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Initial–Boundary Value Problems for the Korteweg–de Vries Equation

T. R. MARCHANT AND N. F. SMYTH

Department of Mathematics, University of Wollongong, PO Box 1144, Wollongong, NSW 2500, Australia

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Exact and approximate solutions of the initial–boundary value problem for the Korteweg–de Vries equation on the semi-infinite line are found. These solutions are found for both constant and time-dependent boundary values. The form of the solution is found to depend markedly on the specific boundary and initial value. In particular, multiple solutions and nonsteady solutions are possible. The analytical solutions are compared with numerical solutions of the Korteweg–de Vries equation and are found to be in good agreement.

1. Introduction

THE Korteweg–de Vries equation forms one of a group of equations governing nonlinear dispersive wave motion which are solvable on the infinite line by the technique of inverse scattering (see e.g. Newell, 1985). In particular, inverse scattering allows the solution of the initial value problem for the Korteweg–de Vries equation,

$$\left.\begin{array}{ll} u_t + 6uu_x + u_{xxx} = 0 & (-\infty < x < \infty,\ t > 0), \\ u(x, 0) = u_0(x) & (-\infty < x < \infty), \end{array}\right\} \tag{1.1}$$

when $u_0(x)$ is suitably bounded as $x \to \pm\infty$. The technique of inverse scattering has recently been applied to the initial–boundary value problem for the Korteweg–de Vries equation on the semi-infinite line (Fokas & Ablowitz, 1989),

$$\left.\begin{array}{ll} u_t + 6uu_x + u_{xxx} = 0 & (0 < x < \infty,\ t > 0), \\ u(x, 0) = u_i & (0 < x < \infty), \\ u(0, t) = u_b(t) & (t > 0). \end{array}\right\} \tag{1.2}$$

The solution of (1.2) was reduced to the solution of a nonlinear singular integrodifferential equation for the scattering data, this equation being difficult to solve in general. To enable a simple inverse scattering solution to be obtained, Camassa & Wu (1989) used an alternative approach which required values of u_x and u_{xx} at $x = 0$. However, these values are not known *a priori*. These initial–boundary value problems form a physically important class of problems. For example, the problem (1.2) models the generation of waves in a shallow channel by a wave-making device.

An alternative approach to finding solutions of (1.2) is to use approximate methods of solution of the Korteweg–de Vries equation. In the present work, modulation theory as applied to the Korteweg–de Vries equation by Whitham

(1965, 1974) will be used. Smyth (1987) used modulation theory to construct a solution of (1.2) in the special case when $u_i = 0$ and $u_b > 0$ and constant. This modulation theory solution was a partial undular bore. A partial undular bore is a modulated cnoidal wave with waves of modulus $m = 1$ at its leading edge, so that the leading edge is basically solitons, and waves of modulus $m_0 > 0$ at $x = 0$. The boundary condition in (1.2) is satisfied in the mean. This solution was found to be in good agreement with the numerical solutions of Chu *et al.* (1983). An important feature of the solution was that a steady state did not form, but that waves were continuously generated at $x = 0$, these waves propagating away from $x = 0$.

In the present work, solutions of (1.2) for $u_i \neq 0$ and $u_b = u_b(t)$, with u_b both positive and negative, will be found. The form of the solution will be found to depend markedly on u_i and u_b (see Fig. 1). In particular, depending on the values of u_i and u_b, the solution can either approach a steady state or remain unsteady. Also, for certain values of u_i and u_b both steady and unsteady solutions are theoretically possible. Numerical evidence shows that, in these cases, the steady-state solution is the one attained. An extension of the modulation theory solution of Smyth (1987) is used to construct the solution of (1.2) for one of the cases when the solution remains unsteady. Both the steady and unsteady solutions are found to be in good agreement with numerical solutions of (1.2). A simple approximation to the unsteady solution as a train of solitons is also made. This is found to be a reasonable approximation for small $u_b - u_i$ which only breaks down as the soliton amplitude increases owing to the neglect of u_x and u_{xx} at $x = 0$.

2. Steady boundary conditions

In this section, we consider the initial–boundary value problem for the Korteweg–de Vries equation (1.2) with u_b and u_i both constant. Previous consideration of (1.2) has centred on the special case of $u_b > 0$ (both constant and time dependent) and $u_i = 0$. Chu *et al.* (1983) produced numerical solutions for this special case which showed the development of an unsteady wavetrain in $x > 0$ which consisted of soliton-like waves, while Camassa & Wu (1989) used inverse scattering to obtain an approximate solution. Smyth (1987) used modulation theory for the Korteweg–de Vries equation to develop a solution for constant $u_b > 0$ and $u_i = 0$, this solution being a partial undular bore with solitons at the leading edge and cnoidal waves of modulus m_0 (>0) at the trailing edge at $x = 0$.

In this work, we place no restriction on the values of u_b and u_i and find a rich variety of solution types occurring in the u_b versus u_i plane (see Fig. 1). Each type of solution will be explained in turn and compared with the numerical solution of (1.2). The finite-difference scheme of Vliengenthart (1971) is used to calculate the numerical solutions.

2.1 *Modulation Theory*

The Korteweg–de Vries equation on the infinite line $-\infty < x < \infty$ has the well-known periodic wave solution

$$u = \beta + 2a\left[m^{-1} - 1 - \frac{\mathrm{E}(m)}{m\mathrm{K}(m)} + \mathrm{cn}^2\left(\frac{\mathrm{K}(m)\theta}{\pi}\right)\right], \tag{2.1}$$

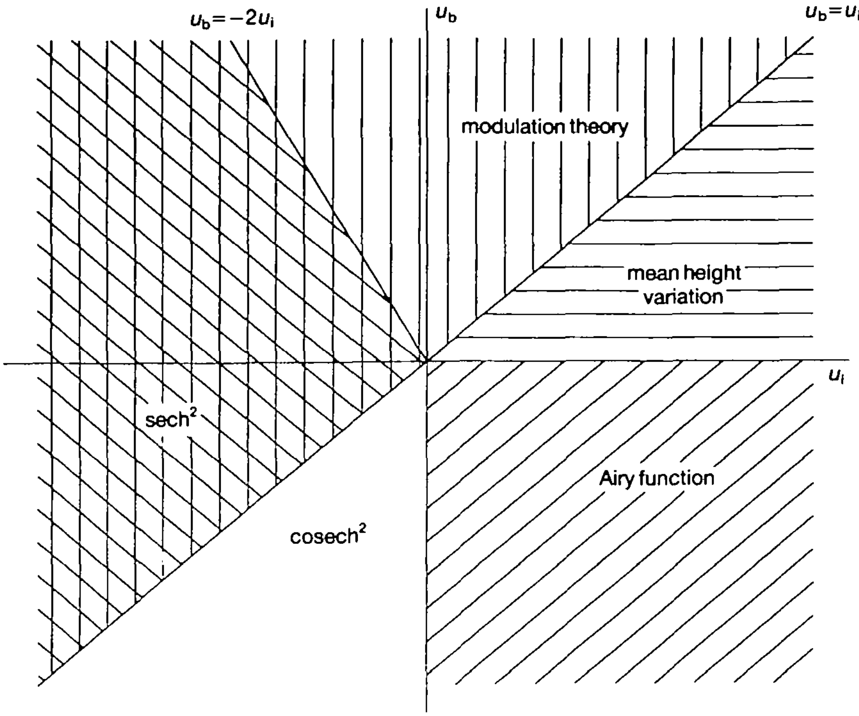

FIG. 1. Division of the u_b versus u_i plane into the different solution regimes.

