last two are $x = 2 \pm \sqrt{8}$; so that the four roots sought will be,

$$1.\ x = 1 + \sqrt{5}, \qquad 2.\ x = 1 - \sqrt{5},$$
$$3.\ x = 2 + \sqrt{8}, \qquad 4.\ x = 2 - \sqrt{8}.$$

Consequently, the four factors of our equation will be $(x - 1 - \sqrt{5}) \times (x - 1 + \sqrt{5}) \times (x - 2 - \sqrt{8}) \times (x - 2 + \sqrt{8})$, and their actual multiplication produces the given equation; for the first two being multiplied together, give $x^2 - 2x - 4$, and the other two give $x^2 - 4x - 4$: now, these products multiplied together, make $x^4 - 6x^3 + 24x + 16$, which is the same equation that was proposed.

## CHAP. XIV.

### *Of the* Rule *of* Bombelli *for reducing the* Resolution *of* Equations *of the* Fourth Degree *to that of* Equations *of the* Third Degree.

765. We have already shewn how equations of the third degree are resolved by the rule of Cardan; so that the principal object, with regard to equations of the fourth degree, is to reduce them to equations of the third degree. For it is impossible to resolve, generally, equations of the fourth degree, without the aid of those of the third; since, when we have determined one of the roots, the others always depend on an equation of the third degree. And hence we may conclude, that the resolution of equations of higher dimensions presupposes the resolution of all equations of lower degrees.

766. It is now some centuries since Bombelli, an Italian, gave a rule for this purpose, which we shall explain in this chapter *.

Let there be given the general equation of the fourth degree, $x^4 + ax^3 + bx^2 + cx + d = 0$, in which the letters $a$, $b$, $c$, $d$, represent any possible numbers; and let us suppose that this equation is the same as $(x^2 + \frac{1}{2}ax + p)^2 - (qx + r)^2 = 0$, in which it is required to determine the letters $p$, $q$, and $r$, in order that we may obtain the equation

* This rule rather belongs to Louis Ferrari. It is improperly called the Rule of Bombelli, in the same manner as the rule discovered by Scipio Ferreo has been ascribed to Cardan. F. T.

proposed. By squaring, and ordering this new equation, we shall have

$$\begin{aligned} x^4 + ax^3 + \tfrac{1}{4}a^2x^2 &+ apx + p^2 \\ 2px^2 &- 2qrx - r^2 \\ - q^2x^2&. \end{aligned}$$

Now, the first two terms are already the same here as in the given equation; the third term requires us to make $\frac{1}{4}a^2 + 2p - q^2 = b$, which gives $q^2 = \frac{1}{4}a^2 + 2p - b$; the fourth term shews that we must make $ap - 2qr = c$, or $2qr = ap - c$; and, lastly, we have for the last term $p^2 - r^2 = d$, or $r^2 = p^2 - d$. We have therefore three equations which will give the values of $p$, $q$, and $r$.

767. The easiest method of deriving those values from them is the following: if we take the first equation four times, we shall have $4q^2 = a^2 + 8p - 4b$; which equation, multiplied by the last, $r^2 = p^2 - d$, gives

$$4q^2r^2 = 8p^3 + (a^2 - 4b)p^2 - 8dp - d(a^2 - 4b).$$

Farther, if we square the second equation, we have $4q^2r^2 = a^2p^2 - 2acp + c^2$. So that we have two values of $4q^2r^2$, which, being made equal, will furnish the equation

$$8p^3 + (a^2 - 4b)p^2 - 8dp - d(a^2 - 4b) = a^2p^2 - 2acp + c^2,$$

or, bringing all the terms to one side, and arranging,

$$8p^3 - 4bp^2 + (2ac - 8d)p - a^2d + 4bd - c^2 = 0,$$

an equation of the third degree, which will always give the value of $p$ by the rules already explained.

768. Having therefore determined three values of $p$ by the given quantities $a$, $b$, $c$, $d$, when it was required to find only one of those values, we shall also have the values of the two other letters $q$ and $r$; for the first equation will give $q = \sqrt{(\frac{1}{4}a^2 + 2p - b)}$, and the second gives $r = \dfrac{ap - c}{2q}$.

