Theorem 4.6. *Let $u \in H^s(\Omega)$, $s \geqslant 3$ be a solution to (3.4) and let $u_h \in V_h$ of order $k \geqslant 2$ be the discrete solution to (4.3). Then with $r = \min\{s, k+1\}$ the error $e = u - u_h$ satisfies*

$$\|e\|_{a_h,*} \lesssim h^{r-2}\|u\|_{r,\Omega}, \tag{4.51}$$
$$\|e\|_{\Omega} \lesssim h^{r-\max\{0,3-k\}}\|u\|_{r,\Omega}. \tag{4.52}$$

Mind punctuation in all equations throughout the document !

Remark. Be aware that for $k = 2$ the estimate (4.52) is suboptimal with 1 order.

Proof. We will divide the proof into two steps.
Step 1. We want to prove that $\|e\|_{a_h,*} \lesssim h^{r-2}\|u\|_{r,\Omega}$. Decompose $e = u - u_h$ intro $e = e_h + e_\pi$, where we denote the discrete error $e_h = C_h^e u - u_h$ and the interpolation error $e_\pi = u - C_h^e u$. We can then observe that

$$\begin{aligned}
\|u - u_h\|_{a_h} &\leqslant \|u - C_h^e u + C_h^e u - u_h\|_{a_h,*} \\
&\leqslant \|u - C_h^e u\|_{a_h,*} + \|C_h^e u - u_h\|_{a_h,*} \\
&\lesssim \|e_\pi\|_{a_h,*} + \|e_h\|_{A_h}.
\end{aligned} \tag{4.53}$$

Using Lemma 4.4, is it clear that $\|e_\pi\|_{a_h,*} \lesssim h^{r-2}\|u\|_{r,\Omega}$ is already fulfilled, hence, it remains to estimate e_h. From Lemma 4.2 and 4.3, the weak Galerkin orthogonality and Assumption EP2 (4.50) is it natural to arrive at,

$$\begin{aligned}
\|e_h\|_{A_h}^2 &\lesssim a_h(C_h^e u - u_h, e_h) + g_h(C_h^e u - u_h, e_h) \\
&= a_h(C_h^e u - u, e_h) + a_h(u - u_h, e_h) + g_h(C_h^e u - u_h, e_h) \\
&= a_h(C_h^e u - u, e_h) + g_h(C_h^e u, e_h).
\end{aligned} \tag{4.54}$$

Hence, now utilizing the Assumption EP2 (4.50) is it clear that

$\|w\|_{nc}$ EP2

$$\begin{aligned}
a_h(C_h^e u - u, e_h) + g_h(C_h^e u, e_h) &\lesssim \|C_h^e u - u\|_{a_h,*}\|e_h\|_{a_h} + |C_h^e u|_{g_h} |e_h|_{g_h} \\
&\lesssim \|C_h^e u - u\|_{a_h,*}\|e_h\|_{a_h} + h^{r-2}\|e_h\|_{r,\Omega} |e_h|_{g_h} \\
&\lesssim (\|C_h^e u - u\|_{a_h,*} + h^{r-2}\|e_h\|_{r,\Omega})\|e_h\|_{A_h} \\
&\lesssim h^{r-2}\|u\|_{r,\Omega}\|e_h\|_{A_h}.
\end{aligned} \tag{4.55}$$

Here we noticed that $\|e_h\|_{a_h} + |e_h|_{g_h} \lesssim \|e_h\|_{A_h}$, and used that $\|C_h^e u - u\|_{a_h,*} \lesssim h^{r-2}\|u\|_{r,\Omega}$ from Lemma 4.4.
~~Finally,~~ combining (4.54) and (4.55) and dividing by $\|e_h\|_{A_h}$ is it clear that $\|e_h\|_{A_h} \lesssim h^{r-2}\|u\|_{r,\Omega}$. Hence, the first part of the proof is complete.

