# 1 What Is The Problem

This text is a showcase for the `mpll` tool, or *Metaperlualatex* tool. This is a document written in LaTeX, and using Perl functions, MetaPost drawings and Lua functions (a few).

Yet, this document is not an artificial document whose only purpose is to pack significant *Metaperlualatex* features in a meaningless text. It is also an explanation of a real phenomenon which can be observed in December (in temperate areas in the Northern hemisphere). During the fall, the sun rises later and later and sets sooner and sooner. No surprise there. During the spring, the sun rises earlier and earlier and sets later and later. No surprise there either. But from 14th December until 1st January the sun rises later and later and it also sets later and later. In other words, the day gradually replaces the night during the evening, while the night still gradually replaces the day during the morning, as can be seen in the table below.

| Day | rise | set | var.morn | var.even |
|---|---|---|---|---|
| 2011-12-05 | 07:45:11 | 15:55:06 | 586 | -269 |
| 2011-12-12 | 07:53:14 | 15:53:16 | 483 | -110 |
| 2011-12-19 | 07:59:08 | 15:54:10 | 354 | 54 |
| 2011-12-26 | 08:02:34 | 15:57:48 | 206 | 218 |
| 2012-01-02 | 08:03:18 | 16:03:58 | 44 | 370 |
| 2012-01-09 | 08:01:20 | 16:12:19 | -118 | 501 |
| 2012-01-16 | 07:56:46 | 16:22:24 | -274 | 605 |
| 2012-01-23 | 07:49:50 | 16:33:45 | -416 | 681 |
| 2012-01-30 | 07:40:49 | 16:45:56 | -541 | 731 |

On the other side, this document is not a scientific publication with strict computations of celestial movements. It is rather an explanation for an intuitive understanding of some mostly unknown aspects of Earth's movement.

For a more precise explanation, yet still readable by a layman, you can refer to the French-written book "Histoire de l'Heure en France", by Jacques Gapaillard, published by Vuibert – ADAPT.

As said above, the document applies to temperate areas in the Northern hemisphere. Around the equator, the variation of daylight is not sufficient for the reasoning to be valid. In the polar areas, the continous night in winter and the continuous daylight in summer play havoc with the reasoning. And in temperate areas of the Southern hemisphere, it might be sufficient to shift all dates by six months, so the winter falls in June and the spring falls in September. The algorithm used to compute the sunrise and the sunset is Paul Schlyter's algorithm.

# 2 Solar Noon And The Variations Of Sunrise And Sunset

## 2.1 The Various Types Of Noon

If you explain to a XXIth Century person that there are two notions of "noon", the astronomical noon, which is the instant where the sun is at its highest in the sky and is aligned with the local meridian, and the mean noon (or rather, "mean time noon"), which can be obtained with a watch or a clock, if you explain that the offset between these two instants varies from a day to the other, this XXIth century person, if leaving in Great Britain for example, will answer "Of course, during the summer the astronomical noon is one hour behind the mean noon, while they are the same during winter." After a little thinking, this person will add that this also depends on the place you live in: "This is for Kent, although in Cornwall, the astronomical noon is half-an-hour behind the mean noon in winter and one hour and a half minutes behind the mean noon in summer."

This person confuses the "legal hour" with the "mean time" and the "mean noon" with the "astronomical noon". If we attempt to explain him that, for Kent in winter, the difference between "astronomical noon" and "mean noon" may vary from 14 minutes in one direction and 17 minutes in the other, this person will no longer understand.

On the other side, if you explain to a XVIIIth century person that the astronomical noon oscillates on both sides of the mean noon and that this gap may reach one quarter of an hour, it is more likely that this person will approve, especially if he works with clocks or sundials. This person would not understand the notion of "legal time", because it did not exist at the time, but he would understand the irregularities of the crossing of the meridian by the sun. In this time, it was well-known that the interval between two successive times when the sun crosses the meridian was not a constant. Yet, as J. Gapaillard explains in his book, even in this time, there were many people who erroneously thought that the apparent movement of the sun around the Earth was constant within one second.

In the first part, we posit, without explanation, that the astronomical noon varies with respect to the mean noon and we explain how this results in the paradoxical variation of sunrise and sunset in December. In the second part, the document explains why there exists a variation between the astronomical noon and the mean noon.

