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Abstract

Euler's *Vollständige Anleitung zur Algebra* (1770) is the prototype of a successful textbook on elementary algebra. The selection of problems by Euler displays a great familiarity with the typical recreational and practical problems of Renaissance and 16th-century algebra books. A detailed study into the sources of Euler revealed that he copied most of his problems from Christoff Rudolff's *Coss*, which was first published in 1525 and reissued in 1553 by Michael Stifel. Why would Euler found his popular textbook on algebra on a book published 250 years earlier? Part of the motivation could be sentimental. Euler was taught mathematics by his father using Stifel's edition of the *Coss*, and the young Euler spent several years studying the problems from the book. However, we propose an explanation based on the evolving rhetorical function of problems in algebra textbooks since the first printed book on algebra by Pacioli (1494). We discern six stages in the evolution from abbacus problem solving to algebraic theory. The first theory emerged through the extraction of general principles from the practice of problem solving. The algebra textbooks of the 18th century close a circle of continuous rhetorical development by using problems for practicing general principles and applying the algebraic language. Euler's *Algebra* is a prime example of the new rhetoric of problems still prominent in today's textbooks.

Introduction

Euler's *Vollständige Anleitung zur Algebra* is not only the most popular textbook on elementary algebra, but also, with the exception of Euclid's *Elements*, it is the most widely printed book on mathematics (Truesdell 1972). After 240 years this algebra textbook is still in print today, available in several languages and editions. Such enormous success poses to significant questions. What makes a book on algebra a good textbook? Hundreds of books on algebra were published before Euler's. Why did this book become a standard for any textbook on mathematics? We will provide an answer to these questions from a historical and epistemological viewpoint. We will do so by tracing the changing meaning of an algebraic problem in books published before Euler and the role of rhetoric in textbooks.

Publication history

Euler's *Vollständige Anleitung zur Algebra* was published in two volumes by the Academy of Sciences in St. Petersburg in 1770. It was translated into Russian (1768-9), Dutch (1773), French (1774), Latin (1790), English (1797, 1822) and Greek (1800). The popular German edition from Reclam Verlag sold no less than 108,000 copies between 1883 and 1943 (Reich 1992). The publication history is complex and this has lead to some misunderstandings. Euler wrote his algebra originally in German. Based on internal evidence, Fellmann dates the manuscript at 1765/1766 (Fellmann 1983, Fellmann 1995, 108). This was the time when he returned from Berlin to St. Petersburg and some years before he went completely blind. It was first published in a Russian version, translated by his students Peter Inokhodtsev and Ivan Yudin. As the first published edition was in Russian, some mistakenly claimed that Euler wrote this algebra in Russian. The first part on determinate analysis has the Eneström (1910) index 387^{A} (Euler 1768). The second part is on Diophantine equations and has index number 387^{B} (Euler 1769). In 1812, another Russian edition in two parts was published, translated from the French edition. The French edition was translated under the auspices of d'Alembert by Johann III Bernoulli, and published in Lyon in 1774.[1] This translation was appended with a large section by Lagrange on indeterminate analysis (2, 370-662). Bernoulli changed the order of the original version, moving parts of the second volume into the first. This French edition was the basis of several other translations such as the English and Latin and the additions by Lagrange were also translated into German. The English translation follows the French edition very faithful, even translating the footnotes added by Bernoulli. Another French edition was published by Jean-Guillaume Garnier (Euler 1807) which was again translated into English in 1824 by Charles Taylor. In the

1 The French edition does not mention which of the Bernoulli's was responsible for the translation. However, Jean-Marie Bruyset added a note in the second edition of 1795 (f. $aiii^{V}$) stating that M. Bernoulli was the director of the Berlin observatory, which points at Johann III Bernoulli (1744-1807).

Weber edition of Euler's *Opera Omnia*, the *Algebra* is in the first volume of the first series (Euler 1911). It keeps the original format by Euler but adds a German translation of Lagrange's additions.

Christoff Rudolff's influence

Euler shows, by his selection of problems in the *Algebra*, a great familiarity with the typical recreational and practical problems of Renaissance and 16th-century algebra books. I succeeded in establishing the source of Euler's problems when a digital version of Stifel's edition of Rudolff's Coss became available.[2] A fist glance made it immediately evident that most of Euler's 51 problems (discussed below) are taken from this. Rudolff's original edition was the first printed book entirely devoted to algebra and was published in 1525 in Strassburg under the title *Behend vnnd Hubsch Rechnung durch die kunstreichen regeln Algebre so gemeincklich die Coss genennt werden*. Stifel used many problems from Rudolff in the *Arithematica Integra* of 1544 but found the work so important that he decided to publish his own annotated edition in 1553, *Die Coss Christoffs Rudolffs mit schönen Exempeln der Coss*. The following three examples of textual evidence leave little doubt about Euler's source:

Rudolff, *Coss*, 1553	**Euler, *Opera Omnia* I, II**
Ich hab kaufft etlich eln tuch, und ye 5 eln fur 7 fl. verkauft wider ye 7 eln fur 11 fl. und gewin 100 fl uber das haubt gut. Wie vil ist dess tuch geweses? (f. 209r, problem 50)	Ich habe gekauft etliche Ellen Tuch und für jede 5 Ellen gegeben 7 Rthl. Ich habe wieder verkauft je 7 Ellen für 11 Rthl. und gewonnen 100 Rthl. über das Hauptguth: wie viel ist des Tuchs gewesen? (p. 224)
Ein wechsler hat zweyerley müntz. Der ertsen thun 20 stuck ein floren. Der andern müntz thun 30 stuck ein floren. Nu kompt einer der wil haben der zweyerley müntz 27 stuck fü ein floren. Ist die frag wie vil er yeder müntz nemen soll? (f. 246r, problem 97)	Ein Wechsler hat zweyerley Müntze; von der ersten gehen a Stück einen Rthl. von den zweyten Sorte b Stück. Nun kommt einer und will c Stück vor einen Rthl. haben. wie muß ihm der Wechsler von jeder Sorte geben? (p. 224)
Drei haben ein hauss kaufft fur 100 fl. Begert der erst vom andern 1/2 seyns gelts, so hette er das hauss alleyn zu bezalen. Der ander begat vom dritten 1/3 seynes gelts das er das hauss alleyn könte bezalen. Der dritt begert vom ersten 1/4 seyns gelts das er mochte das hauss alleyn bezalen. Wie vil hat yeder gelt gehabt? (f. 216r, problem 123)	Drey haben ein Haus gekauft für 100 Rthl. der erste begehrt vom andern ½ seines Gelds so könnte er das Haus allein bezahlen; der andere begehrt vom dritten 1/3 seines Geldes, so könnte er das Haus allein bezahlen. Der dritte begehrt vom ersten 1/4 seines Gelds so möchte er das Haus allein bezahlen. Wie viel hat jeder Geld gehabt? (p. 235)

2 http://www.ub.uni-bielefeld.de/diglib/rechenbuecher/coss/ at the University of Bielefeld. For a comprehensive argumentation see Heeffer 2007.

The first volume of Euler's *Algebra* on determinate equations contains 59 numbered problems. Two thirds of these can be directly matched with problems from Rudolff. Some are literal reproductions, as the examples given above. For other problems, the values have been changed or the problem has been reformulated slightly. The second part on indeterminate equations contains 59 problems also and here the affinity is somewhat less prominent but many problems can still be attributed to Rudolff. The second volume deals with, what we now call diophantine analysis. However, Euler's terminology is rooted in the older tradition and he titles the second part as "Von den sogenannten regel Coeci", referring to the *regula coecis* or *regula virginum* which appears frequently in arithmetic and algebra books of the 16th century. This terminology was abandoned in later translations.

Having determined the source for Euler's problems, the question remains about his motive for going back 240 years. The answer lies in his first exposure to mathematical problems and could thus be a sentimental one. In the Russian Euler archives at St. Petersburg there exists a manuscript containing a short autobiography dictated by Euler to his son Johann Albrecht on the first of December, 1767 (Fellmann 1995, 11). There he states that his father Paulus Euler taught him at young age the basics of mathematics with the use of the Stifel edition of Christoff Rudolff's *Coss* (Stifel 1553). Euler practiced mathematics for several years using this book, studying over four hundred algebra problems.[3] When he decided to write an elementary textbook on algebra, he must have had in mind his first mathematics book. The book was to be used for self study, in the same way that he had used Rudolff's book. As the many examples from Rudolff had assisted Euler to practice his algebraic skills, so would he also include many *aufgaben* related to the solution of equations. So while the motivation to use a 16th-century book may have been partly sentimental, the recognized educational value of algebraic problem solving was an important factor.

