For convenience, will we define also the interior of the active set as $\mathcal{T}_{int}$.

To be consistent $\mathcal{T}_h^{int}$ with notation $\mathcal{F}_h^{int}$.

$$\boxed{\mathcal{T}_{int}} = \{T \in \mathcal{T}_h \mid T \cap \mathrm{Int}(\Omega) \neq \emptyset\}\,.$$

Hence, we have that the active set is the union of the interior and cut elements, $\mathcal{T}_h = \mathcal{T}_{int} \cup \mathcal{T}_\Gamma$. For an illustration, see Figure 12.

$\mathcal{T}_h^{int} \cup \mathcal{T}_h^{\Gamma}$

4.2 Cut continuous interior penalty methods

As Ω is static is it easy to observe that having a polynomial basis on the full mesh, $\widetilde{\mathcal{T}}_h$, is not necessary. Hence, we define the polynomial space ~~using~~ only on the active set $\mathcal{T}_h$ from (4.1),

$$V_h = \left\{ v \in C^0\left(\mathcal{T}_h\right) : v_T = v|_T \in \mathcal{P}^k\left(T\right), \forall T \in \mathcal{T}_h \right\}.$$

not defined set. Rather write $C(\Omega_h)$ and define $\Omega_h = \bigcup_{T \in \mathcal{T}_h} T$.

Function space on meshes ($C^0(\mathcal{T}_h)$) are often understood as broken spaces, only defined element-wise.

Furthermore, drawing on the principles outlined in Section 3, we can indeed recall two CIP formulations for the biharmonic equation: the Hessian formulation (3.15) and the Laplace formulation (3.18). that will be required later ...

To make sure the problem is stabilized will we add a hypothetical consistent symmetric positive semi-definite bilinear ghost-penalty term $g_h : V_h \times V_h \to \mathbb{R}$ ~~in addition~~ to our bilinear. That is, we define the discrete problem to ~~s.t.~~ be: Remove all such abbreviations!

$$\text{Find } u_h \in V_h \text{ such that } \quad A_h(u_h, v) := a_h(u_h, v) + g_h(u_h, v) = l_h(v) \quad \forall v \in V_h. \tag{4.2}$$

Here ~~is~~ stands for $a_h(\cdot,\cdot)$ ~~denoted as~~ either $a_h^L(\cdot,\cdot)$ or $a_h^H(\cdot,\cdot)$.

don't do "do" → "..section, we provide.."

~~We will~~ in this section do a full proof for the Hessian formulation, however, the proofs ~~should~~ does not differ to much ~~for the Laplace formulation~~. For simplification will we use the notation $a_h(u,v) = a_h^H(u,v)$ and $l_h(v) = l_h^H(v)$ for the rest of the stability and convergence analysis. "for the Laplace formulation"

Keep in mind that in contrast to the standard CIP methods, ~~this~~ "our proposed" method is defined on an unfitted mesh. As we will see in the analysis, the ghost penalty is a method to ensure numerical stability on cut cell $\mathcal{T}_\Gamma$. The main reason why this numerical instability is happening for a unfitted mesh is that when a cell is badly cut, see examples in Figure 14. In other words, when a cell is "badly cut," it means that it is intersected by the boundary Γ in such a way that only a very small part of the interior of an element T intersects with the physical domain Ω , i.e. $|\Omega \cap T|_d \ll h^d$. This can lead to "both stability issues and" a very poor condition number of the ~~local~~ system matrix ~~corresponding to such a cell, causing numerical instability~~.

The ghost penalty stabilization technique is designed to tackle this issue. Essentially, this approach introduces additional terms into the finite element method that penalize jumps in the discrete solution and its gradients across cell interfaces, typically in the cut cells. This penalty not only improves the conditioning of the system matrix but also enhances the robustness of the method with respect to the location of the boundary inside each cell. However, to make this possible, we assume a so-called fat-intersection property, which will be relevant in Section 4.5.

Our first assumption as as follows; for a $T \in \mathcal{T}_\Gamma$ there always exists a patch $\omega(T)$ which contains T and an element T' with a so-called fat intersection $|T' \cap \Omega|_d \gtrsim |T'|_d$, where $|\cdot|_d$ is the measure of an element of dimensions $d = 2, 3$. For an illustration, see Figure 13.

