

## What is a Zipfian Distribution?

- A discrete distribution ( $\mathbf{p} = (p_1, p_2, \dots, p_k)$ ) over  $k$  alphabet items where for some decay  $s > 0$  and permutation function  $\pi$ , the  $i^{th}$  largest probability, denoted  $p_{\pi(i)}$ , satisfies

$$p_{\pi(i)} \propto i^{-s}$$

- $s > 0$  is called the decay.  $\pi$  gives the order of the alphabet items in decreasing order of probability

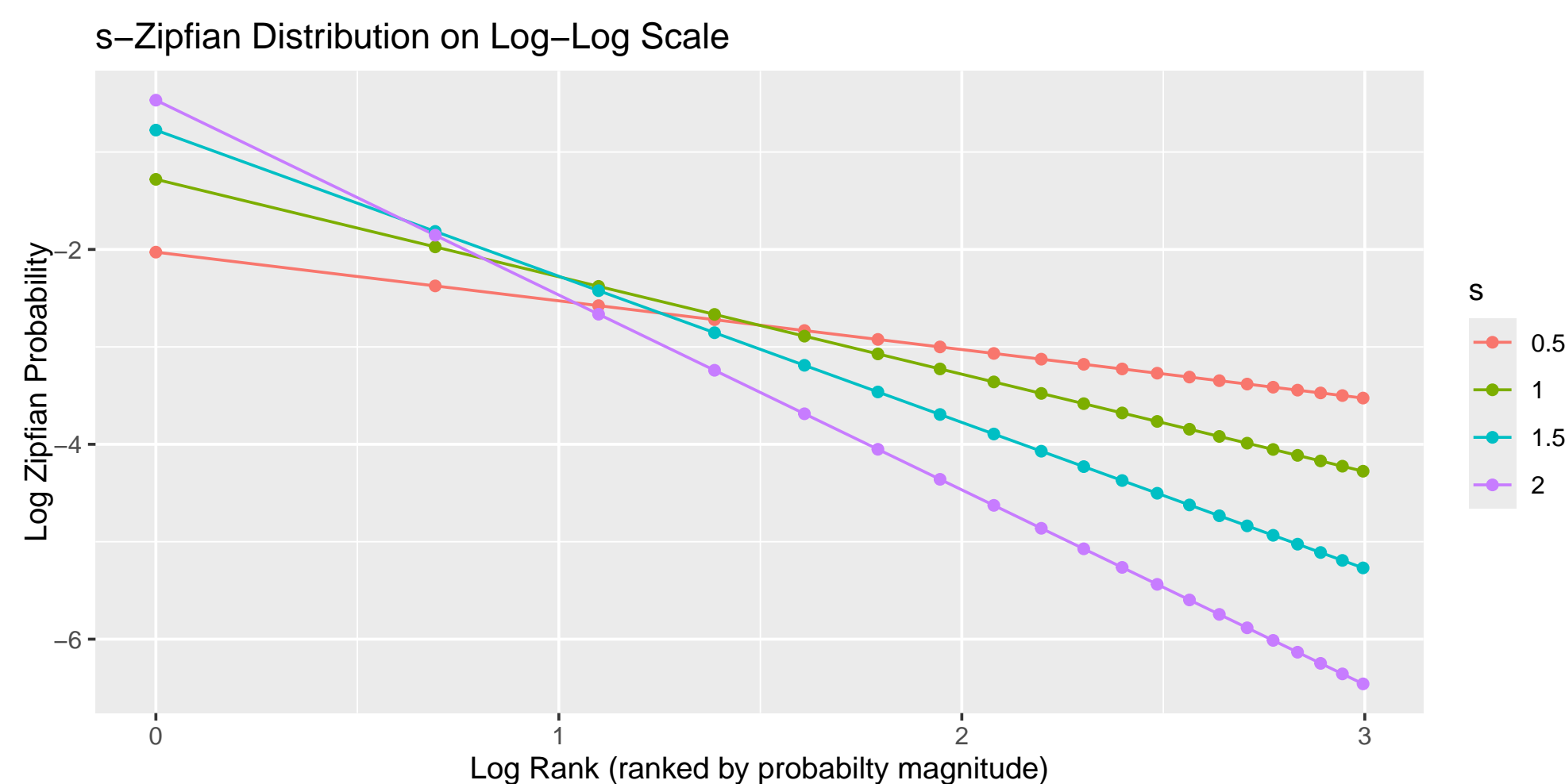


Figure 1. The top 20 probabilities in decreasing order for the  $k = 100$  and  $s \in \{.5, 1, 1.5, 2\}$  Zipfian distributions

