Optimized Schwarz Method for the Linearized KdV equation and for coupling a NSWE-Serre model

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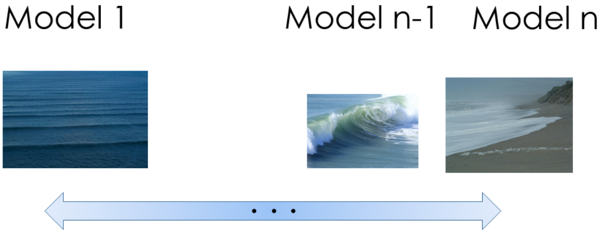
WTE2016


Model 1
Model n-1
Model n
. . .

RL1.6 on Resource Assesment and Site Characerization

Advanced modelling for marine energy

Model 1

Model n-1

Model n

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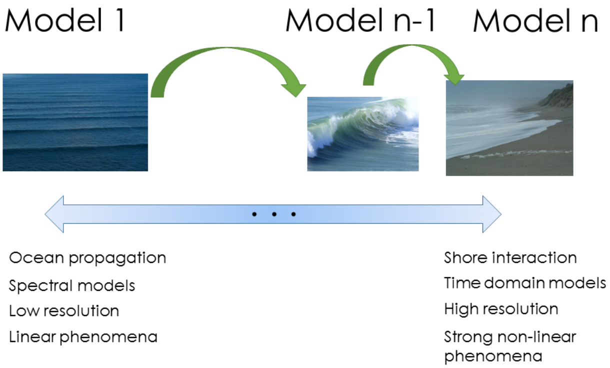
Ocean propagation
Spectral models
Low resolution
Linear phenomena

Shore interaction
Time domain models
High resolution
Strong non-linear phenomena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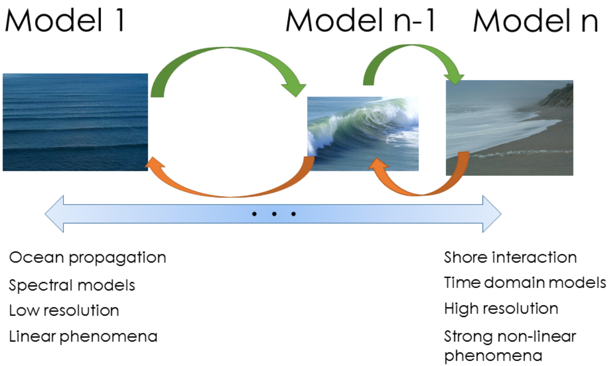
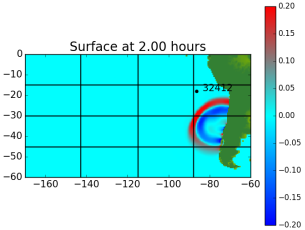


Question

Can we make these different models communicate with each other?

Truncated domain error: A tsunami simulation

Nonlinear long wave model (GeoClaw from the U. Washington)



Challenges

- Long simulations are required for statistics estimates
- Transport of the numerical error from deep to shallow water: shoaling

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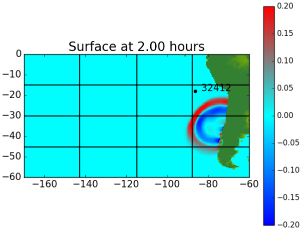

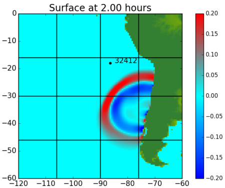

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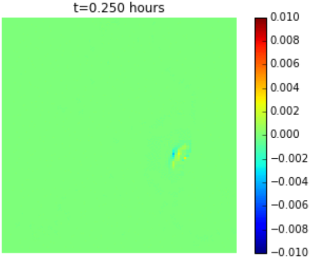

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Our work

First results

- Develop a self coupling methodology: Domain Decomposition Methods (DDM)

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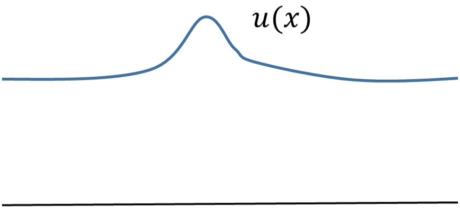
Ongoing work

