Safety and stability analysis of FollowerStopper

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***Abstract*—In this paper we prove that the velocity controller, FollowerStopper, is safe and string stable. FollowerStopper is a controller that is meant to be implemented on an autonomous vehicle or in an adaptive cruise control (ACC) system. It takes as inputs the autonomous vehicle's velocity, relative distance to the car in front, relative velocity to the car in front, and a desired velocity. It then commands a new velocity for the autonomous vehicle which will either be the desired velocity or some lower velocity if that is necessary to maintain a safe distance to the car in front. Through mathematical proof, simulation in Simulink, and hardware in the loop implementation on a real autonomous vehicle through Robot Operating System (ROS) and Gazebo, several results are achieved. It is found that, given a maximum LiDAR range of 81 m, there is a maximum permissible safe speed of the car based on its maximum deceleration, and the car is programmed so that it will never be less than 1 m from the vehicle in front. It is shown that a vehicle with FollowerStopper in a singular lane without merges will not crash and that it will be string stable, effectively dissipating human-caused traffic waves if enough vehicles are deployed in the traffic flow.**

I. Introduction and Related Work

In order to maintain safety while driving, humans need a certain amount of safe distance between their car and the car directly in front of them so that if the car in front stops abruptly, they can react and stop in time to prevent a crash. From here forward the car which is of interest shall be termed as the autonomous vehicle (AV) and the car directly in front will be termed as the lead vehicle. If the cars are traveling at a slow speed, the AV can follow at a closer distance because the car will not travel as far during the time it takes to react to the lead and to brake. When the number of cars on a section of highway increases, the car density increases, often termed as congestion. When highways are congested, cars must travel closer together than they normally would, so drivers must drive slower than the speed limit.

It was shown that humans, once the congestion reaches a certain threshold, will inevitably cause traffic jams even if there are no traffic triggers, termed bottlenecks, such as lane changes, merges, tunnels, or other physical hindrances [1]. The reason for the formation of these "phantom traffic jams" is that humans are only concerned with maintaining safety, but are not typically not concerned about dissipating traffic. When a driver brakes, for example, the driver behind will typically brake harder, and this chain of events will continue until cars must come to a complete stop. It is even proposed that bottlenecks lead to traffic jams because such events will cause the car density to exceed the threshold [1].

Throughput is the number of cars that pass through a given area over a certain time. The best alternative to traffic jams, meaning the situation which will allow for the greatest throughput, is for all of the cars to follow the same optimal velocity. In doing so, there would be no hard braking or quick acceleration, further providing benefits of improved fuel economy, less wear on the brake pads and engine, and preventing the frustration and stress that accompanies road rage. The throughput is highest when setting the optimal velocity of the autonomous vehicle to be the average speed of the traffic wave ahead [2].

With the development of new ACC and autonomous systems, a larger percentage of cars on the road will have some degree of automation. It will take many decades for all cars on the road to be autonomous, so it is critical to inspect the impact that just a few autonomous vehicles will have on the overall traffic flow. Stern et al demonstrated that even a small percentage of autonomous vehicles ($< 5\%$) could have a substantial effect in reducing traffic [2]. Some researchers suspect that between 2020 and 2040 there could be the development of highway lanes solely for autonomous vehicles [4], at which point designers could rely on vehicle-to-vehicle communication with the formation of high-density platoons. Until then, however, it is necessary to design autonomous vehicles and ACC systems with the ability to safely interact with imperfect and unpredictable human drivers. To optimize the situation, the autonomous systems should be safer than human drivers and should do as much as they can to reduce traffic.

II. Setup

Suppose there is a line of vehicles in a straight infinite lane that never has any lane changes or other bottlenecks, as pictured in Figure 1. In Figure 1, the middle car is vehicle i. The vehicle to the right is vehicle $i-1$, and the trend continues so that the vehicle at the very front of the line of vehicles will be vehicle 1. The vehicle to the left is vehicle $i+1$, and the trend continues so that the vehicle at the very end of the line of n vehicles will be vehicle n. The x-position of 0 is an arbitrary location, but for ease it can be thought of as the starting position of the last vehicle in the line such that at all times $t \geq 0$, the position of every vehicle $x_i \geq 0$.

