related methods, sometimes considered equivalent, named Extended FEM and Trace FEM [74, 75]. This has been successfully implemented for the BH problem for the mixed formulation [76] and the CIP formulation [77, 74]. However, both implementations are considering an interface problem between two domains. Specifically for the CH problem, the mixed formulation [78] has been shown to be successful. Aggregated unfitted finite element method (AgFEM) is a close relative to CutFEM and has also shown to be promising [79, 80]. The method is an alternative way to the ghost penalty, which instead applies a so-called cell aggregation with respect to a cut cell (assuming each cell has enough support with interior elements) and, thus, the badly cut cells are removed, ensuring robustness and well-posedness.

1.3 Outline of the report

In this article, we propose an novel stabilized unfitted cut continuous interior penalty method (CutCIP) specifically for the BH problem, which incorporates the CutFEM methodology in combination with a CIP formulation. We will follow the work theoretical procedure as presented in the DG Poisson formulation proposed by [81], but instead apply the CIP BH formulation while taking account to analytical results provided in [61, 54].

The main results in this thesis is as follows. In Section 3 the basic construction of the CIP BH formulations and the related properties is established. Subsequently, in Section 4 we propose the corresponding CutCIP BH method, but also provide a theoretical proof to show that the stability and convergence properties from the original CIP method are conserved. In Section 5 we briefly extend the method to handle the CH problem. Finally, Section 6 is dedicated to numerical experiments.

methods / notation for what? Be more precise, e.g.:

2 Mathematical Background

"the mathematical background to formulate and analyze

We will in this section ~~will~~ we review standardized and well-know methods and notations. ~~Generally is the notation followed from~~ [82, Chapter 1].

where we follow the presentation in

finite element based discretizations of partial differential"

2.1 Notation

We will in this report assume Ω to be a compact and open set in $\mathbb{R}^d$. Let $p \in \mathbb{R}$, $1 \leqslant p \leqslant \infty$, and define the space $L^p(\Omega)$ to be the set of all measurable functions $u : \Omega \mapsto \mathbb{R}$ such that $|f|^p$ is Lebesgue integrable, i.e,

$$L^p(\Omega) = \left\{ u : \Omega \mapsto \mathbb{R} \mid \int_\Omega |u|^p \, d\Omega < \infty \right\}.$$

Let $u \in L^p(\Omega)$. We define the integral norm of order p to be

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p \, dx \right)^{\frac{1}{p}}.$$

Since $p = 2$ is frequently used in this report, we also define for convenience a compact notation $\|u\|_\Omega = \|u\|_{L^2(\Omega)}$. Recall that $L^2(\Omega)$ is a Hilbert space if it is equipped with a inner product of two functions $u, v \in L^2(\Omega)$ such that $(u, v)_\Omega = (u, v)_{L^2(\Omega)} = \int_\Omega uv dx$. The following definition for derivatives is employed,

$$\partial^\alpha u = \frac{\partial^{|\alpha|} u}{\partial^{\alpha_1} x_1 \partial^{\alpha_2} x_2}, \quad \text{for } \alpha = (\alpha_1, \alpha_2) \text{ and } f \in C^{|\alpha|}(\Omega). \tag{6}$$

For d dimensions of order k we define the multi-index $\alpha = (\alpha_1, \dots, \alpha_d)$ with the absolute value $|\alpha| = \sum_{i=1}^{d} \alpha_i = k$ s.t.

$$\partial^\alpha u = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}} u$$

Let $m \geq 0$ be an integer and let $1 \leqslant p \leqslant \infty$ be a real number. Then the Sobolev space $H^m(\Omega)$ are defined by

$$H^m(\Omega) = \left\{ u \in L^2(\Omega) \mid \partial^\alpha u \in L^2(\Omega) \, \forall \alpha : |\alpha| \leqslant m \right\}.$$

