this latter term to such as are composed of two or more of the smallest numbers of either kind; so that the simple numbers of the first kind will be 2, 3, 5, 11, 29; and the compound numbers of the same class will be 8, 12, 14, 18, 20, 27, 30, 32, 35, 40, 45, 48, 50, &c.

The simple numbers of the second class will be 1, 7, 31 and all the rest of this class will be compound numbers, namely, 4, 6, 9, 10, 15, 16, 22, 24, 25, 28, 33, 36, 40, 42, 49.

CHAP. XII.

Of the Transformation *of the* Formula $ax^2 + cy^2$ *into* Squares, *and higher* Powers.

181. We have seen that it is frequently impossible to reduce numbers of the form $ax^2 + cy^2$ to squares; but whenever it is possible, we may transform this formula into another, in which $a = 1$.

For example, the formula $2p^2 - q^2$ may become a square; for, as it may be represented by

$$(2p + q)^2 - 2(p + q)^2,$$

we have only to make $2p + q = x$, and $p + q = y$, and we shall get the formula $x^2 - 2y^2$, in which $a = 1$, and $c = 2$. A similar transformation always takes place, whenever such formulæ can be made squares. Thus, when it is required to transform the formula $ax^2 + cy^2$ into a square, or into a higher power, (provided it be even) we may, without hesitation, suppose $a = 1$, and consider the other cases as impossible.

182. Let, therefore, the formula $x^2 + cy^2$ be proposed, and let it be required to make it a square. As it is composed of the factors $(x + y\sqrt{-c}) \times (x - y\sqrt{-c})$, these factors must either be squares, or squares multiplied by the same number. For, if the product of two numbers, for example, pq, must be a square, we must have $p = r^2$, and $q = s^2$; that is to say, each factor is of itself a square; or $p = mr^2$, and $q = ms^2$; and therefore these factors are squares multiplied both by the same number. For which reason, let us make $x + y\sqrt{-c} = m(p + q\sqrt{-c})^2$; it will follow that $x - y\sqrt{-c} = m(p - q\sqrt{-c})^2$, and we shall have $x^2 + cy^2 = m^2(p^2 + cq^2)^2$, which is a square.

Farther, in order to determine x and y, we have the equations $x + y\sqrt{-c} = mp^2 + 2mpq\sqrt{-c} - mcq^2$, and $x - y\sqrt{-c} = mp^2 - 2mpq\sqrt{-c} - mcq^2$; in which x is necessarily equal to the rational part, and $y\sqrt{-c}$ to the irrational part; so that $x = mp^2 - mcq^2$, and $y\sqrt{-c} = 2mpq\sqrt{-c}$, or $y = 2mpq$; and these are the values of x and y that will transform the expression $x^2 + cy^2$ into a square, $m^2(p^2 + cq^2)^2$, the root of which is $mp^2 + mcq^2$.

183. If the numbers x and y have not a common divisor, we must make $m = 1$. Then, in order that $x^2 + cy^2$ may become a square, it will be sufficient to make $x = p^2 - cq^2$, and $y = 2pq$, which will render the formula equal to the square $(p^2 + cq^2)^2$.

Or, instead of making $x = p^2 - cq^2$, we may also suppose $x = cq^2 - p^2$, since the square x^2 is still left the same.

Besides, the same formulæ having been already found by methods altogether different, there can be no doubt with regard to the accuracy of the method which we have now employed. In fact, if we wish to make $x^2 + cy^2$ a square, we suppose, by the former method, the root to be $x + \frac{py}{q}$, and find $x^2 + cy^2 = x^2 + \frac{2pxy}{q} + \frac{p^2y^2}{q^2}$.

Expunge the x^2, divide the other terms by y, multiply by q^2, and we shall have

$$cq^2y = 2pqx + p^2y\text{; or } cq^2y - p^2y = 2pqx.$$

Lastly, dividing by $2pq$, and also by y, there results $\frac{x}{y} = \frac{cq^2 - p^2}{2pq}$. Now, as x and y, as well as p and q, are to have no common divisor, we must make x equal to the numerator, and y equal to the denominator, and hence we shall obtain the same results as we have already found, namely, $x = cq^2 - p^2$, and $y = 2pq$.