Now, these three values being determined for each given case, the four roots of the proposed equation may be found in the following manner:

This equation having been reduced to the form $(x^2 + \frac{1}{2}ax + p)^2 - (qx + r)^2 = 0$, we shall have

$$(x^2 + \tfrac{1}{2}ax + p)^2 = (qx + r)^2,$$

and, extracting the root, $x^2 + \frac{1}{2}ax + p = qx + r$, or $x^2 + \frac{1}{2}ax + p = -qx - r$. The first equation gives $x^2 = (q - \frac{1}{2}a)x - p + r$, from which we may find two roots; and the second equation, to which we may give the form $x^2 = -(q + \frac{1}{2}a)x - p - r$, will furnish the two other roots.

769. Let us illustrate this rule by an example, and suppose that the equation

$$x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$$

was given. If we compare it with our general formula (at the end of Art. 767.), we have $a = - 10$, $b = 35$, $c = - 50$, $d = 24$; and, consequently, the equation which must give the value of $p$ is

$$8p^3 - 140p^2 + 808p - 1540 = 0, \text{ or}$$
$$2p^3 - 35p^2 + 202p - 385 = 0.$$

The divisors of the last term are 1, 5, 7, 11, &c.; the first of which does not answer; but making $p = 5$, we get $250 - 875 + 1010 - 385 = 0$, so that $p = 5$; and if we farther suppose $p = 7$, we get $686 - 1715 + 1414 - 385 = 0$, a proof that $p = 7$ is the second root. It remains now to find the third root; let us therefore divide the equation by 2, in order to have $p^3 - \frac{35}{2}p^2 + 101p - \frac{385}{2} = 0$, and let us consider that the coefficient of the second term, or $\frac{35}{2}$, being the sum of all the three roots, and the first two making together 12, the third must necessarily be $\frac{11}{2}$.

We consequently know the three roots required. But it may be observed that one would have been sufficient, because each gives the same four roots for our equation of the fourth degree.

770. To prove this, let $p = 5$; we shall then have, by the formula, $\sqrt{(\frac{1}{4}a^2 + 2p - b)}$, $q = \sqrt{(25 + 10 - 35)} = 0$, and $r = \dfrac{-50+50}{0} = \frac{0}{0}$. Now, nothing being determined by this, let us take the third equation,

$$r^2 = p^2 - d = 25 - 24 = 1,$$

so that $r = 1$; our two equations of the second degree will then be:

$$1.\ x^2 = 5x - 4, \qquad 2.\ x^2 = 5x - 6.$$

The first gives the two roots

$$x = \tfrac{5}{2} \pm \sqrt{\tfrac{9}{4}}, \text{ or } x = \frac{5 \pm 3}{2},$$

that is to say, $x = 4$ and $x = 1$.

The second equation gives

$$x = \tfrac{5}{2} \pm \sqrt{\tfrac{1}{4}} = \frac{5 \pm 1}{2},$$

that is to say, $x = 3$, and $x = 2$.

But suppose now $p = 7$, we shall have

$$q = \sqrt{(25 + 14 - 35)} = 2, \text{ and } r = \frac{-70+50}{4} = -5,$$

whence result the two equations of the second degree,

$$1.\ x^2 = 7x - 12, \qquad 2.\ x^2 = 3x - 2;$$

the first gives $x = \frac{7}{2} \pm \sqrt{\frac{1}{4}}$, or $x = \frac{7 \pm 1}{2}$,

so that $x = 4$, and $x = 3$; the second furnishes the root

$$x = \tfrac{3}{2} \pm \sqrt{\tfrac{1}{4}} = \frac{3 \pm 1}{2},$$

and, consequently, $x = 2$, and $x = 1$; therefore by this second supposition the same four roots are found as by the first.

Lastly, the same roots are found, by the third value of $p$, $= \frac{11}{2}$: for, in this case, we have

$$q = \sqrt{(25 + 11 - 35)} = 1, \text{ and } r = \frac{-55+50}{2} = -\tfrac{5}{2};$$

so that the two equations of the second degree become,

$$1.\ x^2 = 6x, \qquad 2.\ x^2 = 4x - 3.$$

Whence we obtain from the first, $x = 3 \pm \sqrt{1}$, that is to say, $x = 4$, and $x = 2$; and from the second, $x = 2 \pm \sqrt{1}$, that is to say, $x = 3$, and $x = 1$, which are the same roots that we originally obtained.

