Homework 5

Xupeng Chen

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1

i. The likelihood function is

$$L(\theta;\boldsymbol{x})=\prod_{i=1}^{n}f(x;\theta)=$$

$$\prod_{i=1}^{n}\frac{1}{\theta}I_{\theta\le x_i\le 2\theta}=\frac{1}{\theta^n}I_{x(1)\ge\theta}I_{x(n)\le 2\theta}$$

To maximize this function,we should minimize θ.\ because

$$\frac{x_{(n)}}{2}\le\theta\le x_{(1)}$$

so the MLE of θ is $\frac{x(n)}{2}$.

ii. it's not the unbiased estimation

$$E(x_{(n)})=\int_{\theta}^{2\theta}x\cdot n(\frac{x-\theta}{\theta})^{n-1}\frac{1}{\theta}dx=\frac{n}{\theta^n}\int_{\theta}^{2\theta}x(x-\theta)^{n-1}dx=\frac{1}{\theta^n}\int_{\theta}^{2\theta}xd(x-\theta)^n dx$$
$$=\frac{1}{\theta^n}[x(x-\theta)^n|_{\theta}^{2\theta}-\int_{\theta}^{2\theta}(x-\theta)^n dx]=\frac{1}{\theta^n}(2\theta^{n+1}-\frac{\theta^{n+1}}{n+1})=\frac{2n+1}{n+1}\theta.$$

so we have one unbiased estimate of θ: $\frac{n+1}{2n+1}x_{(n)}$

2

The log-likelihood:

$$\log L(a,\sigma;\boldsymbol{x})=\sum_{i=1}^{n}\log f(x_i;a,\sigma)=$$

$$\sum_{i=1}^{n}(-\log 2\sigma-\frac{|x_i-a|}{\sigma})=-n\log 2\sigma-\frac{\sum_{i=1}^{n}|x-a|}{\sigma}.$$

$$\because\ \frac{\partial\log L(a,\sigma;\boldsymbol{x})}{\partial a}=\begin{cases}n, & a<x_{(1)}\\ n-2i, & x_{(i)}<a<x_{(i+1)},i=1,2,\cdots,n-1\\ -n, & a>x_{(n)}\\ \text{not exist}, & a=x_{(i)},i=1,2,\cdots,n\end{cases}$$

$$\therefore a_0=median(x_i)=\begin{cases}x_{(\frac{n+1}{2})}, & n\text{ is odd}\\ \frac{1}{2}(x_{(\frac{n}{2})}+x_{(\frac{n}{2}+1)}), & n\text{ is even}\end{cases}$$
$$\text{maximize }\log L(a,\sigma;x)\text{ with any determined }\sigma.$$

$$\frac{\partial \log L(a,\sigma;\boldsymbol{x})}{\partial \sigma} = -\frac{n}{\sigma} + \frac{\sum_{i=1}^{n} |x-a|}{\sigma^2} = 0 \Rightarrow \sigma = \frac{\sum_{i=1}^{n} |x-a|}{n}.$$

$$\frac{\partial^2 \log L(a,\sigma;\boldsymbol{x})}{\partial \sigma^2}\Big|_{\sigma=\frac{\sum_{i=1}^{n}|x-a|}{n}} = (\frac{n}{\sigma^2} - \frac{2\sum_{i=1}^{n} |x-a|}{\sigma^3})\Big|_{\sigma=\frac{\sum_{i=1}^{n}|x-a|}{n}} = -\frac{n^3}{(\sum_{i=1}^{n} |x-a|)^2} < 0.$$

$$\text{Let} a_0 = median(x_i),\ \sigma_0 = \frac{\sum_{i=1}^{n} |x - median(x_i)|}{n},$$
$$\because \forall \sigma_1 > 0, -\infty < a_1 < \infty$$
$$\text{from the previous condition we have } L(a_1, \sigma_1; \boldsymbol{x}) \le L(a_0, \sigma_0; \boldsymbol{x}).$$
$$\text{so } a_0 = median(x_i), \sigma_0 = \frac{\sum_{i=1}^{n} |x - median(x_i)|}{n} \text{ maximize } \log L(a,\sigma;\boldsymbol{x}),$$
$$\text{so the MLE of } (a,\sigma) \text{ is } (median(x_i), \frac{\sum_{i=1}^{n} |x - median(x_i)|}{n}).$$

