1

→ I typically prefer to use v_h and not only $v \in V_h$ to make 100% that the reader understands that we are talking about inverse estimate which hold only for discrete functions.

We aware that these inequalities also holds for the first order, that is.

$$
\|\partial_n v\|_{\mathcal{T}_h \cap \Gamma} \lesssim h^{-\frac{1}{2}} \|\nabla v\|_{\mathcal{T}_h}, \tag{35}
$$

$$
\|\partial_n v\|_{\mathcal{F}_h \cap \Omega} \lesssim h^{-\frac{1}{2}} \|\nabla v\|_{\mathcal{T}_h}. \tag{36}
$$

In fact, combining the second order inequalities we get the following identity.

$$
h\|\partial_{nn} v\|^2_{\mathcal{F}_h \cap \Omega} + h\|\partial_{nn} v\|^2_{\mathcal{T}_h \cap \Gamma} \lesssim \|D^2 v\|^2_{\mathcal{T}_h}. \tag{37}
$$

For more information about the derivations of the inequalities, see [81].

We may introduce our first assumption on the ghost penalty. → Here it is important that you state that we follow the abstract approach from [Gurkan Massing 2019] CutDG paper.

Assumption ~~4.1~~ (EP1). *The ghost penalty g_h extends the H^1 norm s.t.* H^2 ∇

Don't give it a number and a special label that is superfluous.

$$
\|D^2 v\|^2_{\mathcal{T}_h} \lesssim \|D^2 v\|^2_{\Omega} + |v|^2_{g_h}.
$$

Combing the results we get the following convenient corollary.

Corollary 4.1. *Let g_h satisfy Assumption 4.1 then*

$$
h\|\partial_{nn} v\|^2_{\mathcal{F}_h \cap \Omega} + h\|\partial_{nn} v\|^2_{\mathcal{T}_h \cap \Gamma} \lesssim \|D^2 v\|^2_{\Omega} + |v|^2_{g_h} \lesssim \|v\|^2_{A_h}
$$

Use consecutive numbering! Same numbering for all lemma, Corollary, Theorem etc. Makes it easy to find.

Proof. The first inequality is a direct result of (37) and Assumption 4.1. The second inequality is simply a result of the definition (31). □

Lemma 4.1. *The discrete form A_h is coercive, that is,*

$$
\|v\|^2_{A_h} \lesssim A_h(v, v) \forall v \in V_h.
$$

space

Proof. Let $v \in V^h$. Observe that

V_h

$$
A_h(v, v) = a_h(v, v) + |v|^2_{g_h}.
$$

Thus, since the second term already is part of the $\|\cdot\|_{A_h}$ norm is a good start to focus on ~~the~~ a_h term, that is,

simplify to $\alpha^{\frac{1}{2}}$ assume that $\alpha > 0$!

$$
a_h(v, v) = \| |\alpha|^{\frac{1}{2}} \cdot v\|^2_{\Omega} + \|D^2 v\|^2_{\Omega} + 2(\{\!\{\partial_{nn} v\}\!\}, [\![\partial_n v]\!])_{\mathcal{F}_h \cap \Omega} + 2(\partial_{nn} v, \partial_n v)_{\Gamma}
$$
$$
+ \frac{\gamma}{h}\| [\![\partial_n v]\!] \|^2_{\mathcal{F}_h} + \frac{\gamma}{h}\|\partial_n v\|^2_{\Gamma}.
$$

Don't forget proper punctuation.

Don't use future tense if you are just about to do it.