The problems

Although Euler's *Algebra* is separated from Rudolff's *Coss* by more than two centuries of algebraic practice, the structure of both works is not very different. We will limit our discussion to the first volume. Euler starts with operations on simple quantities, or what he calls *einfachen Grössen*. These include integers, fractions, irrational and imaginary numbers and logarithms. The second section deals with compound quantities, which refers to polynomials as well as

3 "vi ich bey Zeiten von meinem Vater den ersten Unterricht erhielt; und weil derselbe einer von den Discipeln des weltberühmten Jacobi Bernoulli gewesen, so tractete er mir sogleich die erste Gründe der Mathematic beizubringen, und bediente sich zu diesem End des Christophs Rudolphs Coss mit Micheals Stiefels Anmerckungen, worinnen ich mich einige Jahr mit allem Fleisch übte", Fellmann 1995, 11.

irrational binomials such as $1+\sqrt{a}$. The third section is on proportions and progressions, both arithmetical and geometrical. In the second part of the first volume, the different types of equations and their resolution methods are given.[4] Rudolff deals with roughly the same subjects but his organization reflects more the tradition of medieval algorisms.[5] For each of the different species, whole numbers, fractions, etc., he first gives the numeration and then discusses the possible operations which he calls algorithms. The rest of Rudolff's book consists of eight sections on the eight rules of algebra. These correspond with linear equations, the six Arab types of quadratic equations and the cubic equation with only the cube term. As the subdivision of quadratic equations in separate rules disappeared in the early 17th century, Euler's arrangement is different. He has separate sections on linear problems in one unknown, linear equations in multiple unknowns, the pure quadratic equation, the mixed quadratic, the pure cubic and the complete cubic equation. Euler uses the adjective *pure* for equations with only the square of cube term present. While the division between determinate and indeterminate problems is common since the 18th century, the distinction between pure and complete equations is a relic of Rudolff's *Coss*. The pure quadratic equation is one of the Arab rules and the pure cubic was the only cubic known to be solvable by Rudolff. We do not find this distinction in other 18th-century works such as Newton (1707), Wolff (1732) or Simpson (1745).

While the first sections include some examples as illustration, the problems are all contained in the second part on equations, as with Rudolff. Euler includes a total of 59 problems. The third chapter dealing with linear equations in one unknown lists 21 problems. They clearly show how Euler went sequentially through Rudolff (1553) and selected suitable examples from this book. The problems are practically in the same order as Rudolff's.[6] They include the well-known legacy problems, two cups and a cover, allegation, division and overtaking problems.[7] The fourth chapter contains linear problems in more than one unknown, including the mule and ass problem, doubling each other's money and men who buy a horse.[8] The fifth chapter is on the pure quadratic with five problems all taken from Rudolff.[9] The sixth has ten problems

4 The organization of the English editions is different. The second part of the German edition becomes the fourth section in the English.
5 Rudolff obviously does not discuss imaginary numbers and logarithms as Euler does. The discussion on cubic equations is added by Stifel and do not appear in the original 1525 edition. Both these inventions, the roots of cubic equations and imaginary numbers, where introduced by Cardano (1545).
6 Starting with Euler's problem 8, the correspondence is as follows: 8-16, 9-9, 10 and 11 are variations on 9, 12 to 21 correspond with Rudolff's 24, 26, 6, 50, 53, 59, 68, 97, 98 and 110 respectively.
7 For an overview and classification of these problems, see Tropfke 1980 and Singmaster 2004.
8 Euler's problem 3 to 7 correspond with Rudolff's 132, 112, 122, 123 and 128 respectively.
9 Euler's first three are from the fifth rule, problems 2, 4 and 11. The fourth is problem 240 from the first rule and the last is problem 20 from the second rule.

on the mixed quadratic equation, of which nine are taken from Rudolff.[10] Chapter eight, on the extraction of roots of binomials, has five problems, none from Rudolff. Finally, the chapter of the pure cubic has five problems, two from Rudolff, and on the complete cubic there are six problems, of which four are from Stifel's additional chapter. While Euler also treats logarithms and complex numbers, he included no problems on these subjects.

The English edition of John Hewlett adds 51 "problems for practice".[11] It is not clear where they originate from, as they do not appear in the French edition (Euler 1774).[12] It seems doubtful that the bible translator Hewlett (1811) added the problems himself. In any case, they were not selected by Euler.

Phases in rhetorical development of treatises on algebra

The medieval tradition

One of the first Latin problem collections found in the Western world is attributed to Alcuin of York under the title *Propositiones ad Acuendos Juvenes* or *Problems to Sharpen the Youth*. The text is dated of around 800 and consists of 53 numbered problems with their solution. As an example let us look at problem 16 on *Propositio de duobus hominibus boves ducentibus*, appearing twice in the *Patrologia Latina* :[13]

> "Two men were leading oxen along a road, and one said to the other: 'Give me two oxen, and I'll have as many as you have'. Then the other said: 'Now you give me two oxen,

10 The first is the seventh problem from the seventh rule. The next are from Rudolff's fifth rule, problems 2, 4, 11, 17, 18, 19, 26 and 33 respectively.
11 Twenty five linear problems in chapter 3, six quadratic problems in chapter 5, twelve problems on binomials in chapter 6, three problems in chapter 15 and five in chapter 16. For some unknown reason they were deleted from the 1810 edition, and added again with the third edition.
12 I checked several French editions, except the one from Jean Guillaume Garnier (1807).
13 Translation from Singmaster 1992. The declamatory style becomes apparent from the original Latin text, Folkerts 1978, 53: "Duo homines ducebant boves per viam, e quibus unus alteri dixit: Da mihi boves duos, habebo tot boves, quot et tu habes. At ille ait: Da mihi, inquit, et tu duos boves, et habebo duplum quam tu habes. Dicat, qui velit, quot boves fuerunt, quot unusquisque habuit. Solutio. Prior, qui dari sibi duos rogavit, boves habebat IIII. At vero, qui rogabatur, habebat VIII. Dedit quippe rogatus postulanti duos, et habuerunt uterque sex. Qui enim prior acceperat, reddidit duos danti priori, qui habebat sex, et habuit VIII, quod est duplum a quattuor, et illi remanserunt IIII, quod est simplum ab VIII".

> and I'll have double the number you have'. How many oxen were there, and how many did each have? Solution. The one who asked for two oxen to be given him had 4, and the one who was asked had 8. The latter gave two oxen to the one who requested them, and each then had 6. The one who had first received now gave back two oxen to the other who had 6 and so now had 8 which is twice 4, and the other was left with 4 which is half 8".

The rhetorical structure of these problems is that of a dialogue between a master and his students and is typical for the function of *quaestiones* since Antiquity. Rhyme and cadence in riddles and stories provided mnemonic aids and facilitated the oral tradition of problem solving. Many of the older problems are put in verse. Some best known examples are "Going to St-Yves" using the geometric progression $7 + 7^2 + 7^3 + 7^4$ (Tropfke 1980, 629). We know also many problems in rhyme from Greek epigrams[14] such as Archimedes cattle problem (Hillion an Lenstra 1999), the ass and mule problem from Euclid (Singmaster 1999) and age problems (Tropfke 1980, 575-576). During the Middle Ages complete algorisms were written this way, taking over 500 verses (Karpinski and Waters 1928, Waters 1929). Even without rhyme, problems were cast into a specific cadence to make it easier to learn by heart. The 53 problems of Alcuin, clearly show a character of declamation, specific for the medieval system of learning by rote. Before algebra was introduced in Europe, medieval students were required to calculate the solution to problems mentally and to memorize rules and examples. The solution depends on precepts which provide no explanation, but are easy to remember rules for solving similar problems.

The structure of a problem as a dialogue between master and student is also explicitly present in early Hindu mathematical writings. These treatises consist of long series of verses in which a master challenges a student with problems. An example from in the *Gaṇitasārasangraha* of Mahāvīra is as follows:

> "Here (in this problem), 120 gold pieces are divided among 4 servants in the proportional parts of ½, 1/3 , ¼ and 1/6. O arithmetician, tell me quickly what they obtained". (Padmavathamma and Rangācārya, 2000, 247, stanza 80 ½)

The student is addressed as friend, arithmetician or a learned man and is defied in solving difficult problems. In one instance, Brāhmagupta states in his *Brāhmasphutasiddhānta* of 628 AD that:

> "He, who tells the number of [elapsed] days from the number of days added to past revolutions, or to the residue of them, or to the total of these, or from their sum, is a person versed in the pulverizer". (Colebrook 1817, 8)

Thus someone who is able to solve this problem on lunar revolutions, should have memorized the

14 The most comprehensive collection of Greek epigram problems I have come across is in the second book of *Mathesis Biceps* on algebra by Caramuel 1670.

verses describing the *Kuttaka* or pulverizer method for solving indeterminate problems. Literally stated, the memorization of the rules formulated in *stanzas* by the master is a precondition for problem solving. Hindu algebra is based on the reformulation of problems to a format for which a memorized rule can be applied. The rhetorical function of problems in medieval, as well as Hindu texts, is to provide templates for problem solving which can be applied in similar circumstances.