184. This solution will hold good, whether the number c be positive or negative; but, farther, if this number itself had factors, as, for instance, the formula $x^2 + acy^2$, we should not only have the preceding solution, which gives $x = acq^2 - p^2$, and $y = 2pq$, but this also, namely, $x = cq^2 - ap^2$, and $y = 2pq$; for, in this last case, we have, as in the other,

$$x^2 + acy^2 = c^2q^4 + 2acp^2q^2 + a^2p^4 = (cq^2 + ap^2)^2;$$

which takes place also when we make $x = ap^2 - cq^2$, because the square x^2 remains the same.

This new solution is also obtained from the last method, in the following manner:

If we make $x + y\sqrt{-ac} = (p\sqrt{a} + q\sqrt{-c})^2$, and $x - y\sqrt{-ac} = (p\sqrt{a} - q\sqrt{-c})^2$, we shall have $x^2 + acy^2 = (ap^2 + cq^2)^2$, and, consequently, equal to a square. Farther, because

$$x + y\sqrt{-ac} = ap^2 + 2pq\sqrt{-ac} - cq^2, \text{ and}$$
$$x - y\sqrt{-ac} = ap^2 - 2pq\sqrt{-ac} - cq^2,$$

we find $x = ap^2 - cq^2$, and $y = 2pq$.

It is farther evident, that if the number ac be resolvible into two factors, in a greater number of ways, we may also find a greater number of solutions.

185. Let us illustrate this by means of some determinate formulæ; and, first, if the formula $x^2 + y^2$ must become a square, we have $ac = 1$; so that $x = p^2 - q^2$, and $y = 2pq$; whence it follows that $x^2 + y^2 = (p^2 + q^2)^2$.

If we would have $x^2 - y^2 = \square$; we have $ac = -1$; so that we shall take $x = p^2 + q^2$, and $y = 2pq$, and there will result $x^2 - y^2 = (p^2 - q^2)^2 = \square$.

If we would have the formula $x^2 + 2y^2 = \square$, we have $ac = 2$; let us therefore take $x = p^2 - 2q^2$, or $x = 2p^2 - q^2$, and $y = 2pq$, and we shall have

$$x^2 + 2y^2 = (p^2 + q^2)^2, \text{ or } x^2 + 2y^2 = (2p^2 + q^2)^2.$$

If, in the fourth place, we would have $x^2 - 2y^2 = \square$, in which $ac = -2$, we shall have $x = p^2 + 2q^2$, and $y = 2pq$; therefore $x^2 - 2y^2 = (p^2 - 2q^2)^2$.

Lastly, let us make $x^2 + 6y^2 = \square$. Here we shall have $ac = 6$; and, consequently, either $a = 1$, and $c = 6$, or $a = 2$, and $c = 3$. In the first case, $x = p^2 - 6q^2$, and $y = 2pq$; so that $x^2 + 6y^2 = (p^2 + 6q^2)^2$; in the second, $x = 2p^2 - 3q^2$, and $y = 2pq$; whence

$$x^2 + 6y^2 = (2p^2 + 3q^2)^2.$$

186. But let the formula $ax^2 + cy^2$ be proposed to be transformed into a square. We know beforehand, that this cannot be done, except we already know a case, in which this formula really becomes a square; but we shall find this given case to be, when $x = f$, and $y = g$; so that $af^2 + cg^2 = h^2$; and we may observe, that this formula can be transformed into another of the form $t^2 + acu^2$, by making