## What Problem Do We Consider?

- $\mathbf{p}$  is  $s$ -Zipfian for some  $s > 0$ ,  $k \in \mathbb{N}$  and some permutation function  $\pi$ .
- We must learn  $\mathbf{p}$  only from  $n$  i.i.d samples  $Y_1, Y_2, \dots, Y_n$
- High Dimensional Inference:  $k = n^\beta$  for some  $\beta > 0$ .
- We seek an *estimator* for  $\mathbf{p}$ , denoted  $\hat{\mathbf{p}}$ , that is close to  $\mathbf{p}$  for large  $n$

## How Do We Compare Estimators?

Let  $\mathcal{P}_{s,k}$  denote the  $k$ -dimensional  $s > 0$  Zipfian distributions

- A good estimator  $\hat{\mathbf{p}}$  will minimize maximum risk over  $\mathcal{P}_{s,k}$  as  $n$  grows:

$$\sup_{\mathbf{p} \in \mathcal{P}_{s,k}} \mathbb{E}_{\mathbf{p}} L_1(\hat{\mathbf{p}}, \mathbf{p}) \approx \inf_{\hat{\mathbf{p}} \in \text{Estimators}} \sup_{\mathbf{p} \in \mathcal{P}_{s,k}} \mathbb{E}_{\mathbf{p}} L_1(\hat{\mathbf{p}}, \mathbf{p})$$

- For  $\mathbf{p}, \mathbf{q}$ ,  $L_1(\mathbf{p}, \mathbf{q}) := \sum_{j=1}^k |p_j - q_j|$

## Why is this problem important?

Because high-dimensional Zipfian or near Zipfian distributions occur in a variety of natural settings. Examples include City Size and Word Frequency distributions.

