# Introduction to Quantum Information Theory

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**Abstract**

Will finish this when i actually know what the jeebuz is going on in the class.

# Contents

# 1 Vectors and Vector Spaces

## 1.1 Vector Spaces

In modern mathematics, an area of importance is first defining what field the problems are being asked in. When speaking on vectors and vector spaces, it is easiest to work in two main fields of numbers.

### 1.1.1 Relevant Vector Spaces

1. $\mathbb{R}$ : **All Real Numbers** $(\infty, -\infty)$

   This includes all real numbers from zero to infinity, something of note is that this number field is **not algebraically closed**, meaning not all numbers can be produced from combinations of other numbers in the field. An example of this is $x^2 + 1 = 0$, meaning $x = 1 * i$

2. $\mathbb{C}$ : **All Complex Numbers**

   This number field **is algebraically closed**. This means that every possible number in the field can be produced from combinations of other numbers in the field. This makes sense as two complex numbers multiplied together can produce 1. A completely real number 2. A complex number with a real and imaginary component 3. A complex number with only an imaginary component.

## 1.2 Complex Number Operations

*In the following sections, we will make the assumption* $a, b, c, d, e \in \mathbb{R}$

### 1.2.1 Addition

$$(a + bi) + (c + di) = (a + c) + (b + d)i \tag{1.1}$$

### 1.2.2 Multiplication

$$(a + bi) * (c + di) = (ac + adi) + (bci - bd) \tag{1.2}$$

### 1.2.3 Conjugation

$$\overline{a + bi} = (a + bi)^* = a - bi \tag{1.3}$$

### 1.2.4 Modulus

$$|z| = |a + bi| = \sqrt{a^2 + b^2} \tag{1.4}$$

## 1.3 Common Vector Space Notation and Definitions

This section is to lay out some of the most important pieces of notation and definitions for the complex mathematics required for this course.

### 1.3.1 Linear Algebra Terms

**Definition 1.1** An element $|v\rangle \in \mathbb{C}^n$ is called a **(ket) vector** and is expressed as a **column** of $n$ complex numbers. The integer $n$ is called the **dimension** of the vector space $\mathbb{C}^n$. ⧫

**Definition 1.2** A **linear combination** of $|v_1\rangle, ..., |v_k\rangle \subset \mathbb{C}^n$ is just a single vector in the form $\lambda_1 |v_1\rangle + \lambda_2 |v_2\rangle + ... + \lambda_k |v_k\rangle$ for some $\lambda_1, \lambda_2, ..., \lambda_k \in \mathbb{C}$. ⧫

**Example 1.3** Create a **linear combination** of $|v_1\rangle = \begin{bmatrix} i \\ 1 \end{bmatrix}, |v_2\rangle = \begin{bmatrix} -1 \\ i+1 \end{bmatrix}$

To create a linear combination, we simply must add two complex number constants $\lambda_1, \lambda_2$ in the form shown in Definition 1.2, $\boldsymbol{\lambda_1 |v_1\rangle + \lambda_2 |v_2\rangle}$.

$$\lambda_{final} |v_{final}\rangle = \lambda_1 |v_1\rangle + \lambda_2 |v_2\rangle \qquad \begin{aligned} \lambda_1 &= 1 \\ \lambda_2 &= 3 + 2i \end{aligned}$$

$$\begin{aligned} \lambda_{final} |v_{final}\rangle &= 1 \begin{bmatrix} i \\ 1 \end{bmatrix} + (3+2i) \begin{bmatrix} -1 \\ i+1 \end{bmatrix} \\ &= \begin{bmatrix} i \\ 1 \end{bmatrix} + \begin{bmatrix} -3-2i \\ 1+5i \end{bmatrix} \\ &= \begin{bmatrix} -3-i \\ 3+5i \end{bmatrix} \end{aligned} \tag{1.5}$$

⧫

**Definition 1.4** Two vectors $|v_1\rangle, |v_2\rangle$ are **linearly independent** if the only way to create the 0 vector from a linear combination is through setting all complex constants $\lambda_1, \lambda_2, ..., \lambda_k$ to zero in the linear combination formula described in Definition 1.2. ⧫

**Definition 1.5** A **subspace** of complex plane $\mathbb{C}$ is a set $\mathbf{M} \in \mathbb{C}^n$ that satisfies three properties:

- $|0\rangle \in \mathbf{M}$
- $|v\rangle + |u\rangle \in \mathbf{M}$ for all $|v\rangle, |u\rangle$
- $c|v\rangle \in \mathbf{M} \;\; \forall \;\; |v\rangle \in \mathbf{M},\; c \in \mathbb{C}$ ⧫

**Definition 1.6** The **span** of a set $\mathbf{S}$ is the smallest linear subspace $\in \mathbf{S}$. ⧫

**Theorem 1.7** ***(Span of a set is always a subspace theorem)***
*Let* $\mathbf{s} = \{|v_1\rangle, |v_2\rangle, ..., |v_k\rangle\} \subset \mathbb{C}^n$*, then* $span(\mathbf{s})$ *is a subspace of* $\mathbb{C}^n$

**Proof.** 1. Note that $|0\rangle = 0|v_1\rangle + 0|v_2\rangle + ... + 0|v_k\rangle \in span(\mathbf{s})$.

2. Let $|v\rangle, |u\rangle \in span(\mathbf{s})$ be arbitrary.

   Since $|v\rangle, |u\rangle \in span(\mathbf{s})$, there exists scalars $c_1, c_2, ..., c_k$ and $d_1, d_2, ..., d_k$ such that

$$\begin{aligned} |v\rangle &= c_1|v_1\rangle, ..., c_k|v_k\rangle \\ |u\rangle &= d_1|v_1\rangle, ..., d_k|v_k\rangle \end{aligned}$$

   then,

$$\begin{aligned} |v\rangle + |u\rangle &= (c_1|v_1\rangle, ..., c_k|v_k\rangle) + d_1|u_1\rangle, ..., d_k|u_k\rangle \\ &= (c_1 + d_1)|v_1\rangle + ... + (c_k + d_k)|v_k\rangle \ \in \ span(\mathbf{s}) \end{aligned}$$

   This is the true because all of these are linear combinations consisting of a complex constant and a ket vector, which coincides with the Linear Combination Definition 1.2.