We define the underlying norms for $v \in V_h$ as

$$\|v\|_{a_h}^2 = \alpha \|v\|_{\mathcal{T}_h \cap \Omega}^2 + \|D^2 v\|_{\mathcal{T}_h \cap \Omega}^2 + \|h^{-\frac{1}{2}} [\![\partial_n v]\!] \|_{\mathcal{F}_h \cap \Omega}^2 + \|h^{-\frac{1}{2}} \partial_n v\|_\Gamma^2, \tag{4.3}$$

$$|v|_{g_h}^2 = g(v, v), \tag{4.4}$$

$$\|v\|_{A_h}^2 = \|v\|_{a_h}^2 + |v|_{g_h}^2, \tag{4.5}$$

$$\tag{4.6}$$

and for $v \in V \oplus V_h$ we get, also introduce

$$\|v\|_{a_h,*}^2 = \|v\|_{a_h}^2 + \|h^{\frac{1}{2}} \{\!\{\partial_{nn} v\}\!\} \|_{\mathcal{F}_h \cap \Omega}^2 + \|h^{\frac{1}{2}} \partial_{nn} v\|_\Gamma^2 \,. \tag{4.7}$$

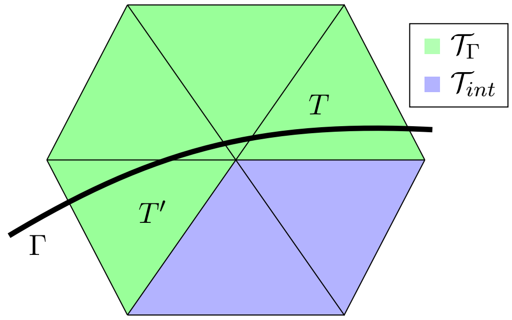

Figure 13: Illustration of the fat intersection property. Let $T \in \mathcal{T}_\Gamma$. It shows a patch $\omega(T)$ that contains elements $T \in \mathcal{T}_\Gamma$ and $T' \in \mathcal{T}_{int} \cup \mathcal{T}_\Gamma = \mathcal{T}_h$, with T' having a sufficiently large intersection with Ω.

Remark. Note that it holds that $\mathcal{T}_h \cap \Omega = \Omega$ and $\mathcal{T}_h \cap \Gamma = \Gamma$. Depending on context, we choose the best suitable notation.

Remark. The necessity to define the supplementary terms in the $\|\cdot\|_{ah,*}$ may raise certain questions. The reason is because when v is continious, i.e. $v \in V$, the local inverse estimates 2.15 does not hold for $\| \{\!\{\partial_{nn} v\}\!\} \|_{\mathcal{F}_h \cap \Omega}$ and $\|\partial_{nn} v\|_\Gamma$ when evaluating $a_h(v, v)$. Hence, this leads necessity adding the additional terms into the norm. when we later need to bound $a_h(u - C_{gh}, v_h)$ as part of the a priori error estimate derivation.

4.3 Stability estimate

In section

~~Recall the Subsection~~ 2.7 ~~where~~ we discussed standard local inverse estimates, which plays a crucial role in the theoretical analysis of the classical CIP method for biharmonic problems. Similarly for cut elements is it easy to see that this must hold,

$$\|\partial_{nn} v_h\|_{F \cap \Omega} \lesssim \|\partial_{nn} v_h\|_F \lesssim \|h_T^{-\frac{1}{2}} D^2 v_h\|_T. \tag{4.8}$$

A useful variant is the following inequality ~~that is,~~

"For the proposed unfitted version",

$$\|\partial_{nn} v_h\|_{\Gamma \cap T} \lesssim h^{-\frac{1}{2}} \|D^2 v_h\|_T. \tag{4.9}$$

This is essential motivation so don't put it into a remark.

