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The interaction of solitary-wave solutions of a model equation for long waves in dispersive media is examined numerically. It is found that the waves do not emerge from the interaction unscathed. Instead, two new solitary waves, having slightly different amplitudes from the original waves, together with a small dispersive tail are generated as a result of the interaction.

I. INTRODUCTION

The Korteweg–de Vries equation

$$u_t + u_x + (u^2)_x + u_{xxx} = 0 \tag{1}$$

was originally derived as a model for the unidirectional propagation of certain kinds of water waves. In this context, $u = u(t, x)$ represents the wave amplitude and t and x are, respectively, proportional to time and distance, with x increasing in the direction of propagation of the waves. The equation admits a family of solitary-wave solutions having the property that the result of the nonlinear interaction of a pair of unequal solitary waves leaves the waves unaltered, except for a phase shift. This so-called soliton property was first observed in numerical studies made by Zabusky and Kruskal[1] and the proof of it was one of the triumphs of the inverse-scattering method for solving partial-differential equations (see Miura[2] for a summary of results). For water waves, an alternative model has been proposed by Peregrine[3] and by Benjamin *et al.*,[4] namely

$$u_t + u_x + (u^2)_x - u_{xxt} = 0 . \tag{2}$$

Equation (2) has solitary-wave solutions, similar to those for (1), of the form

$$u(t,x) = A \operatorname{sech}^2\left\{\left(\frac{A}{4A+6}\right)^{1/2}\left[x - \left(1 + \frac{2}{3}A\right)t\right]\right\}, \tag{3}$$

where $A > 0$ is a parameter specifying the amplitude (and speed) of the wave.

Since (1) and (2) purport to model the same physical situation, it might be expected that the solutions to the two equations display similar properties. Indeed, it has been shown by the authors[5] that the solutions of Eqs. (1) and (2), corresponding to the same small-amplitude, long-wavelength, initial data, remain within order a^2 of each other on a time scale of order $a^{-3/2}$. Here, a is the maximum amplitude of the initial profile and, in keeping with the assumptions underlying the derivation of these models, the wavelength is of order $a^{-1/2}$. Thus, in the regime where the equations are expected to model surface water waves, this theorem shows that either (1) or (2) can be used, with similar accuracy, to describe the wave field.

Although these models usually arise when trying to describe a physical situation in which the wave amplitudes are small, the soliton property for (1) is not a small-amplitude phenomenon and is of interest in its own right. There is, however, no analog of inverse-scattering theory for (2). Indeed, recent results of Olver[6] and McLeod and Olver[7] suggest that no such theory can exist for (2), at least in the way we currently understand the inverse-scattering procedure. Consequently, there are no techniques to analytically study the interaction of solitary waves for (2), but such interactions have been investigated numerically by Eilbeck and McGuire[8] and by Abdulloev *et al.*[9] The former study suggested that the solitary waves emerged with a phase shift, but were unaltered in shape, to within the accuracy of their computations. On the other hand, the latter study reports that a very small "rarefaction" wave is also produced as a result of the interaction.

The results of our own numerical computations, broadly speaking, confirm those of Abdulloev *et al.*[9] in showing that a third wave emerges from the interaction of two solitary waves for Eq. (2). Our purpose here is to describe a particular interaction and show the details of the evolution of the "dispersive tail" that appears behind, and eventually separates from, the smaller solitary wave. First, we give a brief outline of the numerical scheme used to integrate Eq. (2).

II. THE NUMERICAL SCHEME

As shown in Benjamin *et al.*[4] the solution to (2) satisfies

$$u_t = K * (u + u^2) , \tag{4}$$

where $K(x) = \frac{1}{2}\operatorname{sgn}(x)\exp(-|x|)$. The numerical scheme used to integrate (2) consists of discretizing Eq. (4). The spatial discretization approximates the convolution integral in (4) using the trapezoidal rule with derivative end correction,[10] the derivative being computed via

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an appropriate difference quotient. (The derivative correction is made only at the point corresponding to the discontinuity in K.) The time discretization is simply a fourth-order Runge–Kutta method. The result is a scheme for which one can prove fourth-order accuracy in both space and time. Utilizing the exact solitary-wave solution of the equation, we have checked this property in a detailed convergence study. Although there are no stability limitations on the convergence of the scheme, the choice $\Delta t = \Delta x$ appears to be close to optimal in terms of accuracy achieved for a given amount of work. Because of special properties of K, the work estimate per time step for the scheme is reduced to $O(n)$, where n is the number of spatial points.