$$t = \frac{afx + cgy}{h}, \text{ and } u = \frac{gx - fy}{h}; \text{ for if}$$

$$t^2 = \frac{a^2f^2x^2 + 2acfgxy + c^2g^2y^2}{h^2}, \text{ and}$$

$$u^2 = \frac{g^2x^2 - 2fgxy + f^2y^2}{h^2}, \text{ we have}$$

$$t^2 + acu^2 = \frac{a^2f^2x^2 + c^2g^2y^2 + acg^2x^2 + acf^2y^2}{h^2} =$$

$$\frac{ax^2(af^2 + cg^2) + cy^2(af^2 + cg^2)}{h_2};$$

also, since $af^2 + cg^2 = h^2$, we have $t^2 + acu^2 = ax^2 + cy^2$. Thus, we have given easy rules for transforming the expression $t^2 + acu^2$ into a square, to which we have now reduced the formula proposed, $ax^2 + cy^2$.

187. Let us proceed farther, and see how the formula $ax^2 + cy^2$, in which x and y are supposed to have no common divisor, may be reduced to a cube. The rules already given are by no means sufficient for this; but the method which we have last explained applies here with the greatest success: and what is particularly worthy of observation, is, that the formula may be transformed into a cube, whatever numbers a and c are; which could not take place with regard to squares, unless we already knew a case, and which does not take place with regard to any of the other even powers; but, on the contrary, the solution is always possible for the odd powers, such as the third, the fifth, the seventh, &c.

188. Whenever, therefore, it is required to reduce the formula $ax^2 + cy^2$ to a cube, we may suppose, according to the method which we have already employed, that

$$x\sqrt{a} + y\sqrt{-c} = (p\sqrt{a} + q\sqrt{-c})^3, \text{ and}$$
$$x\sqrt{a} - y\sqrt{-c} = (p\sqrt{a} - q\sqrt{-c})^3;$$

the product $(ap^2 + cq^2)^3$, which is a cube, will be equal to the formula $ax^2 + cy^2$. But it is required, also, to determine rational values for x and y, and fortunately we succeed. If we actually take the two cubes that have been pointed out, we have the two equations

$$x\sqrt{a} + y\sqrt{-c} = ap^3\sqrt{a} + 3ap^2q\sqrt{-c} - 3cpq^2\sqrt{a} - cq^3\sqrt{-c}, \text{ and}$$
$$x\sqrt{a} - y\sqrt{-c} = ap^3\sqrt{a} - 3ap^2q\sqrt{-c} - 3cpq^2\sqrt{a} + cq^3\sqrt{-c};$$

from which it evidently follows, that

$$x = ap^3 - 3cpq^2, \text{ and } y = 3ap^2q - cq^3.$$

For example, let two squares x^2, and y^2, be required, whose sum, $x^2 + y^2$, may make a cube. Here, since $a = 1$, and $c = 1$, we shall have $x = p^3 - 3pq^2$, and $y = 3p^2q - q^3$, which gives $x^2 + y^2 = (p^2 + q^2)^3$. Now, if $p = 2$, and $q = 1$, we find $x = 2$, and $y = 11$; wherefore

$$x^2 + y^2 = 125 = 5^3.$$

189. Let us also consider the formula $x^2 + 3y^2$, for the purpose of making it equal to a cube. As we have, in this case, $a = 1$, and $c = 3$, we find

$$x = p^3 - 9pq^2, \text{ and } y = 3p^2q - 3q^3,$$

whence $x^2 + 3y^2 = (p^2 + 3q^2)^3$. This formula occurs very frequently; for which reason we shall here give a Table of the easiest cases.

p	q	x	y	$x^2 + 3y^2$
1	1	8	0	$64 = 4^3$
2	1	10	9	$343 = 7^3$
1	2	35	18	$2197 = 13^3$
3	1	0	24	$1728 = 12^3$
1	3	80	72	$21952 = 28^3$
3	2	81	30	$9261 = 21^3$
2	3	154	45	$29791 = 31^3$

