Project Thesis
Solving Cahn Hilliard Equation

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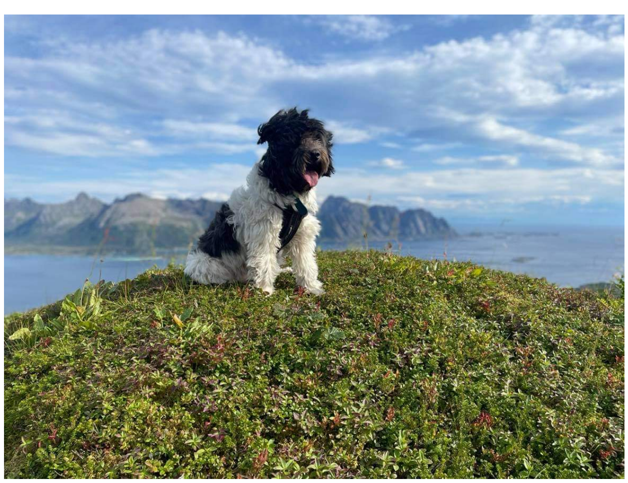

1 Introduction

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PLAN FOR REPORT

1) Introduction

2) DG for poission problem
- classical DG
- (Sability/ apriori error analysis)
- (numerical experiments)
- HDG for Possion Equation

3) Biharmonic Equation (Main part)
- (CIP for biharmonic equation)
- Hybridized CIP for biharmonic equation
- (Stability/ Error Estimate)
- Numerical Experiments
    * Manufactured solution
    * Convergence rate
    * Condition number (h^-4)

4) Cahn-Hilliard Equation
    - Combine CIP Biharmonic with Cahn Hilliard

5) Possible Extensions:
    Compare with mixed formulation
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2 DG for Possion Problem

2.1 Possion Problem

Lets define the problem

$$\begin{aligned} -\varepsilon\nabla u &= f \quad \text{in } \Omega \\ u &= u_D \quad \text{on } \Gamma_D \\ \partial_n u &= g \quad \text{on } \Gamma_N \\ \partial_n u + \beta u &= h \quad \text{on } \Gamma_R \end{aligned}$$

Here is $\partial\Omega = \Gamma_D \cup \Gamma_N \cup \Gamma_R$.

2.2 Classical DG

2.3 Hybrid DG Method

We want to write this on a weak form. Let the spaces we work on be

$$H^1(\mathcal{T}_h) = \left\{u \in L^2(\Omega), u \in H^1(T)\, \forall T \in \mathcal{T}_h\right\}$$

.

For the problem to be discontinuous do we define the trial and test function to be $u \in H^1(\Omega)$ and $v \in H^1(\mathcal{T}_h)$. Thus,

$$-\sum_{T\in\mathcal{T}_h}\int_T \varepsilon\nabla^2 u \cdot v dx = \sum_{T\in\mathcal{T}_h}\left\{\int_T \varepsilon\nabla u\nabla v dx - \int_{\partial T}\varepsilon\cdot\partial_n u\cdot v ds\right\} = \sum_{T\in\mathcal{T}_h}\int_T f\cdot v dx. \tag{1}$$

But we want to introduce the shorter notation equivalently such that

$$\sum_{T\in\mathcal{T}_h}\left\{\varepsilon(\nabla u, \nabla v)_T - \varepsilon\langle\partial_n u, v\rangle_{\partial T}\right\} = \sum_{T\in\mathcal{T}_h}(f, v). \tag{2}$$

Called it " Continuous interior penalty method "
= CIP

3 C^0 Interior Penalty Method for the Biharmonic Equation

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain and $\partial\Omega$ be its corresponding boundary. Let the fourth order biharmonic equation have the form,

$$\begin{aligned} \Delta^2 u + \alpha u &= f \quad \text{in } \Omega \\ \partial_n u &= g_1(x) \quad \text{on } \partial\Omega\,. \\ \partial_n \nabla^2 u &= g_2(x) \quad \text{on } \partial\Omega. \end{aligned} \tag{10}$$

Here is Δ^2 the biharmonic operator, also known as the bilaplacian. We will assume for now that $u \in H^4(\Omega)$, α is a nonnegative constant and $f \in L_2(\Omega)$. We may consider the functions g_1 and g_2 as time independent boundary conditions. Such problems as (10) are often associated with the Cahn-Hilliard model [2] for phase seperation. As a matter of fact, the major difference is that (10) has no time dependencie. However, depending on how Cahn-Hilliard model is time discretized numerically can (10) naturally arise. I refer to [3] for more informastion on this.

Before we start constructing a numerical method, we might want to introduce the basic weak formulation of (10). Now, let the solution space be on the form,

$$V = \left\{ v \in H^2(\Omega) : \partial_n v = g_1 \text{ on } \partial\Omega \right\}..$$

Consider the weak formulation to solve for a $u \in V$ such that

$$a(u,v)_\Omega = F(v). \quad \forall v \in V, \tag{11}$$

where the terms are computed as

$$\begin{aligned} a(u,v)_\Omega &= \int_\Omega \left(D^2u : D^2v + \alpha uv \right) dx, \\ F(v)_\Omega &= (f,v)_\Omega - \langle g_2, v\rangle_{\partial\Omega} + \langle \nabla g_1, \nabla v\rangle_{\partial\Omega}\,. \end{aligned}$$

We denote D^2 as the Hessian matrix operator. In fact, the solution is unique for $\alpha > 0$. However, for $\alpha = 0$ must we assume the solvability condtion,

$$\int_\Omega f\,dx = \int_{\partial\Omega} g_2\,ds.$$

This condition easily arise when using the substitution $v = 1$ in (11). To handle this, can we extended the solution space

$$V^* = \begin{cases} V & \alpha > 0 \\ \{ v \in V : v(p_*) = 0 \}, & \alpha = 0 \end{cases}$$

where p_* is a corner of the polygonal domain Ω. Thus, the unique solution in $v \in V^*$ belongs to $H^3(\Omega)$ and we get the follwing elliptic regularity estimate [4],

Why is setting test function $v(p_*) = 0$ on corners interesting?

$$|u|_{H^3(\Omega)} \le C_\Omega \left(\|f\|_{L_2(\Omega)} + (1 + \alpha^{\frac{1}{2}}) \cdot \|w\|_{H^4(\Omega)} \right), \quad w \in H^4(\Omega)\,. \tag{12}$$

This regularity estimate may be important for further usecases in terms of error analysis.

