CHAPTER IX.

Of the Nature *of* Equations *of the* Second Degree.

689. What we have already said sufficiently shews, that equations of the second degree admit of two solutions; and this property ought to be examined in every point of view, because the nature of equations of a higher degree will be very much illustrated by such an examination. We shall therefore retrace, with more attention, the reasons which render an equation of the second degree capable of a double solution; since they undoubtedly will exhibit an essential property of those equations.

690. We have already seen, indeed, that this double solution arises from the circumstance that the square root of any number may be taken either positively, or negatively; but, as this principle will not easily apply to equations of higher degrees, it may be proper to illustrate it by a distinct analysis. Taking, therefore, for an example, the quadratic equation, $x^2=12x-35$, we shall give a new reason for this equation being resolvible in two ways, by admitting for x the values 5 and 7, both of which will satisfy the terms of the equation.

691. For this purpose it is most convenient to begin with transposing the terms of the equation, so that one of the sides may become 0; the above equation consequently takes the form

$$x^2-12x+35=0;$$

and it is now required to find a number such, that, if we substitute it for x, the quantity $x^2-12x+35$ may be really equal to nothing; after which, we shall have to shew how this may be done in two different ways.

692. Now, the whole of this consists in clearly shewing, that a quantity of the form $x^2-12x+35$ may be considered as the product of two factors. Thus, in reality, the quantity of which we speak is composed of the two factors $(x-5)\times(x-7)$; and since the above quantity must become 0, we must also have the product $(x-5)\times(x-7)=0$; but a product, of whatever number of factors it is composed, becomes equal to 0, only when one of those factors is reduced to 0. This is a fundamental principle, to which we must pay particular attention, especially when equations of higher degrees are treated of.

693. It is therefore easily understood, that the product

$(x-5)\times(x-7)$ may become 0 in two ways: first, when the first factor $x-5=0$; and also, when the second factor $x-7=0$. In the first case, $x=5$, in the second $x=7$. The reason is therefore very evident, why such an equation $x^2-12x+35=0$, admits of two solutions; that is to say, why we can assign two values of x, both of which equally satisfy the terms of the equation; for it depends upon this fundamental principle, that the quantity $x^2-12x+35$ may be represented by the product of two factors.

694. The same circumstances are found in all equations of the second degree: for, after having brought the terms to one side, we find an equation of the following form $x^2-ax+b=0$, and this formula may be always considered as the product of two factors, which we shall represent by $(x-p)\times(x-q)$, without considering what numbers the letters p and q represent, or whether they be negative or positive. Now, as this product must be $=0$, from the nature of our equation, it is evident that this may happen in two cases; in the first place, when $x=p$; and in the second place, when $x=q$; and these are the two values of x which satisfy the terms of the equation.

695. Let us here consider the nature of these two factors, in order that the multiplication of the one by the other may exactly produce x^2-ax+b. By actually multiplying them, we obtain $x^2-(p+q)x+pq$; which quantity must be the same as x^2-ax+b, therefore we have evidently $p+q=a$, and $pq=b$. Hence is deduced this very remarkable property; that in every equation of the form $x^2-ax+b=0$, the two values of x are such, that their sum is equal to a, and their product equal to b: it therefore necessarily follows, that, if we know one of the values, the other also is easily found.

696. We have at present considered the case, in which the two values of x are positive, and which requires the second term of the equation to have the sign $-$, and the third term to have the sign $+$. Let us also consider the cases, in which either one or both values of x become negative. The first takes place, when the two factors of the equation give a product of this form, $(x-p)\times(x+q)$; for then the two values of x are $x=p$, and $x=-q$; and the equation itself becomes

$$x^2+(q-p)x-pq=0;$$

the second term having the sign $+$ when q is greater than p, and the sign $-$ when q is less than p; lastly, the third term is always negative.