771. Let there now be proposed the equation

$$x^4 - 16x - 12 = 0,$$

in which $a = 0$, $b = 0$, $c = -16$, $d = -12$; and our equation of the third degree will be

$$8p^3 + 96p - 256 = 0, \text{ or } p^3 + 12p - 32 = 0,$$

and we may make this equation still more simple, by writing $p = 2t$; for we have then

$$8t^3 + 24t - 32 = 0, \text{ or } t^3 + 3t - 4 = 0.$$

The divisors of the last term are 1, 2, 4; whence one of the roots is found to be $t = 1$; therefore $p = 2$, $q = \sqrt{4} = 2$, and $r = \frac{16}{4} = 4$. Consequently, the two equations of the second degree are

$$x^2 = 2x + 2, \text{ and } x^2 = -2x - 6;$$

which give the roots

$$x = 1 \pm \sqrt{3}, \text{ and } x = -1 \pm \sqrt{-5}.$$

772. We shall endeavour to render this resolution still more familiar, by a repetition of it in the following example. Suppose there were given the equation

$$x^4 - 6x^3 + 12x^2 - 12x + 4 = 0,$$

which must be contained in the formula

$$(x^2 - 3x + p)^2 - (qx + r)^2 = 0,$$

in the former part of which we have put $-3x$, because $-3$ is half the coefficient $-6$, of the given equation. This formula being expanded, gives

$$x^4 - 6x^3 + (2p + 9 - q^2)x^2 - (6p + 2qr)x + p^2 - r^2 = 0;$$

which, compared with our equation, there will result from that comparison the following equations:

$$1.\ 2p + 9 - q^2 = 12,$$
$$2.\ 6p + 2qr = 12,$$
$$3.\ p^2 - r^2 = 4.$$

The first gives $q^2 = 2p - 3$;
the second, $2qr = 12 - 6p$, or $qr = 6 - 3p$;
the third, $r^2 = p^2 - 4$.

Multiplying $r^2$ by $q^2$, and $p^2 - 4$ by $2p - 3$, we have

$$q^2r^2 = 2p^3 - 3p^2 - 8p + 12;$$

and if we square the value of $qr$, we have

$$q^2r^2 = 36 - 36p + 9p^2;$$

so that we have the equation

$$2p^3 - 3p^2 - 8p + 12 = 9p^2 - 36p + 36, \text{ or}$$
$$2p^3 - 12p^2 + 28p - 24 = 0, \text{ or}$$
$$p^3 - 6p^2 + 14p - 12 = 0,$$

one of the roots of which is $p = 2$; and it follows that $q^2 = 1$, $q = 1$, and $qr - r = 0$. Therefore our equation will be $(x^2 - 3x + 2)^2 = x^2$, and its square root will be $x^2 - 3x + 2 = \pm x$. If we take the upper sign, we have $x^2 = 4x - 2$; and taking the lower sign, we obtain $x^2 = 2x - 2$, whence we derive the four roots $x = 2 \pm \surd 2$, and $x = 1 \pm \surd -1$.

## CHAP. XV.

### *Of a new* Method *of resolving* Equations *of the* Fourth Degree.

773. The rule of Bombelli, as we have seen, resolves equations of the fourth degree by means of an equation of the third degree; but since the invention of that Rule,

another method has been discovered of performing the same resolution: and, as it is altogether different from the first, it deserves to be separately explained *.

774. We suppose that the root of an equation of the fourth degree has the form, $x = \sqrt{p} + \sqrt{q} + \sqrt{r}$, in which the letters $p$, $q$, $r$, express the roots of an equation of the third degree, $z^3 - fz^2 + gz - h = 0$; so that $p + q + r = f$; $pq + pr + qr = g$; and $pqr = h$. This being laid down, we square the assumed formula, $x = \sqrt{p} + \sqrt{q} + \sqrt{r}$, and we obtain

$$x^2 = p + q + r + 2\sqrt{pq} + 2\sqrt{pr} + 2\sqrt{qr};$$

and, since $p + q + r = f$, we have

$$x^2 - f = 2\sqrt{pq} + 2\sqrt{pr} + 2\sqrt{qr};$$

we again take the squares, and find

$$x^4 - 2fx^2 + f^2 = 4pq + 4pr + 4qr + 8\sqrt{p^2qr} + 8\sqrt{pq^2r} + 8\sqrt{pqr^2}.$$