## 2.2 Real Life Example

The table below give the sunrise and sunset hours, as well as the astronomical noon, in Greenwich in March 2011. Using Greenwich allows to equate the UTC time with the mean time (the formerly well-known GMT), within a few seconds. The table gives the time of sunrise, of sunset, of noon and the variations (in seconds) with the previous day. Also shown is the variation of the day length.

| Day | rise | noon | set | var.morn | var.even | var.length | var.noon |
|---|---|---|---|---|---|---|---|
| 2011-03-16 | 06:12:11 | 12:09:20 | 18:06:29 | -136 | 103 | 239 | -16 |
| 2011-03-17 | 06:09:56 | 12:09:03 | 18:08:11 | -135 | 102 | 237 | -16 |
| 2011-03-18 | 06:07:40 | 12:08:46 | 18:09:53 | -136 | 102 | 238 | -17 |
| 2011-03-19 | 06:05:23 | 12:08:29 | 18:11:35 | -137 | 102 | 239 | -17 |
| 2011-03-20 | 06:03:07 | 12:08:11 | 18:13:16 | -136 | 101 | 237 | -17 |
| 2011-03-21 | 06:00:50 | 12:07:54 | 18:14:58 | -137 | 102 | 239 | -17 |
| 2011-03-22 | 05:58:34 | 12:07:36 | 18:16:39 | -136 | 101 | 237 | -17 |
| 2011-03-23 | 05:56:17 | 12:07:19 | 18:18:21 | -137 | 102 | 239 | -17 |
| 2011-03-24 | 05:54:00 | 12:07:01 | 18:20:02 | -137 | 101 | 238 | -18 |
| 2011-03-25 | 05:51:43 | 12:06:43 | 18:21:43 | -137 | 101 | 238 | -18 |
| 2011-03-26 | 05:49:26 | 12:06:25 | 18:23:24 | -137 | 101 | 238 | -18 |
| 2011-03-27 | 05:47:10 | 12:06:07 | 18:25:05 | -136 | 101 | 237 | -17 |
| 2011-03-28 | 05:44:53 | 12:05:49 | 18:26:46 | -137 | 101 | 238 | -18 |
| 2011-03-29 | 05:42:36 | 12:05:31 | 18:28:26 | -137 | 100 | 237 | -18 |
| 2011-03-30 | 05:40:20 | 12:05:13 | 18:30:07 | -136 | 101 | 237 | -17 |

Let us consider 21st March. On this date, the daylight increases by 239 seconds, that is, a bit less than 4 minutes. The solar noon switch from 12:08:11 to 12:07:54 so it shifts by -17.5 seconds. One half of the daylight increase applies to the sunrise (which comes earlier) and the other half to sunset (which comes later). On the other side, the variation of solar noon applies in full to both sunrise and sunset, and with the same direction. So, since the solar noon shifts by -17.5 seconds and the daylight increases by 239 seconds, sunrise moves by

$$-17.5 - \frac{239}{2} = -137$$

that is, it moves backward from 06:03:07 to 06:00:50 while sunset moves by

$$-17.5 + \frac{239}{2} = +102$$

that is, it moves forward from 18:13:16 to 18:14:58. There is a difference, but most persons do not notice, since the daylight variation is much greater than the solar noon shift.

Now, let us consider 23th December.

| Day | rise | noon | set | var.morn | var.even | var.length | var.noon |
|---|---|---|---|---|---|---|---|
| 2011-12-16 | 07:56:54 | 11:55:10 | 15:53:26 | 51 | 7 | -44 | 29 |
| 2011-12-17 | 07:57:41 | 11:55:39 | 15:53:38 | 47 | 12 | -35 | 29 |
| 2011-12-18 | 07:58:26 | 11:56:09 | 15:53:52 | 45 | 14 | -31 | 29 |
| 2011-12-19 | 07:59:08 | 11:56:39 | 15:54:10 | 42 | 18 | -24 | 30 |
| 2011-12-20 | 07:59:47 | 11:57:09 | 15:54:31 | 39 | 21 | -18 | 30 |
| 2011-12-21 | 08:00:23 | 11:57:39 | 15:54:56 | 36 | 25 | -11 | 30 |
| 2011-12-22 | 08:00:55 | 11:58:09 | 15:55:24 | 32 | 28 | -4 | 30 |
| 2011-12-23 | 08:01:25 | 11:58:40 | 15:55:55 | 30 | 31 | 1 | 30 |
| 2011-12-24 | 08:01:51 | 11:59:10 | 15:56:30 | 26 | 35 | 9 | 30 |
| 2011-12-25 | 08:02:14 | 11:59:40 | 15:57:07 | 23 | 37 | 14 | 30 |
| 2011-12-26 | 08:02:34 | 12:00:11 | 15:57:48 | 20 | 41 | 21 | 30 |
| 2011-12-27 | 08:02:50 | 12:00:41 | 15:58:32 | 16 | 44 | 28 | 30 |
| 2011-12-28 | 08:03:03 | 12:01:11 | 15:59:19 | 13 | 47 | 34 | 30 |
| 2011-12-29 | 08:03:13 | 12:01:41 | 16:00:09 | 10 | 50 | 40 | 30 |
| 2011-12-30 | 08:03:19 | 12:02:10 | 16:01:02 | 6 | 53 | 47 | 29 |