The second case, in which both values of x are negative, occurs when the two factors are

$$(x+p) \times (x+q);$$

for we shall then have $x=-p$, and $x=-q$; the equation itself therefore becomes

$$x^2+(p+q)x+pq=0.$$

in which both the second and third terms are affected by the sign $+$.

697. The signs of the second and the third terms consequently shew us the nature of the roots of any equation of the second degree. For let the equation be $x^2 \ldots . ax \ldots . b=0$. If the second and third terms have the sign $+$, the two values of x are both negative; if the second term have the sign $-$, and the third term $+$, both values are positive: lastly, if the third term also have the sign $-$, one of the values in question is positive. But, in all cases whatever, the second term contains the *sum* of the two values, and the third term contains their *product*.

698. After what has been said, it will be easy to form equations of the second degree containing any two given values. Let there be required, for example, an equation such, that one of the values of x may be 7, and the other -3. We first form the simple equations $x=7$, and $x=-3$; whence, $x-7=0$, and $x+3=0$; these give us the factors of the equation required, which consequently becomes $x^2-4x-21=0$. Applying here, also, the above rule, we find the two given values of x; for if $x^2=4x+21$, we have, by completing the square, &c. $x=2\pm\sqrt{25}=2\pm5$; that is to say, $x=7$, or $x=-3$.

699. The values of x may also happen to be equal. Suppose, for example, that an equation is required, in which both values may be 5. Here the two factors will be $(x-5)\times(x-5)$, and the equation sought will be $x^2-10x+25=0$. In this equation, x appears to have only one value; but it is because x is twice found $=5$, as the common method of resolution shews; for we have $x^2=10x-25$; wherefore $x=5\pm\sqrt{0}=5\pm0$, that is to say, x is in two ways $=5$.

700. A very remarkable case sometimes occurs, in which both values of x become imaginary, or impossible; and it is then wholly impossible to assign any value for x, that would satisfy the terms of the equation. Let it be proposed, for example, to divide the number 10 into two parts, such that their product may be 30. If we call one of those parts x, the other will be $10-x$, and their product will be

$10x-x^2=30$; wherefore $x^2=10x-30$, and $x=5\pm\surd-5$, which, being an imaginary number, shews that the question is impossible.

701. It is very important, therefore, to discover some sign, by means of which we may immediately know whether an equation of the second degree be possible or not.

Let us resume the general equation $x^2-ax+b=0$. We shall have $x^2=ax-b$, and $x=\frac{1}{2}a\pm\surd(\frac{1}{4}a^2-b)$. This shews, that if b be greater than $\frac{1}{4}a^2$, or $4b$ greater than a^2, the two values of x are always imaginary, since it would be required to extract the square root of a negative quantity; on the contrary, if b be less than $\frac{1}{4}a^2$, or even less than 0, that is to say, if it be a negative number, both values will be possible or real. But, whether they be real or imaginary, it is no less true, that they are still expressible, and always have this property, that their sum is equal to a, and their product equal to b. Thus, in the equation $x^2-6x+10=0$, the sum of the two values of x must be 6, and the product of these two values must be 10; now, we find, 1. $x=3+\surd-1$, and 2. $x=3-\surd-1$, quantities whose sum is 6, and the product 10.

702. The expression which we have just found may likewise be represented in a manner more general, and so as to be applied to equations of this form, $fx^2\pm gx+h=0$; for this equation gives

$$x^2=\mp\frac{gx}{f}-\frac{h}{f}, \text{ and } x=\mp\frac{g}{2f}\pm\surd\left(\frac{g^2}{4f^2}-\frac{h}{f}\right), \text{ or } \ldots\ldots$$

$x=\dfrac{\mp g\pm\surd(g^2-4fh)}{2f}$; whence we conclude, that the two values are imaginary, and consequently, the equation impossible, when $4fh$ is greater than g^2; that is to say, when, in the equation $fx^2-gx+h=0$, four times the product of the first and the last term exceeds the square of the second term: for the product of the first and the last term, taken four times, is $4fhx^2$, and the square of the middle term is g^2x^2; now, if $4fhx^2$ be greater than g^2x^2, $4fh$ is also greater than g^2, and, in that case, the equation is evidently impossible; but in all other cases, the equation is possible, and two real values of x may be assigned. It is true, they are often irrational; but we have already seen, that, in such cases, we may always find them by approximation: whereas no approximations can take place with regard to imaginary expressions, such as $\surd-5$; for 100 is as far from being the value of that root, as 1, or any other number.