We ~~will~~ first focus on the $\boxed{\text{symmetry terms}}$. Using Cauchy-Schwarz we observe that

not easy to identify / recall symmetry terms, since $w_h = v$. 2

$$
(\{\!\{\partial_{nn} v\}\!\}, [\![\partial_n v]\!])_{\mathcal{F}_h \cap \Omega} \geq -\|h^{\frac{1}{2}} \{\!\{\partial_{nn} v\}\!\} \|_{\mathcal{F}_h \cap \Omega} \|h^{-\frac{1}{2}} [\![\partial_n v]\!] \|_{\mathcal{F}_h \cap \Omega}
$$

$$
(\partial_{nn} v, \partial_n v)_{\Gamma} \geq -\|h^{\frac{1}{2}} \partial_{nn} v\|_{\Gamma} \|h^{-\frac{1}{2}} \partial_n v\|_{\Gamma}
$$

Using inverse-inequalities (33) and (34) and the Corollary 4.1 can we easily observe that

$h^{\frac{1}{2}}$ $\mathcal{F}_h$

$$
\| \{\!\{\partial_{nn} v\}\!\} \|^2_{\mathcal{T}_h \cap \Omega} \leqslant C_1 \|D^2 v\|^2_{\mathcal{T}_h} \leqslant C(\|D^2 v\|^2_{\Omega} + |v|^2_{g_h})
$$

$$
\|\partial_{nn} v\|^2_{\Gamma} \leqslant C_2 \|D^2 v\|^2_{\mathcal{T}_h} \leqslant C(\|D^2 v\|^2_{\Omega} + |v|^2_{g_h})
$$

Use $\lesssim$ (\lesssim) instead. (Define it in the beginning of your thesis) This will transform you mathematical life. :-)

Thus, by applying Youngs ε-inequality. $2ab \leqslant \varepsilon^{-1} a^2 + \varepsilon b^2$, ~~is it natural to~~ we see that.

$$
-C_1^{\frac{1}{2}} \|D^2 v\|_{\mathcal{T}_h} \|h^{-\frac{1}{2}} [\![\partial_n v]\!] \|_{\mathcal{F}_h \cap \Omega} \geq -\frac{1}{\varepsilon} C(\|D^2 v\|^2_{\Omega} + |v|^2_{g_h}) - \varepsilon \|h^{-\frac{1}{2}} [\![\partial_n v]\!] \|^2_{\mathcal{F}_h \cap \Omega}
$$

$$
-C_2^{\frac{1}{2}} \|D^2 v\|_{\mathcal{T}_h} \|h^{-\frac{1}{2}} \partial_n v\|_{\Gamma} \geq -\frac{1}{\varepsilon} C(\|D^2 v\|^2_{\Omega} + |v|^2_{g_h}) - \varepsilon \|h^{-\frac{1}{2}} \partial_n v\|^2_{\Gamma}
$$

① First discuss first order case, that is the most relevant one as terms $\|\partial_{n_w} v\|_T$ and $\|[\![\partial_{n_w} v]\!]\|_{\mathcal{F}\cap\Omega}$ appear in relevant norm. These are also the most common known.
Also provide references.
Then you can either state the general inverse estimates $\Big(\|\partial^\alpha v\|_F^2 \lesssim \|\partial^\beta v\|_T^2$ and the second orders you need as a particular case, or you just state the " " " ".

② Use clearer descriptions, references to terms you would like to treat.
Here e.g. you could easily change the order in which you present terms in the equations, e.g.

$$(4.11)\quad a_h(v,v) = \|\alpha^{\frac{1}{2}} v\|_\Omega^2 + - \qquad + 2(\{\!\{\partial_{nn} v\}\!\}, [\![\partial_n v]\!])_{\mathcal{F}_h} + 2(\partial_{nn} v, \partial_n v)_T .$$

Then you refer to relevant terms, by " the last two terms in (4.11).

Also please turn on equation numbers for all equations now, it does not help us with the editing revision if you do it at the end / right before submission ☺.

Combining these ideas do we end up with the following inequality,

$$a_h(v,v) \geq \| \, |\alpha|^{\frac{1}{2}} \, v\|_\Omega^2 + \|D^2 v\|_\Omega^2 - \frac{1}{\varepsilon} 4C(\|D^2 v\|_\Omega^2 + |v|_{g_h}^2) + (\gamma - 2\varepsilon)\left(\|h^{-\frac{1}{2}} \, [\![\partial_n v]\!] \, \|_{\mathcal{F}_h \cap \Omega}^2 + \|h^{-\frac{1}{2}} \partial_n v\|_\Gamma^2\right)$$