~~*Remark.*~~ It may be natural to instead look at $\|\partial_{nn} v_h\|_{\Gamma \cap T} \lesssim h^{-\frac{1}{2}} \|D^2 v_h\|_{T \cap \Omega}$, this however, this cannot hold for ~~a~~ an arbitary cut configutration for an unfitted mesh. To demonstrate, let $\varepsilon \ll 1$ be a small length. For the examples provided in Figure 14 we have two cases: i) $|\Gamma \cap \Omega|_{d-1} \lesssim \varepsilon h^{d-1}$ and $|T \cap \Omega|_{d-1} \lesssim \varepsilon h^{d-1}$, and ii) $|\Gamma \cap \Omega|_{d-1} \lesssim h^{d-1}$ and $|T \cap \Omega|_{d-1} \lesssim \varepsilon h^{d-1}$. The first first case impacts the condition number since it is introducing almost vanishing entries in the system matrix from (2.11). ~~While for~~ the second case is bad for inverse estimates, and, thus, problematic for proving "discard" coercivity. To recover, one in fact, on the other hand, must incorporate the full element T into the inverse estimate "as done in (4.8) and (4.9)."

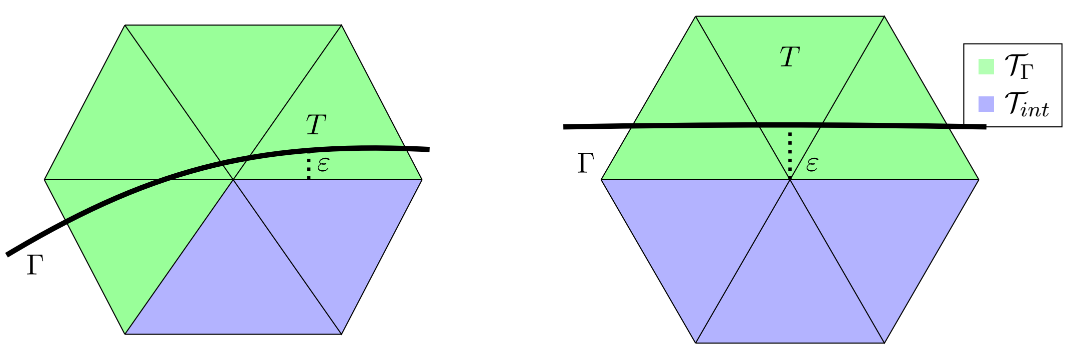

Figure 14: Illustration of two examples of bad cut cells with an arbitrary small length $\varepsilon \ll 1$. Let $T \in \mathcal{T}_\Gamma$ be a cut cell. On the left example, is it clear that $|\Gamma \cap T| \lesssim h^{d-1}$ and $|\Omega \cap T| \lesssim \varepsilon h^d$. However, on the right example is it clear that $|\Gamma \cap T| \lesssim \varepsilon h^{d-1}$ and $|\Omega \cap T| \lesssim \varepsilon h^d$.

" work with norms which are defined on "

Since the inequalities above holds for all elements locally is it natural to ~~extend~~ it ~~to~~ the full mesh $\mathcal{T}_h$. T~~his implies that~~

$$\|\partial_{nn} v_h\|_{\mathcal{T}_h \cap \Gamma} \lesssim h^{-\frac{1}{2}} \|D^2 v_h\|_{\mathcal{T}_h}, \tag{4.10}$$

$$\|\partial_{nn} v_h\|_{\mathcal{F}_h \cap \Omega} \lesssim h^{-\frac{1}{2}} \|D^2 v_h\|_{\mathcal{T}_h}. \tag{4.11}$$

We aware that these inequalities also holds for the first order, that is.

$$\|\partial_{n} v_h\|_{\mathcal{T}_h \cap \Gamma} \lesssim h^{-\frac{1}{2}} \|\nabla v\|_{\mathcal{T}_h}, \tag{4.12}$$

$$\|\partial_{n} v_h\|_{\mathcal{F}_h \cap \Omega} \lesssim h^{-\frac{1}{2}} \|\nabla v_h\|_{\mathcal{T}_h}. \tag{4.13}$$

In fact, combining the second order inequalities we get the following identity.

$$h\|\partial_{nn} v_h\|^2_{\mathcal{F}_h \cap \Omega} + h\|\partial_{nn} v_h\|^2_{\mathcal{T}_h \cap \Gamma} \lesssim \|D^2 v_h\|^2_{\mathcal{T}_h} \quad \forall v_h \in V_h. \tag{4.14}$$

For more information about the derivations of the inequalities, see discussion in [81, Section 2.4].

We may introduce our first assumption on the ghost penalty. Inspired by the expansion for the H^1-norm as detailed in [81, Equation 2.23], we adopt an analogous approach for the H^2-norm in this scenario.

