Assignment 8: Butterworth Filter

Ivan Arevalo

5613567

Description:

The purpose of this assignment is to understand and implement a simple procedure for the design of a digital high-pass Butterworth filter. Butterworth filters are one of the most widely applied techniques in filter design. They are known to be a popular implementation of the low-pass filter designs in practice as an approximation of the ideal low-pass filter. Moreover, low pass Butterworth filters have been the best starting point in designing all kinds of filters both analog and digital. These filters were made with a few requirements in mind. Some of these are that they need to be in rational polynomial form, be of finite order, have real coefficients to implement with real components, and must be a stable filter. With these in mind a general formula for the magnitude of the low-pass Butterworth filter is:

$$\left| H_n(j\omega) \right|^2 = \frac{1}{1 + (\frac{\omega}{\omega_o})^{2n}}$$

Where n is the order of the filter, and w_o is the cutoff frequency. The design challenge consists in finding the correct values for the order and cut-off frequency of the filter given a few design characteristics. For this assignment we are given the following:

a. *Max. passband attenuation:* $\alpha_{max} = 0.5\ dB$
b. *Passband frequency:* $\Omega_p = 3\pi/4$
c. *Min. stopband attenuation:* $\alpha_{min} = 20\ dB$
d. *Stopband frequency:* $\Omega_s = \pi/2$

Method:

Step 1 of our design procedure starts by determining the order of the filter. To do this we will need to use a relationship between the order of filter n and the design characteristics mentioned above. We start by showing the relationship between the allowed magnitude attenuation in the pass and stop bands respectively:

$$0 \geq 10\,log\,\frac{1}{1 + (\frac{\omega}{\omega_o})^{2n}} \geq -\alpha_{max}$$

$$10\,log\,\frac{1}{1 + (\frac{\omega}{\omega_o})^{2n}} \leq -\alpha_{min}$$

We then derive the relationship between pass and stop band frequencies with the attenuation factors and the order of the filter:

$$\omega_p \;\leq\; \omega_o \,(\, 10^{\alpha_{max}/10} \;-1\,)^{\frac{1}{2n}}$$

$$\omega_s \;\geq\; \omega_o \,(\, 10^{\alpha_{min}/10} \;-1\,)^{\frac{1}{2n}}$$

Relating these two and solving for n gives us:

$$n \;\geq\; \frac{log\,\frac{10^{\alpha_{min}/10}\;-1}{10^{\alpha_{max}/10}\;-1}}{2\;log\,\frac{\omega_s}{\omega_p}}$$

We now need to solve for w_s *and* w_p from our given values of pass-band and stop-band frequencies.

$$w_s \;=\; tan(\tfrac{\pi-\Omega_s}{2}) \;=\; 1$$

$$w_p \;=\; tan(\tfrac{\pi-\Omega_p}{2}) \;=\; 0.414$$

Given our values, for α_{min} *and* α_{max} are 20dB and 0.5 respectively we get that n rounds up to 4.

Step 2 consists in determining the cutoff frequency of our filter given the design characteristics provided. To do this we need to know the range of possible cutoff frequencies that satisfies our design. Solving for w_o in both equations above, we get:

$$\frac{\omega_p}{(\,10^{\alpha_{max}/10}\;-1\,)^{\frac{1}{2n}}} \;\leq\; \omega_o \;\leq\; \frac{\omega_s}{(\,10^{\alpha_{min}/10}\;-1\,)^{\frac{1}{2n}}}$$

$$0.538 \;\leq w_o \;\leq 0.563$$

I have chosen 0.55 as my filter cutoff frequency.

