

# THE BEAUTIFUL GEOMETRY OF APERIODIC TESSELLATIONS

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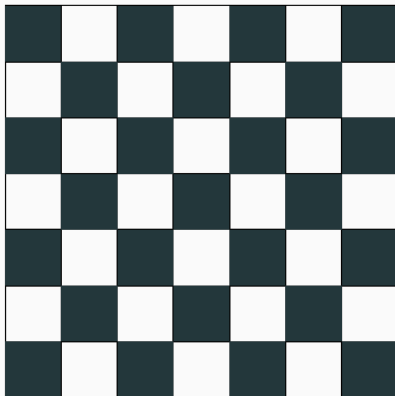
Jesse Bettencourt

March 30, 2015

Supervised by Miroslav Lovric

WHAT IS A TESSELLATION?

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Checkerboard Tessellation

# WHAT IS A TESSELLATION?

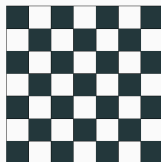
## Definition:

A **tessellation**  $\mathcal{T}$  of the space  $\mathbb{E}^n$  is a countable family of closed sets,  $T$ , called tiles:

$$\mathcal{T} = \{T_1, T_2, \dots\}$$

such that

1.  $\mathcal{T}$  has **no overlaps**:  $\overset{\circ}{T}_i \cap \overset{\circ}{T}_j = \emptyset$  if  $i \neq j$
2.  $\mathcal{T}$  has **no gaps**:  $\bigcup_{i=1}^{\infty} T_i = \mathbb{E}^n$



## Definition:

Let  $\{T_1, T_2, \dots\}$  be the set of tiles of tessellation  $\mathcal{T}$ , partitioned into a set of equivalence classes by **criterion**  $\mathcal{M}$ . The set,  $\mathcal{P}$ , of representatives of these equivalence classes is called the **protoset** for  $\mathcal{T}$  with respect to  $\mathcal{M}$ .

## Example:

**Criterion:**  $\mathcal{M} = \{\text{Colour of the tile. Only opposite colours may touch.}\}$

**Protoset:**

$$\mathcal{P} = \left\{ \blacksquare, \square \right\}$$

## Definition:

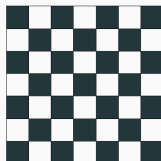
If  $\mathcal{T}$  is a tessellation with proto-set  $\mathcal{P}$ , then we say that  $\mathcal{P}$  admits  $\mathcal{T}$ .

## Example:

We say

$$\mathcal{P} = \{ \blacksquare, \square \}$$

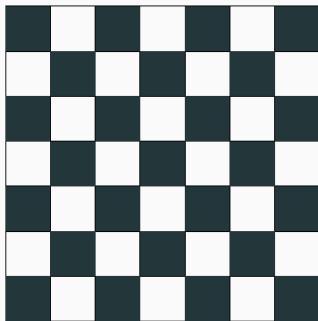
admits



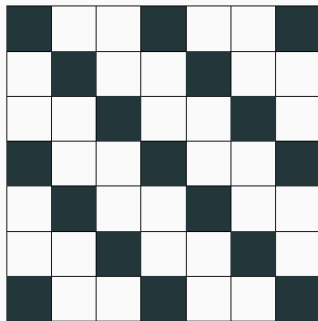
## PROTOSETS CAN ADMIT MULTIPLE TESSELLATIONS

$$\mathcal{P} = \{ \blacksquare, \square \}$$

admits both



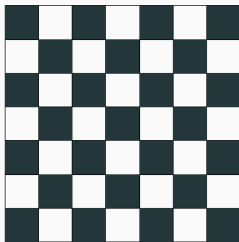
and



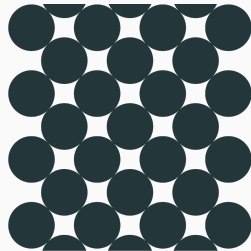
# MATCHING RULES CORRESPOND TO DEFORMED PROTOSETS

Edge deformations can force matching rules.

$$\mathcal{P} = \{ \blacksquare, \square \}$$



$$\mathcal{P} = \{ \bullet, \text{deformed square} \}$$



Matching Rules  $\implies$  Deformed Protoset



## DESCRIBING TESSELLATIONS

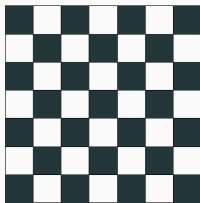
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## Definition:

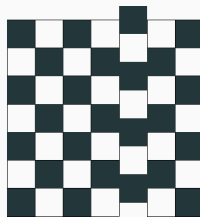
A tessellation is said to be **periodic** if it admits translational symmetry in two directions.

