Homework 11

Xupeng Chen

2017/12/19

1.(i)

$$\text{We can assume that } X_i \sim B(p),\ \text{p is unknown.}$$

$$\text{So the number of young adults listening to that program } \sum_{i=1}^{n} X_i \sim B(n,p).$$

ii.

$$\text{binomial test: } H_0 : p \le p_0, H_1 : p > p_0.$$
$$\text{Use the program we can get p-value=0.0001 in this test, supports } H_1.$$
$$\text{So the claim made by the manager is supported.}$$

2

i.

$$\text{X are counts of the head and } X \sim B(n,\theta).$$

$$L(\theta_0;x) = \binom{n}{x}\theta_0^x(1-\theta_0)^{n-x}, \because \theta_{MLE} = \frac{x}{n}, \therefore \sup_{\theta\in\Theta} L(\theta;x) = \binom{n}{x}(\frac{x}{n})^x(\frac{n-x}{n})^{n-x},$$

$$\lambda = \frac{n^n\theta_0^x(1-\theta_0)^{n-x}}{x^x(n-x)^{n-x}},$$

$$\frac{\partial(-\log\lambda)}{\partial x} = \log x + 1 - \log(n-x) - 1 - \log\theta_0 + \log(1-\theta_0) = \log[\frac{x(1-\theta_0)}{\theta_0(n-x)}],$$

$$\text{so we have: } \begin{cases} \frac{\partial(-\log\lambda)}{\partial x} < 0, & x < n\theta_0 \\ \frac{\partial(-\log\lambda)}{\partial x} = 0, & x = n\theta_0 \\ \frac{\partial(-\log\lambda)}{\partial x} > 0, & x > n\theta_0 \end{cases}$$

$$\therefore \lambda < \lambda_0 \Leftrightarrow -\log\lambda > C_1 \Leftrightarrow |x - n\theta_0| > C_2.$$

$$Let P_{\theta_0}(|X - n\theta_0| > C_2) = \alpha, P_{\theta_0}(X > n\theta_0 + C_2) = \frac{\alpha}{2}.$$

$$\text{then find the smallest } C_2\ s.t. P_{\theta_0}(X > n\theta_0 + C_2) = \sum_{X=n\theta_0+C_2}^{n}\binom{n}{X}\theta_0^X(1-\theta_0)^{n-X} \le \frac{\alpha}{2}.$$

$$P_\theta(X > 57) = 0.067,\ and\ P_\theta(X > 58) = 0.044, \therefore C_2 = 8.$$

$$\text{The likelihood ratio test is } \begin{cases} 1, & X > 58 \text{ or } X < 42 \\ 0, & otherwise \end{cases}$$

$$\because x = 60 > 58, \therefore H_0 \text{ is rejected.}$$

(ii)

$$\text{According to CLT, } \frac{X - n\theta}{\sqrt{n\theta(1-\theta)}} \xrightarrow[n\to\infty]{d} N(0,1).$$

$$Let P_{\theta_0}(|X - n\theta_0| > C_2) = \alpha, \text{then } C_2 = z_{\alpha/2}\sqrt{n\theta_0(1-\theta_0)} = 8.224268.$$

$$\because x = 60 > 58.224268, \therefore H_0 \text{ is rejected.}$$

3.

Let X_i be the i-th women's blood pressure difference after and before the usage of the pill.

$$\text{According to CLT , } \frac{\sqrt{n}(\bar{X}-\mu)}{S} \xrightarrow[n\to\infty]{d} N(0,1), S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2.$$

$$\text{Assume } X_i \text{ approxiamtely } i.i.d. \sim N(\mu,\sigma^2).$$

$$L(\mu,\sigma^2;\boldsymbol{x}) = (2\pi\sigma^2)^{-n/2}\exp[\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2],$$

$$\begin{cases} \frac{\partial \log L(\mu,\sigma^2;\boldsymbol{x})}{\partial\mu} &= \frac{n}{\sigma^2}(\bar{x}-\mu) \\ \frac{\partial \log L(\mu,\sigma^2;\boldsymbol{x})}{\partial(\sigma^2)} &= -\frac{n}{2\sigma^2}+\frac{1}{2\sigma^4}\sum_{i=1}^{n}(x_i-\mu)^2. \end{cases}$$

