THE BEAUTIFUL GEOMETRY OF APERIODIC TESSELLATIONS

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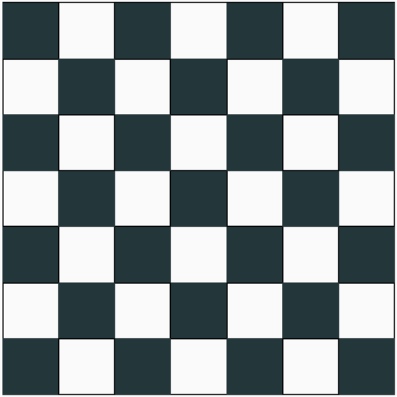
WHAT IS A TESSELLATION?

A SIMPLE TESSELLATION



Checkerboard Tessellation

WHAT IS A TESSELLATION?

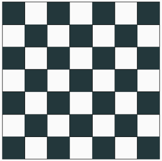
Definition:

*A **tessellation** $\mathcal{T}$ of the space $\mathbb{E}^n$ is a countable family of closed sets, T, called* tiles*:*

$$\mathcal{T} = \{T_1, T_2, \dots\}$$

such that

1. *$\mathcal{T}$ has **no overlaps**: $\mathring{T}_i \cap \mathring{T}_j = \emptyset$ if $i \neq j$*
2. *$\mathcal{T}$ has **no gaps**: $\bigcup_{i=1}^{\infty} T_i = \mathbb{E}^n$*



COMPONENTS OF A TESSELLATION

Definition:

Let $\{T_1, T_2, \ldots\}$ be the set of tiles of tessellation $\mathcal{T}$, partitioned into a set of equivalence classes by ***criterion*** *$\mathcal{M}$. The set, $\mathcal{P}$, of representatives of these equivalence classes is called the* ***protoset*** *for $\mathcal{T}$ with respect to $\mathcal{M}$.*

Example:

Criterion: $\mathcal{M} = \{$*Colour of the tile. Only opposite colours may touch.*$\}$

Protoset: $$\mathcal{P} = \{ \blacksquare , \square \}$$

PROTOSETS ADMIT TESSELLATIONS

Definition:

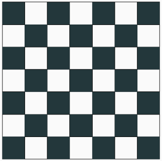
If $\mathcal{T}$ is a tessellation with protoset $\mathcal{P}$, then we say that $\mathcal{P}$ admits $\mathcal{T}$.

Example:

We say

$$\mathcal{P} = \{ \blacksquare , \square \}$$

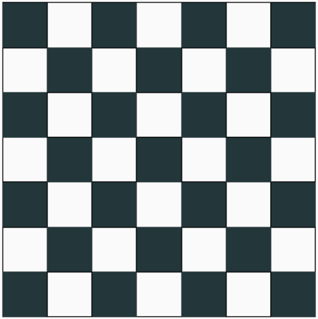
admits



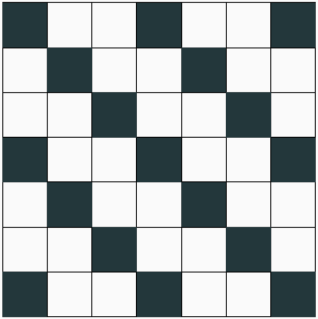
PROTOSETS CAN ADMIT MULTIPLE TESSELLATIONS

$$\mathcal{P} = \{ \blacksquare , \square \}$$

admits both



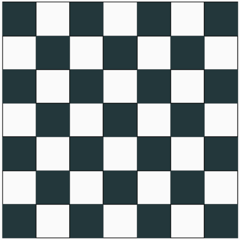
and



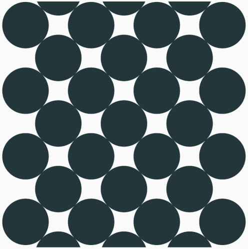
Edge deformations can force matching rules.