The abbacus tradition

While the medieval tradition of riddles or problems with standard recipes was carried through to 16th-century arithmetic books, a new tradition of algebraic problem solving emerged in Renaissance Italy. The *Catalogue* by Warren van Egmond (1980) provides ample evidence of a continuous thriving of algebraic practice from the 14th till the 16th centuries.[15] Over two hundred manuscripts present us with a rare insight in the practice of teaching the basics of arithmetic to sons of merchants in the abbacus schools of major towns in Renaissance Italy.[16] The more skilled of these abbacus masters drafted treatises on algebraic problem solving in the vernacular. These consist typically of a short introduction on the basic operations on polynomials and the rules for solving problems (resolving equations). The larger part of these treatises is devoted to the algebraic solution of problems. We can state that the algebraic practice of the abbacus tradition is the rhetorical formulation of problems using an unknown. The solution typically depends on the reformulation of the problems in terms of the hypothetical unknown. The right choice of the unknown is half of the solution to the problem. Once the several sought quantities are expressed in that unknown, the analytic method consists of manipulating the polynomials and applying the rules of algebra (resolution of equations) to the point of the resolution of a value for the unknown.
As an example of the rhetoric of algebraic problem solving let us look at the major abbacus master of the 14th century, Antonio de' Mazzinghi (Problem 9, Arrighi 1967, 28-29:

Find two numbers which, multiplying one with the other gives 8, and [adding] their squares gives 27.	Truova 2 numeri che, multiplichato l'uno per l'altro, faccino 8 e i loro quadrati sieno 27.

After the problem text is given, the solution typically starts with the hypothetical definition of an unknown: "Suppose that the first quantity is one *cosa*". The skill of abbacus master and the elegance of the problem-solving method depends mostly on the clever choice of the unknown. Maestro Antonio not only is most skilful in this, he also was the very first to introduce multiple

15 For an overview of the texts and methods within this tradition, see Franci and Rigatelli 1985 and Høyrup 2007. "Abbacus" is spelled with double b to distinguish the art of calculation with Hindu-Arabic numerals taught by maestri d'abbaco from the material calculating device called the abacus.
16 Algebra was not part of the curriculum in abbacus schools, but served as a trade secret for a lineage of abbacus masters.

unknowns for solving difficult problems in an elegant way:

> "Ma per aguagliamenti dell'algibra anchora possiamo fare; e questo è che porremo che lla prima quantità sia una chosa meno la radice d'alchuna quantità, l'altra sia una chosa più la radice d'alchuna quantità. Ora multiplicherai la prima quantità in sè et la seconda quantità in sè et agugnerai insieme et araj 2 censi et una quantità non chonosciuta, la quale quantità non chonosciuta è quel che è da 2 censj infino in 27, che v'è 27 meno 2 censj, dove la multiplichatione di quella quantità è 13 1/2 meno i censo".

Instead of using the *cosa* for one of the numbers, or two unknowns for the two numbers, Maestro Antonio here uses $x-\sqrt{y}$ and $x+\sqrt{y}$. Squaring these two numbers gives $x^2-2x\sqrt{y}+y$ and $x^2+2x\sqrt{y}+y$ respectively. Adding them together results in $2x^2+2y$, which is equal to 27. The auxiliary unknown thus is $13\frac{1}{2}-x^2$.

> "Adunque la minore parte è una chosa meno la radice di 13 1/2 meno uno censo, l'altra è una chosa più radice di 13 1/2 meno i censo. E diraj che abbia trovato 2 quantità e' quadrati delle qualj insieme agunte sono 27 et l'una è una chosa meno radice di 13 1/2 meno uno censo, l'altra è una chosa più radice di 13 1/2 meno 1 censo. Ora è da vedere se multiplichato l'uno per l'altro fanno 8; dove multiplicheraj una chosa meno radice di 13 1/2 meno 1 censo per una chosa più radice di 13 1/2 meno uno censo. E quando multiplicheraj prima 1 chosa per una chosa, fanno uno censo; e di poj multiplicha i chosa per più radice di 13 1/2 meno 1 censo e una chosa per meno radice di 13 1/2 meno uno censo, fanno 0; e di poj multiplicha meno radice di 13 1/2 meno 1 censo per più radice di 13 1/2 meno 1 censo, fanno uno censo meno 13 1/2. E agugnj al censo di prima, fanno 2 censi meno 13 1/2. Et questi sono igualj a 8, dove raguaglieraj le parti et araj a ragugnere a ognj parte 13 1/2 e aremo che 2 censi sono igualj a 21 1/2".

The two numbers then can be expressed as $x-\sqrt{13\frac{1}{2}-x^2}$ and $x+\sqrt{13\frac{1}{2}-x^2}$ effectively eliminating the auxiliary unknown. Multiplying the two together gives $x^2-\left(13\frac{1}{2}-x^2\right)$ which is equal to 8, resulting in the equation $2x^2=21\frac{1}{2}$.

> "Dove arrecha a uno censo et arremo che uno censo sia igualj a 10 3/4. Adunque la chosa vale la radiche di 10 3/4 e il censo vale il suo quadrato cioè 10 3/4; onde la prima parte, che troviamo ch'era una chosa più radice di 13 1 /2 meno i censo, trarraj 10 3/4 di 13 1/2, rimanghono 2 3/4. E dirai che lla prima parte è la radice di 10 3/4 più radice di 2 3/4 e l'altra, che fu una chosa meno la radice di 13 1/2 meno i censo, trarrai 10 3/4 di 13 1/2, rimanghono 2 3/4 e diraj che ll'altra parte fusse la radice di 10 3/4 meno la radice di 2 3/4".

The equation can hence be expressed in a format for which a standard rule applies, namely , one of the six cases of quadratic problems from the Arabs. Applying the rule gives a value for the *cosa* of $\sqrt{10\frac{3}{4}}$. This results in the two numbers $\sqrt{10\frac{3}{4}}+\sqrt{2\frac{3}{4}}$ and $\sqrt{10\frac{3}{4}}-\sqrt{2\frac{3}{4}}$ which is the solution to the problem.

The abbacus masters have mostly abandoned the geometric proofs of Arab algebra and instead demonstrate the validity of the solution by a test. Multiplying the two numbers together gives 8 and adding their squares results in 27:

> "E chosì abbiamo trovato 2 quantità ch' e' loro quadrati sono 27 e la multiplichatione dell'una nell'altra è 8, chome volavamo. E l'una quantità è la radice di 10 3/4 meno la radice di 2 3/4, cioè la minore, e la maggiore è radice di 10 3/4 più radice di 2 3/4; chome trovamo prima".

This text fragment from the end of the 14th century is exemplary for the abbacus tradition. Algebraic practice consists of analytical problem solving. The rhetorical structure depends on the reformulation of the given problem in terms of the *cosa* and applying the analytical method to arrive at a value for the unknown. The sought quantities can then easily be determined. A test subsitituting the values of the quantities in the original problem provides proof of the validity of the solution.

The beginning of algebraic theory: from Pacioli to Cardano

By the end of 15th century we observe a change in the rhetorical structure of algebra treatises. While the solution to problems still remains the major focus of the texts, authors pay more attention to the introductory part. While a typical abbacus text on algebra was limited to thirty or forty *carta*, the new treatises easily fill hundreds of folios. Two trends contribute to more comprehensive approach: the use of the algorism as a rhetorical basis for an introductory theory and the extraction of general principles from practice.

The amalgamation of the algorism with the abbacus text

The algorism, as grown from the first Latin translations of Arab adoptions of Hindu reckoning, describes the Hindu-Arabic numerals and the basic operations of addition, subtraction, multiplication and division. In later texts we also find doubling and root extraction as separate operations. These operations are applied to natural numbers, fractions and occasionally also hexadecimal numbers. Through the influence of Boetian arithmetic, some algorisms also include sections on proportions and progressions. Whereas we find this structure also in abacus texts on arithmetic, the treatises on algebra have a different character. The introductory part extends on early Arab algebra with the six rules for solving quadratic problems, lengthened by some derived rules. By the end of 15th century algebraic treatises also incorporate the basic operations on arithmetic and broaden the discussion on whole numbers and fractions with irrational binomials and cossic numbers. We witness this evolution in Italy as well as in Germany. The culmination of this evolution is reflected in the *Practica Arithmeticae* of Cardano (1539). Cardano begins his book with the numeration of whole numbers, fractions, and surds (irrational numbers) as in the algorisms. He then adds *de numeratione denominationum* placing expressions in the power of an unknown in the same league with other numbers, which is completely new. In doing so he shows that the expansion of the number concept has progressed to the point of accepting powers of the unknown as one of the four basic types of numbers. He further discusses the basic operations in separate chapters and applies each operation to the four types. Also, he applies root extraction to powers of an unknown in the same way as done for whole numbers (chapter 21). He continues by constructing aggregates of cossic numbers with whole numbers, fractions or surds (chapter 33 to 36). As an example of the aggregation of cossic numbers with surds, he shows how $\sqrt{3}$ multiplied with $4x^2 + 5x$ gives $\sqrt{48x^4 + 120x^3 + 75x^2}$. Though Cardano was not the first, his *Practica Arithmeticae* is a prime example of the adoption of the algorism for the rhetorical structure of the new text books on algebra, and functioned as an model for later authors. Cossic numbers were in this way fully integrated with the numeration of the species of number and served as a completion of the operations of arithmetic.