Equipped with the inner product for $u, v \in H^m(\Omega)$,

$$(u, v)_{H^m(\Omega)} = \sum_{|\alpha| \leqslant m} \int_\Omega \partial^\alpha u \partial^\alpha v dx,$$

with the corresponding norm $\|u\|^2_{H^m(\Omega)} = \|u\|^2_{L^2(\Omega)} + \sum_{k=1}^{m} |u|^2_{H^k(\Omega)}$. Here the seminorm is defined such that, $|u|^2_{H^k(\Omega)} = \sum_{|\alpha|=k} \|\partial^\alpha u\|^2_\Omega$. We will often use the shorthand notation $\|u\|_{k,\Omega} = \|u\|_{H^k(\Omega)}$ and $|u|_{k,\Omega} = |u|_{H^k(\Omega)}$.

Given the context of Sobolev spaces, we consider the functions $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$. We can denote Greens theorem, which links integrals over a volume and its boundary, as follows,

$$(\Delta u, v)_\Omega = -(\nabla u, \nabla v)_\Omega + (u, \partial_n v)_\Gamma$$

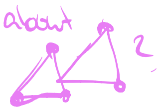
This identity serves as an essential tool for the calculations done.

2.2 Computational Domains

Assume that $\Omega \subset \mathbb{R}^d$ is a ~~compact set~~ an open and bounded domain with a boundary Γ. In standard FEM methods a key assumption is that the set Ω is a polyhedra. This is useful since a polyhedra can be fully covered by a collection of polyhedra and, hence, motivating us to define a fitted mesh. We define a fitted mesh $\mathcal{T}$ of the domain Ω to be a collection of disjoint polyhedra $\{T\}$ forming a partition of Ω s.t $\overline{\Omega} = \bigcup_{T \in \mathcal{T}} T$, for illustration see Figure 3. Here we say that each $T \in \mathcal{T}$ is a mesh element or an element. The mesh size is defined as the maximum diameter $h := h_{max}$ of any polyhedra in the mesh $\mathcal{T} = \{T\}$, that is, $h_{max} = \max_{T \in \mathcal{T}} h_T$ s.t. $h_T = \operatorname{diam}(T) = \max_{x_1, x_2 \in T} \operatorname{dist}(x_1, x_2)$ Hence, ~~motivating us to~~ we use use the notation $\mathcal{T}_h$ for a mesh $\mathcal{T}$ with size h.

please take my previous comments into account! Must be closed polyhedra $\{T\}$ with disjoint interior.

Still not correct. What about



2.

In 3d: If $T_1 \neq T_2$ and $T_1 \cap T_2 \neq \emptyset$, then $T_1 \cap T_2$ is either a common vertex, edge or face.

For simplicity will we restrict ourself to simplicial and quadrilateral elements. A mesh $\mathcal{T}_h$ in $\mathbb{R}^d$ is said to be matching if for all neighbouring elements $T_1, T_2 \in \mathcal{T}_h$ such that the intersection $T_1 \cap T_2$ is a manifold of dimensions $(d-1)$, then $T_1 \cap T_2$ a entire facet of both T_1 and T_2.

Let the chunkiness parameter $c_T := h_T / r_T$, where r_T is the largest ball that be inscribed inside a element $T \in \mathcal{T}_h$. A mesh is said to be shape regular if $c_T \leqslant c$ is independent of T and h. We also say that the mesh is quasi-uniform only if it is shape regular and $h_{\max} \leqslant c h_{\min}$. For a more complete description of meshes, see [83, Chapter 8].

In this thesis will we assume that a mesh $\mathcal{T}_h$ is matching, shape regular and quasi-uniform unless specified. The fact that the mesh is conform makes is a useful property since the interface between mesh elements has come into contact in the sense that it is either a vertex or a facet. This with the combination of shape regularity and quasi-uniformity is a major key to prove important inequalities in broken Sobolev spaces [82, Chapter 1.4.1]. Hence, the assumptions are very handy when proving convergence.