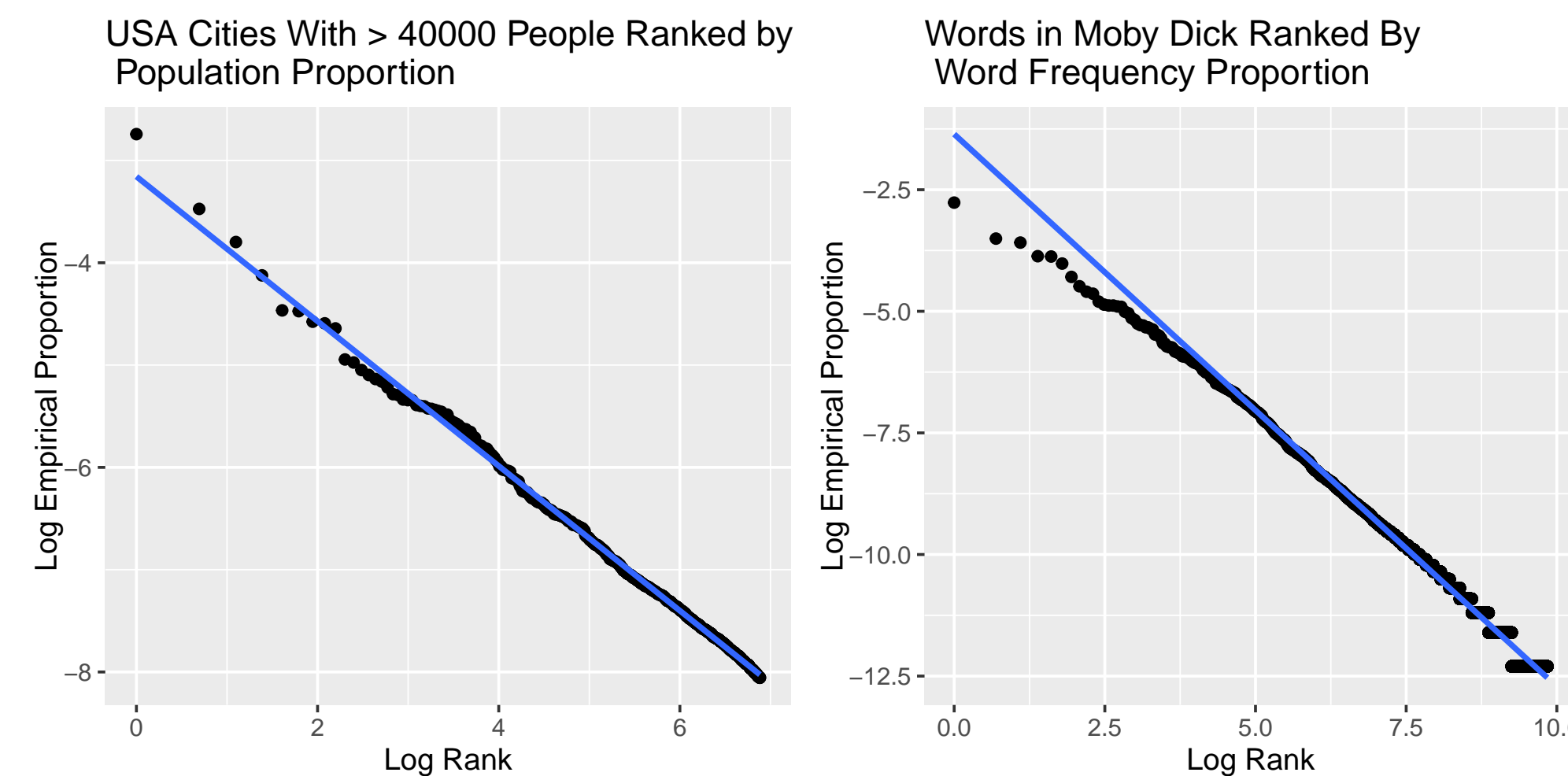


Figure 2. Double Log Plots for City Size Proportions in the USA (left,  $n = 125, 145, 895$  people,  $k = 976$  cities) and Word Frequency Proportions in Moby Dick (right,  $n = 219, 422$ ,  $k = 18, 811$  words). The Blue line represents a Zipfian Probability Law fit using Ordinary Least Squares

## Our Approach

Is  $s$  known before data collection?

- If Yes: Use **Sort and Snap**: Sort the alphabet items by empirical frequency, and assign the  $i^{th}$  largest  $s$ -Zipfian probability as the estimate for the alphabet item with  $i^{th}$  largest frequency
- If No: **Adaptive Sort and Snap**: Estimate  $s$  first via approximate Maximum Likelihood. Then apply Sort and Snap with given estimate of  $s$ .

Through theory and experiment, we compare **Sort and Snap** (SS) and **Adaptive Sort and Snap** (SSA) to SOTA (state of the art) estimators.

Estimator	Usable for $s$ unknown?
Sort and Snap (SS)	No
Adaptive Sort and Snap (SSA)	Yes
Empirical Proportions (EPE)	Yes
Good-Turing (EPE-GT) [3]	Yes
Absolute Discounting (AD) [2]	Yes
Braess-Sauer (BS) [1]	Yes

Table 1. Estimators Under Comparison

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## Results

### Highlighted Theoretical Results

In the  $s$  known case:

- SS is near minimax and outperforms SOTA for  $\beta < \frac{1}{B(s)}$  where  $B(s) := \max(1, s) + 2$
- SS achieves an *exponential decay* in worst case risk when  $\beta < \frac{1}{B(s)}$  of general form  $n^{-f(\beta,s)} \exp(-n^{1-\beta B(s)})$
- An adaptation of SS that uses SS only for top  $\frac{1}{B(s)}$  ranks and EPE otherwise is near minimax when  $\beta > \frac{1}{B(s)}$  and  $s > 2$

Here we highlight experiments in which the worst case risk of the estimators is compared at various  $(s, \beta)$ .