3. Let $|w\rangle \in span(\mathbf{s})$ and $c \in \mathbb{C}$. If this is true, then there exists $\lambda_1, \lambda_2, ..., \lambda_k$ such that $|v\rangle = \lambda_1|v_1\rangle, ..., \lambda_k|v_k\rangle$. Consider:

$$\begin{aligned} c|w\rangle &= c|v\rangle \\ &= C(\lambda_1|v_1\rangle, ..., \lambda_k|v_k\rangle) \\ &= (C\lambda_1)|v_1\rangle + ... + (C\lambda_k)|v_k\rangle \end{aligned} \tag{1.6}$$

   **Equation 1.6** shows that after distributing **C** in, we have another linear combination, which must exist inside of $span(\mathbf{s})$, and since $span(\mathbf{s})$ satisfies 1,2,3, $span(\mathbf{s})$ must be a subspace. ■

**Theorem 1.8** ***Basis Theorem***
*If* $\boldsymbol{S}$ *is a set of* $\mathbf{n}$ *linearly independent vectors in* $\mathbb{C}^{\mathbf{n}}$*, then* $span(\mathbf{S}) = \mathbb{C}$*.*

**Definition 1.9** A **basis** of a subspace $M \subseteq \mathbb{C}^n$ is a set of vectors **S** such that

1. S is linearly independent.
2. $span(\mathbf{S}) = M$, or in plain English "S spans M" ♦

**Definition 1.10** The **dimension** of a space is the number of vectors in it. ♦

**Example 1.11 Standard basis of different complex vector spaces.**

- Standard Basis for $\mathbb{C}^2$: $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$

- Standard Basis for $\mathbb{C}^n$: $\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}, \begin{bmatrix}0\\1\\\vdots\\0\end{bmatrix}, \ldots, \begin{bmatrix}0\\0\\\vdots\\1\end{bmatrix}$ ♦

**Exercise 1.12 (Vector Space Exercise)** Below are some exercises designed to put the concepts in this section to work, in order to further reinforce learning.

1. Consider S $= \{|v_1\rangle, |v_2\rangle, |v_3\rangle\}$, where $|v_1\rangle = \begin{bmatrix}1\\-1\end{bmatrix}, |v_2\rangle = \begin{bmatrix}i\\1\end{bmatrix}, |v_3\rangle = \begin{bmatrix}0\\i\end{bmatrix}$

   (a) Give a linear combination of the vectors in S.

$$\lambda|v\rangle = 1\begin{bmatrix}1\\-1\end{bmatrix} + 1\begin{bmatrix}i\\1\end{bmatrix} - (1+200i)\begin{bmatrix}0\\i\end{bmatrix} \tag{1.7}$$

   (b) Determine if $\begin{bmatrix}1+i\\200-i\end{bmatrix}$ is in Span($S$).

$$\alpha|v1\rangle + \beta|v_2\rangle + \gamma|v_3\rangle = \begin{bmatrix}1+i\\200-i\end{bmatrix}, \qquad \alpha = 1, \beta = 1, \gamma = ?$$

$$1\begin{bmatrix}1\\-1\end{bmatrix} + 1\begin{bmatrix}i\\1\end{bmatrix} - \gamma\begin{bmatrix}0\\i\end{bmatrix} = \begin{bmatrix}1+i\\200-i\end{bmatrix} \text{Choose alpha and beta to obtain} \begin{bmatrix}1+i\\0\end{bmatrix}$$

$$\begin{bmatrix}1+i\\0\end{bmatrix} - \gamma\begin{bmatrix}0\\i\end{bmatrix} = \begin{bmatrix}1+i\\200-i\end{bmatrix} \qquad \text{Now we must isolate } \gamma$$

$$\begin{bmatrix}0\\\gamma * i\end{bmatrix} = \begin{bmatrix}0\\-200+i\end{bmatrix} \qquad \text{We now set the matrix as an equality}$$

$$\gamma * i = -200 + i$$

$$\gamma = -(1+200i)$$

$$1\begin{bmatrix}1\\-1\end{bmatrix} + 1\begin{bmatrix}i\\1\end{bmatrix} - (1+200i)\begin{bmatrix}0\\i\end{bmatrix} = \begin{bmatrix}1+i\\200-i\end{bmatrix}$$

   Since we were able to create a linear combination of the vectors that satisfy the given vector, it is in Span($S$).

   (c) Describe Span($S$) geometrically.

   Span($S$) is every possible vector $\in \mathbb{C}^2$ ♦

**Exercise 1.13 (Linear Independence Exercise)** Find the condition under which the following two vectors are linearly independent.

$$|v_1\rangle = \begin{bmatrix} x \\ y \\ 3 \end{bmatrix} \in \mathbb{R}, \qquad |v_2\rangle = \begin{bmatrix} 2 \\ x - y \\ 1 \end{bmatrix} \in \mathbb{R}$$

$$\begin{bmatrix} x \\ y \\ 3 \end{bmatrix} = \alpha \begin{bmatrix} 2 \\ x - y \\ 1 \end{bmatrix} \tag{1.8}$$

First, we must make this linearly dependent so we know when x and y fail linear independence

$$\begin{aligned} \alpha = 3 \quad \rightarrow \quad & \begin{bmatrix} x \\ y \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ x - y \\ 1 \end{bmatrix} \\ & \begin{bmatrix} x \\ y \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 3(x - y) \\ 3 \end{bmatrix} \end{aligned}$$

Now we need to make this an equality to solve for x and y.

$$\begin{aligned} x = 6 \quad \rightarrow \quad y &= 3(x - y) \\ y &= 3x - 3y \\ 4y &= 18 \\ y &= 9/2 (\text{or } 4.5) \end{aligned}$$

♦

**Exercise 1.14 (Basis Exercise)** Show that the set formed by the following vectors is a basis for $C^3$

$$|V_1\rangle = \begin{bmatrix}1\\1\\1\end{bmatrix}, \ |V_1\rangle = \begin{bmatrix}1\\0\\1\end{bmatrix}, \ |V_1\rangle = \begin{bmatrix}1\\-1\\-1\end{bmatrix}$$

There are two things we must prove in order for a set of vectors to form a basis: linear independence and an equal number of vectors as the dimension of the space.
First we will prove linear independence:

**1. Linear Independence Test**

$$\begin{aligned}
&\begin{bmatrix}1&1&1&0\\1&0&-1&0\\1&1&-1&0\end{bmatrix}\text{First, we build augmented matrix}\\
\xrightarrow{R_2-R_3}&\begin{bmatrix}1&1&1&0\\0&-1&0&0\\1&1&-1&0\end{bmatrix}\\
\xrightarrow{R_3-R_1}&\begin{bmatrix}1&1&1&0\\0&-1&0&0\\0&0&-2&0\end{bmatrix}\\
\xrightarrow{-1*R_2,\,-\frac{1}{2}*R_3}&\begin{bmatrix}1&1&1&0\\0&1&0&0\\0&0&1&0\end{bmatrix}\\
\xrightarrow[R_1-R_3]{R_1-R_2}&\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix}
\end{aligned} \tag{1.9}$$

Thus, we have proved that if the only way to obtain the [0] matrix is through $\lambda_1, ..., \lambda_k = 0$ as mentioned in Definition 1.4, we know that the vectors are **linearly independent**.

