which result is accurate, and confirms the rule. But if we had supposed $a = 1$, $b = 1$, and $c = -1$, we should have found for the cube of $1 + 1 - 1$, that is of 1,

$1 + 3 - 3 + 3 - 6 + 3 + 1 - 3 + 3 - 1 = 1$, which is a still farther confirmation of the rule.

CHAP. XII.

Of the Expression *of* Irrational Powers *by* Infinite Series.

361. As we have shewn the method of finding any power of the root $a + b$, however great the exponent may be, we are able to express, generally, the power of $a + b$, whose exponent is undetermined; for it is evident that if we represent that exponent by n, we shall have by the rule already given (Art. 348 and the following):

$$(a + b)^n = a^n + \frac{n}{1} a^{-1}b + \frac{n}{1} \cdot \frac{n-1}{2} a^{n-2}b^2 + \frac{n}{1} \cdot \frac{n-1}{2} \cdot$$
$$\frac{n-2}{3} a^{n-3}b^3 + \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} a^{n-4}b^4 + \&c.$$

362. If the same power of the root $a - b$ were required, we need only change the signs of the second, fourth, sixth, &c. terms, and should have

$$(a - b)^n = a^n - \frac{n}{1} a^{n-1}b + \frac{n}{1} \cdot \frac{n-1}{2} a^{n-2}b^2 - \frac{n}{1} \cdot \frac{n-1}{2} \cdot$$
$$\frac{n-2}{3} a^{n-3}b^3 + \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} a^{n-4}b^4 - \&c.$$

363. These formulas are remarkably useful, since they serve also to express all kinds of radicals; for we have shewn that all irrational quantities may assume the form of powers whose exponents are fractional, and that $\sqrt[2]{a} = a^{\frac{1}{2}}, \sqrt[3]{a} = a^{\frac{1}{3}}$, and $\sqrt[4]{a} = a^{\frac{1}{4}}$, &c.: we have, therefore,

$$\sqrt[2]{(a + b)} = (a + b)^{\frac{1}{2}}; \quad \sqrt[3]{(a + b)} = (a + b)^{\frac{1}{3}};$$
$$\text{and } \sqrt[4]{(a + b)} = (a + b)^{\frac{1}{4}}, \&c.$$

Consequently, if we wish to find the square root of $a + b$, we have only to substitute for the exponent n the fraction $\frac{1}{2}$, in the general formula, Art. 361, and we shall have first, for the coefficients,

$$\frac{n}{1} = \tfrac{1}{2}; \; \frac{n-1}{2} = -\tfrac{1}{4}; \; \frac{n-2}{3} = -\tfrac{3}{6}; \; \frac{n-3}{4} = -\tfrac{5}{8}; \; \frac{n-4}{.\,5} =$$

$$-\tfrac{7}{10}; \; \frac{n-5}{6} = -\tfrac{9}{12}. \quad \text{Then,} \; a^n = a^{\frac{1}{2}} = \sqrt{a} \; \text{and} \; a^{n-1} =$$

$\dfrac{1}{\sqrt{a}}; \; a^{n-2} = \dfrac{1}{a\sqrt{a}}; \; a^{-3} = \dfrac{1}{a^2\sqrt{a}}$, &c. or we might express

those powers of a in the following manner: $a^n = \sqrt{a}; \; a^{n-1}$

$$= \frac{\sqrt{a}}{a}; \; a^{n-2} = \frac{a^n}{a^2} = \frac{\sqrt{a}}{a^2}; \; a^{n-3} = \frac{a^n}{a^3} = \frac{\sqrt{a}}{a^3}; \; a^{n-4} = \frac{a^n}{a^4} =$$

$\dfrac{\sqrt{a}}{a^4}$, &c.

364. This being laid down, the square root of $a + b$ may be expressed in the following manner:

$$\sqrt{(a + b)} = \sqrt{a} + \tfrac{1}{2}b\frac{\sqrt{a}}{a} - \tfrac{1}{2} \cdot \tfrac{1}{4}b^2\frac{\sqrt{a}}{aa} + \tfrac{1}{2} \cdot \tfrac{1}{4} \cdot \tfrac{3}{6}b^3\frac{\sqrt{a}}{a^3}$$

$$-\tfrac{1}{2} \cdot \tfrac{1}{4} \cdot \tfrac{3}{6} \cdot \tfrac{5}{8}b^4\frac{\sqrt{a}}{a^4}, \text{&c.}$$

365. If a therefore be a square number, we may assign the value of $\sqrt{a}$, and, consequently, the square root of $a + b$ may be expressed by an infinite series, without any radical sign.

Let, for example, $a = c^2$, we shall have $\sqrt{a} = c$; then

$$\sqrt{(c^2 + b)} = c + \tfrac{1}{2} \cdot \frac{b}{c} - \tfrac{1}{8} \cdot \frac{b^2}{c^3} + \tfrac{1}{16} \cdot \frac{b^3}{c^5} - \tfrac{5}{128} \cdot \frac{b^4}{c^7},$$

&c.