703. We have farther to observe, that any quantity of

the second degree, $x^2 \pm ax \pm b$, must always be resolvible into two factors, such as $(x \pm p) \times (x \pm q)$. For, if we took three factors, such as these, we should come to a quantity of the third degree; and taking only one such factor, we should not exceed the first degree. It is therefore certain, that every equation of the second degree necessarily contains two values of x, and that it can neither have more nor less.

704. We have already seen, that when the two factors are found, the two values of x are also known, since each factor gives one of those values, by making it equal to 0. The converse also is true, viz. that when we have found one value of x, we know also one of the factors of the equation; for if $x=p$ represents one of the values of x, in any equation of the second degree, $x-p$ is one of the factors of that equation; that is to say, all the terms having been brought to one side, the equation is divisible by $x-p$; and farther, the quotient expresses the other factor.

705. In order to illustrate what we have now said, let there be given the equation $x^2+4x-21=0$, in which we know that $x=3$ is one of the values of x, because $(3 \times 3)+(4 \times 3)-21=0$; this shews, that $x-3$ is one of the factors of the equation, or that $x^2+4x-21$ is divisible by $x-3$, which the actual division proves. Thus,

$$
\begin{array}{r l}
x-3) & x^2+4x-21 \ (x+7 \\
& \underline{x^2-3x} \\
& 7x-21 \\
& \underline{7x-21} \\
& 0.
\end{array}
$$

So that the other factor is $x+7$, and our equation is represented by the product $(x-3) \times (x+7)=0$; whence the two values of x immediately follow, the first factor giving $x=3$, and the other $x=-7$.

CHAPTER X.

Of Pure Equations *of the* Third Degree.

706. An equation of the third degree is said to be *pure*, when the cube of the unknown quantity is equal to a known

quantity, and when neither the square of the unknown quantity, nor the unknown quantity itself, is found in the equation; so that

$$x^3=125; \text{ or, more generally, } x^3=a,\ x^3=\frac{a}{b}, \text{ \&c.}$$

are equations of this kind.

707. It is evident how we are to deduce the value of x from such an equation, since we have only to extract the cube root of both sides. Thus, the equation $x^3=125$ gives $x=5$, the equation $x^3=a$ gives $x=\sqrt[3]{a}$, and the equation $x^3=\frac{a}{b}$ gives $x=\sqrt[3]{\frac{a}{b}}$, or $x=\frac{\sqrt[3]{a}}{\sqrt[3]{b}}$. To be able, therefore, to resolve such equations, it is sufficient that we know how to extract the cube root of a given number.

708. But in this manner, we obtain only one value for x: and since every equation of the second degree has two values, there is reason to suppose that an equation of the third degree has also more than one value. It will be deserving our attention to investigate this; and, if we find that in such equations, x must have several values, it will be necessary to determine those values.