190. If the question were not restricted to the condition, that the numbers x and y must have no common divisor, it would not be attended with any difficulty; for if $ax^2 + cy^2$ were required to be a cube, we should only have to make $x = tz$, and $y = uz$, and the formula would become $at^2z^2 + cu^2z^2$; which we might make equal to the cube $\frac{z^3}{v^3}$, and should immediately find $z = v^3(at^2 + cu^2)$. Consequently, the values sought of x and y would be $x = tv^3(at^2 + cu^2)$, and $y = uv^3(at^2 + cu^2)$, which, beside the cube v^3, have also the quantity $at^2 + cu^2$ for a common divisor; so that this solution immediately gives

$$ax^2 + cy^2 = v^6(at^2 + cu^2)^2 \times (at^2 + cu^2) = v^6(at^2 + cu^2)^3,$$

which is evidently the cube of $v^2(at^2 + cu^2)$.

191. This last method, which we have made use of, is so much the more remarkable, as we are brought to solutions, which absolutely required numbers rational and integer, by means of irrational, and even imaginary quantities; and, what is still more worthy of attention, our method cannot be applied to those cases, in which the irrationality vanishes. For example, when the formula $x^2 + cy^2$ must become a cube, we can only infer from it, that its two irrational factors, $x + y\sqrt{-c}$, and $x - y\sqrt{-c}$, must likewise be cubes; and since x and y have no common divisor, these factors cannot have any. But if the radicals were to disappear, as in the case of $c = -1$, this principle would no

longer exist; because the two factors, which would then be $x + y$, and $x - y$, might have common divisors, even when x and y had none; as would be the case, for example, if both these letters expressed odd numbers.

Thus, when $x^2 - y^2$ must become a cube, it is not necessary that both $x + y$, and $x - y$, should of themselves be cubes; but we may suppose $x + y = 2p^3$, and $x - y = 4q^3$; and the formula $x^2 - y^2$ will undoubtedly become a cube, since we shall find it to be $8p^3q^3$, the cube root of which is $2pq$. We shall farther have $x = p^3 + 2q^3$, and $y = p^3 - 2q^3$. On the contrary, when the formula $ax^2 + cy^2$ is not resolvible into two rational factors, we cannot find any other solutions beside those which have been already given.

192. We shall illustrate the preceding investigations by some curious examples.

Question 1. Required a square, x^2, in integer numbers, and such, that, by adding 4 to it, the sum may be a cube. The condition is answered when $x^2 = 121$; but we wish to know if there are other similar cases.

As 4 is a square, we shall first seek the cases in which $x^2 + y^2$ becomes a cube. Now, we have found one case, namely, if $x = p^3 - 3pq^2$, and $y = 3p^2q - q^3$: therefore, since $y^2 = 4$, we have $y = \pm 2$, and, consequently, either $3p^2q - q^3 = + 2$, or $3p^2q - q^3 = - 2$. In the first case, we have $q(3p^2 - q^2) = 2$, so that q is a divisor of 2.

This being laid down, let us first suppose $q = 1$, and we shall have $3p^2 - 1 = 2$; therefore $p = 1$; whence $x = 2$, and $x^2 = 4$.

If, in the second place, we suppose $q = 2$, we have $6p^2 - 8 = \pm 2$; admitting the sign $+$, we find $6p^2 = 10$, and $p^2 = \frac{5}{3}$; whence we should get an irrational value of p, which could not apply here; but if we consider the sign $-$, we have $6p^2 = 6$, and $p = 1$; therefore $x = 11$: and these are the only possible cases; so that 4, and 121, are the only two squares, which, added to 4, give cubes.

193. *Question* 2. Required, in integer numbers, other squares, beside 25, which, added to 2, give cubes.

Since $x^2 + 2$ must become a cube, and since 2 is the double of a square, let us first determine the cases in which $x^2 + 2y^2$ becomes a cube; for which purpose we have, by Article 188, in which $a = 1$, and $c = 2$, $x = p^3 - 6pq^2$, and $y = 3p^2q - 2q^3$; therefore, since $y = \pm 1$, we must have $3p^2q - q^3$, or $q(3p^2 - 2q^2) = \pm 1$; and, consequently, q must be a divisor of 1.