**2. Dimension Test**

Now we must prove that the dimension of the space is equal to the number of the vectors, and since we are working with $C^3$ and we have $|V_1\rangle, |V_2\rangle, |V_3\rangle$ we know that our space has three dimensions and we have three vectors, therefore we have just proven that **the given set of vectors forms a basis for $\mathbb{C}^3$.** ⧫

## 1.4 Inner/Outer Products, and Norms of Vectors

**Definition 1.15** The **dual space of** $\mathbb{C}^{\mathbf{n}}$ ($\mathbb{C}^{\mathbf{n}^*}$) is a space of row vectors where there are n entries from $\mathbb{C}$. ♦

**Definition 1.16** A **transpose** is an operation that every column into a row vector and vice versa without changing the order. ♦

**Example 1.17 (Example of transpose operation)**

$$|V\rangle^\dagger = \begin{bmatrix} 7 \\ 8i \\ \pi + 3i \\ 0 \end{bmatrix}^\dagger \tag{1.10}$$

$$\langle V| = \begin{bmatrix} 7 & 8i & \pi + 3i & 0 \end{bmatrix} \tag{1.11}$$

It can be seen that the only difference between **1.10** and **1.11** is that we have converted the column to a row, this is accomplished through the **dagger (†)** being applied to a matrix or vector. ♦

**Definition 1.18** Given $|V\rangle, |W\rangle$, the **inner product** of $|V\rangle, |W\rangle$ is $\langle W|V\rangle$ which can also be written as $\langle W| * |V\rangle$.
*It should be noted that the inner product of a bra and ket vector will always yield a constant.* ♦

**Example 1.19 (Example of an Inner Product on $\mathbb{R}^n$)**

$$|V\rangle = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, |W\rangle = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} \tag{1.12}$$

$$\langle W|V\rangle = \begin{bmatrix} 5 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} = 5 - 2 + 3 = 6 \tag{1.13}$$

*It should be noted that the inner product on $\mathbb{R}^n$ is just the usual dot product.* ♦

**Example 1.20 (Example of an Inner Product on $\mathbb{C}^n$)**

$$|V\rangle = \begin{bmatrix} i \\ 1 \end{bmatrix}, |W\rangle = \begin{bmatrix} -i \\ -1 \end{bmatrix} \tag{1.14}$$

$$\langle W|V\rangle = \begin{bmatrix} -i & -1 \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} = 1 - 1 = \mathbf{0} \tag{1.15}$$

*The inner product yielding 0 means that the two vectors are **orthogonal**.* ♦

**Definition 1.21** The **norm (or magnitude)** of $|V\rangle \in \mathbb{C}^n$ is

$$\||V\rangle\| \coloneqq \sqrt{\langle V|V\rangle} \tag{1.16}$$

♦

**Example 1.22 (Example of taking the norm of a complex vector)**

$$v = \begin{bmatrix}1, 1\end{bmatrix} \in \mathbb{R}^2$$

$$\||V\rangle\| \coloneqq \sqrt{\begin{bmatrix}1 & 1\end{bmatrix} | \begin{bmatrix}1 \\ 1\end{bmatrix}} = \sqrt{1+1} = \sqrt{2}$$

**Two points of note:**
1. Can $\langle W|V\rangle$ be complex? **YES**
2. Can $\langle V|V\rangle = \||V\rangle\|^2 \in \mathbb{C}$? **YES** ♦

**Definition 1.23** A set of nonzero vectors $\mathrm{S} = \{|V_1\rangle, |V_2\rangle, ..., |V_k\rangle\} \leq \mathbb{C}^n$ is called **an orthogonal set** if $\langle \mathbf{v_i}|\mathbf{v_j}\rangle = \mathbf{0}$ if $\mathbf{i} \neq \mathbf{j}$. ♦

**Theorem 1.24 (Orthogonality and Linear Independence Theorem)** *In this theorem, we will use what we have learned so far in order to gain a deeper understanding into the properties of orthogonality in a vector space.*

***If S*** $= |V_1\rangle, ..., |V_1\rangle$ ***is an orthogonal vector set of nonzero vectors in*** $\mathbb{C}^n$***, then S is linearly independent.***

**Proof.** Let $c_1, ..., c_k \in \mathbb{C}$ s.t. $c_1 |v_1\rangle, c_2 |v_2\rangle, ..., c_k |v_k\rangle = |0\rangle$.
**Goal**: Show $c_1, ..., c_k = 0$ using assumption S is orthogonal set.

Let $j \in \{1, ..., k\}$, then

$$\begin{aligned} &\langle v_j| (c_1 |v_1\rangle, ..., c_k |v_k\rangle) = \langle v_j|0\rangle \\ &c_1 \langle v_j|v_1\rangle + ... + c_k \langle v_j|v_k\rangle \\ &c_j \langle v_j|v_j\rangle = \text{Since S is an orthogonal set} \end{aligned} \tag{1.17}$$

Since $\langle v_j|v_j\rangle$ can't possibly be 0 because we know that $j \neq 0$, it means that $c_k$ must be zero, which as stated in the Linear Independence Definition, means that S is linearly independent.■

**Definition 1.25** A basis is **orthonormal** if two conditions are satisfied:

1. S is an orthogonal set of $n$ unit vectors.
2. Span(S) $\in \mathbb{C}^n$

In plain English, an orthonormal basis is a basis in which there are exactly the number of vectors as the dimension of the space (from the basis definition) as well as well as all of those vectors being orthogonal to each other. ♦

**Exercise 1.26 (Basis Practice)** In this example, there will be multiple transformations completed to gain a better understanding of what a basis is, and what it means to be orthogonal.

1.
$$|b_1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \qquad |b_2\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix} \tag{1.18}$$

   (a) Show that $\mathcal{B}$ is an orthonormal basis for $\mathbb{C}^2$.

      i. Check orthogonality:

$$\langle b_1|b_2\rangle = \frac{1}{\sqrt{2}}(\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} = 1 - 1 = \mathbf{0}$$

      These two vectors are orthogonal since their inner product is 0.

      ii. Check that they are unit vectors: (is the norm one?)

$$\||b_1\rangle\| = \sqrt{\langle b_1|b_1\rangle} = \frac{1}{\sqrt{2}}\sqrt{1+1} = \frac{\sqrt{2}}{\sqrt{2}} = \mathbf{1}$$
$$\||b_2\rangle\| = \sqrt{\langle b_2|b_2\rangle} = \frac{1}{\sqrt{2}}\sqrt{1^2+(-1)^2} = \frac{\sqrt{2}}{\sqrt{2}} = \mathbf{1}$$

   This means that, since the vectors are orthogonal and unit vectors, they are orthonormal, forming an orthonormal basis for $\mathbb{C}^2$.