$$\mathcal{P} = \{ \blacksquare , \square \}$$

$$\mathcal{P} = \{ \bullet , $$  $$ \}$$



$$\longrightarrow$$



Matching Rules $\implies$ Deformed Protoset

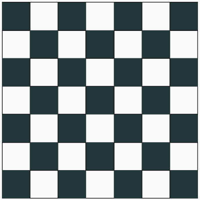
DESCRIBING TESSELLATIONS

Definition:

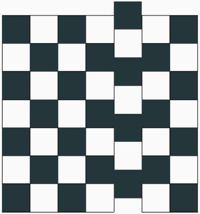
A tessellation is said to be ***periodic*** *if it admits translational symmetry in two directions.*

Definition:

A tessellation is said to be ***non-periodic*** *if it admits no translational symmetries.*



Periodic



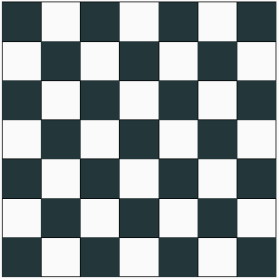
Non-Periodic

THE APERIODIC QUESTION

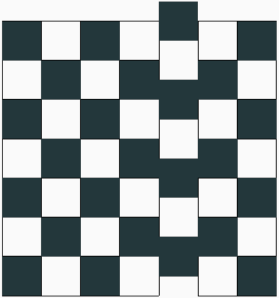
Checkerboard protoset,

$$\mathcal{P} = \{ \blacksquare , \square \}$$

admits both **periodic** and **non-periodic** tessellations.



Periodic



Non-Periodic

Checkerboard protoset,

$$\mathcal{P} = \{ \blacksquare , \square \}$$

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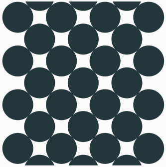
Some protosets,

$\mathcal{P} = \{ \bullet ,$  $\}$

admit only **periodic** tessellations.



Checkerboard protoset,

$$\mathcal{P} = \{ \blacksquare, \square \}$$

admits both **periodic** and **non-periodic** tessellations.

Some protosets,

$$\mathcal{P} = \{ \bullet, $$  $$ \}$$

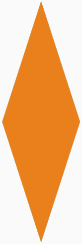
admit only **periodic** tessellations.

Are there any protosets,

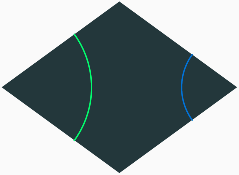
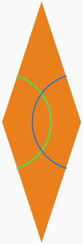
$$\mathcal{P} = \{?\}$$

that admit only **non-periodic** tessellations?

AN APERIODIC TESSELLATION

$\mathcal{P} = \{$  ,  $\}^*$

$*$ With matching rules

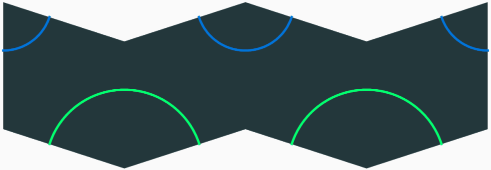
$\mathcal{P} = \{$  ,  $\}$

admits only **non-periodic** tessellations, called Penrose Tessellations.

Cannot be constructed through local procedures.

Example:

This arrangement

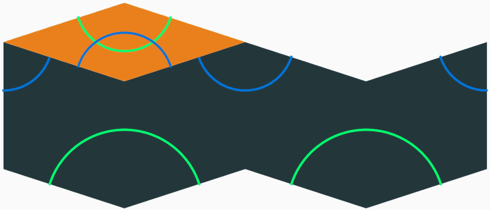


follows the matching rules.

Cannot be constructed through local procedures.

