Proof. **Estimate** (4.27)**.** We see that $|A_h(v_h, w_h)| \lesssim |a_h(v_h, w_h)| + |g_h(v_h, w_h)|$. By assumption the ghost penalty $g_h(\cdot,\cdot)$ is positive semi-definite, thus fulfills the Cauchy-Schwarz inequality,

$$|g_h(v_h, w_h)| \lesssim |v_h|_{g_h} |w_h|_{g_h}. \tag{4.29}$$

Hence, $|g_h(v_h, w_h)| \lesssim \|v_h\|_{A_h} \|w_h\|_{A_h}$by definition of $A_h(\cdot,\cdot)$. It remains to show that the bilinear term $a_h(\cdot,\cdot)$ is bounded. We numerate the terms in this fashion.

$$\begin{aligned}
a_h(v_h, w_h) \leqslant\ & \overbrace{(\alpha v_h, w_h)_{\mathcal{T}_h \cap \Omega}}^{\mathrm{I}} + \overbrace{(D^2 v_h, D^2 w_h)_{\mathcal{T}_h \cap \Omega}}^{\mathrm{II}} \quad \text{— see my comments on p. 31.} \\
& + \overbrace{(\{\!\{\partial_{nn} v_h\}\!\}, [\![\partial_n w_h]\!])_{\mathcal{F}_h \cap \Omega}}^{\mathrm{III}} + \overbrace{(\{\!\{\partial_{nn} w_h\}\!\}, [\![\partial_n v_h]\!])_{\mathcal{F}_h \cap \Omega}}^{\mathrm{IV}} + \overbrace{\frac{\gamma}{h} ([\![\partial_n v_h]\!], [\![\partial_n w_h]\!])_{\mathcal{F}_h \cap \Omega}}^{\mathrm{V}} \\
& + \overbrace{(\partial_{nn} v_h, \partial_n w_h)_\Gamma}^{\mathrm{VI}} + \overbrace{(\partial_{nn} w_h, \partial_n v_h)_\Gamma}^{\mathrm{VII}} + \overbrace{\frac{\gamma}{h} (\partial_n v_h, \partial_n w_h)_\Gamma}^{\mathrm{VIII}} \\
& = \mathrm{I} + \ldots + \mathrm{VIII}
\end{aligned} \tag{4.30}$$

The strategy is to bound each term individually using the Cauchy-Schwarz inequality (2.4). From this is it easy to see that $|\mathrm{I}| + |(\mathrm{II})| \lesssim \|v_h\|_{a_h} \|w_h\|_{a_h}$. To the terms III and IV we apply the inequality (4.17) from the Corollary 4.1 to see that,

$$|\mathrm{III}| \lesssim \|h^{\frac{1}{2}} \partial_{nn} v_h\|_{\mathcal{F}_h \cap \Omega} \|h^{-\frac{1}{2}} [\![\partial_n w_h]\!]\|_{\mathcal{F}_h \cap \Omega} \lesssim \|v_h\|_{A_h} \|w\|_{a_h}. \tag{4.31}$$

The interior penalty can we easily see that,

$$|\mathrm{V}| \lesssim \|h^{-\frac{1}{2}} [\![\partial_n v_h]\!]\|_{\mathcal{F}_h \cap \Omega} \|h^{-\frac{1}{2}} [\![\partial_n w_h]\!]\|_{\mathcal{F}_h \cap \Omega} \lesssim \|v_h\|_{a_h} \|w_h\|_{a_h}. \tag{4.32}$$

The remaining terms terms VI and VII can again be handles by Corollary 4.1, leading to

$$|\mathrm{VI}| \lesssim \|h^{\frac{1}{2}} \partial_{nn} v_h\|_\Gamma \|h^{-\frac{1}{2}} \partial_n w_h\|_\Gamma \lesssim \|v_h\|_{A_h} \|w_h\|_{a_h} \tag{4.33}$$