As before, we use the shorthand notation, $\|v\|_{\mathcal{T}_h} = \|v\|_{L^2(\mathcal{T}_h)}$ and $(v,w)_{\mathcal{T}_h} = (v,w)_{L^2(\mathcal{T}_h)}$. That is,

$$\|v\|^2_{L^2(\mathcal{F}_h)} = \sum_{F\in\mathcal{F}_h} \|v\|^2_{L^2(F)} \quad \text{and} \quad (v,w)_{L^2(\mathcal{F}_h)} = \sum_{T\in\mathcal{F}_h} (v,w)_{L^2(F)}.$$

Again, we often use the more compact notation $\|v\|_{\mathcal{F}_h} = \|v\|_{L^2(\mathcal{F}_h)}$ and $(v,w)_{\mathcal{F}_h} = (v,w)_{L^2(\mathcal{F}_h)}$. A very useful lemma when working with estimates on broken Sobolev spaces is that a if a function is continuous, then the jump between the mesh elements is zero. A function $v \in H^1(\mathcal{T}_h)$ belongs to $H^1(\Omega)$ if and only if $[\![v]\!] = 0$ for $F \in \mathcal{F}_h^{int}$.

2.4 Useful ~~inverse estimates~~ inequalities

← These are NOT inverse estimates. "Inverse" is reserved to inequalities which you can derive for discrete function space and in some sense go the opposite direction than usually.

We first some the standard inequalities.

(i) A fundamental property of the inner-product the so-called Cauchy-Schwarz inequality

$$(u,v)_{H^m(\Omega)} \leqslant \|u\|_{H^m(\Omega)}\|v\|_{H^m(\Omega)} \forall u,v \in H^m(\Omega)$$

(ii) For any $a,b > \mathbb{R}$ the well known Youngs ε-inequality is on the form,

$$2ab \leqslant \varepsilon a^2 + \frac{1}{\varepsilon}b^2.$$

As said before these hold only for discrete function spaces and need to be stated after you introduced FEM.

Choose any triangle $T \in \mathcal{T}_h$ and let $v \in \mathcal{P}(T)$. Then does the local inverse estimate hold,

$$|v|_{H^l(T)} \lesssim h^{m-l}\, |v|_{H^m(\Omega)}\,.$$

for $l \leqslant m$. For proof, see [83, Lemma 12.1]. An essential example is the following inequality.

$$\|D^2 v\|^2_T \lesssim h^{-1}\|\nabla v\|_T \lesssim h^{-2}\|v\|_T$$

Another very useful inequality is the so-called trace inequality which connect the relationship of evaluating the norm on element T and with any of the corresponding facets $F \in \partial T$. The general form is $\|v\|_F \lesssim h^{-\frac{1}{2}}\|v\|_T$, see [83, Lemma 12.8]. Let $\partial_n v = \nabla v\ n$ and $\partial_{nn} v = n^T D^2 v\ n$, and keeping in mind that the normal vector has a unit length, can we evidently apply the trace inverse inequality to conclude,

$$\|\partial_n v\|_F \leqslant \|\nabla v\|_F \leqslant h^{-\frac{1}{2}}\|\nabla v\|_T,$$
$$\|\partial_{nn} v\|_F \leqslant \|D^2 v\|_F \leqslant h^{-\frac{1}{2}}\|D^2 v\|_T.$$

2.5 Lax-Milgram lemma

The intention is to introduce a abstract framework which can handle various types of partial differential equations (PDE). Let $\mathcal{A} : X \to Y$ be a abstract linear operator encoding the structure of any linear PDE, including boundary conditions and X, Y are spaces of functions. Then we denote the abstract strong formulation as the equation

$$\mathcal{A}u = f. \tag{7}$$

Where the function $f : \Omega \subset \mathbb{R}^n \mapsto \mathbb{R}$. We assume that the function $u : \Omega \to \mathbb{R}$ satisfies the relation (7) pointwise so that $\mathcal{A}u(x) = f(x) \forall x \in \Omega$. We will discover that Sobolev spaces are specifically engineered to study these kinds of problems.