Example:

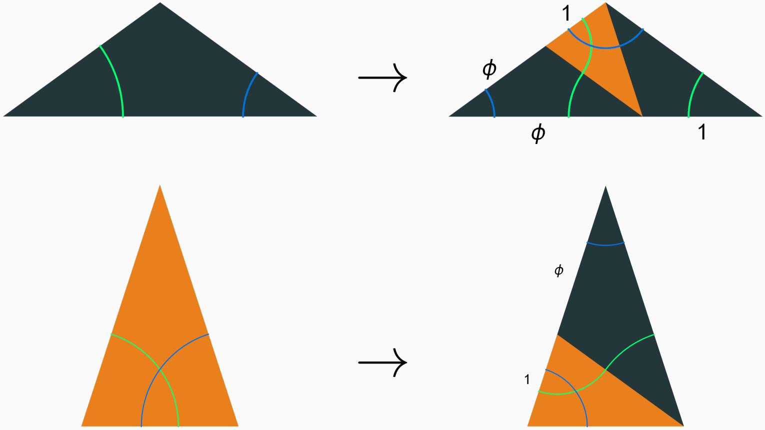
This arrangement



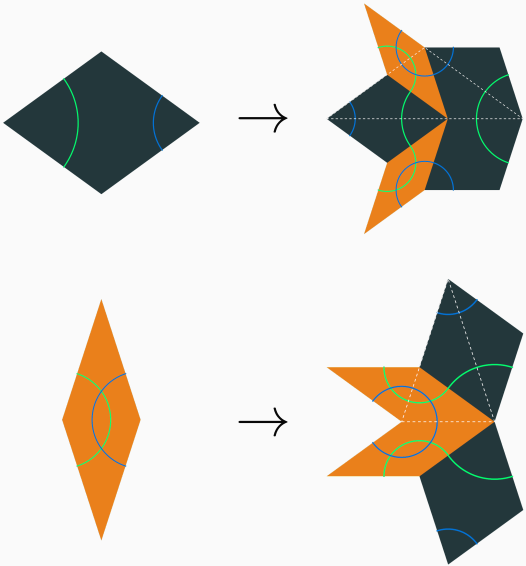
follows the matching rules, but is a mistake:

CONSTRUCTION BY SUBSTITUTION

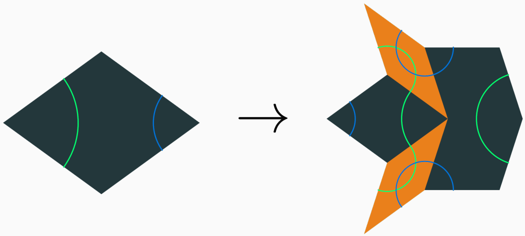
First, consider the Rhombs as joined triangles:

