

# Lecture 13 – Fourier transform

Prof. João Fernando Mari

[joaofmari.github.io](https://joaofmari.github.io)

*joaof.mari@ufv.br*

- The Fourier transform
- The pair of Fourier transforms
- Analyzing the Fourier Transform equation
- The Discrete Fourier Transform
- DFT computation

# The Fourier transform

- The Fourier transform of a continuous function  $f(t)$  is defined as:

$$\mathfrak{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-j2\pi\mu t} dt$$

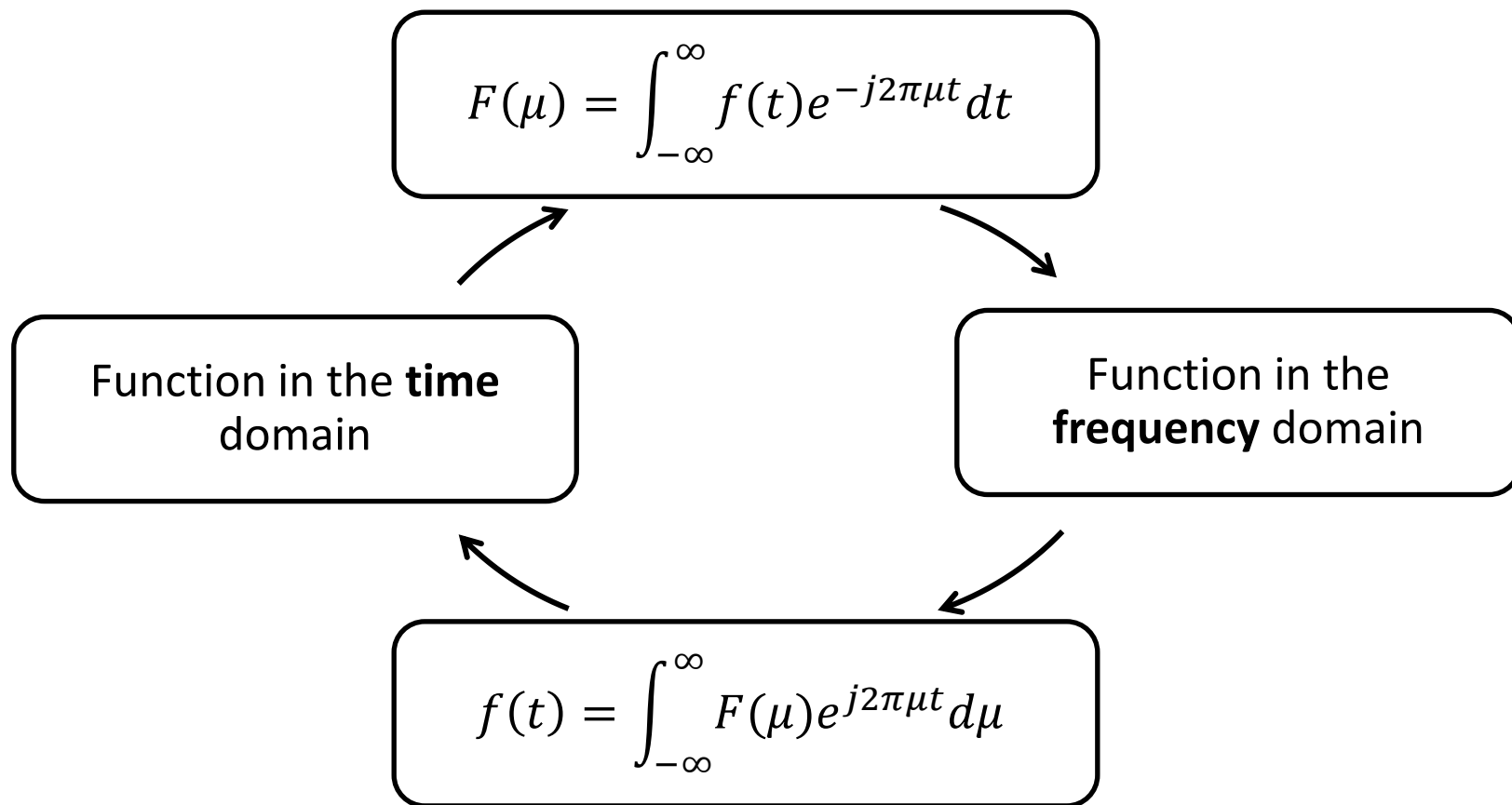
- As  $\mathfrak{F}\{f(t)\}$  is a function of just  $\mu$ , since  $t$  is eliminated by integration, the Fourier transform of  $f(t)$  can be expressed as:

$$F(\mu) = \int_{-\infty}^{\infty} f(t)e^{-j2\pi\mu t} dt$$

- Given  $F(\mu)$ , we can obtain  $f(t)$  again using the inverse Fourier transform,  $f(t) = \mathfrak{F}^{-1}\{F(\mu)\}$ , expressed as:

$$f(t) = \int_{-\infty}^{\infty} F(\mu)e^{j2\pi\mu t} d\mu$$

# The pair of Fourier transforms



# Analyzing the Fourier Transform equation

- Using the Euler's formula we can rewrite...