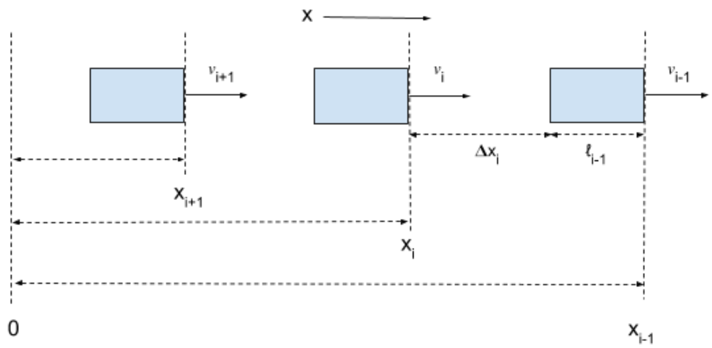

Fig. 1. Line of vehicles in a straight lane.

A. Variables

- Relative distance: The relative distance Δx_i for vehicle i is the distance between the front bumper of vehicle i and the back bumper of vehicle $i-1$. That is,

$$\Delta x_i = x_{i-1} - x_i - l_{i-1}, \quad (1)$$

  where x_{i-1} is the position of the front bumper of vehicle $i-1$, x_i is the position of the front bumper of vehicle i, and l_{i-1} is the length of vehicle $i-1$.
- Relative velocity: The relative velocity Δv_i for vehicle i is the velocity difference between vehicle $i-1$ and vehicle i. That is,

$$\Delta v_i = v_{i-1} - v_i, \quad (2)$$

  where v_{i-1} is the velocity of vehicle $i-1$ and v_i is the velocity of vehicle i.
- Reference velocity: Also known as the desired velocity or optimal velocity, the reference velocity r is the velocity at which the autonomous vehicle desires to travel. It is typically the average velocity over the length of a traffic wave, and is found by dividing the total distance traveled by a vehicle in front of the AV by the time. That is,

$$r = \frac{d_{total}}{t_{total}} \quad (3)$$

- Spacing error: The spacing error ϵ_i is the difference between the desired relative distance and the actual relative distance, giving,

$$\epsilon_i = \Delta x_{ides} - \Delta x_i. \quad (4)$$

B. Definitions

- Safe: A scenario is safe if for every vehicle i such that $1 < i \leq n$, $\Delta x_i > 0$ m. This means that the relative distance, or the distance between the front bumper of the subject vehicle and the back bumper of the vehicle directly in front will be greater than 0.
- Individual vehicle stable: A velocity control law is considered to be individual vehicle stable if the spacing error of the AV approaches 0 with time if the lead vehicle travels at a constant velocity.
- String stable: When a lead vehicle accelerates or decelerates in front of an individual vehicle stable AV, the spacing error will momentarily be nonzero. Suppose there is an infinite line of autonomous vehicles with one lead vehicle in front. The velocity control law is considered to be string stable if, during acceleration or deceleration of the lead vehicle, the spacing error decreases with each successive vehicle as they react to the change in velocity [3]. Additionally, all spacing errors must be in the same direction, either all negative or all positive. INCLUDE DESCRIPTION OF STRING STABLE VS STRING UNSTABLE GRAPH

C. Infrastructure

The FollowerStopper velocity controller is first modeled in Simulink and then used in conjunction with ROS to test in the physics-based simulation engine, Gazebo. After demonstrating success in Gazebo, the velocity controller is implemented onto the Cognitive and Autonomous Test (CAT) Vehicle at the University of Arizona using a hardware in the loop (HIL) configuration as described in Bhadani et all [5].