Now, $4pq + 4pr + 4qr = 4g$; so that the equation becomes $x^4 - 2fx^2 + f^2 - 4g = 8\sqrt{pqr} \times (\sqrt{p} + \sqrt{q} + \sqrt{r})$; but $\sqrt{p} + \sqrt{q} + \sqrt{r} = x$, and $pqr = h$, or $\sqrt{pqr} = \sqrt{h}$; wherefore we arrive at this equation of the fourth degree, $x^4 - 2fx^2 - 8x\sqrt{h} + f^2 - 4g = 0$, one of the roots of which is $x = \sqrt{p} + \sqrt{q} + \sqrt{r}$; and in which $p$, $q$, and $r$, are the roots of an equation of the third degree,

$$z^3 - fz^2 + gz - h = 0.$$

775. The equation of the fourth degree, at which we have arrived, may be considered as general, although the second term $x^3y$ is wanting; for we shall afterwards shew, that every complete equation may be transformed into another, from which the second term has been taken away.

Let there be proposed the equation $x^4 - ax^2 - bx - c = 0$, in order to determine its root. This we must first compare with the formula, in order to obtain the values of $f$, $g$, and $h$; and we shall have,

1. $2f = a$, and, consequently, $f = \frac{a}{2}$;

2. $8\sqrt{h} = b$, so that $h = \frac{b^2}{64}$;

3. $f^2 - 4g = -c$, or $\frac{a^2}{4} - 4g + c = 0$,

or $\frac{1}{4}a^2 + c = 4g$; consequently, $g = \frac{1}{16}a^2 + \frac{1}{4}c$.

* This method was the invention of Euler himself. He has explained it in the sixteenth volume of the Ancient Commentaries of Petersburg. F. T.

776. Since, therefore, the equation

$$x^4 - ax^2 - bx - c = 0,$$

gives the values of the letters $f$, $g$, and $h$, so that

$$f = \tfrac{1}{2}a,\ g = \tfrac{1}{16}a^2 + \tfrac{1}{4}c, \text{ and } h = \tfrac{1}{64}b^2, \text{ or } \sqrt{h} = \tfrac{1}{8}b,$$

we form from these values the equation of the third degree $z^3 - fz^2 + gz - h = 0$, in order to obtain its roots by the known rule. And if we suppose those roots, 1. $z = p$, 2. $z = q$, 3. $z = r$, one of the roots of our equation of the fourth degree must be, by the supposition, Art. 774,

$$x = \sqrt{p} + \sqrt{q} + \sqrt{r}.$$

777. This method appears at first to furnish only one root of the given equation; but if we consider that every sign $\sqrt{}$ may be taken negatively, as well as positively, we shall immediately perceive that this formula contains all the four roots.

Farther, if we chose to admit all the possible changes of the signs, we should have eight different values of $x$, and yet four only can exist. But it is to be observed, that the product of those three terms, or $\sqrt{pqr}$, must be equal to $\sqrt{h} = \frac{1}{8}b$, and that if $\frac{1}{8}b$ be positive, the product of the terms $\sqrt{p}$, $\sqrt{q}$, $\sqrt{r}$, must likewise be positive; so that all the variations that can be admitted are reduced to the four following:

1. $x = \sqrt{p} + \sqrt{q} + \sqrt{r}$,
2. $x = \sqrt{p} - \sqrt{q} - \sqrt{r}$,
3. $x = -\sqrt{p} + \sqrt{q} - \sqrt{r}$.
4. $x = -\sqrt{p} - \sqrt{q} + \sqrt{r}$.

In the same manner, when $\frac{1}{8}b$ is negative, we have only the four following values of $x$:

1. $x = \sqrt{p} + \sqrt{q} - \sqrt{r}$,
2. $x = \sqrt{p} - \sqrt{q} + \sqrt{r}$,
3. $x = -\sqrt{p} + \sqrt{q} + \sqrt{r}$,
4. $x = -\sqrt{p} - \sqrt{q} - \sqrt{r}$.