709. Let us consider, for example, the equation $x^3=8$, with a view of deducing from it all the numbers, whose cubes are, respectively, 8. As $x=2$ is undoubtedly such a number, what has been said in the last chapter shews that the quantity $x^3-8=0$, must be divisible by $x-2$: let us therefore perform this division.

$$\begin{array}{l}
x-2)\ x^3-8\ (x^2+2x+4 \\
\underline{x^3-2x^2} \\
2x^2-8 \\
\underline{2x^2-4x} \\
4x-8 \\
\underline{4x-8} \\
0.
\end{array}$$

Hence it follows, that our equation, $x^3-8=0$, may be represented by these factors;

$$(x-2)\times(x^2+2x+4)=0.$$

710. Now, the question is, to know what number we are to substitute instead of x, in order that $x^3=8$, or that $x^3-8=0$; and it is evident that this condition is answered, by supposing the product which we have just now found equal to 0: but this happens, not only when the first

factor $x-2=0$, which gives us $x=2$, but also when the second factor
$x^2+2x+4=0$. Let us, therefore, make
$x^2+2x+4=0$; then we shall have $x^2=-2x-4$, and thence $x=-1\pm\sqrt{-3}$.

711. So that beside the case, in which $x=2$, which corresponds to the equation $x^3=8$, we have two other values of x, the cubes of which are also 8; and these are,

$x=-1+\sqrt{-3}$, and $x=-1-\sqrt{-3}$, as will be evident, by actually cubing these expressions;

$$\begin{array}{lcl}
\begin{array}{r} -1+\sqrt{-3} \\ -1+\sqrt{-3} \\ \hline 1-\sqrt{-3} \quad\;\; \\ -\sqrt{-3}-3 \\ \hline -2-2\sqrt{-3} \\ -1+\;\sqrt{-3} \\ \hline 2+2\sqrt{-3} \quad\;\; \\ +2\sqrt{-3}+6 \\ \hline 8. \end{array} & \begin{array}{c} \\ \\ \\ \\ \text{square} \\ \\ \\ \\ \text{cube.} \end{array} & \begin{array}{r} -1-\sqrt{-3} \\ -1-\sqrt{-3} \\ \hline 1+\sqrt{-3} \quad\;\; \\ +\sqrt{-3}-3 \\ \hline -2+2\sqrt{-3} \\ -1-\;\sqrt{-3} \\ \hline 2-2\sqrt{-3} \quad\;\; \\ +2\sqrt{-3}+6 \\ \hline 8. \end{array}
\end{array}$$

It is true, that these values of x are imaginary, or impossible; but yet they deserve attention.

712. What we have said applies in general to every cubic equation, such as $x^3=a$; namely, that beside the value $x=\sqrt[3]{a}$, we shall always find two other values. To abridge the calculation, let us suppose $\sqrt[3]{a}=c$, so that $a=c^3$, our equation will then assume this form, $x^3-c^3=0$, which will be divisible by $x-c$, as the actual division shews:

$$\begin{array}{l}
x-c)\; x^3-c^3\; (x^2+cx+c^2 \\
\qquad\;\; \underline{x^3-cx^2} \\
\qquad\qquad\quad cx^2-c^3 \\
\qquad\qquad\quad \underline{cx^2-c^2x} \\
\qquad\qquad\qquad\quad c^2x-c^3 \\
\qquad\qquad\qquad\quad \underline{c^2x-c^3} \\
\qquad\qquad\qquad\qquad 0.
\end{array}$$

Consequently, the equation in question may be represented by the product $(x-c)\times(x^2+cx+c^2)=0$, which is in fact $=0$, not only when $x-c=0$, or $x=c$, but also when $x^2+cx+c^2=0$. Now, this expression contains two other values of x; for it gives

$x^2=-cx-c^2$, and $x=-\frac{c}{2}\pm\surd(\frac{c^2}{4}-c^2)$, or............

$x=\frac{-c\pm\surd-3c^2}{2}$; that is to say, $x=\frac{-c\pm c\surd-3}{2}$
$=\frac{-1\pm\surd-3}{2}\times c$.

713. Now, as c was substituted for $\sqrt[3]{a}$, we conclude, that every equation of the third degree, of the form $x^3=a$, furnishes three values of x expressed in the following manner:

$$1.\ x=\sqrt[3]{a},$$

$$2.\ x=\frac{-1+\surd-3}{2}\times\sqrt[3]{a},$$

$$3.\ x=\frac{-1-\surd-3}{2}\times\sqrt[3]{a}.$$

This shews, that every cube root has three different values; but that one only is real, or possible, the two others being impossible. This is the more remarkable, since every square root has two values, and since we shall afterwards see, that a biquadratic root has four different values, that a fifth root has five values, and so on.

