To "the" "inequality" only IV is a symmetry term. Just avoid symmetry here.

The strategy is to bound each term individually using Cauchy-Schwarz (2.4). From this is it easy to see that $|(\mathrm{I})| + |(\mathrm{II})| \lesssim \|v_h\|_{a_h}\|w_h\|_{a_h}$. For the terms ~~symmetrical terms~~ (III) and (IV) we apply ~~the inverse inequality (4.11)~~ "inequality (4.15) from Corollary 4.1 to see that"

$$|(\mathrm{III})| \lesssim \|h^{\frac{1}{2}}\partial_{nn}v_h\|_{\mathcal{F}_h\cap\Omega}\|h^{-\frac{1}{2}}[\![\partial_n w_h]\!]\|_{\mathcal{F}_h\cap\Omega} \lesssim \|v_h\|_{A_h}\|w\|_{a_h}. \tag{4.30}$$

~~Here we used that $\|h^{\frac{1}{2}}\partial_{nn}v_h\|_{\mathcal{F}_h\cap\Omega} \lesssim \|v_h\|_{A_h}$ thanks to Corollary 4.1~~. The interior penalty terms ~~can we easily see that~~, can be bounded by

$$|(\mathrm{V})| \lesssim \|h^{-\frac{1}{2}}[\![\partial_n v_h]\!]\|_{\mathcal{F}_h\cap\Omega}\|h^{-\frac{1}{2}}[\![\partial_n w_h]\!]\|_{\mathcal{F}_h\cap\Omega} \lesssim \|v_h\|_{a_h}\|w_h\|_{a_h}. \tag{4.31}$$

"The ~~It~~ remaining ~~to handle the symmetry~~ terms (VI) and (VII) can again be handled using Corollary 4.1, leading to"

$$|(\mathrm{VI})| \lesssim \|h^{\frac{1}{2}}\partial_{nn}v_h\|_{\Gamma}\|h^{-\frac{1}{2}}\partial_n w_h\|_{\Gamma} \lesssim \|v_h\|_{A_h}\|w_h\|_{a_h} \tag{4.32}$$

~~Again, here we used the Corollary 4.1~~. Finally, using the definition of the norm is it easily to see that,

$$|(\mathrm{VIII})| \lesssim \|\partial_n v_h\|_{\Gamma}\|\partial_n w_h\|_{\Gamma} \lesssim \|v_h\|_{a_h}\|w_h\|_{a_h}.$$

Hence, we can conclude

$$|a_h(v_h, w_h)| \leqslant \|v_h\|_{a_h}\|w_h\|_{a_h} \forall v_h, w_h \in V_h. \tag{4.33}$$

Therefore, since $\|\cdot\|_{a_h} \lesssim \|\cdot\|_{A_h}$, it has been demonstrated that $a_h(\cdot,\cdot)$ is bounded within the $\|\cdot\|_{Ah}$ norm.

In boldface: Estimate (4.26)

~~**Step 2.** The goal is to prove (4.26)~~. Let $v \in V_h \oplus V$ and $w_h \in V_h$. The only difference is that since v is ~~continious~~ we cannot apply to Corollary 4.1 on the estimates (4.30) and (4.32). However, this is not a problem since $\|h^{\frac{1}{2}}\{\!\{\partial_{nn}v\}\!\}\|_{\mathcal{F}_h\cap\Omega}$ and $\|h^{\frac{1}{2}}\partial_{nn}v\|_{\Gamma}$ are terms in the norm $\|v\|_{a_h,*}$. Thus, we know that

$$|a_h(v, w_h)| \leqslant \|v\|_{a_h,*}\|w_h\|_{A_h} \quad \forall v \in V_h \oplus V \text{ and } w_h \in V_h \tag{4.34}$$

□

v can have a contribution from V where no inverse estimate can be used to bound $\{\partial_{nn} v\}$.