Extracting general principles from algebraic practice

For a second trend in the expansion of an introductory theory in algebraic treatises we can turn to Pacioli. It has long been suspected that Pacioli based his *Summa de arithmetica geometria proportioni et proportionalita* of 1494 on several manuscripts from the abbacus tradition. These claims have been substantiated for large parts of the *Geometry* during the past decades. Ettore Picutti has shown that "all the 'geometria' of the *Summa*, from the beginning on page 59v. (119 folios), is the transcription of the first 241 folios of the Codex Palatino 577" (cited in Simi and Rigatelli 1993, 463). Margaret Daly Davis (1977) has shown that 27 of the problems on regular bodies in Pacioli's *Summa* are reproduced from Piero's *Trattao d'abaco* almost word for word. Franci and Rigatelli (1985) claim that a detailed study of the sources of the *Summa* would yield many surprises. Yet, for the part dealing with algebra, no hard evidence for plagiarism has been given. While studying the history of problems involving numbers in geometric progression (GP), I found that a complete section of the *Summa* is based on the *Trattato di Fioretti* of Maestro Antonio. Interestingly, this provides us with a rare insight in Pacioli's restructuring of old texts, and as such, in the shift in the rhetoric of algebra books. We will go into some detail as this instance is exemplary for other 16th-century works.[17]

Pacioli discusses thirty problems on numbers in GP (from 35 problems in distinction 6, treaty 6, article 14), before he treats algebra itself. Most of these problems correspond with problems from Maestro Antonio, often with the same values. More importantly, most original problem solving methods are reproduced literally by Pacioli, including one rare instance using two unknowns and one which Antonio calls "without algebra". Relevant for our discussion are two introductory sections preceding the problems. Pacioli gives some theoretical principles on three numbers in GP in the section called *De tribus quantitatibus continue proportionalium*

17 A similar shift can be observed within the evolution of Pacioli's work himself. Elsewhere, I compared the Perugia manuscript by Pacioli (1478) with his Summa of 1494 (Heeffer 2010).

(distinction 6, treaty 6, article 12, f. 88^{v}).[18] Another section on keys, lists theoretical principles on four numbers in GP under the heading *De clavibus seu evidentiis quantitatum continue proportionalium* (distinction 6, treaty 6, article 11, f. 88^{r}). Pacioli does not explain how these principles are derived. He only gives some numerical examples. However, a close comparison with the *Trattato di Fioretti* shows that several are extracted from Maestro Antonio's solution. Let us look at one example involving three numbers in GP with their sum given and an additional condition.

Pacioli, f. 91^{r}	**Maestro Antonio (Arrighi 1967, 15)**
Famme de 13 tre parti continue proportionali che multiplicata la prima in laltre dui, la seconda in laltre dui, la terça in laltre dui, e queste multiplicationi gionti asiemi facino 78.	Fa' di 19, 3 parti nella proportionalità chontinua che, multiplichato la prima chontro all'altre 2 e lla sechonda parte multiplichato all'altre 2 e lla terza parte multiplichante all'altre 2, e quelle 3 somme agunte insieme faccino 228. Adimandasi qualj sono le dette parti.

In modern notation, the general structure of the problem is as follows:

$$\frac{x}{y}=\frac{y}{z}$$
$$x+y+z=a$$
$$x(y+z)+y(x+z)+z(x+y)=b$$

Maestro Antonio is the first to treat this problem and uses values a = 19 and b = 228. Expanding the products and summing the terms gives $2xy+2xz+2yz=228$, but as $y^2=xz$, we can write this also as $2xy+2y^2+2yz=228$, or $2y(x+y+z)=228$. Given the sum of 19 for the three terms, these results in 6 for the middle term. Antonio then proceeds to find the other terms with the procedure of dividing a number into two extremes such that their product is equal to the square of the middle term. Pacioli solves the problem in exactly the same way. However, the rhetorical structure is quite different. Maestro Antonio performs an algebraic derivation on a particular case. Instead, Pacioli justifies the same step as an application of a more general principle, defined as a general key:

> "Questa solverai per la 14ª chiave. Laqual dice che stu partirai la su*m*ma de ditte multiplicationi, cioe 78 per lo doppio de 13. El qual 13 sera la summa de ditte q*uan*tita ne virra la seconda parte". (Pacioli 1494, f. 91^{r})

18 I have used the 1523 edition but the numbering of pages and sections is practically identical with the original.

The fourteenth key he is referring to has been previously formulated as follows (Pacioli 1494, f. 89[v]):

> On three quantities in continuous proportion, when multiplying each with the sum of the other two and adding these products together. Then divide this by double the sum of these three quantities and this always gives the second quantity.

> De 3 quantita continue proportionali che multiplicata ciascuna in laltre doi e quelli multiplicationi gionti insiemi. E poi questo partito nel doppio de la summa de ditte 3 quantita e sempre laverimento sera la 2ª quantita.

This particular key is one of several variations on the algebraic derivation of Maestro Antonio, each presented as a general principle. In modern notation:

$$\frac{x(y+z)+y(x+z)+z(x+y)}{2(x+y+z)}=y \qquad (1.1)$$

Pacioli also lists $\frac{x(y+z)+y(x+z)+z(x+y)}{2y}=x+y+z$,

$\frac{x(y+z)+y(x+z)+z(x+y)}{2y}=x+y+z$, and

$\frac{x(y+z)+y(x+z)+z(x+y)}{2y}=x+y+z$, (Key 13)

These "general principles" are presented without any argumentation except for numerical examples as a test. Pacioli continues to solve Antonio's problem by again applying a general rule for the division of a number into two parts proportional to the mean term. This rule is also extracted from previous algebraic practice. We can be certain that Pacioli mined Antonio's treatise for general principles such as this, because they are used nowhere else than for solving the problems taken from Antonio. The only sense we can give to the definition of a general principle, which is used only one time, is precisely rhetorical. Pacioli has chosen to present some typical derivations as general rules which are later applied to solve problems in a clear and concise way. Even with the body of evidence against him, we should be careful in accusing Pacioli of plagiarism. At best, we observe here an appropriation of problems and methods. The restructuring of material and the shift in rhetoric is in itself an important aspect in the development of 16th-century textbooks on algebra. Pacioli raised the testimonies of algebraic problem solving from the abbacus masters to the next level of scientific discourse, the textbook. When writing the *Summa*, Pacioli had already almost twenty years of experience in teaching mathematics at several universities. His restructuring of abbacus problem solving methods is undoubtedly inspired by this teaching experience.[19] Cardano's *Practica Arithmeticae* continues to build on this evolution and the two works together will shape the structure of future treatises on algebra.

19 Pacioli is often wrongfully considered an abbacist (e.g. Biagioli 1989). In fact, he enjoyed the social status of a well-paid university professor. Between 1477 and 1514, he taught mathematics at the universities of Perugia, Zadar (Croatia), Florence, Pisa, Naples and Rome (Taylor 1942).

Algebra as a model for demonstration

The two decades following Cardano's *Practica Arithmeticae* were the most productive in the development towards a symbolic algebra. Cardano (1545) himself secured his fame by publishing the rules for solving the cubic equation in his *Ars Magna* and introduced operations with two equations. In Germany, Michael Stifel (1544) produces his *Arithmetica Integra* which serves as a model of clarity and method for many authors during the following two centuries. Stifel also provided significant improvements in algebraic symbolism, which have been essential during the 16th century. He was followed by a Johannes Scheubel (1550) who included an influential introduction to algebra in his edition of the first six books on Euclid's *Elements*. This introduction was published separately in the subsequent year in Paris as the *Algebrae compendiosa* (Scheubel 1551) and reissued two more times. In France, Jacques Peletier (1554) published the first French work entirely devoted to algebra, heralding a new wave of French algebraists after the neglected Chuquet (1484) and de la Roche (1520). Johannes Buteo (1559) built further on Cardano, Stifel and Peletier to develop a method for solving simultaneous linear equations, later perfected by Guillaume Gosselin (1577). In 1560, an anonymous short Latin work on algebra was published in Paris. It appeared to be of the hand of Petrus Ramus and was later edited and republished by Schöner (1586, 1592). The work depended on Scheubel's book to such a measure that Ramus refrained from publishing it under his own name. In Flanders, Valentin Mennher published a series of books between 1550 and 1565, showing great skill in the application of algebra for solving practical problems. England saw the publication of the first book treating algebra by Robert Recorde (1557). This *Whetstone of witte* was based on the German books of Stifel and more importantly Scheubel. It introduced the equation sign as a result of the completion of the concept of an equation. It is not possible to review here all this books and we will have to limit ourselves to some general trends and changes in the rhetorical structure of the 16th-century algebra textbook.

