March 14, 2015

Abstract

1 Introduction

1.1 Theory of Periodic Tiling

In 1619, Kepler discovered the third law of planetary motion which he described in his book *Harmonices Mundi* (The Harmonies of the World). In this work, Kepler also provides the first mathematical treatment of plane tiling by regular polygons. Primarily, he shows that only 3 regular polygons; triangles, squares, and hexagons; are able to completely tile the plane, that is, without overlap or gaps (Fig. 1). Further, Kepler experimented with tilings produced by combinations of polygon tiles (Fig. 2). While patterns of this kind have been known to the ancients, featured widely in art and architecture, it was Kepler who first provided a mathematical framework with which to discuss these tiling patterns. One aspect of this treatment is the symmetry of the tiling.

We can describe the tiling patterns generally by the symmetries which they admit. In planar tilings there are four fundamental types of symmetry:

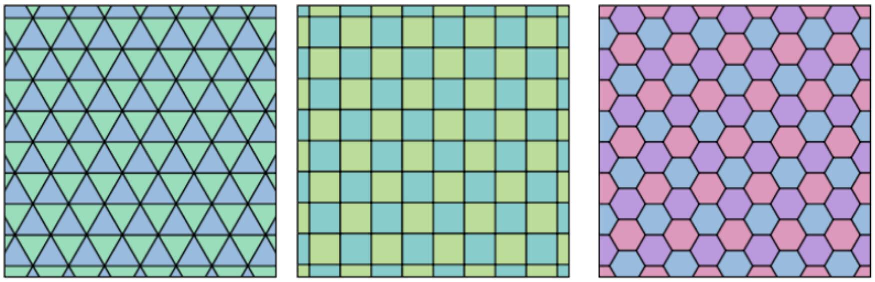
1. A tiling has **rotational** symmetry if it can be rotated a non-trivial angle about a point and overlap itself identically.

2. A tiling has **reflection** symmetry if it can be mirrored across a line and overlap itself identically.

3. A tiling has **translational** symmetry if it can be shifted by some non-trivial distance in a direction and overlap itself identically.

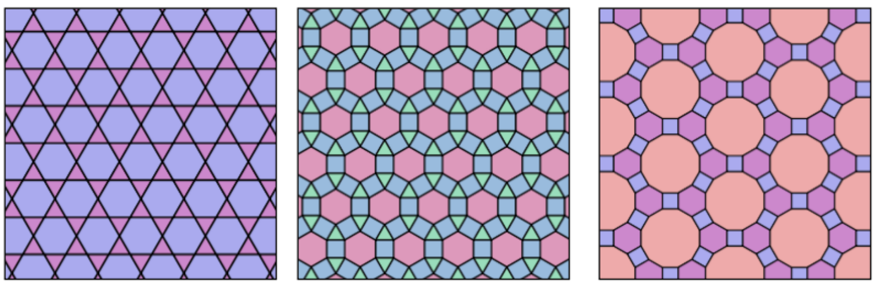
4. A tiling has **glide** symmetry if it can be translated some distance, then reflected about a line, and overlap itself identically.

Using the notion of translational symmetry, we can introduce a new property of plane tiling, periodicity. A tiling is said to be **periodic** if it admits translational symmetry. The tilings in Fig. 1 and Fig. 2 are all periodic, but is it possible to make a tiling pattern which is not periodic? Well, actually it's quite easy. Consider the square tiling in Fig. 1b but with each column shifted up and down by some random decimal amount. This tiling of jostled squares



(a) Tiling by triangles (b) Tiling by squares (c) Tiling by hexagons

Figure 1: Regular tiling of the plane using single polygons



(a) Trihexagonal (b) Rhombitrihexagonal (c) Truncated trihexagon

Figure 2: Periodic tiling of the plane using combination of polygons

would completely tile the plane, but since the squares are all randomly shifted there's no way we could translate the entire plane so that it overlaps itself. This is an example of a tiling pattern which does not have the periodic property, we call these patterns aperiodic.

1.2 History of Aperiodic Tiling

We've seen that square tiles can be used to produce both periodic and aperiodic tiling, depending on the rules used to tile. This is also true for the triangles of Fig. 1a. However, the hexagons of Fig. 1c cannot be arranged to tile aperiodically. Think of these hexagons as 'locked in' to each other, and unlike the squares and triangles there is no way to jostle the tiles into an aperiodic pattern. So squares and triangles can create both periodic and aperiodic tilings, but patterns using only hexagon tiles must be periodic. Of course, as seen in Fig. 2, we can make patterns with sets of multiple tiles which force periodicity like the hexagons (Fig. 2c) or may allow both periodic and aperiodic tiling (consider Fig. 2a randomly jostled diagonally). This may lead you to a logical following question, are there a set of tiles which, when used to tile a plane, must be arranged aperiodically? In 1966, Robert Berger found a set of such [width=]Checkerboard

Figure 3: Rhombitrihexagonal

tiles that force their tiling to be aperiodic, though his set required 20,426 tiles! Later, Berger was able to reduce the number of required tiles to 104. An even smaller set of six aperiodic tiles was discovered in 1971 by Raphael M. Robinson. Finally, Roger Penrose in 1973 reduced the required number of tiles to two (Fig. **??**)! Any tiling from the two Penrose tiles, darts and kites, will be aperiodic. An example of a small section of this tiling can be seen in Fig. **??**.

2 Penrose Tiling

2.1 Definitions

We will begin our discussion of Penrose tilings with some definitions given by Senechal [**?**]. First, a formal definition of a tiling of Euclidean n-space, $\mathbb{E}^n$:

Definition 1. *A* ***tiling*** $\mathcal{T}$ *of the space* $\mathbb{E}^n$ *is a countable family of closed sets,* T*, called tiles:*

$$\mathcal{T} = \{T_1, T_2, \ldots\}$$

such that

1. $\mathcal{T}$ *has no overlaps:* $\mathring{T}_i \cap \mathring{T}_j = \emptyset$ *if* $i \neq j$
2. $\mathcal{T}$ *has no gaps:* $\bigcup_{i=1}^{\infty} T_i = \mathbb{E}^n$

Here $\mathring{T}$ denotes the interior of tile T. Further, we assume that a tile is the closure of its interior, and that tiles have positive volume. These assumptions allow, for example, a line segment to be a tile in $\mathbb{E}^1$ but not in $\mathbb{E}^2$. Notice that this definition of tiling neither restricts the shape of the individual tiles nor the number of unique tiling shapes.

Definition 2. *Let* $\{T_1, T_2, \ldots\}$ *be the set of tiles of tiling* $\mathcal{T}$*, partitioned into a set of equivalence classes by criterion* $\mathcal{M}$*. The set,* $\mathcal{P}$*, of representatives of these equivalence classes is called the* ***protoset*** *for* $\mathcal{T}$ *with respect to* $\mathcal{M}$*.*

For example, consider an infinite black-and-white checkerboard. Each tile in the checkerboard is either a black or white square, and the tiling is given by the matching rule that black squares may only share edges with white squares, and vice-versa.

3 Collins Walks on Penrose Rhombs