(b) Find the coordinates of $|x\rangle = \begin{bmatrix}-21\end{bmatrix}$ relative to basis $\mathcal{B}$ (i.e. find the scalars such that...)

$$c_1, c_2 \in \mathbb{C} \text{ such that } c_1 |b_1\rangle + c_2 |b_2\rangle = |x\rangle$$

$$\langle b_1| (c_1 |b_1\rangle + c_2 |b_2\rangle) = \langle b_1|x\rangle$$
$$c_1 \langle b_1|b_1\rangle + c_2 \langle b_1|b_2\rangle = \langle b_1|x\rangle$$

$$b_1, b_2 \text{ are orthogonal so } \langle b_1|b_2\rangle = 0 \text{ so}$$

$$c_1 = \langle b_1|x\rangle$$

$$\boxed{c_1 = \frac{1}{\sqrt{2}} \begin{bmatrix}1 & 1\end{bmatrix} \begin{bmatrix}-2 \\ 1\end{bmatrix} = \frac{\mathbf{-1}}{\sqrt{\mathbf{2}}}}$$

$$\langle b_2| (c_1 |b_1\rangle + c_2 |b_2\rangle) = \langle b_2|x\rangle$$
$$c_1 \langle b_2|b_1\rangle + c_2 \langle b_2|b_2\rangle = \langle b_2|x\rangle$$

$$b_1, b_2 \text{ are orthogonal so } \langle b_2|b_1\rangle = 0 \text{ so}$$

♦

$$c_2 = \langle b_2|x\rangle$$

$$\boxed{c_2 = \frac{1}{\sqrt{2}} \begin{bmatrix}1 & -1\end{bmatrix} \begin{bmatrix}-2 \\ 1\end{bmatrix} = \frac{\mathbf{-3}}{\sqrt{\mathbf{2}}}}$$

**Definition 1.27** Define $P_i \coloneqq |f_i\rangle \langle f_i|$ to be the **projection operator** onto span($|k_i\rangle$) ⧫

**Proposition 1.28** *Let $\mathcal{B} = \{|f_1\rangle, ..., |f_n\rangle\}$ be an orthonormal basis for $\mathbb{C}^n$ then...*

1. $P_i(|v\rangle) \in span(|f_1\rangle)$
2. $|v\rangle - P_i|v\rangle$ *is orthogonal to* $|f_i\rangle$
3. ${P_i}^2 = P_i * P_i = P_i$ *Projection will still be the same.*
4. $P_iP_j = 0$ *shadow on one plane will be "nothing" in the perspective of another plane.*
5. $P_i^\dagger = P_i$ *self-adjoint*
6. $\sum_{i=1}^n P_i = I_n$

**Example 1.29 (Outer Product Example)** Given $P_2 = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, what is $P_1 + P_2$

$$\begin{aligned} P_1 &= |f_1\rangle \langle f_1| \\ &= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} * \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\ P_1 + P_2 &= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

This is an interesting result, it shows that if you add all of the projections in a space you will get the identity matrix, proof available elsewhere as I am too lazy to type it out. ⧫

**Why is projection useful?**: It allows us to find $c_1, ..., c_k$ for $|x\rangle = c_1|f_1\rangle + ... + c_k|f_k\rangle$ when all $|f\rangle$ are orthogonal to each other.

**Theorem 1.30 (The Gram Schmidt Process)** *Let $|b_1\rangle, ..., |b_k\rangle$ be the basis for a subspace $M$ in $\mathbb{C}^n$. Define the set of vectors $|f_1\rangle, ..., |f_k\rangle$ as follows:*

***Step 1:***
*Define $|f_1\rangle \coloneqq |b_1\rangle$.*

***Step i+1:***
*for i = 1:k-1*

$$|f_{i+1}\rangle \coloneqq |b_{i+1}\rangle - (\sum_{j=1}^{i} \frac{|f_j\rangle \langle f_j|}{\langle f_j|f_j\rangle}) |b_{i+1}\rangle = |b_{i+1}\rangle - (\sum_{j=1}^{i} \frac{\langle f_j|b_{i+1}\rangle}{\langle f_j|f_j\rangle}) |f_j\rangle \tag{1.19}$$

***Done***

***Now normalization step***
*for i = 1 : k*

$$|f_i\rangle \coloneqq \frac{1}{\| |f_i\rangle \|} |f_i\rangle \tag{1.20}$$

***Done, now we know that $|f_1\rangle, ... |f_k\rangle$ is a orthonormal basis for M***

*A special thank you for Dr. Hamidi and Dr. Ismert for giving this psuedocode function for Gram Schmidt, I have pulled from it heavily in this page as it is the best way to explain this code.*

# 2 Matrices and Linear Transformations

## 2.1 Matrices

**This section assumes a basic knowledge of matrix operations, and should be a basic overview.**

**Definition 2.1** $\mathbf{M_{mn(}}\mathbb{C})$ = set of all mxn matrices with complex entries. ♦

**Definition 2.2** $\mathbf{M_n}(\mathbb{C}) \equiv \mathbf{M_{nn}}(\mathbb{C})$ meaning that it is just a square matrix with nxn size. ♦

**Example 2.3 (Pauli Matrices)**

$$\begin{aligned} \sigma_x &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ \sigma_y &= \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \qquad \in M_2 \\ \sigma_z &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \end{aligned} \tag{2.21}$$

*These three matrices are called the* ***Pauli matrices*** *the the basic matrices we will use for quantum computations in the future.* ♦

**Definition 2.4** The **Identity Matrix** $(I_n)$ is the matrix $M_n$ whose columns are the standard basis $|e_1\rangle, ..., |e_n\rangle$ for $\mathbb{C}_n$.

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.22}$$

♦

**Theorem 2.5** *$M_n$ has a well defined matrix multiplication, in general, if $A \in M_{mn}, B \in M_{nk}$.*

*The matrix $AB = [A\,|b_1\rangle \;\; ... \;\; A\,|b_k\rangle]$*

**Example 2.6 (Pauli Matrices in action)** For this example, we will be scaling the $\sigma_y$ Pauli matrix.

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & -3i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 1\,|a_1\rangle & 3\,|a_2\rangle \end{bmatrix} \tag{2.23}$$

This shows that in general, multiplying a row matrix by a diagonal matrix will scale the row matrix by the respective diagonal matrix value. ♦

## 2.2 Eigenvectors and Eigenvalues

### 2.2.1 Eigenvalues/Eigenvectors

**Definition 2.7** An **eigenvalue** for $A \in M_n$ is a complex number $\lambda \in \mathbb{C}$ such that there is a nonzero vector $|x\rangle \in \mathbb{C}^n$ satisfying

$$A\,|x\rangle = \lambda\,|x\rangle \tag{2.24}$$

♦

**Definition 2.8** A mmatrix $A \in M_n$ is **diagnolizable** if and only if:

1. There exists a diagonal matrix D and an invertable matrix P such that $A = PDP^{-}1$ if and only if
2. there exists a basis for $C^n$ consisting of eigenvalues for A. ♦

**Example 2.9** Consider:

$$A = \begin{bmatrix} I_2 & 0 \\ 0 & \sigma_y \end{bmatrix} \equiv \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \end{bmatrix} \tag{2.25}$$

1. Get eigenvalues and some normalized eigenvectors:

$$\begin{aligned} &\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix} \\ =&\begin{bmatrix} 1-\lambda & 0 & 0 & 0 \\ 0 & 1-\lambda & 0 & 0 \\ 0 & 0 & \lambda & -i \\ 0 & 0 & i & -\lambda \end{bmatrix} \\ =&(1-\lambda)(1-\lambda)[\lambda^2 - 1] = 0 \\ &\lambda = 1, 1, 1, -1 \end{aligned} \tag{2.26}$$

2. Now plug in eigenvalue and find eigenvector:

$$A|e_1\rangle = |e_1\rangle$$
$$A|e_2\rangle = |e_2\rangle$$
$$A - \lambda * I = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -i \\ 0 & 0 & i & -1 \end{bmatrix}$$
$$\xrightarrow{R_4 + iR_3} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & i \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\xrightarrow[R_1 \leftrightarrow R_3]{-1R_1} \begin{bmatrix} 0 & 0 & 1 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now we must use this and transfer it into an equation to solve for $x_3$, $x_4$

$x_3 = ix_4$

Choose $x_4 = 1$

$$|x\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ i \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ i \end{bmatrix} \tag{2.27}$$

These first two arrays are just chosen because there are no $x_1, x_2$ values in the eigenvector array, so we choose two unit vectors.