Definition 2.1 (Linear bounded functional)**.** *Let X be a Hilbert space. Furthermore, we define the dual space X' to be the space of linear and bounded functionals $F : X \mapsto \mathbb{R}$, i.e.,*

$$X' = \{F : X \to \mathbb{R} \mid F \text{ is linear and bounded}\}$$

Problem 2.1 (Abstract linear problem)**.** *Assume X and Y to be two Hilbert spaces. Let the vector space $\mathcal{L}(X,Y)$ be all linear bounded operators spanned from X to Y. We define the abstract linear problem as follows; find $u \in X$ s.t.*

$$a(u,v) = l(v) := \langle f, v\rangle_{X',X} \, \forall v \in X$$

Where $a \in \mathcal{L}(X,X,\mathbb{R})$ is a bounded bilinear form and $f \in X' := \mathcal{L}(X,\mathbb{R})$ is a bounded linear form associated with the abstract strong formulation (7). Here we denote by $\langle \cdot,\cdot\rangle_{X'.X}$ the duality pairing between X' and X.

Definition 2.2 (Coercivity and Boundedness)**.** *Let X be a Hilbert space and let $a(\cdot,\cdot) \in \mathcal{L}(X,X,\mathbb{R})$. Recall that the bilinear form $a(\cdot,\cdot)$ is coercive on X if there exists an constant $C_1 > 0$ such that*

$$a(v,v) \geq C_2\|v\|_X \quad \forall v \in X$$

The bilinear form $a(\cdot,\cdot)$ is said to bounded if there exists an constant C_2

$$a(u,v) \leqslant C_1\|u\|_X\|v\|_X$$

Lemma 2.1 (Lax-Milgram)**.** *The abstract linear problem 2.1 is well-posed if $a(\cdot,\cdot)$ is bounded and coercive. Moreover, the following a priori estimate holds true.*

$$\|v\|_X \lesssim \|f\|_{X'}$$

Proof. The problem can easily be proved using a special case of the Banach–Nečas–Babuška theorem. See [82, Lemma 1.4] □

2.6 Finite element method

The finite element method (FEM) is a numerical method to solve partial differential equation by finding an approximation of the Problem 2.1. Let X_h be a finite-dimensional (polynomial) approximation space on the mesh $\mathcal{T}_h$. We say that a method is conform if $X_h \subset X$ and non-conform if $X_h \not\subset X$. We define the approximate problem as follows.

Problem 2.2 (The approximate problem)**.** *Find $u_h \in X_h$ s.t.*

What is X_h? here

$$a_h(u_h,v) = l_h(v) := \langle f, v\rangle \, \forall v \in X_h$$

We denote the functional $a_h : X_h \times X_h \to \mathbb{R}$ as an consistent approximation of $a : X \times X \to \mathbb{R}$, and similarly for the right-hand side $l_h : \times_h \to \mathbb{R}$ as an approximation of $l : X \to \mathbb{R}$. In this report will we generally specify any discrete space $X_h(\mathcal{T}_h)$ as the C^0 continuous polynomial space $\mathcal{P}^k(\mathcal{T}_h)$, of order k.

First local than global.

Definition 2.3 (Broken polynomial spaces)**.** *Let $\mathcal{T}_h$ be a mesh of $\Omega \in \mathbb{R}^d$. Let $\mathcal{P}^k(T)$ be the space of all polynomials of order k in the mesh element T in $\mathcal{T}_h$. We define the broken polynomial space as*

$$\mathcal{P}^k(\mathcal{T}_h) := \left\{ v \in L^2(\mathcal{T}_h) \mid v|_T \in \mathcal{P}^k(T) \quad \forall T \in \mathcal{T}_h \right\}.$$