Develop a coupled model between the Non Linear Shallow Water Equations (NSWE) and the Serre equations

A key ingredient: *Transparent* Boundary Conditions (TBC)



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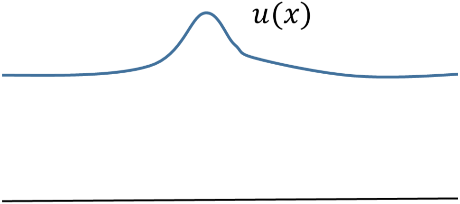

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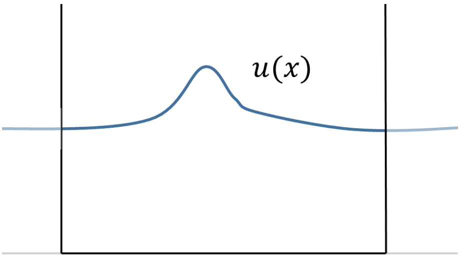

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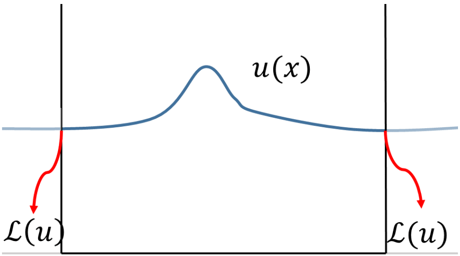

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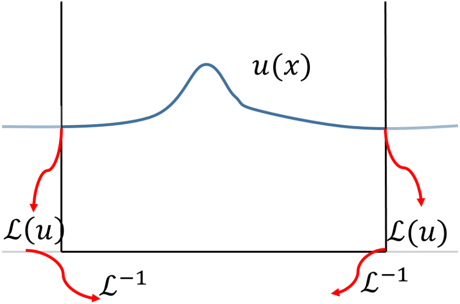

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5. *Un-apply* it and get new equations for the boundaries

Transparent Boundary Conditions (TBC's)

Exact TBC's (Besse et al, 2015)

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- Solving these equations is also *very* hard (non-local operator)

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- Not suited for numerical computations!

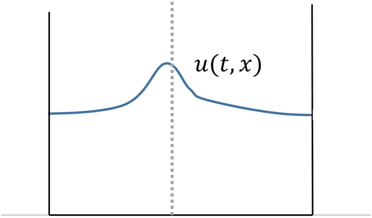
Approximate TBC's

Result

Using the approximation $\lambda(s) \approx c$ we obtain the following approximate TBC's

$$\begin{cases} u(t, \mathit{left}) - cu_x(t, \mathit{left}) + c^2 u_{xx}(t, \mathit{left}) = 0 \\ u(t, \mathit{right}) - c^2 u_{xx}(t, \mathit{right}) = 0 \\ u_x(t, \mathit{right}) + cu_{xx}(t, \mathit{right}) = 0 \end{cases}$$

Using the TBC's for a DDM: The Additive Schwarz Method



DDM Methodology

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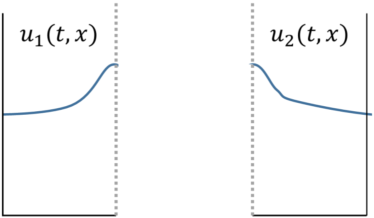
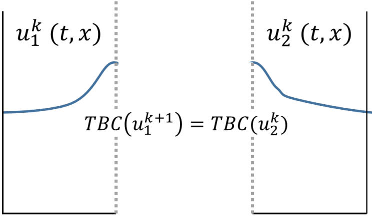


DDM Methodology

- Solve each problem separately, using the adjacent domain last information.

