Homework2

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1

It is true that a nonlinear transformation of a normal distribution is no longer normal. as $n \to \infty, Var(\theta|y) \to 0$ can still hold local linearity property. So for local one-to-one transformation, as $n \to \infty, \phi|y \sim Normal$

2

assume for simplicity that the posterior distribution is continuous and that needed moments exist.

a

$$
\begin{gathered}
E(L(a|y)) = \int (\theta - a)^2 P(\theta|y) d\theta \\
\Rightarrow \frac{dE}{d\theta} = 2 \int (\theta - a) p(\theta|y) d\theta = 0 \\
\Rightarrow \int \theta[\theta|y) d\theta - a \cdot \int p(\theta|y) d\theta d = 0 \\
\Rightarrow a = E(\theta|y)
\end{gathered}
$$

b

$$
\begin{aligned}
&E\left(L(a|y) = \int 1\theta - \alpha|p(\theta|y)d\theta\right. \\
&= \int_{-\infty}^{a} (a - \theta) p(\theta|y) d\theta + \int_{0}^{\infty} (\theta - a) p(\theta|y) d\theta
\end{aligned}
$$

We can have:

$$
\begin{gathered}
\frac{dE}{da} = \int_{-\infty}^{a} p_{(\theta|y)} d\theta - \int_{a}^{\infty} p(\theta|y) d\theta = 0 \\
\text{so that a is median of } \theta|y
\end{gathered}
$$

c

similarly we have:

$$
\frac{dE}{da} = k_1 \int_{-\infty}^{a} p_{(\theta|y)} d\theta - k_0 \int_{a}^{\infty} p(\theta|y) d\theta = 0
$$

We have that \$p (| y) \$ is $\frac{k_0}{k_0+k_1}$ quantile, from c we can also have the conclusio of b

3

Unbiasedness: $\mathrm{E}(\mathrm{E}(\theta|y)|\theta) = \theta$

$$
\begin{gathered}
E(\theta \mathrm{E}(\theta|y)) = \mathrm{E}[\mathrm{E}(\theta \mathrm{E}(\theta|y)|\theta)] = \mathrm{E}\left[\theta^2\right] \\
\text{At the same time} \\
E(\theta \mathrm{E}(\theta|y)) = \mathrm{E}[\mathrm{E}(\theta \mathrm{E}(\theta|y)|y)] = \mathrm{E}\left[\mathrm{E}(\theta|y)^2\right]
\end{gathered}
$$

It mush follows that $\mathrm{E}\left[(\mathrm{E}(\theta|y)-\theta)^2\right]=0$ which assumes θ is constant.

4

$$p(\mu,\sigma^2|y) \propto \sigma^{-n-2} exp\{-\frac{1}{2\sigma^2}\left[(n-1)s^2+n(\bar{y}-\mu)^2\right]\}$$

$$logp(\mu,\sigma^2|y) = -\frac{(n+2)}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\left[(n-1)s^2+n(\bar{y}-\mu)^2\right]$$

$$\frac{d}{d\mu}\log p = \frac{n\left(y^2-\mu\right)}{\sigma^2}$$

$$\frac{d}{d\sigma^2}\log p = -\left(\frac{n}{2}+1\right)\cdot\frac{1}{\sigma^2}+\frac{1}{2\left(\sigma^2\right)^2}\left[(n-1)s^2+n(\bar{y}-\mu)^2\right]$$

So we have $\hat{\mu}=\bar{y},\ \hat{\sigma}^2=\frac{(n-1)s^2}{n+2}$ To derive $I(\theta)$,calculate derivatives of logp with respecitives to $d\mu^2, d(\sigma^2)^2, d\mu d\sigma^2$

$$I(\hat{\theta}) = \begin{bmatrix} \frac{n}{\sigma^2} & 0 \\ 0, & \frac{(n+2)^3}{2(n-1)^2s^4} \end{bmatrix}$$

$$p(\mu,\sigma^2|y) \sim N\left(\begin{pmatrix}\hat{\mu}\\ \hat{\sigma}^2\end{pmatrix}\cdot I^{-1}(\theta)\right)$$

$$= N\left(\begin{pmatrix}\bar{y}\\ \frac{(n-1)s^2}{n+2}\end{pmatrix}\cdot\begin{bmatrix} \frac{\sigma^2}{n} & 0 \\ 0, & \frac{2(n-1)^2s^4}{(n+2)^3} \end{bmatrix}\right)$$