THE BEAUTIFUL GEOMETRY OF APERIODIC TESSELLATIONS

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WHAT IS A TESSELLATION?

A SIMPLE TESSELLATION

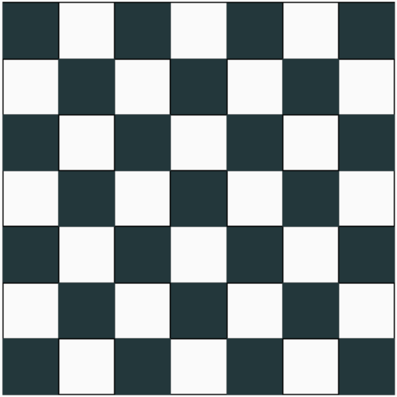


Figure 1: Checkerboard Tessellation

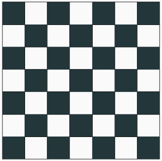
Definition:

*A **tessellation** $\mathcal{T}$ of the space $\mathbb{E}^n$ is a countable family of closed sets, T, called* tiles*:*

$$\mathcal{T} = \{T_1, T_2, \ldots\}$$

such that

1. $\mathcal{T}$ *has **no overlaps**:* $\mathring{T}_i \cap \mathring{T}_j = \emptyset$ *if* $i \neq j$
2. $\mathcal{T}$ *has **no gaps**:* $\bigcup_{i=1}^{\infty} T_i = \mathbb{E}^n$



COMPONENTS OF A TESSELLATION

Definition:

Let $\{T_1, T_2, \ldots\}$ be the set of tiles of tessellation $\mathcal{T}$, partitioned into a set of equivalence classes by ***criterion*** *$\mathcal{M}$. The set, $\mathcal{P}$, of representatives of these equivalence classes is called the* ***protoset*** *for $\mathcal{T}$ with respect to $\mathcal{M}$.*

Example:

Criterion: $\mathcal{M} = \{\textit{Colour of the tile. Only opposite colours may touch.}\}$

Protoset:

$$\mathcal{P} = \{ \blacksquare , \square \}$$

PROTOSETS ADMIT TESSELLATIONS

Definition:

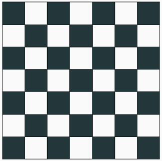
If $\mathcal{T}$ is a tessellation with protoset $\mathcal{P}$, then we say that $\mathcal{P}$ admits $\mathcal{T}$.

Example:

We say

$$\mathcal{P} = \{ \blacksquare, \square \}$$

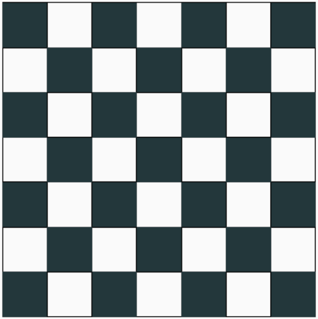
admits



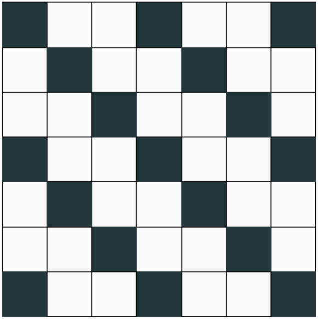
PROTOSETS CAN ADMIT MULTIPLE TESSELLATIONS

$$\mathcal{P} = \{ \blacksquare , \square \}$$

admits both

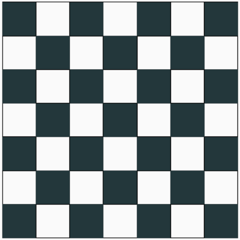


and

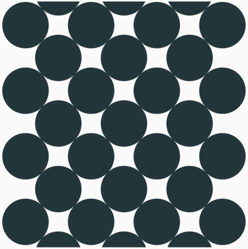


Edge deformations can force matching rules.

$$\mathcal{P} = \{ \blacksquare , \square \} \qquad \mathcal{P} = \{ \bullet ,$$  $$\}$$



$$\to$$



Matching Rules $\implies$ Deformed Protoset

DESCRIBING TESSELLATIONS

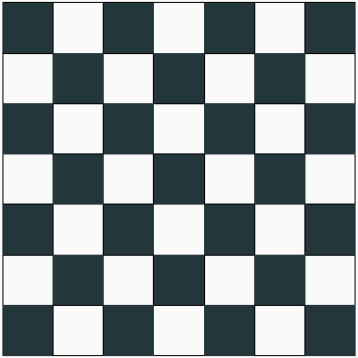
Definition:

A tessellation is said to be ***symmetric*** *under a transformation if that transformation maps the tiling to itself identically.*

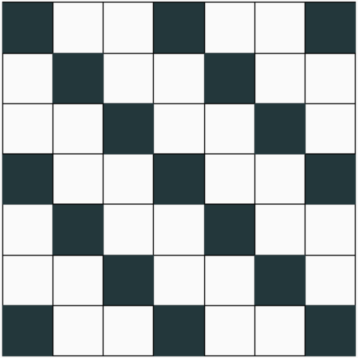
Rotational Symmetry: $\mathcal{T}$ can be rotated a non-trivial angle about a point and overlap itself identically

Translational Symmetry: $\mathcal{T}$ can be shifted by some non-trivial distance in a direction and overlap itself identically.

