1 Numerical solution of The Non linear Shallow Water Equations

The one dimensional Non Linear Shallow Water Equations with flat bottom read in conservation form

$$
\begin{aligned}
h_t + (hu)_x &= 0 \\
(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x &= 0
\end{aligned}
\tag{1}
$$

where subscripts denote partial derivatives respect to time and space variables t and x; h denotes the water column height; u the horizontal velocity; $g = 9.81$ the gravity acceleration. As a conservation law, the system (**??**) can be written as

$$
U_t + F(U)_x = 0 \tag{2}
$$

where $U = (h, hu)^T$, $F(U) = (hu, hu^2 + \frac{1}{2}gh^2)$. Weak solutions are approximated using a Finite Volume scheme. After averaging the system (2) in a cell $\Omega_i = [x_i - \Delta x/2, x_i + \Delta x/2]$, and defining $\overline{U} = \frac{1}{\Delta x}\int_{\Omega_i} U(x)dx$, then a semidiscrete approximation to (2) is

$$
\overline{U}_t + \frac{1}{\Delta x}\left(F(U_{i+1/2}) - F(U_{i-1/2})\right) = 0 \tag{3}
$$

where $U_{i\pm 1/2}$ corresponds to the values of the conserved variables at the interface of each cell. The system (3) is integrated in time using an Euler scheme with CFL condition

$$
\Delta t = CFL \frac{\Delta x}{\max_i(|u_i| + c_i)} \tag{4}
$$

with $CFL = 0.45$.

1.1 Riemann problem

At each time-step the the values at each interface $U^* = U_{i+1/2}$ of system (**??**) are obtained from the solution to the Riemann problem of the non-conservative form of (2) between the two neighbor states $U_L = U_i$ and $U_R = U_{i+1}$

$$
\begin{aligned}
U_t + A(U)U_x &= 0 \\
U(t = 0, x) &= \begin{cases} U_l & , \text{ if } x \leq 0. \\ U_r & , \text{ if } x > 0 \end{cases}
\end{aligned}
\tag{5}
$$

where A is the jacobian matrix of $F(U)$. The solution to this Riemann problem is found using the approximate Riemann solver of Roe that is described in reference [1]. It consists first of a change of variables that allows to write (5) for $h > 0$ as

$$
\begin{aligned}
V_t + C(V)V_x &= 0 \\
V(t=0,x) &= \begin{cases} V_l & , \text{ if } x \leq 0. \\ V_r & , \text{ if } x > 0 \end{cases}
\end{aligned} \tag{6}
$$

with $V = (2c, u)^T$ and $C(V) = \begin{pmatrix} u & c \\ c & u \end{pmatrix}$. Second, instead of using the exact formulation, a linearized problem is solved using $C(\hat{V})$ in place of $C(V)$, with $\hat{V} = (V_L + V_R)/2$. The matrix $C(\hat{V})$ is diagonalizable and thus, a decoupled system can be obtained in the form

$$
\begin{aligned}
(w_1)_t + \hat{\lambda}_1 (w_1)_x &= 0 \\
(w_2)_t + \hat{\lambda}_2 (w_2)_x &= 0 \\
(w_1, w_2)^T (t=0,x) &= \begin{cases} ((w_1)_L, (w_2)_L)^T & , \text{ if } x \leq 0. \\ ((w_1)_L, (w_2)_L)^T & , \text{ if } x > 0 \end{cases}
\end{aligned} \tag{7}
$$

where $\hat{\lambda}_1 = \hat{u} - \hat{c}$, $\hat{\lambda}_2 = \hat{u} + \hat{c}$, $w_1 = u - 2c$, $w_2 = u + 2c$ and $(w_1)_L = u_L - 2c_L, (w_2)_L = u_L - 2c_L$, $(w_1)_R = u_R - 2c_R, (w_2)_R = u_R - 2c_R$. Writing $W = (w_1, w_2)$ and noticing that $\hat{\lambda}_1 \leq \hat{\lambda}_2$, the solution can be found for three separate cases:

- If $\lambda_1 > 0$, then $W^* = W_L$
- If $\lambda_1 \leq 0$ and $\lambda_2 > 0$, $W^* = ((w_R)_1, (w_L)_2)^T$
- If $\lambda_2 \leq 0$, $W^* = W_R$

and values at the interface can then be recovered setting the inverse transformation

$$
\begin{aligned}
u^* &= \frac{1}{2}(w_1^* + w_2^*) \\
h^* &= \frac{1}{16g}(w_2^* - w_1^*)^2
\end{aligned} \tag{8}
$$

A third step is necessary, which consists on an entropy fix to select only weak solutions that are physically consistent. This is simply obtained by setting $W^* = \hat{W}$ whenever $(\lambda_1)_L < 0$ and $(\lambda_1)_r > 0$, or $(\lambda_2)_L < 0$ and $(\lambda_2)_R > 0$.

1.2 Second order Finite Volume Scheme

To obtain second order convergence for smooth solutions a MUSCL (Monotonic Upstream-Centered) scheme is used. This means that instead of solving a Riemann problem between $U_L = U_i$ and $U_R = U_{i+1}$ one must solve for $U_L = U_{i,r}$ and $U_R = U_{i+1,l}$, where $U_{i,r} = U_i + \frac{\Delta x}{2} s$, $U_{i,l} = U_i - \frac{\Delta x}{2} s$ $s = minmod(s_L, s_R)$, $s_L = \frac{U_i - U_{i-1}}{\Delta x}$, $s_R = \frac{U_{i+1} - U_i}{\Delta x}$ and

$$minmod(s_1, s_2) = \begin{cases} min(s_1, s_2) & \text{if } s_1 > 0 \; and \; s_2 > 0 \\ max(s_1, s_2) & \text{if } s_1 < 0 \; and \; s_2 < 0 \\ 0 & elsewhere \end{cases} \tag{9}$$

References

[1] F. Marche, P. Bonneton, P. Fabric, and N. Seguin. Evaluation of well-balanced bore-capturing schemes for 2d wetting and drying processes. *International Journal for Numerical Methods in Fluids*, 53:867–894, 2006.