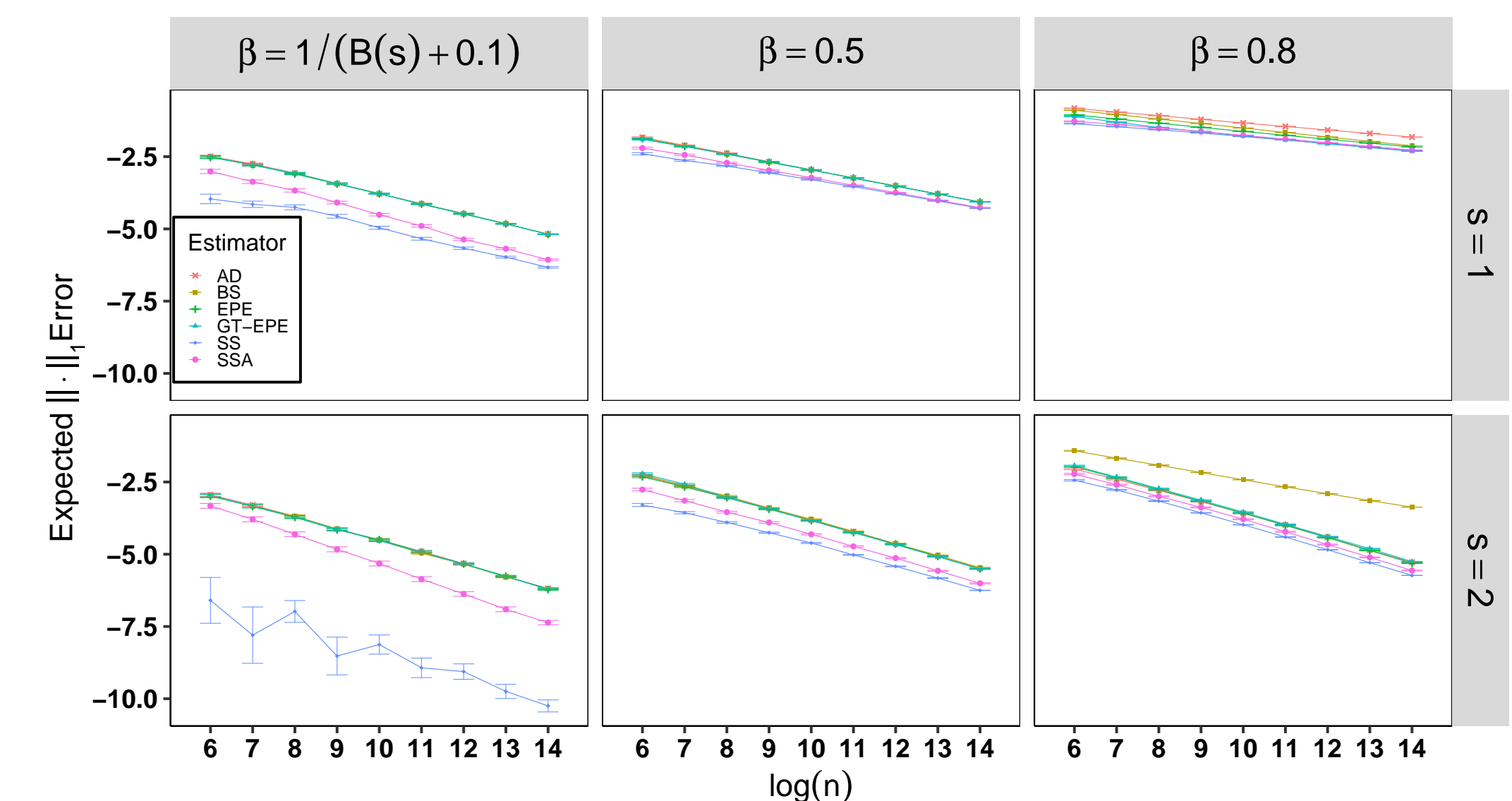


Figure 3. Monte Carlo Study Using  $M = 300$  iterations to compare expected risk under a variety of  $(s, \beta)$  settings. Note  $B(s) := \max(s, 1) + 2$ , which is an intermediate dimension at which Sort and Snap no longer has exponential decay in its risk

## Summary of Conclusions

- The above simulations, and our theoretical results suggest SS and SSA are competitive with SOTA methods for all  $(s, \beta)$ , and superior when  $\beta$  is small relative to  $s$ .
- Future work will investigate
  - SSA for more complex distributions (Zipf-Mandelbrot) that occur in nature
  - How to incorporate SSA into a model-selection strategy when the model is unknown.