The daylight increases by 1 second, while the solar noon moves by 30.5 seconds, shifting from 11:58:09 to 11:58:40 (allowing for rounding errors). The sunrise moves by:

$$+30.5 - \frac{1}{2} = +30$$

that is, the sunrise moves by 30 seconds, shifting from 08:00:55 to 08:01:25; while the sunset moves by:

$$+30.5 + \frac{1}{2} = +31$$

that is, moves by 31 seconds, shifting from 15:55:24 to 15:55:55. In this case, the solar noon variation is much greater that the half of the daylight variation, so the sunrise and the sunset move in the same direction.

### 2.3 Mea Culpa

In the previous section, I made a logical mistake. I have equated the solar noon with the precise instant located midway between sunrise and sunset. This is nothing more than an approximation. Let use go back to the March table.

| Day | rise | noon | set | var.morn | var.even | var.length | var.noon |
|---|---|---|---|---|---|---|---|
| 2011-03-20 | 06:03:07 | 12:08:11 | 18:13:16 | -136 | 101 | 237 | -17 |
| 2011-03-21 | 06:00:50 | 12:07:54 | 18:14:58 | -137 | 102 | 239 | -17 |
| 2011-03-22 | 05:58:34 | 12:07:36 | 18:16:39 | -136 | 101 | 237 | -17 |

On 20th March, the daylight lasts 12 h 10 mn 09 s, that is, 43 809 seconds. On 21st March, it lasts 12 h 14 mn 08 s, that is, 44 048 seconds and on 22nd March, it lasts 12 h 18 mn 5 s, that is, 44 285 seconds. This does not mean that on 20th March at 23 h 59, the daylight duration is 43 809 seconds and that on 21st March at 00 h 01 it becomes suddenly 44 048 seconds, The daylight duration is a continuous function. And since it varies by 239 seconds in 24 hours, it varies by 60 seconds in 6 hours. This gives:

| | | |
|---|---|---|
| 20 mars, noon | 43 809 | 12 h 10 mn 9 s |
| 20 mars, 18 h | 43 869 | 12 h 11 mn 9 s |
| 21 mars, 6 h | 43 988 | 12 h 13 mn 8 s |
| 21 mars, noon | 44 048 | 12 h 14 mn 8 s |
| 21 mars, 18 h | 44 108 | 12 h 15 mn 8 s |
| 22 mars, 6 h | 44 225 | 12 h 17 mn 5 s |
| 22 mars, noon | 44 285 | 12 h 18 mn 5 s |

So, on 21st March, sunrise occurs 6 h 6 mn 34 s before solar noon and sunset occurs 6 h 7 mn 34 s seconds after solar noon. From this, we infer that solar noon really occurs at 12 h 7 mn 24 s instead of 12 h 7 mn 54 s as printed in the table above.

The difference between solar noon and the middle of the daylight is proportional both to the daylight variation (maximum at the equinoxes) and to the daylight duration (maximum at the summer solstice). So the maximum difference occurs around 19th April, when the daylight variation is still approximately equal to its maximum value and when the daylight have notably increased since the equinox.

| Day | rise | noon | set | var.morn | var.even | var.length | var.noon |
|---|---|---|---|---|---|---|---|
| 2011-04-18 | 04:58:12 | 12:00:04 | 19:01:57 | -128 | 100 | 228 | -14 |
| 2011-04-19 | 04:56:05 | 11:59:51 | 19:03:38 | -127 | 101 | 228 | -13 |
| 2011-04-20 | 04:53:59 | 11:59:38 | 19:05:18 | -126 | 100 | 226 | -13 |

On this date, at noon, the daylight lasts 50853 seconds (that is, 14 h 7 mn 33 s) and it variation is 228 seconds. So, at sunrise, the daylight is shorter by $\frac{228\times 7}{24} = 66$ seconds and at sunset it is longer by 66 seconds likewise. The morning lasts $\frac{50853-66}{2} = 25393$ seconds and the afternoon lasts $\frac{50853+66}{2} = 25459$ seconds. The end result is that solar noon occurs at 11:59:18 while the middle of daylight occurs at 11:59:51.