It is easy to get that

$$\sup_{\theta\in\Theta} L(\theta;\boldsymbol{x}) = [\frac{2\pi}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2]^{-n/2}\exp(-\frac{n}{2})$$

$$\sup_{\theta\in\Theta_0} L(\theta;\boldsymbol{x}) = \begin{cases} [\frac{2\pi}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2]^{-n/2}\exp(-\frac{n}{2}), & \bar{x}\ge\mu_0 \\ [\frac{2\pi}{n}\sum_{i=1}^{n}(x_i-\mu_0)^2]^{-n/2}\exp(-\frac{n}{2}), & \bar{x}<\mu_0 \end{cases}$$

$$\lambda = \frac{\sup_{\theta\in\Theta_0} L(\theta;x)}{\sup_{\theta\in\Theta} L(\theta;x)} = \begin{cases} 1, & \bar{x}\ge\mu_0 \\ [\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{\sum_{i=1}^{n}(x_i-\mu_0)^2}]^{-n/2}, & \bar{x}<\mu_0 \end{cases}.$$

$$\lambda<\lambda_0 \Leftrightarrow \frac{\sqrt{n}(\bar{x}-\mu_0)}{\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2}} < C.$$

$$Let P_{\mu_0}(\frac{\sqrt{n}(\bar{x}-\mu_0)}{\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2}} < C) = \alpha, \text{ then } C = -t_{n-1,\alpha}.$$

$$\text{The likelihood ratio test is } \varphi(\boldsymbol{x}) = \begin{cases} 1, & \frac{\sqrt{n}(\bar{x}-\mu_0)}{\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2}} < -t_{n-1,\alpha/2} \\ 0, & \text{otherwise} \end{cases}.$$

(ii)

$$\frac{\sqrt{n}(\bar{x}-\mu_0)}{\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2}} = \frac{\sqrt{90}\bar{x}}{s}, t_{n-1,\alpha/2} = t_{89,0.05}.$$

$$If \frac{\sqrt{90}\bar{x}}{s} < -t_{89,0.05}, \text{ then } H_0 \text{ is rejected. Else, } H_0 \text{ is accepted.}$$

4.

i.

$$H_0 : \mu = 2.5, H_1 : \mu \neq 2.5.$$

$$L(\mu_0; \boldsymbol{x}) = (2\pi\sigma^2)^{-n/2} \exp[-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu_0)^2],$$

$$\sup_{\theta \in \Theta} L(\theta; \boldsymbol{x}) = (2\pi\sigma^2)^{-n/2} \exp[-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \bar{x})^2],$$

$$\lambda = \frac{L(\theta_0; x)}{\sup_{\theta \in \Theta} L(\theta; x)} = \exp[-\frac{n}{2\sigma^2} (\bar{x} - \mu_0)^2],$$

$$\lambda < \lambda_0 \Leftrightarrow \frac{n}{2\sigma^2} (\bar{x} - \mu_0)^2 > C_1 \Leftrightarrow \frac{\sqrt{n}|\bar{x} - \mu_0|}{\sigma} > C_2$$

$$\text{Let } P_{\mu_0}(\frac{\sqrt{n}|\bar{x} - \mu_0|}{\sigma} > C_2) = \alpha, \text{ then } C_2 = z_{\alpha/2}.$$

$$\text{The likelihood ratio test is } \varphi(\boldsymbol{x}) = \begin{cases} 1, & \frac{\sqrt{n}|\bar{x} - \mu_0|}{\sigma} > z_{\alpha/2} \\ 0, & \text{otherwise} \end{cases}$$

$$\frac{\sqrt{n}|\bar{x} - \mu_0|}{\sigma} = 0.8 < 1.959964 = z_{\alpha/2}, \therefore H_0 \text{ is accepted.}$$

(ii)

$$\text{The power function is}$$
$$\beta(\mu) = P_\mu(\frac{\sqrt{n}|\bar{x} - \mu_0|}{\sigma} > z_{\alpha/2})$$
$$= 1 + \Phi(-z_{\alpha/2} + \frac{\sqrt{n}(\mu_0 - \mu)}{\sigma}) - \Phi(z_{\alpha/2} + \frac{\sqrt{n}(\mu_0 - \mu)}{\sigma})$$
$$= 1 + \Phi(98.04 - 40\mu) - \Phi(101.96 - 40\mu)$$

5.

i.