To solve this numerically do we want to introduce the C^0 Interior Penalty Method (C0IP), which is a Discontinious Galerkin method (DG) using C^0 finite elements. There is several reasons why we want to apply C^0 instead of the often used C^1 finite elements for fourth order problems. First and foremost is the C^0 finite elements simpler than obtaining C^1 finite elements. Also, compared to other methods similar to the mixed finite element method for the problem (10), C0IP has in fact preserved the symmetric positive definiteness, which means the stability analysis is more straight forward. Finally and most importantly according to [3] can naive use mixed methods of splitting the boundary conditions of the problem (10) produce wrong solutions if Ω is nonconvex.

Let $w, v \in H^4(T)$ and $\mathcal{T}_h$ the simplicial triangulation of Ω. Using the same ~~method~~ approach as in [3, 4] can we deduce that for every triangle $T \in \mathcal{T}_h$ is it holds that

$$\begin{aligned} (\Delta^2 w, v)_T &= \left\langle \partial_n \nabla^2 w, v \right\rangle_{\partial T} - \left(\nabla(\nabla^2 w), \nabla v \right)_T \\ &= \left(D^2 w, D^2 v \right)_T + \left\langle \partial_n \nabla^2 w, v \right\rangle_{\partial T} - \langle \partial_n \nabla w, \nabla v\rangle_{\partial T} \\ &= \left(D^2 w, D^2 v \right)_T - \langle \partial_{nt} w, \partial_t v\rangle_{\partial T} - \langle \partial_{nn} w, \partial_n v\rangle_{\partial T} + \left\langle \partial_n \nabla^2 w, v \right\rangle_{\partial T} \end{aligned}$$

Let Chapter 3 start here.

Do a similar but more detailed calculation, when you derive the weak form (11) in Chapter 2 (see Comment 1). In particular clarify how we get the Hessian. Use PhD thesis Gu 2012, p. 29 as orientation.

What is ∇^2 vs Δ vs. D^2 ?

① Put this part into a separate chapter 2, so that you have the overall chapter structure

1. Introduction
2. The biharmonic equation
3. A C^0 interior penalty method for the biharmonic equation
4. Numerical experiments
5. Conclusion and outlook

Main points for Chapter 2

- strong form of the your model of the biharmonic eq, i.e. equation (20)
- explain derivation of weak continuous problem (11) in particular you can present / explain all "integration-by-parts" magic and why we have the Hessian $\int_\Omega D^2 w : D^2 v \, dx$ in $a(\cdot,\cdot)$ and not e.g. $(\Delta w, \Delta v)$.

Start Chapter 3 by explaining briefly challenges with numerical discretization

- a conform numerical discretization where $V_h \subseteq V$ requires C^1 finite elements. They exists in 2D but are complicated to implement, no generalization to 3D known.
- Alternative would be to use C^1 B-splines but they are based on tensor product elements and they definition stretch over several elements

$\Rightarrow$ much less flexible when it comes to generate B-spline suitable meshes for complicated domains.

Thus alternatives to conform discretization are often considered.

a) Reformulation as 2nd order system via extra variable $\overline{w} = \Delta w$

b) Non-conform discretization of 4th order problem using C^0 elements. $\Rightarrow$ CIP method you consider here.

This should not be more than 1 page. Afterwards you can start

<u>Section 3.1</u>: Derivation of the CIP method.

1. Define computational domains
   - $\mathcal{T}_h$, T, $h = ?$
   - $\mathcal{F}_h^{ext}$, $\mathcal{F}_h^{int}$, $\mathcal{F}_h$
   - Bilde ao et mesh
2. discrete Function space $V_h = \{ v \in C(\Omega) : v|_T \in P^k(T) \ \forall T \}$
3. Derivation of

3.2 Stability results

3.3 A priori error estimate.

4 HC0IP Method copied from NGSolve

We consider the Kirchhoff plate equation: Find $w \in H^2$, such that

$$\int \nabla^2 w : \nabla^2 v = \int f v$$

A conforming method requires C^1 continuous finite elements. But there is no good option available, and thus there is no H^2 conforming finite element space in NGSolve.

$$\sum_T \nabla^2 w : \nabla^2 v - \int_E \{\nabla^2 w\}_{nn} \, [\partial_n v] - \int_E \{\nabla^2 v\}_{nn} \, [\partial_n w] + \alpha \int_E [\partial_n w][\partial_n v]$$

[Baker 77, Brenner Gudi Sung, 2010]

We consider its hybrid DG version, where the normal derivative is a new, facet-based variable:

$$\sum_T \nabla^2 w : \nabla^2 v - \int_{\partial T} (\nabla^2 w)_{nn} \, (\partial_n v - \widehat{v_n}) - \int_{\partial T} (\nabla^2 v)_{nn} \, (\partial_n w - \widehat{w_n}) + \alpha \int_E (\partial_n v - \widehat{v_n})(\partial_n w - \widehat{w_n})$$