Step 2. We want to show that $\|e\|_\Omega \lesssim h^{r-\max(0,3-k)}\|u\|_{r,\Omega}$. The idea is to apply the so-called Aubin-Nitsche duality trick while being aware of the ghost penalty g_h. Let us denote the following observation. Assume that $e := u - u_h \in L^2(\Omega)$ and $\psi \in H^4(\Omega)$. Let the corresponding dual problem to (3.1) be

This is not an assumption. We assume that $u \in H^s(\Omega)$ $s \geq 3$ and u_h is always in $H^1(\Omega) \subset L^2(\Omega)$, so that $u - u_h \in L^2(\Omega)$ is a consequence (not an assumption) of your other assumptions.

State this after you introduce (4.56) Again not an assumption, but a consequence of your other assumption. You can refer to Brenner 2010

$$\begin{aligned}
\Delta^2\psi &= e \quad \text{in } \Omega \\
\partial_n\psi &= 0 \quad \text{on } \Gamma \\
\partial_n\Delta\psi &= 0 \quad \text{on } \Gamma
\end{aligned} \tag{4.56}$$

This implies that it exists a $\psi \in H^4(\Omega)$ such that $a_h(v, \psi) = (e, v)_\Omega \ \forall v \in V_h$. Hence, we can

and say something like "Thanks to standard regularity results for the biharmonic equation [Brenner], $\psi \in H^4(\Omega)$".

easily observe that

$a(e, \psi)$ (ψ its solution to dual problem since it needs to be in the second slot.

$$
\begin{aligned}
\|e\|_\Omega^2 &= (e, e)_\Omega = (e, \Delta^2 \psi)_\Omega \\
&= \cancel{a_h(\psi, e)} = a_h(u - u_h, \psi) \\
&= a_h(u - u_h, \psi + C_h^e \psi - C_h^e \psi) \\
&= a_h(u - u_h, \psi - C_h^e \psi) + \underbrace{a_h(u - u_h, C_h^e \psi)}_{\neq 0 \text{ but } = g_h(u_h, C_h^e \psi)} \qquad (4.57)\\
&= a_h(u - u_h, \psi - C_h^e \psi) + \\
&\lesssim \underbrace{\|u - u_h\|_{a_h,*}}_{\mathrm{I}} \underbrace{\|\psi - C_h^e \psi\|_{a_h,*}}_{\mathrm{II}}
\end{aligned}
$$

weak Galerkin orthog.

Needs to be extended (1)

Now, for I we simply use the energy a priori estimate

$$
\mathrm{I} \lesssim h^{r-2} \|u\|_{r,\Omega} \tag{4.58}
$$

However, to estimate II we set $\widetilde{r} = \min(4, k+1)$, where 4 comes from the regularity $\psi \in H^4(\Omega)$ and $\|\psi\|_{4,\Omega} \lesssim \|e\|_\Omega$, thus,

$$
\mathrm{II} \lesssim h^{\widetilde{r}-2} \|\psi\|_{\widetilde{r},\Omega} \lesssim h^{\widetilde{r}-2} \|e\|_\Omega. \tag{4.59}
$$

Hence, combining (4.57), (4.58) and (4.59) can we conclude

$$
\|e\|_\Omega \lesssim h^{r+\widetilde{r}-4} \|u\|_{r,\Omega} \tag{4.60}
$$

Having a clear look at $\widetilde{r}$, wee see that

$$
\widetilde{r} = \min(4, k+1) = \begin{cases} 3, & k = 2 \\ 4, & k \geqslant 3 \end{cases} \tag{4.61}
$$

So we have the following estimate,

$$
\|e\|_\Omega \lesssim \|u\|_{r,\Omega} \begin{cases} h^{r-1}, & k = 2 \\ h^{r-2}, & k \geqslant 3 \end{cases} \tag{4.62}
$$

or equivalently $\|e\|_\Omega \lesssim \|u\|_\Omega^{r-\max(0,k-3)}$.

□

4.5 Constructing ghost penalties

We have the following assumptions for the ghost penalty.