where

$$\theta = kx - \omega t, \qquad \omega = 6\beta k + 4ak\left(2m^{-1} - 1 - \frac{3\mathrm{E}(m)}{m\mathrm{K}(m)}\right), \tag{2.2}$$

the so-called cnoidal wave solution. The modulus squared m lies in the range $0 \leq m \leq 1$. In the limit $m \to 1$, the cnoidal wave becomes the soliton solution

$$u = \beta + 2a\,\mathrm{sech}^2\,[a^{\frac{1}{2}}(x - Ut)], \tag{2.3}$$

where

$$U = 6\beta + 4a, \tag{2.4}$$

and, in the limit $m \to 0$, the cnoidal wave becomes the linear wave solution

$$u = \beta + a\cos\theta, \tag{2.5}$$

$$\omega = 6\beta k - ak^3. \tag{2.6}$$

In the linear limit $m \to 0$, we have $a \to 0$ with $a/m \to \frac{1}{2}k^2$. The cnoidal wave solution (2.1) is a very special solution of the Korteweg–de Vries equation. In general, the wavenumber, frequency, amplitude, and mean height of a wavetrain can vary. Modulation theory has been developed (Whitham, 1974) to determine equations for the wavenumber, frequency, amplitude, and mean height of a slowly varying wavetrain. Smyth (1987) used this modulation theory to obtain an approximate solution to (1.2) for the special case constant $u_b > 0$ and $u_i = 0$. Here we extend this modulation theory solution to its full range of validity, $u_b > 0$ and $u_b > u_i$. The modulation equations for the Korteweg–de Vries equation are (Whitham, 1965, 1974)

$$\left.\begin{aligned}
&\mathrm{P}:\quad 2\beta + 4a\frac{\mathrm{K}(m)-\mathrm{E}(m)}{m\mathrm{K}(m)} - 2a - \frac{2a}{m} = \text{const} \quad \text{on}\ \frac{\mathrm{d}x}{\mathrm{d}t} = U - \frac{4a\mathrm{K}(m)}{\mathrm{K}(m)-\mathrm{E}(m)},\\
&\mathrm{Q}:\quad 2\beta + 4a\frac{\mathrm{K}(m)-\mathrm{E}(m)}{m\mathrm{K}(m)} - \frac{2a}{m} = \text{const} \quad \text{on}\ \frac{\mathrm{d}x}{\mathrm{d}t} = U - \frac{4a(1-m)\mathrm{K}(m)}{\mathrm{E}(m)-(1-m)\mathrm{K}(m)},\\
&\mathrm{R}:\quad 2\beta + 4a\frac{\mathrm{K}(m)-\mathrm{E}(m)}{m\mathrm{K}(m)} - 2a = \text{const} \quad \text{on}\ \frac{\mathrm{d}x}{\mathrm{d}t} = U + 4a(1-m)\frac{\mathrm{K}(m)}{m\mathrm{E}(m)},
\end{aligned}\right\} \qquad (2.7)$$

where β is the mean height, a is the amplitude, and $\mathrm{K}(m)$ and $\mathrm{E}(m)$ are complete elliptic integrals of the first and second kinds of modulus squared (parameter) m respectively. The above modulation equations describe the evolution of a cnoidal wave whose amplitude a, mean height β, and wavenumber k are slowly varying.

The phase velocity U and wavenumber k are given by

$$U = 6\beta + 4a\left(\frac{2}{m} - 1 - \frac{3\mathrm{E}(m)}{m\mathrm{K}(\mathrm{m})}\right), \qquad k = \frac{\pi}{\mathrm{K}(m)}\left(\frac{a}{m}\right)^{\frac{1}{2}}. \qquad (2.8)$$

The modulation equations (2.7) have a simple wave solution, first found by Gurevich & Pitaevskii (1974) and Fornberg & Whitham (1978), which is an expansive fan on the characteristic Q. The Riemann invariants associated with the other characteristics, R and P, remain constant throughout the simple wave, while the properties of the simple wave vary throughout the fan of Q characteristics. The simple wave solution is

$$\left.\begin{aligned}
&\beta = (B-A)m + 2A - B + 2(B-A)\frac{\mathrm{E}(m)}{\mathrm{K}(m)},\\
&a = (B-A)m,\\
&k = \frac{\pi}{\mathrm{K}(m)}|B-A|^{\frac{1}{2}},\\
\text{on}\quad &\frac{x}{t} = 2(B-A)m + 4A + 2B - \frac{4(B-A)m(1-m)\mathrm{K}(m)}{\mathrm{E}(m)-(1-m)\mathrm{K}(m)}\\
&\qquad (0 \leqslant m \leqslant 1,\ \ 12A - 6B \leqslant x/t \leqslant 4B + 2A),
\end{aligned}\right\} \qquad (2.9)$$

where β is the mean height, a is the amplitude of the wavetrain, and A and B are constants determined by the boundary conditions. This solution represents a modulated cnoidal wavetrain (often referred to as an undular bore because it links together two different mean levels), where at the front of the bore (where $m = 1$) $\beta = A$ and solitons of amplitude $2(B - A)$ occur, while at the rear of a full bore (where $m = 0$) $\beta = B$ and sinusoidal waves of zero amplitude occur. The characteristics P propagate from the front of the bore and the Riemann invariant on these characteristics is $2A$, while the characteristics R propagate from the rear of the bore and on these characteristics the Riemann invariant is $2B$. From (2.9), we see that we require $B > A$ for the solution to be valid, which means physically that the mean level increases on the passage of a bore.

This simple wave solution is used to construct a solution of (1.2). From the boundary and initial conditions, we see that the appropriate boundary conditions for the simple wave solution (2.9) are

$$\beta = u_i \quad \text{at} \quad m = 1, \qquad \beta = u_b \quad \text{at} \quad m = m_0 \tag{2.10}$$

where $x/t(m = m_0) = 0$ or $x/t(m = m_0) \geqslant 0$.

When the slope of the characteristic at the trailing edge of the bore (when $m = 0$) is positive, a full undular bore occurs which propagates forwards. In this case, $m_0 = 0$, and, since $\beta = A$ at the leading edge and $\beta = B$ at the trailing edge of a full bore, we set $A = u_i$ and $B = u_b$ to satisfy the initial and boundary conditions. Alternatively, a partial undular bore occurs (with modulus $m_0 \leqslant m \leqslant 1$) if some of the characteristics in the simple wave have negative slope, as only forward propagating waves occur for (1.2). The solution for the full undular bore, which occurs if

$$0 \leqslant \tfrac{1}{2}u_b < u_i < u_b, \tag{2.11}$$

is

$$\left.\begin{aligned}
&\beta = (u_b - u_i)m + 2u_i - u_b + 2(u_b - u_i)\frac{\mathrm{E}(m)}{\mathrm{K}(m)},\\
&a = (u_b - u_i)m,\\
&k = \frac{\pi}{\mathrm{K}(m)}(u_b - u_i)^{\frac{1}{2}},\\
\text{on}\quad &\frac{x}{t} = 2(u_b - u_i)m + 4u_i + 2u_b - \frac{4(u_b - u_i)m(1-m)\mathrm{K}(m)}{\mathrm{E}(m) - (1-m)\mathrm{K}(m)} \quad (0 \leqslant m \leqslant 1).
\end{aligned}\right\} \tag{2.12}$$

From (2.12), we see that we require $u_b > u_i$ for a physically valid solution. Taking the $m = 0$ limit for the characteristic slope x/t in (2.12), it can be found that we require $2u_i > u_b$ for this trailing characteristic to have positive slope, so that a full undular bore results. Taking the opposite limit $m = 1$ in the characteristic slope in (2.12), it can be found that we require $2u_b + u_i > 0$ for the slope of this leading characteristic to be positive. Since we require $2u_i - u_b > 0$ and $2u_b + u_i > 0$, we see that for (2.12) to be valid we have the restriction $u_i > 0$. It was noted above that $u_b > u_i$ for the solution (2.12) to be physically valid. Combining these inequalities, we have that the full undular bore solution (2.12) is valid in the region $0 < \frac{1}{2}u_b < u_i < u_b$ of the (u_i, u_b) plane.