Similarly, the global C^0 continuous polynomial space is denoted as

$$\mathcal{P}_c^k(\mathcal{T}_h) := \left\{ v \in C^0(\Omega) \mid v|_T \in \mathcal{P}^k(T) \quad \forall T \in \mathcal{T}_h \right\}.$$

$Q^k(\mathcal{T}_h) :=$

$Q_c^k(\mathcal{T}_h) :=$

Definition 2.4 (Local polynomial space)**.** *Let T be a element in a mesh $\mathcal{T}_h$, $x = [x_1, \ldots, x_d]$ be a vector, and $\alpha = [\alpha_1, \ldots, \alpha_d] \in \mathbb{N}^d$ be a multi index. The local polynomial space $\mathcal{P}^k(T)$ for a simplex is denoted as*

$$\mathcal{P}^k(T) = \operatorname{span}\{x^\alpha \ \textit{for}\ x \in T \ \textit{and}\ 0 \leqslant \alpha_i \leqslant k\}. \tag{8}$$

where x^α is a monomial such that $x^\alpha = x_1^{\alpha_1} \ldots x_d^{\alpha_d}$.

Let T be a cuboid, i.e., $T = \prod_{i=1}^d [z_i^-, z_i^+]$ where $z_i^- < z_i^+$ for $z_i^\pm \in \mathbb{R}$. Then the polynomial space $\mathcal{Q}^k(T)$ in $\mathbb{R}^d$ is defined as the tensor product of 1-dimensional finite elements, i.e.,

Q^k

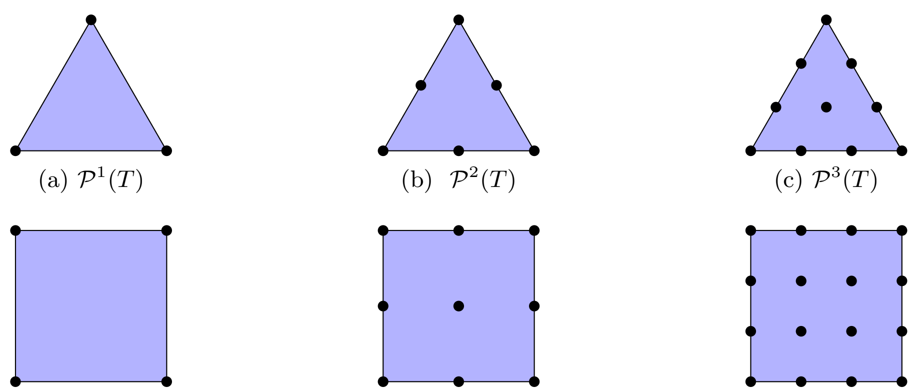
$$\mathcal{Q}_k(T) := \mathcal{P}^k([z_1^-, z_1^+]) \otimes \ldots \otimes \mathcal{P}^k([z_d^-, z_d^+])$$

$V_h = \begin{cases} P_c^k(T_h) \\ Q_c^k(T_h) \end{cases}$ X_h

For more information about the local polynomial spaces, see [83, Chapter 6.4, 7.3]

Following Ciarlet [84, pp.93], the abstract definition of a finite element is defined as the triplet $(T, \mathcal{P}, \Sigma)$. In our case is T a simplex or a quadrilateral geometry and the $\mathcal{P}$ a finite dimensional polynomial space consisting of N shape functions $\{\phi_i\}_{i\in N}$ such as (8). On the other hand, Σ is the so-called dual of $\mathcal{P}$, that is, the set of linear forms $\{\sigma_i\}$ such that. $\sigma_j(\phi_i) = \delta_{ij}$ and $p(x) = \sum_{i\in N} \sigma_i(p) p_i$. If there is a set of points $\{a_i\}_{i\in N}$ in T such that $\sigma_i(p) = p(a_i)\ \forall p \in \mathcal{P}$, then the triple $(T, \mathcal{P}, \Sigma)$ is called a Lagrangian finite element. The set of points $\{a_i\}_{i\in N}$ is called nodes and is associated with the so-called nodal basis of $\mathcal{P}$ such that $\phi(a_i) = \delta_{ij}\ \forall i, j \in N$