This circumstance enables us to determine the four roots in all cases; as may be seen in the following example.

778. Let there be proposed the equation of the fourth degree, $x^4 - 25x^2 + 60x - 36 = 0$, in which the second term is wanting. Now, if we compare this with the general formula, we have $a = 25$, $b = -60$, and $c = 36$; and after that,

$$f = \tfrac{25}{2},\ g = \tfrac{625}{16} + 9 = \tfrac{769}{16}, \text{ and } h = \tfrac{225}{4};$$

by which means our equation of the third degree becomes,

$$z^3 - \tfrac{25}{2}z^2 + \tfrac{769}{16}z - \tfrac{225}{4} = 0.$$

First, to remove the fractions, let us make $z = \frac{u}{4}$; and we shall have $\frac{u^3}{64} - \frac{25u^2}{32} + \frac{769u}{64} - \frac{225}{4} = 0$, and multiplying by the greatest denominator, we obtain

$$u^3 - 50u^2 + 769u - 3600 = 0.$$

We have now to determine the three roots of this equation; which are all three found to be positive; one of them being $u = 9$: then dividing the equation by $u - 9$, we find the new equation $u^2 - 41u + 400 = 0$, or $u^2 = 41u - 400$, which gives

$$u = \tfrac{41}{2} \pm \sqrt{(\tfrac{1681}{4} - \tfrac{1600}{4}} = \frac{41 \pm 9}{2};$$

so that the three roots are $u = 9$, $u = 16$, and $u = 25$.

Consequently, as $z = \frac{u}{4}$ the roots are

$$1.\ z = \tfrac{9}{4},\ 2.\ z = 4,\ 3.\ z = \tfrac{25}{4}.$$

These, therefore, are the values of the letters $p$, $q$, and $r$; that is to say, $p = \frac{9}{4}$, $q = 4$, and $r = \frac{25}{4}$. Now, if we consider that $\sqrt{pqr} = \sqrt{h} = -\frac{15}{2}$, and that therefore this value $= \frac{1}{8}b$ is negative, we must, agreeably to what has been said with regard to the signs of the roots $\sqrt{p}$, $\sqrt{q}$, and $\sqrt{r}$, take all those three roots negatively, or take only one of them negatively; and consequently, as $\sqrt{p} = \frac{3}{2}$, $\sqrt{q} = 2$, and $\sqrt{r} = \frac{5}{2}$, the four roots of the given equation are found to be:

$$\begin{aligned}
1.\ x &= \tfrac{3}{2} + 2 - \tfrac{5}{2} = 1,\\
2.\ x &= \tfrac{3}{2} - 2 + \tfrac{5}{2} = 2,\\
3.\ x &= -\tfrac{3}{2} + 2 + \tfrac{5}{2} = 3,\\
4.\ x &= -\tfrac{3}{2} - 2 - \tfrac{5}{2} = -6.
\end{aligned}$$

From these roots are formed the four factors,

$$(x - 1) \times (x - 2) \times (x - 3) \times (x + 6) = 0.$$

The first two, multiplied together, give $x^2 - 3x + 2$; the product of the last two is $x^2 + 3x - 18$; again multiplying these two products together, we obtain exactly the equation proposed.

779. It remains now to shew how an equation of the fourth degree, in which the second term is found, may be transformed into another, in which that term is wanting: for which we shall give the following rule *.

* An investigation of this rule may be seen in Maclaurin's Algebra, Part II. chap. 3.

Let there be proposed the general equation $y^4 + ay^3 + by^2 + cy + d = 0$. If we add to $y$ the fourth part of the coefficient of the second term, or $\frac{1}{4}a$, and write, instead of the sum, a new letter $x$, so that $y + \frac{1}{4}a = x$, and consequently $y = x - \frac{1}{4}a$: we shall have

