A formalisation of transcendence of e

Jujian Zhang

July 15, 2020

Abstract

The objective of this report is to present formalisations of some basic theorems from transcendental number theory with `Lean` and `mathlib` in the hope that it will serve as a motivation for mathematicians to be more curious about interactive theorem proving. The following theorems are formalised:

1. the set of algebraic numbers is countable, hence transcendental number exists:

```
1  theorem algebraic_set_countable : set.countable algebraic_set
2  theorem transcendental_number_exists : ∃ x : ℝ, transcendental x
```

2. all Liouville's numbers are transcendental:

```
1  theorem liouville_numbers_transcendental :
2    ∀ x : ℝ, liouville_number x → transcendental x
```

3. $\alpha := \sum_{i=0}^{\infty} \frac{1}{10^{i!}}$ is a Liouville's number hence α is transcendental.

```
1  theorem liouville_α : liouville_number α
2  theorem transcendental_α : transcendental α :=
3    liouville_numbers_transcendental α liouville_α
```

4. e is transcendental:

```
1  theorem e_transcendental : transcendental e
```

Disclaimer

The plan is to a self-contained report so that after chapter 2 any reader even without prior exposure to interactive theorem proving will be able to understand 3 where the details of formalisations and proofs reside. This should be relatively straightforward since the author is a `Lean`-dilettantes at best with only a partial picture of the full language. For the same reason, much of the code is perhaps not idiomatic or even plainly bad, thus it is not advisable to use this as a tutorial.

Contents

Chapter 1

Overview

1.1 Interactive theorem proving

Around 1920s, the German mathematician David Hilbert put forward the Hilbert programme to seek:

1. an axiomatic foundation of mathematics;

2. a proof of consistency of the said foundation;

3. Entscheidungsproblem: an algorithm to determine if any proposition is universally valid given a set of axioms.

The first two aims were later proved to be impossible by Gödel and the celebrated incompleteness theorems. Via the completeness of first order logic, the Entscheidungsproblem can also be interviewed as an algorithm for producing proofs using deduction rules. Even without a panacea approach for mathematics, computer still bears advantages against a carbon-based mathematician. Perhaps the most manifested advantage is the accuracy of a computer to execute its command and to recall its memories. Thus came the idea of **interactive theorem proving** — instead of hoping a computer algorithm to spit out some unfathomable proofs, assuming computers are given the ability to check correctness of proofs, human-comprehensible proofs can be verified by machines and thus guaranteed to be free of errors. With a collective effort, all theorems verified this way can be collected in an error-free library such that all mathematicians can utilise to prove further theorems which can then be added to the collection, ad infinitum Boyer et al. [1994]. Curry-Howard isomorphism provided the crucial relationship between mathematical proofs and computer programmes, more specifically relationship between propositions and types, to make such project feasible Kennedy and Kossak [2011]. The idea will be explained in section 2 along with `Lean`.

The proof of "Kepler's conjecture[1]" will serve as an illustrative example of

---

[1] the most efficient way to pack spheres should be hexagonally

utility of interactive theorem proving. As early as 1998, Thomas Hales had claimed a proof Hales [1998], Harrison et al. [2014], however the proof is controversial in the sense that mathematician even with great effort could not guarantee its correctness. A collaborative project using `Isabelle`[2] and `HOL Light`[3] verified the proof around 2014 and hence settled the controversy in 2017 Hales et al. [2017]. There is also Georges Gonthier with his teams using `Coq`[4] who formalised the four colour theorem and Feit-Thompson theorem where the latter is a step to the classification of simple groups Gonthier [2008], Gonthier et al. [2013]. Using `Lean`[5], Buzzard et al. were able to formalise modern notion of perfectoid spaces Buzzard et al. [2020].

1.2 History of transcendental number theory

"Transcendence" as a mathematical jargon first appeared in a Leibniz's 1682 paper where he proved that sin is a transcendental function in the sense that for any natural number n there does not exist polynomials $p_0, \cdots, p_n$ such that

$$p_0(x) + p_1(x)\sin(x) + p_2(x)\sin(x)^2 + \cdots + p_n(x)\sin(x)^n = 0$$

holds for all $x \in \mathbb{R}$ Bourbaki [1998]. The Swiss mathematician Johann Heinrich Lambert in his 1768 paper proved the irrationality of e and π where he also conjectured their transcendence Lambert [2004]. It is until 1844 that Joseph Liouville proved the existence of any transcendental numbers and until 1851 an explicit example of transcendental number is actually given by its decimal expansion:Kempner [1916]

$$\sum_{i=1}^{\infty} \frac{1}{10^{i!}} = 0.11000100000\cdots.$$

However, this construction is still artificial in nature. The first example of a real number proven to be transcendental that is not constructed for the purpose of being transcendental was e. Charles Hermite proved the transcendence of e in 1873 with a method applicable with help of symmetric polynomial to transcendence of π in 1882 and later to be generalised to Lindemann-Weierstrass theorem in 1885 stating that if $\alpha_1, \cdots, \alpha_n$ are distinct algebraic numbers then $e^{\alpha_1}, \cdots, e^{\alpha_n}$ are linearly independent over the algebraic numbers Baker [1990]. The transcendence of π was particularly celebrated because it immediately implied the impossibility of the ancient greek question of squaring the circle, i.e. it is not possible to construct a square, using compass and ruler only, with equal area to a circle. For this question is plainly equivalent to construct $\sqrt{\pi}$ which is not possible for otherwise π is algebraic. Georg Cantor in 1874 proved that