Let us now consider the case of a partial undular bore. Since $\beta = A$ at the leading edge of a bore, whether full or partial, we set $A = u_i$ in (2.9) to satisfy the initial condition. The solution for the partial undular bore, which occurs if

$$u_i \leqslant \tfrac{1}{2}u_b \quad \text{for } u_b > 0 \qquad \text{and} \qquad u_i < u_b \quad \text{for } u_b < 0 \tag{2.13}$$

is then

$$\left.\begin{aligned}
\beta &= (B-u_i)+2u_i-B+2(B-u_i)\frac{\mathrm{E}(m)}{\mathrm{K}(m)},\\
a &= (B-u_i)m,\\
k &= \frac{\pi}{\mathrm{K}(m)}(B-u_i)^{\frac{1}{2}},\\
\text{on}\quad \frac{x}{t} &= 2(B-u_i)m+4u_i+2B-\frac{4(B-u_i)m(1-m)\mathrm{K}(m)}{\mathrm{E}(m)-(1-m)\mathrm{K}(m)}\quad (m_0\leqslant m\leqslant 1),
\end{aligned}\right\}\tag{2.14}$$

where B and m_0 are the solutions of

$$u_b-u_i=(B-u_i)\left(m_0-1+\frac{2\mathrm{E}(m_0)}{\mathrm{K}(m_0)}\right),\tag{2.15a}$$

$$-3u_i=(B-u_i)\left(m_0+1-\frac{2m_0(1-m_0)\mathrm{K}(m_0)}{\mathrm{E}(m_0)-(1-m_0)\mathrm{K}(m_0)}\right).\tag{2.15b}$$

Equation (2.15a) results in $\beta=u_b$ at $x=0$, while (2.15b) causes the slope of the trailing characteristic of the expansion fan to be zero, and hence for this trailing characteristic to lie on the boundary $x=0$. The set of algebraic equations (2.15) is solved numerically for given u_b and u_i. For a full undular bore, $\frac{1}{2}u_b<u_i$, so that the characteristic slope x/t in the $m=0$ limit propagates forward. For a partial undular bore, we require that the characteristic slope in this limit propagates backwards; hence $\frac{1}{2}u_b>u_i$ is required. In addition, for a physically valid solution $B>u_i$, which combined with (2.15a) gives $u_b>u_i$ (because the term in large brackets in (2.15a) lies between 0 and 1). These are the regions of validity in the u_b versus u_i plane as given in (2.13). If $m_0\to 0$, then (2.15) gives $u_b\to B$ and $u_i\to\frac{1}{2}u_b$, which is just the full undular bore described above. Smyth (1987) considered the special case $u_i=0$, for which $m_0=0{\cdot}64$ and $B=1{\cdot}09u_i$. This represents a partial undular bore with solitons of amplitude $2{\cdot}09u_i$ at the leading edge and cnoidal waves of modulus squared $m=0{\cdot}64$ and amplitude $1{\cdot}33u_i$ at the trailing edge $x=0$.

As $u_i\to-\infty$, for fixed u_b, $m_0\to 1$ and $B\to\infty$. Hence this limit represents a train of solitons of infinite amplitude. As $m_0\to 1$, (2.15) gives the solution $u_b\to u_i$ and $B\to-\frac{1}{2}u_i$. Hence this limit represents a train of solitons with finite amplitude ($a=-\frac{3}{2}u_i$ and $u_i<0$). The two special cases involving trains of solitons do not, however, occur in practice (see Section 2.5 for details of the physically realized solution), as the lead characteristic of the undular bore has $x/t\to 0$ (in the limit $m=1$). Hence the generation time of the soliton train becomes very large and in practice is not generated at all (see Fig. 1 for details of this region).

Figure 2 shows the numerical solution of (1.2) for $u_b=0{\cdot}2$ and $u_i=0{\cdot}1$ at $t=60$. Also shown is the envelope $\beta+a$, which is the prediction of the modulation theory solution given in (2.12). There is an excellent comparison except near the lead soliton, where a slight difference exists.

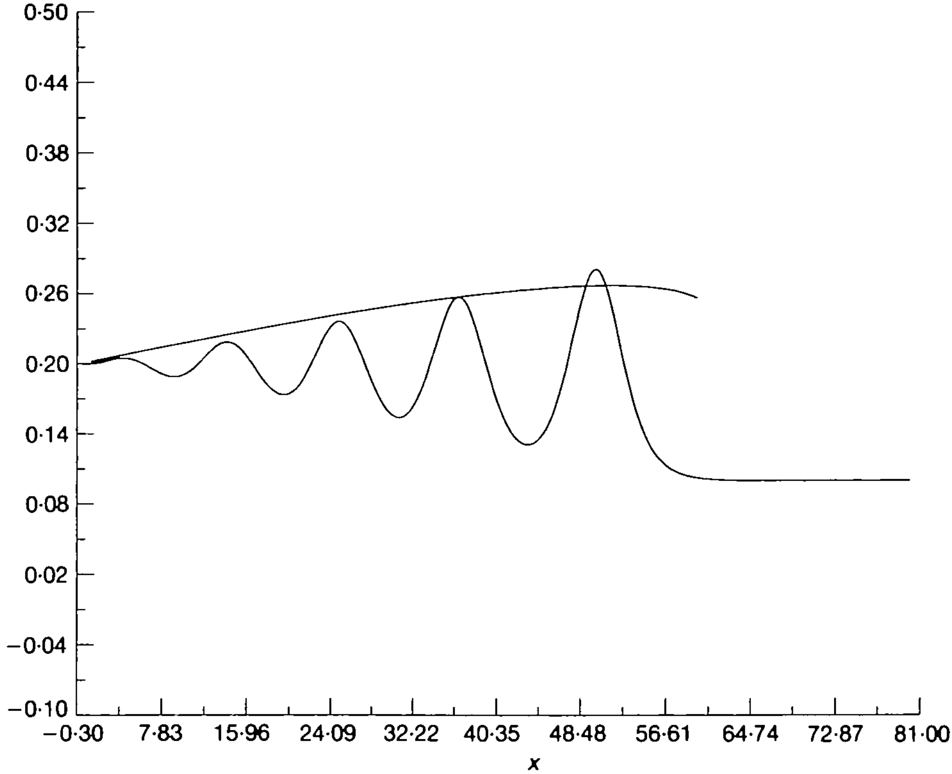

FIG. 2. Numerical solution of (1.2) for $u_b = 0{\cdot}2$ and $u_i = 0{\cdot}1$ after $t = 60$. Also shown is the envelope due to modulation theory.