Finally, using the definition of the norm it is easy to see that

$$|\mathrm{VIII}| \lesssim \|\partial_n v_h\|_\Gamma \|\partial_n w_h\|_\Gamma \lesssim \|v_h\|_{a_h} \|w_h\|_{a_h}.$$

Hence, we can conclude

$$|a_h(v_h, w_h)| \leqslant \|v_h\|_{a_h} \|w_h\|_{a_h} \forall v_h, w_h \in V_h. \tag{4.34}$$

Therefore, since $\|\cdot\|_{a_h} \lesssim \|\cdot\|_{A_h}$, it has been demonstrated that $a_h(\cdot,\cdot)$ is bounded within the $\|\cdot\|_{Ah}$ norm.

Estimate (4.28) **.** Let $v \in V_h \oplus V$ and $w_h \in V_h$. The only difference is that since v can have a contribution from V where no inverse estimate can be used to bound $\{\!\{\partial_{nn} v\}\!\}$, hence, we cannot apply to Corollary 4.1 on the estimates (4.31) and (4.33). However, this is not a problem since $\|h^{\frac{1}{2}} \{\!\{\partial_{nn} v\}\!\}\|_{\mathcal{F}_h \cap \Omega}$ and $\|h^{\frac{1}{2}} \partial_{nn} v\|_\Gamma$ are terms in the norm $\|v\|_{a_h,*}$. Thus, we know that

$$|a_h(v, w_h)| \leqslant \|v\|_{a_h,*} \|w_h\|_{A_h} \quad \forall v \in V_h \oplus V \text{ and } w_h \in V_h \tag{4.35}$$

□

Mention main point: estimates are geometrically robust, thanks to the ghost penalty i.e. they are not affected by the particular cut configuration | how the boundary cut the background mesh.

4.4 A priori error estimate

Nice introductionary paragraph!

For the proposed method, we want to derive a priori error estimate with respect to both the $\|\cdot\|_{a_h,*}$-norm and the $\|\cdot\|_{\Omega}$-norm. First, we ~~We will~~ construct a suitable (quasi-)interpolation operator, here we use the Clement quasi interpolation operator which in contrast to the standard Lagrange nodal interpolation ~~iterator~~ operator is also defined for low regularity function $u \in L^2(\Omega)$. In combination with discrete coercivity this allows ~~you~~ we to derive an a priori error estimate in the energy norm. Finally, we use a standard duality argument, also known as Aubin-Nitsche trick, to derive the $L^2(\Omega)$-error estimate.

Recall that for $v \in H^1(\mathcal{T}_h)$ the inequalities,

$$\|\nabla v\|_{\partial T} \lesssim h_T^{-\frac{1}{2}} \|\nabla v\|_T + h^{\frac{1}{2}} \|D^2 v\|_T, \tag{4.36}$$

$$\|\nabla v\|_{\Gamma \cap T} \lesssim h^{-\frac{1}{2}} \|\nabla v\|_T + h_T^{\frac{1}{2}} \|D^2 v\|_T, \tag{4.37}$$

holds $\forall T \in \mathcal{T}_h$, for proof see [93, Lemma 4.2].

Assume that Ω has a boundary Γ in C^1, then there exists a bounded extension operator,

$$(\cdot)^e : H^m(\Omega) \to H^m(\mathbb{R}^d), \tag{4.38}$$

for all $v \in H^m(\Omega)$ which satisfies

$$\begin{aligned} v^e|_{\Omega} &= v, \\ \|v^e\|_{m,\mathbb{R}^d} &\lesssim \|v\|_{m,\Omega}. \end{aligned} \tag{4.39}$$

For more information, see [94, Theorem 9.7] and [95, p.181, p.185]. For the notation we simply write $v := v^e$ for $v \in \mathbb{R}^d \setminus \Omega$.