Some examples of the accuracy achieved are as follows: Using the mesh $\Delta t = \Delta x = 0.025$, the maximum error in u that arose when computing the evolution of a single solitary wave was less than 0.96×10^{-3} when $A = 6.0$ and was less than 0.21×10^{-6} when $A = 1.5$, at time $t = 31.35$. When the waveform had evolved for a time 62.70, the maximum errors were less than 0.34×10^{-2} when $A = 6.0$ and less than 0.47×10^{-6} when $A = 1.5$. Using a fourth-order interpolation of the data points of the computed waveform, we also determined the amplitude and the location of the crest of the numerical solution, enabling us to identify separately the errors in the phase and the height of the wave. Consequently, the "shape" of the computed waveform can be compared with that of a theoretical solitary wave (3) whose crest has the same height and location as that of the numerical solution. The computed solutions had slightly smaller amplitudes and a small phase lag when compared with the theoretical solution. So, for example, at $t = 31.35$ and with $\Delta t = \Delta x = 0.025$, the height errors and phase lags were, respectively, less than 0.37×10^{-4} and 0.46×10^{-3}, when $A = 6.0$, and less than 0.30×10^{-7} and 0.48×10^{-6} when $A = 1.5$; the maximum difference between the computed waveform and a solitary wave of the same amplitude (the "shape error") was less than 0.59×10^{-5} for $A = 6.0$, and was less than 0.29×10^{-7} for $A = 1.5$. At $t = 62.70$ the height and phase errors, respectively, had degraded to 0.70×10^{-4} and 0.16×10^{-2} for $A = 6.0$, and to 0.37×10^{-7} and 0.11×10^{-5} for $A = 1.5$. For both waves the "shape error" was nearly independent of time, after an initial settling down period, suggesting that, although the computed wave underwent a continual attrition, its shape remained very close to that of a theoretical solitary wave [Eq. (3)] of suitably chosen amplitude.

More complete details of the numerical scheme, together with rigorous error bounds, are to be reported in a study by the present authors,[5] in which the predictions of Eq. (2) are compared with the results of some laboratory experiments concerning the unidirectional propagation of surface waves of large wavelength in a uniform channel.

III. NUMERICAL RESULTS

In the experiments to be described, the initial wave profiles consisted of a pair of solitary waves, of the form (3), disposed so that their crests were well separated, with the larger wave behind the smaller one. The separation of the crests was chosen so that the overlap of the two waveforms was less than 10^{-9}. The width of the data field, which was continually adjusted throughout the experiment, was initially chosen so that the amplitude in the tails of the waves was less than 10^{-9}.

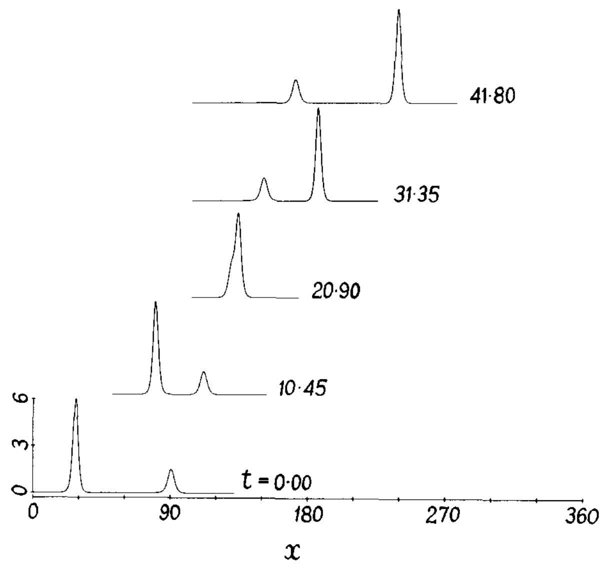

FIG. 1. The interaction of solitary waves of amplitudes 1.5 and 6.0, from time $t = 0.0$ to $t = 41.80$.