This inequality is useful, since if we apply it on the $\|\cdot\|_{A_h}$ we have a extra ghost penalty term s.t.,

$$\begin{aligned} A_h(v,v) &= a(v,v) + |v|_{g_h}^2 \\ &\geq \| \, |\alpha|^{\frac{1}{2}} \, v\|_\Omega^2 + (1 - \frac{1}{\varepsilon} 4C)(\|D^2 v\|_\Omega^2 + |v|_{g_h}^2) \\ &+ (\gamma - 2\varepsilon)\left(\|h^{-\frac{1}{2}} \, [\![\partial_n v]\!] \, \|_{\mathcal{F}_h \cap \Omega}^2 + \|h^{-\frac{1}{2}} \partial_n v\|_\Gamma^2\right). \end{aligned}$$

So due to consistent use of $\gtrsim$ now if the hidden constant is $=1$.

$\gamma = 32C$ otherwise it looks like you set C and not γ.

Setting $\varepsilon = 8C$ and $C = \frac{\gamma}{32}$ we ~~simplify the problem to~~ "arrive at the desired estimate",

$$\begin{aligned} A_h(v,v) &\geq \| \, |\alpha|^{\frac{1}{2}} \, v\|_\Omega^2 + \frac{1}{2}(\|D^2 v\|_\Omega^2 + |v|_{g_h}^2) \\ &+ \frac{\gamma}{2}\left(\|h^{-\frac{1}{2}} \, [\![\partial_n v]\!] \, \|_{\mathcal{F}_h \cap \Omega}^2 + \|h^{-\frac{1}{2}} \partial_n v\|_\Gamma^2\right) \\ &\gtrsim \|v\|_{A_h}^2 \end{aligned}$$

~~Hence, proof is complete~~. ☐

Lemma 4.2. *The discrete form A_h is bounded, that is,*

$$A_h(v,w) \lesssim \|v\|_{A_h} \|\cancel{v}\|_{A_h} \forall w \in V_h \quad \forall v,w \in V_h \tag{38}$$

w

Moreover, for $v \in V_h + V$ and $w \in V_h$ the discrete form a_h satisfies

$$a_h(v,w) \lesssim \|v\|_{a_h,*} \|w\|_{A_h} \, . \tag{39}$$

$a_h(\cdot,\cdot)$

Simply. Write Step 1. in boldface. Don't use Enumerate indentation?) does not look nice and you waste too much space.

Proof. We will divide the proof in two steps.

1) The goal is to prove the inequality (38).

$$|A_h(v,w)| \lesssim |a_h(v,w)| + |g_h(v,w)|$$

By assumption is the ghost penalty g_h positive semi-definite, thus, it fulfills the Cauchy-Schwartz inequality

"Schwarz" never with a t.

Fix punctuation for all displayed estimates!

$$|g_h(v,w)| \lesssim |v|_{g_h} |w|_{g_h} \, .$$

Hence, by definition ~~is~~ of A_h, $|g_h(v,w)| \lesssim \|v\|_{A_h} \|w\|_{A_h}$. ~~Now~~ it remains to show that the bilinear term a_h is bounded.

too large?

$$\begin{aligned} |a_h(v,w)| \leq & \left|(\alpha v, w)_{\mathcal{T}_h \cap \Omega}\right| + \left|(D^2 v, D^2 w)_{\mathcal{T}_h \cap \Omega}\right| \\ & + \left|(\{\!\{\partial_{nn} v\}\!\}, [\![\partial_n w]\!])_{\mathcal{F}_h \cap \Omega}\right| + \left|(\{\!\{\partial_{nn} v\}\!\}, [\![\partial_n w]\!])_{\mathcal{F}_h \cap \Omega}\right| \\ & + |(\partial_{nn} v, \partial_n w)_\Gamma| + |(\partial_{nn} w, \partial_n v)_\Gamma| \\ & + \frac{\gamma}{h}\left|([\![\partial_n v]\!], [\![\partial_n w]\!])_{\mathcal{F}_h \cap \Omega}\right| + \frac{\gamma}{h}|(\partial_n v, \partial_n w)_\Gamma| \end{aligned} \tag{40}$$

The strategy is to bound each term individually using Cauchy-Schwartz. We can easily see that $\left|(\alpha v, w)_{\mathcal{T}_h \cap \Omega}\right| \lesssim \|v\|_{a_h} \|w\|_{a_h}$ and that $\left|(D^2 v, D^2 w)_{\mathcal{T}_h \cap \Omega}\right| \lesssim$

No need to give such a (too) detailed account.

