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CUT FINITE ELEMENT METHOD FOR THE CAHN-HILLIARD EQUATION

And why it is quite cool stuff

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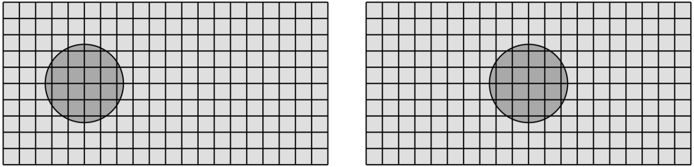
I will present a method to solve PDE's on an unfitted mesh



- Fairly new class of methods! 5-10 years old.
- Can handle smooth boundaries and complex geometries!
- Possible to apply moving domains without re-meshing!

Maybe I should introduce myself

- Isak Hammer, 27 year old, Lofoten
- Graduate student in Industrial Mathematics
- Department of Mathematical Sciences (IMF)
- Specialization in
  1. Finite element methods
     1.1 Writing master thesis on unfitted methods for Cahn-Hilliard.
  2. Optimal control problems for PDEs
     2.1 Side projects on biomembrane dynamics

Master student for one the CFD groups at IMF

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- Research aligned towards biophysics problems
- 1 Postdoc
- 2 PhD candidates
- 1 Master's student

Introduction

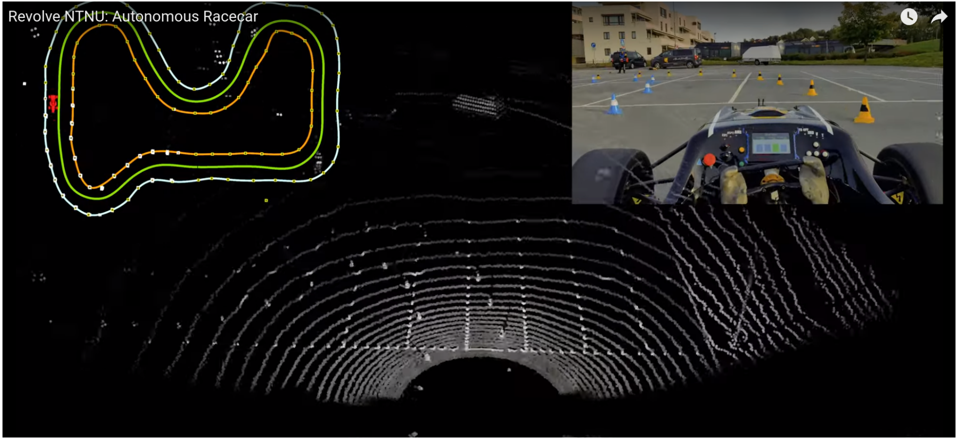
- Offshore apprentice for 2 years
- Worked with downhole electric instruments find reservoir properties
    1. Reservoir permeability and pressure
    2. Geological characteristics as rock dating, porosity etc.

Introduction

- Robotics Engineer for 2 years
- Revolve NTNU
- Autonomous Systems
    1. Vehicle Dynamics
    2. Nonlinear MPC and trajectory optimization
    3. Unknown path planning
    4. Linux, C++, and Python (Casadi, acados, ROS, Docker etc)

Introduction

Introduction



Live demonstration of the system: here

Contents

Plan for today

1. Introduction - 2 min
2. The Cahn Hilliard Equation - 3 min
3. Introduction to CutFEM - 4 min
4. Results (hehehe) - 2 min
5. Questions - 5 min

My goal is to give you a taste of modern finite element methods!

Great we have a solution

What is the problem?

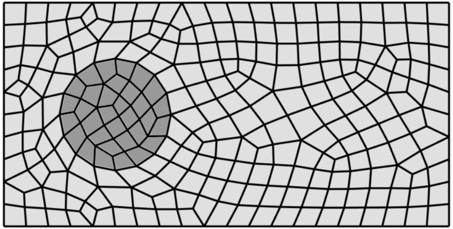
- It is suboptimal on moving domains $\Omega(t)$.
- And only works if Ω can be fully covered by by the mesh

$$\Omega = \bigcup_i T_i$$

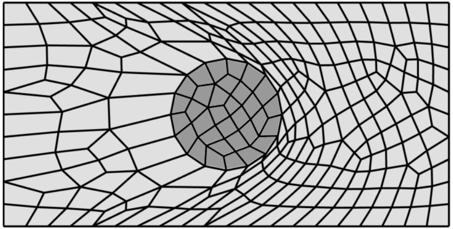
Thus, cannot handle smooth boundaries.