3. Finally, normalize the vector.

$$|x\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ i \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ 1 \\ i \end{bmatrix}$$

4. Final answer

$$\text{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} * i \\ 0 & 0 & \frac{1}{\sqrt{2}} * i & \frac{1}{\sqrt{2}} \end{bmatrix}, \text{D} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$ ♦

### 2.2.2 Eigenvectors and Eigenvalues Big Picture

1. Self-Adjoint (Hermitian)

$$\begin{aligned} A \in M_n &\text{ is Hermitian if} \\ A^\dagger &= A \\ {\sigma_x}^\dagger &= \sigma_x \\ {\sigma_y}^\dagger &= \sigma_y \\ {\sigma_z}^\dagger &= \sigma_z \end{aligned} \tag{2.28}$$

   Additionally, all Hermitian matrices are diagnolizable, meaning their eigenvalues form a orthonormal basis.

2. Normal Matrices

$$A * A^\dagger = A^\dagger * A \leftrightarrow \text{eigenvectors form orthonormal basis} \tag{2.29}$$

3. Positive semi-definite matrix $M_n$ is:

$$\begin{aligned} &\forall \left|x\right\rangle \in \mathbb{C}^n, \text{we have} \\ &\langle x|A|x\rangle \geq 0 \text{ where } A\left|x\right\rangle = y \end{aligned} \tag{2.30}$$

4. (Unitary Matrices) The following are all equivalent:
   - $U \in M_n$ is unitary
   - $U^\dagger = U^{-1}$ or $(U^{-1} * U = U * U^{-1} = I_n)$
   - $U * U^\dagger = I_n$ or $U^\dagger * U = I_n$ *note: $U^\dagger$ not always equal to $U$, but they are normal.*
   - The column of U form an orthonormal basis for $\mathbb{C}^n$.
   - $\forall \left|x\right\rangle, \left|y\right\rangle \in \mathbb{C}^n$ and $\langle U_x|U_y\rangle = \langle x|y\rangle$ meaning unitary matrices are rotation matrices.

# 3 Introduction to Quantum Theory

Now that we have laid out the mathematical framework necessary to understand this subject, it is time to lay our the physics necessary to complete the basis (no pun intended) for Quantum Information Theory.

## 3.1 Quantum Theory Axioms

These axioms are humans attempts at creating rules and methods of abstractions to describe and understand

### 3.1.1 Axiom 1.1: A vector state x is a unit vector in a complex Hilbert space.

We want to figure out the photon's (particle) polarization, to figure this out we are going to shoot it at a vertically polarized filter. Classically, $|x\rangle$ should be polarized $\uparrow$ or $\rightarrow$, if this is the case we expect two outcomes:

1. If $|x\rangle$ goes through filter, then $|x\rangle$ is $\uparrow$.
2. If $|x\rangle$ is deflected, then $|x\rangle$ is $\rightarrow$.

### 3.1.2 Axiom 1.2: Linear combinations (or superposition) of the physical states are allowed to act as x vectors.

Physical States (Choices made by humans to describe quantum mechanics):

$$\uparrow = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \rightarrow = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

*Something of note, these two vectors form a basis for $\mathbb{C}^2$ and all combinations are:*

$$|x\rangle = \alpha \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ where } \|\alpha\|^2 + \|\beta\|^2 = 1$$

To keep track of this choice, we make the matrix $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ representing vertically polarized filter.

*Another note, the e-vals of that matrix are $|v\rangle, |h\rangle$.*

**Example 3.1** $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ might be a nice to represent a horizontally polarized filter. ♦

### 3.1.3 Axiom 2: Observable

An observable of a state $|x\rangle$ corresponds to a Hermitian matrix A. $|x\rangle$ is in state $|u\rangle$, an e-vect for A with probability $|\langle u|x\rangle|^2$

**Definition 3.2** Expectation value of A (or mean value) of observable associated to A after measurements with respect to many copies of $|\psi\rangle$ is **the weighted average of the expected outcomes**.

$$\begin{aligned} \langle A \rangle_\psi &= \sum_{i=1}^{n} |\langle u_i|\psi\rangle|^2 \\ |\psi\rangle &= \sum_{i=1}^{n} c_i |u_i\rangle \text{ where } u_i \text{ is an e-basis from A} \end{aligned}$$

♦

### 3.1.4 Axiom 3: The time dependence of a state is governed by the Schrödinger equation

$$i\bar{h}\frac{\delta|\psi|}{\delta t} = H|\psi\rangle \tag{3.31}$$

$\bar{h}$ is reduced Planck's constant
H is Hermitian matrix corresponding to energy of the system **Hamiltonian**.

When H is time invariant (constant), the Schrödinger equation becomes:

$$|\psi(t)\rangle = e^{\frac{-itH}{\bar{h}}}|\psi(0)\rangle \tag{3.32}$$

## 3.2 More Introductory Quantum Concepts

Now that the basic framework of what a quantum state is, it is now time to apply these concepts and build on them to further characterize what a quantum system is, and why it is important to us. Prepare yourself, because we are jumping right into an example applying the previous axioms to characterize a *real world* quantum system.

**Example 3.3** Consider a physical system with Hamiltonian $H = \frac{\hbar}{2}\omega\sigma_x$ and suppose $|\psi(0)\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (The first column of $\sigma_z$.)

- Find the wave function $|\psi(t)\rangle, t > 0$, which by the Shrödinger equation is:

$$\begin{aligned}
|\psi(t)\rangle &= \exp\frac{-itH}{\hbar} * |\psi\rangle \\
&= \exp\left(i(\frac{-t}{\hbar})(\frac{\hbar}{2}\omega\sigma_x)\right)\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \exp i(\frac{t}{2}\omega)\sigma_x)\begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{where } \frac{t}{2}\omega \text{ is } \alpha \\
&= \exp i(\alpha)\sigma_x)\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= [\cos(\frac{t}{2}\omega)I + i\sin(\frac{t}{2}\omega)\sigma_x]\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \cos(\frac{t}{2}\omega) & i\sin(\frac{t}{2}\omega) \\ i\sin(\frac{t}{2}\omega) & \cos(\frac{t}{2}\omega) \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
|\psi(t)\rangle &= \begin{bmatrix} \cos(\frac{\omega t}{2}) \\ i\sin\frac{\omega t}{2} \end{bmatrix}
\end{aligned} \tag{3.33}$$

- Find the probability for the system to have outcome +1 upon measurement of $\sigma_z$

$$\begin{aligned}
P_2|\psi(t)\rangle = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \cos(\frac{\omega t}{2}) \\ i\sin\frac{\omega t}{2} \end{bmatrix} = \begin{bmatrix} \cos(\frac{\omega t}{2}) \\ 0 \end{bmatrix} \\
P_\downarrow(t) = |\cos\frac{\omega t}{2}|^2 = \cos^2\frac{\omega t}{2}
\end{aligned} \tag{3.34}$$

As you can see, we have picked off the cos out of $\psi$ and in order to compute the probability distribution of the wave, we square the final answer, this will look familiar to a probability and statistics class topic of expected values and distributions (because it is... Wow! never thought it would be useful again)

- Find the probability for the system to have outcome -1 upon measurement of $\sigma_z$

$$
\begin{aligned}
P_2\,|\psi(t)\rangle &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\frac{\omega t}{2}) \\ i\sin\frac{\omega t}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ i\sin\frac{\omega t}{2} \end{bmatrix} \\
P_\downarrow(t) &= |i\sin\frac{\omega t}{2}|^2 = \sin^2\frac{\omega t}{2}
\end{aligned}
\tag{3.35}
$$

Same thing as the one above, but notice that this one is picking out the sin in $\psi$ instead using the other diagonal term in the P matrix.