$\|v\|_{a_h}\|w\|_{a_h}$ using Cauchy Schwartz. For the symmetric terms we also apply the inverse inequality (34).

$$\begin{aligned}\left|\left(\{\!\{\partial_{nn}v\}\!\}, [\![\partial_n w]\!]\right)_{\mathcal{F}_h\cap\Omega}\right| &\lesssim \cancel{\|\{\!\{\partial_{nn}v\}\!\}\|_{\mathcal{F}_h\cap\Omega}\|[\![\partial_n w]\!]\|_{\mathcal{F}_h\cap\Omega}}\\ &\lesssim \|h^{\frac{1}{2}}\partial_{nn}v\|_{\mathcal{F}_h\cap\Omega}\|h^{-\frac{1}{2}}[\![\partial_n w]\!]\|_{\mathcal{F}_h\cap\Omega}\\ &\lesssim \|v\|_{A_h}\|w\|_{a_h}\end{aligned}$$

Still easy to understand? Try to put in one line

thanks to

that

Here we used ~~the~~ Corollary 4.1 ~~s.t.~~ $\|h^{\frac{1}{2}}\partial_{nn}v\|_{\mathcal{F}_h\cap\Omega}\lesssim\|v\|_{A_h}$. The boundedness interior penalty inequality is showed in ~~this~~ same manner,

$$\frac{\gamma}{h}\left|\left([\![\partial_n v]\!], [\![\partial_n w]\!]\right)_{\mathcal{F}_h\cap\Omega}\right| \lesssim \|h^{-\frac{1}{2}}[\![\partial_n v]\!]\|_{\mathcal{F}_h\cap\Omega}\|h^{-\frac{1}{2}}[\![\partial_n w]\!]\|_{\mathcal{F}_h\cap\Omega} \lesssim \|v\|_{a_h}\|w\|_{a_h}.$$

Now it remains to handle boundary terms,

$$|(\partial_{nn}v,\partial_n w)_\Gamma| \lesssim \|h^{\frac{1}{2}}\partial_{nn}v\|_\Gamma\|h^{-\frac{1}{2}}\partial_n w\|_\Gamma \lesssim \|v\|_{A_h}\|w\|_{a_h}$$

Again, here we used the Corollary 4.1. Finally, using the definition of the norm is it easily to see that,

$$\frac{\gamma}{h}|(\partial_n v,\partial_n w)_\Gamma| \lesssim \gamma\|h^{-\frac{1}{2}}\partial_n v\|_\Gamma\|h^{-\frac{1}{2}}\partial_n w\|_\Gamma \lesssim \|v\|_{a_h}\|w\|_{a_h}.$$

Obviously is $\|v\|_{a_h}\lesssim\|v\|_{A_h}$. Hence, we have showed that all terms in a_h is bounded in the $\|\cdot\|_{A_h}$ norm.

Step 2.

No enumeration environment with indentation

ii) The goal is to prove (39) using many of the same ideas as in the first part. Let $v\in V_h+V$ and $v\in V_h$. Next step is to show that the bilinear term a_h is bounded.