$$F(\mu) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t} dt$$

- as:

$$F(\mu) = \int_{-\infty}^{\infty} f(t) \cos(2\pi\mu t) - j \operatorname{sen}(2\pi\mu t) dt$$

- $F(\mu)$  is the function  $f(t)$  itself multiplied by sinusoidal terms with frequencies defined by the values of  $\mu$ .
  - The variable  $t$  (time) is eliminated by integration.
  - In fact  $t$  can represent any continuous variable: time, space, etc.
    - The units of the frequency variable depend on the unit defined for  $t$ :
      - If  $t$  represents time and is in seconds:  $\mu$  represents cycles/s (Hz)
      - If  $t$  represents space and is in meters:  $\mu$  represents cycles/meter

# The Discrete Fourier Transform

- Given the continuous nature of the Fourier transform, it cannot be implemented on a computer.

- The discrete Fourier transform is:

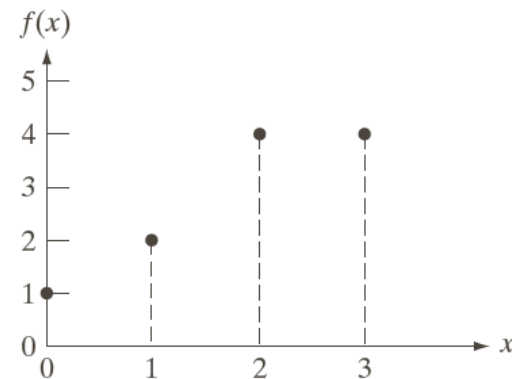
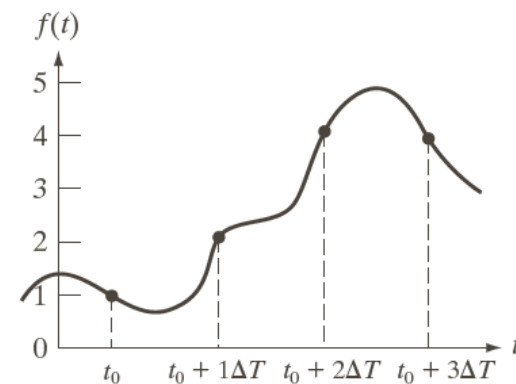
$$F(u) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi ux/M}, \quad u = 0, 1, 2, \dots, M-1$$

- The inverse discrete Fourier transform is:

$$f(x) = \frac{1}{M} \sum_{u=0}^{M-1} F(u) e^{j2\pi ux/M}, \quad x = 0, 1, 2, \dots, M-1$$

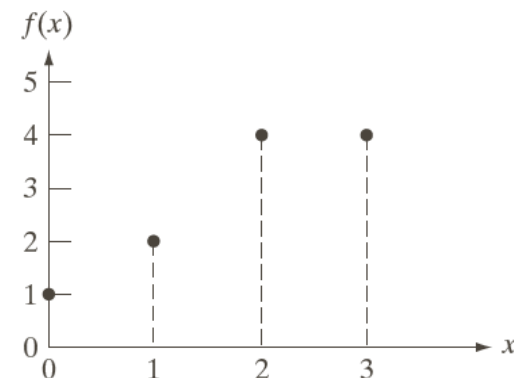
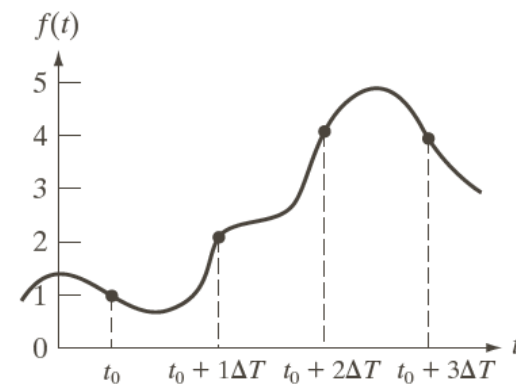
# DFT computation

- DFT:  $F(u) = \sum_{x=0}^{M-1} f(x) e^{-\frac{j2\pi ux}{M}}$
- $F(0) = \sum_{x=0}^3 f(x) = [f(0) + f(1) + f(2) + f(3)]$
- $F(0) = 1 + 2 + 4 + 4 = 11$
- $|F(0)| = \sqrt{(11)^2 + (0)^2} = 11.0$