Step 3 involves finding the poles of the transfer function. To find the poles of the function, we must go back to the beginning of our derivation where we define the magnitude of the filter based on the requirements for a general low-pass butterworth filter. From here we'll find the poles of our desired function.

$$\left| H_n(j\omega) \right|^2 = \frac{1}{1+(\frac{\omega}{\omega_O})^{2n}} = \frac{1}{1+((\frac{j\omega}{j\omega_O})^2)^n}$$

Given Hermitian Symmetry:

$$\left| H_n(j\omega) \right|^2 = H_n(j\omega)\, H_n^*(j\omega) = H_n(j\omega)\, H_n(-j\omega)$$

$$H_n(s)\, H_n(-s) = \frac{1}{1+(-1)^n(\frac{s}{\omega_O})^{2n}}$$

Equating denominator to 0 and finding the poles of the expression for **n even**:

$$1 + (-1)^n (\frac{s}{\omega_O})^{2n} = 0$$

$$(\frac{s}{\omega_O})^{2n} = 1 = exp(j2k\pi)$$

$$s = \omega_o\, exp(j2k\pi/2n) \qquad k = 0, 1, \dots 2n\text{-}1$$

Equating denominator to 0 and finding the poles of the expression for **n odd**:

$$(\frac{s}{\omega_O})^{2n} = -1 = exp(j(2k+1)\pi)$$

$$s = \omega_o\, exp(j(2k+1)\pi/2n) \qquad k = 0, 1, \dots 2n\text{-}1$$

We now need to find the poles that correspond only to $H_n(s)$. These are the poles that make the filter stable, in other words, those left of the jw axis.

$$H_n(s)\,H_n(-s) = \frac{\omega_o^{2n}}{[(s-s_1)(s+s_1)]\,[(s-s_2)(s+s_2)]\,\cdots\,[\,(s-s_n)(s+s_n)]}$$

$$= \frac{\omega_o^{n}}{(s-s_1)(s-s_2)\,\cdots\,(s-s_n)} \cdot \frac{\omega_o^{n}}{(s+s_1)(s+s_2)\,\cdots\,(s+s_n)}$$

$$H_n(s) = \frac{\omega_o^{n}}{(s-s_1)(s-s_2)\,\cdots\,(s-s_n)}$$

The poles of this transfer function correspond to the following:

For **n odd**

$$s = \omega_o \exp(\pm j\varphi_k) = \omega_o \exp(j(\pi \pm \theta_k)) \qquad k = 0, 1, \ldots (n-1)/2$$

$$= \omega_o \exp(j(\pi \pm 2k\pi)/2n))$$

$$= -\,\omega_o\ [\cos\theta_k \pm j\sin\theta_k]$$

For **n even**

$$s = -\,\omega_o\ [\cos\theta_k \pm j\sin\theta_k] \qquad k = 0, 1, \ldots n/2 - 1$$

Given our derived expressions for poles when n is even and odd, we can construct two table of values for different order Butterworth filters as shown below:

n	$\Delta\theta$	*Butterworth angles* θ_k	*pole locations*
1	180^o	$\theta = 0^o$	$s = -\,\omega_o$
2	90^o	$\theta = \pm 45^o$	$s = -\,\omega_o \exp(j45^o)$
3	60^o	$\theta = 0^o, \pm 60^o$	$s = -\,\omega_o, -\,\omega_o \exp(j60^o)$
4	45^o	$\theta = \pm 22.5^o, \pm 67.5^o$	$s = -\,\omega_o \exp(j22.5^o), -\,\omega_o \exp(j67.5^o)$
5	36^o	$\theta = 0^o, \pm 36^o, \pm 72^o$	$s = -\,\omega_o, -\,\omega_o \exp(j36^o), -\,\omega_o \exp(j72^o)$

n	*transfer functions H(s)*
1	$H(s) = \frac{\omega o}{s + \omega o}$
2	$H(s) = \frac{\omega_o^2}{(s^2 + 1.414\,\omega_o s + \omega_o^2)}$
3	$H(s) = \frac{\omega_o^3}{(s + \omega_o)(s^2 + \omega_o s + \omega_o^2)}$
4	$H(s) = \frac{\omega_o^4}{(s^2 + 1.848\,\omega_o s + \omega_o^2)(s^2 + 0.765\,\omega_o s + \omega_o^2)}$
5	$H(s) = \frac{\omega_o^5}{(s + \omega_o)(s^2 + 1.618\,\omega_o s + \omega_o^2)(s^2 + 0.618\,\omega_o s + \omega_o^2)}$

Given this table and our value for w_o, we can derive our low-pass Butterworth filter transfer function.

$$H(s) = \frac{(0.55)^4}{(s^2+1.848(0.55)s+(0.55)^2)(s^2+0.765(0.55)s+(0.55)^2)}$$