The evolution of such a profile, with solitary-wave amplitudes of 6.0 and 1.5, is shown in Fig. 1. To a first approximation, the waves appear to have emerged from their interaction unaltered, except for a phase shift. However, on closer inspection, it was apparent that, following the collision, a tail of very small amplitude had emerged from the trailing face of the smaller solitary wave. This is indicated in Fig. 2, where the amplitude scale of Fig. 1 has been magnified by a factor of 100. (Note that the overshoot of the solitary waves has been truncated in this figure.) As shown in Fig. 3, this tail eventually separated from the smaller soli-

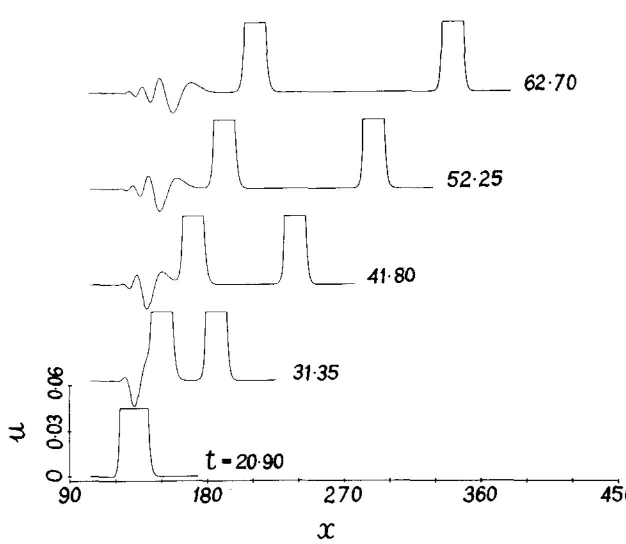

FIG. 2. The interaction of solitary waves of amplitudes 1.5 and 6.0, with the amplitude scale enlarged by a factor of 100, from t=20.90 to t=62.70.

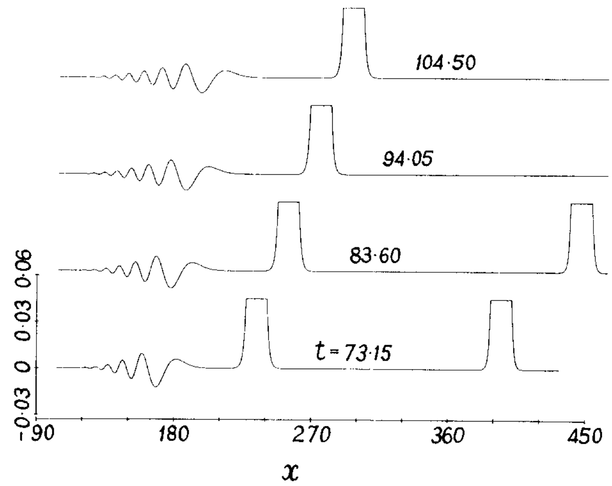

FIG. 3. The interaction of solitary waves of amplitudes 1.5 and 6.0, with the amplitude scale enlarged by a factor of 100, from $t=73.15$ to $t=104.50$.

tary wave, with its maximum amplitude decreasing and the number of oscillations increasing, as a function of time, suggesting it was of a dispersive nature. This conclusion was also supported by the property that the speed c of the leading crest of the tail was found to be slightly less than the "long-wave" speed (i.e., $c<1$). These results were obtained using the mesh $\Delta t=\Delta x=0.025$.

The oscillatory tail shown in Figs. 2 and 3 reached a maximum amplitude, in the negative excursion nearest the smaller solitary wave, of 0.0167 at the time $t=33.375$. The parameters for this experiment were chosen to allow a direct comparison with the work of Abdulloev *et al.*,[9] who reported a "rarefaction-wave" amplitude of 0.17×10^{-2}. However, there is presumably a typographical error in the exponent, since they later refer to the amplitude of the rarefaction wave, relative to the large solitary wave, as being 0.3%, implying an intended amplitude of 0.17×10^{-1}, in agreement with our results.

To verify that the discrete solution has converged to the correct values in the region of the oscillatory tail, we compared the wave profiles computed using different values of Δt and Δx with those determined when $\Delta t=\Delta x=0.0125$. At $t=41.80$, the greatest difference in the region of the oscillatory tail was less than 0.56×10^{-4} with $\Delta t=\Delta x=0.05$ and less than 0.30×10^{-5} with $\Delta t=\Delta x=0.025$, in accordance with the fourth-order convergence rate of the scheme.