2.2 *Train of Solitons*

It was shown in Section 2.1 that the modulation theory solution of (1.2) for $u_b > 0$ and $u_b > u_i$ consists of a modulated cnoidal wave with modulus m varying between m_0 and 1. Now, for $u_i = 0$, we have $m_0 = 0{\cdot}64$, so as a simple approximation, we may take the wavetrain to be a train of solitons, for which $m_0 = 1$. A similar approximation was used by Grimshaw & Smyth (1986) to analyse the wavetrain generated by the resonant flow of a fluid over topography. Let us therefore assume that the solution of (1.2) at time t consists of N uniform solitons of amplitude a and spacing h.

The mass and energy conservation equations for the Korteweg–de Vries equation are

$$\frac{\partial}{\partial t}(u - u_i) + \frac{\partial}{\partial x}(3u^2 + u_{xx}) = 0, \tag{2.16}$$

$$\frac{\partial}{\partial t}(\tfrac{1}{2}u^2 - \tfrac{1}{2}u_i^2) + \frac{\partial}{\partial x}(2u^3 + uu_{xx} - \tfrac{1}{2}u_x^2) = 0, \tag{2.17}$$

respectively. Integrating these equations gives

$$\int_0^\infty (u - u_i)\,\mathrm{d}x = [3u_b^2 + u_{xx}(0, t) - 3u_i^2]t, \tag{2.18}$$

$$\tfrac{1}{2}\int_0^\infty (u^2 - u_i^2)\,\mathrm{d}x = [2u_b^3 + u_b u_{xx}(0, t) - \tfrac{1}{2}u_x^2(0, t) - 2u_i^3]t, \tag{2.19}$$

on using the boundary and initial conditions in (1.2). The background level u_i has been added to the time derivatives in (2.16) and (2.17) to ensure that the integrals in (2.18) and (2.19) converge, since $u \to u_i$ as $x \to \infty$. We are assuming that the solution of (1.2) consists of N uniform solitons. Now the soliton solution of the Korteweg–de Vries equation is

$$u = u_i + a\,\mathrm{sech}^2\,\{(\tfrac{1}{2}a)^{\frac{1}{2}}[x - (2a + 6u_i)t]\}. \tag{2.20}$$

We thus see that the integrals in (2.18) and (2.19) are

$$\int_0^\infty (u - u_i)\,\mathrm{d}x = 2\sqrt{2}Na^{\frac{1}{2}},$$

$$\tfrac{1}{2}\int_0^\infty (u^2 - u_i^2)\,\mathrm{d}x = N(\tfrac{2}{3}\sqrt{2}a^{\frac{3}{2}} + 2\sqrt{2}u_i a^{\frac{1}{2}}). \tag{2.21}$$

Equations (2.18) and (2.19) involve the values of u_x and u_{xx} at $x = 0$, which are not known *a priori*, but are determined as part of the solution. To use (2.18) and (2.19) to determine the amplitude a of the solitons, estimates of $u_x(0, t)$ and $u_{xx}(0, t)$ need to be made. We shall assume here that $u_x(0, t) = u_{xx}(0, t) = 0$. It is expected that this will be a reasonable assumption for $u_b - u_i$ not too large. The same approximation was made by Camassa & Wu (1989) so that inverse scattering as applied to (1.2) yielded solutions in a simple manner. Using this assumption, we find from (2.18), (2.19), and (2.21) that

$$a = \frac{(2u_b + u_i)(u_b - u_i)}{u_b + u_i}. \tag{2.22}$$

The speed of the solitons, from (2.20), is

$$U = 2a + 6u_i. \tag{2.23}$$

Hence the wavenumber k of the solitons is

$$k = \frac{N}{(6u_i + 2a)t}, \tag{2.24}$$

which becomes

$$k = \frac{3(u_b - u_i)^{\frac{1}{2}}(u_b + u_i)^{\frac{5}{2}}}{8\sqrt{2}(2u_b + u_i)^{\frac{1}{2}}(u_b^2 + u_b u_i + u_i^2)} \tag{2.25}$$

on using (2.18) and (2.21).

Since $a > 0$, the amplitude expression (2.22) is valid for $u_b > u_i$ and $u_b > -u_i$. We furthermore require that the soliton speed $U > 0$. From (2.22) and (2.23), it can be shown that $U > 0$ for $u_b > u_i$ and $u_b > -u_i$. The modulation theory solution of Section 2.1 was found to be valid for $u_b > 0$ and $u_b > u_i$. The limit $u_b = u_i$, for which $a = 0$ according to both (2.12) and (2.22), is then the same for both solutions. However, the limit $u_b \to -u_i$ for (2.22), for which $a \to \infty$, does not agree with the modulation theory solution. It can be found from (2.15) that, as u_i becomes more negative, the amplitude of the modulated cnoidal wave increases. The assumption made above that $u_x(0, t)$ and $u_{xx}(0, t)$ are negligible then becomes less valid. If better estimates of $u_x(0, t)$ and $u_{xx}(0, t)$ for $u_i < 0$ could be made, then the agreement between the modulation theory solution and the present soliton approximation would be better.

Figure 3 shows the soliton amplitude versus u_i for $u_b = 0{\cdot}2$. Compared are the soliton theory solution (– – –), modulation theory (——), and numerical results (◇) for $-\frac{1}{2}u_b \leq u_i \leq u_b$. This corresponds to the region where an undular bore (full or partial) occurs; for $u_i < -\frac{1}{2}u_b$ the steady-state sech^2 solution will occur (see Fig. 1 and Section 2.5). Throughout most of the region, a good comparison is obtained between all three solutions. As $u_i \to -u_b$ though, the soliton theory predicts an amplitude which goes to infinity, which is inaccurate due to the neglect of $u_x(0, t)$ and $u_{xx}(0, t)$.

