Inriamericwaves
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Contents

1 Domain decomposition method for the Serre equations

Considering that the study and derivation of transparent boundary conditions is much more developed and known for linear partial differential equations [3], we will consider in the section linearized versions of the Serre equations. The linearization will be performed around constant average water height and velocity, denoted respectively by h_0 and u_0. Thus, the Serre equations can be written as

$$\begin{cases} h_t + u_0 h_x + h_0 u_x = 0 \\ u_t + u_0 u_x + g h_x - \frac{h_0^2}{3}(u_{xxt} + u_0 u_{xxx}) = 0 \end{cases} \tag{1}$$

1.1 Linearization with $u_0 = 0$ (for the complete equation)

Firstly, we will consider the case $u_0 = 0$, which gives, from (1),

$$\begin{cases} h_t + h_0 u_x = 0 \\ u_t + g h_x - \frac{h_0^2}{3} u_{xxt} = 0 \end{cases} \tag{2}$$

1.1.1 Derivation of the IBCs following [2] (rate of convergence of the ASM)

The derivation of the IBCs will be made analogously as in [2]. We will consider the domain $\Omega = \mathcal{R}$ divided in two (possibly overlapped) subdomains,

$$\Omega_1 =]-\infty, L], \qquad \Omega_2 = [0, \infty[$$

so L denotes the size of the overlap.

We will consider that only one interface boundary condition is imposed, denoted by the operator

$$\mathcal{B}_i U(t,x) = \alpha U_x(t,x) + \Lambda_i(U(t,x)) \tag{3}$$

where $U = (h,u)^T$, i indicates the subdomain and Λ_i is an linear operator of symbol $\psi_i(s)$ in the Laplace domain.

Thus, the additive Schwarz method is written as

$$\begin{cases} h_t^{1,n+1} + h_0 u_x^{1,n+1} = 0, \ t \geq 0, \ x \in \Omega_1 \\ u_t^{1,n+1} + g h_x^{1,n+1} - \frac{h_0^2}{3} u_{xxt}^{1,n+1} = 0, \ t \geq 0, \ x \in \Omega_1 \\ \alpha U_x^{1,n+1}(t,L) + \Lambda_1(U_1^{n+1}(t,L)) = \alpha U_2^n(t,L) + \Lambda_1(U_2^n(t,L)), \ t \geq 0 \end{cases} \tag{4}$$

$$\begin{cases} h_t^{2,n+1} + h_0 u_x^{2,n+1} = 0, \ t \geq 0, \ x \in \Omega_2 \\ u_t^{2,n+1} + g h_x^{2,n+1} - \frac{h_0^2}{3} u_{xxt}^{2,n+1} = 0, \ t \geq 0, \ x \in \Omega_2 \\ \alpha U_x^{2,n+1}(t,0) + \Lambda_2(U_2^{n+1}(t,0)) = \alpha U_1^n(t,0) + \Lambda_2(U_1^n(t,0)), \ t \geq 0 \end{cases} \tag{5}$$

Performing the Laplace transform of (4) and (5), one obtains

$$\begin{cases} s\hat{h}^{1,n+1} + h_0 \hat{u}_x^{1,n+1} = 0, \ s \in \mathcal{C}, \ s > 0, \ x \in \Omega_1 \\ s\hat{u}^{1,n+1} + g\hat{h}_x^{1,n+1} - s\frac{h_0^2}{3}\hat{u}_{xx}^{1,n+1} = 0, \ s \in \mathcal{C}, \ s > 0, \ x \in \Omega_1 \\ \alpha \hat{U}_x^{1,n+1}(s,L) + \psi_1 \hat{U}^{1,n+1}(s,L) = \alpha \hat{U}_x^{2,n}(s,L) + \psi_1 \hat{U}^{2,n}(s,L), \ s \in \mathcal{C} \end{cases} \tag{6}$$

$$\begin{cases} s\hat{h}^{2,n+1} + h_0 \hat{u}_x^{2,n+1} = 0, \ s \in \mathcal{C}, \ s > 0, \ x \in \Omega_1 \\ s\hat{u}^{2,n+1} + g\hat{h}_x^{2,n+1} - s\frac{h_0^2}{3}\hat{u}_{xx}^{2,n+1} = 0, \ s \in \mathcal{C}, \ s > 0, \ x \in \Omega_1 \\ \alpha \hat{U}_x^{1,n+1}(s,0) + \psi_2 \hat{U}^{2,n+1}(s,0) = \alpha \hat{U}_x^{1,n}(s,0) + \psi_2 \hat{U}^{1,n}(s,0), \ s \in \mathcal{C} \end{cases} \tag{7}$$