Giovanna Cifoletti (1993) is one of the few who wrote on the rhetoric of algebra and specifically on this period. She attributes a high importance to Peletier's restructuring of the algebra textbook. However, we have shown that the merger of the algorism with the practical treatises of the abbacus tradition was initiated by the end of the 15th-century, culminating in Cardano (1539). This trend cannot be attributed to Peletier, as proposed by Cifoletti. On the other hand, Peletier was an active participant in the humanist reform program which aimed not only at language and literature but also at science publications. His works on arithmetic (1549), algebra (1554) and geometry (1557) make explicit references to this program and reflections on the rhetoric of mathematics teaching. Cifoletti (1993, 225) demonstrates how Peletier intentionally evokes the context of the author as the classical Orator in order to approach a textbook from the point of view of rhetoric. He recognizes the weak point of his predecessors, explicitly referring to Stifel and Cardano, namely the demonstration of mathematical facts. His ideal model for mathematical demonstration is the rules of logic represented under the form of a syllogism. In his introduction to Euclid's

Elements he considers the application of syllogisms in mathematical proof as analogous with that of a lawyer at the court house, applying the rules of rhetoric:[20]

> "Que si quelqu'un recherche curieusement, pourquoi en la démonstration des propositions ne se fait voir la forme du syllogisme, mais seulement y apparoissent quelques membres concis du syllogisme, que celui là sache, que ce seroit contre la dignité de la science, si quand on la traite à bon escient, il falloit suivre ric à ric les formules observées aux écoles. Car l'Advocat, quand il va au barreau, il ne met pas sur ses doigts ce que le Professeur en Rhétorique lui a dicté: mais il s'étudie tant qu'il peut, encore qu'il soit fort bien recours des preceptes de Rhétorique, de faire entendre qu'il ne pense rien moins qu'à la rhétorique".

So, how did Peletier apply his understanding of rhetoric in his *Algebre*? Cifoletti (1993, 239) points at the contamination of the rhetorical notion of *questio* and the algebraic notion of problem, initiated by Ramus and Peletier, and fully apparent in the *Regulae* of Descartes. She goes so far to identify the algebraic equation with the rhetorical *quaestio*:

> "But I also think that from the point of view of the history of algebra, so crucial for later theoreticians of Method, *quaestio* has played a fundamental role because it has allowed consideration of the process of putting mathematical matters into the form of equations in a rhetorical mode". (Cifoletti 1993, 245)

In Cicero's writings, the *quaestio* is an important part of rhetorical theory. He distinguishes between the *quaestio finita,* related to time and people, and the *quaestio infinita,* as a question which is not constrained. The *quaestio finita* is also called *causa,* and the alternative name for *quaestio infinita* is *propositum,* related to the Aristotelian notion of *thesis.* Cicero discerns the two types of *propositum*, the first of which is *propositum cognitionis,* theoretical, and the second is *propositum actions,* practical. Both these types of *quaestio infinita* have their role in algebra as the art addresses both theoretical as practical problems.

I believe the rhetorical function of algebra recognized by the authors cited above, is contained more in the development of algebraic symbolism, than in the changing role of *quaestio*. I have argued elsewhere that the period between Cardano (1539) and Buteo (1559) has been crucial for the development of the concept of the symbolic equation (Heeffer 2008). The improved symbolism of Viète, and symbols in general, are the result, rather than the start, of symbolic reasoning. It is precisely Cardano, Stifel, Peletier and Buteo who shaped the concept of the symbolic equation by defining the combinatorial operations which are possible on an equation. The process of representing a problem in a symbolic mode and applying the rules of algebra to arrive at a certain solution, have reinforced the belief in a *mathesis universalis*. Such a universal

20 Peletier 1557, introduction, cited by Cifoletti 1993, 227-228.

mathesis not only allows us to address numerical problems but possibly allows us to solve all problems which we can formulate. The thought originates within the Ramist tradition as part of a broader philosophical discussion on the function and method of mathematics, but the term turns up first in the writings of Adriaan Van Roomen (1597). The idea will flourish in the 17th century with Descartes and Leibniz. A *mathesis universalis* is inseparably connected with the newly invented symbolism. As Archimedes only needed the right lever to be able to lift the world, so did the new algebraist only need to formulate a problem in the right symbolism to solve it. *Nullum non problema solvere*, or "leave no problem unsolved" as Viète would zealously write at the end of the century. Much has been written on the precise interpretation of Descartes' use of the term. The changing rhetoric of algebra textbooks at the second half of the 16th century gives support to the interpretation of Chikara Sasaki, in which *mathesis universalis* can be considered as algebra applied as a model for the normative discipline of arriving at certain knowledge.[21] This is the function described by Descartes in Rule IV of his *Regulae*. Later, Wallis (1657) uses *Mathesis Universalis* as the title for his treatise on algebra and includes a large historical section discussing the uses of symbols in different languages and cultures. As a consequence, the study of algebra delivers us also a tool for reasoning in general.

The generalization of problems to propositions

For the modern reader of 16th-century algebras, it is difficult to understand why it took so long before algebraic problems became formulated in more general terms. Many of the textbooks of mid-16th century contained hundreds of problems often of similar types which were intentionally dispersed throughout the book. It is evident that someone who can solve the general case can solve all individual problems belonging to that case. Not that this evidence, or the need for generality, was unrecognized. For example Cardano (1545) writes: "We have used this variety of examples so that you may understand that the same can be done in other cases" (Witmer 1968, 37).

There is a specific historical reason for the lack of generality. The algebraic symbolism was by 1560 developed to a point that multiple equations of higher degree could be simultaneously formulated without ambiguities. One crucial aspect was missing: the tools for the generalization of the values of the coefficients. This required the generalization of the concept of an equation to a general structure which can be approached under different circumstances. It was Viète who initiated the shift from the solution of problems to the study of the structure of equations and transformations of equations. Let us look at one example as an illustration of the importance of the new symbolism for coefficients. In the *In Artem Analyticem Isagoge*, Viète (1591) studies several problems with numbers in GP, as did Cardano and Stifel before him. But there is an important difference in the approach of Viète. Cardano and Stifel constructed

21 For a discussion of this interpretation and an overview of other positions taken see Sasaki 2003, 189-203.

equations in order to solve specific instances of problems with numbers in GP. Viète, on the other hand, is interested in the relationship between the properties of numbers in GP and the structure of the quadratic and cubic equation. He investigates the circumstances in which one can be transformed into the other. Take three numbers in GP:

$$\frac{a}{b} = \frac{b}{c}$$

Now, suppose that the two extremes are not known but that they differ by a known number. Using the unknown *x* this can be represented as:

$$\frac{a+x}{b} = \frac{b}{x}$$

Multiplying the terms creates the quadratic equation $x^2+ax=b^2$. The important aspect of the transformation of proportions in the equation is that x^2+ax equals a square. Before Viète this crucial property of this quadratic equation could not be represented. Viète therefore introduced the use of the vowels A, E, I, O and U to represent unknowns, and the use of consonants for the constants and coefficients of an equation. Thus the specific property of this equation is preserved using his symbolism in the expression $A^2+AB=B^2$. The equation represents a class of problems which now can be formulated in general terms as:[22]

> "Given the mean of three magnitudes in continuous proportion and the difference between the extremes, find the extremes".

I have intentionally reversed the line of reasoning from the text, as written by Viète, because this is the most likely path the generalization was established. Viète formulated the general problem from the transformation of properties of numbers in GP into the structure of the quadratic equation. Although the problem in the individual case does appear in previous writings, Viète did not generalize the problem from such instances.[23]

However, others after Viète, show the inclination to reformulate classic problems in more general terms. Christopher Clavius, the great reformer of mathematics teaching, publishes an *Algebra* of his own in Rome in 1608. Unexpectedly, he ignores most of the achievements and improvements in symbolism of the second half of the 16th century and goes back to Stifel's *Arithmetica Integra as* a model for structure and for most of his large problem collection. In doing so, he takes less care in mentioning his sources than Stifel did. A problem from Stifel

22 "Data media trium proportionalium et differentia extremorum, invenire extremas", Viete 1591, 233.