$$\text{The joint likelihood function is}$$
$$L(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2; \boldsymbol{x}, \boldsymbol{y}) = (2\pi)^{-\frac{m+n}{2}} (\sigma_1^2)^{-\frac{m}{2}} (\sigma_2^2)^{-\frac{n}{2}} \exp[-\frac{1}{2\sigma_1^2} \sum_{i=1}^{m} (x_i - \mu_1)^2 - \frac{1}{2\sigma_2^2} \sum_{i=1}^{n} (y_i - \mu_2)^2],$$

$$\log L(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2; \boldsymbol{x}, \boldsymbol{y}) = -\frac{m+n}{2} \log 2\pi - \frac{m}{2} \log \sigma_1^2 - \frac{n}{2} \log \sigma_2^2 - \frac{1}{2\sigma_1^2} \sum_{i=1}^{m} (x_i - \mu_1)^2 - \frac{1}{2\sigma_2^2} \sum_{i=1}^{n} (y_i - \mu_2)^2$$

$$Let \begin{cases} \frac{\partial \log L(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2; \boldsymbol{x}, \boldsymbol{y})}{\partial \mu_1} = 0 \\ \frac{\partial \log L(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2; \boldsymbol{x}, \boldsymbol{y})}{\partial \mu_2} = 0 \\ \frac{\partial \log L(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2; \boldsymbol{x}, \boldsymbol{y})}{\partial (\sigma_1^2)} = 0 \\ \frac{\partial \log L(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2; \boldsymbol{x}, \boldsymbol{y})}{\partial (\sigma_2^2)} = 0 \end{cases},$$

$$\text{we can get the only maximun point } (\bar{x}, \bar{y}, \frac{1}{m}\sum_{i=1}^{m}(x_i-\bar{x})^2, \frac{1}{n}\sum_{i=1}^{n}(y_i-\bar{y})^2),$$

$$\therefore \sup_{\sigma_1\neq\sigma_2} L(\mu_1,\mu_2,\sigma_1^2,\sigma_2^2;\boldsymbol{x},\boldsymbol{y}) = (2\pi)^{-\frac{m+n}{2}}[\frac{1}{m}\sum_{i=1}^{m}(x_i-\bar{x})^2]^{-\frac{m}{2}}[\frac{1}{n}\sum_{i=1}^{n}(y_i-\bar{y})^2]^{-\frac{n}{2}}\exp(-\frac{m+n}{2}).$$

$$\text{Similarly, } \sup_{\sigma_1=\sigma_2=\sigma} L(\mu_1,\mu_2,\sigma^2;\boldsymbol{x},\boldsymbol{y}) = (2\pi)^{-\frac{m+n}{2}}\{\frac{1}{m+n}[\sum_{i=1}^{m}(x_i-\bar{x})^2+\sum_{i=1}^{n}(y_i-\bar{y})^2]\}^{-\frac{m+n}{2}}\exp(-\frac{m+n}{2}),$$

$$\lambda = \frac{\{\frac{1}{m+n}[\sum_{i=1}^{m}(x_i-\bar{x})^2+\sum_{i=1}^{n}(y_i-\bar{y})^2]\}^{-\frac{m+n}{2}}}{[\frac{1}{m}\sum_{i=1}^{m}(x_i-\bar{x})^2]^{-\frac{m}{2}}[\frac{1}{n}\sum_{i=1}^{n}(y_i-\bar{y})^2]^{-\frac{n}{2}}}.$$

$$\lambda<\lambda_0 \Leftrightarrow \frac{(n-1)\sum_{i=1}^{m}(x_i-\bar{x})^2}{(m-1)\sum_{i=1}^{n}(y_i-\bar{y})^2} < C_1 or \frac{(n-1)\sum_{i=1}^{m}(x_i-\bar{x})^2}{(m-1)\sum_{i=1}^{n}(y_i-\bar{y})^2} > C_2.$$

$$\text{Under assumption of } H_0, \frac{(n-1)\sum_{i=1}^{m}(x_i-\bar{x})^2}{(m-1)\sum_{i=1}^{n}(y_i-\bar{y})^2} \sim F_{m-1,n-1},$$

$$\therefore C_1 = F_{m-1,n-1,1-\alpha/2}, C_2 = F_{m-1,n-1,\alpha/2} \text{ make } P(\lambda<\lambda_0)=\alpha.$$

The likelihood ratio test is

$$\varphi(\boldsymbol{x}) = \begin{cases} 1, & \frac{(n-1)\sum_{i=1}^{m}(x_i-\bar{x})^2}{(m-1)\sum_{i=1}^{n}(y_i-\bar{y})^2} < F_{m-1,n-1,1-\alpha/2} \text{ or } \frac{(n-1)\sum_{i=1}^{m}(x_i-\bar{x})^2}{(m-1)\sum_{i=1}^{n}(y_i-\bar{y})^2} > F_{m-1,n-1,\alpha/2} \\ 0, & otherwise \end{cases}$$

$$Inthisquestion, \frac{s_x^2}{s_y^2} = \frac{4}{9} > 0.2295034 = F_{8,9,0.025}, \therefore H_0 \text{ is accepted.}$$