We see, therefore, that there is no number, whose square root we may not extract in this manner; since every number may be resolved into two parts, one of which is a square represented by c^2. If, for example, the square root of 6 be required, we make $6 = 4 + 2$, consequently, $c^2 = 4$, $c = 2$, $b = 2$, whence results

$$\sqrt{6} = 2 + \tfrac{1}{2} - \tfrac{1}{16} + \tfrac{1}{64} - \tfrac{5}{1024}, \text{&c.}$$

If we take only the two leading terms of this series, we shall have $2\tfrac{1}{2} = \tfrac{5}{2}$, the square of which, $\tfrac{25}{4}$, is $\tfrac{1}{4}$ greater than 6; but if we consider three terms, we have $2\tfrac{7}{16} = \tfrac{39}{16}$, the square of which, $\tfrac{1521}{256}$, is still $\tfrac{15}{256}$ too small.

366. Since, in this example, $\tfrac{5}{2}$ approaches very nearly to the true value of $\sqrt{6}$, we shall take for 6 the equivalent quantity $\tfrac{25}{4} - \tfrac{1}{4}$; thus $c^2 = \tfrac{25}{4}$; $c = \tfrac{5}{2}$; $b = \tfrac{1}{4}$; and calculating only the two leading terms, we find $\sqrt{6} = \tfrac{5}{2} + \tfrac{1}{2} \cdot$

$\dfrac{-\tfrac{1}{4}}{\tfrac{5}{2}} = \tfrac{5}{2} - \tfrac{1}{2} \cdot \dfrac{\tfrac{1}{4}}{\tfrac{5}{2}} = \tfrac{5}{2} - \tfrac{1}{20} = \tfrac{49}{20}$; the square of which

fraction being $\frac{2401}{400}$, it exceeds the square of $\sqrt{6}$ only by $\frac{1}{400}$.

Now, making $6 = \frac{2401}{400} - \frac{1}{400}$, so that $c = \frac{49}{20}$ and $b = -\frac{1}{400}$; and still taking only the two leading terms, we have $\sqrt{6} = \frac{49}{20} + \frac{1}{2} \cdot \dfrac{-\frac{1}{400}}{\frac{49}{20}} = \frac{49}{20} - \frac{1}{2} \cdot \dfrac{\frac{1}{400}}{\frac{49}{20}} = \frac{49}{20} - \frac{1}{1960}$ $= \frac{4801}{1960}$, the square of which is $\frac{23049601}{3841600}$; and 6, when reduced to the same denominator, is $= \frac{23049600}{3841600}$; the error therefore is only $\frac{1}{3841600}$.

367. In the same manner, we may express the cube root of $a + b$ by an infinite series; for since $\sqrt[3]{(a+b)} = (a+b)^{\frac{1}{3}}$, we shall have in the general formula, $n = \frac{1}{3}$, and for the coefficients,

$\dfrac{n}{1} = \frac{1}{3}; \dfrac{n-1}{2} = -\frac{1}{3}; \dfrac{n-2}{3} = -\frac{5}{9}; \dfrac{n-3}{4} = -\frac{2}{3}; \dfrac{n-4}{5} = -\frac{11}{15}$, &c. and, with regard to the powers of a, we shall have

$a^n = \sqrt[3]{a}; \; a^{n-1} = \dfrac{\sqrt[3]{a}}{a}; \; a^{n-2} = \dfrac{\sqrt[3]{a}}{a^2}; \; a^{n-3} = \dfrac{\sqrt[3]{a}}{a^3}$, &c. then

$\sqrt[3]{(a + b)} = \sqrt[3]{a} + \frac{1}{3} \cdot b\dfrac{\sqrt[3]{a}}{a} - \frac{1}{9} \cdot b^2\dfrac{\sqrt[3]{a}}{a^2} + \frac{5}{81} \cdot b^3\dfrac{\sqrt[3]{a}}{a^3} - \frac{10}{243}$

$\cdot b^4\dfrac{\sqrt[3]{a}}{a^4}$, &c.

368. If a therefore be a cube, or $a = c^3$, we have $\sqrt[3]{a} = c$, and the radical signs will vanish; for we shall have

$\sqrt[3]{(c^3 + b)} = c + \frac{1}{3} \cdot \dfrac{b}{c^2} - \frac{1}{9} \cdot \dfrac{b^2}{c^5} + \frac{5}{81} \cdot \dfrac{b^3}{c^8} - \frac{10}{243} \cdot \dfrac{b^4}{c^{11}}$ $+$, &c.

369. We have therefore arrived at a formula, which will enable us to find, *by approximation,* the cube root of any number; since every number may be resolved into two parts, as $c^3 + b$, the first of which is a cube.

If we wish, for example, to determine the cube root of 2, we represent 2 by $1 + 1$, so that $c = 1$ and $b = 1$; consequently, $\sqrt[3]{2} = 1 + \frac{1}{3} - \frac{1}{9} + \frac{5}{81}$, &c. The two leading terms of this series make $1\frac{1}{3} = \frac{4}{3}$, the cube of which $\frac{64}{27}$ is too great by $\frac{10}{27}$: let us therefore make $2 = \frac{64}{27} - \frac{10}{27}$, we have $c = \frac{4}{3}$ and $b = -\frac{10}{27}$, and consequently $\sqrt[3]{2} = \frac{4}{3} + \frac{1}{3} \cdot \dfrac{-\frac{10}{27}}{\frac{16}{9}}$: these two terms give $\frac{4}{3} - \frac{5}{72} = \frac{91}{72}$, the cube of which is $\frac{753571}{373248}$: but, $2 = \frac{746496}{373248}$, so that the error is $\frac{7075}{373248}$; and in this way we might still approximate, the faster in proportion as we take a greater number of terms *.

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* In the Philosophical Transactions for 1694, Dr. Halley has given a very elegant and general method for extracting roots of