Starting from Lemma 2.8, assume $v \in H^s(\Omega)$ and let $r = \min(s, k+1)$. Revisit the definition of V_h from (4.2), which is a polynomial of degree k. We can then employ the combination of the Clément interpolator with the extension operator to create $C_h^e : H^m(\mathbb{R}^d) \to V_h$, such that $C_h^e v := C_h v^e$. Next, recall that $\sum_T \|v\|_{s,\omega(T)} \leqslant C\|v\|_{s,\mathcal{T}_h}$ where C is some constant decided by shape regularity of the mesh and the maximal number of different patches a single element can belong to. This also holds for the inequality $\sum_T \|v\|_{s,\omega(F)} \leqslant C\|v\|_{s,\mathcal{T}_h}$. The following estimates are thereby established.

Right, These two follow directly from (2.17)

$$\|v - C_h^e v\|_{l,\mathcal{T}_h} \lesssim h^{r-l} \sum_{T\in\mathcal{T}_h} \|v\|_{r,\omega(T)} \lesssim h^{r-l-\frac{1}{2}} \|v\|_{r,\Omega}, \quad 0 \leqslant l \leqslant r, \tag{4.40}$$

$$\|v - C_h^e v\|_{l,\mathcal{F}_h} \lesssim h^{r-l-\frac{1}{2}} \sum_{T\in\mathcal{T}_h} \|v\|_{r,\omega(F)} \lesssim h^{r-l-\frac{1}{2}} \|v\|_{r,\Omega}, \quad 0 \leqslant l \leqslant r - \frac{1}{2}, \tag{4.41}$$

This on the other hands requires a proof.

$$\|v - C_h^e v\|_{l,\Gamma} \lesssim h^{r-l-\frac{1}{2}} \sum_{T\in\mathcal{T}_h} \|v\|_{r,\omega(T)} \lesssim h^{r-l-\frac{1}{2}} \lesssim \|v\|_{r,\Omega}, \quad 0 \leqslant l \leqslant r - \frac{1}{2}. \tag{4.42}$$

the (there is only one)

Lemma 4.4. *Let $u \in H^s(\Omega)$ for $s \geqslant 3$ be a exact solution to (3.4) and let k be the polynomial order of V_h. Set $r = \min(s, k+1)$, then we have the ~~final a priori estimates~~* interpolation estimates

$$\|u - C_h u\|_{a_h,*} \lesssim h^{r-2} \|u\|_{r,\Omega}. \tag{4.43}$$

But don't state (4.42) explicitly, rather provide the proof as part of the proof of Lemma 4.4.

Don't use overbrace here, I only used them in last document for myself, but the numbering I is already sufficient.

Proof. By definition,

as you did down there.

$$
\begin{aligned}
\|u - C_h^e u\|_{a_h,*}^2 = \alpha\, & \overbrace{\|(u - C_h^e u)\|_{\mathcal{T}_h \cap \Omega}^2}^{\mathrm{I}} + \overbrace{\|D^2(u - C_h^e u)\|_{\mathcal{T}_h \cap \Omega}^2}^{\mathrm{II}} \\
& + \overbrace{\gamma \|h^{-\frac{1}{2}} [\![\partial_n (u - C_h^e u)]\!] \|_{\mathcal{F}_h \cap \Omega}^2}^{\mathrm{III}} + \overbrace{\gamma \|h^{-\frac{1}{2}} \partial_n (u - C_h^e u)\|_{\Gamma}^2}^{\mathrm{IV}} \\
& + \overbrace{\|h^{\frac{1}{2}} \{\!\{\partial_{nn}(u - C_h^e u)\}\!\} \|_{\mathcal{F}_h \cap \Omega}^2}^{\mathrm{V}} + \overbrace{\|h^{\frac{1}{2}} \partial_{nn}(u - C_h^e u)\|_{\Gamma}^2}^{\mathrm{VI}} \\
= {} & \mathrm{I} + \ldots + \mathrm{VI}.
\end{aligned}
\tag{4.44}
$$