# DFT computation

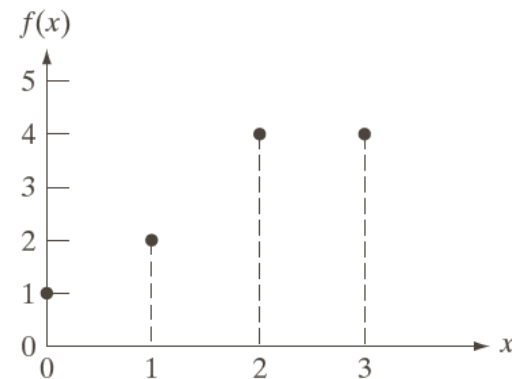
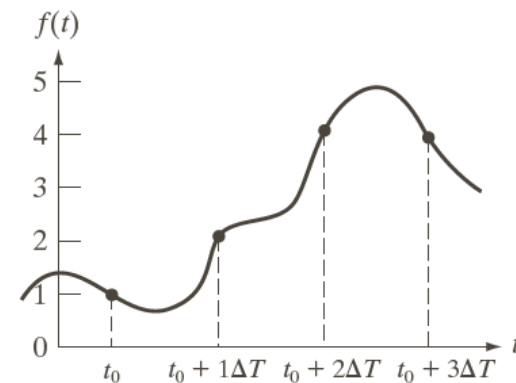
- DFT:  $F(u) = \sum_{x=0}^{M-1} f(x) e^{-\frac{j2\pi ux}{M}}$
- $F(1) = \sum_{x=0}^3 f(x) e^{-j2\pi(1)x/4}$
- $F(1) = 1e^{-j2\pi(1)0/4} + 2e^{-j2\pi(1)1/4} + 4e^{-j2\pi(1)2/4} + 4e^{-j2\pi(1)3/4}$
- $F(1) = 1e^0 + 2e^{-j\pi/2} + 4e^{-j\pi} + 4e^{-j3\pi/2} = -3 + 2j$
- $|F(1)| = \sqrt{(-3)^2 + (2)^2} = 3.61$