Assumption (EP1)**.** *The ghost penalty g_h extends the H^2 norm such that*

$$\|D^2 v\|^2_{\mathcal{T}_h} \lesssim \|D^2 v\|^2_{\Omega} + |v|^2_{g_h}.$$

Combing the results we get the following convenient corollary.

Corollary 4.1. *Let g_h satisfy Assumption 4.3, then*

$$h\|\partial_{nn} v_h\|^2_{\mathcal{F}_h \cap \Omega} + h\|\partial_{nn} v_h\|^2_{\mathcal{T}_h \cap \Gamma} \lesssim \|D^2 v_h\|^2_{\Omega} + |v_h|^2_{g_h} \lesssim \|v_h\|^2_{A_h} \quad \forall v_h \in V_h \tag{4.15}$$

From this is it also clear that

$$\|v_h\|_{a_h,*} \lesssim \|v_h\|_{A_h} \quad \forall v_h \in V_h \tag{4.16}$$

Proof. The the first result (4.15) is a direct result of (4.14), Assumption 4.3 and the definition of $\|\cdot\|_{A_h}$. The second result (4.16) is simply a observation that the terms in (4.15) appears in $\|\cdot\|_{a_h,*}$, hence, the inequality follows. □

Now add a remark where you state the alternative versions of EP1 $\left(\|\Delta^2 v\|_{\mathcal{T}_h} \lesssim \|\Delta^2 v\|^2 + |v|^2_{g_h} \right)$ for the laplace formulation (4.15).

Lemma 4.2. *The discrete form A_h is coercive, that is,*

$$\|v_h\|^2_{A_h} \lesssim A_h(v_h, v_h) \quad \forall v_h \in V_h.$$

Proof. Let $v_h \in V_h$. Observe that

$$A_h(v_h, v_h) = a_h(v_h, v_h) + |v_h|^2_{g_h}. \tag{4.17}$$

Mind punctuation in all displayed equation

Firstly, the ghost penalty term is already a part of the $\|\cdot\|_{A_h}$ norm, hence, it only remains to ~~check~~ "bound" the $a_h(\cdot,\cdot)$ term. "properly from below".

$$\begin{aligned} a_h(v_h, v_h) = {} & \alpha\|v_h\|^2_{\Omega} + \|D^2 v_h\|^2_{\Omega} + \frac{\gamma}{h}\| [\![\partial_n v_h]\!] \|^2_{\mathcal{F}_h} + \frac{\gamma}{h}\|\partial_n v_h\|^2_{\Gamma} \\ & \boxed{+} 2(\{\!\{\partial_{nn} v_h\}\!\}, [\![\partial_n v_h]\!])_{\mathcal{F}_h \cap \Omega} + 2(\partial_{nn} v, \partial_n v_h)_{\Gamma} \end{aligned} \tag{4.18}$$

always add a little indent in splitted equations (" & \quad + 2 ... ")

We first focus on the last two terms in (4.18). Using Cauchy-Schwarz (2.4), we observe that

$$\begin{aligned} (\{\!\{\partial_{nn} v_h\}\!\}, [\![\partial_n v_h]\!])_{\mathcal{F}_h \cap \Omega} &\geq -\|h^{\frac{1}{2}} \{\!\{\partial_{nn} v_h\}\!\}\|_{\mathcal{F}_h \cap \Omega} \|h^{-\frac{1}{2}} [\![\partial_n v_h]\!]\|_{\mathcal{F}_h \cap \Omega} \\ (\partial_{nn} v_h, \partial_n v_h)_{\Gamma} &\geq -\|h^{\frac{1}{2}} \partial_{nn} v_h\|_{\Gamma} \|h^{-\frac{1}{2}} \partial_n v_h\|_{\Gamma} \end{aligned} \tag{4.19}$$

\geqslant (everywhere)

wrong word order, English grammar works quite differently than Norwegian, verb is never in 1. position (Except in questions combined with auxiliary verbs like "can")