23 As this is a very easy problem in arithmetic, it does not appear in Cardano or Stifel. However, Jean de Murs treats it algebraically in his *Quadripartitum* (f. 48r, L'Huillier 1990). De Billy gives de construction as well as the algebraic derivation (101). A version with the sum of the extremes appears in the first vernacular treatise on algebra by Jacobus de Florentia, chapter 19 (Høyrup 2007, 324-331).

on three numbers in GP given their sum and the mean term is formulated by Calvius, without acknowledgment, as follows:

Stifel 1544, f. 280^{v}	**Clavius 1608, 292**
Divide 283 in tres terminos continue proportionales, quorum medius faciat 78.	Datum numerum in tres numeros continuae proportionales partiri, quorum medius datus sit, cuius tamen quadratus maior non sit quadrato semissis illius numeri, qui relinquitur, detracto dato medio ex numero.

Stifel gives a specific problem, Clavius defines the general one. Stifel uses the unknown for the first term. The third is thus $283 - x - 78$. As the product of the two extremes equals the square of the middle term, Stifel arrives at the equation $205\,x - x^2 = 6084$. Remarkably for the time (1545) Stifel recognizes that this quadratic equation has two roots, namely 169 and 36, giving the values of the two extremes. After the general formulation, Clavius starts from exactly the same problem ("Sit numerus 283 dividendus in tres continuae proportionales quorum medius sit 78") to arrive at the same equation. As the general solution for the roots of the equation

$$x^2 = bx - c \,, \quad \text{is} \quad x = \frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 - c},$$

Clavius gives this as a general rule. He thus generalizes by formulating specific problems from Stifel in general terms and by rephrasing the solution to the equation as a general recipe for solution.

Table 1: *Terms for questions and demonstration in algebra textbooks*

Author	**Year**	**Examples**	**Problems**	**Proofs**
Pacioli	1494	Exemplis	Quaestioni	mostrare
de la Roche	1520		questions application de regle	(none)
Rudolff	1525		Exempla der Coss oder enigmata	
Ghaligai			(not named)	
Vanden Houcke	1537		(not named)	(none)
Cardano	1539		Questionis	
Stifel	1543	exempla	exemplum	pictura exempli
Scheubel	1551			
Peletier	1554	exemple		
Mennher	1556	examples	exemples sur la premier equation	demonstration geometrique
Buteo	1559		Problema Quaestio	
Ramus	1560	exempla	Exempla	(none)
Bombelli	1572	Essempio	Problema	Dimonstratione
Gosselin	1577		Problema	Demonstratio
Petri	1583		Exempelen op die eerste vergelijckinge	
Stevin	1586		Question	Construction Demonstration
Clavius	1608		Aenigmata	
Henrion	1620		Question	
de Billy			Propositio - Canon	Construction
Descartes	1639		Quaestio sive aequatio	
Wallis	1657		Theoremata – problemata	
Caramuel			Quaestiones	
Kersey	1673		Question - resolution Proposition - Canon	Composition
Newton	1707	Example	Problem	

This method of generalization is continued and completed by Jacques de Billy (1643), who treats no less than 270 problems on numbers in GP in his *Nova Geometriae Clavis Algebra*. For each problem, he gives a general formulation, a construction method, an algebraic derivation and a general canon. De Billy abandons the terms "problem" and *quaestio* and uses *propositio* instead. The general formulation of problems thus allows him to dispose of problems altogether and move to general propositions which constitute a new body of mathematical theory. The general recipes of Clavius, formulated as canons by de Billy, become propositions and *lemmata* themselves which are used as justifications of steps in further propositions. A reasoning built on references to previously proven propositions and *lemmata* changes the rhetoric of problem solving to that of concise rhetoric of justification based on the logic of syllogisms, previously aimed for by Peletier. The generalization of problems achieved thus more than one had hoped for. Not only provided the solution to a general problem a rule that can be applied in all similar cases. It also constituted a body of mathematical knowledge that could be referred to in a rhetorical exposition, strengthening its persuasive power.

An attempt at an axiomatic theory

The method of de Billy, of generalizing problems and turning their solution into canons which are universally applicable and may be used in the derivation of other propositions, was taken over by a new wave of algebraists in England. From the several citations it becomes apparent that de Billy was well appreciated in England.[24] This is unexpected, because before 1650 he published only a concise introduction to algebra in French (de Billy 1637) and the Latin treatise on numbers in GP (de Billy 1643). The books were not issued again in France. In England however, William Leybourn (1660) added a translation of de Billy's *Abrégé des préceptes d'algèbre* as the fourth part of his *Arithmetic*, first published in 1657. This popular work was reprinted several times into the 18th century. But it is de Billy's other work which influenced the rhetoric of English algebra textbooks in the later in the 17th century. In England, the need for rigor in the demonstration of algebraic reasoning was felt more directly. The prime model for truthful reasoning was, without doubt, Euclidian geometry, constructing theorems which follow from axioms by deductive reasoning. Before the 17th century, algebra was considered more as a practice, to be performed by those skilled in the art. It required experience and knowledge of many rules, which had their own name such as the *regula alligationis*.[25] The idea of a universal

24 de Billy is cited by Kersey 1673, f. b3r.

25 Widman (1489) list more than twenty rules. Also Pacioli (1494) used many names, which was ridiculed by Cardano (1539, Opera IV, 79), who showed that one can turn any generalized derivation into a rule and give a name to it: "Est etiam regula de Modo a me appellata, quoniam ex ipsa habent regulae infinitae in rebus maxime mercantilibus, et potes replere librum ex ipsis in uno mense, diversarum operationum, quae omnes regulae diversae videbuntur: et ita Frater Lucas, Borgias, Fortunatus, fecerunt libros per Neotericis instruendis, et ita tu lector poteris quotidie novas regular et inusitatis fabricare. Modus est talis solve quaestionem quamvis per algebra deinde detrahe la co. et serva operationes easdem in terminis suis, et erit regula generalis".

mathesis rendered knowledge of such rules superfluous. Algebra was basically not different from geometry or arithmetic (Wallis 1657, 85). Algebra starts from some simple facts which can be formulated as axioms. All other knowledge about algebraic theorems can be derived from these axioms by deduction. John Wallis introduces the term axioms in relation to algebra in an early work, called *Mathesis Universalis*, included in his *Operum mathematicorum* (1657, 85). With specific reference to Euclid's *Elements*, he gives nine *Axiomata*, also called *communes notationes* (from Cicero and Proclus's *communes notiones* or *communes sententiae*, see Cifoletti 2006).

Table 2: *Common notions or axioms of algebra*

1	Due eidem sunt aequalia, sunt et inter se aequalia	if $A = C$ and $B = C$ then $A = C$
2	Si aequalibus aequalia addantur, tota sunt aequalia	if $A = B$ then $A + C = B + C$
3	Si ab aequalibus aequalia subducantur, reliqua sunt aequalia	if $A = B$ then $A - C = B - C$
4	Si inequalibus aequalia addantur tota sunt inequalia	if $A \neq B$ then $A + C \neq B + C$
5	Si inaequalibus aequalia auserantur, reliqua sunt inaequalia	if $A \neq B$ then $A - C \neq B - C$
6	Quae eiusdem sunt dupliciae sunt inter se aequalia	$2A = A + A$
7	Quae eiusdem sunt dimidia, sunt aequalia inter se	$A/2 = A - A/2$
8	Totum est maius sua parte	
9	Totum aequatur partibus suis omnibus simul sumptis	The total equals the sum of its parts.

Some years later, John Kersey (1673, Book IV, 179) expanded on this and formulated 29 axioms "or common notions, upon which the force of inferences or conclusions, about the equality, majority and minority of quantities compared to one another, doth chiefly depend". Although using many more axioms, he basically reformulates those from Wallis. The method of constructing theorems or canons and the belief in the infallibility of the chain of reasoning becomes apparent from Kersey's explication of the difference between the analytic and the synthetic approach in the introduction:

> "Algebra which first assumes the quantity sought, whether it be a number or a line in a question, as if it were known, and then, with the help of one or more quantities given, proceeds by undeniable consequences, until that quantity which at first was but assumed or supposed to be known, is found to some quantity certainly known, and is therefore known also. Which analytical way of reasoning produceth in conclusion, either a theorem declaring some property, proportion or equality, justly inferred from things given or granted in a proposition, or else a canon directing infallibly how that may be

found out or done which is desired; and discovers demonstrations of the certainty of the resulting theorem or canon, in the synthetical method, or way of composition, by steps of the analysis, or resolution". (Kersey 1673, f. B2v)