## Definition:

A tessellation is said to be **non-periodic** if it admits no translational symmetries.



Periodic

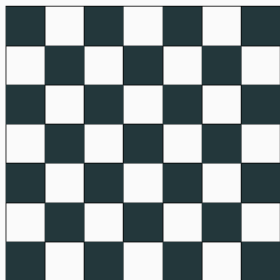


Non-Periodic

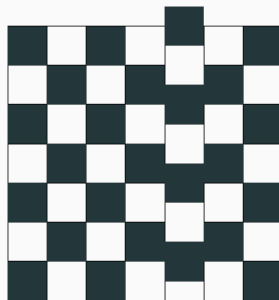
Checkerboard protoset,

$$\mathcal{P} = \{ \blacksquare, \square \}$$

admits **both** periodic and **non-periodic** tessellations.



Periodic



Non-Periodic

Checkerboard protoset,

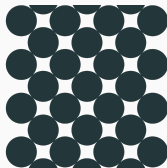
$$\mathcal{P} = \{ \blacksquare, \square \}$$

admits **both** periodic and non-periodic tessellations.

Some protosets,

$$\mathcal{P} = \{ \bullet, \square \}$$

admit **only** periodic tessellations.



Checkerboard protoset,

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admits **both** periodic and non-periodic tessellations.

Some protosets,

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admit **only** periodic tessellations.

Are there any protosets,

$$\mathcal{P} = \{ ? \}$$

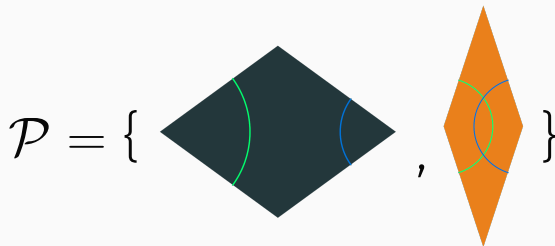
that admit **only** non-periodic tessellations?

## AN APERIODIC TESSELLATION

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$$\mathcal{P} = \{ \text{dark blue rhombus}, \text{orange rhombus} \}^*$$

\* With matching rules



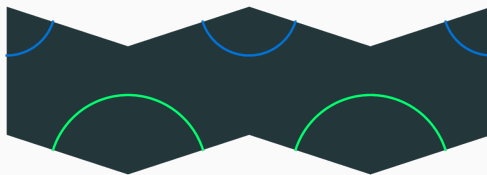
admits **only non-periodic** tessellations, called Penrose Tessellations.



Cannot be constructed through **local** procedures.

**Example:**

*This arrangement*



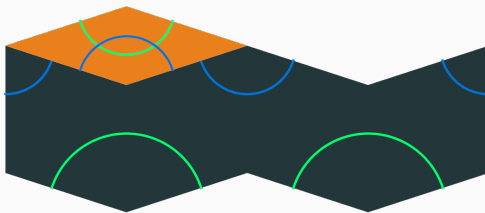
*follows the matching rules.*

# NON-LOCALITY OF THE PENROSE TESSELLATIONS

Cannot be constructed through **local** procedures.

**Example:**

*This arrangement*



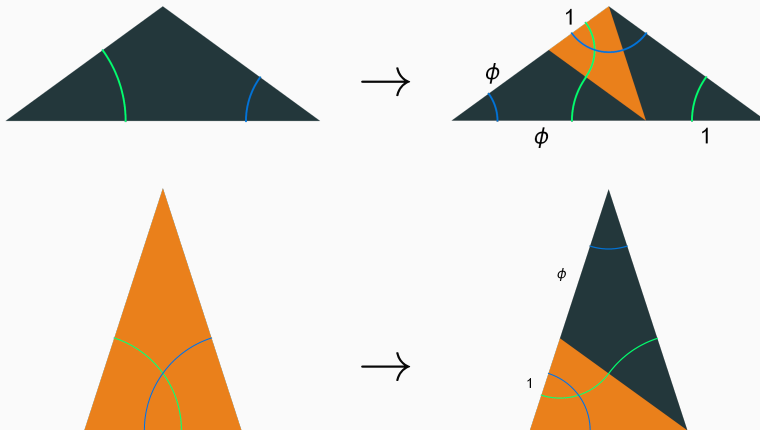
*follows the matching rules, but is a **mistake**:*