Moving Domains

- Potentially very costly re-meshing procedures.
- Ill-conditioned if mesh is too bad



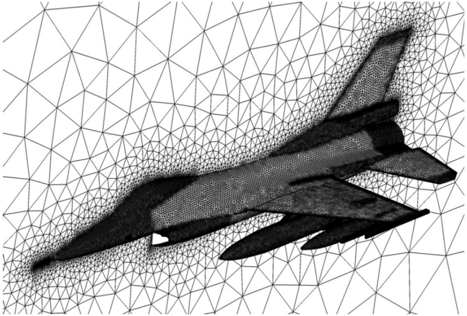
(a) Initial mesh.



(b) Transformed mesh.

Other problems of unstructured mesh

- Unstructured mesh is difficult to parallelize
- Cannot handle smooth boundaries. Some application may actually require smooth boundaries (shape optimization etc).



Ways to solve this problem

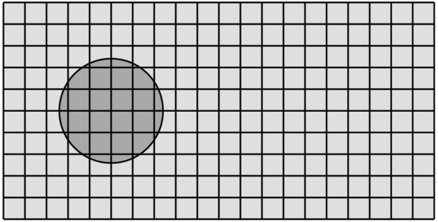
- Change the geometry.
  1. Time-dependent mesh elements on moving domains.
  2. Delete and add nodes when necessary.
  3. Full re-mesh generation.
  4. Probably many more clever methods . . .
- Utilize the geometry instead of modifying it
  1. Method that can handle smooth boundaries.
  2. Do some smart transformations

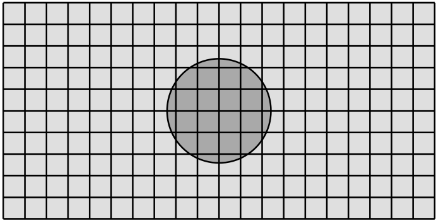
Question

How do you approach this problem?

Cut finite element method

Method to solve PDE's with moving domains on a unfitted mesh!

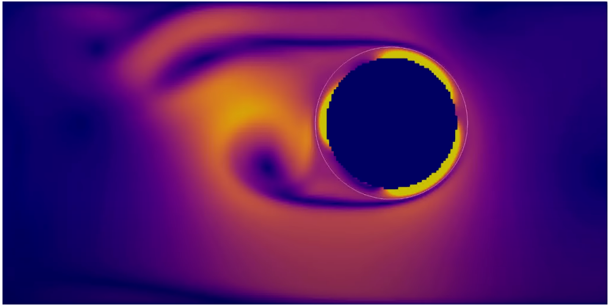




- Here we considering an smooth boundary Γ in C^2
- No re-meshing on moving domains, only new configuration of cut cells.
- Potential to handle very complex geometries

Example of complex domains

Navier-Stokes on moving domains



Link to video

Computational Domains

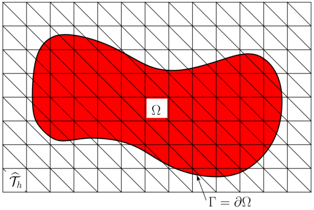


Figure: Physical domain

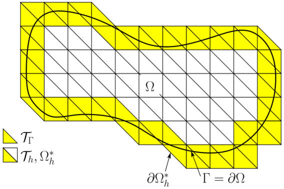
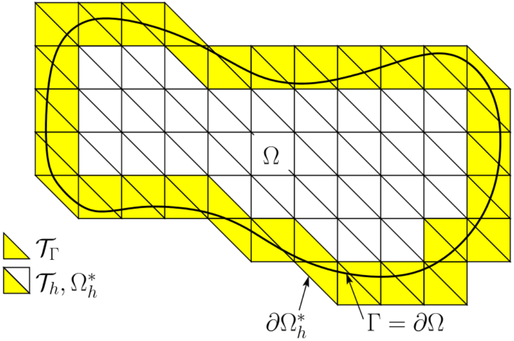


Figure: Cut cells

- We define a background mesh $\widetilde{\mathcal{T}}_h$
- An active submesh $\mathcal{T}_h \subset \widetilde{\mathcal{T}_h}$ containing physical domain Ω.
- Cut cells $\mathcal{T}_\Gamma \subset \mathcal{T}_h$ is the mesh elements that intersects with the boundary Γ.