$y^2 = x^2 - \frac{1}{2}ax + \frac{1}{16}a^2$, $y^3 = x^3 - \frac{3}{4}ax^2 + \frac{3}{16}a^2x - \frac{1}{64}a^3$,

and, lastly, as follows:

$$\begin{array}{rl} y^4 = & x^4 - ax^3 + \frac{3}{8}a^2x^2 - \frac{1}{16}a^3x + \frac{1}{256}a^4 \\ ay^3 = & \qquad ax^3 - \frac{3}{4}a^2x^2 + \frac{3}{16}a^3x - \frac{1}{64}a^4 \\ by^2 = & \qquad\qquad bx^2 - \frac{1}{2}abx + \frac{1}{16}a^2b \\ cy = & \qquad\qquad\qquad cx - \frac{1}{4}ac \\ d = & \qquad\qquad\qquad\qquad d \end{array}$$

And hence, by addition,

$$\left.\begin{array}{r} x^4 + 0 - \frac{3}{8}a^2x^2 + \frac{1}{8}a^3x - \frac{3}{256}a^4 \\ bx^2 - \frac{1}{2}abx + \frac{1}{16}a^2b \\ cx - \frac{1}{4}ac \\ d \end{array}\right\} = 0.$$

We have now an equation from which the second term is taken away, and to which nothing prevents us from applying the rule before given for determining its four roots. After the values of $x$ are found, those of $y$ will easily be determined, since $y = x - \frac{1}{4}a$.

780. This is the greatest length to which we have yet arrived in the resolution of algebraic equations. All the pains that have been taken in order to resolve equations of the fifth degree, and those of higher dimensions, in the same manner, or, at least, to reduce them to inferior degrees, have been unsuccessful: so that we cannot give any general rules for finding the roots of equations, which exceed the fourth degree.

The only success that has attended these attempts has been the resolution of some particular cases; the chief of which is that, in which a rational root takes place; for this is easily found by the method of divisors, because we know that such a root must be always a factor of the last term. The operation, in other respects, is the same as that we have explained for equations of the third and fourth degree.

781. It will be necessary, however, to apply the rule of Bombelli to an equation which has no rational roots.

Let there be given the equation $y^4 - 8y^3 + 14y^2 + 4y - 8 = 0$. Here we must begin with destroying the second term, by adding the fourth of its coefficient to $y$, supposing $y - 2 = x$, and substituting in the equation, instead of $y$, its new value $x + 2$, instead of $y^2$, its value $x^2 + 4x + 4$; and doing the same with regard to $y^3$ and $y^4$, we shall have,

$$\begin{array}{rl} y^4 = & x^4 + 8x^3 + 24x^2 + 32x + 16 \\ -8y^3 = & \quad\; - 8x^3 - 48x^2 - 96x - 64 \\ 14y^2 = & \qquad\qquad 14x^2 + 56x + 56 \\ 4y = & \qquad\qquad\qquad\quad 4x + \;\,8 \\ -\;8 = & \qquad\qquad\qquad\qquad\quad -\;8 \\ \hline & x^4 + 0 \;\; - 10x^2 - \;\;4x + \;\;8 = 0. \end{array}$$

This equation being compared with our general formula, gives $a = 10$, $b = 4$, $c = -8$; whence we conclude, that $f = 5$, $g = \frac{17}{4}$, $h = \frac{1}{4}$, and $\sqrt{h} = \frac{1}{2}$; that the product $\sqrt{pqr}$ will be positive; and that it is from the equation of the third degree, $z^3 - 5z^2 + \frac{17}{4}z - \frac{1}{4} = 0$, that we are to seek for the three roots $p$, $q$, $r$. (Art. 774.)

782. Let us first remove the fractions from this equation, by making $z = \frac{u}{2}$, and we shall thus have, after multiplying by 8, the equation $u^3 - 10u^2 + 17u - 2 = 0$, in which all the roots are positive. Now, the divisors of the last term are 1 and 2; if we try $u = 1$, we find $1 - 10 + 17 - 2 = 6$; so that the equation is not reduced to nothing: but trying $u = 2$, we find $8 - 40 + 34 - 2 = 0$, which answers to the equation, and shews that $u = 2$ is one of the roots. The two others will be found by dividing by $u - 2$, as usual; then the quotient $u^2 - 8u + 1 = 0$ will give $u^2 = 8u - 1$, and $u = 4 \pm \sqrt{15}$. And since $z = \frac{1}{2}u$, the three roots of the equation of the third degree are,

$$1,\ z = p = 1,$$

$$2,\ z = q = \frac{4 + \sqrt{15}}{2},$$

$$3,\ z = r = \frac{4 - \sqrt{15}}{2}.$$

783. Having therefore determined $p$, $q$, $r$, we have also their square roots; namely, $\sqrt{p} = 1$,

$$\sqrt{q} = \frac{\sqrt{(8 + 2\sqrt{15})}}{2}\,^*, \text{ and } \sqrt{r} = \frac{\sqrt{(8 - 2\sqrt{15})}}{2}.$$