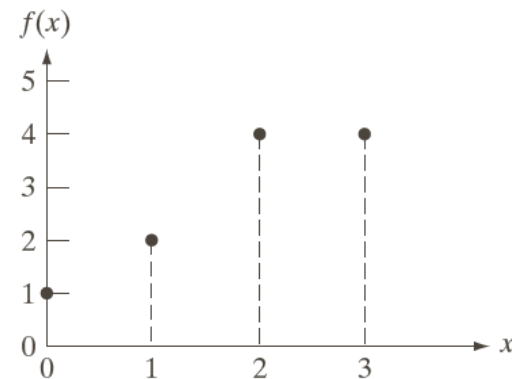
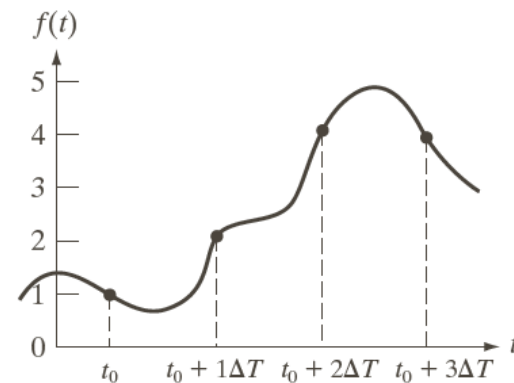
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- DFT:  $F(u) = \sum_{x=0}^{M-1} f(x) e^{-\frac{j2\pi ux}{M}}$
- $F(2) = \sum_{x=0}^3 f(x) e^{-j2\pi(2)x/M}$
- $F(2) = -(1 + 0j)$
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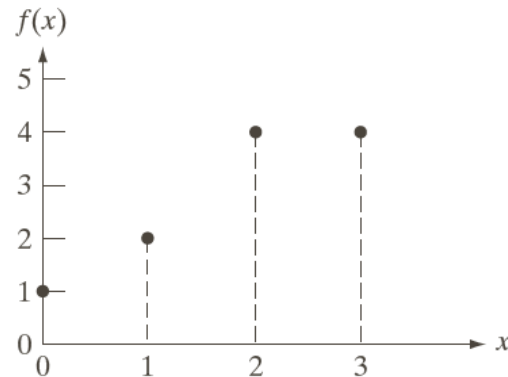
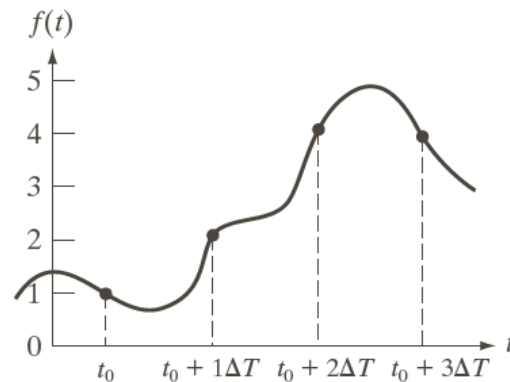