4.4 A priori error estimate

For the proposed method, we want to derive a priori error estimate with respect to both the $\|\cdot\|_{a_h,*}$-norm and the $\|\cdot\|_{\Omega}$-norm. We will construct a suitable (quasi-)interpolation operator, here we use the Clement quasi-interpolation operator which in contrast to the standard Lagrange nodal interpolation iterator is also defined for low regularity function in $u \in L^2(\Omega)$. In combination with discrete coercivity this allows you to derive an a priori error estimate in the energy norm. Finally, we use a standard duality argument, i.e. Aubin-Nitsche trick, to derive the L^2 error estimate.

Recall that for $v \in H^1(\mathcal{T}_h)$ these inequalities holds $\forall T \in \mathcal{T}_h$ ~~such that~~ "also known as"

$$\|v\|_{\partial T} \lesssim h_T^{-\frac{1}{2}}\|v\|_T + h^{\frac{1}{2}}\|\nabla v\|_T,$$
$$\|v\|_{\Gamma\cap T} \lesssim h^{-\frac{1}{2}}\|v\|_T + h_T^{\frac{1}{2}}\|\nabla v\|_T,$$

equation numbers?

for proof see [92, Lemma 4.2].

Assume that Ω has a boundary Γ in C^1, then there ~~it does~~ exists a bounded extension operator,

$$(\cdot)^e : H^m(\Omega) \to H^m(\mathbb{R}^d),$$

for all $v \in H^m(\Omega)$ which satisfies

$$v^e|_{\Omega} = v,$$
$$\|v^e\|_{m,\mathbb{R}^d} \lesssim \|v\|_{m,\Omega}. \tag{4.35}$$

write explicitly $V_h = \mathbb{P}_c^k(\mathcal{T}_h)$ have to remind the reader on the "the notation" polynomial order k.

For more information, see [93, Theorem 9.7] and [94, p.181, p.185]. To simplify, we ~~use the notation~~ simply write $v := v^e$ for $v \in \mathbb{R}^d \setminus \Omega$.

Starting from ~~Recall the~~ Lemma 2.8, we ~~construct~~ combine the Clément's interpolator ~~combined~~ with the extension operator ~~such that~~ "to construct" $C_h^e : H^m(\mathbb{R}^d) \to \boxed{V_h}$ such that $C_h^e v := C_h v^e$. We can immediately observe that the interpolation satisfies the following global error estimates. Let $v \in H^s(\Omega)$ and set $r = \min(s, k+1)$, ~~that is,~~ then

$$\|v - C_h^e v\|_{l,\mathcal{T}_h} \lesssim h^{r-l} \sum_{T\in\mathcal{T}_h} \|v\|_{r,\omega(T)}, \quad 0 \leqslant l \leqslant r, \tag{4.36}$$

$$\|v - C_h^e v\|_{l,\mathcal{F}_h} \lesssim h^{r-l-\frac{1}{2}} \sum_{T\in\mathcal{T}_h} \|v\|_{r,\omega(F)}, \quad 0 \leqslant l \leqslant r - \frac{1}{2}, \tag{4.37}$$

$$\|v - C_h^e v\|_{l,\Gamma} \lesssim h^{r-l-\frac{1}{2}} \sum_{T\in\mathcal{T}_h} \|v\|_{r,\omega(T)}, \quad 0 \leqslant l \leqslant r - \frac{1}{2}. \tag{4.38}$$

Next, we recall ~~and arguing~~ that $\sum_T \|v\|_{s,\omega(T)} \leqslant C\|v\|_{s,\mathcal{T}_h}$ where C is some constant ~~decided by the maximum number of elements in a patch $\omega(T)$ for all $T \in \mathcal{T}_h$~~ depending on shape regularity of mesh. This also holds for the inequality $\sum_T \|v\|_{s,\omega(F)} \leqslant C\|v\|_{s,\mathcal{T}_h}$. Hence, we have $\boxed{\text{perhaps an even more useful}}$ set of inequalities.