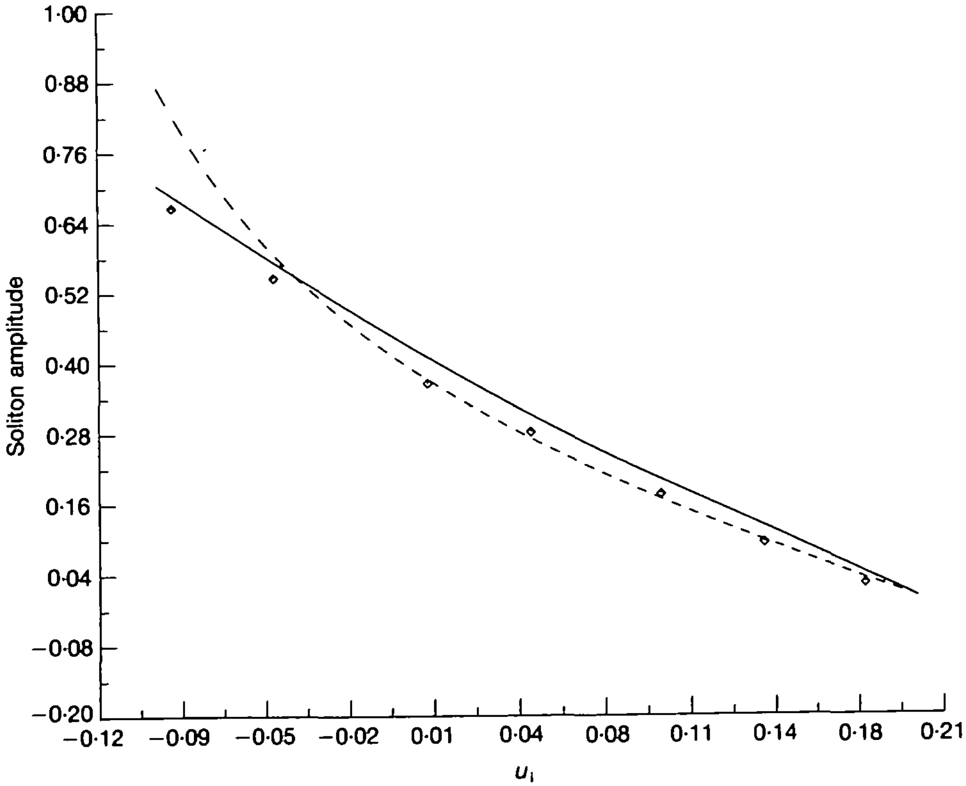

FIG. 3. Soliton amplitude versus u_i for $u_b = 0{\cdot}2$. Compared are soliton theory (– – –), modulation theory (——), and numerical solutions (◇).

2.3 *Mean Height Variation*

The solution of (1.2) will now be found for the portion of the u_b versus u_i plane for which $0 \leq u_b < u_i$. This solution is found to consist of a mean height variation with no oscillatory component and was first obtained by Fornberg & Whitham (1978). They considered the special case $u_b = 0$ and found an approximate solution by ignoring the dispersive term in the Korteweg–de Vries equation. This was done because from numerical evidence there were no oscillations in the solution. The difference between the mean height variation considered here and the solution of Section 2.1 is that here the solution has no waves: it is just a variation in β.

The mean height variation solution can be derived in two ways. The first, which was the method used by Fornberg & Whitham (1978), is to ignore the dispersive term in the Korteweg–de Vries equation. We consider the solution of

$$u_t + 6uu_x = 0, \tag{2.26}$$

subject to the boundary conditions of (1.2). This has solution

$$u = \begin{cases} u_b & (0 \leq x/t \leq 6u_b), \\ 6x/t & (6u_b \leq x/t \leq 6u_i), \\ u_i & (x/t > 6u_i), \end{cases} \tag{2.27}$$

which is valid for

$$0 \leq u_b < u_i. \tag{2.28}$$

The solution (2.27) can also be derived from the modulation equations (2.7) when the amplitude $a \equiv 0$. In this case, the modulation equations reduce to the single equation

$$\beta = \text{const} \quad \text{on} \quad \frac{dx}{dt} = 6\beta, \tag{2.29}$$

which has solution (2.27).

Figure 4 shows a comparison of the numerical solution of (1.2) (——) and the solution (2.27) ($\times$) for $u_b = 0{\cdot}1$ and $u_i = 0{\cdot}3$. There is a slight variation between the numerical solution and (2.27) which occurs because (2.27) has discontinuous derivatives at $x/t = 6u_b$ and $6u_i$. These derivatives are smoothed out in the numerical solution by the dispersive term u_{xxx}.

2.4 *An Airy Function Solution*

The solution considered here is valid in the region

$$u_i > 0 \quad \text{and} \quad u_b < 0, \tag{2.30}$$

and is related to the similarity solution of the Korteweg–de Vries equation (see Rosales, 1978). The similarity solution obtained by Rosales is of the form

$$u = (3t)^{-\frac{2}{3}} f\left(\frac{x}{(3t)^{\frac{1}{3}}}\right) \tag{2.31}$$

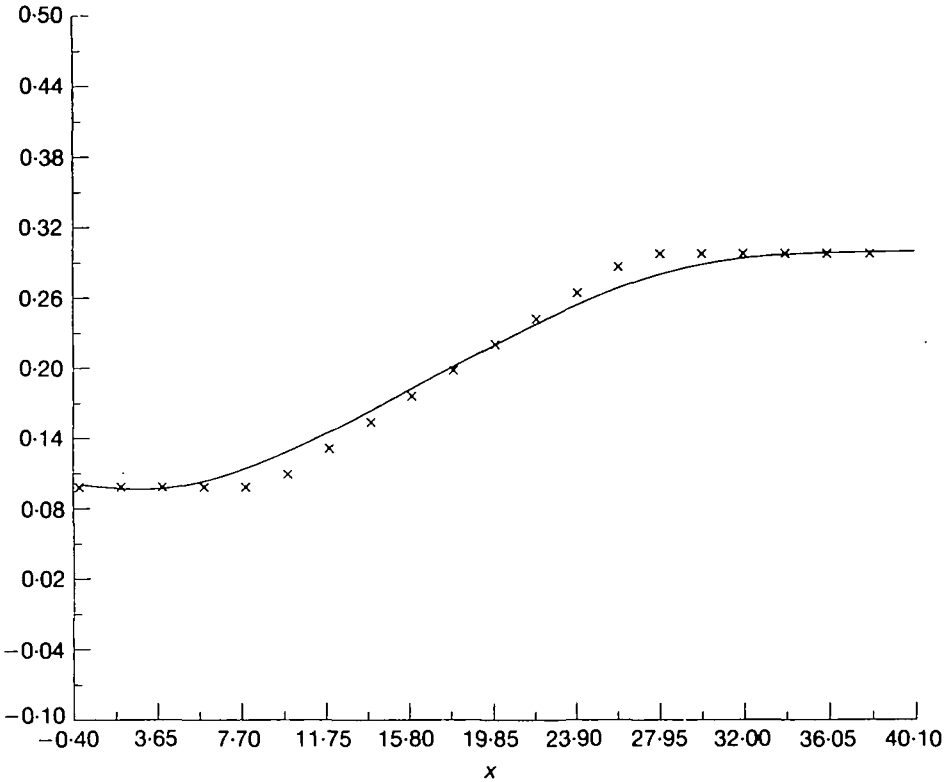

FIG. 4. Numerical solution (——) of (1.2) for $u_b = 0{\cdot}1$ and $u_i = 0{\cdot}3$ after $t = 15$. Also shown is the mean height variation (2.27) (×).

and cannot satisfy the time-independent boundary condition of (1.2) at $x = 0$. However, if (1.2) is linearized, then a similarity solution which satisfies the boundary condition of (1.2) can be found. This linearization is valid for $u_i - u_b$ small.

We thus consider

$$u_t + u_{xxx} = 0, \tag{2.32}$$

and take the similarity variables

$$u = f(z), \qquad z = x/(3t)^{\frac{1}{3}}, \tag{2.33}$$

which gives

$$\left.\begin{aligned} &f''' - zf' = 0, \\ &f = u_i \quad \text{as} \quad z \to \infty, \\ &f = u_b \quad \text{at} \quad z = 0, \end{aligned}\right\} \tag{2.34}$$

subject to the transformed boundary conditions.