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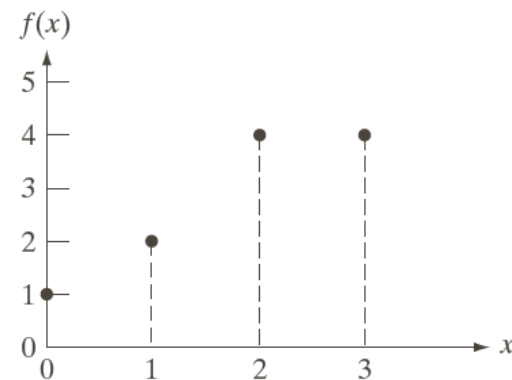
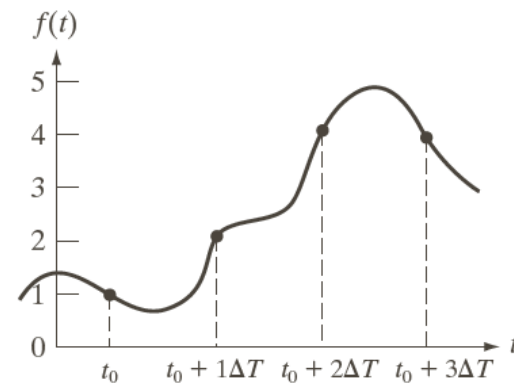
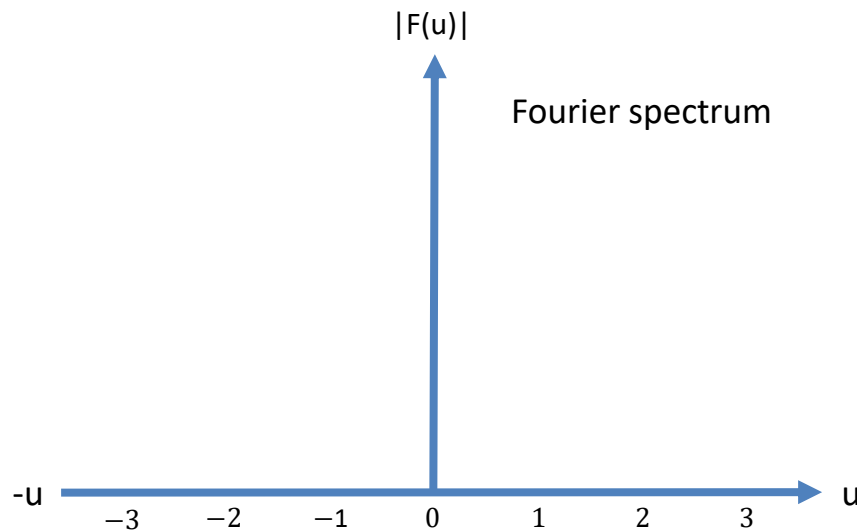
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- $F(2) = \sum_{x=0}^3 f(x) e^{-j2\pi(2)x/M}$
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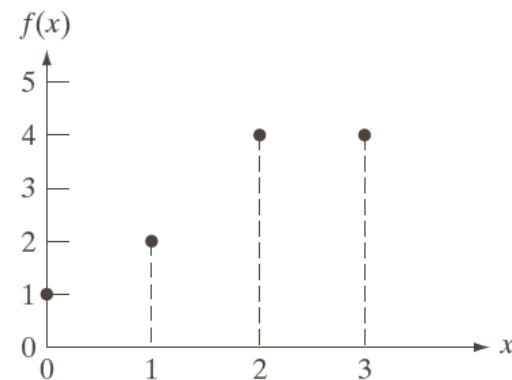
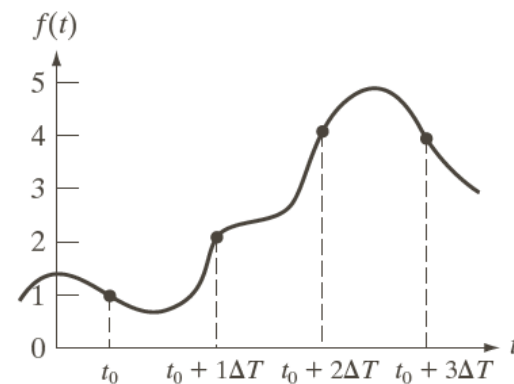
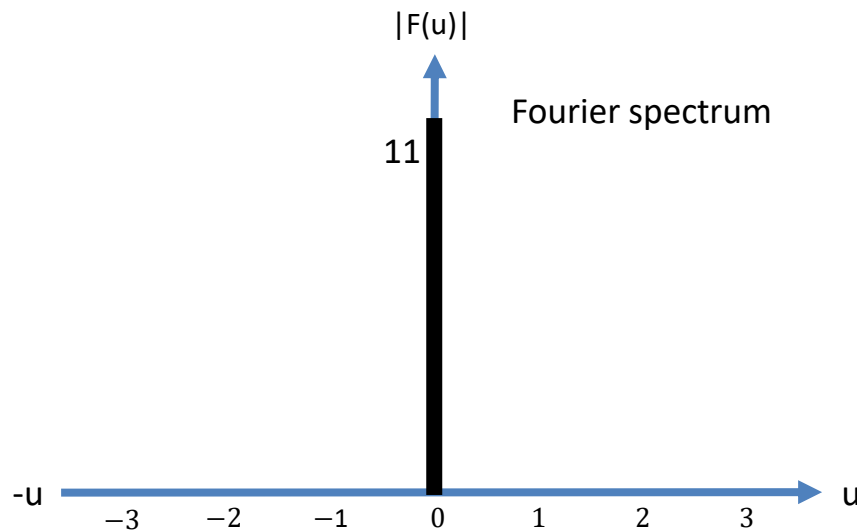
# DFT computation