Constructing the method



Observation

1. The interior is pretty nice to deal with.
2. The boundary must be parametrized somehow.
   - Level-set functions $\varphi(x) = 0$ is one way.
   - Splines is also possible
3. How do we deal with Dirichlet conditions?
4. How do we deal with elements with "bad" cuts?
5. Can we show that the problem is still well-posed?

Recall Poisson problem

Recall the formulation $(\nabla u, \nabla v)_\Omega - (\partial_n u, v)_\Gamma = (f, v)_\Omega$

Problem

- Dirichlet conditions is embedded in the function space,
  $V_g = \{v \in H^1(\Omega) \mid u = g \text{ on } \Gamma\}$
- But is difficult to handle when Γ is smooth.

Can we impose the Dirichlet conditions naturally?

Yes! We add a penalty on the boundary $\mu(u - g, v)_\Gamma$,

$$(\nabla u, \nabla v)_\Omega - (\partial_n u, v)_\Gamma + \mu(u, v)_\Gamma = (f, v)_\Omega + \mu(g, v)_\Gamma.$$

For symmetry we can also add $(u - g, \partial_n v)_\Gamma$,

$$(\nabla u, \nabla v)_\Omega - (\partial_n u, v)_\Gamma + (-u, \partial_n v)_\Gamma + \mu(u, v)_\Gamma = (f, v)_\Omega + \mu(g, v)_\Gamma + (-g, \partial_n v)_\Gamma.$$

Poisson formulation on a smooth boundary

Recall that $\mathcal{T}_h$ is the active mesh, that is, all trianges intersection with the interior of the domain Ω.

Definitions

Let $V_h := \mathcal{P}^k(\mathcal{T}_h) \cap C^0(\Omega)$.We denote the bilinear form $a_h : V_h \times V_h \to \mathbb{R}$ and the linear form $l_h : V_h \to \mathbb{R}$ to be,

$$
\begin{aligned}
a_h(u,v) &:= (\nabla u, \nabla v)_\Omega - (\partial_n u, v)_\Gamma - (u, \partial_n v)_\Gamma + \mu(u,v)_\Gamma \\
l_h(v) &:= (f,v)_\Omega + \mu(g,v)_\Gamma - (g, \partial_n v)_\Gamma
\end{aligned}
$$

Problem Statement

We want to find a $u \in V_h$ s.t. $a_h(u,v) = l_h(v) \quad \forall v \in V_h$.

Recall Lax Milgram

Theorem

$a_h(u, v) = l_h(v)$ *well-posed if both of these statements holds;*

- *The bilinear form is bounded,*

$$|a_h(v, w)| \leq C_1 \|v\|_{a_h} \|w\|_{a_h} \quad \forall v, w \in V_h.$$

- *The bilinear form is coercive (one-to-one),*

$$a_h(v, v) \geq C_2 \|v\|_{a_h}^2 \quad \forall v \in V_h.$$

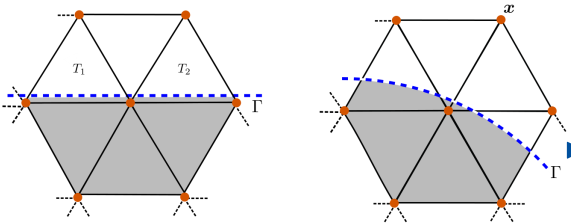
Is the new system well-posed?

- The Dirchlet conditions problem is in good shape for smooth domains!
- But from basic FEM theory it is now necessarry to apply

$$h^{\frac{1}{2}}\|\partial_n v\|_{\Gamma\cap T} \leq C\|v\|_{\Omega\cap T}$$

  to obtain well-posedness!
- But what about integration on very bad cut elements?
  - The relation between length $|F \cap \Omega|$ and volume $|T \cap \Omega|$ is very different on cut elements, thus, the norm is unbounded if the cut is bad.

The problem with the bad cuts



Observation

- Bad cuts makes it hard to justify length $F \cap \Omega$ vs area $|T \cap \Omega|$. Thus, the necessarry

$$h^{\frac{1}{2}} \|\partial_n v\|_{\Gamma \cap T} \leq C \|v\|_{\Omega \cap T}$$

become unbounded. Hence, the system is ill-conditoned.