The solutions of (6) and (7) have the form

$$\hat{U}^{i,n+1}(s,x) = \overline{U}^{i,n+1}(s) e^{\lambda x} \tag{8}$$

Replacing in (6) and (7) gives $M\overline{U}^{i,n+1} = 0$, where

$$M = \begin{pmatrix} s & \lambda h_0 \\ \lambda g & s\left(1 - \frac{h_0^2\lambda^2}{3}\right) \end{pmatrix} \tag{9}$$

We have nontrivial solutions $\overline{U}^{i,n+1}$ only if $det M = 0$, i.e,

$$\lambda = \pm\sqrt{\frac{s^2}{h_0\left(g + \frac{s^2 h_0}{3}\right)}} \tag{10}$$

Considering that the solutions $\hat{U}^{i,n+1}$ must vanish on $\pm\infty$, we have

$$\hat{U}^{1,n+1} = \overline{U}^{1,n+1} e^{|\lambda| x} \qquad \hat{U}^{2,n+1} = \overline{U}^{2,n+1} e^{-|\lambda| x} \tag{11}$$

The coefficients $\overline{U}^{i,n+1}$ are determined using the boundary conditions in (6) and (7). We firstly solve (7), and we get

$$\alpha \hat{U}_x^{2,n+1}(s,0) + \psi_2 \hat{U}^{2,n+1}(s,0) = \overline{U}^{2,n+1}(-\alpha|\lambda| + \psi_2) = \alpha \hat{U}_x^{1,n}(s,0) + \psi_2 \hat{U}^{1,n}(s,0)$$

so

$$\hat{U}^{2,n+1}(s,x) = \frac{\alpha \hat{U}_x^{1,n}(s,0) + \psi_2 \hat{U}^{1,n}(s,0)}{-\alpha|\lambda| + \psi_2} e^{-|\lambda| x} = \frac{\alpha|\lambda| + \psi_2}{-\alpha|\lambda| + \psi_2} \hat{U}^{1,n}(s,0) e^{-|\lambda| x} \tag{12}$$

Similarly, solving (6), we obtain

$$\hat{U}^{1,n+1}(s,x) = \frac{\alpha \hat{U}_x^{2,n}(s,L) + \psi_1 \hat{U}^{2,n}(s,L)}{\alpha|\lambda| + \psi_1} e^{|\lambda|(x-L)} = \frac{-\alpha|\lambda| + \psi_1}{\alpha|\lambda| + \psi_1} \hat{U}^{2,n}(s,L) e^{|\lambda|(x-L)} \tag{13}$$

and, using (12) in (13), we get

$$\hat{U}^{1,n+1}(s,0) = \frac{-\alpha|\lambda| + \psi_1}{\alpha|\lambda| + \psi_1} \frac{\alpha|\lambda| + \psi_2}{-\alpha|\lambda| + \psi_2} e^{-2|\lambda| L} \hat{U}^{1,n-1}(s,0) \tag{14}$$

Similarly, using (13) in (12),

$$\hat{U}^{2,n+1}(s,L) = \frac{-\alpha|\lambda| + \psi_1}{\alpha|\lambda| + \psi_1} \frac{\alpha|\lambda| + \psi_2}{-\alpha|\lambda| + \psi_2} e^{-2|\lambda| L} \hat{U}^{2,n-1}(s,L) \tag{15}$$

We can thus define the rate of convergence of the Schwarz method as

$$\rho(s,L) = \frac{-\alpha|\lambda| + \psi_1}{\alpha|\lambda| + \psi_1} \frac{\alpha|\lambda| + \psi_2}{-\alpha|\lambda| + \psi_2} e^{-2|\lambda| L} \tag{16}$$

In the case $\alpha = 0$, $\psi_1 = \psi_2 = 1$, we recover the classical ASM (with Dirichlet interface boundary condition), and the rate of convergence is

$$\rho(s, L)_{classical} = e^{-2|\lambda|L} \tag{17}$$

We can see that, in this case, the ASM converges only if there is an overlapping $L > 0$.