* This expression for the square root of $q$ is obtained by multiplying the numerator and denominator of $\frac{4 + \sqrt{15}}{2}$ by 2, and extracting the root of the latter, in order to remove the surd: Thus, $\frac{4 + \sqrt{15}}{2} \times 2 = \frac{8 + 2\sqrt{15}}{4}$; and $\frac{\sqrt{(8 + 2\sqrt{15})}}{\sqrt{4}}$ $= \frac{\sqrt{(8 + 2\sqrt{15}}}{2}$.

But we have already seen, (Art. 675, and 676), that the square root of $a \pm \sqrt{b}$, when $\sqrt{(a^2 - b)} = c$, is expressed by $\sqrt{(a \pm \sqrt{b})} = \sqrt{(\frac{a+c}{2})} \pm \sqrt{(\frac{a-c}{2})}$: so that, as in the present case, $a = 8$, and $\sqrt{b} = 2\sqrt{15}$; and consequently, $b = 60$, and $c = \sqrt{(a^2 - b)} = 2$, we have $\sqrt{(8 + 2\sqrt{15})} = \sqrt{5} + \sqrt{3}$, and $\sqrt{(8 - 2\sqrt{15})} \ldots\ldots = \sqrt{5} - \sqrt{3}$. Hence, we have $\sqrt{p} = 1$, $\sqrt{q} = \frac{\sqrt{5} + \sqrt{3}}{2}$, and $\sqrt{r} = \frac{\sqrt{5} - \sqrt{3}}{2}$; wherefore, since we also know that the product of these quantities is positive, the four values of $x$ will be:

1. $x = \sqrt{p} + \sqrt{q} + \sqrt{r} = 1 + \frac{\sqrt{5} + \sqrt{3} + \sqrt{5} - \sqrt{3}}{2} \ldots\ldots\ldots$
$= 1 + \sqrt{5}$,

2. $x = \sqrt{p} - \sqrt{q} - \sqrt{r} = 1 + \frac{-\sqrt{5} - \sqrt{3} - \sqrt{5} + \sqrt{3}}{2} \ldots\ldots$
$= 1 + \sqrt{5}$;

3. $x = -\sqrt{p} + \sqrt{q} - \sqrt{r} = -1 + \frac{\sqrt{5} + \sqrt{3} - \sqrt{5} + \sqrt{3}}{2} \ldots\ldots$
$= -1 + \sqrt{3}$,

4. $x = -1\sqrt{p} - \sqrt{q} + \sqrt{r} = -1 + \frac{-\sqrt{5} - \sqrt{3} + \sqrt{5} - \sqrt{3}}{2}$.
$= -1 - \sqrt{3}$.

Lastly, as we have $y = x + 2$, the four roots of the given equation are:

1. $y = 3 + \sqrt{5}$, 2. $y = 3 - \sqrt{5}$,
3. $y = 1 + \sqrt{3}$, 4. $y = 1 - \sqrt{3}$.

QUESTIONS FOR PRACTICE.

1. Given $z^4 - 4z^3 - 8z + 32 = 0$, to find the values of $z$. *Ans.* $4, 2, -1 + \sqrt{-3}, -1 - \sqrt{-3}$.

2. Given $y^4 - 4y^3 - 3y^2 - 4y + 1 = 0$, to find the values of $y$. *Ans.* $\frac{-1 \pm \sqrt{-3}}{2}$, and $\frac{5 \pm \sqrt{21}}{2}$.

3. Given $x^4 - 3x^2 - 4x = 3$, to find the values of $x$.
*Ans.* $\frac{1 \pm \sqrt{13}}{2}$, and $\frac{-1 \pm \sqrt{-3}}{2}$.