Report on APNUM-D-17-00138

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Title: Optimized Schwarz Waveform Relaxation method for the linearized KdV equation
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In this manuscript, the authors present an optimized Schwarz Waveform Relaxation method (OSWR) with the interface boundary conditions (IBCs) constructed based on the approximation of the exact transparent boundary conditions (TBCs) between the subdomains for the linearized KdV equation without the advective term. The form of the approximate conditions is simple for convenience of calculations and accelerating convergence. Small corrections to the scheme in monodomain insure the convergence. Finally, they obtain the regression expressions that provide the optimal coefficients via a lot of experiments. However, as a consequence of the following remarks, the algorithm the authors proposed is not the optimized Schwarz waveform relaxation method actually, hence I do not recommend the publication.

Major comments:

1. In Subsection 3.2 the authors proposed "optimized Schwarz waveform relaxation" for the monodomain problem in the continuous sense. However, the algorithm the authors proposed is not the optimized Schwarz waveform relaxation method actually. From equation (11)-(13) (and also the explanation of α_j^i in equation (14): the converged solution in the previous time step), the authors did not provide the correct form of optimized Schwarz waveform relaxation method. I strongly recommend the authors to read the reference on waveform relaxation and Schwarz waveform relaxation carefully, and also review the both algorithms in Introduction.

If the author propose the correct algorithm, they also need to consider the following problems.

2. In Subsection 3.1 the authors intended to introduce the optimal Schwarz waveform relaxation algorithm for the monodomain problem in the continuous sense. However, what is the index n in equation (10), $u_i^{n,k}$ and so on. In the continuous sense, there is no index n actually. In addition, it is better to introduce the algorithms directly to the model problem and provide the convergence theorem in continuous sense.
   Now it's well known that the additive Schwarz method is not equivalent to the parallel Schwarz method for the overlapping case. Then it should be the parallel Schwarz (waveform relaxation) method. The authors should also review the relevant reference on (optimized) Schwarz method in Introduction.

3. In Subsection 3.2, in equation (11)-(13), the index n is again confused, and what is t_0 and L? The authors need to provide the original problem with IBC as external boundaries and domain Ω, and then present the decomposition of the domain (in addition, it's better to consider the general decomposition, i.e., decomposing $\Omega = (-a, b)$ into $\Omega_1 = (-a, 0)$ and $\Omega_2 = (0, b)$, where $a \neq b$).
   Could you further provide the regularity theorem of OSWR?

4. The authors obtained the optimized parameters numerically in Subsection 3.3. Generally, you need to obtain optimized parameters c_L and c_R theoretically in Subsection 3.2.

5. On page 3, the authors said that "Therefore, our objective lays on the convergence of the DDM to the solution of the same problem in the monodomain, independently of the errors on the external boundaries." What about the relevance between convergence in the monodomain and convergence in the whole domain $\mathbb{R}$ when applying these operators as IBCs (change the boundary conditions inside) in the Schwarz waveform relaxation method? How does the convergence remains true?

6. In equation (6), the coefficients "c_L" and "c_R" are considered as possibly different, while on page 12, you consider $c_L = c_R = c$ to simplify the tests and avoid expensive computations. Thus, How will this affect performance of your algorithm?

Minor comments:

1. Page 2: "computation is not doable in general [1]. [12] and [3] propose numerical approximation": propose $\to$ proposed.
2. There are mixture of abbreviation and full name. On page 4, "transparent boundary conditions" should be "TBC"; On page 8, "Schwarz Waveform Relaxation method" should be "SW".
3. Page 5: $c_L, c_R \in \{-10, -1, -0.1, 0, 0.1, 1, 10\}^2 \to (c_L, c_R) \in \{-10, -1, -0.1, 0, 0.1, 1, 10\}^2$.
4. Page 6: "of them. Therefore, one must find functions that satisfies the PDE in each...", satisfies $\to$ satisfy.
5. Page 6: the denotation of $\mathcal{B}_i$ should be provided just following equation (10).
6. Page 7: "where ∂n_i is the outward normal to Ω_i on Γ , and the D2N (Dirichlet to Neumann": D2N $\to$ $D2N$.
7. Page 8: "given by is (11)-(12) is called Schwarz Waveform Relaxation method [4].": the first "is" should be deleted.
8. Page 9: the notation u_j^i is not coincide with u_i^kin previous two subsections.
9. Page 12: How do you choose the initial guess u_i^0 in the numerical tests?
10. Page 15: "Partial conclusion": conclusion $\to$ conclusions.