

$$\|v - C_h^e v\|_{l,\mathcal{T}_h} \lesssim h^{r-l} \sum_{T\in\mathcal{T}_h} \|v\|_{r,\omega(T)} \lesssim h^{r-l-\frac{1}{2}}\|v\|_{r,\Omega}, \quad 0 \leqslant l \leqslant r, \tag{4.41}$$

$$\|v - C_h^e v\|_{l,\mathcal{F}_h} \lesssim h^{r-l-\frac{1}{2}} \sum_{T\in\mathcal{T}_h} \|v\|_{r,\omega(F)} \lesssim h^{r-l-\frac{1}{2}}\|v\|_{r,\Omega}, \quad 0 \leqslant l \leqslant r - \frac{1}{2}. \tag{4.42}$$

$\hookrightarrow \partial\mathcal{T}_\omega$ $\quad$ $\omega(\partial T$

Lemma 4.4. *Let $u \in H^s(\Omega)$ for $s \geqslant 3$ be the exact solution to (3.4) and let k be the polynomial order of V_h. Set $r = \min(s, k+1)$, then we have the interpolation estimates*

$$\|u - C_h u\|_{a_h,*} \lesssim h^{r-2}\|u\|_{r,\Omega}. \tag{4.43}$$

$\|\sigma^+ - C_h^e \sigma^+\|_{1,\mathcal{T}_\omega}$

$$\|v_h\|_{T_1}^2 \leqslant C\|v_h\|_{T_2}^2 + g_{F_1}^{L^2}(v_h, v_h) \tag{4.83}$$
$$\leqslant C(C(\|v_h\|_{T_3}^2 + g_{F_2}^{L^2}(v_h, v_h)) + g_{F_1}^{L^2}(v_h, v_h)) \tag{4.84}$$
$$\lesssim \|v_h\|_{T_l}^2 + \sum_{i=1}^{l-1} g_{F_i}^{L^2}(v_h, v_h) \tag{4.85}$$
$$\lesssim \|v_h\|_{T_l \cap \Omega}^2 + \sum_{i=1}^{l-1} g_{F_i}^{L^2}(v_h, v_h) \tag{4.86}$$

Here the last steps arise from the fact that $\|v_h\|_{T_l} \lesssim \|v_h\|_{T_l \cap \Omega}$, which is a consequence of the fat intersection property. Summation over the cut elements $\mathcal{T}_h^\Gamma$ implies,

$$\|v_h\|_{\mathcal{T}_h^\Gamma}^2 \lesssim \|v_h\|_{\mathcal{T}_h^\Gamma \cap \Omega}^2 + \sum_{i=1}^{l-1} g_{F_l}^{L^2}(v_h, v_h) \tag{4.87}$$
$$= \|v_h\|_{\mathcal{T}_h^\Gamma \cap \Omega}^2 + \sum_{j=0}^{k} h^{2j+1}([\![\partial_n^j v_h]\!], [\![\partial_n^j v_h]\!])_{\mathcal{F}_h^g} \tag{4.88}$$

And as a trivial extension this now also holds for the active mesh $\mathcal{T}_h$, that is,

$$\|v_h\|_{\mathcal{T}_h}^2 \lesssim \|v_h\|_{\mathcal{T}_h \cap \Omega}^2 + \sum_{j=1}^{k} h^{2j+1}([\![\partial_n^j v_h]\!], [\![\partial_n^j v_h]\!])_{\mathcal{F}_h^g}. \tag{4.89}$$

Hence, (4.80a) holds and the first part of the proof is complete.

Estimate (4.80b) **and** (4.80b)**.** We will simply use the exact same procedure as **Estimate** (4.80a) for the estimates with (4.71a) and (4.71b). Hence, proof is complete. □

Finally, we now have the tools we need to construct an candidate for the ghost penalty for which satisfies all assumptions.

