1 Splitting metrics

1.1 Geography-based

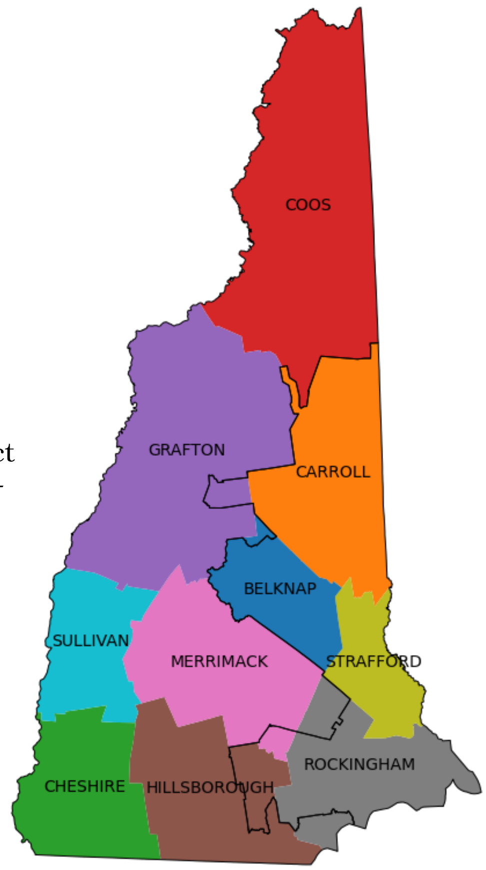
Traditional metrics of locality splitting are calculated using a map of locality and district boundaries. Notably, they do not depend at all on where people live within these boundaries. Here are three examples of geography-based splitting metrics.

Localities split

This is the simplest way to measure locality splitting. Just count the number of localities that are split into multiple districts.

Locality-district intersections

One shortcoming of the previous metric is that a locality split into five different districts is treated exactly the same as a locality split in two. An alternative is to calculate the number of locality-district pairs that intersect. This way, splitting a locality into five districts is punished much more harshly than splitting it into two.

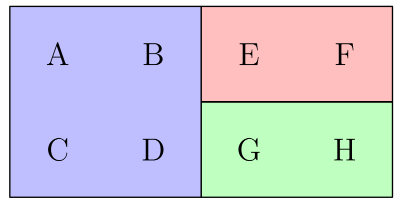


Map of the 10 counties and 2 congressional districts of New Hampshire. The district boundaries split 5 of the 10 counties, and there are 15 county-district intersecting pairs. Note that while Rockingham County is split into three pieces, they only count as two pairs (since two of the pieces are in the same district).

1.2 Population-based

Split pairs

The “split pairs” metric calculates among all pairs of people in the same locality, what proportion of them are split into two different districts. As a simple example, let’s say that a small, rural locality called Alphabetville has 8 people: A, B, C, D, E, F, G, and H. Suppose that A, B, C, and D are in one district, E and F are in another, and G and H are in yet another.

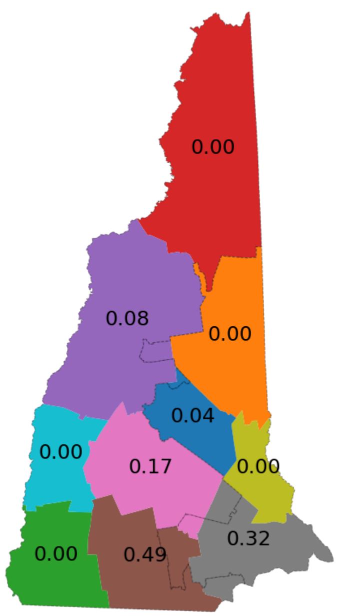


Alphabetville

Then the following pairs of people are split into different districts:

AE, AF, AG, AH, BE, BF, BG, BH, CE, CF, CG, CH, DE, DF, DG, DH, EG, EH, FG, FH

while the following pairs are not: AB, AC, AD, BC, BD, CD, EF, GH.