The rhetoric of the algebra textbooks in the second half of the 17th century clearly shifts through the adoption of the Euclidian style of demonstration. An illustrative example is Wallis's conclusion of the chapter on numbers in GP with a very terse synopsis. The idea may have been taken from a book published some years before by Richard Balam (1653). The last chapter of this book is devoted to problems in GP and its aim is clearly to construct a body of algebraic knowledge from a general formulation of this type of problems. Starting from eight "special rules" he constructs 28 cases depending on which of the elements are given. Balam (1653, 152) concludes: "By this work thus performed, 164 new equations are found, which are so many theorem". Wallis expands on the scope of Balam's 8 rules and from the following definitions, he formulates 15 *theoremata* and 57 *problemata*:

A : smaller extreme
V : larger extreme
Z: reciprocal
T: number of terms (*t* when used as power)
R: common ratio
D: distance of the first term (*d* when used as power), $D = T - 1$
S: sum of the terms

The theorems include the classic sum of all the terms as $S = \frac{AR^t - A}{R-1}$ (theorem 5) as well as the less expected $R = \frac{S-A}{S-V}$ (theorem 9). The problems are formulated in general form and thus correspond with the propositions of de Billy. They all are formulated by considering some of the elements given, and asking for the other ones. For example problem 57 considers $\frac{V}{A}, T, Z$ as given and asks for R, A, V, VA, Z (Wallis 1657, 315). For each of these elements, Wallis lists the problem numbers where it can be derived from, in this case (10, 50, 4, 1, 12). Thus *R* follows from problem 10 which gives $\frac{V}{A} = R^d$, *A* from problem 50 as $A = \frac{R-1}{R^t - 1}S$, which in turn follows from 4, 1 and 12, *V* follows from problem 4 or $V = \frac{V}{A}A$, *VA* follows from problem 1, namely $VA = V.A$, and finally Z follows from problem 12, which is $Z = TV A$. Thus, nothing is taken for granted, not even $VA = V.A$, which is formulated as a derived proposition. Whenever it is used, Wallis refers to the proposition. Everything is linked into a chain of derivations, sound, rigorous, without compromises, as in Euclidean geometry.

The attempt to grasp the foundations of algebraic reasoning in basic axioms was pursued until the early 18th century. Before Euler in Germany, the most influential writer of textbooks on mathematics was Christian Wolff (1713-5). His *Elementa matheseos universae* was originally issued in two volumes. The first one treats the traditional disciplines arithmetic, algebra, geometry and trigonometry. A later addition added a wide variety of practical mathematics, from optics and astronomy to fortification and pyrotechnics. With the Basel edition (Wolff 1732) this standard textbook was enlarged to five volumes, reprinted and adapted several times in the 18th century. Immanuel Kant owned a copy of the first edition and was intimitaly acquainted with Wolff's work (Warda 1922, 07026). The book had an important influence on Kant's conception of the synthetic a priori in his *Critique of Pure Reason* (Shabel 2003). Especially Kant's view, on the role of algebra in symbolic construction, as based on the manipulation of geometrically constructible objects, is strongly influenced by the way Wolff conceived of algebra. The part on algebra in the *Elementa* was also published separately in a *Compendium* (Wolff 1742) and translated into English (Wolff 1739) and German. Wolff starts his *Compendium* with an introduction to the *methodo mathematica* describing the axiomatic method. In the introduction to arithmetic, preceding the algebra, he gives eight axioms "on which the general way of calculation is founded", corresponding with these of Wallis (1657).[26] He adds:

> "The delivering of these may seem superfluous, but it will be found that they are of great help to the understanding of Algebra, giving a clear idea of the way of reasoning that is used therein". (Wolff 1739, 3)

While the axioms define the basic properties of quantities and, as such, belong to the realm of arithmetic, they are considered functional for the study of algebra. Wolff gives many problems, always formulated in the general way, leading to a general solution and illustrated by a numerical example. The solution is often presented as a theorem. For example, problem 61 asks for three numbers in GP, being given the ratio *m* and *a* as the product of the first with the square of the third (Wolff 1739, 130). Using *x* for the first term, the second will be *mx* and the third m^2x. Therefore:

$$a = m^4x^3 \quad \text{and thus} \quad x = \sqrt[3]{\frac{a}{m^4}}$$

The theorem is formulated as:

> "The cube of the 1st term in a continued geometric progression, is to the product of the square into the first, in a quadruplicate ratio of the first of the second".

While the axiomatic approach was abandoned in the most common textbooks after Euler, the attempts by Wallis, Kersey and Wolff were extended into the 19th century through some lesser-known works. Perkins (1842, 60) lists "four axioms used in solving equations". Ingrid Hupp (1998) studied a tradition of three university professors teaching mathematics at the University of Würzburg. Franz Huberti (1762), Franz Trentel (1774) and Andreas Metz (1804) all continued Wolff's approach to express the essentials of algebra and arithmetic by axioms. Their motivation may have been more didactical than in pursuance of a *mathesis universalis*. The axiomatic

26 Wolff 1742, 6-9, in the 1774 Vienna edition, I used. The English translation gives seven axioms.

method brings rigor, clarity and brevity to the mathematical discipline, all too much inundated by numerous individual rules and recipes. Metz uses these properties of the axiomatic structure of algebra explicitly as an argument to include it in an elementary textbook on arithmetic:

> "Die schönsten und kürzesten Beweise der wichtigen Lehrsätze sowohl als die leichtesten Auflösungen der Aufgaben in der Mathematik (besonders, wie ich hinzu setze, in der Lehre von Proportionen, Progressionen und Logaritmen, wo sogar manche Aufgaben ohne Algebra nicht lösbar sind) [gründen] sich auf eine, wenigstens etwas genauere Kenntnis der Gleichungen [...] Dieser Vortheile nicht verlustig zu werden, soll also hier von der ersten Anfangsgründen dieser materie nur so viel gesagt werden als Kürze, Gründlichkeit und Deutlichkeit in der Folge verlangen". (Metz 1804, 159-160, cited by Hupp 1998, 60-61)

Huberti is the most complete, listing 14 axioms. Six of these include the earlier axioms from Wolff, but Huberti (1762, 5) adds axioms which express the basic operations on an equation, such as: equal quantities can be multiplied or divided by the same factor ("Si aequalia per eandem quantitatem multiplices, producta sunt aequalia"), equal quantities can be given a power ("quantitates aequales, elevatae ad eundum potentiae gradum, sunt aequales"), and a term can be moved to the other side while changing the sign ("Si duae quantitates efficiant aliquam summam, major terminus est aequalis summae mulctatae termino minore, et minor terminus est aequalis summae mulctatae termino majore"). While Wolff defined axioms but never used them in his *Algebra*, Huberti and to a larger extent Trentel and Metz occasionally apply the axioms in derivations (Hupp 1998, 87-93).

Though the axiomatic method, found in algebra textbooks until the early 19th century, does not match the standards of mathematical logic emerging at the late 19th century, the axiomatic model of Euclidian geometry is used rhetorically to arrive at "undeniable consequences". The purpose of algebra moves from the solution of numerical problems to the construction of a body of certain mathematical knowledge formulated by means of theorems and derived by rigorous deduction. Importantly, problems are the main instrument in this rhetorical transition. The whole body knowledge, in the form of theorems, is derived from generalized problems. The changing role of problems has facilitated the rhetorical transition of algebra textbooks.

Practicing the algebraic language

Taking the body of algebraic knowledge for granted, the rhetoric of problems in algebra textbooks shifts again during the 18th century. Newton's *Arithmetica universalis* is a good example. From the inventory of his library we know which books he owned on algebra and arithmetic (Harrisson 1978). The two copies of Oughtred's *Clavis* (Oughtred 1652, 1667) and the standard work of Kersey (1673) appear to be the ones that had the most weight on the *Arithmetica universalis*.

Helena Pycior describes how John Collins persistently tried to find and publish an algebra textbook in English suitable for use at universities (Pycior 1997, chapters 3 and 7). The only existing algebra in 1660 was Oughtred (1652) and this abstruse Latin work was not considered appropriate to expound on the algebraic achievements of the 17th century. Looking at foreign textbooks he found the *Algebra* of Gerard Kinckhuysen (1661) best suited for the task. He had the book translated into Latin and asked Newton in 1669 to write a commentary (*Observations on the Algebra of Gerard Kinckhuysen*, Whiteside 1968, 2, 364-447). Although Newton was very critical of Kinckhuysen, especially on the particular way he solved problems, he would use several of these problems in his own *Arithmetica universalis* published three decades later.[27]

Newton's introduction on the difference between the synthetic and analytic method echoes that from Kersey cited above. He also follows Oughtred's *Clavis* in the view of algebra as leading to universal truth. Everything derived through algebra can be considered a theorem.[28] Although Newton recognizes the universality of the method, he does not use axioms with respect to algebra, as done by Kersey. Also problems have a very different role in Newton's *Arithmetica*. In Kersey's *Algebra* the theorems are formulated as the result of problem solving. Newton uses far fewer problems than in previous algebra textbooks, and these problems serve no function in the construction of a body of theory. The sixteen numbered problems on arithmetic are given as an illustration and for practicing the algebraic language:[29]

> "Let the learner proceed to *exercise* or put in *practice* these operations, by bringing problems to aequations and lastly, let him learn or contemplate the nature and resolution of aequations".