Using the TBC's for a DDM: The Additive Schwarz Method



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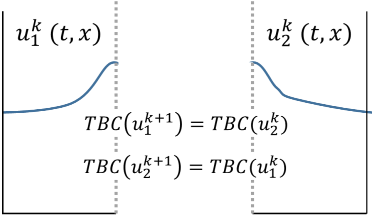
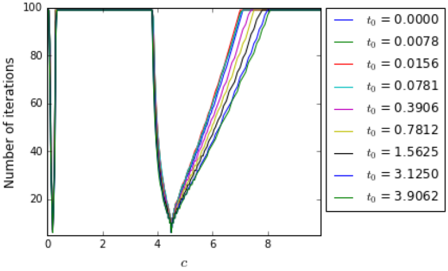


DDM Methodology

- Solve each problem separately, using the adjacent domain last information.
- Iterate until convergence criteria.
- How do we determine the c_L and c_R constants ?

DDM: Determination of optimal $c_L = c_R = c$

- For a fixed mesh discretization (in time and space), fixed interface location and different time-steps (initial condtion)



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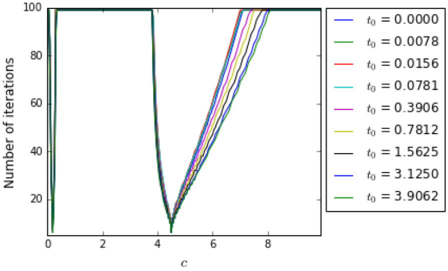

Remark

- There exist optimal values for c (e.g ≈ 4.5) where convergence is observed with (only) 5-7 iterations.

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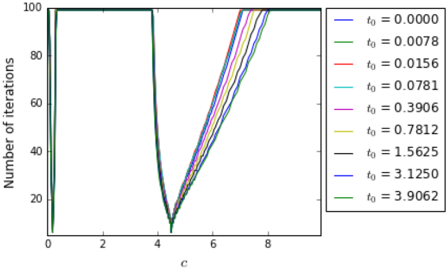

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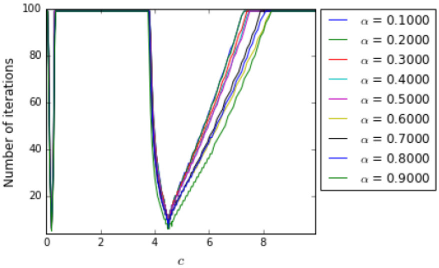
- There exist optimal values for c (e.g ≈ 4.5) where convergence is observed with (only) 5-7 iterations.
- There is no observed dependency on the initial condition.

DDM: Dependency on the location of the interface



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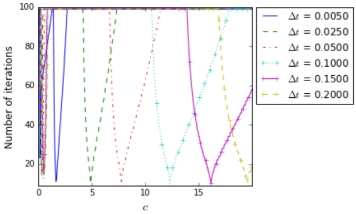
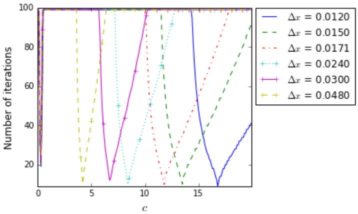
DDM: Dependency on the discretization size: Δx, Δt



(a) Fixed $\Delta x = \frac{12}{250}$



(b) Fixed $\Delta t = 0.02$

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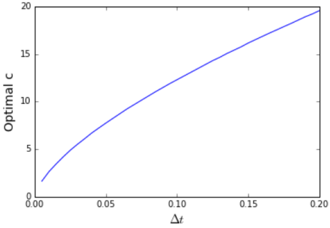

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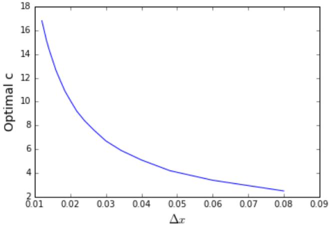

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Result

There is a strong dependance on Δx and Δt.