- **DFT:**  $F(u) = \sum_{x=0}^{M-1} f(x) e^{-\frac{j2\pi ux}{M}}$



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# DFT computation

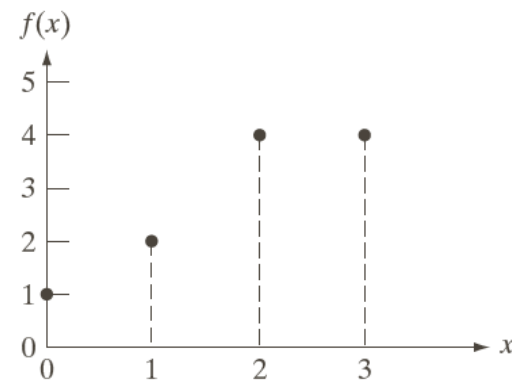
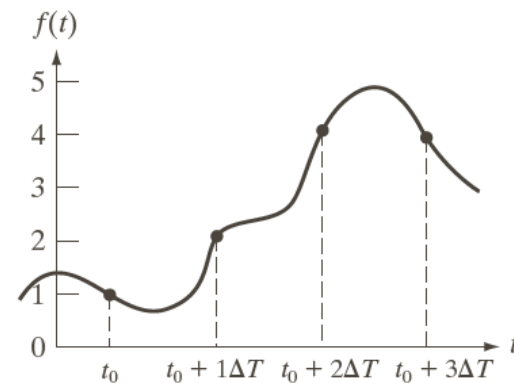
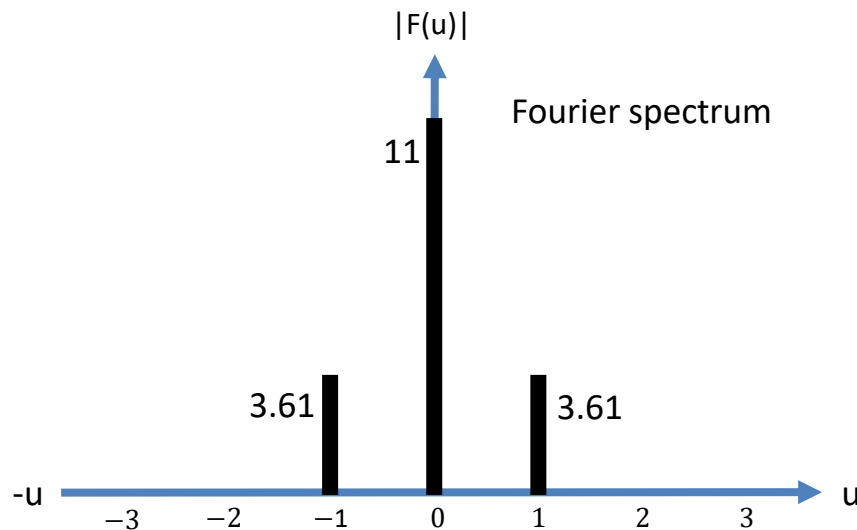
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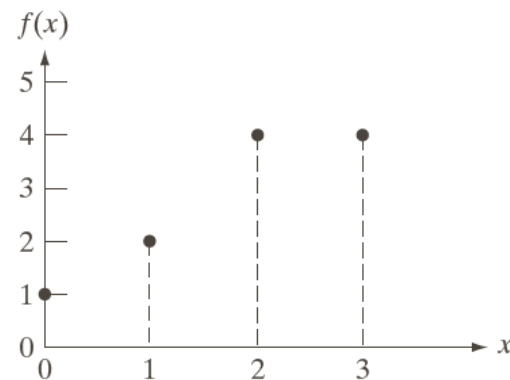
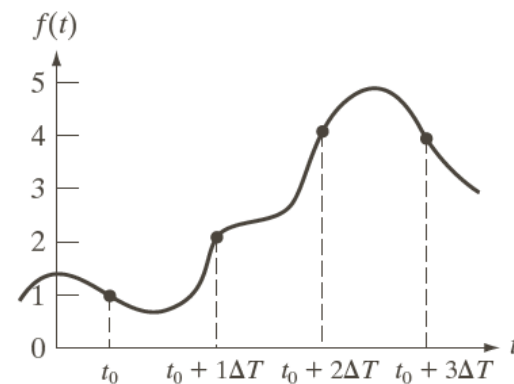
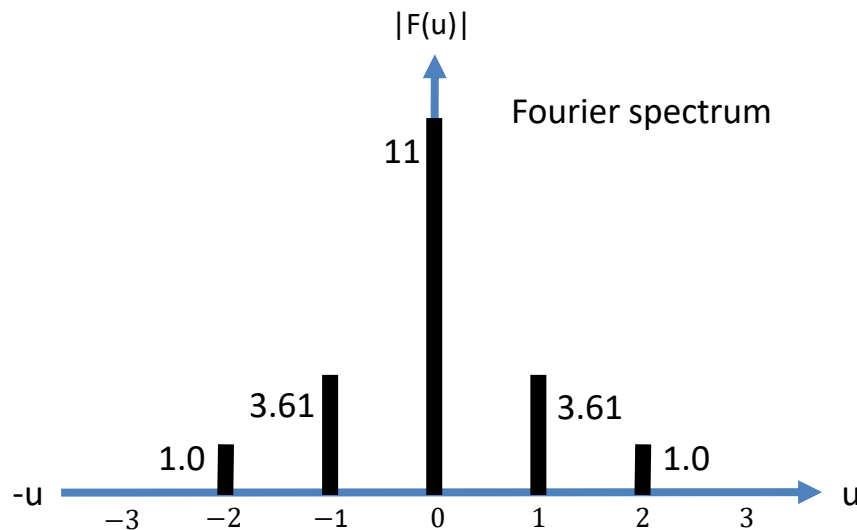
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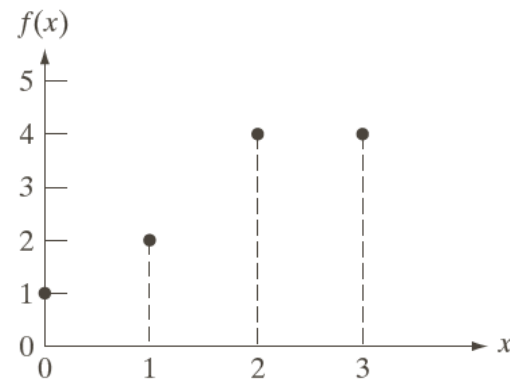
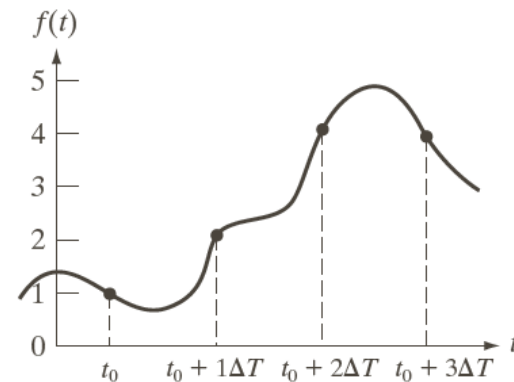