The strategy is to bound each term individually. By initially focusing on the first two terms and employing equation (4.40), we can easily observe

$$
\begin{aligned}
\mathrm{I} + \mathrm{II} &\lesssim \|u - C_h^e u\|_{\mathcal{T}_h}^2 + \|D^2(u - C_h^e u)\|_{\mathcal{T}_h}^2 \\
&\lesssim (h^{2r} + h^{2(r-2)})\|u\|_{r,\mathcal{T}_h}^2 \lesssim h^{2(r-2)}\|u\|_{r,\mathcal{T}_h}^2.
\end{aligned}
\tag{4.45}
$$

From (2.5) is it clear that $\| [\![\partial_n u]\!] \|_{\mathcal{F}_h} \lesssim \|\nabla u\|_{\partial \mathcal{T}_h}$. Hence, first applying the trace inequality (4.36) and then (4.40) is it clear that,

$$
\begin{aligned}
\mathrm{III} &\lesssim h^{-1}\|\nabla(u - C_h^e)\|_{\partial\mathcal{T}_h}^2 \lesssim h^{-2}\|\nabla(u - C_h^e u)\|_{\mathcal{T}_h}^2 + \|D^2(u - C_h^e)\|_{\mathcal{T}_h}^2 \\
&\lesssim (h^{2(r-1)-2} + h^{2(r-2)})\|u\|_{r,\mathcal{T}_h}^2 \lesssim h^{2(r-2)}\|u\|_{r,\mathcal{T}_h}^2
\end{aligned}
\tag{4.46}
$$

And for the boundary term we apply estimate (4.42)

$$
\mathrm{IV} \lesssim h^{-1}\|\nabla(u - C_h^e u)\|_{\Gamma}^2 \lesssim h^{2(r-2)}\|u\|_{r,\mathcal{T}_h}
\tag{4.47}
$$

Good, use Version 2, that provides a proof for (4.42)

Version 2. And for the boundary term we apply (4.37) and then (4.40)

$$
\begin{aligned}
\mathrm{IV} &\lesssim h^{-1}\|\nabla(u - C_h^e u)\|_{\Gamma}^2 \lesssim h^{-2}\|\nabla(u - C_h^e u)\|_{\mathcal{T}_h}^2 + \|D^2(u - C_h^e u)\|_{\mathcal{T}_h}^2 \\
&\lesssim h^{2(r-2)}\|u\|_{r,\mathcal{T}_h}^2
\end{aligned}
\tag{4.48}
$$

Again, from (2.5) is it clear that $\| \{\!\{\partial_{nn} u\}\!\} \|_{\mathcal{F}_h} \lesssim \|D^2 u\|_{\partial\mathcal{T}_h}$, thus we see that,

$$
\mathrm{V} \lesssim h\|D^2(u - C_h^e u)\|_{\partial\mathcal{T}_h}^2 \lesssim h^{2(r-2)}\|u\|_{r,\mathcal{T}_h}^2.
\tag{4.49}
$$

The final term we use the estimate (4.42),

Provide short proof similar to (Version 2)

$$
\boxed{\mathrm{VI} \lesssim h\|D^2(u - C_h^e u)\|_{\Gamma}^2 \lesssim h^{2(r-\frac{5}{2})+1}\|u\|_{r,\Gamma}^2 \lesssim h^{2(r-2)}\|u\|_{r,\mathcal{T}_h}^2.}
\tag{4.50}
$$

Hence, we have $\|u - C_h^e u\|_{a_h,*} \lesssim h^{r-2}\|u\|_{r,\mathcal{T}_h}^2$. ☐

Lemma 4.5 (Weak galerkin orthogonality)**.** *Let $u \in H^s(\Omega)$, $s \geqslant 3$ be the exact solution to* (3.4) *and $u_h \in V_h$ is a discrete solution to* (4.3)*. Then is*

$$
a_h(u - u_h, v_h) = g_h(u_h, v_h) \quad \forall v_h \in V_h.
$$

Proof. From the definition of the problem (4.3) and utilizing that for $u \in H^s(\Omega)$ we have the identity $A_h(u, v_h) = a_h(u, v_h) = l(v_h)\ \forall v_h \in V_h$. Consequently, it follows that

$$
l(v_h) = A_h(u_h, v_h) = a_h(u, v_h) = a_h(u_h, v_h) + g_h(u_h, v_h) \quad \forall v_h \in V_h.
$$