The solution of (2.34) is

$$f(z) = 3(u_i - u_b)\int_0^z \mathrm{Ai}(t)\,\mathrm{d}t + u_b, \tag{2.35}$$

where Ai(z) is the Airy function. Figure 5 compares the numerical solution of (1.2) (——) and the solution (2.34) (- - -) for

$$u_b = -0{\cdot}05 \quad \text{and} \quad u_i = +0{\cdot}05. \tag{2.36}$$

It can be seen that this is a good approximation for small $u_i - u_b$, in which case the nonlinear term is not important.

The solution of (1.2) for $u_i > 0$ and $u_b < 0$ when $u_i - u_b$ is not small is not clear. As stated above, the nonlinear similarity solution of Rosales (1978), which was of the form (2.31), is not valid as it cannot satisfy the boundary condition at $x = 0$. Furthermore, a nonlinear similarity solution

$$u = f\left(\frac{x}{(3t)^{\frac{1}{3}}}\right) \tag{2.37}$$

is not a solution of (1.2) owing to the term uu_x.

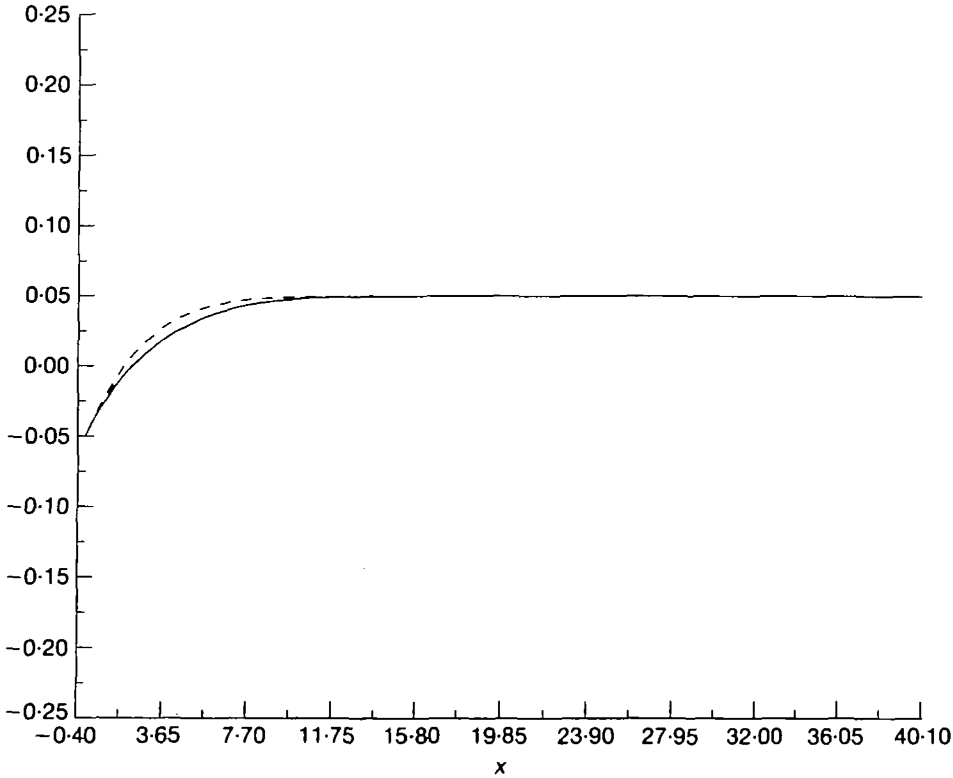

FIG. 5. Numerical solution (——) of (1.2) for $u_b = -0{\cdot}05$ and $u_i = +0{\cdot}05$ after $t = 10$. Also shown is the Airy function solution (2.35) (- - -).

2.5 *Solutions Involving Hyperbolic Functions*

In the region

$$u_i < 0 \quad \text{and} \quad u_b \leqslant -2u_i, \tag{2.38}$$

exact steady solutions to (1.2) can be found. These solutions are

$$u(x) = u_i - 3u_i \operatorname{sech}^2[(-\tfrac{3}{2}u_i)^{\frac{1}{2}}x + \alpha_0] \quad \text{where} \quad \alpha_0 = \operatorname{sech}^{-1}\left(\frac{u_b - u_i}{-3u_i}\right)^{\frac{1}{2}}, \tag{2.39}$$

which is valid for $u_i < 0$ and $u_i < u_b \leqslant -2u_i$ and

$$u = u_i + 3u_i \operatorname{cosech}^2[(-\tfrac{3}{2}u_i)^{\frac{1}{2}}x + \alpha_0] \quad \text{where} \quad \alpha_0 = \operatorname{cosech}^{-1}\left(\frac{u_b - u_i}{3u_i}\right)^{\frac{1}{2}}, \tag{2.40}$$

which is valid for $u_b < u_i < 0$. Figure 6 shows a comparison of the numerical solution of (1.2) at $t = 10$ (——) and the steady solution (2.39) ($\times$) for $u_b = 0{\cdot}2$ and $u_i = -0{\cdot}12$, while Fig. 7 shows the numerical solution of (1.2) at $t = 10$ (——) and the steady solution (2.40) ($\times$) for $u_b = -0{\cdot}4$ and $u_i = -0{\cdot}2$. There is excellent agreement. The different solutions of (1.2) found in different regions of the u_b

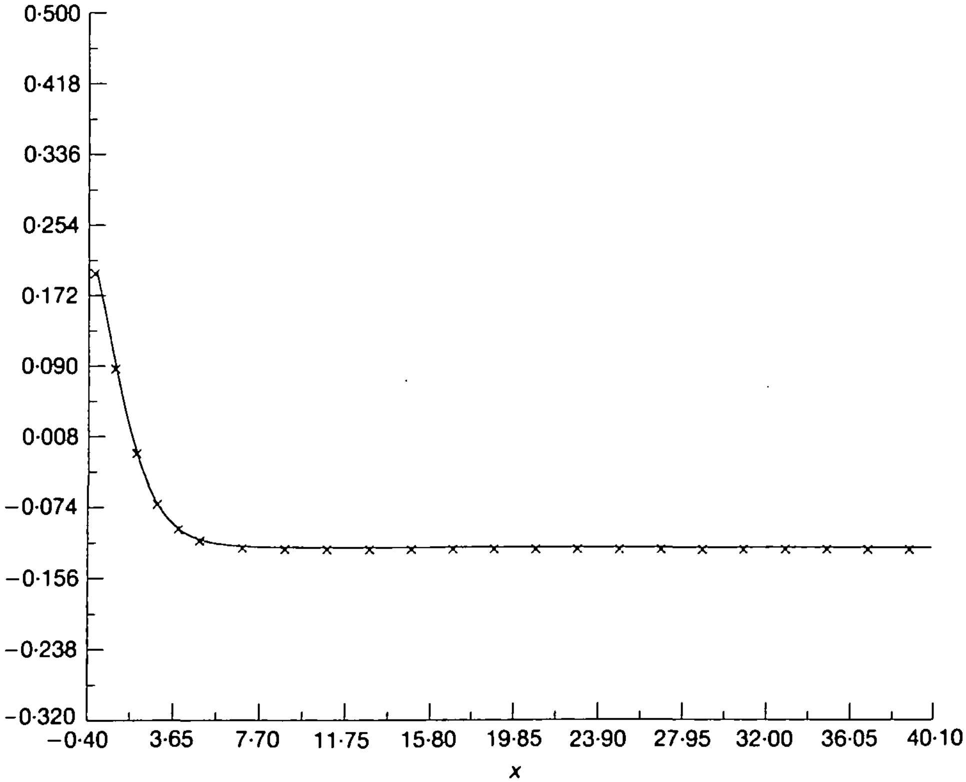

FIG. 6. Numerical solution (——) of (1.2) for $u_b = 0{\cdot}2$ and $u_i = -0{\cdot}12$ after $t = 10$. Also shown is the sech^2 solution (2.39) ($\times$).