The overlapping is not necessary in the general case (16). Indeed, choosing $\psi_1 = \alpha|\lambda|$ and $\psi_2 = -\alpha|\lambda|$, the rate of convergence is $\rho(s, L) \equiv 0$, and the Schwarz method converges after two iterations [**?**]. In this case, the operators for the TBCs, applied respectively in the right boundary of Ω_1 and in the left boundary of Ω_2, are

$$B_1(U) = U_x + |\lambda|U, \qquad B_2(U) = U_x - |\lambda|U$$

1.1.2 Derivation of the IBCs following [1] (via the derivation of TBCs)

In an alternative way, we will follow the approach proposed by [1] to derive the IBCs for the linearized Serre equation (2), and use them as IBCs for the ASM. Being a simpler approach, we will use it in the derivation of IBCs considering other linearizations of the Serre equations.

If we want to solve the problem (2) in the finite domain $[a, b]$, the TBCs are constructed by solving it in the complementary of Ω. Thus, we will solve

$$\begin{cases} h_t + u_0 h_x + h_0 u_x = 0, \ t > 0, x < a \text{ or } x > b \\ u_t + g h_x - \frac{h_0^2}{3} u_{xxt} = 0, \ t > 0, x < a \text{ or } x > b \\ u \longrightarrow 0, \ x \longrightarrow \pm\infty \end{cases} \tag{18}$$

The approach follows the same arguments as done above. We solve (18) in the Laplace domain, which gives a solution in the form (8), with the two roots λ_i of the respective characteristic polynomial given by (10). To force the solution to vanish in $\pm\infty$, we have the solutions

$$\hat{U}(s, x) = \overline{U}_{-} e^{|\lambda(s)|x}, x < a,$$
$$\hat{U}(s, x) = \overline{U}_{+} e^{-|\lambda(s)|x}, x > b$$

Thus, from these two solutions we can obtain the following TBCs for solving (2) in $\Omega = [a, b]$:

$$\hat{U}_x(s,a) - |\lambda(s)|\hat{U}(s,a) = 0$$
$$\hat{U}_x(s,b) + |\lambda(s)|\hat{U}(s,b) = 0$$

respectively for the left and the right boundary of Ω. Approximating $|\lambda(s)|$ by a constant $c > 0$ and using these TBCs as IBCs for the ASM, we obtain the operators

$$B_1(U) = U_x + |c|U, \qquad B_2(U) = U_x - |c|U$$

1.2 Linearization with $u_0 \neq 0$ (for the complete equation)

Following the approach of [1], we will derive TBCs for the linearized equation :

$$\begin{cases} h_t + u_0 h_x + h_0 u_x = 0 \\ u_t + +u_0 u_x + g h_x - \frac{h_0^2}{3}(u_{xxt} + u_0 u_{xxx}) = 0 \end{cases} \tag{19}$$

We will solve this equation in the complementary of the domain of interest, $\Omega = [a,b]$:

In the Laplace space, (19) is written as

$$\begin{cases} s\hat{h} + u_0 \hat{h}_x + h_0 \hat{u}_x = 0 \\ s\hat{u} + u_0 u_x + g\hat{h}_x - \frac{h_0^2}{3}(s\hat{u}_{xxt} + u_0 \hat{u}_{xxx}) = 0 \end{cases} \tag{20}$$

Considering a solution in the form $\hat{U}(s,x) = \overline{U}(s)e^{\lambda(s)x}$, and replacing in (20), we get the linear system $M\overline{U} = 0$, with

$$M = \begin{pmatrix} s + \lambda u_0 & \lambda h_0 \\ g\lambda & s + u_0\lambda - \frac{h0^2}{3}(\lambda^2 s + \lambda^3 u_0) \end{pmatrix}$$

We have nontrivial solution if and only if $detM = 0$, i.e.,

$$s^2 + 2su_0\lambda + \left(-gh_0 - \frac{h_0^2 s^2}{3} + u_0^2\right)\lambda^2 - \frac{2}{3}h_0^2 u_0 s\lambda^3 - \frac{1}{3}h_0^2 u_0^2 \lambda^4 = 0 \tag{21}$$