The function of problems in Newton's textbook is thus a complete shift from previous works on algebra. Also, the nature of the problems is different. Newton includes problems which were not seen again since the first half of the 16th century. Take for example the following simple arithmetical problem:

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27 Pycior (1993, 179) cites a letter to Collins in which Newton complains that Kinckhuysen does not solve his problems "by any general analytic method", "proper to instruct a learner" and that his examples are as valuable as "Acrostick's & such kind of artificiall poetry" is in learning Ovid's poetry. However, Newton reproduces problems 13 and 14 from Kinckhuysen's third and fourth, and generalized problem 15 (Newton 1720, 81-83).

28 Newton (1707, 1) writes: "Arithmetica quidem definite et particulariter, algebraica autem indefinitè et universaliter; ita et enuntiata serè omnia quae in haec computatione habentur, et praesertim conclusiones".

29 Newton (1707, 2): " Deinde has operationes, reducendo problemata ad aequationes, exerceat, et ultimo naturam et resolutionem aequationum contempletur" (translation from Newton 1720, 2; italics mine).

"Problem IV: A person being willing to distribute some money among some beggars, wanted eight Pence to give three Pence a piece to them; he therefore gave to each two Pence, and had three Pence remaining over and above. To find the number of beggars". (Newton 1720, 71)

Using x for the number of beggars, the sum of money equals $3x - 8$ when giving three each or $2x + 3$ when giving two each. Both these expression are equal, so $x = 11$. The generalization of this problem to a theorem would be trivial and is not the function of problems in Newton's *Arithmetica*. These problems only serve the purpose of practicing the art of "translating out of the English, or any other tongue it is proposed in, into the algebraical language, that is, into characters fit to denote our conceptions of the relations of quantities" (Newton 1720, 69-70). In fact, the changing function of problems allowed Newton to incorporate this problem again in a textbook.[30] This problem, better known in the formulation of handing out figs to children, was popular during the Middle Ages and the Renaissance. It probably originated from Hindu sources and was traditionally solved by a recipe, as formulated in the *Bīja-Ganita* of Bhāskarācārya (c. 1150, Colebrook 1817, 188). With the general form $ax + b = cx - d = y$ it can be solved as $y = \frac{ad + bc}{a - c}$, as well as $x = \frac{b + d}{c - a}$.[31] Both these solutions appear as separate recipes in medieval sources. These problems functioned as vehicles for the transmission of arithmetical recipes before the advent of algebra. It is one of Widman's many rules called *regula augmenti et decrementi* (Widman 1489, f. 117).[32] The problems appeared in the 16th century for the last time in Mennher (1550, f. Cviii^r, 1565, f. Tviii^v). After that, such simple problems were not interesting enough to be included in the program of the French algebraists of constructing a body of mathematical theory from algebraic problem solving. With the changing rhetoric of problems in the 18th century, simple problems reaffirm their function, now for exercising and practicing the new symbolism. Formulating simple problems in algebraic equations is a required deftness for 18th-century men of science. Algebra has turned into a language which learned men cannot afford to neglect. Problems happen to be the primary tools in textbooks to acquire the necessary skills in symbolic algebra.

The changed role of problems became the new standard in 18th-century textbooks. Thomas Simpson adopted the rhetoric of problems as practice in his popular *Treatise of Algebra*. He included a large number of recreational and practical problems popular during the Renaissance. The purpose of the many word problems is to practice the process of abstraction and to identify the essential algebraic structure of problems:

30 For a more extensive discussion of the following argument see my "How Algebra spoiled Renaissance Recreation Problems", forthcoming.
31 See Heeffer 2010b for a discussion of this rule and its transmission from India to Europe.
32 Kaunzner 1969, 77, not to be confused with the rule of false position which is sometimes also named as such.

> "This being done, and the several quantities therein concerned being denoted by proper symbols, let the true sense and meaning of the question be translated from the verbal to a symbolic form of expression; and the conditions, thus expressed in algebraic terms, will, if it be properly limited, give as many equations as are necessary to its solution". (Simpson 1809, 75)

Simpson gives 75 determinate problems in the section *The Application of Algebra to the Resolution of Numerical Problems*. Several of these were not seen anymore in algebra textbooks of a century before him. An example is the lazy worker problem, which was very popular during the 15th century ("Der faule Arbeiter", Tropfke 1980, 603). A man receives *a* pence for every day he works and has to return *b* for every day he fails to turn up. At the end of a period of *c* days he is left with value *d*. How many days did he work? This simple problem leads to two linear equations in two unknowns:

$$\begin{matrix} x+y=c \\ ax-by=d \end{matrix} \text{ with solutions } x=\frac{bc+d}{a+b} \text{ and } \frac{ac-d}{a+b}$$

The early formulations of the problem often had $d = 0$ and applied the recipe of dividing the product *bc* by the sum $a + b$, without any explanation, let alone an algebraic derivation (e.g. Borghi, 1484, Ff 111^v-112^r). It disappeared from algebra books by 1560 because it did not function within the rhetoric of that time. While books on algebra in the 16th and 17th centuries were *the* testimonies of mathematical scholarship, from the late 17th century, new algebraic methods more and more divulged in scientific periodicals as the *Acta Eruditorum* in Leipzig, the *Philosophical Transactions* in London and the *Histoire de l'Académie royale des sciences* in Paris. With some expections, as Cramer (1750), the algebra books of the 18th century are primarily intended as textbooks, as part of the mathematics curriculum. Simpson (1740) is an early example. He introduces simple problems as the lazy worker again, not so much because the problem is interesting in itself, but in order to practice the translation and interpretation of word problems. It is within this new rhetoric that we have to situate Euler's *Algebra*. What Euler did not state himself was made clear by the publisher:

> "We present the lovers of Algebra a work, of which a Russian translation appeared two years ago. The object of the celebrated author was to compose an Elementary Treatise, by which the beginner, without any other assistance might make himself complete master of Algebra". (Euler, 1822, xxiii)

The rhetoric of problems is emphasized over and over again throughout the book: "To illustrate this method by examples" (Euler 1822, 207, §609) and "in order to illustrate what has been said by an example" (Euler 1822, 256, §726). Euler's book was the most successful of all algebra textbooks ever. By appropriating the problems from the antique book of Rudolff his father used for teaching him mathematics, Euler appealed to a large audience. His lucid accounts, such as the explanation why the quadratic equation has two roots (Euler 1822, 244-248), are illustrated with practical and recreational problems to practice the translation into algebraic language.

Conclusion

The examination of algebra textbooks from the point of view of the changing rhetoric of problems provides us with some interesting insights. Different ways of presenting problems have played a crucial role in the transformation of early abbacus manuscripts on algebra into the typical 18th-century textbook. While algebra consisted originally of problem solving only, an expansion through the amalgamation of medieval algorisms with abbacus texts was the first step towards the modern textbook. Pacioli's appropriation of abbacus texts in his *Summa* initiated an important restructuring of algebraic derivations into a theoretical introduction and its application in problem solving. The extension of the number concept and the treatment of operations on irrational binomials and polynomials by Cardano set a new standard for algebra textbooks by his *Practica Arithmeticae*. Humanists such as Ramus and Peletier were inspired by the developments within rhetoric to restructure algebra books and paid more attention to the art of demonstration in algebraic derivations. The emergence of symbolic algebra in the mid-16th century contributed to the idea of a *mathesis universalis*, as a normative discipline for arriving at certain knowledge. By the end of the 16th-century the change of focus to the study of the structure of equations led to a more general formulation of problems. The solutions to general problems yielded theorems, propositions and canons, which constituted an extensive body of algebraic knowledge. The rhetoric of 17th-century textbooks adopted the Euclidian style of demonstration to provide rigor in demonstration. The algebra textbooks of the 18th century abandoned the constructive role of problems in producing mathematical knowledge. Instead, problems were used only for illustration and for practicing the algebraic language. Recreational problems from the Renaissance, which disappeared from books for almost two centuries, acquired the new function of exercises in transforming problems into equations.

Euler's *Algebra* is the textbook *par excellence* intended for self-study. The book revives many older problems but presents them as exercises for translating word problems into symbolic equations. This new established role of problems in algebra text books explains why Euler found in Rudolff's *Coss* a suitable repository of examples. The difficulties Euler encountered as a young man studying Rudolff's book receive a special treatment, such as in the section on negative quantities and the explanation why a quadratic equation always has two roots. Euler's algebra was conceived from the beginning as a textbook to completely ("Volständig") master elementary algebra. These qualities that were well recognized explain its enormous success as a textbook.

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