(ii)

$$\mathcal{F}_{m-1,n-1}(x) \text{ is the c.d.f. of F-distribution } F_{m-1,n-1}.$$

$$\because \frac{(n-1)\sigma_2^2\sum_{i=1}^{m}(x_i-\bar{x})^2}{(m-1)\sigma_1^2\sum_{i=1}^{n}(y_i-\bar{y})^2} \sim F_{m-1,n-1},$$

$$\therefore \text{ the power function is}$$

$$\beta(\sigma_1,\sigma_2) = P_{\sigma_1,\sigma_2}(\frac{(n-1)\sum_{i=1}^{m}(x_i-\bar{x})^2}{(m-1)\sum_{i=1}^{n}(y_i-\bar{y})^2} < F_{m-1,n-1,1-\alpha/2}) + P_{\sigma_1,\sigma_2}(\frac{(n-1)\sum_{i=1}^{m}(x_i-\bar{x})^2}{(m-1)\sum_{i=1}^{n}(y_i-\bar{y})^2} > F_{m-1,n-1,\alpha/2})$$

$$= P_{\sigma_1,\sigma_2}(\frac{(n-1)\sigma_2^2\sum_{i=1}^{m}(x_i-\bar{x})^2}{(m-1)\sigma_1^2\sum_{i=1}^{n}(y_i-\bar{y})^2} < \frac{\sigma_2^2}{\sigma_1^2}F_{m-1,n-1,1-\alpha/2}) + P_{\sigma_1,\sigma_2}(\frac{(n-1)\sigma_2^2\sum_{i=1}^{m}(x_i-\bar{x})^2}{(m-1)\sigma_1^2\sum_{i=1}^{n}(y_i-\bar{y})^2} > \frac{\sigma_2^2}{\sigma_1^2}F_{m-1,n-1,\alpha/2})$$

$$= 1 + \mathcal{F}_{m-1,n-1}(\frac{\sigma_2^2}{\sigma_1^2}F_{m-1,n-1,1-\alpha/2}) - \mathcal{F}_{m-1,n-1}(\frac{\sigma_2^2}{\sigma_1^2}F_{m-1,n-1,\alpha/2}).$$

$$\text{so } \beta(2,3) = 1 + \mathcal{F}_{8,9}(\frac{9}{4}F_{8,9,0.025}) - \mathcal{F}_{8,9}(\frac{9}{4}F_{8,9,0.975}) = 0.1838922.$$

6.

i.

$$\text{The likelihood function is } L(\theta;\boldsymbol{x}) = \prod_{i=1}^{n} \frac{1}{1-\theta} I_{\theta \le x_i \le 1} = \frac{1}{(1-\theta)^n} I_{\theta \le x(1)},$$

$$\sup_{\theta\in\Theta} L(\theta;\boldsymbol{x}) = \frac{1}{(1-x_{(1)})^n},$$

$$\sup_{\theta\in\Theta_0} L(\theta;\boldsymbol{x}) = \begin{cases} \frac{1}{(1-x(1))^n}, & \theta_0 \le x_{(1)} \\ 0, & \theta_0 > x_{(1)} \end{cases},$$

$$\lambda = \frac{\sup_{\theta\in\Theta_0} L(\theta;x)}{\sup_{\theta\in\Theta} L(\theta;x)} = \begin{cases} 1, & \theta_0 \le x_{(1)} \\ 0, & \theta_0 > x_{(1)} \end{cases}.$$

$$\text{The likelihood ratio test is } \varphi(\boldsymbol{x}) = \begin{cases} 1, & \theta_0 > x_{(1)} \\ 0, & \theta_0 \le x_{(1)} \end{cases}$$

ii.

$$\text{The p.d.f. of } X_{(1)} \text{ is } f(x) = n(1 - \frac{x-\theta}{1-\theta})^{n-1} \cdot \frac{1}{1-\theta} = \frac{n}{(1-\theta)^n}(1-x)^{n-1}, \theta < x < 1.$$

$$\text{If } \theta_0 > \theta, \text{ then } P_\theta(\theta_0 > X_{(1)}) = \int_{\theta}^{\theta_0} f(x)dx = 1 - (\frac{1-\theta_0}{1-\theta})^n.$$

$$\text{If } \theta_0 \le \theta, \text{ then } P_\theta(\theta_0 > X_{(1)}) = 0.$$

$$\text{The power function is } \beta(\theta) = P_\theta(\theta_0 > X_{(1)}) = \begin{cases} 1 - (\frac{1-\theta_0}{1-\theta})^n, & \theta < \theta_0 \\ 0, & \theta \ge \theta_0 \end{cases}.$$