5.3.1 Modified Euler Method

Numerical solution of Initial Value Problem:

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = f(t, Y) \Leftrightarrow Y(t_{n+1}) = Y(t_n) + \int_{t_n}^{t_{n+1}} f(t, Y(t))\,\mathrm{d}t.$$

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Approximate integral using the trapezium rule:

$$Y(t_{n+1}) \approx Y(t_n) + \frac{h}{2}\left[f(t_n, Y(t_n)) + f(t_{n+1}, Y(t_{n+1}))\right], \quad t_{n+1} = t_n + h.$$

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Use Euler's method to approximate $Y(t_{n+1}) \approx Y(t_n) + hf(t_n, Y(t_n))$ in trapezium rule:

$$Y(t_{n+1}) \approx Y(t_n) + \frac{h}{2}\left[f(t_n, Y(t_n)) + f(t_{n+1}, \color{red}{Y(t_n) + hf(t_n, Y(t_n))})\right].$$

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Hence the **modified Euler**'s scheme

$$y_{n+1} = y_n + \frac{h}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_n + hf(t_n, y_n))\right] \Leftrightarrow \begin{cases} K_1 = hf(t_n, y_n) \\ K_2 = hf(t_{n+1}, y_n + K_1) \\ y_{n+1} = y_n + \dfrac{K_1 + K_2}{2} \end{cases}$$

5.3.1 Modified Euler Method — Local truncation error (1/3)

Local truncation error due to the approximation:

$$Y(t_{n+1}) \approx Y(t_n) + \frac{1}{2}(K_1 + K_2)$$

where $K_1 = hf(t_n, Y(t_n))$ and $K_2 = hf(t_n + h, Y(t_n) + K_1)$.

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Taylor Series of $f(t_n + h, Y(t_n) + K_1)$ in two variables:

$$K_2 = h\left[f(t_n, Y(t_n)) + h\frac{\partial}{\partial t}f(t_n, Y(t_n)) + K_1\frac{\partial}{\partial Y}f(t_n, Y(t_n)) + O\left(h^2, K_1^2\right)\right].$$

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Since $K_1 = hf(t_n, Y(t_n)) = O(h)$,

$$\frac{1}{2}(K_1 + K_2) = hf(t_n, Y(t_n)) \\ + \frac{h^2}{2}\left[\frac{\partial}{\partial t}f(t_n, Y(t_n)) + f(t_n, Y(t_n))\frac{\partial}{\partial Y}f(t_n, Y(t_n))\right] + O\left(h^3\right),$$

Expression to be compared with Taylor expansion of $Y(t_{n+1})$

Taylor Series of $Y(t_{n+1}) = Y(t_n + h)$:

$$Y(t_n + h) = Y(t_n) + hY'(t_n) + \frac{h^2}{2} Y''(t_n) + O\left(h^3\right).$$

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Substitute $Y'(t_n) = f(t_n, Y(t_n))$ and

$$Y''(t_n) = \frac{\mathrm{d}}{\mathrm{d}t} f(t, Y(t))\bigg|_{t_n} = \frac{\partial}{\partial t} f(t_n, Y(t_n)) + \frac{\mathrm{d}}{\mathrm{d}t} Y(t_n) \frac{\partial}{\partial Y} f(t_n, Y(t_n)),$$

$$= \frac{\partial}{\partial t} f(t_n, Y(t_n)) + f(t_n, Y(t_n)) \frac{\partial}{\partial Y} f(t_n, Y(t_n)),$$

Taylor Series of $Y(t_{n+1}) = Y(t_n + h)$:

$$Y(t_n + h) = Y(t_n) + hY'(t_n) + \frac{h^2}{2} Y''(t_n) + O\left(h^3\right).$$

Substitute $Y'(t_n) = f(t_n, Y(t_n))$ and

$$\begin{aligned} Y''(t_n) = \left.\frac{\mathrm{d}}{\mathrm{d}t} f(t, Y(t))\right|_{t_n} &= \frac{\partial}{\partial t} f(t_n, Y(t_n)) + \frac{\mathrm{d}}{\mathrm{d}t} Y(t_n) \frac{\partial}{\partial Y} f(t_n, Y(t_n)), \\ &= \frac{\partial}{\partial t} f(t_n, Y(t_n)) + f(t_n, Y(t_n)) \frac{\partial}{\partial Y} f(t_n, Y(t_n)), \end{aligned}$$