Proposition 4.11 (Face-based ghost penalty)**.** *Let $k \geqslant 2$ be the order of the polynomial basis in V_h . For any set of positive parameters $\{\gamma_j\}_{j=0}^k$, the ghost penalty defined as*

$$g_h(v_h, w_h) := \sum_{j=1}^{k} \sum_{F \in \mathcal{F}_h^g} \gamma_j h_F^{2j-3}([\![\partial_n^j v_h]\!], [\![\partial_n^j w_h]\!])_F \text{ for any } v_h, w_h \in V_h, \tag{4.90}$$

satisfies the Assumption ***EP1*** *and* ***EP2****.*

Proof. From Lemma 4.10 is it clear that $\|D^2 v_h\|_{\mathcal{T}_h} \lesssim \|D^2 v_h\|_\Omega + |v_h|_{g_h}$, hence, Assumption **EP1** holds. Therefore, we only need to verify Assumption **EP2**, which states that $|C_h^e v|_{g_h} \lesssim h^{r-2}\|v\|_{r,\Omega}$. Let $v \in H^s(\Omega)$, $s \geqslant 3$, and $r = \min\{s, k+1\}$. We can see that from definition is,

$$\begin{aligned} |C_h^e v|_{g_h}^2 &= \sum_{j=0}^{k} \gamma_j h^{2j-3} \|[\![\partial_n^j C_h^e v]\!]\|_{\mathcal{F}_h^g}^2 \\ &= \sum_{j=0}^{r-1} \gamma_j h^{2j-3} \|[\![\partial_n^j C_h^e v]\!]\|_{\mathcal{F}_h^g}^2 + \sum_{j=r}^{k} \gamma_j h^{2j-3} \|\, [\![\partial_n^j C_h^e v]\!]\, \|_{\mathcal{F}_h^g}^2 = \mathrm{I} + \mathrm{II} \end{aligned} \tag{4.91}$$

$\partial_n^j (C_h^e - v^e)$

Here we added a zero term $v^e \in H^s(\Omega_h)$ since jump vanishes for the for the first $r-1$ terms, i.e. $\left[\!\left[\partial_n^j v^e\right]\!\right] = 0 \; \forall s \leqslant r-1$.

1) **Version 1.**

$$\sum_{j=0}^{k} \gamma_j h^{2j-3} \| [[\partial_n^j C_h^e v]] \|^2_{\mathcal{F}_h^g} \lesssim \sum_{j=0}^{k} h^{2j-3} \|\partial_n^j C_h^e v\|^2_{\partial\mathcal{T}_h} \lesssim \sum_{j=0}^{k} h^{2j-3} h^{2(r-j)-1} \|C_h^e v\|^2_{r,\Omega} \lesssim h^{2(r-2)} \|C_h^e v\|^2_{r,\Omega} \tag{4.92}$$

2) **Version 2a of I.** What we can see is that the first term can easily be estimated using the a priori estimate (4.42) ,

$$\begin{aligned} \mathrm{I} &\lesssim \sum_{j=0}^{r-1} h^{2j-3} \| [[\partial_n^j (C_h^e v - v^e)]] \|^2_{\mathcal{F}_h^g} \\ &\lesssim \sum_{j=0}^{r-1} h^{2j-3} h^{2r-j-1} \|v\|^2_{r,\Omega} \\ &\lesssim \sum_{j=0}^{r-1} h^{2r+j-4} \|v\|^2_{r,\Omega} \lesssim h^{3r-5} \|v\|^2_{r,\Omega} \end{aligned} \tag{4.93}$$

Not sure if this estimate makes sense since we must extend to the full space $\partial\mathcal{T}_h$, see (2.9)

3) **Version 2b of I.** Using the trace inequality (4.36), i.e.

$$\|D^j v\|^2_{\partial T} \lesssim h^{-\frac{1}{2}} \|D^j v\|^2_T + h^{\frac{1}{2}} \|D^{j+1} v\|^2_T, \quad v \in H^{j+1}(T) \tag{4.94}$$

such that

$$\begin{aligned} \mathrm{I} &\lesssim \sum_{j=0}^{r-1} h^{2j-3} \|D^j (C_h^e v - v^e)\|^2_{\partial\mathcal{T}_h} \\ &\lesssim \sum_{j=0}^{r-1} h^{2j-3} (h^{-1} \|D^j (C_h^e v - v^e)\|^2_{\mathcal{T}_h} + h \|D^{j+1} (C_h^e v - v^e)\|^2_{\mathcal{T}_h}) \\ &\lesssim \sum_{j=0}^{r-1} h^{2j-3} (h^{2(r-j)-2} + h^{2(r-j+1)}) \|v\|^2_{r,\Omega} \\ &\lesssim \sum_{j=0}^{r-1} (h^{2r-5} + h^{2r-1}) \|v\|^2_{r,\Omega} \end{aligned} \tag{4.95}$$

Here we applied (4.41).