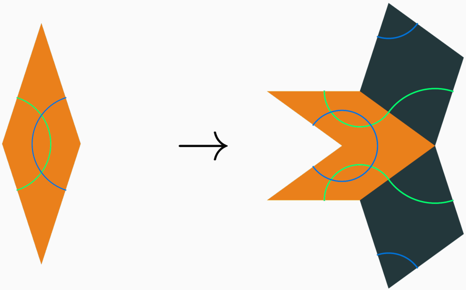


SUBSTITUTION RULES ON PENROSE RHOMBS

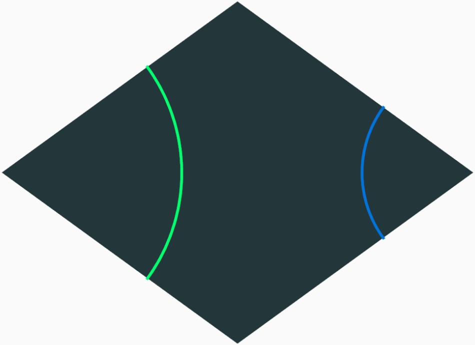


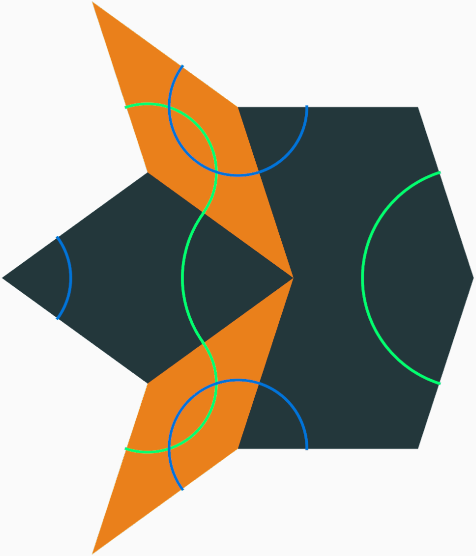
SUBSTITUTION RULES ON PENROSE RHOMBS



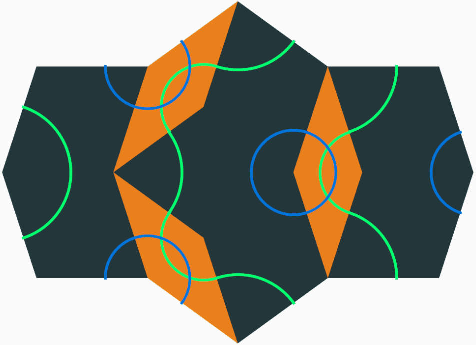


START WITH A SEED: THICK RHOMBUS

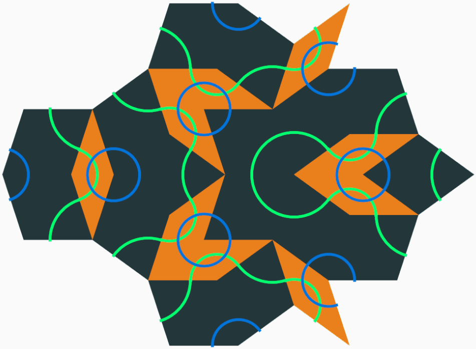




APPLY SUBSTITUTION RULES AGAIN



APPLY SUBSTITUTION RULES AGAIN AND AGAIN



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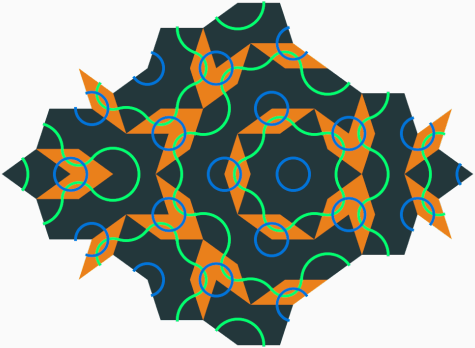
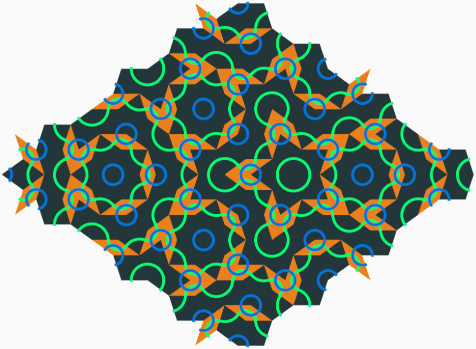
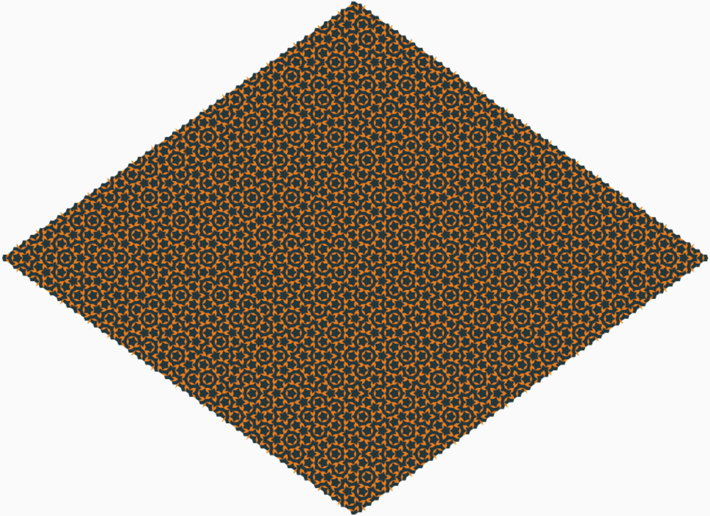