to get

$$\begin{aligned} Y(t_n + h) &= Y(t_n) + hf(t_n, Y(t_n)) \\ &+ \frac{h^2}{2} \left[\frac{\partial}{\partial t} f(t_n, Y(t_n)) + f(t_n, Y(t_n)) \frac{\partial}{\partial Y} f(t_n, Y(t_n)) \right] + O\left(h^3\right). \end{aligned} \tag{5.10}$$

Now, the equations

$$\begin{aligned} Y(t_{n+1}) = Y(t_n) &+ hf(t_n, Y(t_n)) \\ &+ \frac{h^2}{2}\left[\frac{\partial}{\partial t} f(t_n, Y(t_n)) + f(t_n, Y(t_n))\frac{\partial}{\partial Y} f(t_n, Y(t_n))\right] + O\left(h^3\right) \end{aligned}$$

5.3.1 Modified Euler Method — Local truncation error (3/3)

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and

$$\begin{aligned} \frac{1}{2}(K_1 + K_2) = &\, hf(t_n, Y(t_n)) \\ &+ \frac{h^2}{2}\left[\frac{\partial}{\partial t} f(t_n, Y(t_n)) + f(t_n, Y(t_n)) \frac{\partial}{\partial Y} f(t_n, Y(t_n))\right] + O\left(h^3\right) \end{aligned}$$

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$$\begin{aligned} \frac{1}{2}(K_1 + K_2) &= hf(t_n, Y(t_n)) \\ &\quad + \frac{h^2}{2}\left[\frac{\partial}{\partial t} f(t_n, Y(t_n)) + f(t_n, Y(t_n))\frac{\partial}{\partial Y} f(t_n, Y(t_n))\right] + O\left(h^3\right) \end{aligned}$$

imply that $Y(t_{n+1}) = Y(t_n) + \frac{1}{2}(K_1 + K_2) + O\left(h^3\right)$.

The local truncation error is $\tau_n = O\left(h^3\right)$: the modified Euler method is second order accurate. (A method is conventionally called p^{th} order if the local truncation error is of order $p+1$.)

5.3.2 Second order Runge-Kutta schemes (1/3)

Modified Euler is an example of 2^{nd} order R-K method.

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The **general 2nd order Runge-Kutta** scheme takes the form

$$\begin{cases} K_1 = hf(t_n, y_n); \\ K_2 = hf(t_n + \alpha h, y_n + \beta K_1); \\ y_{n+1} = y_n + a_1 K_1 + a_2 K_2. \end{cases} \tag{5.11}$$

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Repeating the earlier analysis, $K_1 = hf(t_n, Y(t_n))$ and

$$\begin{aligned} K_2 = {} & hf(t_n, Y(t_n)) \\ & + h^2 \left[\alpha \frac{\partial}{\partial t} f(t_n, Y(t_n)) + \beta f(t_n, Y(t_n)) \frac{\partial}{\partial Y} f(t_n, Y(t_n)) \right] + O(h^3); \end{aligned}$$

$$\begin{aligned} \Rightarrow a_1 K_1 + a_2 K_2 = {} & h(a_1 + a_2) f(t_n, Y(t_n)) \\ & + a_2 h^2 \left[\alpha \frac{\partial}{\partial t} f(t_n, Y(t_n)) + \beta f(t_n, Y(t_n)) \frac{\partial}{\partial Y} f(t_n, Y(t_n)) \right] + O(h^3). \end{aligned}$$

Comparing

$$
\begin{aligned}
a_1 K_1 + a_2 K_2 = {} & h(a_1 + a_2) f(t_n, Y(t_n)) \\
& + a_2 h^2 \left[\alpha \frac{\partial}{\partial t} f(t_n, Y(t_n)) + \beta f(t_n, Y(t_n)) \frac{\partial}{\partial Y} f(t_n, Y(t_n)) \right] + O(h^3)
\end{aligned}
$$