The CAT Vehicle is a modified Ford Hybrid Escape with a SICK LMS 291 Front Laser Rangefinder, a Velodyne HDL-64E S2 LiDAR, two Pointgrey Firefly MV FFMV-03M2C cameras, and a Novatel VPS/IMU. The FollowerStopper velocity controller uses distance data from the LiDAR to determine the relative distance to the car in front Δx_i. Using the relative distance data and the sampling frequencye of the LiDAR of 75 Hz, the AV can determine the relative velocity [6]. Assuming that the AV knows its own velocity at all times, it can use the relative velocity data to determine the velocity of the lead vehicle.

INCLUDE INFO ABOUT DELAY

III. FollowerStopper Description

A. Classification

The premise of FollowerStopper is to command exactly the reference velocity r whenever safe because this is the velocity that could dissipate already formed traffic jams and could prevent new traffic jams from forming. If $r > v_{lead}$ and the AV is getting close to the lead, FollowerStopper will command a lower velocity v_{cmd} whenever safety requires, based on the AV's velocity, relative velocity to the lead vehicle, and relative distance to the lead vehicle. Assuming that the two cars start out far enough apart, there are three relative velocity regions.

1) $v_{lead} > r$: $v_{AV} = r$ because the AV does not have to worry about catching up to the lead
2) $v_{lead} = r$: $v_{AV} = r$ because the AV does not have to worry about catching up to the lead
3) $v_{lead} < r$: $v_{AV} \leq r$ because if the AV is far enough away, it can continue to travel at r. Once the AV gets within a specified distance $\Delta x \leq \xi_3$ then it must travel less than r in order to prevent a collision. This distance should take into account that hard braking is a contributing factor to traffic jams, so the AV should begin slowing down well in advance of a crash in

order to be able to decelerate at a comfortable level of deceleration a_{dcmft}.

The Follower Stopper controller from Bhadani et al [6] is partially reproduced below for clarity.

$$v_{cmd} = \begin{cases} 0 & \Delta x \le \xi_1 \\ v^*_{lead}\dfrac{\Delta x - \xi_1}{\xi_2 - \xi_1} & \xi_1 < \Delta x \le \xi_2 \\ v^*_{lead} + (r - v)\dfrac{\Delta x - \xi_2}{\xi_3 - \xi_2} & \xi_2 < \Delta x \le \xi_3 \\ r & \xi_3 < \Delta x \end{cases} \tag{5}$$

where $v^*_{lead} = min(v_{lead}, 0)$. r is the reference velocity as taken from the output of smoothUpParams and v_{cmd} is the command velocity that is sent to the AV.

B. SmoothUpParams

Theoretically, the reference velocity will be given to the AV by means of a roadside controller which takes traffic data and determines the optimal constant velocity. The function smoothUpParams edits the reference velocity once the AV receives it but before it is used by followerStopper in order to prevent harsh and unnecessary acceleration or deceleration. smoothUpParams is a function of the new reference velocity r, the autonomous vehicle velocity v_{AV}, the maximum comfortable acceleration for the autonomous vehicle a_{cmft}, and the maximum comfortable deceleration for the autonomous vehicle a_{dcmft}. Using these inputs, smoothUpParams edits the reference velocity so that it is a reasonable value when it is sent to the followerStopper controller. Suppose, for example, the speed limit were to change from 10 m/s to 15 m/s and no car was in front of the AV. followerStopper would command the AV to travel at the reference velocity of 15 m/s immediately, but this will result in a large acceleration and an overshoot. smoothUpParams will cause the reference velocity to increase at a slow rate so that followerStopper can still command the reference velocity but it will not result in a large acceleration.