Step 4 consists in performing a analog-digital frequency transformation to get a digital low-pass Butterworth transfer function. For simplicity we will use the bilinear transformation method for the frequency transformation with a β value of 1.

$$s = \beta \frac{z-1}{z+1}$$

$$H(z) = \frac{(0.55)^4}{((\frac{z-1}{z+1})^2+1.848(0.55)(\frac{z-1}{z+1})+(0.55)^2)((\frac{z-1}{z+1})^2+0.765(0.55)(\frac{z-1}{z+1})+(0.55)^2)}$$

Step 5 rests in performing a discrete-discrete frequency transformation to transform the digital low-pass Butterworth into a digital high-pass Butterworth filter.

Frequency transformation	*Foster functions*	$\alpha = 0$
lowpass-lowpass (scaling)	$z = \frac{Z - \alpha}{1 - \alpha Z}$	$z = Z$
lowpass-highpass	$z = -\frac{Z + \alpha}{1 + \alpha Z}$	$z = -Z$

Given the above table with two functions of frequency mappings for the discrete-to-discrete case and choosing α equal to zero for simplicity, we can easily derive our digital high-pass Butterworth filter transfer function as follows:

$$H(z) \ = \ \frac{(0.55)^4}{((\frac{-1-z}{1-z})^2+1.848(0.55)(\frac{-1-z}{1-z})+(0.55)^2)((\frac{-1-z}{1-z})^2+0.765(0.55)(\frac{-1-z}{1-z})+(0.55)^2)}$$

To find our discrete zeros and poles, we can factor the bottom polynomials and derived and expression as such:

$$H(z) \ = \ \frac{(0.55)^4}{((\frac{-1-z}{1-z})-s_1)((\frac{-1-z}{1-z})-s_2)((\frac{-1-z}{1-z})-s_3)((\frac{-1-z}{1-z})-s_4)}$$

$$H(z) \ = \ \frac{(0.55(-z+1))^4}{(-1-z-s_1(1-z))(-1-z-s_2(1-z))(-1-z-s_3(1-z))(-1-z-s_1(1-z))}$$

$$H(z) \ = \ \frac{(0.55(-z+1))^4}{(z(-1+s_1)-1-s_1)(z(-1+s_2)-1-s_2)(z(-1+s_3)-1-s_3)(z(-1+s_4)-1-s_4)}$$

Solving for the zeros:

$$-z+1 \ = \ 0 \ \rightarrow z \ = 1 \ = e^{j0}$$

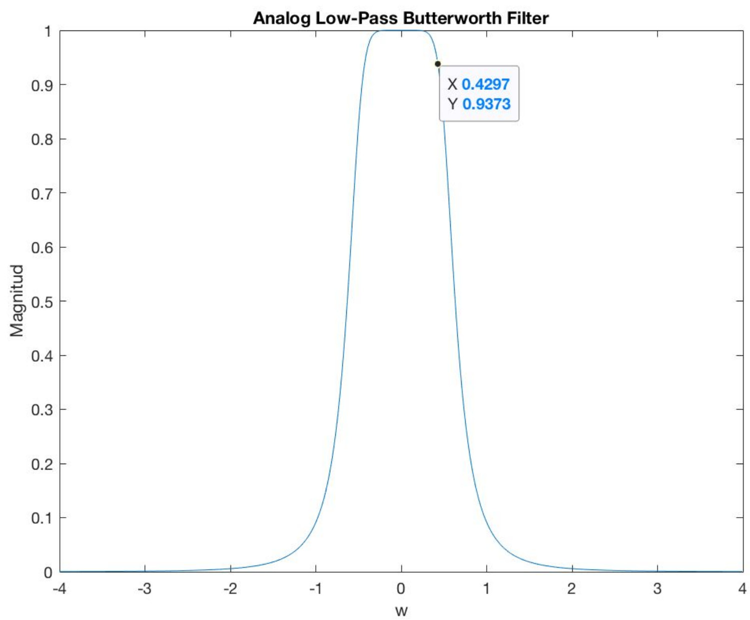
Solving for the poles in general:

