# Homework 1

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## 1 Part I

### 1.1 Exercise 2.1 in Chapter 2

A sample space describing a three-children family is as follows:
S = bbb, bbg, bgb, gbb, bgg, gbg, ggb, ggg, and assume that all eight outcomes are equally likely to occur. Next, let X be the r.v. denoting the number of girls in such a family. Then:
(i) Determine the set of all possible values of X.
(ii) Determine the p.d.f. of X.
(iii) Calculate the probabilities: P(X $\geqslant$ 2), P(X $\leqslant$ 2).
(i)0,1,2,3
(ii)$P(0) = \frac{1}{8}, P(1) = \frac{3}{8}, P(2) = \frac{3}{8}, P(3) = \frac{1}{8}$ (iii)$P(X \geqslant 2) = P(2) + p(3) = \frac{1}{2}, P(X \leqslant 2) = 1 - P(3) = \frac{7}{8}$

### 1.2 Exercise 1.3 in Chapter 3

A chemical company currently has in stock 100 lb of a certain chemical, which it sells to customers in 5 lb packages. Let X be the r.v. denoting the number of packages ordered by a randomly chosen customer, and suppose that the p.d.f. of X is given by: f (1) = 0.2, f (2) = 0.4, f (3) = 0.3, f (4) = 0.1.

| x | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| f(x) | 0.2 | 0.4 | 0.3 | 0.1 |

(i) Compute the following quantities: EX, $EX^2$, and Var(X).
(ii) Compute the expected number of pounds left after the order of the customer in question has been shipped, as well as the s.d. of the number

of pounds around the expected value.
Hint. For part (ii), observe that the leftover number of pounds is the r.v. Y = 100 − 5X.
(i)$EX = 0.2 + 0.8 + 0.9 + 0.4 = 2.3$
$EX^2 = 0.2 + 1.6 + 2.7 + 1.6 = 6.1$
$Var(X) = EX^2 - (EX)^2 = 6.1^2 - 2.3^2 = 0.81$
(ii)$Y = 100 - 5X$
$EY = E(100 - 5X) = 100 - 5EX = 88.5$
$EY^2 = E(100 - 5X)^2 = 10000 - 1000EX + 25EX^2 = 7852.5$
$\sigma(Y) = Var(Y)^{\frac{1}{2}} = (EY^2 - (EY)^2)^{\frac{1}{2}} = (7852.5 - 7832.25)^{\frac{1}{2}} = 4.5$

### 1.3 Exercise 1.11 in Chapter 3

If the r.v. X has p.d.f. $f(x) = 3x^2 – 2x + 1$, for 0 < x < 1, compute the expectation and variance of X.
(i)$EX = \int xf(x)dx = \int_0^1 x(3x^2 - 2x + 1)dx = \frac{3}{4}x^4 - \frac{2}{3}x^3 + \frac{1}{2}x^2|_0^1 = \frac{7}{12}$
(ii)$EX^2 = \int x^2 f(x)dx = \int_0^1 x^2(3x^2 - x + 1)dx = \frac{3}{5}x^5 - \frac{1}{2}x^4 + \frac{1}{3}x^3|_0^1 = \frac{13}{20}$
$Var(X) = EX^2 - EX^2 = \frac{67}{720}$

### 1.4 Exercise 1.41 in Chapter 3

Let X be a r.v. (of the continuous type) with d.f. F given by:

$$F(x) = \begin{cases} 0 & x \geqslant 0 \\ x^3 - x^2 + x & 0 < x < 1 \\ 1 & x \geqslant 1 \end{cases}$$

(i) Show that the p.d.f. of X is given by: f (x) = $3x^2$ − 2x + 1 for 0 < x < 1, and 0 otherwise.
(ii) Calculate the expectation and the variance of X, EX, and $\sigma^2(X)$.
(iii) Calculate the probability P(X > $\frac{1}{2}$).
(i)$Becausep.d.fofXisthedifferentialofd.fF(x), i.e.f(x) = F'(x)$
So, we have:

$$f(x) = \begin{cases} 0 & x \geqslant 0 \\ 3x^2 - 2x + 1 & 0 < x < 1 \\ 0 & x \geqslant 1 \end{cases}$$

(ii)from question 3 we know $EX = \int xf(x)dx = \frac{13}{20} and Var(X) = \frac{67}{720}$ (iii)$P(X > \frac{1}{2}) = \int_{\frac{1}{2}}^{1} xf(x)dx = \int_{\frac{1}{2}}^{1} 3x^2{-}2x + 1dx = x^3 - x^2 + x|_{\frac{1}{2}}^{1} = \frac{5}{8}$

### 1.5 Exercise 2.1 in Chapter 5

For any r.v.'s $X_1, ..., X_n$, set

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \tag{1}$$

and

$$S^2 = \frac{1}{n}\sum_{i=1}^{n} (X_i - \bar{X})^2 \tag{2}$$

and show that: (i)

$$nS^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2 = \sum_{i=1}^{n} X_i^2 - n\bar{X}^2 \tag{3}$$

(ii)If the r.v.'s have common (finite) expectation $\mu$, then (as in relation (14))

$$\sum_{i=1}^{n} (X_i - \mu)^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2 + n(\bar{X} - \mu)^2 = nS^2 + n(\bar{X} - \mu)^2 \tag{4}$$