The roots of (21) (and even their expansions around $s = 0$) have a quite complicated form, but them can be written in the form

$$
\begin{aligned}
\lambda_1(s) &= -\frac{1}{2}(A_1 + A_2) + s\left(-\frac{1}{2u_0} - A_3 - A_4\right) + O(s^2) \\
\lambda_2(s) &= \frac{1}{2}(A_1 - A_2) + s\left(-\frac{1}{2u_0} - A_3 + A_4\right) + O(s^2) \\
\lambda_3(s) &= -\frac{1}{2}(A_1 - A_2) + s\left(-\frac{1}{2u_0} + A_3 - A_4\right) + O(s^2) \\
\lambda_4(s) &= \frac{1}{2}(A_1 + A_2) + s\left(-\frac{1}{2u_0} + A_3 + A_4\right) + O(s^2)
\end{aligned}
$$

Truncating these expressions to order 0, we can see that two of the roots have negative real part, and the other two have positive real part. Let them be named $\lambda_1^-, \lambda_2^- \lambda_1^+, \lambda_2^+$ respectively.

Moreover, from the polynomial (21), we get the relations

$$
\lambda_1^- + \lambda_2^- + \lambda_1^+ + \lambda_2^+ = \frac{\frac{2}{3}h_0^2 u_0 s}{-\frac{1}{3}h_0^2 u_0^2} = \frac{-2s}{u_0} \tag{22}
$$

$$
\lambda_1^- \lambda_2^- \lambda_1^+ \lambda_2^+ = \frac{s^2}{-\frac{1}{3}h_0^2 u_0^2} = \frac{-3s^2}{h_0^2 u_0^2} \tag{23}
$$

The solutions of (19) in the complementary of $[a, b]$ are

$$
\begin{aligned}
\hat{U}(s, x) &= \overline{U}_1^+ e^{\lambda_1^+(s)x} + \overline{U}_2^+ e^{\lambda_2^+(s)x}, x < a, \\
\hat{U}(s, x) &= \overline{U}_1^- e^{\lambda_1^-(s)x} + \overline{U}_2^- e^{\lambda_2^-(s)x}, x > b
\end{aligned}
$$

from which we can derive the TBCs

$$
\begin{aligned}
\hat{u}(s, a) - (\lambda_1^+(s) + \lambda_2^+(s))\hat{u}_x(s, a) + \lambda_1^+(s)\lambda_2^+(s)\hat{u}_{xx}(s, a) = 0 \\
\hat{u}(s, b) - (\lambda_1^-(s) + \lambda_2^-(s))\hat{u}_x(s, b) + \lambda_1^-(s)\lambda_2^-(s)\hat{u}_{xx}(s, b) = 0
\end{aligned}
$$

respectively for the left and the right boundaries of $[a, b]$. Using the relations (22) and (23) We propose to approximate the TBCs using three constants, c_1, c_2, c_3, such that :

$$\lambda_1^- + \lambda_2^- = c_1$$
$$\lambda_1^- \lambda_2^- = c_2$$
$$\lambda_1^+ + \lambda_2^+ = -2c_3 - c_1$$
$$\lambda_1^+ \lambda_2^+ = -\frac{3c_3^2}{c_2 h_0}$$

giving the IBC operators

$$B_1(U) = U - c_1 U_x + c_2 U_{xx}, \qquad B_2(U) = U + (2c_3 + c_1)U_x - \frac{3c_3^2}{c_2 h_0} U_{xx} \quad (24)$$

applied respectively in the right and the left boundaries.