Just as for the Korteweg–de Vries equation, the interaction affected the phases of the two solitary waves in the present case. Thus, with $\Delta t=\Delta x=0.025$, the phase of the larger solitary wave was found to have been advanced by 1.76 at $t=31.35$ and the phase of the smaller solitary wave was retarded by 2.85 at time $t=62.7$. Note that these phase adjustments, relative to the positions the original solitary waves would have had in the absence of an interaction, continued to change very slowly, owing to the fact that the amplitude and speed of the waves had been slightly modified. Therefore, we have given the phase adjustments as soon as the solitary waves had effectively established their "equilibrium" configurations following the interaction: as shown in Fig. 2, the larger solitary wave had separated from the interaction zone much sooner than the smaller solitary wave had separated from the tail. To determine, at these times, how closely the waves conformed to theoretical solitary waves, we compared their shapes (in the manner already indicated) and speeds with waves of the form (3) having the same amplitude as the computed waveforms. At the time $t=31.35$ the leading wave differed in shape and speed from a solitary wave (3) of the same amplitude by less than 0.9×10^{-5} and 0.1×10^{-5}, respectively; the second wave, at $t=62.70$, differed in shape by less than 0.4×10^{-5} and in speed by less than 0.1×10^{-5} when compared with a wave of the form (3). Between these two leading waves and, after separation, between the smaller solitary wave and the dispersive tail, the wave amplitude was less than 0.2×10^{-5}.

These comparisons suggest that, following the interaction, two solitary waves had emerged, so that it is interesting to examine the amplitude of these new waves. At $t=31.35$ the leading wave had an amplitude of approximately 0.17×10^{-4} greater than that of the original solitary wave. This amplitude change was determined with the mesh $\Delta t=\Delta x=0.0125$ and, for comparison, the numerical scheme under the same conditions would have diminished the height of a single solitary wave of amplitude 6.0 by only 0.12×10^{-5}. So, although we cannot be sure from the present results, it would appear that the amplitude of the larger solitary wave is increased by the interaction. By contrast, the smaller wave, at $t=62.70$, was found to have been reduced from its initial height by approximately 0.24×10^{-3}. This amplitude change was determined with the mesh $\Delta t=\Delta x=0.025$, but computations with the finer mesh, carried out to $t=41.80$, gave a height for this

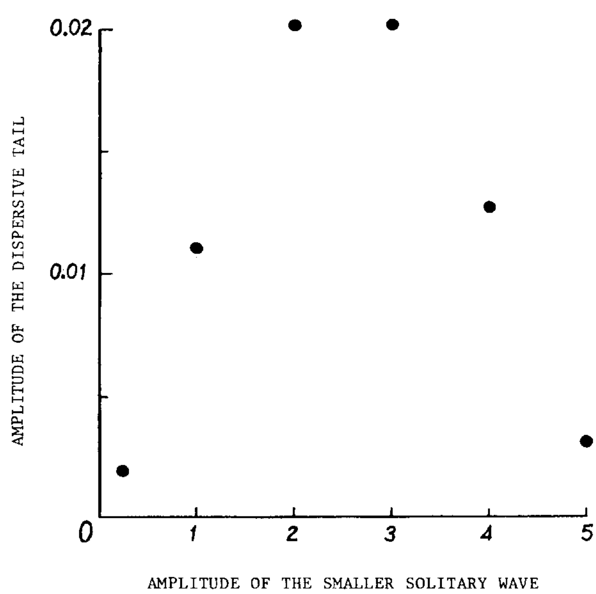

FIG. 4. The amplitude of the dispersive tail as a function of the amplitude of the smaller solitary wave, with the amplitude of the larger solitary wave fixed at 6.0.

wave differing by less than 0.2×10^{-5} from that obtained with the coarser mesh. A single solitary wave of amplitude 1.5, when propagated for a time $t = 62.70$ with the mesh $\Delta t = \Delta x = 0.025$, decreased in height by less than 0.4×10^{-7}. Thus, it would appear from these results that the majority of the energy in the dispersive tail was generated at the expense of the smaller solitary wave.

Since the amplitude of the dispersive tail relative to that of the larger solitary wave was of the order of 0.1%, verifying the shape of the dispersive tail to just three digits required a total of six meaningful digits in the experiment. Owing to the large number of floating-point operations involved in the experiment, allowance for round-off error required the retention of several more digits. The computations described here were made on a CDC7600, which, having approximately 14 digits of accuracy, was capable of meeting the said requirements.

Experiments with other values of the solitary-wave amplitudes have also been carried out and in all cases the interaction of two solitary waves produced the same kind of results as those shown in Figs. 1–3. A series of experiments was performed involving the interaction of a solitary wave of fixed amplitude ($A = 6.0$) with various smaller solitary waves. It was found that the number of discernible oscillations in the dispersive tail, just after it separated from the smaller solitary wave, decreased and the wavelengths of the individual oscillations increased, as the amplitude of the smaller solitary wave was increased. The graph in Fig. 4 shows the maximum value of u found in the dispersive tail as a function of the amplitude of the smaller solitary wave.

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