Therefore let $q = 1$, and we shall have $3p^2 - 2 = \pm 1$.

If we take the upper sign, we find $3p^2 = 3$, and $p = 1$; whence $x = 5$: and if we adopt the other sign, we get a value of p, which being irrational, is of no use; it follows, therefore, that there is no square, except 25, which has the property required.

194. *Question* 3. Required squares, which, multiplied by 5, and added to 7, may produce cubes; or it is required that $5x^2 + 7$ should be a cube.

Let us first seek the cases in which $5x^2 + 7y^2$ becomes a cube. By Article 188, a being equal to 5, and c equal 7, we shall find that we must have $x = 5p^3 - 21pq^2$, and $y = 15p^2q - 7q^3$; so that in our example y being $= \pm 1$, we have $15p^2q - 7q^3 = q(15p^2 - 7q^2) = \pm 1$; therefore q must be a divisor of 1; that is to say, $q = \pm 1$; consequently, we shall have $15p^2 - 7 = \pm 1$; from which, in both cases, we get irrational values for p: but from which we must not, however, conclude that the question is impossible, since p and q might be such fractions, that $y = 1$, and that x would become an integer; and this is what really happens; for if $p = \frac{1}{2}$, and $q = \frac{1}{2}$, we find $y = 1$, and $x = 2$; but there are no other fractions which render the solution possible.

195. *Question* 4. Required squares in integer numbers, the double of which, diminished by 5, may be a cube; or it is required that $2x^2 - 5$ may be a cube.

If we begin by seeking the satisfactory cases for the formula $2x^2 - 5y^2$, we have, in the 188th Article, $a = 2$, and $c = -5$; whence $x = 2p^3 + 15pq^2$, and $y = 6p^2q + 5q^3$: so that, in this case, we must have $y = \pm 1$; consequently,

$$6p^2q + 5q^3 = q(6p^2 + 5q^2) = \pm 1;$$

and as this cannot be, either in integer numbers, or even in fractions, the case becomes very remarkable, because there is, notwithstanding, a satisfactory value of x; namely, $x = 4$; which gives $2x^2 - 5 = 27$, or equal to the cube of 3. It will be of importance to investigate the cause of this peculiarity.

196. It is not only possible, as we see, for the formula $2x^2 - 5y^2$ to be a cube; but, what is more, the root of this cube has the form $2p^2 - 5q^2$, as we may perceive by making $x = 4, y = 1, p = 2$, and $q = 1$; so that we know a case in which $2x^2 - 5y^2 = (2p^2 - 5q^2)^3$, although the two factors of $2x^2 - 5y^2$, namely, $x\sqrt{2} + y\sqrt{5}$, and $x\sqrt{2} - y\sqrt{5}$, which, according to our method, ought to be the cubes of $p\sqrt{2} + q\sqrt{5}$, and of $p\sqrt{2} - q\sqrt{5}$, are not cubes; for, in our case, $x\sqrt{2} + y\sqrt{5} = 4\sqrt{2} + \sqrt{5}$; whereas

$$(p\sqrt{2} + q\sqrt{5})^3 = (2\sqrt{2} + \sqrt{5})^3 = 46\sqrt{2} + 29\sqrt{5},$$
which is by no means the same as $4\sqrt{2} + \sqrt{5}$.

But it must be remarked, that the formula $r^2 - 10s^2$ may become 1, or -1, in an infinite number of cases; for example, if $r = 3$, and $s = 1$, or if $r = 19$, and $s = 6$: and this formula, multiplied by $2p^2 - 5q^2$, reproduces a number of this last form.