C. Designing ξ_1 for safety

According to FollowerStopper, if $\Delta x \le \xi_1$, then $v_{cmd} = 0$. That is, if the relative distance between the AV and the lead is less than or equal to ξ_1, the AV should brake as hard as possible or remain stopped if already at a stop. It is clear then that if $\Delta x < \xi_1$, the AV should already have initiated its emergency braking procedure. The braking will be initiated at $\Delta x = \xi_1$, so to prove that FollowerStopper is safe, we simply need to show that if $\Delta x = \xi_1$, braking as hard as possible will never result in a crash. The metric is computed as:

$$\begin{aligned} \xi_1 = 1 + \frac{1}{2ka_{dmax}}(v_{lead}^2 - kv_{AV}^2) \\ + v_{AV}(1 - \frac{a_{cmft}}{a_{dmax}})\delta + \frac{a_{cmft}}{2}(1 - \frac{a_{cmft}}{a_{dmax}})\delta^2 \end{aligned} \tag{6}$$

where k is the ratio of the maximum deceleration of the lead to the maximum deceleration a_{dmax} of the AV , a_{cmft} is the AV comfortable acceleration, The derivation can be found in the Appendix.

D. Designing ξ_2 for string stability

When $v_{lead} < v_{AV}$, the relative distance Δx will decrease as the AV approaches the lead. FollowerStopper is designed such that in such a situation, once Δx drops below a certain threshold, v_{AV} will decrease until it is equal to v_{lead}. Once $v_{AV} = v_{lead}$, the AV will maintain a distance $\Delta x = \xi_2$. If the relative distance drops below this equilibrium distance, that is $\Delta x < \xi_2$, the velocity of the AV will decrease below v_{lead} in an attempt to recover this distance. The second line of the piecewise FollowerStopper controller in equation 5 demonstrates this concept.

If the distance is greater than the equilibrium distance, that is $\Delta x > \xi_2$, the velocity of the AV will increase in an attempt to recover this distance unless v_{lead} exceeds the reference velocity, at which point $v_{cmd} = r$. Therefore, the AV will spend the majority of its functioning at or above the distance ξ_2. It is important, then, that ξ_2 is a string stable distance. According to [3], string stability is maintained when the time-gap h between two cars is such that $h \ge 2\tau$, where τ is the time constant of any lags in tracking the command velocity, or in other words, it is the delay of the system δ. With this knowledge the metric is computed as:

$$\xi_2 = (3v_{AV} - v_{lead})\delta + \frac{1}{2}(5a_{AV} - a_{lead})\delta^2 \tag{7}$$

E. Designing ξ_3 for comfort

IV. Human car-following models

A. Human Follower 1

Demonstrate safety and string instability

B. Human Follower 2

Demonstrate safety and string instability

V. Simulink Verification

A. Safety verification

B. String stability verification

VI. Results

A. Gazebo interface

B. CAT Vehicle testing

VII. To Do:

Explain that FollowerStopper has previously proven successful at dissipating traffic with the ξ equations. We want to mathematically prove and then use various platforms to verify that FollowerStopper is safe and string stable.

Double check that there isn't something funky with ξ_1 derivation

New picture for ξ_2 derivation

Delay

ξ_3 derivation

REFERENCES

[1] Y. Sugiyama et al, ?Traffic jams without bottlenecks?experimental evidence for the physical mechanism of the formation of a jam,? New Journal of Physics, vol. 10, March 2008.

[2] R. E. Stern, S. Cui, M. L. Monachec R. Bhadani, M. Bunting, M. Churchill, N. Hamilton, R. Haulcy, H. Pohlmann, F. Wu, B. Piccoli, B. Seibold, J. Sprinkle, D. B. Work, ?Dissipation of stop-and-go waves via control of autonomous vehicles: Field experiments,? Transportation Research Part C: Emerging Technologies, vol. 89, pp. 208–221, April 2018

[3] Vehicle Dynamics and Control

[4] CITE

[5]

[6]

[7] G. Eason, B. Noble, and I. N. Sneddon, “On certain integrals of Lipschitz-Hankel type involving products of Bessel functions,” Phil. Trans. Roy. Soc. London, vol. A247, pp. 529–551, April 1955.