---

[2]a theorem prover relies extensively on dependent type theory and Curry-Howard correspondence.
[3]ibid.
[4]ibid.
[5]ibid.

algebraic numbers are countable hence not only did transcendental numbers exist, they exist in a ubiquitous manner – there is a bijection from the set of all transcendental numbers to $\mathbb{R}$ Cantor [1932, 1878].

In 1900, Hilbert proposed twenty-three questions, the 7th of which is regarding transcendental numbers: Is a^b transcendental, for any algebraic number a that is not 0 or 1 and any irrational algebraic number b? The answer is yes by Gelfond-Schneider theorem in 1934 Gelfond [1934]. This has some immediate consequences such that

1. $2^{\sqrt{2}}$ and its square root $\sqrt{2}^{\sqrt{2}}$ are transcendental;

2. e^π is transcendental for $e^\pi = \left(e^{i\pi}\right)^{-i} = (-1)^{-i}$;

3. $i^i = e^{-\frac{\pi}{2}}$ is transcendental etc.

In contrast, none of $\pi \pm e$, $\pi e, \frac{\pi}{e}$, π^π, π^e etc are proven to be transcendental. It is also conjectured that given any n $\mathbb{Q}$−linearly independent $z_1, \cdots, z_n \in \mathbb{C}$, then $\mathrm{trdeg}\left(\mathbb{Q}(z_1, \cdots, zn, e^{z_1}, \cdots, e^{z_n})/\mathbb{Q}\right)$ is at least n Lang [1966].

Chapter 2

Brief introduction to Lean

Chapter 3

Formalisation using Lean

Chapter 4

Further Work

Bibliography

Alan Baker. *Transcendental number theory*. Cambridge university press, 1990.

Nicolas Bourbaki. *Elements of the History of Mathematics*. Springer Science & Business Media, 1998.

Robert Boyer et al. The qed manifesto. *Automated Deduction–CADE*, 12:238–251, 1994.

Kevin Buzzard, Johan Commelin, and Patrick Massot. Formalising perfectoid spaces. In *Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs*, pages 299–312, 2020.

G Cantor. Uber eine eigenschaft des lnbegriffes aller reelen algebraischen zahlen. crelles j. 77 (1874), 258-262; also in gesammelte a bhandlungen, 1932.

Georg Cantor. Ein beitrag zur mannigfaltigkeitslehre. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 1878(84):242–258, 1878.

Aleksandr Gelfond. Sur le septieme probleme de hilbert.   .
                        , (4):623–634, 1934.

Georges Gonthier. Formal proof–the four-color theorem. *Notices of the AMS*, 55(11):1382–1393, 2008.

Georges Gonthier, Andrea Asperti, Jeremy Avigad, Yves Bertot, Cyril Cohen, François Garillot, Stéphane Le Roux, Assia Mahboubi, Russell O'Connor, Sidi Ould Biha, et al. A machine-checked proof of the odd order theorem. In *International Conference on Interactive Theorem Proving*, pages 163–179. Springer, 2013.

Thomas Hales, Mark Adams, Gertrud Bauer, Tat Dat Dang, John Harrison, Hoang Le Truong, Cezary Kaliszyk, Victor Magron, Sean McLaughlin, Tat Thang Nguyen, et al. A formal proof of the kepler conjecture. In *Forum of mathematics, Pi*, volume 5. Cambridge University Press, 2017.

Thomas C Hales. The kepler conjecture. *arXiv preprint math.MG/9811078*, 1998.

John Harrison, Josef Urban, and Freek Wiedijk. History of interactive theorem proving. In *Computational Logic*, volume 9, pages 135–214, 2014.

Aubrey J. Kempner. On transcendental numbers. *Transactions of the American Mathematical Society*, 17(4):476–482, 1916. ISSN 00029947. URL `http://www.jstor.org/stable/1988833`.

Juliette Kennedy and Roman Kossak. *Set Theory, Arithmetic, and Foundations of Mathematics: Theorems, Philosophies*, volume 36. Cambridge University Press, 2011.

M Lambert. Mémoire sur quelques propriétés remarquables des quantités transcendentes circulaires et logarithmiques. In *Pi: A Source Book*, pages 129–140. Springer, 2004.

Serge Lang. *Introduction to transcendental numbers*. Addison-Wesley Pub. Co., 1966.