- Find the expectation value under many measurements of $\sigma_z$:

$$
\begin{aligned}
< \sigma_x >_\psi &= \langle\psi(t)|\sigma_z\psi(t)\rangle = (+1)(\cos^2\frac{\omega t}{2}) + (-1)(\sin^2\frac{\omega t}{2}) \\
&= (\cos^2\frac{\omega t}{2}) - (\sin^2\frac{\omega t}{2})
\end{aligned}
\tag{3.36}
$$

As you can see, we have set matrix $\sigma_z$ against vector $\psi(t)$, giving us the overall expected outcome if we measured wave function $\psi(t)$ in the $\sigma_z$ "direction". ⧫

## 3.3 Multipartite Physical States

### 3.3.1 Tensor Products

**Definition 3.4** Tensor Product Consider vector space H with an inner product $H = H_1 \otimes H_2$ where $\otimes$ is the **tensor product** combining $H_1, H_2$ in such a way that things in $H_1$ effect $H_2$ and vice versa. ♦

A general vector of H is a linear combination of vectors $\{|v_1\rangle \otimes |v_2\rangle : |v_1\rangle \in H_1, |v_2\rangle \in H_2\}$

Now to take the tensor of two matrices where $A \in M_{mn}, B \in M_{pq}$:

$$A \otimes B = A = \begin{bmatrix} A_{11}B & A_{12}B & \dots & A_{1n}B \\ A_{21}B & A_{22}B & & \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & \dots & & A_{KK} \end{bmatrix} \in M_{(mp)(nq)} \tag{3.37}$$

This equation may look overwhelming at first glance, but it can be seen that all it is doing is applying the *elements* of A to the *matrix* B. This means that you will get a matrix that any change in A affects B, and vice versa just like we hoped.

**Example 3.5**

$$\sigma_x \otimes i\sigma_y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix} \tag{3.38}$$

*Understand that this is actually just a 4X4 matrix, and was only written with the extra brakets in order to provide clarity.* ♦

**Rules of tensor products** Now that we have learned a little about tensor products, lets learn some basic rules that apply to this operation:

1. $(\lambda A) \otimes B = A \otimes (\lambda B), \quad \lambda \in \mathbb{C}$ *(Scaling one matrix affects the other and vise verse)*
2. In general $A \otimes B \neq B \otimes A$ *(Generally noncommutative, not always true though)*
3. $(A \otimes B)(C \otimes D) = AC \otimes BD$ *(Distributivity law)*
4. $A \otimes (B + C) = (A \otimes B) + (A \otimes C)$ *(Another distributivity law)*
5. $(A \otimes B) + (C \otimes D) \neq (A + C) \otimes (B + D)$ *(Nonassociative)*
6. $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$ *(Daggers are distributive)*
7. If $A$ and $B$ are invertible, $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$ *(Inverses are distributive)*

**Example 3.6 (Tensor product of two ket vectors)**

$$|1\rangle \otimes |1\rangle =: |11\rangle = \begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$$

$$|1\rangle \otimes |2\rangle =: |12\rangle = \begin{bmatrix}0\\1\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}$$

$$|2\rangle \otimes |1\rangle =: |21\rangle = \begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}$$

$$|2\rangle \otimes |2\rangle =: |22\rangle = \begin{bmatrix}0\\1\end{bmatrix} \otimes \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

*Note:* $\{|11\rangle, |12\rangle, |21\rangle, |22\rangle\}$ *forms a basis for* $\mathbb{C}^4$ *but these vectors live in* $\mathbb{C}^2 \otimes \mathbb{C}^2 \approx \mathbb{C}^4$ ⧫

This example showed a property of tensor products in which if $|v\rangle \in \mathbb{C}^m, |w\rangle \in \mathbb{C}^n$, then $|wv\rangle := |v\rangle \otimes |w\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n \approx \mathbb{C}^{mn}$, so the result is the two dimensions multiplied.

Since we have done so much fun stuff this chapter, lets finish it off with something fun, a proof!

**Theorem 3.7 (Tensor Product Basis)**
*If $H_1$ has ONB $\varepsilon_1 = \{|e_{1,1}\rangle, |e_{1,2}\rangle, \ldots, |e_{1,m}\rangle\}$ and $H_2$ has ONB $\varepsilon_1 = \{|e_{2,1}\rangle, |e_{2,2}\rangle, \ldots, |e_{2,n}\rangle\}$*

*Note: First subscript is the basis where the vector is formed.*

*Then $H \coloneqq H_1 \otimes H_2$ has an orthonormal basis $\{|e_{1,i}\rangle \otimes |e_{2,i}\rangle : 1 \leq i \leq m, 1 \leq j \leq n\}$*

**Proof.** Let $\sum_p \lambda_p |v_p\rangle \otimes |w_p\rangle \in H$
It suffices to show that just one elementary tensor $ketv_p \otimes |w_p\rangle$ belongs to Span($\varepsilon$), so since $\varepsilon_1$ is a basis for $H_1$ and $ketv_p \in H_1$, we can write

$$|v_p\rangle = c_1 |e_{1,1}\rangle + c_2 |e_{1,2}\rangle + \cdots + c_m |e_{1,m}\rangle \text{ for some } c_1, \ldots, c_m \in \mathbb{C}$$

$$|w_p\rangle = d_1 |e_{2,1}\rangle + d_2 |e_{2,2}\rangle + \cdots + d_n |e_{2,n}\rangle \text{ for some } d_1, \ldots, d_n \in \mathbb{C}$$

$$\begin{aligned}
\text{So } |v_p\rangle \otimes |w_p\rangle &= (c_1 |e_{1,1}\rangle + c_2 |e_{1,2}\rangle + \cdots + c_m |e_{1,m}\rangle) \otimes (d_1 |e_{2,1}\rangle + d_2 |e_{2,2}\rangle + \cdots + d_n |e_{2,n}\rangle) \\
&= (\sum_{i=1}^{m} c_i |e_{1,i}\rangle) \otimes (\sum_{j=1}^{n} d_j |e_{2,j}\rangle) \\
&= \sum_{i=1}^{m} \sum_{j=1}^{n} c_i |e_{1,i}\rangle \otimes d_j |e_{2,j}\rangle \\
&= \sum_{i=1}^{m} \sum_{j=1}^{n} c_i d_j (|e_{1,i}\rangle \otimes |e_{2,j}\rangle) \in Span(\varepsilon) \textbf{ by tensor product fact 1}
\end{aligned}$$