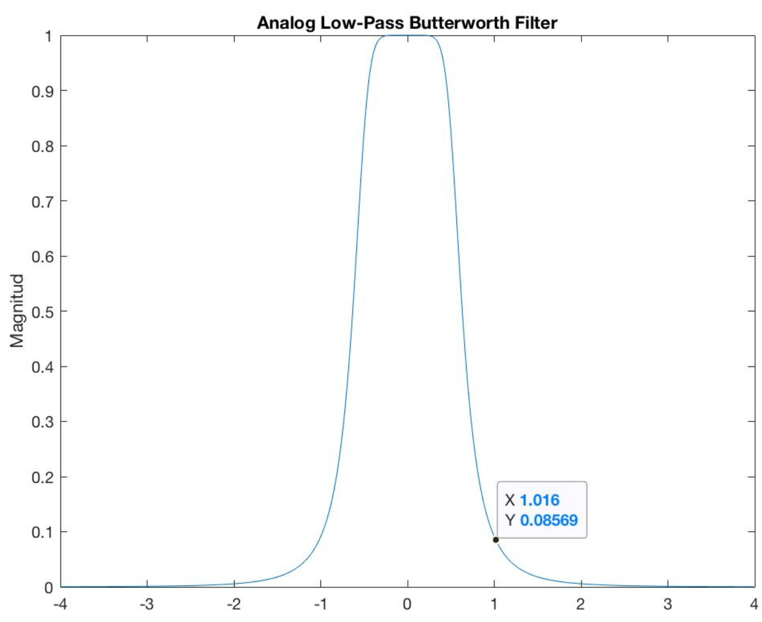
$$z(-1+s)-1-s\ =0\ \rightarrow z=\frac{s+1}{s-1}$$

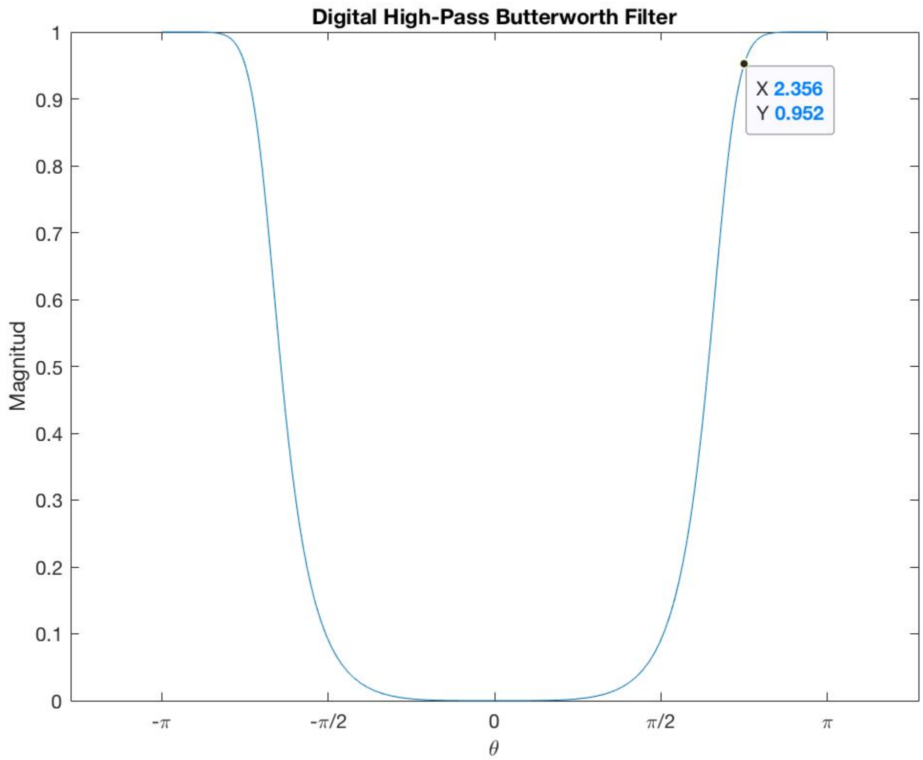
z_1	$\frac{1-(0.55)e^{j22.5^o}}{-(0.55)e^{j22.5^o}-1}=0.351e^{j148.9^o}$
z_2	$\frac{1-(0.55)e^{-j22.5^o}}{(0.55)e^{-j22.5^o}-1}=0.351e^{-j148.9^o}$
z_3	$\frac{1-(0.55)e^{j67.5^o}}{(0.55)e^{j67.5^o}-1}=0.7151932\ e^{j124.4^o}$
z_4	$\frac{1-(0.55)e^{-j67.5^o}}{(0.55)e^{-j67.5^o}-1}0.7151932\ e^{-j124.4^o}$