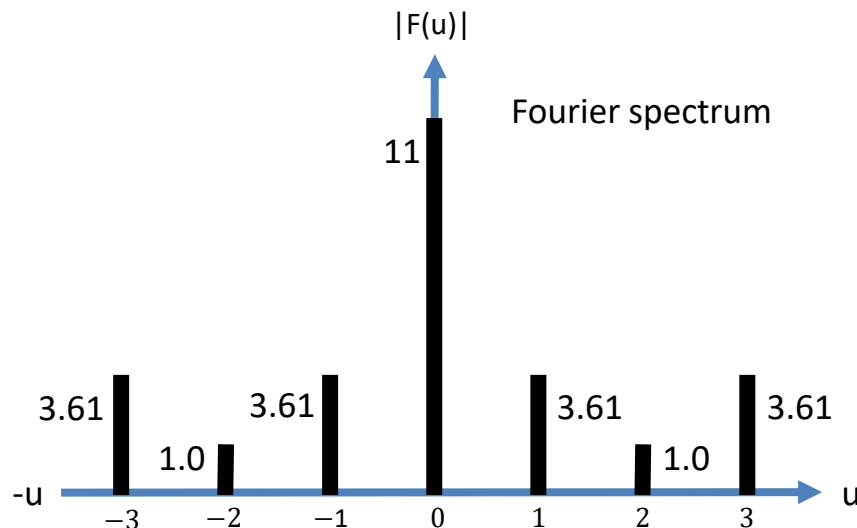
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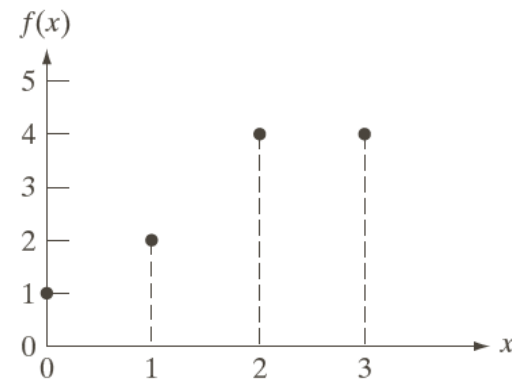
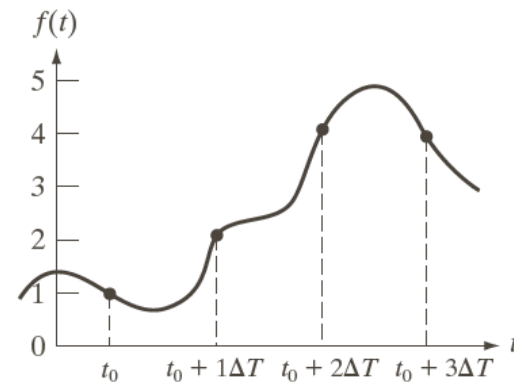
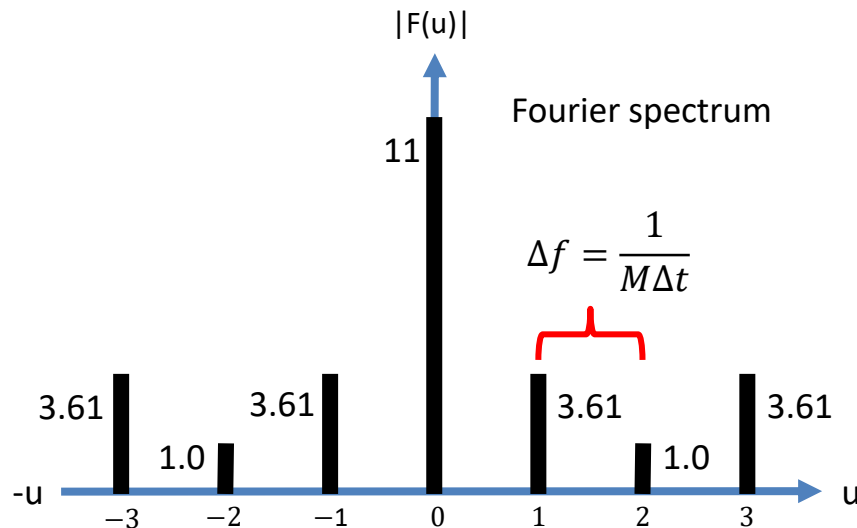
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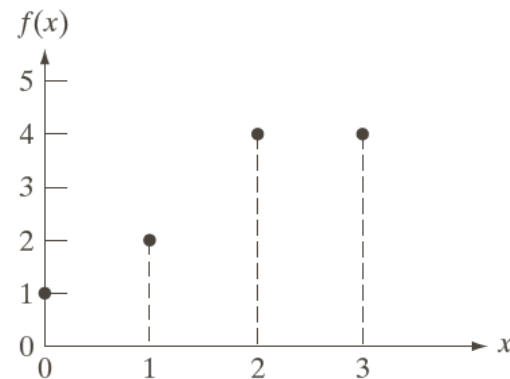
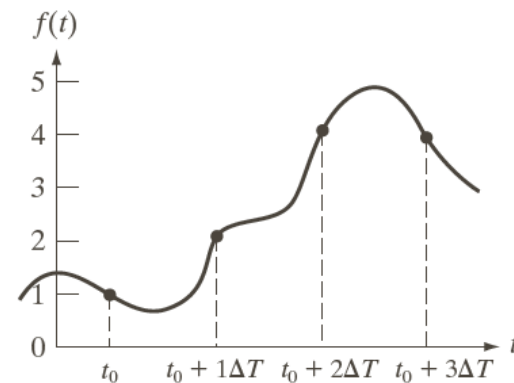
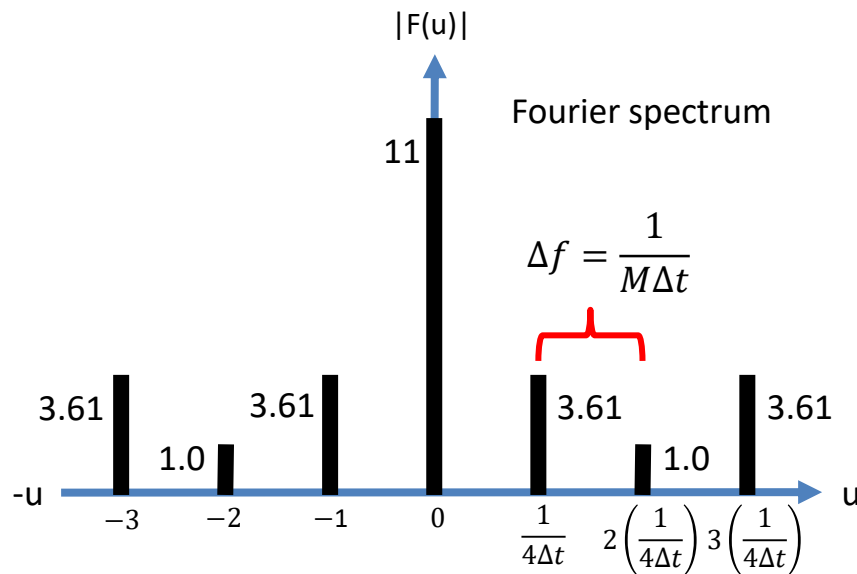
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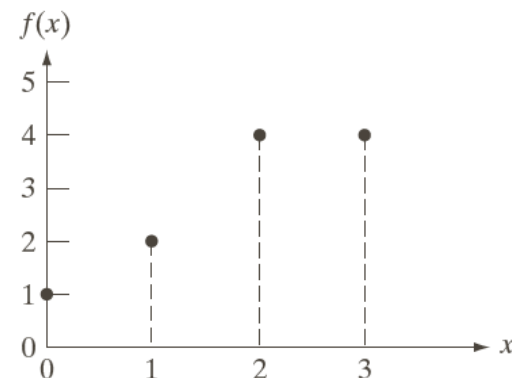
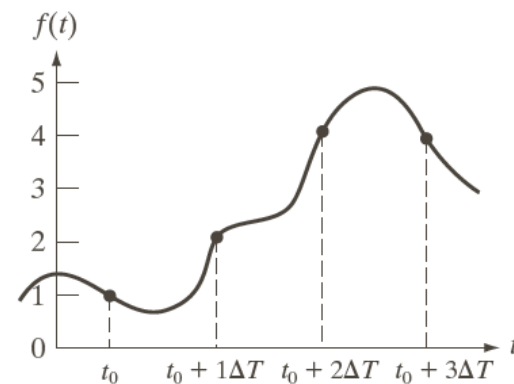
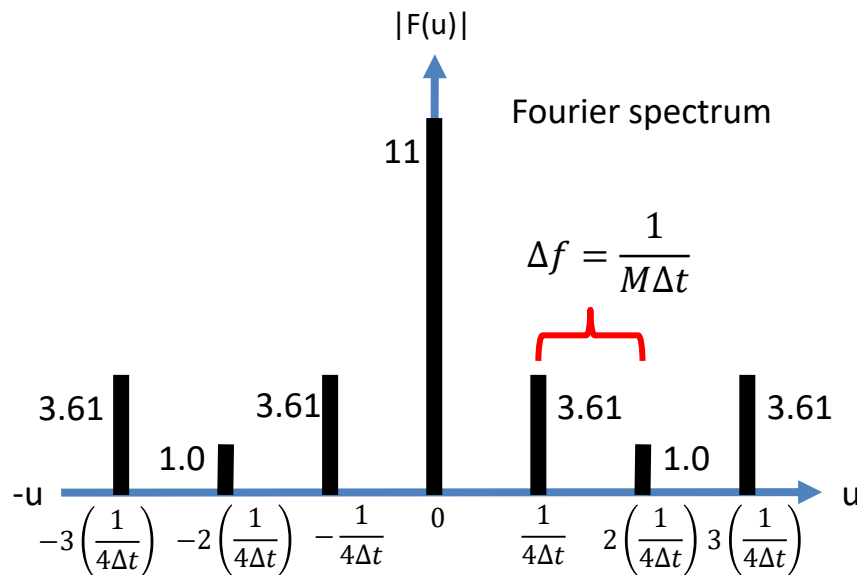
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- MARQUES FILHO, O.; VIEIRA NETO, H. **Processamento digital de imagens**. Brasport, 1999.
  - (*in Brazilian Portuguese*)
  - Available on the author's website (for personal use only)
  - <http://dainf.ct.utfpr.edu.br/~hvieir/pub.html>
- J. E. R. Queiroz, H. M. Gomes. **Introdução ao Processamento Digital de Imagens**. RITA. v. 13, 2006.
  - (*in Brazilian Portuguese*)
  - <http://www.dsc.ufcg.edu.br/~hmg/disciplinas/graduacao/vc-2016.2/Rita-Tutorial-PDI.pdf>

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