3

The log-likelihood function is

$$\log L(\alpha;\boldsymbol{x}) = \sum_{i=1}^{n} \log f(x_i;\alpha) = \sum_{i=1}^{n} (\log\alpha + \log\beta + (\beta-1)\log x_i - \alpha x_i^{\beta})$$
$$= n\log\alpha + n\log\beta + (\beta-1)\sum_{i=1}^{n}\log x_i - \alpha\sum_{i=1}^{n} x_i^{\beta}.$$

$$\text{because } \frac{d\log L(\alpha;\boldsymbol{x})}{d\alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} x_i^{\beta} = 0$$
$$\text{we can derive } \alpha = \frac{n}{\sum_{i=1}^{n} x_i^{\beta}} \text{ and } \frac{d^2\log L(\alpha;\boldsymbol{x})}{d\alpha^2} = -\frac{n}{\alpha^2} < 0,$$
$$\text{then we have the MLE of } \alpha \text{ is } \frac{n}{\sum_{i=1}^{n} x_i^{\beta}}.$$

4

1.

$$\text{because } X_i \sim \text{Exp}(\theta) \text{ so}$$
$$E(\bar{X}) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = E(X_i) = \frac{1}{\theta}.$$
$$E(X_{(1)}) = \int_0^{\infty} x \cdot n[1-(1-e^{-\theta x})]^{n-1}\theta e^{-\theta x}dx = \int_0^{\infty} n\theta x e^{-n\theta x}dx = -\int_0^{\infty} x de^{-n\theta x}$$
$$= -xe^{-n\theta x}|_0^{\infty} + \int_0^{\infty} e^{-n\theta x}dx = \frac{1}{n\theta}, \therefore$$
$$\text{from the property of the expectation: } E(nX_{(1)}) = \frac{1}{\theta}$$

so $\bar{X}$ and $nX_{(1)}$ are both unbiased estimates of $\frac{1}{\theta}$.

2.

$$E(\bar{X}-\frac{1}{\theta})^2 = Var(\bar{X}) = \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) = \frac{1}{n^2}\cdot n\frac{1}{\theta^2} = \frac{1}{n\theta^2}.$$

$$E(nX_{(1)}-\frac{1}{\theta})^2 = \int_0^{\infty}(nx-\frac{1}{\theta})^2\cdot n(e^{-\theta x})^{n-1}\theta e^{-\theta x}dx$$

$$= n^3\theta\int_0^{\infty}x^2e^{-n\theta x}dx - 2n^2\int_0^{\infty}xe^{-n\theta x}dx + \frac{n}{\theta}\int_0^{\infty}e^{-n\theta x}dx = \frac{2}{\theta^2}-\frac{2}{\theta^2}+\frac{1}{\theta^2} = \frac{1}{\theta^2}.$$

$$\text{we have}\frac{1}{\theta^2} \geq \frac{1}{n\theta^2} \rightarrow \mathbf{MSE}(\bar{X}) < \mathbf{MSE}(nX_{(1)})$$

so I will choose $\bar{X}$ rather than $nX_{(1)}$ when considering MSE.

5

we already know that $(n-1)S^2$ is a sufficient statistic of σ^2 and S^2 is an unbiased estimate of σ^2.

$$\text{Let T } = (n-1)S^2,\ h(T) = \frac{T}{n-1} = S^2,\ \text{so } \frac{T}{\sigma^2} \sim \chi^2_{n-1},$$

$$\text{so the p.d.f. of T is} f(t;\sigma^2) = \frac{1}{\Gamma(\frac{n-1}{2})2^{\frac{n-1}{2}}}e^{-\frac{t}{2\sigma^2}}(\frac{t}{\sigma^2})^{\frac{n-3}{2}}\cdot\frac{1}{\sigma^2} =$$

$$\frac{1}{\Gamma(\frac{n-1}{2})2^{\frac{n-1}{2}}(\sigma^2)^{\frac{n-1}{2}}}e^{-\frac{t}{2\sigma^2}}t^{\frac{n-3}{2}}, t > 0.$$