[8] J. Clerk Maxwell, A Treatise on Electricity and Magnetism, 3rd ed., vol. 2. Oxford: Clarendon, 1892, pp.68–73.

[9] I. S. Jacobs and C. P. Bean, “Fine particles, thin films and exchange anisotropy,” in Magnetism, vol. III, G. T. Rado and H. Suhl, Eds. New York: Academic, 1963, pp. 271–350.

[10] K. Elissa, “Title of paper if known,” unpublished.

[11] R. Nicole, “Title of paper with only first word capitalized,” J. Name Stand. Abbrev., in press.

[12] Y. Yorozu, M. Hirano, K. Oka, and Y. Tagawa, “Electron spectroscopy studies on magneto-optical media and plastic substrate interface,” IEEE Transl. J. Magn. Japan, vol. 2, pp. 740–741, August 1987 [Digests 9th Annual Conf. Magnetics Japan, p. 301, 1982].

[13] M. Young, The Technical Writer’s Handbook. Mill Valley, CA: University Science, 1989.

APPENDIX

A. Derivation of ξ_1

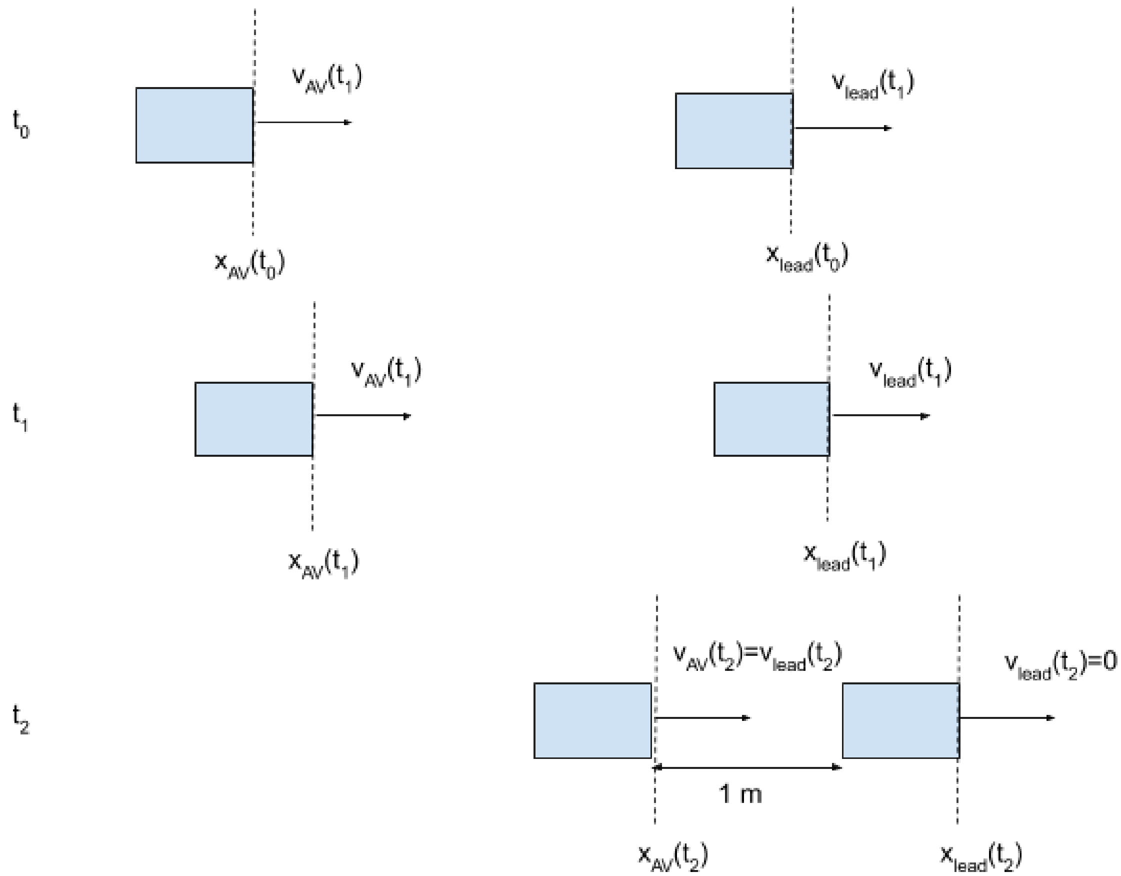

Fig. 2. Emergency breaking progression.