**What is the inner product of H?**

$$\begin{aligned}
&\text{let } |v_1\rangle, |w_1\rangle \in H_1 \text{ and } |v_2\rangle, |w_2\rangle \in H_2 \\
\langle |v_1\rangle \otimes |v_2\rangle \| w_1\rangle \otimes |w_2\rangle\rangle &\coloneqq \langle v_1 v_2 | w_1 w_2\rangle \\
&= \langle v_1 | w_1\rangle_{H_1} \langle v_2 | w_2\rangle_{H_2} \textit{ Inner products of first factors form first set and same for seco}
\end{aligned}$$

Finally we must check that $\varepsilon$ is an orthonormal set.

$$\begin{aligned}
&\text{Let } |e_{1,i}\rangle \otimes |e_{2,j}\rangle, |e_{1,k}\rangle \otimes |e_{2,l}\rangle \in \varepsilon \\
&\text{Then } \langle e_{1,i} e_{2,j} | e_{1,k} e_{2,l}\rangle = \langle e_{1,i} | e_{1,k}\rangle_{H_1} * \langle e_{2,j} | e_{2,l}\rangle_{H_2} \\
&\text{That means this breaks down to when i=k and j=l otherwise they are orthogonal so} \\
&= \| |e_{1,i} e_{2,j}\rangle \| \text{ which must be 1 since they are parallel so } \varepsilon \text{ is an orthonormal set.}
\end{aligned} \tag{3.39}$$

∎

### 3.3.2 Quantum Multipartite Systems

$$H = H_1 \otimes H_2 \otimes \cdots \otimes H_n \text{ is a multipartite system} \tag{3.40}$$

*Note: if n is two then the system is* ***bipartite***

**Definition 3.8 (Bipartite state)** $|\psi\rangle$ state in $H_1 \otimes H_2$ is called a **bipartite state** ♦

**Definition 3.9 (Separability)** A vector $|\psi\rangle \in H = H_1 \otimes H_2$ is **separable** (or an elementary tensor) if $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$ for some $|\psi_1\rangle \in H_1, |\psi_2\rangle \in H_2$ ♦

In plain English, separability states that there are vectors that can be tensored together to create the final tensor, an example below will illustrate this better.

**Example 3.10** Is $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \in \mathbb{C}^4 (\approx \mathbb{C}^2 \otimes \mathbb{C}^2)$ an elementary tensor?

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 * \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ 1 * \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \tag{3.41}$$

It *is* separable. ♦

**Definition 3.11** A vector that is *not* separable is called **entangled**. ♦

**Remark 3.12** A state $\psi = C_{11}|11\rangle + C_{12}|12\rangle + C_{21}|21\rangle + C_{22}|22\rangle$ is separable only if (in $C^2$) rank is one (i.e. only if its rows are scalar multiples of each other), where the coeff matrix is $\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$. ♦

Most of the time, finding if $|\psi\rangle \in H_1 \otimes H_2$ is separable is not as easy as the exercises as above, this is because most of the time we cannot look at a matrix and infer the vectors being tensored together, this is where the next section comes into play, and we use a singular value decomposition.

### 3.3.3 Singular Value Decomposition

First things first, we must know that everything that is about to be shown is only applicable in bipartite systems, anything larger than that is an impossible, and yes that really means impossible, task in most situations because of the properties of quantum systems and their probabilistic nature. If you can figure out an algorithmic way to solve the decomposition of multipartite systems, you will be very famous and more than likely have a decomposition named after you. Now that we have laid that out there, lets continue with singular value decompositions.

The Singular Value Decomposition takes on the form $C = U\Sigma V^\dagger$, this will mean very little, but keep this in mind as you move through the example as this is the motivation to the steps below.

**Definition 3.13** If $A \in M_{mn}$, then the **singular values** for A are the square roots of the e-vals for $A^\dagger A \in M_{nn}$. ♦

**Fact:** For any $A \in M_{mn}$, $A^\dagger A$ is positive semi-definite, meaning it is Hermetian and its eigenvalues are non-negative.

**Example 3.14** Find the singular value decomposition (SVD) for A = $\begin{bmatrix} 1 & 1 \\ 0 & 0 \\ i & i \end{bmatrix}$

1. **Compute eigenvalues and eigenvectors**

$$A^\dagger A = \begin{bmatrix} 1 & 0 & -i \\ 1 & 0 & -i \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ -i & -i \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \tag{3.42}$$

Find the eigenvalues

$$det(A^\dagger A - \lambda I) = 0 \rightarrow \boxed{\lambda = 0, 4}$$

Find the eigenvectors

$$\lambda = 0 \rightarrow A^\dagger A \left|x\right\rangle = \left|0\right\rangle \rightsquigarrow \left|x\right\rangle = \boxed{\frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}}$$

$$\lambda = 4 \rightarrow A^\dagger A \left|x\right\rangle = 4\left|x\right\rangle \rightsquigarrow \left|x\right\rangle = \boxed{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

Now we must take the spectral decomposition to get the e-vec matrix (see below). We are allowed to take the spectral decomposition because we know that $A^\dagger A$ is positive semi definite.

$$A^\dagger A = \boxed{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}} \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\right)^\dagger \tag{3.43}$$

2. **Make sure V's columns are in decreasing order with respect to eigenvalues.**

$$V = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$$

$$\Sigma = \begin{bmatrix}2 & 0\\ 0 & 0\\ 0 & 0\end{bmatrix} \leftarrow \begin{bmatrix}\sqrt{\lambda_1} & 0\\ 0 & \sqrt{\lambda_2}\\ 0 & 0\end{bmatrix} \text{(This is generally how it is)}$$

3. **Construct more e-vectors** $A\left|\lambda_1\right\rangle, A\left|\lambda_2\right\rangle$

$$A\left|\lambda_1\right\rangle = \begin{bmatrix}1 & 1\\ 0 & 0\\ i & i\end{bmatrix}\left(\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\end{bmatrix}\right) = \begin{bmatrix}\frac{2}{\sqrt{2}}\\ 0\\ \frac{2i}{\sqrt{2}}\end{bmatrix} = \sqrt{2}\begin{bmatrix}1\\ 0\\ i\end{bmatrix}$$

$$A\left|\lambda_2\right\rangle = \begin{bmatrix}1 & 1\\ 0 & 0\\ i & i\end{bmatrix}\left(\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ -1\end{bmatrix}\right) = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix} \text{This one is useless as there is no new info}$$

4. **Now use** $\left|u_1\right\rangle = \sqrt{2}\begin{bmatrix}1\\ 0\\ i\end{bmatrix}$ **to create an ONB for** $\mathbb{C}^3$**.**

$\{\left|u_1\right\rangle, \left|u_2\right\rangle, \left|u_3\right\rangle\} \rightsquigarrow \{\left|u_1\right\rangle, \left|2\right\rangle, \left|3\right\rangle\}$ where $\left|2\right\rangle = \begin{bmatrix}0\\ 1\\ 0\end{bmatrix}$ and $\left|3\right\rangle = \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}$

This is linearly independent but not orthogonal, we must use Gram-Schmidt to change that:

5. **Apply the Gram-Schmidt Process to the basis** My poor fingers tire of this, therefore I implore you to go look at the Gram-Schmidt example above to understand how this process works. If you don't believe me, feel free to try it for yourself and email me telling me of my stupidity if it is indeed wrong. Thank you ☺.