Split pairs scores for New Hampshire counties

This makes 20 split pairs out of 28, for a split pairs score of 0.714. If all of the people were placed in the same district, there would be no split pairs, and the score would be 0.

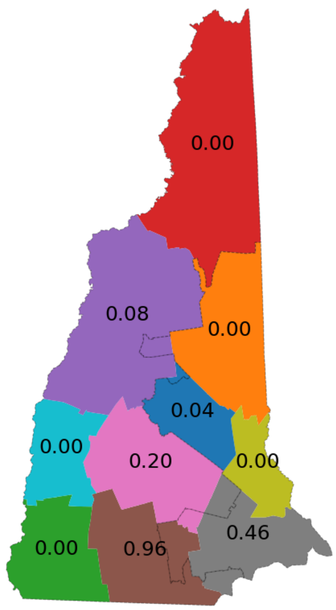
The split pairs score can be summarized through the following hypothetical story: A random voter does not remember his voting district, so he picks a person randomly from his locality and asks what that person’s voting district is. Then he guesses that he lives in the same district. What is the probability of guessing wrong? The split pairs metric.

Effective splits

The “effective splits” metric was proposed recently by Wang, et. al. for measuring community-of-interest splitting. The metric is calculated like this: for a given locality, let $D_1, D_2, \ldots, D_n$ be all the districts that have people in the locality, and let $p_1, p_2, \ldots, p_n$ be the proportion of the locality’s residents that live in each of the n intersecting districts.

Then the effective splits score for the locality is

$$\frac{1}{(p_1)^2 + (p_2)^2 + \cdots + (p_n)^2} - 1.$$



Effective splits scores for New Hampshire counties

Note that if a locality is split into two equally-populated parts, this formula becomes $1/(1/4 + 1/4) - 1 = 1$ effective split. In general, a locality that is split into k equally-populated parts has $k - 1$ effective splits. Splits into unequal parts are punished more lightly.

In our Alphabetville example, the people are split into districts that comprise ½, ¼, and ¼ of the population. This gives an effective splits score of

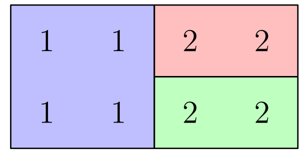
$$\frac{1}{1/4 + 1/16 + 1/16} - 1 = \frac{10}{6} = 1.67.$$

A perceptive reader may notice that the rank order of the effective splits scores by county in New Hampshire (see left) is exactly the same as the split pairs scores. This is not a coincidence! In fact, for real localities with lots of people, it is essentially guaranteed that the rank order of these two scores will be exactly the same. (Proof omitted: if you are nerdy enough to be curious about this, you should be able to work out the math!)

Conditional entropy

Another population-based metric, proposed first by Guth et. al., is based on the mathematical theory of "entropy." Entropy is a principled way to quantify the amount of information inherent in a system. This "conditional entropy" metric quantifies the added information created by the districting plan once the localities are known. This theoretical framework most directly addresses the empirical observation in research that voters in split localities (at least, counties) are less likely to know who represents them. Higher conditional entropy indicates more splitting and more inherent complexity created by the districts, information which the voters will have to learn.

The amount of entropy each person contributes to the metric is given by the formula $\log_2(V_C/V_{CD})$, where V_C is the number of voters in the person's locality and V_{CD} is the number of people in both the person's locality and district. Notice that if a person's locality is kept whole, the value of V_C is the same as V_{CD}, and the amount of entropy added is $\log_2(1) = 0$.

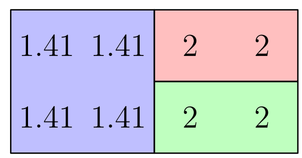


Alphabetville's entropy per person

Returning to Alphabetville, voters A, B, C, and D each have V_C = 8 and V_{CD} = 4, while voters E, F, G, and H each have V_C = 8 and V_{CD} = 2. Thus, the average conditional entropy per person is $1/8 * (4(\log_2 2) + 4(\log_2 4)) = 1/8 * (4(1+2)) = 1.5$.