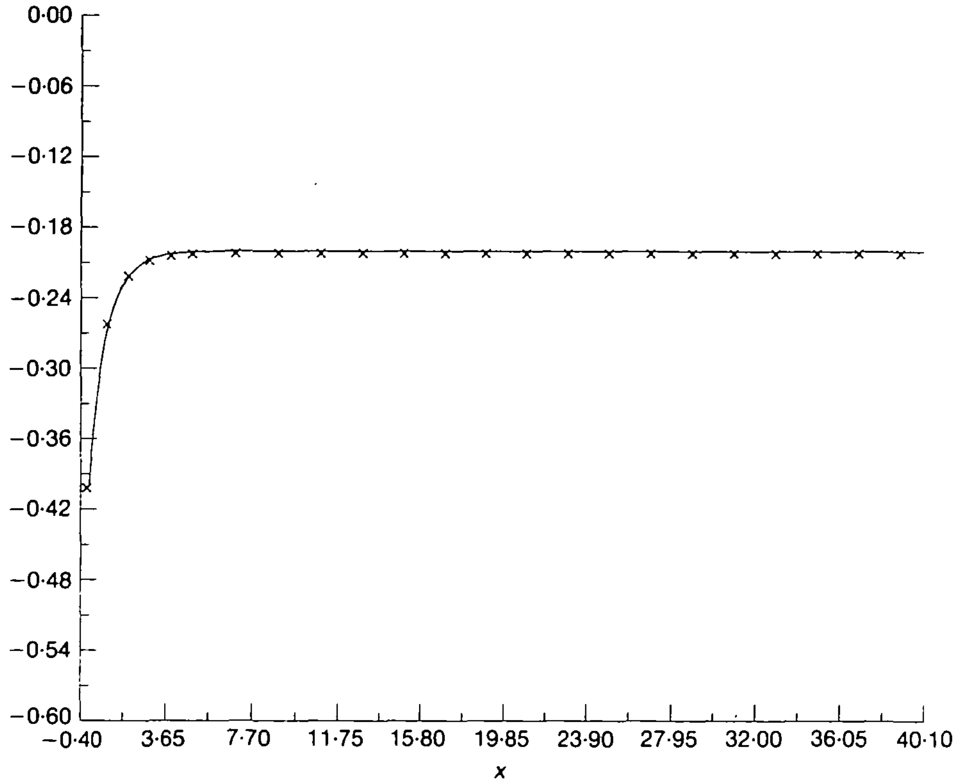

FIG. 7. Numerical solution (——) of (1.2) for $u_b = -0{\cdot}4$ and $u_i = -0{\cdot}2$ after $t = 10$. Also shown is the *cosech² solution (2.40)* (×).

versus u_i plane are shown schematically in Fig. 1. It can be seen from Fig. 1 that there is a region of overlap between the modulation theory solution of Section 2.1 and the sech² solution. This region is

$$u_i < 0 \quad \text{and} \quad u_i \leq u_b \leq -2u_i. \tag{2.41}$$

The modulation theory solution in this region is essentially a train of solitons (as m_0 is near 1). Numerical experimentation shows that the steady sech² solution occurs in this region (i.e. (2.39)) rather than a train of solitons. If a train of solitons in $x > 0$ is used as an initial condition for the numerical solution, it is found that this train of solitons propagates to infinity, leaving the steady solution (2.39). The modulation theory solution is then unstable in some sense in the region (2.41).

3. Time-dependent boundary conditions

In this section, we consider the solutions to (1.2) when u_b is a function of time. Chu *et al.* (1983) considered the numerical solution of (1.2) for $u_i = 0$ and u_b

corresponding to a trapezoidal function, i.e. u_b is turned on at $t=0$, linearly increases to a maximum, then decreases linearly to zero where it is turned off. Their numerical solution for this boundary condition consisted of a train of waves of decreasing amplitude. Camassa & Wu (1989) produced numerical solutions for the trapezoidal $u_b(t)$ and for an exponentially decaying $u_b(t)$. They also calculated an inverse scattering solution for the soliton amplitude. This inverse scattering solution was found to be approximate (within 20%) because u_x and u_{xx} had to be estimated at $x=0$ and because the boundary condition at $x=0$ was incorporated as a time-averaged quantity. In particular, they did not obtain good agreement for small-amplitude solitons.

We shall develop an approximate solution of (1.2) for boundary conditions u_b which are functions of t. For comparison with numerical solutions, we shall use the decreasing u_b given by

$$u_b(t)=\frac{u_0}{1+\alpha t}, \tag{3.1}$$

where u_0 and α are constants. We obtain a theoretical approximation to the solution of (1.2) by considering the soliton ($m\to 1$) limit of the modulation equations for the Korteweg–de Vries equation. We hence assume that the solution is approximately a train of solitons (which is a good approximation for $u_i\leqslant 0$ as $m_0\geqslant 0{\cdot}64$) and that the solution is slowly varying, which is a reasonable assumption if $u_b(t)$ is slowly varying. In this limit, the modulation equations (2.7) are, from Whitham (1974: p. 573),

$$a_t+2aa_x=0, \qquad k_t+(2ak)_x=0. \tag{3.2a, b}$$

The system (3.2a, b) comprises two parabolic equations which can be solved by the method of characteristics since (3.2a) can be solved independently of (3.2b). The solution is

$$a=\text{const}, \quad \frac{\mathrm{d}k}{\mathrm{d}t}=-2ka_x \quad \text{along} \quad x=2at+A, \tag{3.3}$$

where A is constant on a characteristic. This is the solution for a soliton of amplitude a on a background depth of zero. In general, the background depth is u_i, in which case the solution (3.3) is rescaled to

$$a=\text{const}, \qquad \frac{\mathrm{d}k}{\mathrm{d}t}=-2ka_x \quad \text{along} \quad x=(2a+6u_i)t+A. \tag{3.4}$$

To determine the solution (3.4) fully, the values of a and k at $x=0$ are needed. These values cannot be obtained directly from the boundary condition in (1.2). However, the soliton approximation of Section 2.2 can be used to give these values. If we assume that a soliton is created at $x=0$ with an amplitude given by the value of u_b at its time of creation, then this amplitude is given by (2.22), with k given by (2.24). This approximation to the values of a and k at $x=0$ will be valid if $u_b(t)$ is slowly varying. For a constant boundary condition, we see from

(2.22) that the soliton amplitude is

$$a = \frac{(2u_b + u_i)(u_b - u_i)}{u_b + u_i}, \tag{3.5}$$

so for the slowly varying boundary condition (3.1) we let (3.5) be the soliton amplitude at $x = 0$. Hence, at $x = 0$ and $t = \tau$,

$$a = \frac{(2u_0 + u_i + \alpha u_i \tau)(u_0 - u_i - \alpha u_i \tau)}{(1 + \alpha\tau)(u_0 + u_i + \alpha u_i \tau)} \quad \text{and} \quad A = -(2a + 6u_i)\tau. \tag{3.6}$$