Final Orthonormal Basis:

$$\left|u_1\right\rangle = \begin{bmatrix}\frac{1}{\sqrt{2}}\\ 0\\ \frac{i}{\sqrt{2}}\end{bmatrix}$$

$$\left|u_2\right\rangle = \begin{bmatrix}0\\ 1\\ 0\end{bmatrix}$$

$$\left|u_3\right\rangle = \begin{bmatrix}\frac{i}{\sqrt{2}}\\ 0\\ \frac{1}{\sqrt{2}}\end{bmatrix}$$

6. **Form U vector with orthonormal basis calculated above**

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{i}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \text{ is } \textbf{unitary} \tag{3.44}$$

7. **Finally, assemble the SVD!**

$$C = U\Sigma V^\dagger = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{i}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^\dagger \tag{3.45}$$

Whew, that was a hard process.. but now that we have this set of matrices, you can see that we can recover A through these, super useful! ♦

### 3.3.4 Schmidt Decomposition

**Theorem 3.15** *Let $H = H_1 \otimes H_2$ where dim $H_1 = m < infinity$ and dim $H_2 = n < infinity$, every vector $\psi \in H$ admits a Schmidt Decomposition:*

$$\psi = \sum_{j=1}^{r} S_j \left|u_j\right\rangle \otimes \left|v_j\right\rangle \tag{3.46}$$

*where $s_j > 0$ are the* ***Schmidt Coefficients*** *satisfying $\sum_{j=1}^{r} {S_j}^2 = 1$, $\{|u_j\rangle\} \leq H_1$ and $\{|v_j\rangle\} \leq H_2$ are orthonormal, and $r \leq min\{m, n\}$ is called the* ***Schmidt number****.*

*Note: r is just the rank of the coeff matrix*
*Remark: $|\psi\rangle \in H_1 \otimes H_2$ is separable iff it's Schmidt number is 1.*

## 3.4 Mixed States as Density Matrices

In this section, we will learn how to describe quantum systems with density matrices, which we will find has many advantages compared to the ways we have learned to describe systems thus far, lets get into it starting with an exercise.

**Example 3.16** Suppose $|\psi\rangle \in \mathbb{C}^2$ is the state of some quantum system. That means that $\||\psi\rangle\| = 1$ and if we were hoping to measure vertical or horizontal polarization.

Let being vertically polarized by represented by:

$$|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \text{and} \qquad |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

These values should come from the eigenstates from chosen Hermitian matrix
which representsthe apparatus.

$$A = 0\,|0\rangle\langle 0| + 1\,|1\rangle\langle 1| = |1\rangle\langle 1|$$

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \textit{ This can be assigned as the "apparatus"}$$ ♦

To glean information about $\psi$ regarding its polarization, we use the equation

$$\psi = \alpha\,|0\rangle + \beta\,|1\rangle$$

$$\text{where } \|\psi\| = \sqrt{\langle\psi|\psi\rangle} = \sqrt{\|\alpha\|^2 + \|\beta\|^2} = 1$$

We interpret this expression of $\psi$ in the $\{|0\rangle, |1\rangle\}$-base as:

$|\psi\rangle$ is in $|0\rangle$ with $\|\alpha\|^2$ probability and $|\psi\rangle$ is in $|1\rangle$ with $\|\beta\|^2$ probability

**Recall**: $|\psi_1\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$ when we take this, and let it develop over time, the Schrödinger equation states that time development creates a phase shift in the particle vector.
This leads to $|\psi_2\rangle = e^{i\theta}\alpha\,|0\rangle + e^{i\theta}\beta\,|1\rangle$ and it is in $|0\rangle$ with $\|e^{i\theta}\alpha\|^2$ probability and is in $|1\rangle$ with $\|e^{i\theta}\beta\|^2$ probability.

*Note: Something interesting is that based on this is that $|\psi_1\rangle \neq |\psi_2\rangle$ as **vectors** in $\mathbb{C}^2$ but $|\psi_1\rangle = |\psi_2\rangle$ as **states** of the system in "quantum land".*

As we can see, it is going to be really hard to understand what state a particle is in this format, but this is where density matrices come in.
**Consider:**

$$\rho_{\psi_1} = |\psi_1\rangle \langle\psi_1| = (\alpha |0\rangle + \beta |1\rangle)(\alpha^* \langle 0| + \beta^* \langle 1|)$$
$$= |\alpha|^2 |0\rangle \langle 0| + \alpha\beta^* |0\rangle \langle 1| + \alpha^*\beta |1\rangle \langle 0| + |\beta|^2 |1\rangle \langle 1|$$
$$\rho_{\psi_1} = \begin{bmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{bmatrix}$$

Now let's try the same thing with $|\psi_2\rangle$

$$\rho_{\psi_2} = |\psi_2\rangle \langle\psi_2| = e^{i\theta}\alpha(e^{i\theta}\alpha)^* = e^{i\theta}\alpha * e^{-i\theta}\alpha^* = \alpha\alpha^2 = |\alpha|^2 ...$$
$$= \begin{bmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{bmatrix} \checkmark$$

If you continue this you will see that they are the same matrix, meaning that using these matrices we can eliminate the time dependence of the vectors, and still keep the same data about the states.

**Example 3.17** Given $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ or $|\psi\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$

Find $\rho_\psi$ (w.r.t. $\{|0\rangle, |1\rangle\}$):

$$\rho_\psi = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \tag{3.47}$$

♦

Now, a couple observations about density matrices:

- $\rho_\psi \in M_n(\mathbb{C})$
- $\rho_\psi$ has trace 1 (i.e. $\mathrm{Tr}(\rho_\psi) = 1$ where Tr() is the sum of the diagonal matrices)
- $\rho_\psi$ is **Hermetian**, and in fact furthermore **positive semi-definite**.

**Definition 3.18 (Trace)** The **trace** of an nxn matrixis the sum of its diagonal entries. ♦

**Definition 3.19 (Density Matrix)** A **density matrix** $\rho \in M_n$ is a Hermetian, positive semi-definite matrix such that $\mathrm{tr}(\rho) = 1$. ♦

### 3.4.1 Density Matrix Axioms

- Axiom 1': A physical state of a system, whose Hilburt Space $\mathbb{H}$, is completely determined by it's associated density matrix.
- Axiom 2': The mean value of an observable A is $< A >_\psi = tr(\rho A)$
- Axiom 3': The time evolution of density matrix is given by the Louisville - von Nuemann equation

$$i\hbar \frac{d}{dt}\rho = [H, \rho] \tag{3.48}$$