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a_1 K_1 + a_2 K_2 &= h(a_1 + a_2) f(t_n, Y(t_n)) \\
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\end{aligned}
$$

with equation (5.10)

$$
\begin{aligned}
Y(t_{n+1}) &= Y(t_n) + h f(t_n, Y(t_n)) \\
&\quad + \frac{h^2}{2} \left[\frac{\partial}{\partial t} f(t_n, Y(t_n)) + f(t_n, Y(t_n)) \frac{\partial}{\partial Y} f(t_n, Y(t_n)) \right] + O\left(h^3\right),
\end{aligned}
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Comparing

$$a_1 K_1 + a_2 K_2 = h(a_1 + a_2) f(t_n, Y(t_n)) + a_2 h^2 \left[\alpha \frac{\partial}{\partial t} f(t_n, Y(t_n)) + \beta f(t_n, Y(t_n)) \frac{\partial}{\partial Y} f(t_n, Y(t_n)) \right] + O(h^3)$$

with equation (5.10)

$$Y(t_{n+1}) = Y(t_n) + h f(t_n, Y(t_n)) + \frac{h^2}{2} \left[\frac{\partial}{\partial t} f(t_n, Y(t_n)) + f(t_n, Y(t_n)) \frac{\partial}{\partial Y} f(t_n, Y(t_n)) \right] + O\left(h^3\right),$$

one gets a second order scheme $Y(t_{n+1}) = Y(t_n) + a_1 K_1 + a_2 K_2 + O\left(h^3\right)$ if

$$\begin{cases} a_1 + a_2 = 1; \\ \alpha a_2 = \beta a_2 = \dfrac{1}{2}. \end{cases} \tag{5.12}$$

5.3.2 Second order Runge-Kutta Schemes (3/3)

General 2nd order Runge-Kutta scheme:

$$\begin{cases} K_1 = hf(t_n, y_n); \\ K_2 = hf(t_n + \alpha h, y_n + \beta K_1); \\ y_{n+1} = y_n + a_1 K_1 + a_2 K_2. \end{cases} \quad \text{with} \quad \begin{cases} a_1 + a_2 = 1; \\ \alpha a_2 = \beta a_2 = \dfrac{1}{2}. \end{cases}$$

Since we have 3 equations and 4 unknowns $(a_1, a_2, \alpha, \beta)$, there are infinitely many solutions.

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Since we have 3 equations and 4 unknowns $(a_1, a_2, \alpha, \beta)$, there are infinitely many solutions.

The most popular are,

- Modified Euler: $a_1 = a_2 = 1/2$, $\alpha = \beta = 1$.

  $\Rightarrow K_1 = hf(t_n, y_n)$, $K_2 = hf(t_n + h, y_n + K_1)$ and $y_{n+1} = y_n + (K_1 + K_2)/2$.

- Midpoint method: $a_1 = 0$, $a_2 = 1$, $\alpha = \beta = 1/2$.

  $\Rightarrow K_1 = hf(t_n, y_n)$, $K_2 = hf(t_n + h/2, y_n + K_1/2)$ and $y_{n+1} = y_n + K_2$.

- Heun's method: $a_1 = 1/4$, $a_2 = 3/4$, $\alpha = \beta = 2/3$.

  $\Rightarrow K_1 = hf(t_n, y_n)$, $K_2 = hf(t_n + 2h/3, y_n + 2K_1/3)$ and $y_{n+1} = y_n + (K_1 + 3K_2)/4$.

Schemes of the form (5.11) can be extended to higher order methods. The most widely used Runge-Kutta scheme is the 4^{th} order scheme RK4 based on Simpson's rule.

$$
\begin{aligned}
y_{n+1} &= y_n + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4), \\
\text{where } K_1 &= hf(t_n, y_n), \\
K_2 &= hf\left(t_n + \frac{h}{2}, y_n + \frac{K_1}{2}\right), \qquad (5.13) \\
K_3 &= hf\left(t_n + \frac{h}{2}, y_n + \frac{K_2}{2}\right), \\
K_4 &= hf(t_n + h, y_n + K_3).
\end{aligned}
$$

This scheme has local truncation error of order h^5, which can be checked in the same way as the second order scheme, but involves rather messy algebra.