Consider Figure 2, depicting two cars at times t_0, t_1, and t_2. t_0 is the time at which the relative distance is equal to the emergency breaking distance, that is, $\Delta x(t_0) = \xi_1$. t_1 occurs δ seconds after reaching ξ_1, at which point the AV will first be able to react to being within the emergency breaking distance due to the delay. t_2 is the time at which both vehicles will be stopped and the AV will be exactly 1 m behind the lead vehicle. The vehicle to the left is the AV and the vehicle to the right is the human-driven lead. All positions and velocities have been labeled as functions of time, and the relative positions and relative velocities at each time can be computed according to the equations in section II.A. First we will consider the motion of the two vehicles from time t_1 to time t_2. Within this time region in a worst-case scenario, both the AV and lead will constantly decelerate at their maximum decelerations, and in the derivation we will express the lead vehicle maximum deceleration as proportional to the AV maximum deceleration. The minimum distance $\Delta x(t_1)$ which can be considered safe will assume that the AV will end up at a relative distance $\Delta x(t_2) = 1$ m from the lead vehicle. The following derivation uses this idea with the equations of motion to derive a value for $\Delta x(t_1)$.

$$\begin{aligned}
\text{Equation of motion}: v^2 - v_0^2 = 2a\Delta x & \\
\text{Lead vehicle}: & \\
v_{lead}(t_2)^2 - v_{lead}(t_1)^2 &= 2a_{dmaxLEAD}(x_{lead}(t_2) - x_{lead}(t_1)) \\
\text{Suppose } a_{dmaxLEAD} &= ka_{dmax} \\
0 - v_{lead}(t_1)^2 &= 2ka_{dmax}x_{lead}(t_2) - 2ka_{dmax}x_{lead}(t_1)) \\
2a_{dmax}x_{lead}(t_1) - \frac{v_{lead}(t_1)^2}{k} &= 2a_{dmax}x_{lead}(t_2) \\
\text{Autonomous vehicle}: & \\
v_{AV}(t_2)^2 - v_{AV}(t_1)^2 &= 2a_{dmax}(x_{AV}(t_2) - x_{AV}(t_1)) \\
0 - v_{AV}(t_1)^2 &= 2a_{dmax}(x_{lead}(t_2) - l_{lead} - 1 - x_{AV}(t_1)) \\
2a_{dmax}(l_{lead} + 1 + x_{AV}(t_1)) - v_{AV}(t_1)^2 &= 2a_{dmax}x_{lead}(t_2) \\
\text{Setting the equations equal}: & \\
2a_{dmax}x_{lead}(t_1) - \frac{v_{lead}(t_1)^2}{k} &= 2a_{dmax}(l_{lead} + 1 + x_{AV}(t_1)) - v_{AV}(t_1)^2 \\
2a_{dmax}(x_{lead}(t_1) - x_{AV}(t_1) - l_{lead}) &= 2a_{dmax} + \frac{v_{lead}(t_1)^2}{k} - v_{AV}(t_1)^2 \\
\Delta x(t_1) &= 1 + \frac{1}{2a_{dmax}}(\frac{v_{lead}(t_1)^2}{k} - v_{AV}(t_1)^2)
\end{aligned}$$