Square root entropy

In her report to the governor of Pennsylvania in 2018, Moon Duchin proposed a slight modification of the conditional entropy metric, replacing the logarithm with a square root. The purpose of this change was to punish small splits more strongly (more on that in the next chapter). In the square root entropy metric, the formula $\log_2(V_C/V_{CD})$ is changed to $\sqrt{V_C/V_{CD}}$. Notice that this changes the score for non-split localities from 0 to 1.



Alphabetville's square root entropy per person

In the context of our example, voters A, B, C, and D each contribute $\sqrt{2}$ to the score, while voters E, F, G, and H each contribute $\sqrt{4} = 2$. This gives an average of 1.71 per person.

Symmetric splitting metrics: swapping districts with localities

When a locality is much more populous than the required district population, there is no way to avoid splitting the locality. In these cases, some have proposed assessing maps on the extent to which districts are kept within a single locality. (In other words, swap the roles of "locality" and "district" in all the analysis above.) In her report to the Pennsylvania governor, Duchin (2018) calculated county-splitting scores both ways and used the average. This symmetric method has the advantage of explicitly considering the treatment of very populous localities. In practice, we found that it is uncommon for the two methods to disagree about whether a redistricting plan scores better than the state's previous map.

1.3 Choosing which metric to use

Interpretability

The examples in the preceding section demonstrated a clear tradeoff between a metric's interpretability and its ability to incorporate more information. Simple metrics are easier to explain, understand, and visualize on a map. Complicated metrics more effectively reward or punish detailed features of a plan that we might consider important. The value of interpretability certainly depends on the context. A redistricting commission ought to choose a metric that most closely approximates the commission's priorities and its notion of locality splitting, even if the metric is complicated. On the other hand, a citizen writing an op-ed to advocate for her community to not be split might be better served just demonstrating on a map the possibility of keeping the community whole.

How much to punish small splits?

A notable benefit of the population-based splitting metrics is that they treat locality splits differently depending on how many people they affect. For example, all of the population-based metrics give a worse score for splitting a locality's population 50/50 than for splitting the population 90/10. There is no ”correct answer” for how much worse it is to split up more people. However, it is worthwhile to understand the how the metrics implicitly answer this question, based on the relative sizes of the penalties for small and large locality splits. As is shown in the chart below, the geography-based metrics penalize small splits the harshest, followed by the square root entropy metric (as we mentioned, this metric was devised to punish small splits). The effective splits metric penalizes small splits the least. Anyone who is deciding

metric	100/0	90/10	50/50	90/10 pen.	50/50 pen.	ratio
Splits	0	1	1	1	1	100%
Intersections	1	2	2	1	1	100%
Sqrt. entropy	1	1.26	1.41	0.26	0.41	64%
Cond. entropy	0	0.47	1	0.47	1	47%
Split pairs	0	0.18	0.50	0.18	0.50	36%
Eff. splits	0	0.22	1	0.22	1	22%

which metric(s) to use for assessing a districting plan should think about which of these ratios best encapsulates their own belief about how much to penalize small splits.

Population-weighting

To this point, we have explained how splitting metrics are calculated for a single locality. However, in order to assess an entire districting plan, we need a way to incorporate splitting scores for all the relevant localities and determine a single score for the plan.

The simplest way to get a statewide score is to add up all the splitting scores for the individual localities. Another option is to take the population-weighted average, so that localities with more voters have a greater impact on the score. We recommend weighting all localities equally for the metrics that measure "splitting events" (localities split, locality intersections, effective splits) and using population-weighted averages for those that measure voter-level impacts (split pairs, conditional entropy, square root entropy).