(It is not the number of elements in one patch, but the maximal number of different patches, a single element can belong to.)

does not sound scientific, either it is more useful (to you?) or not.

$$\|v - C_h^e v\|_{l,\mathcal{T}_h} \lesssim h^{r-l}\|v\|_{r,\mathcal{T}_h}, \quad 0 \leqslant l \leqslant r, \tag{4.39}$$

$$\|v - C_h^e v\|_{l,\mathcal{F}_h} \lesssim h^{r-l-\frac{1}{2}}\|v\|_{r,\mathcal{T}_h}, \quad 0 \leqslant l \leqslant r - \frac{1}{2}. \tag{4.40}$$

$$\|v - C_h^e v\|_{l,\Gamma} \lesssim h^{r-l-\frac{1}{2}}\|v\|_{r,\mathcal{T}_h}, \quad 0 \leqslant l \leqslant r - \frac{1}{2}. \tag{4.41}$$

entire sentence sounds weird. Just state the these interpolation estimates are crucial in deriving the final a priori error estimate.

Naturally can we see this is the tools we need to construct an estimate for the energy norm.

Lemma 4.4. *Let $u \in H^s(\Omega)$ for $s \geq 3$ be a exact solution to (3.4) and let k be the polynomial order of V_h. Then we have*

$$\|u - C_h u\|_{a_h,*} \lesssim h^{r-2}\|u\|_{r,\Omega}. \quad \text{Set } r = \min(s, k+1) \tag{4.42}$$

Proof. By definition ~~is~~

$$\begin{aligned}\|u - C_h^e u\|_{a_h,*}^2 &= \alpha\|(u - C_h^e u)\|_{\mathcal{T}_h\cap\Omega}^2 + \|D^2(u - C_h^e u)\|_{\mathcal{T}_h\cap\Omega}^2 \\ &\quad + \gamma\|h^{-\frac{1}{2}} [\![\partial_n(u - C_h^e u)]\!] \|_{\mathcal{F}_h\cap\Omega}^2 + \gamma\|h^{-\frac{1}{2}}\partial_n(u - C_h^e u)\|_\Gamma^2 \\ &\quad + \|h^{\frac{1}{2}} \{\!\{\partial_{nn}(u - C_h^e u)\}\!\} \|_{\mathcal{F}_h\cap\Omega}^2 + \|h^{\frac{1}{2}}\partial_{nn}(u - C_h^e u)\|_\Gamma^2.\end{aligned} \tag{4.43}$$

The strategy is to bound each term individually. Starting with the first two terms, we get

no parenthesis needed; just write $2r$

$$\begin{aligned}\alpha\|(u - C_h^e u)\|_{\mathcal{T}_h\cap\Omega}^2 &\lesssim \|(u - C_h^e u)\|_{0,\mathcal{T}_h}^2 \\ &\lesssim h^{2(r-0)}\|u\|_{r,\mathcal{T}_h}^2 \cancel{\lesssim h^{2(r-2)}\|u\|_{r,\mathcal{T}_h}^2} \\ \|D^2(u - C_h^e u)\|_{\mathcal{T}_h\cap\Omega}^2 &\lesssim \cancel{\|u - C_h^e u\|_{2,\mathcal{T}_h}^2 = \|u - C_h^e u\|_{2,\mathcal{T}_h}^2} \\ &\lesssim h^{2(r-2)}\|u\|_{r,\mathcal{T}_h}^2.\end{aligned} \tag{4.44}$$

follows immediately from (4.39)

when you collect all individual estimates you can state the h^{r-2} is the dominant one for $h \to 0$.

Here we simply used (4.39). Recall the first order inverse estimate (4.13) and that $\| [\![\partial_n u]\!] \|_{\mathcal{F}_h} \lesssim \|\partial_{n^+}u^+\|_{\mathcal{F}_h} + \|\partial_{n^-}u^-\|_{\mathcal{F}_h} \lesssim \|\partial_n u\|_{\partial\mathcal{T}_h}^2$, hence, this implies $\| [\![\partial_n u]\!] \|_{\mathcal{F}_h\cap\Omega}^2 \lesssim h^{-1}\|\nabla u\|_{\mathcal{T}_h}^2$. Using this and the inequality (4.39) we therefore can observe, Wow no? :) (1)

$$\begin{aligned}\gamma\|h^{-\frac{1}{2}} [\![\partial_n(u - C_h^e u)]\!] \|_{\mathcal{F}_h\cap\Omega}^2 \lesssim h^{-1}\|\partial_n(u - C_h^e u)\|_{\partial\mathcal{T}_h}^2 \lesssim h^{-2}\|\nabla(u - C_h^e u)\|_{\mathcal{T}_h\cap\Omega}^2 \\ = h^{-2}\|u - C_h^e u\|_{1,\mathcal{T}_h}^2 \lesssim h^{-2}h^{2(r-1)}\|u\|_{r,\mathcal{T}_h}^2 \\ \lesssim h^{2(r-2)}\|u\|_{r,\mathcal{T}_h}^2.\end{aligned} \tag{4.45}$$