Delay is an essential consideration because if the AV is currently at $\Delta x = \Delta x(t_1)$, it needs to initiate emergency breaking, but it will not do as such until δ seconds have passed. Consider the time t_0 at which point the AV recognizes that it must send an emergency braking command in order to initiate emergency braking at the instant it reaches t_1 from the above example. When considering the emergency stopping distance ξ_1, we must plan for the worst case scenario. The worst case scenario is that at time t_0 the lead car begins to accelerate before coming to an immediate stop because the AV might think that it can begin to accelerate as well. Though from time t_0 to time t_1 the accelerations might fluctuate, a worst-case scenario would involve maximum acceleration of the AV and maximum deceleration of the lead. FollowerStopper limits the maximum acceleration to be a_{cmft}. The following derivation, based off of a derivation in [6], incorporates the delay to determine what a safe value of ξ_1 will equal such that if $\Delta x = \xi_1$, the AV will initiate hard braking at exactly $\Delta x = \Delta x(t_1)$. In this setup, $\xi_1 = \Delta x(t_0)$.

$$\begin{aligned}
\text{Equation of motion}: x &= x_0 + v_0 t + \frac{1}{2}at^2 \\
\text{Distance traveled over time ffi}: & \\
x_{lead}(t_1) &= x_{lead}(t_0) + v_{lead}(t_0)\delta + \frac{1}{2}a_{lead}(t_0)\delta^2 \\
x_{AV}(t_1) &= x_{AV}(t_0) + v_{AV}(t_0)\delta + \frac{1}{2}a_{AV}(t_0)\delta^2
\end{aligned}$$

Worst-case scenario, $a_{lead}(t_0) = a_{dmaxLEAD} = ka_{dmax}$, $a_{AV}(t_0) = a_{cmft}$

$$\begin{aligned}
\text{Incorporating relative distance}: & \\
\Delta x(t_1) &= \Delta x(t_0) + (v_{lead}(t_0) - v_{AV}(t_0))\delta + \frac{1}{2}(ka_{dmax} - a_{cmft})\delta^2 \\
\xi_1 = \Delta x(t_0) &= \Delta x(t_1) - (v_{lead}(t_0) - v_{AV}(t_0))\delta - \frac{1}{2}(ka_{dmax} - a_{cmft})\delta^2 \\
&= 1 + \frac{1}{2a_{dmax}}(\frac{v_{lead}(t_1)^2}{k} - v_{AV}(t_1)^2) \\
&\quad -(v_{lead}(t_0) - v_{AV}(t_0))\delta - \frac{1}{2}(ka_{dmax} - a_{cmft})\delta^2
\end{aligned}$$

The above equation for ξ_1 guarantees that a crash is impossible and that the AV will never come within 1 meter of the lead. However, when the AV arrives at time t_0, it will not know $v_{lead}(t_1)$ or $v_{AV}(t_1)$. Both are best approximated using the equation of motion, $v = v_o + at$, with a set to a safe value. In a worst-case scenario, the v_{lead} will be slower than expected and

v_{AV} will be faster than expected. Therefore safe acceleration values will be the same as indicated in the previous derivation. Because all time-dependent values will be expressed at the same time t_0, the notation can be dropped, giving

$$
\begin{aligned}
\xi_1 &= 1 + \frac{1}{2ka_{dmax}}((v_{lead} + ka_{dmax}\delta)^2 - k(v_{AV} + a_{cmft}\delta)^2) - (v_{lead} - v_{AV})\delta - \frac{1}{2}(ka_{dmax} - a_{cmft})\delta^2 \\
&= 1 + \frac{1}{2ka_{dmax}}(v_{lead}^2 + 2v_{lead}ka_{dmax}\delta + k^2a_{dmax}^2\delta^2 - kv_{AV}^2 - 2kv_{AV}a_{cmft}\delta - ka_{cmft}^2\delta^2) \\
&\quad - v_{lead}\delta + v_{AV}\delta - \frac{ka_{dmax}\delta^2}{2} + \frac{a_{cmft}\delta^2}{2} \\
&= 1 + \frac{v_{lead}^2}{2ka_{dmax}} + v_{lead}\delta + \frac{ka_{dmax}\delta^2}{2} - \frac{v_{AV}^2}{2a_{dmax}} - \frac{v_{AV}a_{cmft}\delta}{a_{dmax}} - \frac{a_{cmft}^2\delta^2}{2a_{dmax}} - v_{lead}\delta + v_{AV}\delta - \frac{ka_{dmax}\delta^2}{2} + \frac{a_{cmft}\delta^2}{2} \\
&= 1 + \frac{1}{2ka_{dmax}}(v_{lead}^2 - kv_{AV}^2) + v_{AV}(1 - \frac{a_{cmft}}{a_{dmax}})\delta + \frac{a_{cmft}}{2}(1 - \frac{a_{cmft}}{a_{dmax}})\delta^2
\end{aligned}
$$