CHECKERBOARD SYMMETRIES



Translation Symmetry: 2 Squares
Rotational Symmetry: $\frac{\pi}{2}$



Translation Symmetry: 3 Squares
Rotational Symmetry: π

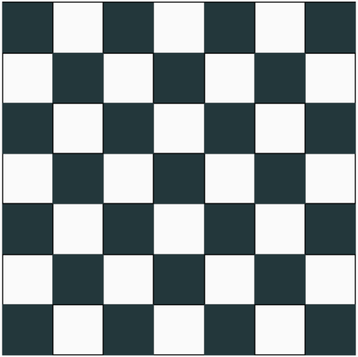
Definition:

A tessellation is said to be ***periodic*** *if it admits translational symmetry in two directions.*

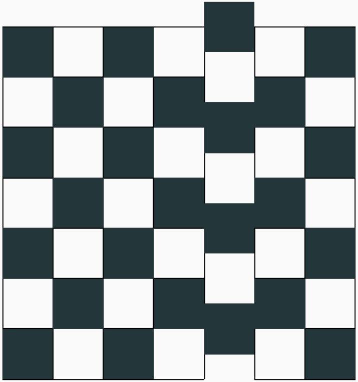
Definition:

A tessellation is said to be ***non-periodic*** *if it admits no translational symmetries.*

CHECKERBOARD PERIODICITY



Periodic



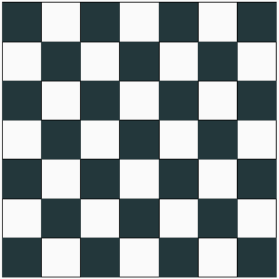
Non-Periodic

THE APERIODIC QUESTION

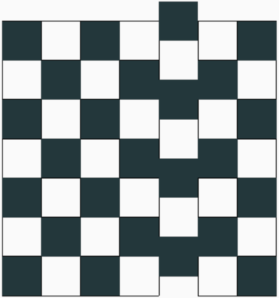
Checkerboard protoset,

$$\mathcal{P} = \{ \blacksquare , \square \}$$

admits both **periodic** and **non-periodic** tessellations.



Periodic



Non-Periodic

THE APERIODIC QUESTION

Checkerboard protoset,

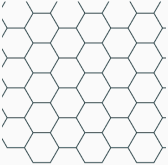
$$\mathcal{P} = \{ \blacksquare \, , \, \square \}$$

admits both **periodic** and **non-periodic** tessellations.

Some protosets,

$\mathcal{P} = \{$  $\}$

admits only **periodic** tessellations.



THE APERIODIC QUESTION

Checkerboard protoset,

$$\mathcal{P} = \{ \blacksquare , \square \}$$

admits both **periodic** and **non-periodic** tessellations.

Some protosets,

$$\mathcal{P} = \{ \hexagon \}$$

admits only **periodic** tessellations.

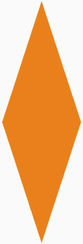
Are there any protosets,

$$\mathcal{P} = \{?\}$$

that admit only **non-periodic** tessellations?

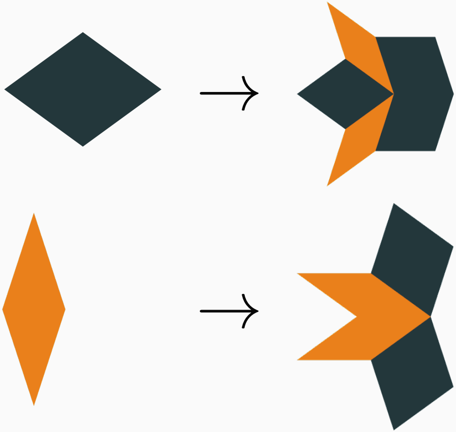
THE PENROSE TESSELLATION

THE PENROSE RHOMBS PROTOSET

$\mathcal{P} = \{$  ,  $\}^*$

$*$ With complicated matching rules

The Penrose Tessellation cannot be constructed **locally**, instead we use **substitution rules**:



START WITH A SEED: FAT RHOMBUS

APPLY SUBSTITUTION RULES

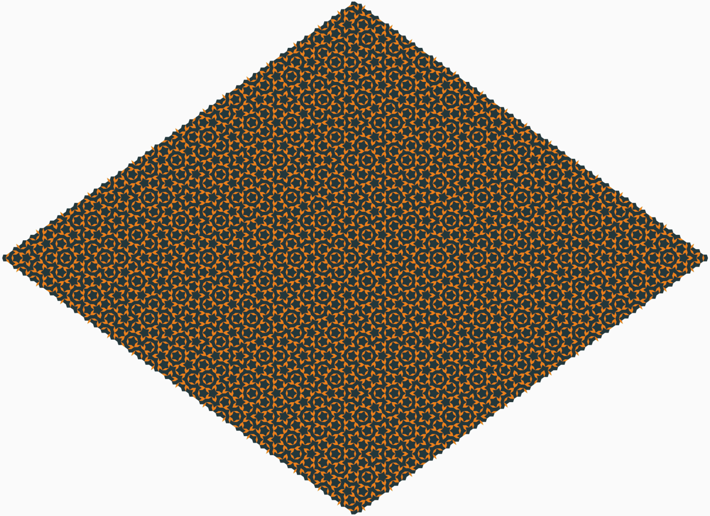
APPLY SUBSTITUTION RULES AGAIN

APPLY SUBSTITUTION RULES AGAIN AND AGAIN

APPLY SUBSTITUTION RULES AGAIN AND AGAIN AND AGAIN

APPLY SUBSTITUTION RULES AGAIN AND AGAIN AND AGAIN AND AGAIN

APPLY SUBSTITUTION RULES 10 TIMES



APPLY SUBSTITUTION RULES 11 TIMES