Notes on the derivatives of Lotka-Volterra equations

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Considering the Lotka-Volterra equations:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\alpha x + \alpha' x^2 + \beta xy \tag{1}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \kappa y - \kappa' y^2 - \lambda xy \tag{2}$$

we can write them as

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y' = \underbrace{ay}_{y_1'} + \underbrace{by^2}_{y_2'} + \underbrace{cxy}_{y_3'} \tag{3}$$

following the logistic model. If $\alpha' = \kappa' = 0$, then we get the simple model.

For the Taylor method, we need to compute the successive derivatives. In order to find an expression that we can compute, let's first write the first derivatives by hand as

$$y^{(n+1)} = \frac{\mathrm{d}y^{(n)}}{\mathrm{d}t} = \frac{\mathrm{d}y_1^{(n)}}{\mathrm{d}t} + \frac{\mathrm{d}y_2^{(n)}}{\mathrm{d}t} + \frac{\mathrm{d}y_3^{(n)}}{\mathrm{d}t} = y_1^{(n+1)} + y_2^{(n+1)} + y_3^{(n+1)} \tag{4}$$

where $y^{(n)}$ is the n-th derivative of y with respect to t.

I am going to calculate the n derivatives of each $y_i^{(n)}$ term separately. The first one, $y_1^{(n)}$, is:

$$\begin{aligned} y_1^{(1)} &= ay \\ y_1^{(2)} &= ay^{(1)} \\ y_1^{(3)} &= ay^{(2)} \\ y_1^{(4)} &= ay^{(3)} \\ y_1^{(5)} &= ay^{(4)} \\ &\vdots \end{aligned}$$

so we can easily see that

$$y_1^{(n+1)} = ay^{(n)} \tag{5}$$

Let's skip the second term for now and focus on the third one, $y_3^{(n)}$:

$$
\begin{aligned}
y_3^{(1)} &= cxy \\
y_3^{(2)} &= c\left[xy^{(1)} + x^{(1)}y\right] \\
y_3^{(3)} &= c\left[xy^{(2)} + 2x^{(1)}y^{(1)} + x^{(2)}y\right] \\
y_3^{(4)} &= c\left[xy^{(3)} + 3x^{(1)}y^{(2)} + 3x^{(2)}y^{(1)} + x^{(3)}y\right] \\
y_3^{(5)} &= c\left[xy^{(4)} + 4x^{(1)}y^{(3)} + 6x^{(2)}y^{(2)} + 4x^{(3)}y^{(1)} + x^{(4)}y\right] \\
&\vdots
\end{aligned}
$$

I didn't found it so trivial to find a general expresion for $y_3^{(n)}$. The approach I followed was, first, to write the terms inside brackets in columns, ordering them by increasing (n) of $y^{(n)}$, as

$$
\begin{aligned}
y_3^{(1)} &= cxy \\
y_3^{(2)} &= c\left[x^{(1)}y + xy^{(1)}\right] \\
y_3^{(3)} &= c\left[x^{(2)}y + 2x^{(1)}y^{(1)} + xy^{(2)}\right] \\
y_3^{(4)} &= c\left[x^{(3)}y + 3x^{(2)}y^{(1)} + 3x^{(1)}y^{(2)} + xy^{(3)}\right] \\
y_3^{(5)} &= c\left[x^{(4)}y + 4x^{(3)}y^{(1)} + 6x^{(2)}y^{(2)} + 4x^{(1)}y^{(3)} + xy^{(4)}\right] \\
&\vdots
\end{aligned}
$$

or, including the zero terms to make it more visual

$$
\begin{aligned}
y_3^{(1)} &= c\left[1xy \quad + 0xy^{(1)} \quad + 0xy^{(2)} \quad + 0xy^{(3)} \quad + 0xy^{(4)}\right] \\
y_3^{(2)} &= c\left[1x^{(1)}y + 1xy^{(1)} \quad + 0xy^{(2)} \quad + 0xy^{(3)} \quad + 0xy^{(4)}\right] \\
y_3^{(3)} &= c\left[1x^{(2)}y + 2x^{(1)}y^{(1)} + 1xy^{(2)} \quad + 0xy^{(3)} \quad + 0xy^{(4)}\right] \\
y_3^{(4)} &= c\left[1x^{(3)}y + 3x^{(2)}y^{(1)} + 3x^{(1)}y^{(2)} + 1xy^{(3)} \quad + 0xy^{(4)}\right] \\
y_3^{(5)} &= c\left[1x^{(4)}y + 4x^{(3)}y^{(1)} + 6x^{(2)}y^{(2)} + 4x^{(1)}y^{(3)} + 1xy^{(4)}\right] \\
&\vdots
\end{aligned}
$$