① That is why I always emphasize that inverse estimate only hold for discrete functions!
But $w - C_{h}^{e} w$ has a non-discrete function component namely w!
So you cannot use inverse estimates here, instead you have to use the trace inequalities right before (4.35) (missing eq number):

$$\| h_w^{-\frac{1}{2}} [\partial_n (w - C_{h}^{e} w)] \|^2_{\mathcal{F}_w \cap \Omega} \lesssim \| h_w^{-\frac{1}{2}} \partial_n (w - C_{h}^{e} w) \|^2_{\partial \mathcal{T}_w}$$

$$\lesssim h_w^{-2} \left(\| \nabla (w - C_{h}^{e} w) \|^2_{\mathcal{T}_w} + \| D^2 (w - C_{h}^{e} w) \|^2_{\mathcal{T}_w} \right)$$

(trace inequality applied to ∇w instead of w.)

$$\lesssim \underbrace{\left(h_w^{2(r-1)-2} + h_w^{2(r-2)} \right)}_{h_w^{2(r-2)}} \| w \|^2_{r,\Omega}$$

$$\lesssim h_w^{2(r-2)} \| w \|^2_{r,\Omega}.$$

no, do same correction as for previous step

And for the boundary term ~~we do a similar procedure~~, but instead use the first order inverse estimate (4.12).

$$\gamma\|h^{-\frac{1}{2}}\partial_n(u - C_h^e u)\|_\Gamma^2 \lesssim h^{-2}\|\nabla(u - C_h^e u)\|_{\mathcal{T}_h}^2 \lesssim h^{-2}\|u - C_h^e u\|_{1,2,\mathcal{T}_h}^2 \\ \lesssim h^{-2}h^{2(r-1)}\|u\|_{r,\mathcal{T}_h}^2 \lesssim h^{2(r-2)}\|u\|_{r,\mathcal{T}_h}^2 \tag{4.46}$$

Recall that $\| \{\!\{u\}\!\} \|_{\mathcal{F}_h} \leqslant \|u^+\|_{\mathcal{F}_h} + \|u^-\|_{\mathcal{F}_h} \lesssim \|u\|_{\partial\mathcal{T}_h}^2$ and the second order inverse inequality (4.11). It is clear by using (4.39) that this holds.

Here as well?

$$\|h^{\frac{1}{2}} \{\!\{\partial_{nn}(u - C_h^e u)\}\!\} \|_{\mathcal{F}_h\cap\Omega}^2 \lesssim h\|\partial_{nn}(u - C_h^e u)\|_{\partial\mathcal{T}_h\cap\Omega}^2 \lesssim hh^{-1}\|D^2(u - C_h^e u)\|_{\mathcal{T}_h}^2 \\ = \|u - C_h^e u\|_{2,\mathcal{T}_h}^2 \lesssim h^{2(r-2)}\|u\|_{r,\mathcal{T}_h} \tag{4.47}$$

Similarly we can easily see by using (4.39) and the second order boundary inverse inequality (4.10) that this must hold,

Here as well? Then should use of inverse estimate in this proof.

$$\|h^{\frac{1}{2}}\partial_{nn}(u - C_h^e u)\|_\Gamma^2 \lesssim \|D^2(u - C_h^e u)\|_{\mathcal{T}_h}^2 \lesssim \|u - C_h^e u\|_{2,\mathcal{T}_h}^2 \lesssim h^{2(r-2)}\|u\|_{r,\mathcal{T}_h}^2 \tag{4.48}$$

Thus, all elements is bounded by $h^{2(r-2)}\|u\|_{r,\mathcal{T}_h}^2$ and the proof is complete. □