## CONSTRUCTION BY SUBSTITUTION

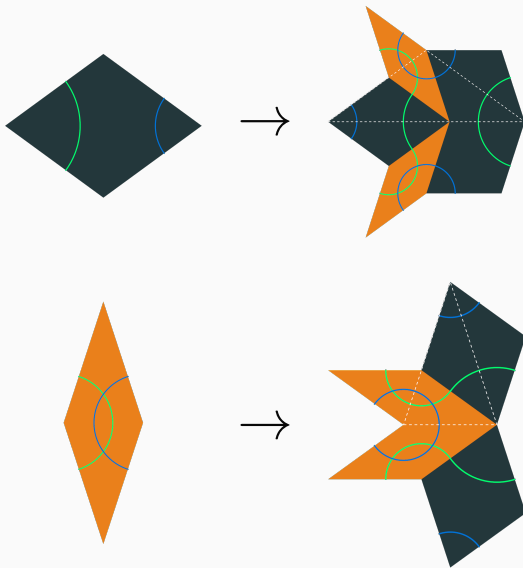
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# SUBSTITUTION RULES ON ROBINSON TRIANGLES

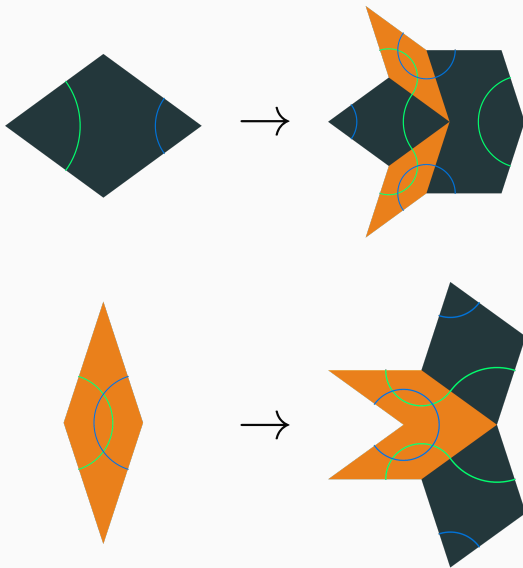
First, consider the Rhombs as joined triangles:



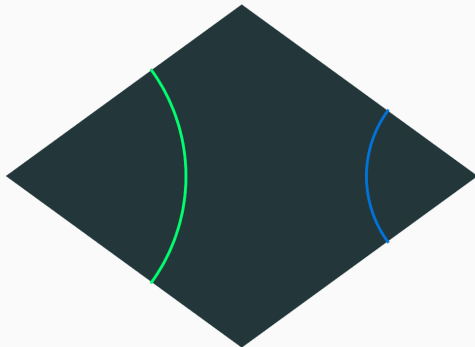
## SUBSTITUTION RULES ON PENROSE RHOMBS

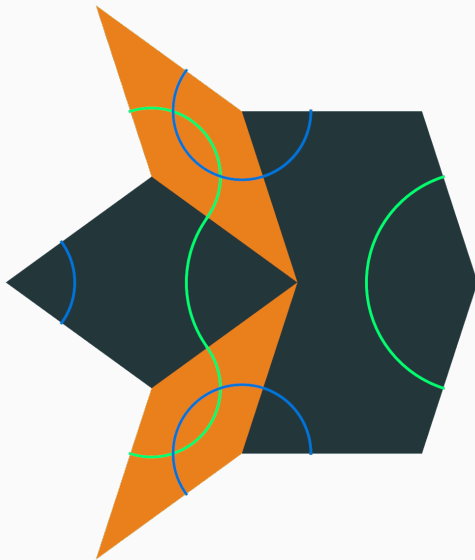


## SUBSTITUTION RULES ON PENROSE RHOMBS



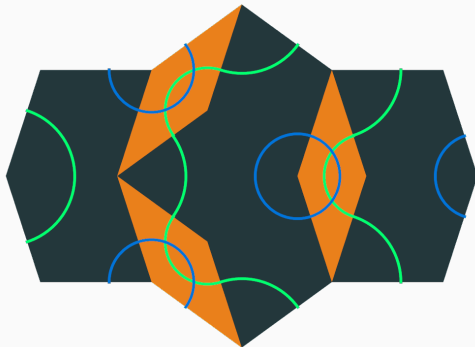
## START WITH A SEED: THICK RHOMBUS



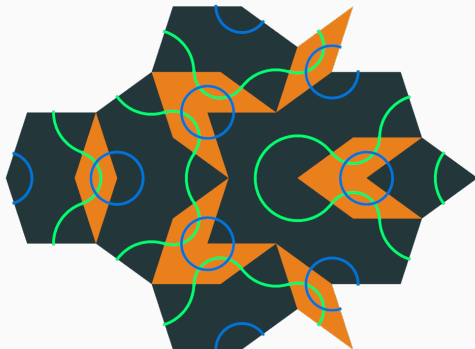




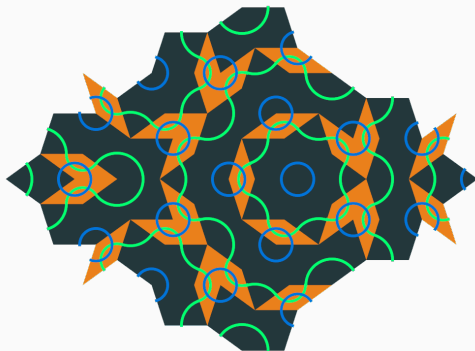
## APPLY SUBSTITUTION RULES AGAIN



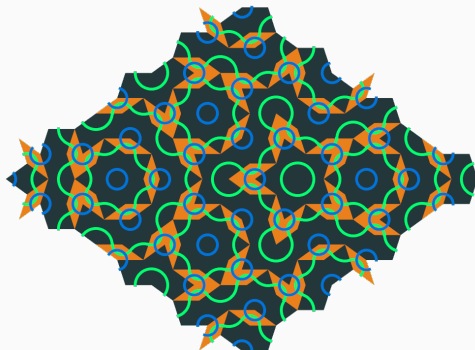
## APPLY SUBSTITUTION RULES AGAIN AND AGAIN



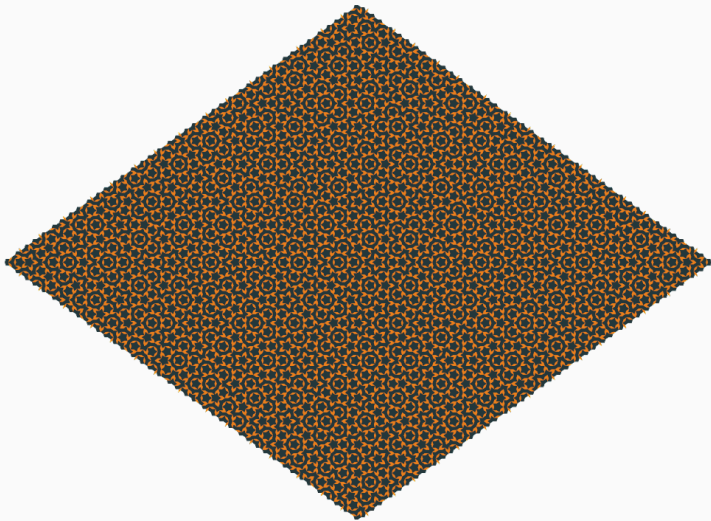
## APPLY SUBSTITUTION RULES AGAIN AND AGAIN AND AGAIN