I can write the coefficients of each column in a table, writing each i-th row for the i-th derivative, $y_3^{(i)}$, which depends on the j-th derivative, $y^{(j)}$, column j-th, by the coefficient i, j.

$$
\begin{array}{c|ccccc}
 & y & y^{(1)} & y^{(2)} & y^{(3)} & y^{(4)} \\
\hline
y^{(1)} & 1 & 0 & 0 & 0 & 0 \\
y^{(2)} & 1 & 1 & 0 & 0 & 0 \\
y^{(3)} & 1 & 2 & 1 & 0 & 0 \\
y^{(4)} & 1 & 3 & 3 & 1 & 0 \\
y^{(5)} & 1 & 4 & 6 & 4 & 1 \\
 & \multicolumn{5}{c}{\underbrace{\hphantom{1 \quad 4 \quad 6 \quad 4 \quad 1}}_{\mathbb{C}}}
\end{array}
$$

Creating a coefficient matrix $\mathbb{C}$, which can be computed as

$$
\mathbb{C}(i+1, j+1) = \mathbb{C}(i, j) + \mathbb{C}(i, j+1), \qquad 1 \leq i, j \leq N-1 \tag{6}
$$

with the conditions

$$\text{First column} \rightarrow \mathbb{C}_{i,1} = 1 \quad 1 \leq i \leq N \tag{7}$$
$$\text{First row} \rightarrow \mathbb{C}_{1,j} = 0 \quad 2 \leq j \leq N \tag{8}$$

I write the third term as

$$y_3^{(n+1)} = c \sum_{k=0}^{n} \mathbb{C}_{n,k}\, x^{(n-k)} y^{(k)} \tag{9}$$

(Note: the indexing of $\mathbb{C}$ in the previous equation starts in 0,0 to simplify formulation)

An improvement would be to write $\mathbb{C}_{i,j}$ in terms of n, but I didn't find a way in a first look, so I'm leaving this at it is.

Now, I go back to the second term, y_2, the last one to compute. If instead of writing it as $y_2 = by^2$ we write it as $y_2 = byy$, it has the same form of y_3 but substituting x for y. Then, just to simplify the code, I won't apply the commutative law for multiplication and calculate the derivatives as in the previous example

$$\begin{aligned}
y_2^{(1)} &= byy \\
y_2^{(2)} &= b\left[y^{(1)}y + yy^{(1)}\right] \\
y_2^{(3)} &= b\left[y^{(2)}y + 2y^{(1)}y^{(1)} + yy^{(2)}\right] \\
y_2^{(4)} &= b\left[y^{(3)}y + 3y^{(2)}y^{(1)} + 3y^{(1)}y^{(2)} + yy^{(3)}\right] \\
y_2^{(5)} &= b\left[y^{(4)}y + 4y^{(3)}y^{(1)} + 6y^{(2)}y^{(2)} + 4y^{(1)}y^{(3)} + yy^{(4)}\right] \\
&\vdots
\end{aligned}$$

Then, we can employ the same formulation as before and write $y_2^{(n)}$ as

$$y_2^{(n+1)} = b \sum_{k=0}^{n} \mathbb{C}_{n,k}\, y^{(n-k)} y^{(k)} \tag{10}$$

Finally, putting together eqs. (5), (10) and (9), I compute the derivatives as

$$\begin{aligned}
y^{(n+1)} &= y_1^{(n+1)} + y_2^{(n+1)} + y_3^{(n+1)} = \\
&= ay^{(n)} + b \sum_{k=0}^{n} \mathbb{C}_{n,k}\, y^{(n-k)} y^{(k)} + c \sum_{k=0}^{n} \mathbb{C}_{n,k}\, x^{(n-k)} y^{(k)}
\end{aligned}$$

$$\boxed{\therefore\; y^{(n+1)} = ay^{(n)} + b \sum_{k=0}^{n} \mathbb{C}_{n,k}\, y^{(n-k)} y^{(k)} + c \sum_{k=0}^{n} \mathbb{C}_{n,k}\, x^{(n-k)} y^{(k)}} \tag{11}$$