As anticipated, the local node configuration of the polynomial space, denoted as $\mathcal{P}^k(T)$, is influenced by the form of T. For the purpose of our discussion, let us represent the polynomial basis for a simplicial element and a quadrilateral element as $\mathcal{P}^k(T)$ and $\mathcal{Q}^k(T)$, both of polynomial order k. In Figure 5 is it illustrated for $k = 1, 2, 3$ in dimension $d = 2$ on how the node configuration evolve.



(a) $\mathcal{P}^1(T)$ (b) $\mathcal{P}^2(T)$ (c) $\mathcal{P}^3(T)$

(d) $\mathcal{Q}^1(T)$ (e) $\mathcal{Q}^2(T)$ (f) $\mathcal{Q}_3(T)$

Figure 5: Illustration of the nodes for the element of a simplex a quadrilateral for dimension $d = 2$ for polynomial orders $k = 1, 2, 3$.

We may introduce the reference element $\hat{T}$ in d dimensions. The reference for a quadrilateral is denoted as $\hat{T} = [0, 1]^d$. The reference for a simplex in is defined by the convex hull spanned by the points $(z_0, e_1, \ldots, e_d)$ where $z_0 := 0$ is the origin and $\{e_i\}_{i=1}^d$ is the standard Cartesian unit basis in $\mathbb{R}^d$. A corresponding reference finite element is defined as $(\hat{T}, \hat{\mathcal{P}}, \hat{\Sigma})$.

Let the mapping $\mathcal{G} : \hat{T} \to T$ an affine mapping, i.e. $\mathcal{G}(x) = Ax + b$. The important property of affine transformations is the preservation of parallelism. Hence, for any two vectors $x, y \in \hat{T}$ that are parallel in the reference element, their images $\mathcal{G}(x)$ and $\mathcal{G}(y)$ will also be parallel. Generally speaking, an affine transformation of the reference simplex is a transformation to any another simplex of the same dimension. However, for

13]. Hence, affine meshes is essential for the error analysis which utilize the interpolation estimates, but it limits us to work on structure mesh if we specifically choose on quadrilatural meshes.

$\mathcal{P}^k_c$

We now have a well-defined discrete global space $X_h = \mathcal{P}^k(\mathcal{T}_h)$ constructed a finite set of basis functions, $\{\phi_i\}_{i=1}^{N}$ associated witht the Lagriangian nodes $\{a_i\}_{i=1}^{N}$, where N is the total degrees of freedom. Let $u_j = u_h(N_j)$, so that $u_h = \sum_j^{N_h} u_j \phi_j$. Then the Problem 2.2 is equivalent to

global space

$\mathcal{P}^k(\mathcal{T}_\omega) =$

$\mathcal{P}^k_c(\mathcal{T}_\omega) :=$

$$\sum_{j=1}^{N_h} u_j a_h(\phi_j, \phi_i) = l(\phi_i), \quad \forall$$

Hence, by letting $\mathbf{u} = [u_j]$, $\mathbf{f} = [(f, \varphi_i)_\Omega]$ and $A = [a_h(\phi_j, \phi_i)]$ can we construct a linear system, $A\mathbf{u} = \mathbf{f}$. Ultimately, the matrix A is shown to be symmetric and positive definite only if $a_h(\cdot,\cdot)$ is well-posed.

To summarize the workflow of solving linear PDEs using the FEM method.

1) Strong formulation: Find $u \in X$ such that $\mathcal{A}u = f$.

2) Abstract linear problem: Find $u \in X$ such that $a(u, v) = l(v)$ $\forall v \in V$.

3) Discrete linear problem: Find $u \in X_h$ such that $a_h(u, v) = l(v)$ $\forall v \in V_h$.