Lemma 4.5 (Weak galerkin orthogonality). *Let $u \in H^s(\Omega)$, $s \geq 3$ be the exact solution to (3.4) and $u_h \in V_h$ is a discrete solution to (4.2). Then is*

$$a_h(u - u_h, v_h) = g_h(u_h, v_h) \quad \forall v_h \in V_h.$$

Proof. From the definition of the problem (4.2) and utilizing that for $u \in H^s(\Omega)$ we have the identity $A_h(u, v_h) = a_h(u, v_h) = l(v_h)\forall v_h \in V_h$. Consequently we have the following sequence of equalities,

space

$$l(v_h) = A_h(u_h, v_h) = a_h(u, v_h) = a_h(u_h, v_h) + g_h(u_h, v_h) \quad u_h, v_h \in V_h.$$

From this, it is clear that $a_h(u - u_h, v_h) = g_h(u_h, v_h)$. □

EP2 is enough

Assumption ~~4.6~~ (EP2). *For $v \in H^s(\Omega)$ and $r = \min\{s, k+1\}$, the semi-norm $|\cdot|_{g_h}$ ~~satisfies the following estimate~~,*

"is weakly consistent in the sense that" $|C_h^e v|_{g_h} \lesssim h^{r-2}\|v\|_{r,\Omega}$.

Theorem 4.7. *Let $u \in H^s(\Omega)$, $s \geq 3$ be a solution to (3.4) and let $u_h \in V_h$ of order $k \geq 2$ be the discrete solution to (4.2). Then ~~for~~ with $r = \min\{s, k+1\}$, the error $e = u - u_h$ satisfies*

$$\|e\|_{a_h,*} \lesssim h^{r-2}\|u\|_{r,\Omega}, \tag{4.49}$$

$$\|e\|_\Omega \lesssim h^{r-\max\{0,3-k\}}\|u\|_{r,\Omega}. \tag{4.50}$$

Remark. Be aware that for $k = 2$ ~~is~~ the estimate (4.50) suboptimal. with 1 order.

Proof. We will divide the proof into two steps.

Step 1. We want to prove that $\|e\|_{a_h,*} \lesssim h^{r-2}\|u\|_{r,\Omega}$. ~~Let~~ Decompose $e = u - u_h$ ~~consist of~~ into $e = e_h + e_\pi$, where we denote the discrete error $e_h = C_h^e u - u_h$ and the interpolation error $e_\pi = u - C_h^e u$. We can then observe that

$$\begin{aligned} \|u - u_h\|_{a_h} &\leq \|u - C_h^e u + C_h^e u - u_h\|_{a_h,*} \\ &\leqslant \|u - C_h^e u\|_{a_h,*} + \|C_h^e u - u_h\|_{a_h,*} \\ &\lesssim \|e_\pi\|_{a_h,*} + \|e_h\|_{A_h} \end{aligned}$$

(because of inverse estimate)

Using Lemma 4.4, is it clear that $\|e_\pi\|_{a_h,*} \lesssim h^{r-2}\|u\|_{r,\Omega}$ is already fulfilled, hence, it remains to estimate ~~check~~ e_h. ~~From~~ Combining Lemma 4.2, 4.3, and the weak Galerkin orthogonality and Assumption 4.6 ~~is it natural~~ we ~~to~~ arrive at,

$$
\begin{aligned}
\|e_h\|_{A_h}^2 &\lesssim a_h(C_h^e u - u_h, e_h) + g_h(C_h^e u - u_h, e_h) \\
&= a_h(C_h^e u - u, e_h) + a_h(u - u_h, e_h) + g_h(C_h^e u - u_h, e_h) \\
&= a_h(C_h^e u - u, e_h) + g_h(C_h^e u, e_h)
\end{aligned}
\tag{4.51}
$$