Does this logical and computation error invalidate the reasoning from the previous section? No, because as you can see, the result is an error no longer than 33 seconds between solar noon and the middle of the daylight. This is still much less than the difference between solar noon and mean noon, which can nearly reach 17 minutes.

In addition, the computation error is nearly zero at the winter solstice, since the daylight variation itself is nearly zero. Therefore, in the December table, the value printed for the solar noon is pretty much the proper one.

## 3 Solar Noon Variation

The problem is now to understand why the solar noon varies from one day to the next, in some cases with a 30-second daily variation.

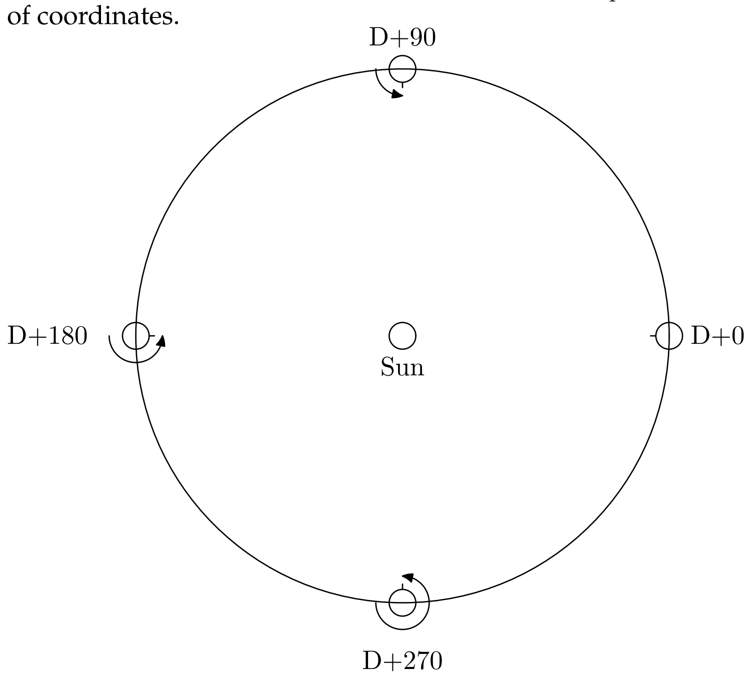
### 3.1 Reference Case

For the movements of Earth around itself and around the sun, let us use the following simplificatons.

1. The year lasts exacty 360 days, instead of 365.25 and some.
2. Earth's orbit around the sun is a perfect circle.
3. The ecliptical plane of Earth's orbit is the same as the equatorial plane. In other words, the axis of Earth's spin is parallel to the axis of Earth revolution around the sun.
4. We do not take into account the big scale variation, such as the variation of the angle between the equatorial plane and the ecliptical plane, or the precession of equinoxes.
5. We do not take into account the influence of other celestial bodies: the Moon, Jupiter, etc. With its mass $1.988 \times 10^{30}$ kg, the sun applies a $4.356 \times 10^{42}$ N to the Earth, while the Moon's attraction is only $1.989 \times 10^{20}$ N and Jupiter applies a $1.248 \times 10^{18}$ N attraction. So we can ignore the influence of other celestial bodies and pay attention only to the Sun attraction.

The first hypothesis and the last two ones are here only to simplify the computation and they will stay in effect for the duration of the whole discussion. The second and third hypothesis will be removed in turn to explain why the solar noon does not coincidate with the mean noon.