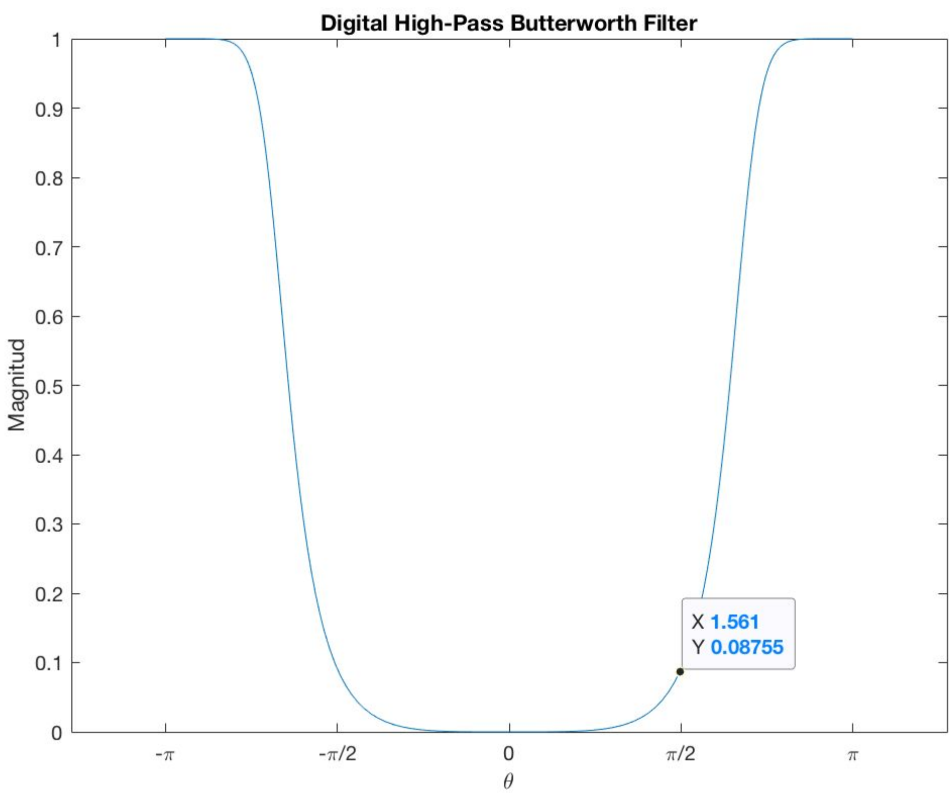
Results:

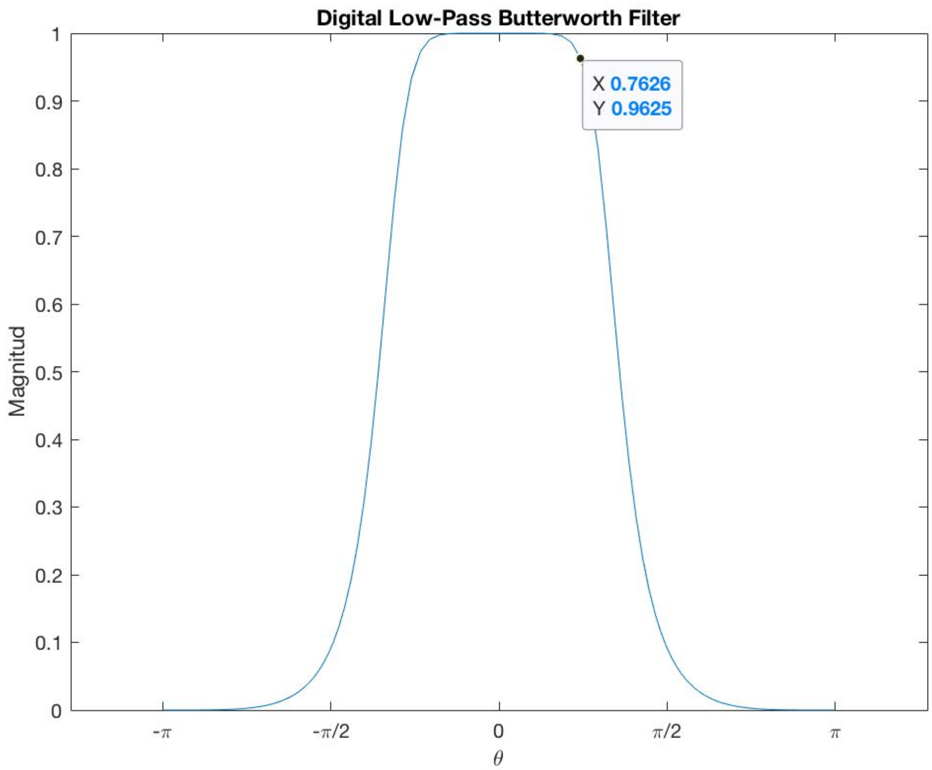
We can now plot the transfer functions in Matlab and provide verification of the design by evaluating the frequency response at the passband and stopband frequencies for each filter respectively.