EP1 The ghost penalty $g_h(\cdot,\cdot)$ extends the H^2 norm such that

$$
\|D^2 v_h\|_{\mathcal{T}_h}^2 \lesssim \|D^2 v_h\|_\Omega^2 + |v_h|_{g_h}^2 \quad \forall v_h \in V_h \tag{4.63}
$$

EP2 For $v \in H^s(\Omega)$ and $r = \min(s, k+2)$, the semi-norm $|\cdot|_{g_h}$ satisfies the following estimate,

$$
|C_h^e v|_{g_h} \lesssim h^{r-2} \|v\|_{r,\Omega}. \tag{4.64}
$$

The goal in this chapter is to engineer an ghost penalty which fulfills these assumptions.

Let k be a positive integer. Recall the multi-index $\alpha = (\alpha_1, \ldots, \alpha_d)$ of order $|\alpha| = \sum_i \alpha_i = k$ with a corresponding component-wise factorial $\alpha! = \alpha_1! \ldots \alpha_d!$. Let $v \in C^k(\Omega)$. The generalization of the normal derivative is denoted as,

$$
\partial_n^k v = \sum_{|\alpha|=k} \frac{\overset{D^\alpha v}{\partial^\alpha v}\, n^\alpha}{\alpha!}, \quad = D^n v\,[n, \ldots\, n] \tag{4.65}
$$

where the component-wise product of the normal vector is $n^\alpha = n_1^{\alpha_1} \ldots n_d^{\alpha_d}$ and the derivative $\partial^\alpha v$ is as defined in Equation (2.1). Remark that $\partial_n^0 v = v$, $\partial_n^1 v = \nabla v\, n = \partial_n v$ and $\partial_n^2 v = \frac{1}{2} n^T D^2 v\, n = \frac{1}{2} \partial_{nn} v$.

The following result is the backbone of the face-based ghost penalty.

$$\|e\|_\Omega^2 = \ldots = a_{\tilde h}(w - u_{\tilde h}, \psi - C_h^e \psi) + s_{\tilde h}(w_{\tilde h}, C_h^e \psi)$$

$$= \quad " \quad " \quad " \quad " \quad + s_{\tilde h}(w_{\tilde h} - C_h^e u, C_h^e \psi) + s_{\tilde h}(C_h^e u, C_h \psi)$$

$$= \mathrm{I} + \mathrm{II} + \mathrm{III}$$

I have you already estimated in (4.58)-(4.59).

$$\mathrm{II} \leq |w_{\tilde h} - C_h^e w|_{s_{\tilde h}} \cdot |C_h^e \psi|_{s_{\tilde h}}$$

$$\lesssim \boxed{\|w_{\tilde h} - C_h^e w\|_{A_{\tilde h}}} \cdot |C_h^e \psi|_{s_{\tilde h}}$$

$$\lesssim \boxed{h^{r-2} \|w\|_{r,\Omega}} \; h^{\tilde r - 2} \|\psi\|_{\tilde r,\Omega} \leq \underbrace{h^{r+\tilde r-4} \|w\|_{r,\Omega} \|e\|_\Omega}_{\text{Same estimate as for I}} .$$

This estimate you have already establish in Step 1 s. (4.54)+(4.55)

EP2 for u, ψ

$$\mathrm{III} \leq |C_h^e w|_{s_{\tilde h}} \, |C_h^e \psi|_{s_{\tilde h}} \lesssim h^{r-2} \|w\|_{r,\Omega} \cdot h^{\tilde r - 2} \|\psi\|_{\tilde r,\Omega}$$

$$\lesssim h^{r+\tilde r-4} \|w\|_{r,\Omega} \cdot \|e\|_\Omega$$

→ Same estimate as for I + II.

In conclusion, we have shown that

$$\|e\|_\Omega^{\cancel{2}} \lesssim h^{r+\tilde r-4} \|w\|_{r,\Omega} \cancel{\|e\|_\Omega} .$$