Hence, we have $a_h(u - u_h, v_h) = g_h(u_h, v_h)$. ☐

Assumption (EP2)**.** *For $v \in H^s(\Omega)$ and $r = \min\{s, k+1\}$, the semi-norm $|\cdot|_{g_h}$ is weakly consistent in the sense that*

$$
|C_h^e v|_{g_h} \lesssim h^{r-2}\|v\|_{r,\Omega}.
\tag{4.51}
$$

Theorem 4.6. *Let $u \in H^s(\Omega)$, $s \geqslant 3$ be a solution to (3.4) and let $u_h \in V_h$ of order $k \geqslant 2$ be the discrete solution to (4.3). Then with $r = \min\{s, k+1\}$ the error $e = u - u_h$ satisfies*

$$\|e\|_{a_h,*} \lesssim h^{r-2}\|u\|_{r,\Omega} \tag{4.52}$$

$$\|e\|_{\Omega} \lesssim h^{r-\max\{0,3-k\}}\|u\|_{r,\Omega} \tag{4.53}$$

Remark. Be aware that for $k = 2$ the estimate (4.53) is suboptimal with 1 order.

Proof. We will divide the proof into two steps.
Step 1. We want to prove that $\|e\|_{a_h,*} \lesssim h^{r-2}\|u\|_{r,\Omega}$. Decompose $e = u - u_h$ intro $e = e_h + e_\pi$, where we denote the discrete error $e_h = C_h^e u - u_h$ and the interpolation error $e_\pi = u - C_h^e u$. We can then observe that

$$\begin{aligned}
\|u - u_h\|_{a_h} &\leqslant \|u - C_h^e u + C_h^e u - u_h\|_{a_h,*} \\
&\leqslant \|u - C_h^e u\|_{a_h,*} + \|C_h^e u - u_h\|_{a_h,*} \\
&\lesssim \|e_\pi\|_{a_h,*} + \|e_h\|_{A_h}
\end{aligned} \tag{4.54}$$

Using Lemma 4.4, is it clear that $\|e_\pi\|_{a_h,*} \lesssim h^{r-2}\|u\|_{r,\Omega}$ is already fulfilled, hence, it remains to estimate e_h. From Lemma 4.2 and 4.3, the weak Galerkin orthogonality and Assumption EP2 (4.51) is it natural to arrive at,

$$\begin{aligned}
\|e_h\|_{A_h}^2 &\lesssim a_h(C_h^e u - u_h, e_h) + g_h(C_h^e u - u_h, e_h) \\
&= a_h(C_h^e u - u, e_h) + a_h(u - u_h, e_h) + g_h(C_h^e u - u_h, e_h) \\
&= a_h(C_h^e u - u, e_h) + g_h(C_h^e u, e_h)
\end{aligned} \tag{4.55}$$

Hence, now utilizing the Assumption EP2 (4.51) is it clear that

$$\begin{aligned}
a_h(C_h^e u - u, e_h) + g_h(C_h^e u, e_h) &\lesssim \|C_h^e u - u\|_{a_h,*}\|e_h\|_{a_h} + |C_h^e u|_{g_h}\,|e_h|_{g_h} \\
&\lesssim \|C_h^e u - u\|_{a_h,*}\|e_h\|_{a_h} + h^{r-2}\|e_h\|_{r,\Omega}\,|e_h|_{g_h} \\
&\lesssim (\|C_h^e u - u\|_{a_h,*} + h^{r-2}\|e_h\|_{r,\Omega})\|e_h\|_{A_h} \\
&\lesssim h^{r-2}\|u\|_{r,\Omega}\|e_h\|_{A_h}.
\end{aligned} \tag{4.56}$$

Here we noticed that $\|e_h\|_{a_h} + |e_h|_{g_h} \lesssim \|e_h\|_{A_h}$, and used that $\|C_h^e u - u\|_{a_h,*} \lesssim h^{r-2}\|u\|_{r,\Omega}$ from Lemma 4.4.