Substituting the expression (3.6) for A into (3.4), we find that τ is determined in terms of x and t by

$$x = \left(\frac{2(2u_0 + u_i + \alpha u_i \tau)(u_0 - u_i - \alpha u_i \tau)}{(1 + \alpha\tau)(u_0 + u_i + \alpha u_i \tau)} + 6u_i\right)(t - \tau) \tag{3.7}$$

for the boundary condition (3.1). In general, (3.7) can be expressed as a cubic polynomial for τ. For the special case in which $u_i = 0$, (3.7) can be solved to give

$$\tau = \frac{4u_0 t - x}{4u_0 + \alpha x}, \tag{3.8}$$

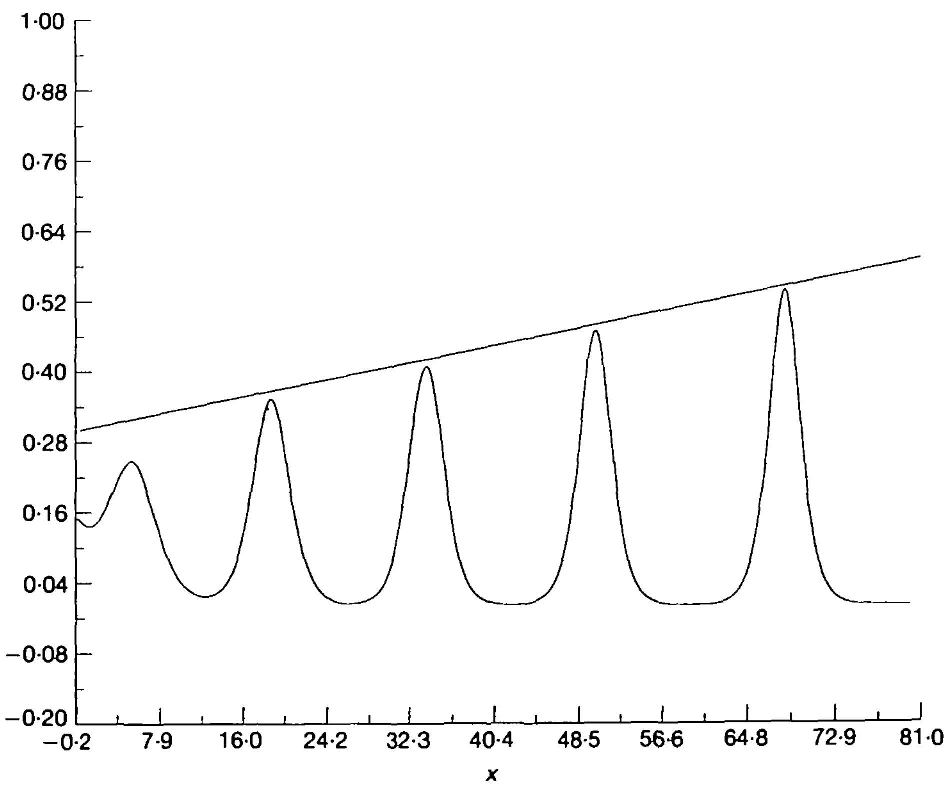

FIG. 8. Numerical solution of (1.2) for $u_b = 0{\cdot}3/(1 + \frac{1}{70}t)$ and $u_i = 0{\cdot}0$ after $t = 70$. Also shown is the envelope (3.9).

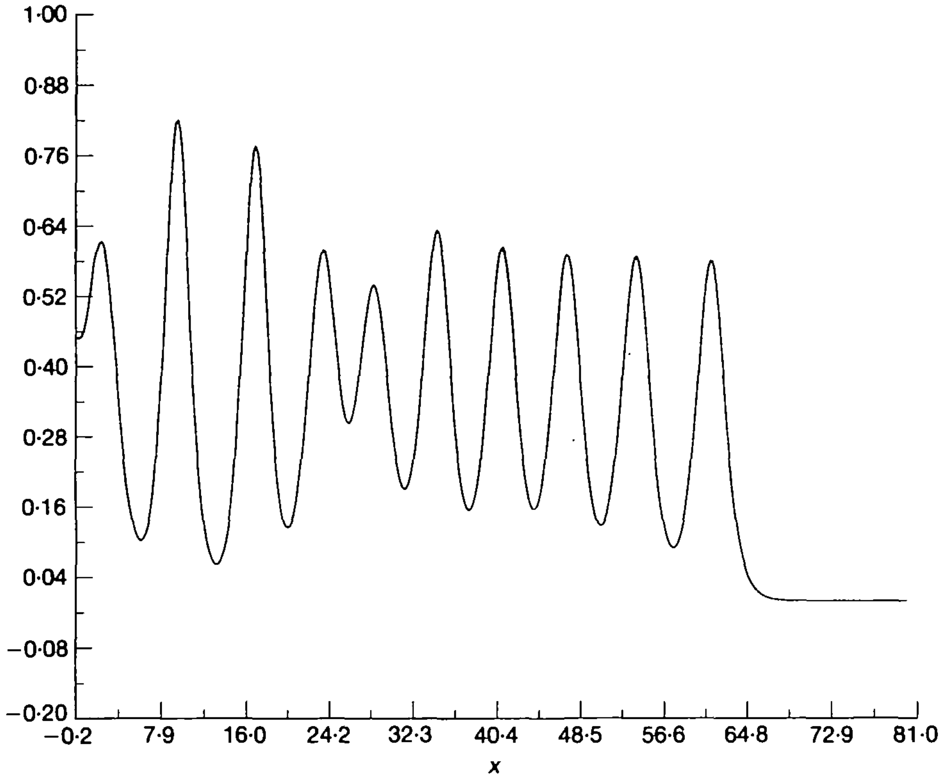

FIG. 9. Numerical solution of (1.2) for $u_b = 0{\cdot}15(1+\frac{1}{40}t)$ and $u_i = 0{\cdot}0$ after $t = 80$.

so that, from (3.6),

$$a = \frac{4u_0 + \alpha x}{2(1+\alpha t)} \quad (0 < x < 4u_0 t). \tag{3.9}$$

Figure 8 shows the numerical solution of (1.2) for $u_b = 0{\cdot}3/(1+\frac{1}{70}t)$ and $u_i = 0$ at $t = 70$. Also shown is the wave envelope $u_i + a$ from our approximate theory (3.7) and (3.8). There is an excellent agreement between the numerical solution and the approximate theory. Figure 9 shows the numerical solution of (1.2) for $u_b = 0{\cdot}15(1+\frac{1}{40}t)$ and $u_i = 0$ at $t = 80$. Here we have a boundary condition at $x = 0$ which is increasing the soliton amplitude. Once the solitons of large amplitude are generated, they overtake the solitons in front of them (since they have lower amplitude and hence lower velocity). Solving the system (3.2a, b) for this increasing boundary condition, we find that the characteristics cross. This corresponds to the interaction and overtaking of the solitons, and so fitting in a shock to resolve the intersection of the characteristics would not be valid in this case. If a non-slowly-varying boundary condition $u_b(t)$ which tends to zero as $t \to \infty$ is used for the numerical solution of (1.2), for example (3.1) with $\alpha = O(1)$, then it is found that the solution u rapidly tends to $u \equiv 0$.

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