If all five simplifications are in effect, Earth turns around the sun in 360 days with a constant speed. So the orbital movement is 1 degree per day. If we temporarily use a set of coordinates centered on earth and using the "fixed stars", the sun moves one degree by day with respect to the "fixed stars". To allow for this 1 degree per day movement and to have the sun at the zenith at noon, Earth's rotation around its axis is 361 degrees per 24 hours. Therefore, on D+90 Earth has rotated for 90 turns and $\frac{1}{4}$, on D+180 Earth has rotated for 180 turns and $\frac{1}{2}$ and on D+270 Earth has rotated for 270 turns and $\frac{3}{4}$. See the picture below, which reverts to an heliocentric set of coordinates.




So, when Earth achieves a circular movement the plane of which is the same as the equator's plane, the solar noon and the mean noon occurs precisely at the same time.

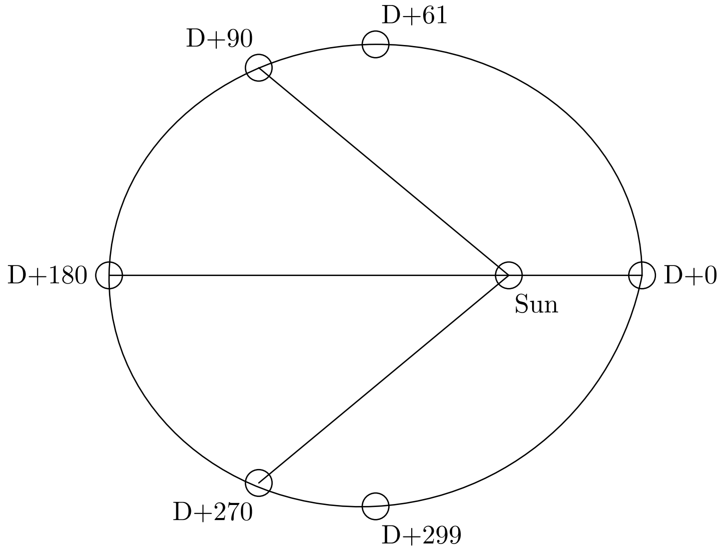
### 3.2 Kepler's Laws

Kepler's first law tells us that Earth's orbit is an ellipse, one focus of which is occupied by the solar system's center of inertia, that is, with a good approximation, the sun.

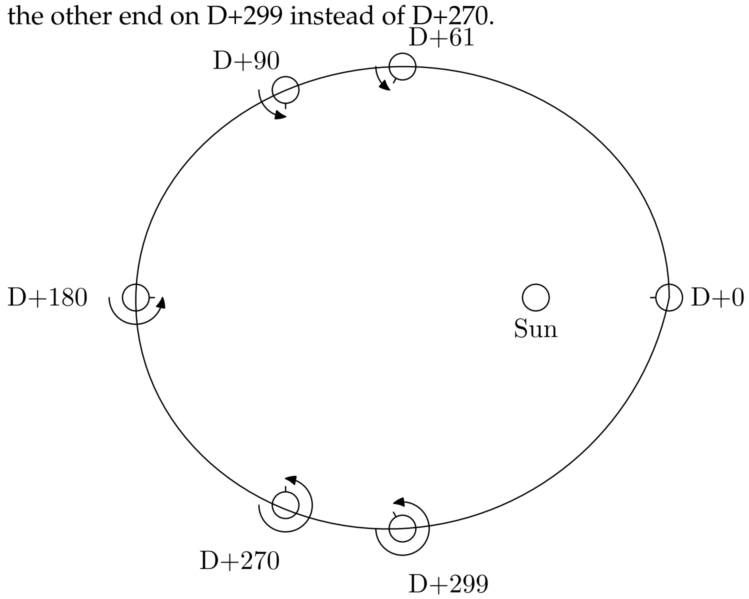
Kepler's second law tells us that the Earth's speed along its sun-focused ellipse is such that the area-speed of the sun-Earth vector is a constant.

Kepler's third law does not bring anything interesting in the discussion.

Let us see what this would give with an ellipse with a high eccentricity (0.5).