$$\frac{df(t;\sigma^2)}{d(\sigma^2)} = \frac{1-n}{2\Gamma(\frac{n-1}{2})2^{\frac{n-1}{2}}(\sigma^2)^{\frac{n+1}{2}}}e^{-\frac{t}{2\sigma^2}}t^{\frac{n-3}{2}} + \frac{1}{\Gamma(\frac{n-1}{2})2^{\frac{n-1}{2}}(\sigma^2)^{\frac{n-1}{2}}}e^{-\frac{t}{2\sigma^2}}t^{\frac{n-3}{2}}\cdot\frac{t}{2\sigma^4} =$$

$$(\frac{1-n}{2\sigma^2}+\frac{t}{2\sigma^4})f(t;\sigma^2)$$

we make $\delta(T)$ be any zero unbiased estimate:

$$E_\sigma(\delta(T)) = \int_{-\infty}^{\infty}\delta(t)f(t;\sigma^2)dt = 0.$$

$$\text{calculate the differentiation in respect to } \sigma^2, \text{we have}$$

$$\int_{-\infty}^{\infty}(\frac{1-n}{2\sigma^2}+\frac{t}{2\sigma^4})\delta(t)f(t;\sigma^2)dt = 0.$$

$$\because \int_{-\infty}^{\infty}\frac{1-n}{2\sigma^2}\delta(t)f(t;\sigma^2)dt = \frac{1-n}{2\sigma^2}\int_{-\infty}^{\infty}\delta(t)f(t;\sigma^2)dt = 0,$$

$$\therefore \int_{-\infty}^{\infty}\frac{t}{2\sigma^4}\delta(t)f(t;\sigma^2)dt = 0, \therefore \int_{-\infty}^{\infty}t\delta(t)f(t;\sigma^2)dt = 0, \rightarrow E_\sigma(T\delta(T)) = 0,$$

$$\therefore E_\sigma(h(T)\delta(T)) = \frac{1}{n-1}E_\sigma(T\delta(T)) = 0$$

so we proved that S^2 is the UMVUE of σ^2.

6

(1)We have proven that $(\bar{X} = \frac{\sum_{i=1}^{n}X_i}{n}, S^2 = \frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{n-1})$ is sufficient and complete for (μ,σ^2).

$$\because E(3\bar{X}+4S^2)=3E(\bar{X})+4E(S^2)=3\mu+4\sigma^2.$$

so according to L-S Theorem, the UMVUE of $3\mu+4\sigma^2$ is $3\bar{X}+4S^2$.

(2) Let $T=(n-1)S^2$, from the question above, the p.d.f. of T is:

$$f(t;\sigma^2)=\frac{1}{\Gamma(\frac{n-1}{2})2^{\frac{n-1}{2}}}e^{-\frac{t}{2\sigma^2}}(\frac{t}{\sigma^2})^{\frac{n-3}{2}}\cdot\frac{1}{\sigma^2},t>0.$$

$$E(\frac{1}{S^2})=E(\frac{n-1}{T})=(n-1)\int_0^\infty\frac{1}{t}f(t;\sigma^2)dt=\frac{n-1}{\sigma^2}\int_0^\infty\frac{1}{\Gamma(\frac{n-1}{2})2^{\frac{n-1}{2}}}e^{-\frac{t}{2\sigma^2}}(\frac{t}{\sigma^2})^{\frac{n-5}{2}}d(\frac{t}{\sigma^2})$$

$$=\frac{n-1}{\sigma^2}\int_0^\infty\frac{1}{(n-3)\Gamma(\frac{n-3}{2})2^{\frac{n-3}{2}}}e^{-\frac{t}{2\sigma^2}}(\frac{t}{\sigma^2})^{\frac{n-3-2}{2}}d(\frac{t}{\sigma^2})=\frac{n-1}{(n-3)\sigma^2}.$$

$$E(\bar{X}^2)=Var(\bar{X})+(E(\bar{X}))^2=\frac{\sigma^2}{n}+\mu^2.$$

$$\because \bar{X} \text{ and } S^2 \text{are independent,}$$

$$\therefore E(\frac{\bar{X}^2}{S^2})=E(\bar{X}^2)E(\frac{1}{S^2})=(\frac{\sigma^2}{n}+\mu^2)\frac{n-1}{(n-3)\sigma^2}=\frac{n-1}{n(n-3)}+\frac{(n-1)\mu^2}{(n-3)\sigma^2},$$

$$\therefore E(\frac{(n-3)\bar{X}^2}{4(n-1)S^2}-\frac{1}{4n})=\frac{\mu^2}{4\sigma^2}.$$