Therefore, let $f^2 - 10g^2 = 1$; and, instead of supposing, as we have hitherto done, $2x^2 - 5y^2 = (2p^2 - 5q^2)^3$, we may suppose, in a more general manner,

$$2x^2 - 5y^2 = (f^2 - 10g^2) \times (2p^2 - 5q^2)^3;$$

so that, taking the factors, we shall have

$$x\sqrt{2} \pm y\sqrt{5} = (f \pm g\sqrt{10}) \times (p\sqrt{2} \pm q\sqrt{5})^3.$$

Now, $(p\sqrt{2} \pm q\sqrt{5})^3 = (2p^3 + 15pq^2)\sqrt{2} \pm (6p^2q + 5q^3)\sqrt{5}$; and if, in order to abridge, we write $\text{A}\sqrt{2} + \text{B}\sqrt{5}$ instead of this quantity, and multiply by $f + g\sqrt{10}$, we shall have $\text{A}f\sqrt{2} + \text{B}f\sqrt{5} + 2\text{A}g\sqrt{5} + 5\text{B}g\sqrt{2}$ to make equal to $x\sqrt{2} + y\sqrt{5}$; whence results $x = \text{A}f + 5\text{B}g$, and $y = \text{B}f + 2\text{A}g$. Now, since we must have $y = \pm 1$, it is not absolutely necessary that $6p^2q + 5q^3 = 1$; on the contrary, it is sufficient that the formula $\text{B}f + 2\text{A}g$, that is to say, that $f(6p^2q + 5q^3) + 2g(2p^3 + 15pq^2)$ becomes $= \pm 1$; so that f and g may have several values. For example, let $f = 3$, and $g = 1$, the formula $18p^2q + 15q^3 + 4p^3 + 30pq^2$ must become ± 1; that is,

$$4p^3 + 18p^2q + 30pq^2 + 15q^3 = \pm 1.$$

197. The difficulty, however, of determining all the possible cases of this kind, exists only in the formula $ax^2 + cy^2$, when the number c is negative; and the reason is, that this formula, namely, $x^2 - acy^2$, which depends on it, may then become 1; which never happens when c is a positive number, because, $x^2 + cy^2$, or $x^2 + acy^2$, always gives greater numbers, the greater the values we assign to x and y. For which reason, the method we have explained cannot be successfully employed, except in those cases in which the two numbers a and c have positive values.

198. Let us now proceed to the fourth degree. Here we shall begin by observing, that if the formula $ax^2 + cy^2$ is to be changed into a biquadrate, we must have $a = 1$; for it would not be possible even to transform the formula into a square (Art. 181); and, if this were possible, we might also give it the form $t^2 + acu^2$; for which reason we shall extend the question only to this last formula, which may be reduced to the former, $x^2 + cy^2$, by supposing $a = 1$. This

being laid down, we have to consider what must be the nature of the values of x and y, in order that the formula $x^2 + cy^2$ may become a biquadrate. Now, it is composed of the two factors $(x + y\sqrt{-c}) \times (x - y\sqrt{-c})$; and each of these factors must also be a biquadrate of the same kind; therefore we must make $x + y\sqrt{-c} = (p + q\sqrt{-c})^4$, and $x - y\sqrt{-c} = (p - q\sqrt{-c})^4$, whence it follows, that the formula proposed becomes equal to the biquadrate $(p^2 + cq^2)^4$. With regard to the values of x and y, they are easily determined by the following analysis:

$x + y\sqrt{-c} = p^4 + 4p^3q\sqrt{-c} - 6cp^2q^2 + c^2q^4 - 4cpq^3\sqrt{-c}$,
$x - y\sqrt{-c} = p^4 - 4p^3q\sqrt{-c} - 6cp^2q^2 + c^2q^4 + 4cpq^3\sqrt{-c}$,
whence, $x = p^4 - 6cp^2q^2 + c^2q^4$; and $y = 4p^3q - 4cpq^3$.

199. So that when $x^2 + y^2$ is a biquadrate, because $c = 1$, we have

$$x = p^4 - 6p^2q^2 + q^4;\ \text{and}\ y = 4p^3q - 4pq^3;$$

so that $x^2 + y^2 = (p^2 + q^2)^4$.