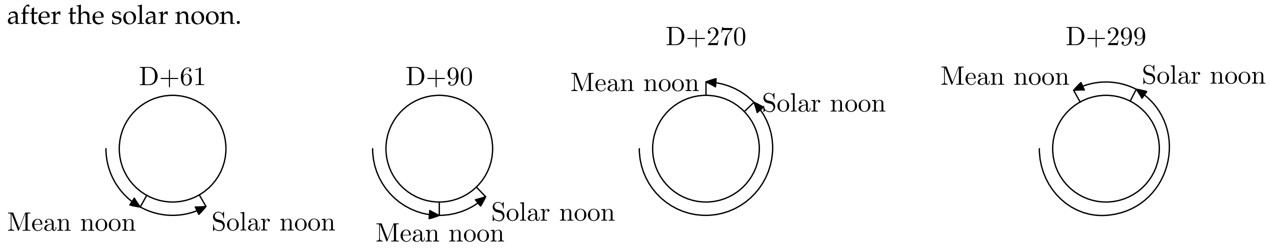



Since the sun is placed in one of the ellipse's foci, on the right of the drawing, we have to shift the D+90 and D+270 points to the left, so that the areas (Sun, D+0, D+90), (Sun, D+90, D+180), (Sun, D+180, D+270) and (Sun, D+270, D+0) are equivalent. So Earth reaches the end of the ellipse's small axis on D+61 instead of D+90, and the other end on D+299 instead of D+270.




On day D+61 at mean noon, Earth has rotated 61 turns plus 61 degrees, but it does not aim at the sun. It must rotate an additional 59 degrees to aim at the sun and reach solar noon. Likewise, on day D+90 at mean noon, Earth has rotated 90 turns plus 90 degrees and has to rotate an additional 46 degrees to reach solar noon.

On the other side (literally), on day D+270, the solar noon occurs before the mean noon: the solar noon occurs after 270 turns plus 224 degrees and the mean noon occurs after 270 turns plus 270 degrees, that is, 46 degrees after the solar noon.

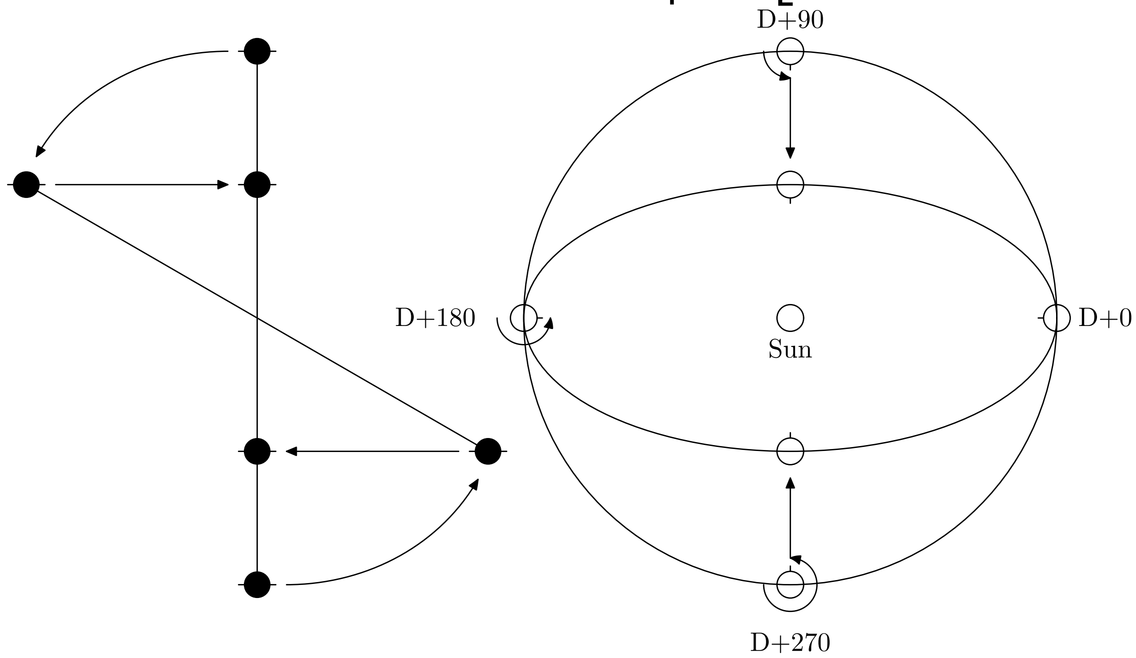



In these examples, the 46- and 59-degree angles are equivalent to a 3- and 4-hour delay respectively. This is very much greater than the delays that happen on the real world Earth. But at the same time, the real world orbit's eccentricity is only 0.02 while the description above posits a 0.5 eccentricity.

## 3.3 Influence of the Orbit Slope

The second simplification is reinstated, while the third simplification is removed. The discussion is a bit harder to follow, because it requires to think in 3-D, which is not convenient on a 2-D sheet of paper.