Finally, combining (4.55) and (4.56) is it clear that $\|e_h\|_{A_h} \lesssim h^{r-2}\|u\|_{r,\Omega}$. Hence, the first part of the proof is complete.

Step 2. We want to show that $\|e\|_\Omega \lesssim h^{r-\max(0,3-k)}\|u\|_{r,\Omega}$. The idea is to apply the so-called Aubin-Nitsche duality trick while being aware of the ghost penalty g_h. Let us ~~denote~~ the following observation. Assume that $e := u - u_h \in L^2(\Omega)$ and $\psi \in H^4(\Omega)$. Let the make corresponding dual problem to (3.1) be

$$\begin{aligned}
\Delta^2\psi &= e \quad \text{in } \Omega, \\
\partial_n\psi &= 0 \quad \text{on } \Gamma, \\
\partial_n\Delta\psi &= 0 \quad \text{on } \Gamma.
\end{aligned} \tag{4.57}$$

This implies that it exists a $\psi \in H^4(\Omega)$ such that $a_h(\psi, v) = (e, v)_\Omega\ \forall v \in V_h$. Hence, we can easily observe that

must be swapped $a_h(v, \psi)$.

(Does not play a role for symmetric problems)

$$\begin{aligned}
\|e\|_\Omega^2 &= (e, e)_\Omega = (e, \Delta^2\psi)_\Omega \\
&= a_h(e, \psi) = a_h(u - u_h, \psi) \\
&= a_h(u - u_h, \psi + C_h^e\psi - C_h^e\psi) \\
&= a_h(u - u_h, \psi - C_h^e\psi) + a_h(u - u_h, C_h^e\psi) \\
&= a_h(u - u_h, \psi - C_h^e\psi) \\
&\lesssim \|u - u_h\|_{a_h,*}\|\psi - C_h^e\psi\|_{a_h,*}\,.
\end{aligned} \tag{4.58}$$

Here we applied the Galerkin orthogonality $a_h(u - u_h, C_h^e \psi) = 0$. Using the a priori estimate (4.4) is it clear that

(r-2)? $h^{\tilde{r}-2}$

$$\|u - u_h\|_{a_h,*} \leqslant h^r \|u\|_{r,\Omega} \quad \text{and} \quad \|\psi - C_h^e \psi\|_{a_h,*} \leqslant h^r \|\psi\|_{4,\Omega}. \tag{4.59}$$

?? No, inverse estimate here? Always only if discrete functions are involved.. (1) See below..

And then standard inverse estimate (2.18) can we see

$$h^r \|u\|_{r,\Omega} \leqslant h^{r-2} \|u\|_\Omega \quad \text{and} \quad h^{\tilde{r}-2} \|\psi\|_{4,\Omega} \leqslant h^{\tilde{r}-2} \|\psi\|_\Omega. \tag{4.60}$$

Here is $r = \max(3, k+1)$ and $\tilde{r} = \max(4, k+1)$. Combining (4.58), (4.59) and (4.60) we have,

$$\|e\|_\Omega^2 \lesssim h^{r-2} \|u\|_\Omega \|\psi\|_\Omega. \tag{4.61}$$

Using that $\|\psi\|_\Omega \leqslant \|e\|_\Omega$ is it easy to see that

$$\|e\|_\Omega \lesssim h^{r-\max(0,k-3)} \|u\|_{r,\Omega} \tag{4.62}$$

□

4.5 Constructing ghost penalties

We have the following assumptions for the ghost penalty.