APPLY SUBSTITUTION RULES AGAIN AND AGAIN AND AGAIN AND AGAIN



## APPLY SUBSTITUTION RULES 10 TIMES



APPLY SUBSTITUTION RULES 11 TIMES



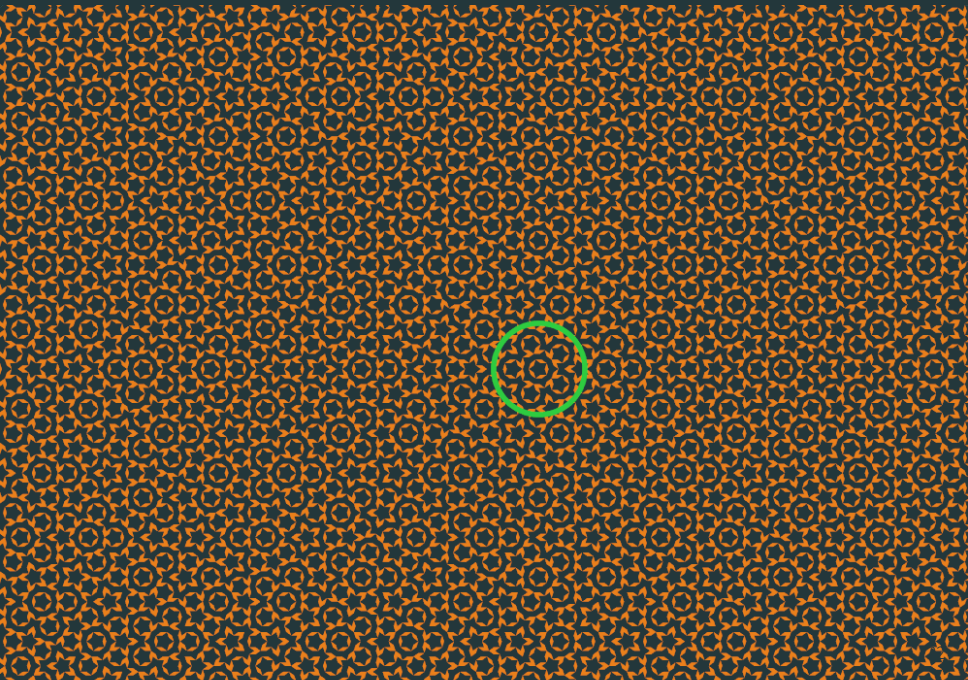
## PENROSE TESSELLATION FEATURES

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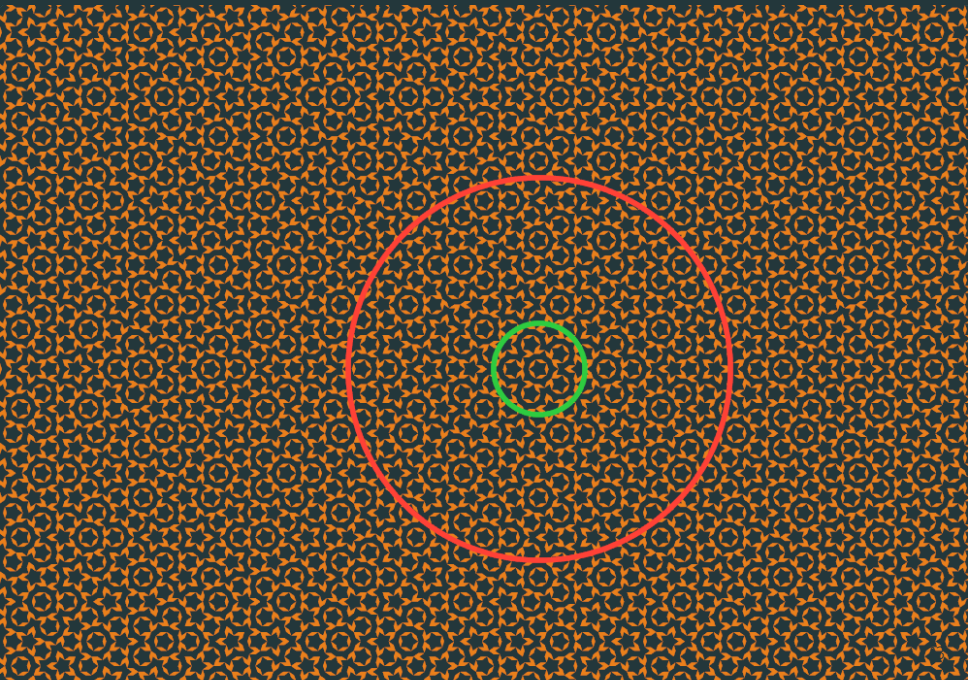
Any pattern with diameter  $d$  will begin repeating within  $2\phi d$  from the perimeter.



