Cholesky decomposition: An example

Mathematical background

A real symmetric matrix $\mathbb{M}$ with dimension N is positive definite if

$$\mathbf{x}^t \mathbb{M}\, \mathbf{x} > 0 \qquad \forall \mathbf{x} \neq \mathbf{0} \in \mathbb{R}^N$$

- All its eigenvalues are larger than 0
- Cholesky decomposition is unique

Semi-positive: ≥ instead of >

Mathematical background

Cholesky decomposition can be understood as a particular LU factorization

$$M_{i,k} = (\mathbb{M})_{i,k} = \sum_{J}^{m} L_i^J L_k^J = \sum_{J}^{m} (\mathbb{L})_{i,J} (\mathbb{L})_{J,k}$$

$$\mathbb{M} = \mathbb{L}\,\mathbb{L}^t$$

Mathematical background

Operationally, iterative procedure, calculating one vector in each step

$$\mathbb{M}^{(J)} = \mathbb{M}^{(J-1)} - \mathbf{L}^{J}(\mathbf{L}^{J})^{t}$$

$$L_i^J = \frac{M_{i,J}^{(J-1)}}{\sqrt{M_{J,J}^{(J-1)}}}$$

Mathematical background

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Optimally, choose maximum diagonal element of $\mathbb{M}^{(n)}$

A worked-out example

Consider the matrix

$$\mathbb{M} \equiv \mathbb{M}^{(0)} = \begin{pmatrix} 36 & -18 & 12 \\ -18 & 34 & -1 \\ 12 & -1 & 21 \end{pmatrix}$$

Its eigenvalues are:

55.6895, 24.9443, 10.3661

Thus it is positive definite and 3 vectors are needed

A worked-out example

$$\mathbb{M}^{(0)} = \begin{pmatrix} 36 & -18 & 12 \\ -18 & 34 & -1 \\ 12 & -1 & 21 \end{pmatrix}$$

For the first vector, the first column is selected:

$$L_i^J = \frac{M_{i,J}^{(J-1)}}{\sqrt{M_{J,J}^{(J-1)}}} \longrightarrow L_i^1 = \frac{M_{i,1}^{(0)}}{\sqrt{M_{1,1}^{(0)}}} \longrightarrow \begin{cases} L_1^1 = \dfrac{36}{\sqrt{36}} = 6 \\ L_2^1 = \dfrac{-18}{\sqrt{36}} = -3 \\ L_3^1 = \dfrac{12}{\sqrt{36}} = 2 \end{cases} \longrightarrow \boldsymbol{L}^1 = \begin{pmatrix} 6 \\ -3 \\ 2 \end{pmatrix}$$

A worked-out example

Now the matrix must be updated:

$$\mathbb{M}^{(1)} = \mathbb{M}^{(0)} - \boldsymbol{L}^1 {\boldsymbol{L}^1}^T$$

$$= \begin{pmatrix} 36 & -18 & 12 \\ -18 & 34 & -1 \\ 12 & -1 & 21 \end{pmatrix} - \begin{pmatrix} 6 \\ -3 \\ 2 \end{pmatrix} \begin{pmatrix} 6 & -3 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 25 & 5 \\ 0 & 5 & 17 \end{pmatrix}$$

A worked-out example

$$\mathbb{M}^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 25 & 5 \\ 0 & 5 & 17 \end{pmatrix}$$

For the second vector, the second column is selected:

$$L_i^J = \frac{M_{i,J}^{(J-1)}}{\sqrt{M_{J,J}^{(J-1)}}} \longrightarrow L_i^2 = \frac{M_{i,2}^{(1)}}{\sqrt{M_{2,2}^{(1)}}} \longrightarrow \begin{cases} L_1^1 = \dfrac{0}{\sqrt{25}} = 0 \\ L_2^1 = \dfrac{25}{\sqrt{25}} = 5 \\ L_3^1 = \dfrac{5}{\sqrt{25}} = 1 \end{cases} \longrightarrow \boldsymbol{L}^2 = \begin{pmatrix} 0 \\ 5 \\ 1 \end{pmatrix}$$

A worked-out example

Now, again, the matrix must be updated:

$$\mathbb{M}^{(2)} = \mathbb{M}^{(1)} - \boldsymbol{L}^2 {\boldsymbol{L}^2}^T$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 25 & 5 \\ 0 & 5 & 17 \end{pmatrix} - \begin{pmatrix} 0 \\ 5 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 5 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 16 \end{pmatrix}$$

A worked-out example

At this moment, it should be clear that the third vector is simply

$$\boldsymbol{L}^3 = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$$

The $\mathbb{L}$ matrix if formed by the $\boldsymbol{L}$ vectors:

$$\mathbb{L} = \begin{pmatrix} \boldsymbol{L}^1 & \boldsymbol{L}^2 & \boldsymbol{L}^3 \end{pmatrix} = \begin{pmatrix} 6 & 0 & 0 \\ -3 & 5 & 0 \\ 2 & 1 & 4 \end{pmatrix}$$