EP1 The ghost penalty g_h extends the H^2 norm such that

$$\|D^2 v_h\|_{\mathcal{T}_h}^2 \lesssim \|D^2 v_h\|_\Omega^2 + |v_h|_{g_h}^2 \quad \forall v_h \in V_h \tag{4.63}$$

EP2 For $v \in H^s(\Omega)$ and $r = \min\{s, k+2\}$, the semi-norm $|\cdot|_{g_h}$ satisfies the following estimate,

$$|\pi_h^e v|_{g_h} \lesssim h^{r-2} \|v\|_{r,\Omega}.$$

The goal in this chapter is to engineer an ghost penalty which fulfills these assumptions.

Let us denote the generalization of the normal derivative,

$$\partial_n^j v = \sum_{|\alpha|=j}^{k} \frac{D^\alpha v(x) n^\alpha}{\alpha!}, \quad |\alpha| = \sum_{i=0}^{d} \alpha_i. \tag{4.64}$$

We denote the multi-index $\alpha = (\alpha_1, \ldots, \alpha_d)$ of order $|\alpha| = \sum_i \alpha_i = k$ and the normal vectors $n^\alpha = n_1^{\alpha_1} \ldots n_d^{\alpha_d}$. Recall the notation for the derivates $D^\alpha v$ [2], that is

$$D^0 v = v, \quad D^1 v = \nabla v \text{ and } D^2 v = J(\nabla v) = \mathrm{Hess}(v). \tag{4.65}$$

where J is the Jacobian operator .

The following result is the backbone of the face-based ghost penalty.

Lemma 4.7. *Let $T_1, T_2 \in \mathcal{T}_h$ be two elements sharing a common face F. Then for $v_h \in V_h$ with polynomial degree k we have*

$$\|v_h\|_{T_1} \lesssim \|v_h\|_{T_2} + \sum_{0 \leqslant j \leqslant k} h^{2j+1} (\llbracket \partial_n^j v_h \rrbracket, \llbracket \partial_n^j v_h \rrbracket)_F \tag{4.66}$$

Proof. See [81, Lemma 2.19]. □

[2] Remark that the operator D^α is related to with the derivate operator ∂^α introduced in (2.1). For instance, for $d = 2$ we have $\alpha = (\alpha_1, \alpha_2)$ such that $D^1 v = \nabla v = \left[\partial^{(1,0)} v, \partial^{(0,1)} v\right]^T$ and $D^2 v = \begin{bmatrix} \partial^{(2,0)} v & \partial^{(1,1)} v \\ \partial^{(1,1)} v & \partial^{(0,2)} v \end{bmatrix}$.

① Start from (4.58)

$$\lesssim \|w - w_h\|_{a_{h,*}} \, \|\varphi - C_h^e \varphi\|_{a_{h,*}} = \mathrm{I} \cdot \mathrm{II}$$

- Now for I we simply use the <u>energy a priori</u> error estimate:

$$\mathrm{I} \lesssim h^{r-2} \|w\|_{r,\Omega} .$$

not $r = \min(s, k+1)$

- Now to estimate $\underline{\mathrm{II}}$, set $\tilde{r} = \min(4, k+1)$ where 4 comes from the regularity of the dual problem ($\varphi \in H^4(\Omega)$, and $\|\varphi\|_{4,\Omega} \lesssim \|e\|$). Thus

$$\mathrm{II} \lesssim h^{\tilde{r}-2} \|\varphi\|_{\tilde{r}} \lesssim h^{\tilde{r}-2} \|e\|_{\Omega}$$

Thus

$$\|e\|_{\Omega}^{\cancel{2}} \lesssim \mathrm{I} \cdot \mathrm{II} \lesssim h^{r-2} \|w\|_{r} \, h^{\tilde{r}-2} \cancel{\|e\|_{\Omega}}$$

- Having a closer look at $\tilde{r}$, we see that

$$\tilde{r} = \min(4, k+1) = \begin{cases} 3 & k = 2 \\ 4 & k \geq 3 \end{cases}$$

So

$$\|e\|_{\Omega} \lesssim h^{r+\tilde{r}-4} \|w\|_{r,\Omega} = \|w\|_{r,\Omega} \cdot \begin{cases} h^{r-1} & k = 2 \\ h^{r-2} & k \geq 3 . \end{cases}$$