Using inverse-inequalities (4.10) and (4.11) and the Corollary 4.1 can we easily observe that
we can

$$
\begin{aligned}
\|h^{\frac{1}{2}}\,\{\partial_{nn}v_h\}\,\|^2_{\mathcal{F}_h\cap\Omega} &\leqslant C_1\|D^2v_h\|^2_{\mathcal{T}_h} \lesssim \|D^2v_h\|^2_{\Omega} + |v_h|^2_{g_h} \\
\|\partial_{nn}v_h\|^2_{\Gamma} &\leqslant C_2\|D^2v_h\|^2_{\mathcal{T}_h} \lesssim \|D^2v_h\|^2_{\Omega} + |v_h|^2_{g_h}
\end{aligned}
\tag{4.20}
$$

Thus, by applying Youngs ε-inequality (2.5), is it natural to see that
Young's it is

$$
\begin{aligned}
-C_1^{\frac{1}{2}}\|D^2v_h\|_{\mathcal{T}_h}\|h^{-\frac{1}{2}}\,[\![\partial_n v_h]\!]\,\|_{\mathcal{F}_h\cap\Omega} &\geq -\frac{1}{\varepsilon}C(\|D^2v_h\|^2_{\Omega} + |v_h|^2_{g_h}) - \varepsilon\|h^{-\frac{1}{2}}\,[\![\partial_n v_h]\!]\,\|^2_{\mathcal{F}_h\cap\Omega} \\
-C_2^{\frac{1}{2}}\|D^2v_h\|_{\mathcal{T}_h}\|h^{-\frac{1}{2}}\partial_n v_h\|_{\Gamma} &\geq -\frac{1}{\varepsilon}C(\|D^2v_h\|^2_{\Omega} + |v_h|^2_{g_h}) - \varepsilon\|h^{-\frac{1}{2}}\partial_n v_h\|^2_{\Gamma}
\end{aligned}
\tag{4.21}
$$

Combining these ideas, ~~so~~ we end up with the following inequality,

too much space

$$
\begin{aligned}
a_h(v_h,v_h) \geq{}& \alpha\|\,v\|^2_{\Omega} + \|D^2v_h\|^2_{\Omega} - \frac{1}{\varepsilon}4C(\|D^2v_h\|^2_{\Omega} + |v_h|^2_{g_h}) \\
&+ (\gamma - 2\varepsilon)\left(\|h^{-\frac{1}{2}}\,[\![\partial_n v_h]\!]\,\|^2_{\mathcal{F}_h\cap\Omega} + \|h^{-\frac{1}{2}}\partial_n v_h\|^2_{\Gamma}\right).
\end{aligned}
\tag{4.22}
$$

by adding

This inequality is useful since ~~if we add~~ a ghost penalty on the left hand side we get,

$$
\begin{aligned}
A_h(v_h,v_h) &= a(v_h,v_h) + |v_h|^2_{g_h} \\
&\gtrsim\ \|\,|\alpha|^{\frac{1}{2}}\ v_h\|^2_{\Omega} + (1 - \frac{1}{\varepsilon}4C)(\|D^2v_h\|^2_{\Omega} + |v_h|^2_{g_h}) \\
&+ (\gamma - 2\varepsilon)\left(\|h^{-\frac{1}{2}}\,[\![\partial_n v_h]\!]\,\|^2_{\mathcal{F}_h\cap\Omega} + \|h^{-\frac{1}{2}}\partial_n v_h\|^2_{\Gamma}\right).
\end{aligned}
\tag{4.23}
$$

Setting $\varepsilon = 8C$ and $\gamma = 32C$ we arrive at the desired ~~state~~, inequality

$$
\begin{aligned}
A_h(v_h,v_h) &\gtrsim \|\,|\alpha|^{\frac{1}{2}}\ v_h\|^2_{\Omega} + \frac{1}{2}(\|D^2v_h\|^2_{\Omega} + |v_h|^2_{g_h}) \\
&\quad + \frac{\gamma}{2}\left(\|h^{-\frac{1}{2}}\,[\![\partial_n v_h]\!]\,\|^2_{\mathcal{F}_h\cap\Omega} + \|h^{-\frac{1}{2}}\partial_n v_h\|^2_{\Gamma}\right)\cdot \\
&\gtrsim \|v_h\|^2_{A_h}.
\end{aligned}
\tag{4.24}
$$

□

bilinear

Lemma 4.3. *The discrete form A_h is bounded, ~~that is~~*

$$
A_h(v_h,w_h) \lesssim \|v_h\|_{A_h}\|w_h\|_{A_h} \quad \forall v_h, w_h \in V_h \tag{4.25}
$$

Moreover, for $v \in V_h \oplus V$ and $w_h \in V_h$ the discrete bilinear form $a_h(\cdot,\cdot)$ satisfies,

no comma (just read inequality as part of the sentence.)

$$
a_h(v,w_h) \lesssim \|v\|_{a_h,*}\|w_h\|_{A_h}. \tag{4.26}
$$

You can condense these even more: just write Estimate (4.25) in boldface.