B. Derivation of ξ_2

As described above, $v_{cmd} = v_{lead}$ when $\Delta x = \xi_2$. When $\Delta x \neq \xi_2$, we do not want v_{AV} to equal v_{lead} because the AV should either be going faster or slower than the lead in order to recover the equilibrium distance ξ_2. It is straightforward then that ξ_2 is the distance at which $v_{cmd} = v_{lead}$. The following derivation uses a similar setup to the derivations above, treating t_1 as the time in which $\Delta x(t_1) = 2\tau v_{AV}(t_1)$ and t_0 as the time at which the AV needs to receive the signal in order to execute the command v_{lead} at t_1.

$$
\begin{aligned}
\text{Equation of motion}: x &= x_0 + v_0 t + \frac{1}{2}at^2 \\
\text{Distance traveled over time ffi}: & \\
x_{lead}(t_1) &= x_{lead}(t_0) + v_{lead}(t_0)\delta + \frac{1}{2}a_{lead}(t_0)\delta^2 \\
x_{AV}(t_1) &= x_{AV}(t_0) + v_{AV}(t_0)\delta + \frac{1}{2}a_{AV}(t_0)\delta^2
\end{aligned}
$$

Assuming constant acceleration is the best prediction, so a_{lead} and a_{AV} should be used as written.

$$
\begin{aligned}
\text{Incorporating relative distance}: & \\
\Delta x(t_1) &= \Delta x(t_0) + (v_{lead}(t_0) - v_{AV}(t_0))\delta + \frac{1}{2}(a_{lead}(t_0) - a_{AV}(t_0))\delta^2 \\
\xi_2 = \Delta x(t_0) &= \Delta x(t_1) - (v_{lead}(t_0) - v_{AV}(t_0))\delta - \frac{1}{2}(a_{lead}(t_0) - a_{AV}(t_0))\delta^2 \\
&= 2\tau v_{AV}(t_1) - (v_{lead}(t_0) - v_{AV}(t_0))\delta - \frac{1}{2}(a_{lead}(t_0) - a_{AV}(t_0))\delta^2
\end{aligned}
$$

In order to calculate ξ_2, all time-dependent variables should be dependent upon t_0. $v_{AV}(t_1)$ can be treated as such through use of the equation of motion, $v = v_0 + at$. Additionally, τ is the delay of the system, so it can be set to $\tau = \delta$. Substituting and rewriting gives the true equilibrium distance value at which point $v_{cmd} = v_{lead}$,

$$
\begin{aligned}
\xi_2 &= 2\tau(v_{AV} + a_{AV}\delta) - (v_{lead} - v_{AV})\delta - \frac{1}{2}(a_{lead} - a_{AV})\delta^2 && (8) \\
&= 2v_{AV}\delta + 2a_{AV}\delta^2 - v_{lead}\delta + v_{AV}\delta - \frac{1}{2}a_{lead}\delta^2 + \frac{1}{2}a_{AV}\delta^2 && (9) \\
&= (3v_{AV} - v_{lead})\delta + \frac{1}{2}(5a_{AV} - a_{lead})\delta^2 && (10)
\end{aligned}
$$