1.3 Linearization with $u_0 \neq 0$ (only for the dispersive equation)

Considering that we solve the Serre equations with a splitting method, we will derive IBCs for each one of the splitting steps. In the following paragraphs, we will consider the dispersive equation :

$$\begin{cases} h_t = 0 \\ u_t - \frac{1}{3h}\left(h^3\left(u_{xt} + u u_{xx} - (u_x)^2\right)\right)_x = 0 \end{cases} \quad (25)$$

We can work only with the second equation of (25). Linearizing it around $h0$ and u_0, we obtain

$$u_t - \frac{h_0^2}{3}(u_{xxt} + u_0 u_{xxx}) = 0 \quad (26)$$

We will solve this equation in the complementary of $[a, b]$ in order to compute its TBcs. In the Laplace space, (26) reads

$$s\hat{u} - \frac{h_0^2}{3}(s\hat{u}_{xx} + u_0 \hat{u}_{xxx}) = 0 \quad (27)$$

which admits a solution in the general form $\hat{u}(s, x) = \overline{u}(\lambda)e^{\lambda(s)x}$. Replacing in (27), we obtain the characteristic polynomial

$$3s - h_0^2 s\lambda^2 - h0^2 u_0 \lambda^3 = 0 \quad (28)$$

The roots of (28) verifies, for $u_0 > 0$ (??????)

$$Re(\lambda_1) > 0, \qquad Re(\lambda_2) < 0, \qquad Re(\lambda_3) < 0$$

and the relations

$$\lambda_1 + \lambda_2 + \lambda_3 = \frac{h_0^2 s}{-h0^2 u_0} = -\frac{s}{u_0} \tag{29}$$

$$\lambda_1\lambda_2\lambda_3 = \frac{3s}{h0^2 u_0} \tag{30}$$

so the solution of (27) has the form

$$\hat{u}(s,x) = \overline{u}_1(s)e^{\lambda_1(s)x}, x < a,$$
$$\hat{u}(s,x) = \overline{u}_2(s)e^{\lambda_2(s)x} + \overline{u}_3(s)e^{\lambda_3(s)x}, x > b$$

from which we can derive the TBCs

$$\frac{1}{\lambda_1(s)}\hat{u}_x(s,a) - u(s,a) = 0 \qquad \frac{1}{\lambda_1^2(s)}\hat{u}_{xx}(s,a) - u(s,a) = 0$$
$$\hat{u}(s,b) - (\lambda_2(s) + \lambda_3(s))\hat{u}_x(s,b) + \lambda_2(s)\lambda_3(s)\hat{u}_{xx}(s,b) = 0$$

Taking into account the relations (29) and (30), we propose to approximate the TBCs using two constants, c_1 and c_2, such that

$$\lambda_1 = c_1$$
$$\lambda_2 + \lambda_3 = -c_2 - c_1$$
$$\lambda_2\lambda_3 = \frac{3c_2}{h_0^2 c_1}$$

giving the IBC operators

$$B_1(u) = u - \frac{1}{c_1}u_x, \qquad B_2(u) = u - \frac{1}{c_1^2}u_{xx}, \qquad B_3(u) = u + (c_2 + c_1)u_x + \frac{3c_2}{h_0^2 c_1}u_{xx} \tag{31}$$

In the DDM, B_1 and B_2 should be applied as IBC in the left boundary of right domain, and B_3 as IBC in the right boundary of left domain.

1.3.1 Tests with the approximate TBCs

In order to study this proposed approximations and find the coefficients that provide the best TBCs (and possibly some dependence on the linearization parameter u_0), we solved the linearized equation (26), with $0 \leq t \leq 10$ for different pairs $(c_1, c_2) \in [-10, 10]^2$ and computed for each case the error

$$e(c_1, c_2) = ||u^{c1,c2} - u^{ref}|| = \sqrt{\Delta t \Delta x \sum_{n=0}^{T} \sum_{i=0}^{N} (u_i^{n,c_1,c_2} - u_i^{n,ref})^2}$$

Two problems were solved, the first one with a wave moving to the right, and the second one with a wave moving to both directions. The initial solution u_0 in both cases is the solitary cnoidal solution, and the movement is forced by the respective h solution of the Serre equations (h is not modified in the dispersive part of the system).

For both tests, we obtained the following conclusions:

- For $c_2 \neq 0$, the error $e(c_1, c_2)$ is independent of c_2.
- $c_2 = 0$ provides much better results than $c_2 = 0$, especially in the first problem.
- The best results are provided by $c_1 \in [-1, 1]$

With $c_2 = 0$, the TBCs are written as

$$\frac{1}{c_1} u_x(t, a) - u(t, a) = 0 \qquad \frac{1}{c_1^2} u_{xx}(t, a) - u(t, a) = 0$$
$$u(t, b) + c_1 u_x(t, b) = 0$$

References

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