4) Linear system of equations: Solve $A\mathbf{u} = \mathbf{f}$.

2.7 Cléments interpolation

Our goal is to to utilize interpolation estimates to compute convergence rates. An important tool in the process is the so-called Cléments interpolation operator, C_h. It is used for interpolation on non smooth functions by applying an regularization on so-called macroelements. However, we need to define affine operations on so-called macroelements before we can proceed with the error estimates.

A patch for a element $\omega(T)$ is denoted as the set of elements in $\mathcal{T}_h$ sharing at least one vertex with $T \in \mathcal{T}_h$. Similarly, a patch of a facet $\omega(F)$ is defined as the set of all elements in $\mathcal{T}_h$ sharing at least one vertex with $F \in \mathcal{F}_h$. For an illustrative example of patches in a two-dimensional triangular mesh, please refer to Figure 8.

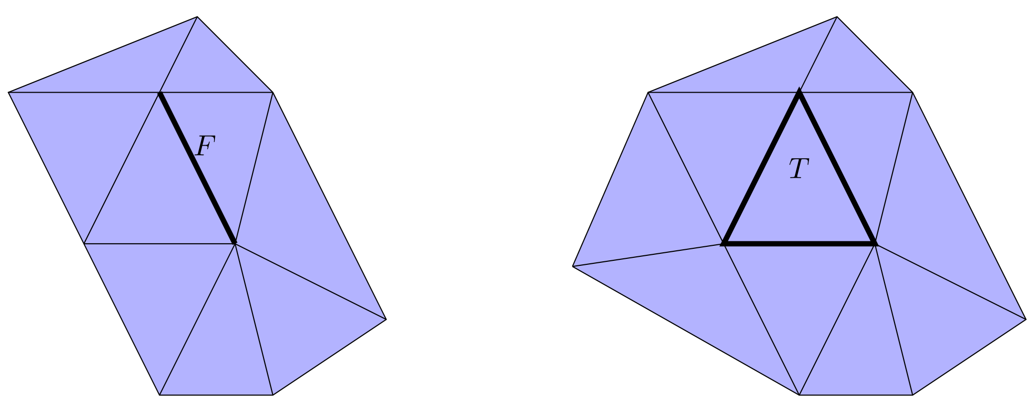


Figure 8: Illustration of the patch $\omega(F)$ on the left-hand side and $\omega(T)$ on the right-hand side.

Let the set $\{a_i\}_{i\in N}$ be all Lagrange nodes on the mesh $\mathcal{T}_h$. Associated with each node a_i we denote the macroelement A_i to consist of all elements containing a_i. Let n_{cf} be the number of configurations for the macroelement, then we define the index

$\alpha > 0 \qquad \alpha(x) \geq \alpha_0 > 0$

2) The Laplace formulation is

$$
\begin{aligned}
a_h^L(u,v) &= (\alpha u, v)_\Omega + (\Delta u, \Delta v)_\Omega \\
&\quad - \left(\{\!\{\Delta u\}\!\}, [\![\partial_n v]\!]\right)_{\mathcal{F}_h^{int}} - \left(\{\!\{\Delta v\}\!\}, [\![\partial_n u]\!]\right)_{\mathcal{F}_h^{int}} + \frac{\gamma}{h}\left([\![\partial_n u]\!], [\![\partial_n v]\!]\right)_{\mathcal{F}_h^{int}} \\
&\quad - (\Delta u, \partial_n v)_\Gamma - (\Delta v, \partial_n u)_\Gamma + \frac{\gamma}{h}(\partial_n u, \partial_n v)_\Gamma \\
l_h^L(v) &= (f,v)_\Omega - (g_2, v)_\Gamma - (g_1, \Delta v)_\Gamma + \frac{\gamma}{h}(g_1, \partial_n v)_\Gamma
\end{aligned}
\tag{14}
$$