No need to repeat to same argument from ( ) again.

$$\begin{aligned}|a_h(v,w)| \leqslant &\left|(\alpha v,w)_{\mathcal{T}_h\cap\Omega}\right| + \left|\left(D^2v,D^2w\right)_{\mathcal{T}_h\cap\Omega}\right|\\ &+\left|\left(\{\!\{\partial_{nn}v\}\!\}, [\![\partial_n w]\!]\right)_{\mathcal{F}_h\cap\Omega}\right| + \left|\left(\{\!\{\partial_{nn}w\}\!\}, [\![\partial_n v]\!]\right)_{\mathcal{F}_h\cap\Omega}\right|\\ &+|(\partial_{nn}v,\partial_n w)_\Gamma| + |(\partial_{nn}w,\partial_n v)_\Gamma|\\ &+\frac{\gamma}{h}\left|\left([\![\partial_n v]\!], [\![\partial_n w]\!]\right)_{\mathcal{F}_h\cap\Omega}\right| + \frac{\gamma}{h}|(\partial_n v,\partial_n w)_\Gamma|\end{aligned} \tag{41}$$

We can easily observe from the first part that this must holds,

$$\left|(\alpha v,w)_{\mathcal{T}_h\cap\Omega}\right| + \left|\left(D^2v,D^2w\right)_{\mathcal{T}_h\cap\Omega}\right| \lesssim \|v\|_{a_h,*}\|w\|_{A_h}.$$

And for the symmetric interior terms,

$$\begin{aligned}\left|\left(\{\!\{\partial_{nn}v\}\!\}, [\![\partial_n w]\!]\right)_{\mathcal{F}_h\cap\Omega}\right| &\lesssim \|h^{\frac{1}{2}}\{\!\{\partial_{nn}v\}\!\}\|_{\mathcal{F}_h\cap\Omega}\|h^{-\frac{1}{2}}[\![\partial_n w]\!]\|_{\mathcal{F}_h\cap\Omega} \lesssim \|v\|_{a_h,*}\|w\|_{A_h},\\ \left|\left(\{\!\{\partial_{nn}w\}\!\}, [\![\partial_n v]\!]\right)_{\mathcal{F}_h\cap\Omega}\right| &\lesssim \|h^{\frac{1}{2}}\{\!\{\partial_{nn}w\}\!\}\|_{\mathcal{F}_h\cap\Omega}\|h^{-\frac{1}{2}}[\![\partial_n v]\!]\|_{\mathcal{F}_h\cap\Omega} \lesssim \|w\|_{A_h}\|v\|_{a_h,*}.\end{aligned}$$

Remark that for $\|h^{\frac{1}{2}}\{\!\{\partial_{nn}v\}\!\}\|_{\mathcal{F}_h\cap\Omega}$ is the norm incorporated in the definition of $\|\cdot\|_{a_h,*}$, but for $\|h^{\frac{1}{2}}\{\!\{\partial_{nn}w\}\!\}\|_{\mathcal{F}_h\cap\Omega}$ was the Corollary 4.1 applied. The jump terms $\|h^{-\frac{1}{2}}[\![\partial_n v]\!]\|_{\mathcal{F}_h\cap\Omega}$ is incorporated in the $\|\cdot\|_{a_h}$ norm, thus, this also holds,

$$\frac{\gamma}{h}\left|\left([\![\partial_n v]\!], [\![\partial_n w]\!]\right)_{\mathcal{F}_h\cap\Omega}\right| \lesssim \|h^{-\frac{1}{2}}[\![\partial_n v]\!],\|_{\mathcal{F}_h\cap\Omega}\|h^{-\frac{1}{2}}[\![\partial_n w]\!]\|_{\mathcal{F}_h\cap\Omega} \lesssim \|v\|_{a_h,*}\|w\|_{A_h}$$

Finally, the boundary terms,

$$\begin{aligned}(\partial_{nn}v,\partial_n w)_\Gamma &\lesssim \|h^{\frac{1}{2}}\partial_{nn}v\|_\Gamma\|h^{-\frac{1}{2}}\partial_n w\|_\Gamma \lesssim \|v\|_{a_h,*}\|w\|_{A_h},\\ (\partial_{nn}w,\partial_n v)_\Gamma &\lesssim \|h^{\frac{1}{2}}\partial_{nn}w\|_\Gamma\|h^{-\frac{1}{2}}\partial_n v\|_\Gamma \lesssim \|w\|_{A_h}\|v\|_{a_h,*},\end{aligned}$$