According to L-S Theorem, the UMVUE of $\frac{\mu^2}{4\sigma^2}$ is $\frac{(n-3)\bar{X}^2}{4(n-1)S^2}-\frac{1}{4n}$.

7

Let $T=\sum_{i=1}^n X_i^2$, then

$$\hat{\sigma}=\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n+1}{2})}\sqrt{\frac{T}{2}}\text{and}\frac{T}{\sigma^2}\sim\chi_n^2.$$

$$The p.d.f. of T is f(t;\sigma^2)=\frac{1}{\Gamma(\frac{n}{2})2^{\frac{n}{2}}}e^{-\frac{t}{2\sigma^2}}(\frac{t}{\sigma^2})^{\frac{n-2}{2}}\cdot\frac{1}{\sigma^2},t>0.$$

$$E(\hat{\sigma})=\int_0^\infty\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n+1}{2})}\sqrt{\frac{t}{2}}\cdot\frac{1}{\Gamma(\frac{n}{2})2^{\frac{n}{2}}}e^{-\frac{t}{2\sigma^2}}(\frac{t}{\sigma^2})^{\frac{n-2}{2}}\cdot\frac{1}{\sigma^2}dt=\sigma\int_0^\infty\frac{1}{\Gamma(\frac{n+1}{2})2^{\frac{n+1}{2}}}e^{-\frac{t}{2\sigma^2}}(\frac{t}{\sigma^2})^{\frac{n+1-2}{2}}d(\frac{t}{\sigma^2})=\sigma.$$

$\because$ T is a sufficient and complete statistic of σ and $\hat{\sigma}=\hat{g}(T)$ is an unbiast estimate of σ

$\therefore \hat{\sigma}$ is the UMVUE of σ.

$$\log f(x;\sigma) = -\log\sqrt{2\pi} - \log\sigma - \frac{x^2}{2\sigma^2}, \frac{\partial \log f(x;\sigma)}{\partial\sigma} = -\frac{1}{\sigma} + \frac{x^2}{\sigma^3}.$$

$$I(\sigma) = E_\sigma[\frac{\partial \log f(X;\sigma)}{\partial\sigma}]^2 = E_\sigma(\frac{1}{\sigma^2} - \frac{2X^2}{\sigma^4} + \frac{X^4}{\sigma^6}) = \frac{1}{\sigma^2} - \frac{2E(X^2)}{\sigma^4} + \frac{E(X^4)}{\sigma^6}$$

$$\text{the m.g.f of } N(0,\sigma^2) \text{ is} M(t) = e^{\frac{\sigma^2 t^2}{2}},$$

$$\text{so we have } E(X^2) = \frac{d^2 M(t)}{dt^2}|_{t=0} = \sigma^2, E(X^4) = \frac{d^4 M(t)}{dt^4}|_{t=0} = 3\sigma^4, \text{ so } I(\sigma) = \frac{2}{\sigma^2}.$$

$$\text{and } Var(\hat{\sigma}) = E(\hat{\sigma}^2) - E^2(\hat{\sigma}) = \frac{\Gamma^2(\frac{n}{2})}{2\Gamma^2(\frac{n+1}{2})}E(T) - \sigma^2 = \frac{n\Gamma^2(\frac{n}{2})}{2\Gamma^2(\frac{n+1}{2})}\sigma^2 - \sigma^2.$$

$$\text{if set } g(\sigma) = \sigma, \text{we can calculate the efficiency is:}$$

$$e_{\hat{g}}(\sigma) = \frac{1}{nI(\sigma)Var(\hat{\sigma})} = \frac{\Gamma^2(\frac{n+1}{2})}{n^2\Gamma^2(\frac{n}{2}) - 2n\Gamma^2(\frac{n+1}{2})}.$$