For the second term are we allowed to use the basic inverse estimate (2.26) on the Clément operator,

$$\begin{aligned} \mathrm{II} &\lesssim \sum_{j=r}^{k} h^{2j-3} \|\partial_n^j C_h^e v\|^2_{\partial\mathcal{T}_h} \lesssim \sum_{j=r}^{k} h^{2j-3} h^{2(r-j)-1} \|C_h^e v\|^2_{r,\Omega} \\ &\lesssim h^{2(r-2)} \|C_h^e v\|^2_{r,\Omega} \end{aligned} \tag{4.96}$$

Hence, the estimate $|C_h^e v|^2_{g_h} \lesssim h^{2(r-2)} \|v\|^2_{r,\Omega}$ holds. $\square$

correct

5 Numerical results

To validate the proposed CutCIP method for the biharmonic problem will we investigate several numerical results both for the Laplace and Hessian formulation. From the theoretical optimal a priori estimates presented in presented in Theorem 4.6 will we provide examples where it holds. We also demonstrate the effect ghost penalty of stabilization by translating the domain to trigger badly cut cells. Finally, we provide numerical validation of the expected convergence for the Cahn-Hilliard problem. We propose the following penalty parameters $\gamma = 20$ for the Hessian formulation (3.15) and the Laplace formulation (3.18), and $\gamma_1, \gamma_2 = 10, 0.5$ for the corresponding ghost penalty (4.90).

Condition numbers are essential to solve linear systems because they help us assess the accuracy and stability of the system's solutions. A large condition number indicates that the system is ill-conditioned, meaning the solution can be highly sensitive to small changes in the input data, potentially leading to inaccurate results. This underline the importance checking the conditional stability of cut cells, hence, motivating to do a so-called translation test with and without ghost penalty.

5.1 Convergence study for the cut finite element method method for the biharmonic problem

For the convergence study will we consider a square background domain $\widetilde{\Omega}$ with side lengths L and a physical domain $\Omega \subset \widetilde{\Omega}$ on the form $\Omega = \{(x,y) \mid \phi(x,y) \leqslant 0\}$, where $\phi : \mathbb{R}^2 \to \mathbb{R}$ is a given level set function. We will consider two cases; a circular domain,

$$\phi(x,y) = x^2 + y^2 - 1 \tag{5.1}$$

and a flower shaped domain,

$$\phi(x,y) = \sqrt{x^2 + y^2} - r_0 - r_1 \cos(\operatorname{atan2}(y,x)) \text{ where } r_0 = 0.3L \text{ and } \; r_1 = 0.1L. \tag{5.2}$$

For an illustration of the flower domain, see Figure 16.

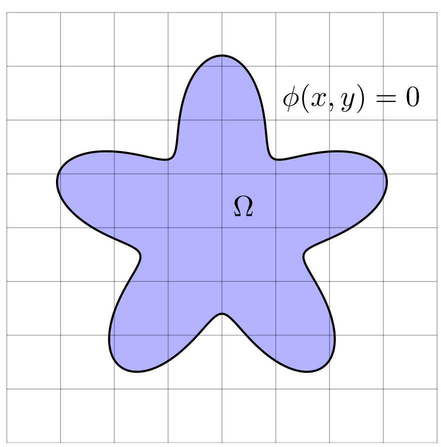

Figure 16: Illustration of the flower domain associated with the level set function (5.2).

We want to test spatial convergence for the method by doing a numerous of mesh refinements. Let $\widetilde{\mathcal{T}}_h^j$ be the associated regular square mesh of the background domain $\widetilde{\Omega}$ with the mesh size $h^j = L/2^{3+j}$ for the side length $L = 2.7$ and refinements $j = 1, \ldots, 8$. For the circular domain is this illustrated in the Figure 17.

On each mesh $\widetilde{\mathcal{T}}_h^k$ we compute a numerical solution u_h^k, hence, motivating us to define the convergence rate. Let u be the exact solution, then do we define the so-called Experimental Order