APPLY SUBSTITUTION RULES AGAIN AND AGAIN AND AGAIN AND AGAIN



APPLY SUBSTITUTION RULES 10 TIMES

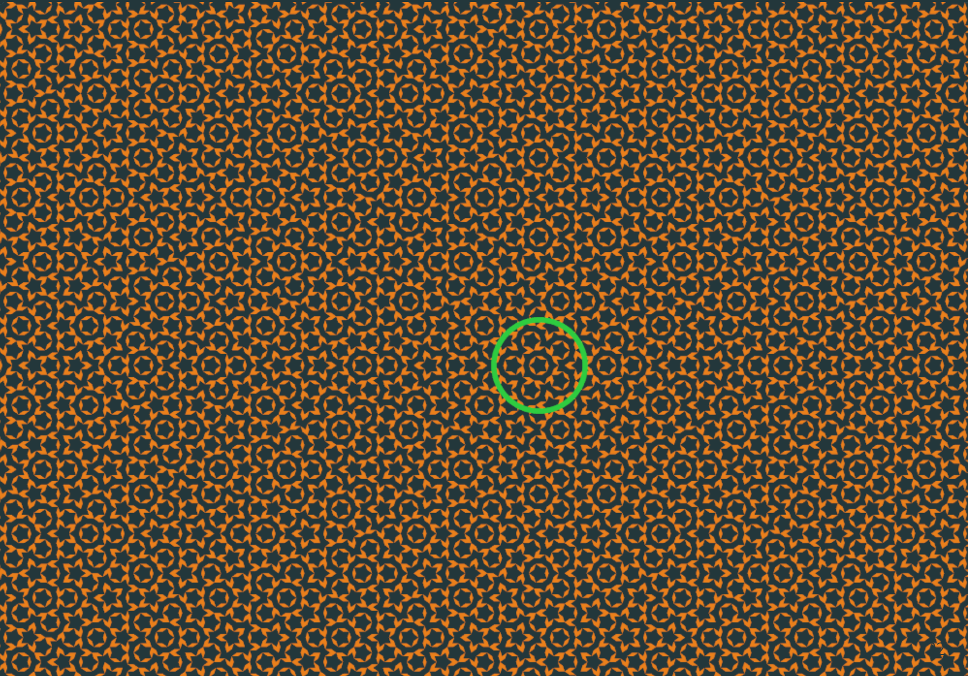


APPLY SUBSTITUTION RULES 11 TIMES

PENROSE TESSELLATION FEATURES

Any pattern with diameter d will begin repeating within $2\phi d$ from the perimeter.