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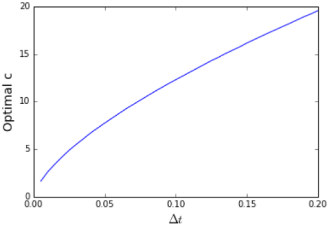

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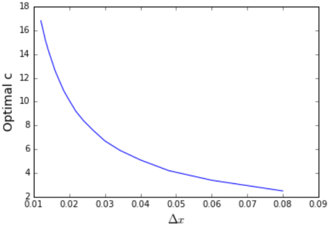

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Result

There is a strong dependance on Δx and Δt. Furthermore the regression

$$c_{opt}(\Delta t, \Delta x) = \kappa + \alpha(\Delta t)^{2/3} + \beta \frac{1}{\Delta x} + \gamma \frac{\Delta t^{2/3}}{\Delta x}$$

fits the numerical experiments with $R^2 = 0.9999894$

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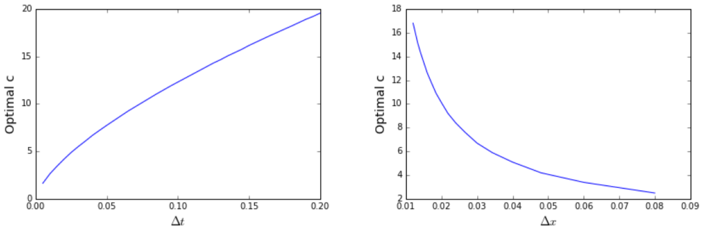

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fits the numerical experiments with $R^2 = 0.9999894$ for $\kappa = 0.0775$, $\alpha = -0.3353$, $\beta = -0.0012$, $\gamma = 2.7407$

A coupled shallow water (NSWE) - Serre model

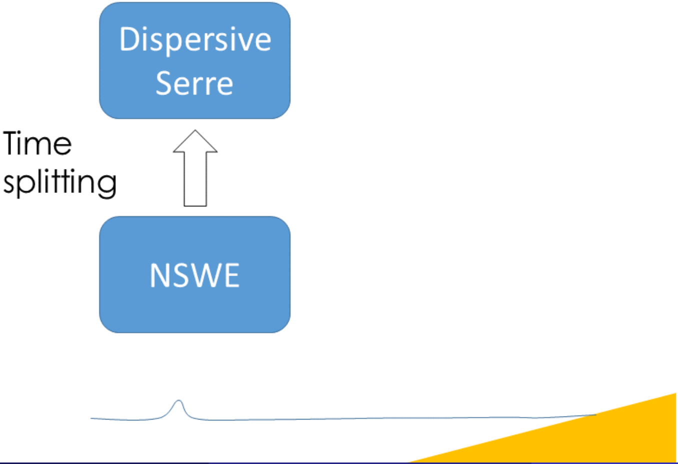
The Serre equations

$$h_t + (hu)_x = 0$$

$$u_t + uu_x + gh_x - \frac{1}{3h}\left(h^3\left(u_{xt} + uu_{xx} - (u_x)^2\right)\right)_x = 0$$

- $h =$ the water column height
- $u =$ depth averaged horizontal velocity (1D)

Coupling methodology



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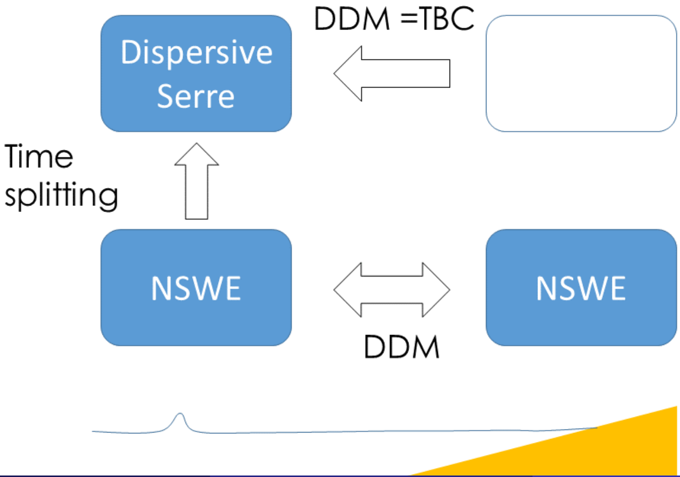

Coupling methodology

Numerical method

- Fractional splitting scheme for non-linear/dispersive terms (Bonneton et al, 2011).
- Well-balanced fourth order accurate Finite Volume NSWE model (Berthon and Marche,2008)
- Fourth order Finite Difference model for the dispersive equation

Three domain coupling: NSWE-Serre-NSWE