Homework 4

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according to the n-dimensional exponential family definition, if X = $(x_1, x_2, ..., x_n)$ is n-dimensional exponential family, it's p.d.f can be written as:

$$exp[X^T \Lambda M(\theta) + S(X) + Q(\theta)]$$

where Λ is a n $\times$ n nonsingular constant matrix

$$X \sim N(\mu, \Sigma), \Sigma > 0 so: \; f_\theta(X) = (2\pi)^{-\frac{n}{2}} |\Sigma|^{-\frac{1}{2}} exp\{-\frac{1}{2}(X-\mu)^T \Sigma^{-1}(X-\mu)\} = exp[ln(2n)^{-\frac{n}{2}} \mathring{u} |\Sigma|^{-\frac{1}{2}}] \cdot exp\{-\frac{1}{2}[X^T \Sigma^{-1} X - X^T$$

because Σ is symmetric positive defeinite metrices : $\mu^T \Sigma^{-1} X = X^T \Sigma^{-1} \mu$

so we can write it as:\

$$f_\theta(X) = exp[ln(2n)^{-\frac{n}{2}} |\Sigma|^{-\frac{1}{2}} - \frac{1}{2}\mu^T \Sigma^{-1} \mu] \mathring{u} exp\{\mu^T \Sigma^{-1} X - \frac{1}{2} X^T \Sigma^{-1} X\}$$

$$C(\theta) = exp[ln(2n)^{-\frac{n}{2}} |\Sigma|^{-\frac{1}{2}} - \frac{1}{2}\mu^T \Sigma^{-1} \mu] Q_1(\theta) = \mu^T \Sigma^{-1} X^T \Sigma^{-1} X = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i^2 where \; a_{ij} \; are \; elements \; of \; \Sigma^{-1} Q_2(\theta) = \sum_{i=1}^{n}$$

we can write $f_\theta(X)$ as:

$$f_\theta(X) = C(\theta) exp Q_1(\theta) T_1(X) + Q_2(\theta) T_2(X) h(x)$$

so normal family are exponential family

2

The answer is no. the log of the p.d.f. is:

$$log f_\theta(X) = log(\alpha\beta) + (\beta - 1)X - \alpha X^\beta$$

we can see that in the last term X and β can't be seperated, so it can't be written as exponential family's form. Specialy if β equals 1, the distribution becomes exponential distribution, then it is an exponential family. But Weibull distribution is not.

3

the joint distribution is:

$$p(x_1, x_2, ...x_n; \theta) = (2\pi\theta^2)^{-\frac{n}{2}} exp\{-\frac{1}{2\theta^2} \sum_{i=1}^{n} (x_i - \theta)^2\} = (2\pi\theta^2)^{-\frac{n}{2}} exp\{-\frac{n\theta^2}{2\theta^2}\} exp\{-\frac{1}{2\theta^2}(\sum_{i=1}^{n} {x_i}^2 - 2\theta \sum_{i=1}^{n} x_i)\} = (2\pi\theta^2)^{-\frac{n}{2}} exp$$

let:

$$t_1 = \sum_{i=1}^{n} x_i t_2 = \sum_{i=1}^{n} x_i^2 g(t_1, t_2, \theta) = (2\pi\theta^2)^{-\frac{n}{2}} exp\{-\frac{n}{2}\} exp\{-\frac{1}{2\theta^2}(t_2 - 2\theta t_1)\},\ h(X) = 1$$

from factorized theorem, $T = (t_1, t_2) = (\sum_{i=1}^{n} x_i, \sum_{i=1}^{n} x_i^2)$ is sufficient statistics. Further we can see it is one to one correspondence to statistics $(\bar{x}, s^2)$, so $\bar{X}$ is sufficient statistic of θ

4

(1) Firstly we show that it is sufficient:\ From the conclusion in question 3, we know that $(\bar{X}, S_X^2)$ is the sufficient statistics of (a, σ^2), and $(\bar{Y}, S_Y^2)$ is the sufficient statistics of (b, σ^2),and:

$$S_X^2 = \frac{1}{m-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$$

is the sufficient statistic of σ^2

$$S_Y^2 = \frac{1}{n-1} \sum_{i=1}^{n} (Y_i - \bar{Y})^2$$

is the sufficient statistic of σ^2

$$so\ S^2 = \frac{1}{m+n-2}[(m-1)S_X^2 + (n-1)S_Y^2]$$

is the sufficient statistic of σ^2

(2) Secondly we show that it is sufficient:\ We have know that normal distribution is the exponential family, its natural form is:

$$f(x, \phi) = C^*(\phi) exp\{\phi_1 T_1(x) + \phi_2 T_2(x)\} h(x), h(x) \equiv 1, \phi_1 = a/\sigma^2, \phi_2 = -\frac{1}{2\sigma^2}, \phi = (\phi_1, \phi_2).\text{natural parameter space is :}$$

From the first proof we can say that $(\bar{X}, \bar{Y}, S^2)$ is the sufficient and complete statistics of (a, b, σ^2)