DENSITY OF PATTERNS



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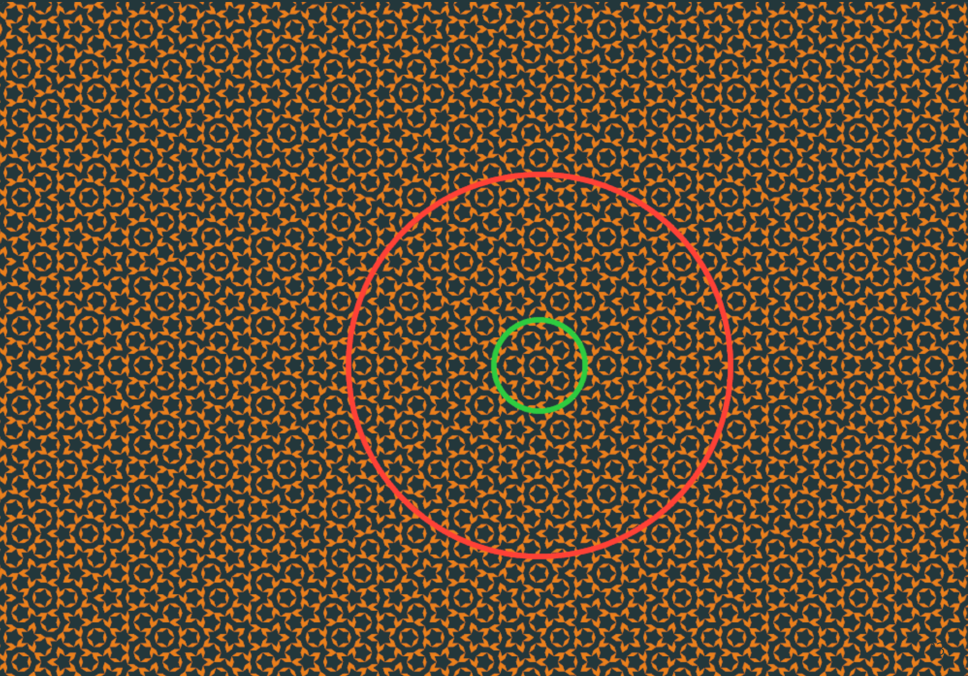

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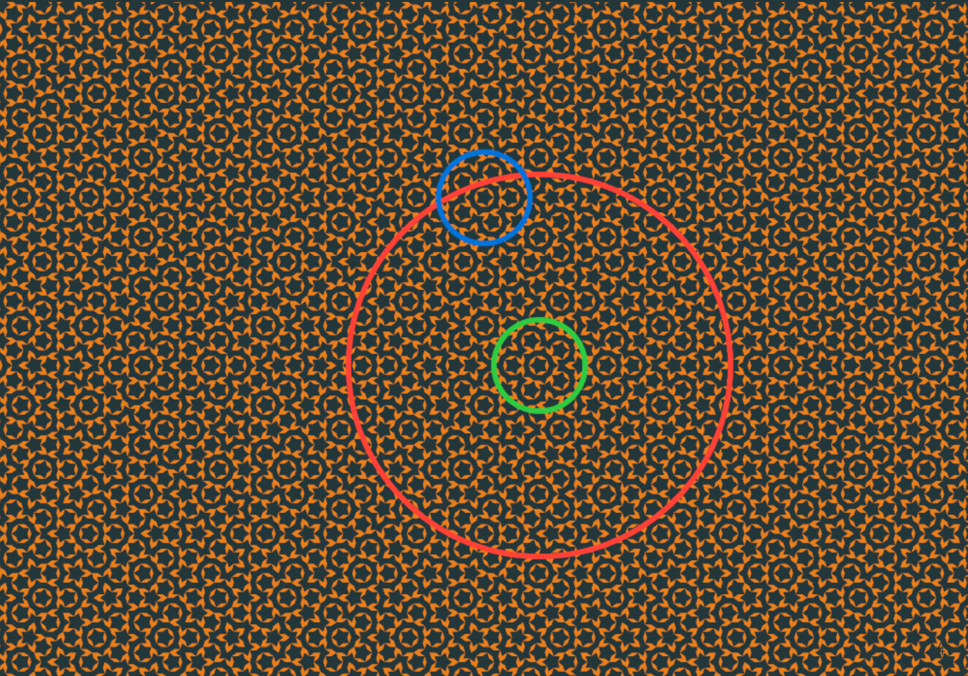