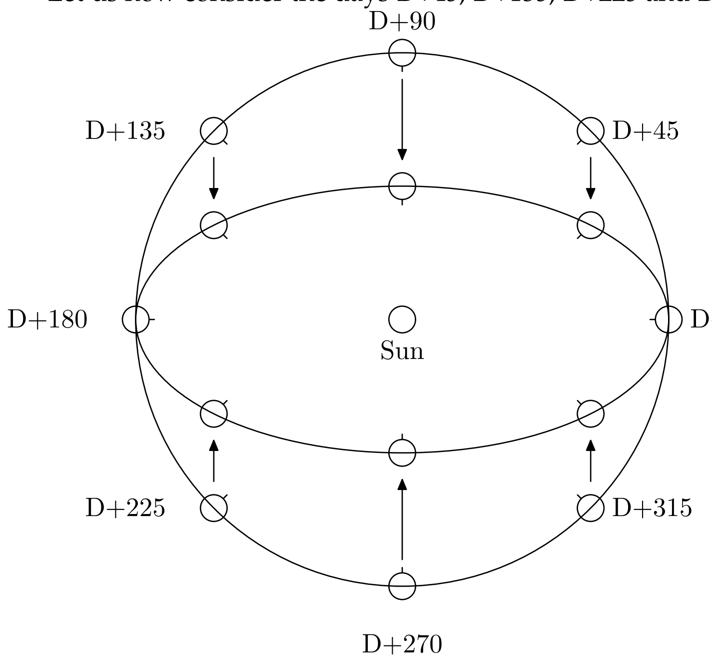
Let us start from the five simplifications are in effect. Then, we rotate the orbital plane *while keeping the equatorial plane parallel to itself*, then we project the new orbit orthogonally on the former orbit's plane, *while still keeping the equatorial plane parallel to itself*.



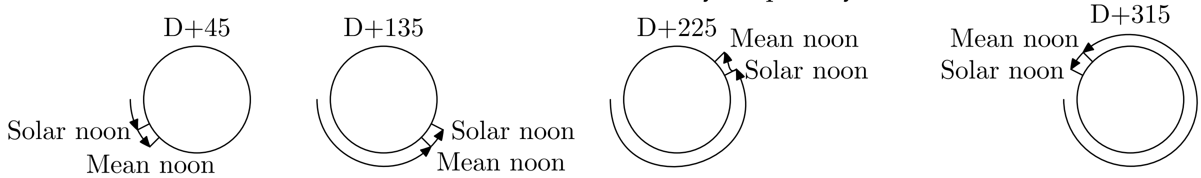

The end result is that the orbit's circle is contracted along the vertical axis, which gives an ellipse. All the while, Earth's rotation around its pole axis stays the same, since the pole-to-pole axis was constantly kept parallel to itself.

Since we use an exaggerated slope of 60 degrees, the circle is contracted by half. With the real world 23-degree slope, the contraction would be only 8%.

Let us now consider the days D+45, D+135, D+225 and D+315.




With this transformation, we notice that mean noon and solar noon occurs at the same time for D+0, D+90, D+180 and D+270. But this is not the case with the other days, especially D+45, D+135, D+225 and D+315.




We see that the solar noon occurs before the mean noon from D+0 to D+90 and then again from D+180 to D+270, while the mean noon occurs before the solar noon from D+90 to D+180 and again from D+270 to D+360. Instead of a periodical variation with a 1-year period, we have now a variation with a 1-half-year period.

### 3.4 In Summary

With the values used in the discussion, 0.5 eccentricity and 60-degree slope, merging the two phenomenons is difficult. But with the real world values, eccentricity less than 0.02 and 23-degree slope, we can assimilate the variations to sine waves with a rather small amplitude and add them. Attention, the D+0 days are not the same for both phenomenons. For the eccentricity, D+0 is 3rd of January, when Earth is at its perihelion. For the angle between the equatorial and orbital plane, D+0 is either equinox, around 21st March and 21st September.

The real variation curve is given in page 32 of "Histoire de l'Heure en France", by Jacques Gapaillard, published by Vuibert – ADAPT. The equation of time (gap between the solar noon and the mean noon) spans from a delay of 14 mn and 15 s of the solar noon with respect to the mean noon, to a 16 mn and 10 s advance of the solar noon with respect to the mean noon.

## 4 Annex