5

the joint distribution is:

$$p(x_1, x_2, ...x_n; \mu) = \prod_{i=1}^{n} \frac{1}{2\theta} e^{-\frac{|x_i|}{\theta}} = \frac{1}{(2\theta)^n} e^{-\frac{1}{\theta}\sum_{i=1}^{n}|x_i|}$$

$$let\ T = \sum_{i=1}^{n} |X_i|,\ then\ t = \sum_{i=1}^{n} |x_i|$$

$$p(x_1, x_2, ...x_n; \mu) = \frac{1}{(2\theta)^n} e^{-\frac{1}{\theta}t} let\ g(t; \theta) = \frac{1}{(2\theta)^n} e^{-\frac{1}{\theta}t}, h(x) = 1 \text{ is irrelevent with parameter } \theta$$

so from the factorized theorem we know that $T = \sum_{i=1}^{n} |X_i|$ is the sufficient statistic of θ\ Also we know the natural parameter space $\Theta^* = \theta : \theta > 0$ has interior point, so $T = \sum_{i=1}^{n} |X_i|$ is the complete statistic of θ too.

6

(i) Firstly we have he joint p.d.f. is:

$$\prod_{i}^{n} p(x_i;\theta) = \begin{cases} (\frac{1}{\theta})^n & \theta \le x \le 2\theta \\ 0 & else \end{cases} = (\frac{1}{\theta})^n I_{\{\theta < x_{(1)} < x_{(n)} < 2\theta\}}$$

let T $= (t_1, t_2) = (x_{(1)}, x_{(n)})$, let $g(t,\theta) = (\frac{1}{\theta})^n I_{\{\theta < x_{(1)} < x_{(n)} < 2\theta\}}, h(x) = 1$,from the factoreized theorem we know $(x_{(1)}, x_{(n)})$ is the sufficient statistics of θ

(ii)Secondly we show that it's not complete we can prove it by finding a function $\phi(x)$ which enables $E_\theta \phi(T) = 0$ but $\phi(T) \not\equiv 0$

let $Y_i = X_i/\theta$, then Y_i i.i.d $\sim U(1,2)$ irrelevent with $\theta Z = X_{(n)}/X_{(1)} = Y_{(n)}/Y_{(1)}$ is irrelevent with θLet a b satisfies: $P(Z <$

so T(X $= (X_{(1)}, X_{(n)})$ is not complete.

7

(i) the joint p.d.f. is:\

$$p(x_1, x_2, ..., x_n; \lambda, \mu) = \prod_{i=1}^{n} \frac{1}{\lambda} e^{-\frac{x_i - \mu}{\lambda}} Y_{x_i > \mu} = \frac{1}{\lambda^n} e^{-\frac{\sum_{i=1}^{n} x_i - n\mu}{\lambda}} I_{x_1, x_2, ..., x_n > \mu} = \frac{1}{\lambda^n} e^{-\frac{\sum_{i=1}^{n} x_i - n\mu}{\lambda}} I_{x_{(1)} > \mu}$$

let $(T_1, T_2) = (X_{(1)}, \sum_{i=1}^{n} X_{(i)})$\

$(t_1, t_2) = (x_{(1)}, \sum_{i=1}^{n} x_{(i)}) p(x_1, x_2, ..., x_n; \lambda, \mu) = \frac{1}{\lambda^n} e^{-\frac{t_2 - n\mu}{\lambda}} I_{t_1 > \mu} g(t_1, t_2; \lambda, \mu) = \frac{1}{\lambda^n} e^{-\frac{t_2 - n\mu}{\lambda}} I_{t_1 > \mu}, h(x) = 1$ is independent with

so according to factorized theorem, $(T_1, T_2) = (X_{(1)}, \sum_{i=1}^{n} X_{(i)})$ is the sufficient statistics for (λ, μ)\

(ii)Let $Y_1 = X_{(1)}, Y_i = X_{(i)} - X_{(1)}, i = 2, 3, ..., n$\

$$Y = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ -1 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \dots & 1 \end{bmatrix} X |J| = 1 f_{X_1, X_2, ..., X_N}(X_{(1)}, X_{(2)}, ..., X_{(n)}) = n! f(x; \lambda, \mu) \text{so } f_{Y_1, Y_2, ..., Y_N}(y_{(1)}, y_{(2)}, ..., y_{(n)}) = \frac{n!}{\lambda^n} exp$$

we can have:

$$f_{Y_1, Y_2, ..., Y_N}(y_{(1)}, y_{(2)}, ..., y_{(n)}) = f_{Y_2, Y_3, ..., Y_N}(y_{(2)}, y_{(3)}, ..., y_{(n)}) f_{Y_1}(y_1)$$

so Y_1 is independent with $\sum_{i=2}^{n} Y_i$, i.e. $X_{(1)}$ is independent with $\sum_{i=1}^{n}(X_{(i)} - X_{(1)})$