Hence, now utilizing the Assumption ~~4.6~~ EQ is it clear that

$$
\begin{aligned}
a_h(C_h^e u - u, e_h) + g_h(C_h^e u, e_h) &\lesssim \|C_h^e u - u\|_{a_h,*}\|e_h\|_{a_h} + |C_h^e u|_{g_h}\,|e_h|_{g_h} \\
&\lesssim \|C_h^e u - u\|_{a_h,*}\|e_h\|_{a_h} + h^{r-2}\|e_h\|_{r,\Omega}\,|e_h|_{g_h} \\
&\lesssim (\|C_h^e u - u\|_{a_h,*} + h^{r-2}\|e_h\|_{r,\Omega})\|e_h\|_{A_h} \\
&\lesssim h^{r-2}\|u\|_{r,\Omega}\|e_h\|_{A_h}.
\end{aligned}
\tag{4.52}
$$

Here we noticed that $\|e_h\|_{a_h} + |e_h|_{g_h} \lesssim \|e_h\|_{A_h}$, ~~We also argued~~ and used that $\|C_h^e u - u\|_{a_h,*} \lesssim h^{r-2}\|u\|_{r,\Omega}$ from Lemma 4.4.

Finally, combining (4.51) and (4.52) is it clear that $\|e_h\|_{A_h} \lesssim h^{r-2}\|u\|_{r,\Omega}$. Hence, the first part of the proof is complete.

Step 2. We want to show that $\|e\|_\Omega \lesssim h^{r-\max(0,3-k)}\|u\|_{r,\Omega}$. The idea is to apply the so-called Aubin-Nitsche duality trick while being aware of the ghost penalty g_h. Let us denote the following observation. Assume that $e := u - u_h \in L^2(\Omega)$ and $\psi \in H^4(\Omega)$. Let the corresponding dual problem to (3.1) be

$$
\begin{aligned}
\Delta^2\psi &= e \quad \text{in } \Omega \\
\partial_n\psi &= 0 \quad \text{on } \Gamma \\
\partial_n\Delta\psi &= 0 \quad \text{on } \Gamma
\end{aligned}
\tag{4.53}
$$

This implies that it exists a $\psi \in H^4(\Omega)$ such that $a_h(\psi, v) = (e, v)_\Omega\ \forall v \in V_h$. Hence, we can easily observe that

$$
\begin{aligned}
\|e\|_\Omega^2 &= (e, e)_\Omega = (e, \Delta^2\psi)_\Omega \\
&= a_h(e, \psi) = a_h(u - u_h, \psi) \\
&= a_h(u - u_h, \psi + C_h^e\psi - C_h^e\psi) \\
&= a_h(u - u_h, \psi - C_h^e\psi) + a_h(u - u_h, C_h^e\psi) \\
&= a_h(u - u_h, \psi - C_h^e\psi) \\
&\lesssim \|u - u_h\|_{a_h,*}\|\psi - C_h^e\psi\|_{a_h,*}
\end{aligned}
\tag{4.54}
$$

Here we applied the Galerkin orthogonality $a_h(u - u_h, C_h^e\psi) = 0$. Using the a priori estimate (4.4) is it clear that

$$
\|u - u_h\|_{a_h,*} \leqslant h^r\|u\|_{r,\Omega} \quad \text{and} \quad \|\psi - C_h^e\psi\|_{a_h,*} \leqslant h^r\|\psi\|_{4,\Omega}. \tag{4.55}
$$

And then standard inverse estimate (2.13) can we see

$$
h^r\|u\|_{r,\Omega} \leqslant h^{r-2}\|u\|_\Omega \quad \text{and} \quad h^{\widetilde{r}-2}\|\psi\|_{4,\Omega} \leqslant h^{\widetilde{r}-2}\|\psi\|_\Omega. \tag{4.56}
$$

Here is $r = \max(3, k+1)$ and $\widetilde{r} = \max(4, k+1)$. Combining (4.54), (4.55) and (4.56) we have,

$$
\|e\|_\Omega^2 \lesssim h^{r-2}\|u\|_\Omega\|\psi\|_\Omega. \tag{4.57}
$$

Using that $\|\psi\|_\Omega \leqslant \|e\|_\Omega$ is it easy to see that

$$
\|e\|_\Omega \lesssim h^{r-\max(0,k-3)}\|u\|_{r,\Omega} \tag{4.58}
$$

$\square$