In ordinary calculations, indeed, we employ only the first of those values, because the other two are imaginary; as we shall shew by some examples.

714. *Question* 1. To find a number, whose square, multiplied by its fourth part, may produce 432.

Let x be that number; the product of x^2 multiplied by $\frac{1}{4}x$ must be equal to the number 432, that is to say, $\frac{1}{4}x^3=432$, and $x^3=1728$; whence, by extracting the cube root, we have $x=12$.

The number sought therefore is 12; for its square 144, multiplied by its fourth part, or by 3, gives 432.

715. *Question* 2. Required a number such, that if we divide its fourth power by its half, and add $14\frac{1}{4}$ to the product, the sum may be 100.

Calling that number x, its fourth power will be x^4; dividing by the half, or $\frac{1}{2}x$, we have $2x^3$; and adding to that $14\frac{1}{4}$, the sum must be 100. We have therefore $2x^3+14\frac{1}{4}=100$; subtracting $14\frac{1}{4}$, there remains $2x^3=\frac{343}{4}$; dividing by 2, gives $x^3=\frac{343}{8}$, and extracting the cube root, we find $x=\frac{7}{2}$.

716. *Question* 3. Some officers being quartered in a

country, each commands three times as many horsemen, and twenty times as many foot-soldiers, as there are officers. Also a horseman's monthly pay amounts to as many florins as there are officers, and each foot-soldier receives half that pay; the whole monthly expense is 13000 florins. Required the number of officers.

If x be the number required, each officer will have under him $3x$ horsemen and $20x$ foot-soldiers. So that the whole number of horsemen is $3x^2$, and that of foot-soldiers is $20x^2$.

Now, each horseman receiving x florins per month, and each foot-soldier receiving $\frac{1}{2}x$ florins, the pay of the horsemen, each month, amounts to $3x^3$, and that of the foot-soldiers, to $10x^3$; consequently, they all together receive $13x^3$ florins, and this sum must be equal to 13000 florins: we have therefore $13x^3=13000$, or $x^3=1000$, and $x=10$, the number of officers required.

717. *Question* 4. Several merchants enter into partnership, and each contributes a hundred times as many sequins as there are partners: they send a factor to Venice, to manage their capital, who gains, for every hundred sequins, twice as many sequins as there are partners, and he returns with 2662 sequins profit. Required the number of partners.

If this number be supposed $=x$, each of the partners will have furnished $100x$ sequins, and the whole capital must have been $100x^2$; now, the profit being $2x$ for 100, the capital must have produced $2x^3$; so that $2x^3=2662$, or $x^3=1331$; this gives $x=11$, which is the number of partners.

718. *Question* 5. A country girl exchanges cheeses for hens, at the rate of two cheeses for three hens; which hens lay each $\frac{1}{2}$ as many eggs as there are cheeses. Farther, the girl sells at market nine eggs for as many sous as each hen had laid eggs, receiving in all 72 sous; how many cheeses did she exchange?

Let the number of cheeses $=x$, then the number of hens, which the girl received in exchange, will be $\frac{3}{2}x$, and each hen laying $\frac{1}{2}x$ eggs, the number of eggs will be $=\frac{3}{4}x^2$. Now, as nine eggs sell for $\frac{1}{2}x$ sous, the money which $\frac{3}{4}x^2$ eggs produce is $\frac{1}{24}x^3$, and $\frac{1}{24}x^3=72$. Consequently, $x^3=24\times72=8\times3\times8\times9=8\times8\times27=1728$; whence $x=12$; that is to say, the girl exchanged twelve cheeses for eighteen hens.