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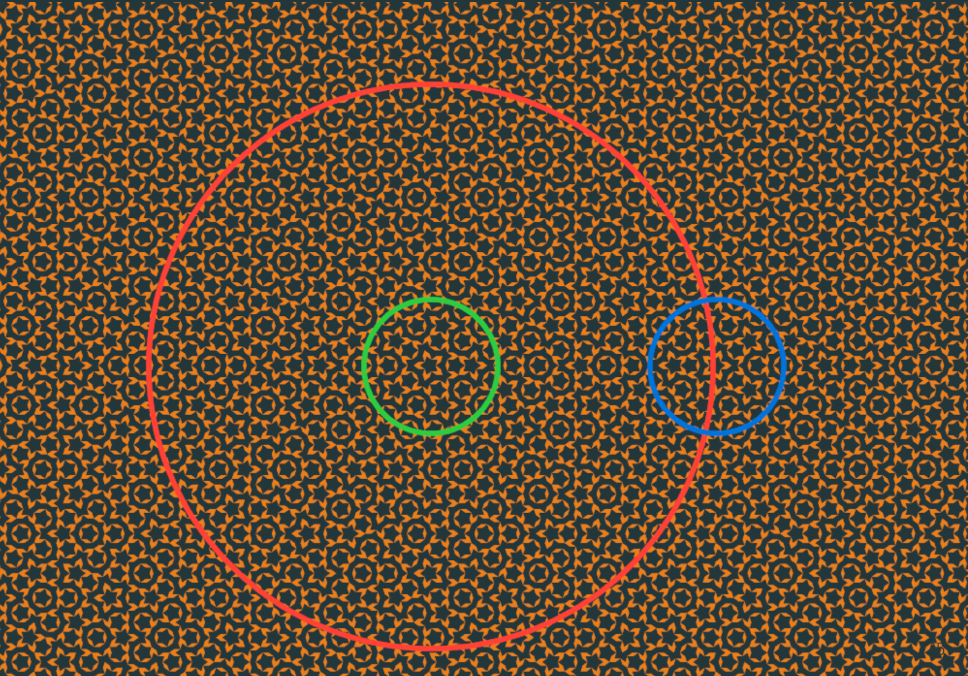
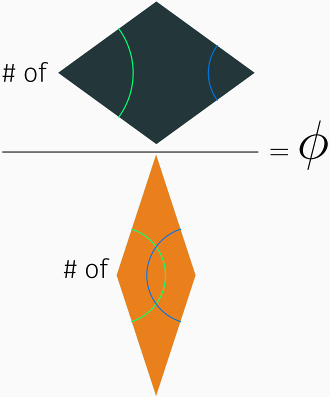


Penrose Tessellations can admit five-fold rotational symmetry.

(Impossible for periodic tessellations)

RATIO OF THICK TO THIN RHOMBS



Questions?

github.com/jessebett/PenroseTilingThesis

BIBLIOGRAPHY

J H Conway and K M Knowles.
Quasiperiodic tiling in two and three dimensions.
Journal of Physics A: Mathematical and General, 19(17):3645–3653, 1986.

N G De Bruijn.
Algebraic theory of Penrose's non-periodic tilings of the plane. I.
In *Indagationes Mathematicae (Proceedings)*, volume 84, pages 39–52. Elsevier, 1981.

N.G. de Bruijn.
Updown generation of Penrose patterns.
Indagationes Mathematicae, 1(2):201–219, June 1990.

Martin Gardner.
Penrose Tiles to Trapdoor Ciphers: And the Return of Dr Matrix.
Cambridge University Press, 1997.

T. Ogawa and R. Collins.
Chains, Flowers, Rings and Peanuts: Graphical Geodesic Lines and Their Application to Penrose Tiling.
Visual Mathematics, (4), 1999.

R. Penrose.
Pentaplexity A Class of Non-Periodic Tilings of the Plane.
The Mathematical Intelligencer, 2(1):32–37, March 1979.

Marjorie Senechal.
Quasicrystals and geometry.
CUP Archive, 1996.