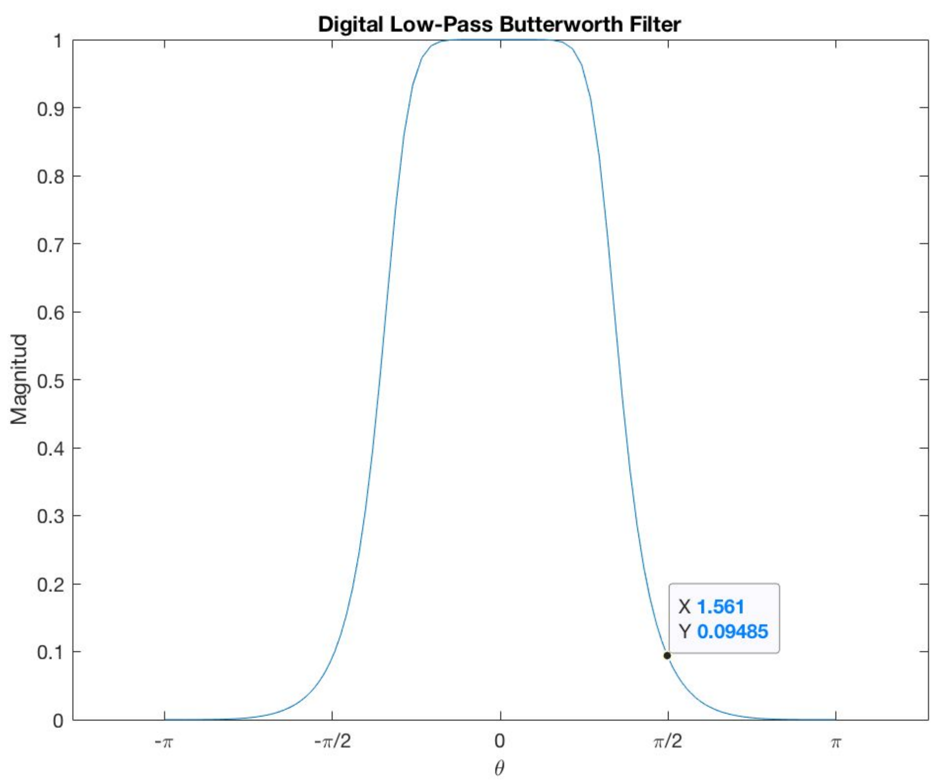





Digital High-Pass Butterworth Filter
Magnitud
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-π
-π/2
0
π/2
π
θ
X 2.356
Y 0.952


Digital High-Pass Butterworth Filter
Magnitud
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-π
-π/2
0
π/2
π
θ
X 1.561
Y 0.08755


Digital Low-Pass Butterworth Filter
Magnitud
θ
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
-π
-π/2
0
π/2
π
X 0.7626
Y 0.9625


Digital Low-Pass Butterworth Filter
Magnitud
θ
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
-π
-π/2
0
π/2
π
X 1.561
Y 0.09485