Proof. **Step 1.** The goal is to prove the inequality (4.25).

$$
|A_h(v_h,w_h)| \lesssim |a_h(v_h,w_h)| + |g_h(v_h,w_h)| \tag{4.27}
$$

By assumption is the ghost penalty g_h positive semi-definite, thus, it fulfills the Cauchy-Schwarz inequality ~~(2.4)~~

avoid collision of math environments

$$
|g_h(v_h,w_h)| \lesssim |v_h|_{g_h}\,|w_h|_{g_h}\,. \tag{4.28}
$$

Hence, by definition of $A_h(\cdot,\cdot)$, $|g_h(v_h,w_h)| \lesssim \|v_h\|_{A_h}\|w_h\|_{A_h}$. It remains to show that the bilinear term $a_h(\cdot,\cdot)$ is bounded. We numerate the terms in this fashion.

$$
\begin{aligned}
a_h\,(v_h,w_h) \leqslant{}& \overbrace{(\alpha v_h,w_h)_{\mathcal{T}_h\cap\Omega}}^{\mathrm{I}} + \overbrace{\left(D^2v_h,D^2w_h\right)_{\mathcal{T}_h\cap\Omega}}^{\mathrm{II}} \\
&+ \overbrace{(\{\!\{\partial_{nn}v_h\}\!\}\,,[\![\partial_n w_h]\!])_{\mathcal{F}_h\cap\Omega}}^{\mathrm{III}} + \overbrace{(\{\!\{\partial_{nn}w_h\}\!\}\,,[\![\partial_n v_h]\!])_{\mathcal{F}_h\cap\Omega}}^{\mathrm{IV}} + \overbrace{\frac{\gamma}{h}\,([\![\partial_n v_h]\!]\,,[\![\partial_n w_h]\!])_{\mathcal{F}_h\cap\Omega}}^{\mathrm{V}} \\
&+ \overbrace{(\partial_{nn}v_h,\partial_n w_h)_{\Gamma}}^{\mathrm{VI}} + \overbrace{(\partial_{nn}w_h,\partial_n v_h)_{\Gamma}}^{\mathrm{VII}} + \overbrace{\frac{\gamma}{h}\,(\partial_n v_h,\partial_n w_h)_{\Gamma}}^{\mathrm{VIII}} \\
={}& (\mathrm{I}) + \ldots + (\mathrm{VIII}).
\end{aligned}
\tag{4.29}
$$

parenthesis are superfluous, they don't add clarity just write I + ... VIII.

To "the" "inequality" → only IV is a symmetry term. Just avoid symmetry here.

The strategy is to bound each term individually using Cauchy-Schwarz (2.4). From this is it easy to see that $|(\mathrm{I})| + |(\mathrm{II})| \lesssim \|v_h\|_{a_h} \|w_h\|_{a_h}$. ~~For~~ the terms ~~symmetrical terms~~ (III) and (IV) we apply ~~the inverse inequality (4.11)~~ "inequality (4.15) from Corollary 4.1 to see that"

$$|(\mathrm{III})| \lesssim \|h^{\frac{1}{2}} \partial_{nn} v_h\|_{\mathcal{F}_h \cap \Omega} \|h^{-\frac{1}{2}} [\![\partial_n w_h]\!] \|_{\mathcal{F}_h \cap \Omega} \lesssim \|v_h\|_{A_h} \|w\|_{a_h}. \tag{4.30}$$