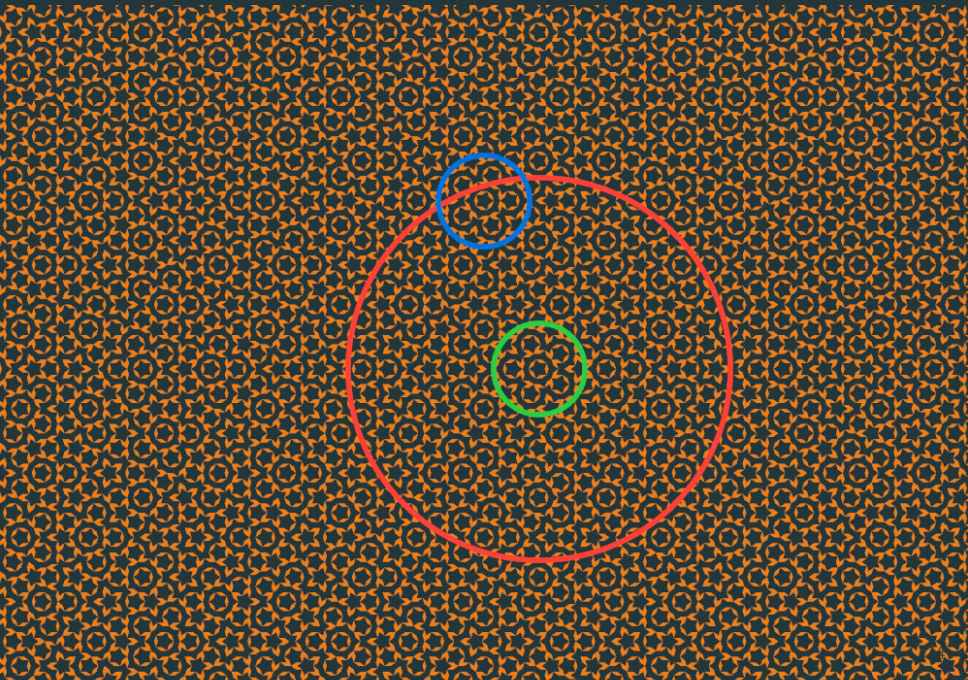
## DENSITY OF PATTERNS



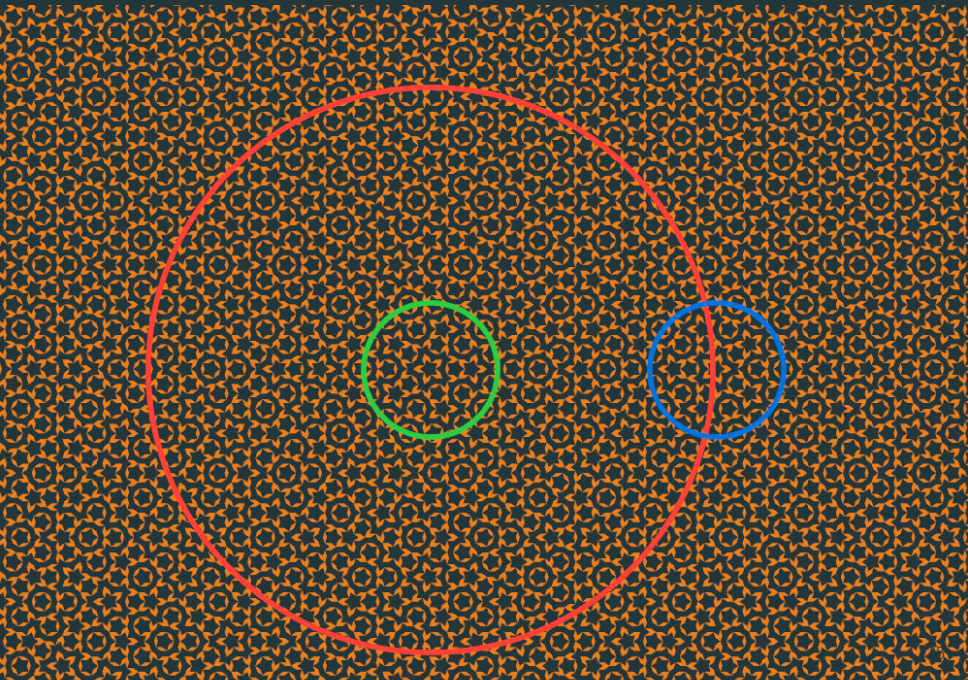
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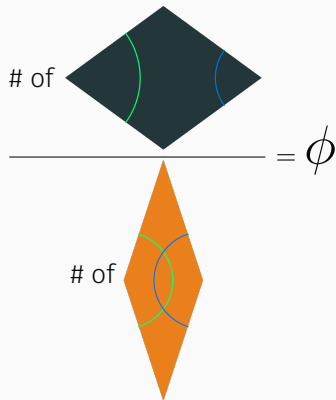
## DENSITY OF PATTERNS



Penrose Tessellations can admit **five-fold** rotational symmetry.

(Impossible for periodic tessellations)

## RATIO OF THICK TO THIN RHOMBS



Questions?



`github.com/jessebett/PenroseTilingThesis`



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**Quasiperiodic tiling in two and three dimensions.**  
*Journal of Physics A: Mathematical and General*, 19(17):3645–3653, 1986.



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