- We are forced to extend the norm s.t.

$$h^{\frac{1}{2}} \|\partial_n v\|_{\Gamma \cap T} \leq C \|v\|_{T}$$

But we are integrating outside of our domain :(

Ghost penalty

The solution of the inverse estimate problem is simple! We add a **ghost penalty** $g_h(u,v)$ to handle the bad cuts as an regularization!

$$h^{\frac{1}{2}}\|\partial_n v\|_{\Gamma\cap T} \leq C\|v\|_T + g_h(u,v)$$

The goal is to regulate the ill-conditioned problem!

- We give it the necessarry assumptions for Lax-Milgram to hold!
- Same strategy is used to to obtain optimal convergence!
- We then engineer the g_h given the assumptions!

Hence, we end up with this stabilized problem formulation

Stabilized Poission Problem

Let $A_h(u,v) := a_h(u,v) + g_h(u,v)$. We want to find a $u \in V_h$ s.t.

$$A_h(u,v) = l_h(v) \quad \forall v \in V_h$$

The reality is more nasty

Proposition 4.2 (Discrete coercivity)
The discrete form A_h is coercive, that is,

$$A_h(v,v) \gtrsim \|v\|_{A_h}^2 \quad \forall v \in V_h. \tag{4.17}$$

Proof. The proof of this statement follows the presentation in [6,16], but was first given for $\mathbb{P}_1$ elements in [9].

First take $v \in V_h$ and set $u = v$, and then using the Cauchy-Schwartz inequality in addition to the inequality $2ab \leqslant a^2\epsilon + b^2/\epsilon$ for real numbers a, b and ϵ. This yields,

$$\begin{aligned} A_h(v,v) &= (\nabla v, \nabla v)_{\mathcal{T}_h \cap \Omega} - (\partial_n v, v)_\Gamma - (v, \partial_n v)_\Gamma + \gamma/h(v,v)_\Gamma + g_h(v,v) \\ &= (\nabla v, \nabla v)_\Omega - 2(h^{1/2}\partial_n v, h^{-1/2}v)_\Gamma + \gamma/h(v,v)_\Gamma + g_h(v,v) \\ &\geqslant \|\nabla v\|_\Omega^2 - 2\|h^{1/2}\partial_n v\|_\Gamma \|h^{-1/2}v\|_\Gamma + \gamma/h\|v\|_\Gamma^2 + |v|_{g_h}^2 \\ &\geqslant \|\nabla v\|_\Omega^2 - \epsilon\|h^{1/2}\partial_n v\|_\Gamma^2 - \frac{1}{\epsilon}\|h^{-1/2}v\|_\Gamma^2 + \gamma\|h^{-1/2}v\|_\Gamma^2 + |v|_{g_h}^2. \end{aligned} \tag{4.18}$$

Then using the inverse estimate from Proposition 4.1 enables bounding the flux over the boundary by the gradient over the whole active mesh. This yields

$$A_h(v,v) \geqslant \|\nabla v\|_\Omega^2 - \epsilon C_\Gamma \|\nabla v\|_{\mathcal{T}_h}^2 + (\gamma - \epsilon^{-1})\|h^{-1/2}v\|_\Gamma^2 + |v|_{g_h}^2. \tag{4.19}$$

Further, we use the requirement put on g_h in Assumption 1 and collect the terms

$$\begin{aligned} A_h(v,v) &\geqslant \|\nabla v\|_\Omega^2 - \epsilon C_\Gamma C_g \left(\|\nabla v\|_\Omega^2 + |v|_{g_h}^2\right) + (\gamma - \epsilon^{-1})\|h^{-1/2}v\|_\Gamma^2 + |v|_{g_h}^2 \\ &= (1 - \epsilon C_\Gamma C_g)\|\nabla v\|_\Omega^2 + (\gamma - \epsilon^{-1})\|h^{-1/2}v\|_\Gamma^2 + (1 - \epsilon C_\Gamma C_g)|v|_{g_h}^2. \end{aligned} \tag{4.20}$$

If we let $\epsilon = 1/(C_\Gamma C_g)$ and $\gamma = 4C_\Gamma C_g$, we can finally assert that

$$A_h(v,v) \geqslant C\|v\|_{A_h}^2, \tag{4.21}$$

from some constant $C > 0$ for all $v \in V_h$. □

But I hope at least you have learned something new :)

Questions?