With the corresponding energy norms

$$
\begin{aligned}
\|v\|_{a_h}^2 &\overset{\alpha}{=} \|v\|_\Omega^2 + \|\Delta v\|_\Omega^2 + \|h^{-\frac{1}{2}}[\![\partial_n v]\!]\|_{\mathcal{F}_h^{int}}^2 + \|h^{-\frac{1}{2}}\partial_n v\|_\Gamma^2, \quad v \in V_h \\
\|v\|_{a_h,*}^2 &= \|v\|_{a_h}^2 + \|h^{\frac{1}{2}}\{\!\{\partial_{nn} v\}\!\}\|_{\mathcal{F}_h^{int}}^2 + \|h^{\frac{1}{2}}\partial_{nn} v\|_\Gamma^2, \quad v \in V \oplus V_h.
\end{aligned}
\tag{15}
$$

Remark. It should be noted that the Hessian formulation has a substantial limitation in that it is only valid for homogeneous Neumann conditions. This constraint arises from the challenges associated with imposing g_1 via the tangential derivative terms in Equation (19) during the proof of Lemma 3.1. From a physical perspective, this is not problematic as it aligns with the boundary conditions of the original CH problem (3). However, from the standpoint of numerical validation, the homogeneous Neumann condition enforces strict rules on the design of manufactured solutions on arbitrary domains. Consequently, the examples illustrated in section 6 are only demonstrated on simple domains. This particular constraint does not apply to the Laplace formulation.

The Hessian formulation is well investigated by Susanne Brenner in several papers for [53, 54, 55] with a corresponding analysis and numerical validation. Similarly, variants of the Laplace formulation can be found here [61, 57]. In these article there is good also evidence that both formulation have the following expected a priori estimates. Let $u \in H^s(\Omega)$, and $u_h \in V_h$ of order k. Then with $r = \min\{s, k+1\}$ the a priori estimates are

$r-2$

$$
\begin{aligned}
\|u - u_h\|_{a_h,*} &\lesssim h^{r-1}\|u\|_{r,\Omega} \\
\|u - u_h\|_\Omega &\lesssim h^r \|u\|_{r,\Omega}
\end{aligned}
$$

3.2 Detailed Construction of Hessian and Laplacian Formulations

Out goal is to derive the Hessian formulation.

Lemma 3.1. *Assume the homogeneous Neumann conditions, $g_1(x) = 0$. Let $u \in H^4(\Omega)$ be the solution to (10). And let $V = \left\{v \in H^1(\Omega) \mid v\,|_T \in H^m(T)\ \forall T \in \mathcal{T}_h\right\}$. Then does the following identity hold.*

$$
\left(\Delta^2 u, v\right)_\Omega = \left(D^2 u, D^2 v\right)_\Omega + (g_2, v)_\Gamma - \left(\{\!\{\partial_{nn} u\}\!\}, [\![\partial_n v]\!]\right)_{\mathcal{F}_h^{int}} - (\partial_{nn} u, \partial_n v)_\Gamma \tag{16}
$$

Proof. We will start constructing a local theory for a triangle K and then extend it to the full mesh $\mathcal{T}_h$. Using Greens Theorem is it obvious that $\left(\Delta^2 u, v\right)_T = (\partial_n \Delta u, v)_{\partial T} - (\nabla(\Delta u), \nabla v)_T$. We can expand the second term in the following way.

$$
\begin{aligned}
(\nabla(\Delta u), \nabla v)_T &= \sum_{i=1}^d (\Delta \partial_{x_i} u, \partial_{x_i} v)_T = \sum_{i=1}^d (\nabla \cdot (\nabla \partial_{x_i} u), \partial_{x_i} v)_T \\
&= \sum_{i=1}^d (\partial_n \partial_{x_i} u, \nabla \partial_{x_i} v)_{\partial T} - (\nabla \partial_{x_i} u, \nabla \partial_{x_i} v)_T = (\partial_n \nabla u, \nabla v)_{\partial T} - (D^2 u, D^2 v)_T
\end{aligned}
$$