Suppose, for example, $p = 2$, and $q = 1$; we shall then find $x = 7$, and $y = 24$; whence $x^2 + y^2 = 625 = 5^4$.

If $p = 3$, and $q = 2$, we obtain $x = 119$, and $y = 120$, which gives $x^2 + y^2 = 13^4$.

200. Whatever be the even power into which it is required to transform the formula $ax^2 + cy^2$, it is absolutely necessary that this formula be always reducible to a square; and for this purpose, it is sufficient that we already know one case in which it happens; for we may then transform the formula, as has been seen, into a quantity of the form $t^2 + acu^2$, in which the first term t^2 is multiplied only by 1; so that we may consider it as contained in the expression $x^2 + cy^2$; and in a similar manner, we may always give to this last expression the form of a sixth power, or of any higher even power.

201. This condition is not requisite for the odd powers; and whatever numbers a and c be, we may always transform the formula $ax^2 + cy^2$ into any odd power. Let the fifth, for instance, be demanded; we have only to make

$$x\sqrt{a} + y\sqrt{-c} = (p\sqrt{a} + q\sqrt{-c})^5,\ \text{and}$$
$$x\sqrt{a} - y\sqrt{-c} = (p\sqrt{a} - q\sqrt{-c})^5,$$

and we shall evidently obtain $ax^2 + cy^2 = (ap^2 + cq^2)^5$. Farther, as the fifth power of $p\sqrt{a} + q\sqrt{-c}$ is $= a^2p^5\sqrt{a} + 5a^2p^4q\sqrt{-c} - 10acp^3q^2\sqrt{a} - 10acp^2q^3\sqrt{-c} + 5c^2pq^4\sqrt{a} + c^2q^5\sqrt{-c}$, we shall, with the same facility, find

$$x = a^2p^5 - 10acp^3q^2 + 5c^2pq^4,\ \text{and}$$
$$y = 5a^2p^4q - 10acp^2q^3 + c^2q^5.$$

If it is required, therefore, that the sum of two squares, such as $x^2 + y^2$, may be also a fifth power, we shall have $a = 1$, and $c = 1$; therefore, $x = p^5 - 10p^3q^2 + 5pq^4$; and $y = 5p^4q - 10p^2q^3 + q^5$; and, farther, making $p = 2$, and $q = 1$, we shall find $x = 38$, and $q = 41$; consequently,

$$x^2 + y^2 = 3125 = 5^5.$$

CHAP. XIII.

Of some Expressions *of the* Form $ax^4 + by^4$, *which are not reducible to* Squares.

202. Much labor has been formerly employed by some mathematicians to find two biquadrates, whose sum or difference might be a square, but in vain; and at length it has been demonstrated, that neither the formula $x^4 + y^4$, nor the formula $x^4 - y^4$, can become a square, except in these evident cases; first, when $x = 0$, or $y = 0$, and, secondly, when $y = x$. This circumstance is the more remarkable, because it has been seen, that we can find an infinite number of answers, when the question involves only simple squares.

203. We shall give the demonstration to which we have just alluded; and, in order to proceed regularly, we shall previously observe, that the two numbers x and y may be considered as prime to each other: for, if these numbers had a common divisor, so that we could make $x = dp$, and $y = dq$, our formulæ would become $d^4p^4 + d^4q^4$, and $d^4p^4 - d^4q^4$: which formulæ, if they were squares, would remain squares after being divided by d^4; therefore, the formulæ $p^4 + q^4$, and $p^4 - q^4$, also, in which p and q have no longer any common divisor, would be squares; consequently, it will be sufficient to prove, that our formulæ cannot become squares in the case of x and y being prime to each other, and our demonstration will, consequently, extend to all the cases, in which x and y have common divisors.

204. We shall begin, therefore, with the sum of two biquadrates; that is, with the formula $x^4 + y^4$, considering x and y as numbers that are prime to each other: and we have to prove, that this formula becomes a square only in the cases above-mentioned; in order to which, we shall enter