(i)$\sum_{i=1}^{n} (X_i - \bar{X})^2$
$= \sum_{i=1}^{n} X_i^2 + n\bar{X}^2 - 2\sum_{i=1}^{n} X_i\bar{X}$
$= \sum_{i=1}^{n} X_i^2 + n\bar{X}^2 - 2n\bar{X}^2$
$= \sum_{i=1}^{n} X_i^2 - n\bar{X}^2$
(ii)$\sum_{i=1}^{n} (X_i - \mu)^2$
$= \sum_{i=1}^{n} X_i^2 + n(EX)^2 - 2n\bar{X}EX$
$= \sum_{i=1}^{n} X_i^2 - n\bar{X}^2 + n(EX)^2 - 2n\bar{X}EX + n\bar{X}^2$
$= \sum_{i=1}^{n} (X_i - \bar{X})^2 + n(\bar{X} - \mu)^2$
and we also have:
$nS^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2$

$sowehave:$
$\sum_{i=1}^{n}(X_i-\mu)^2$
$=\sum_{i=1}^{n}(X_i-\bar{X})^2+n(\bar{X}-\mu)^2$
$=nS^2+n(\bar{X}-\mu)^2$

# 2 Part II

## 2.1 Exercise 2.3 in Chapter 5

If the independent r.v.'s X and Y are distributed as B(m, p) and B(n, p), respectively:
(i) What is the distribution of the r.v. X + Y?
(ii) If m = 8, n = 12, and p = 0.25, what is the numerical value of the probability: P(5 $\leqslant$ X + Y $\leqslant$ 15)?

(i) X+Y $\sim$ B(m+n,p) We can prove that bernoulli distribution is addititive:

$$\begin{aligned}
P(X+Y=k) &= \sum_{i=0}^{k} P(X=i, Y=k-i) \\
&= \sum_{i=0}^{k} P(X=i)P(Y=k-i) && \text{by independence} \\
&= \sum_{i=0}^{k} \binom{n}{i} p^i(1-p)^{n-i} \binom{m}{k-i} p^{k-i}(1-p)^{m-k+i} \\
&= p^k(1-p)^{n+m-k} \sum_{i=0}^{k} \binom{n}{i}\binom{m}{k-i} \\
&= \binom{n+m}{k} p^k(1-p)^{n+m-k}
\end{aligned}$$

So that X+Y $\sim$ B(n+m,p)
(ii) According to (i), X+Y $\sim$ B(20,0.25)
P(5 $\leqslant$ X + Y $\leqslant$ 15) = $\sum_{k=5}^{15} {}^{20}C_k\ 0.25^k 0.75^{20-k} = 0.25^{20}(\sum_{k=5}^{15} {}^{20}C_k 3^{20-k}) =$ 0.58515
I use a python program to calculate the result:
from scipy.misc import comb
result = 0
for i in range(5,16):

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result = result+0.25**20*comb(20,i)*3**(20-i)
print result
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### 2.2 Exercise 2.8 in Chapter 5

The r.v.'s $X_1, ..., X_n$ are independent and Xi $\sim P(\lambda_i)$:
(i) What is the distribution of the r.v. X = $X_1 + \cdots + X_n$?
(ii) If X = $\frac{1}{n}(X_1 + \cdots + X_n)$, calculate the E$\bar{X}$ and the Var( $\bar{X}$) in terms of $\lambda_1, ..., \lambda_n$, and n.
(iii) How do the E$\bar{X}$ and the Var($\bar{X}$) become when the $X_i$'s in part (i) are distributed as P(λ)?

(i) X $\sim$ P($\sum_{i=1}^{n} \lambda_i$) First we can prove that Poisson distribution P($\lambda$) is additive

$$\begin{aligned}
P(X+Y=k) &= \sum_{i=0}^{k} P(X+Y=k, X=i) \\
&= \sum_{i=0}^{k} P(Y=k-i, X=i) \\
&= \sum_{i=0}^{k} P(Y=k-i)P(X=i) \\
&= \sum_{i=0}^{k} e^{-\mu}\frac{\mu^{k-i}}{(k-i)!}e^{-\lambda}\frac{\lambda^i}{i!} \\
&= e^{-(\mu+\lambda)}\frac{1}{k!}\sum_{i=0}^{k}\frac{k!}{i!(k-i)!}\mu^{k-i}\lambda^i \\
&= e^{-(\mu+\lambda)}\frac{1}{k!}\sum_{i=0}^{k}\binom{k}{i}\mu^{k-i}\lambda^i \\
&= \frac{(\mu+\lambda)^k}{k!}\cdot e^{-(\mu+\lambda)}
\end{aligned}$$

So X+ Y$\sim$ P($\mu + \lambda$)
And we can extend the conclusion to n variables which is the anwser of (i)
(ii)for $X_i \sim$ P($\lambda_i$)
E$X_i$ = λ, Var($X_i$) = λ
$E\bar{X} = \frac{1}{n}E\sum_{i=1}^{n} X_i = \frac{1}{n}\sum_{i=1}^{n} EX_i = \frac{1}{n}(\lambda_1 + \lambda_2 + ... + \lambda_n)$ $Var(\bar{X}) =$

$\frac{1}{n^2} Var(\sum_{i=1}^{n} X_i) = \frac{1}{n^2}(\lambda_1 + \lambda_2 + ... + \lambda_n)$
(iii)according to (ii):
$E\bar{X} = \lambda, Var(\bar{X}) = \frac{\lambda}{n}$