~~Here we used that $\|h^{\frac{1}{2}} \partial_{nn} v_h\|_{\mathcal{F}_h \cap \Omega} \lesssim \|v_h\|_{A_h}$ thanks to Corollary 4.1~~. The interior penalty terms ~~can we easily see that~~, can be bounded by

$$|(\mathrm{V})| \lesssim \|h^{-\frac{1}{2}} [\![\partial_n v_h]\!] \|_{\mathcal{F}_h \cap \Omega} \|h^{-\frac{1}{2}} [\![\partial_n w_h]\!] \|_{\mathcal{F}_h \cap \Omega} \lesssim \|v_h\|_{a_h} \|w_h\|_{a_h}. \tag{4.31}$$

"The" ~~It~~ remain"ing" ~~to handle the symmetry~~ terms (VI) and (VII) can again be handled using Corollary 4.1, leading to"

$$|(\mathrm{VI})| \lesssim \|h^{\frac{1}{2}} \partial_{nn} v_h\|_{\Gamma} \|h^{-\frac{1}{2}} \partial_n w_h\|_{\Gamma} \lesssim \|v_h\|_{A_h} \|w_h\|_{a_h} \tag{4.32}$$

~~Again, here we used the Corollary 4.1~~. Finally, using the definition of the norm is it easily to see that

$$|(\mathrm{VIII})| \lesssim \|\partial_n v_h\|_{\Gamma} \|\partial_n w_h\|_{\Gamma} \lesssim \|v_h\|_{a_h} \|w_h\|_{a_h}.$$

Hence, we can conclude

$$|a_h(v_h, w_h)| \leqslant \|v_h\|_{a_h} \|w_h\|_{a_h} \forall v_h, w_h \in V_h. \tag{4.33}$$

Therefore, since $\|\cdot\|_{a_h} \lesssim \|\cdot\|_{A_h}$, it has been demonstrated that $a_h(\cdot,\cdot)$ is bounded within the $\|\cdot\|_{Ah}$ norm.

In boldface: Estimate (4.26)

~~**Step 2.** The goal is to prove (4.26)~~. Let $v \in V_h \oplus V$ and $w_h \in V_h$. The only difference is that since v is ~~continious~~ we cannot apply to Corollary 4.1 on the estimates (4.30) and (4.32). However, this is not a problem since $\|h^{\frac{1}{2}} \{\!\{\partial_{nn} v\}\!\} \|_{\mathcal{F}_h \cap \Omega}$ and $\|h^{\frac{1}{2}} \partial_{nn} v\|_{\Gamma}$ are terms in the norm $\|v\|_{a_h,*}$. Thus, we know that

$$|a_h(v, w_h)| \leqslant \|v\|_{a_h,*} \|w_h\|_{A_h} \quad \forall v \in V_h \oplus V \text{ and } w_h \in V_h \tag{4.34}$$

↳ v can have a contribution from V where no inverse estimate can be used to bound $\{\partial_{nn} v\}$.

□

4.4 A priori error estimate

For the proposed method, we want to derive a priori error estimate with respect to both the $\|\cdot\|_{a_h,*}$-norm and the $\|\cdot\|_{\Omega}$-norm. We will construct a suitable (quasi-)interpolation operator, here we use the Clement quasi interpolation operator which in contrast to the standard Lagrange nodal interpolation iterator is also defined for low regularity function in $u \in L^2(\Omega)$. In combination with discrete coercivity this allows you to derive an a priori estimate in the energy norm. Finally, we use a standard duality argument, i.e. Aubin-Nitsche trick, to derive the L^2 error estimate.

Recall that for $v \in H^1(\mathcal{T}_h)$ these inequalities holds $\forall T \in \mathcal{T}_h$ such that

$$\|v\|_{\partial T} \lesssim h_T^{-\frac{1}{2}} \|v\|_T + h^{\frac{1}{2}} \|\nabla v\|_T,$$
$$\|v\|_{\Gamma \cap T} \lesssim h^{-\frac{1}{2}} \|v\|_T + h_T^{\frac{1}{2}} \|\nabla v\|_T,$$

for proof see [92, Lemma 4.2].

Assume that Ω has a boundary Γ in C^1, then it does exist an bounded extension operator,

$$(\cdot)^e : H^m(\Omega) \to H^m(\mathbb{R}^d),$$

for all $v \in H^m(\Omega)$ which satisfies

$$v^e|_{\Omega} = v,$$
$$\|v^e\|_{m,\mathbb{R}^d} \lesssim \|v\|_{m,\Omega}. \tag{4.35}$$