A Numerical Study of Mesh Adaptivity in Multiphase Flows with Non-Newtonian Fluids

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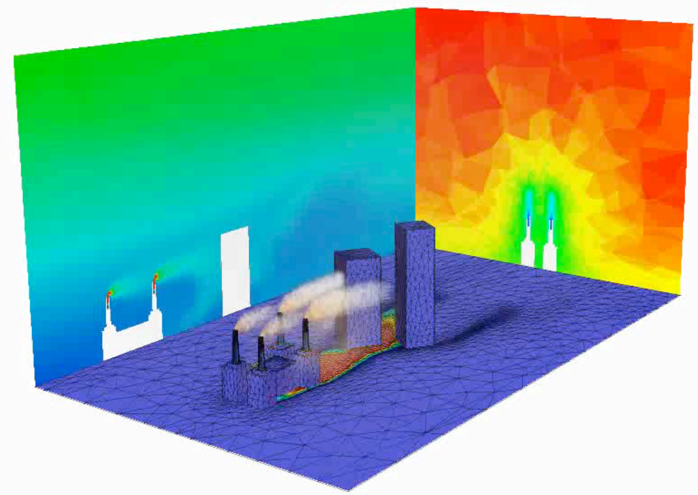
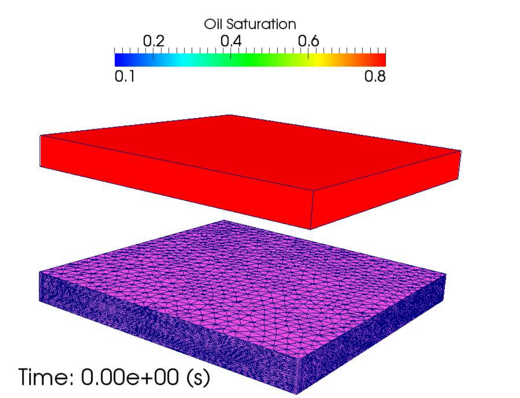
Slides available from jrper.github.io/talks/APS-DFD2014

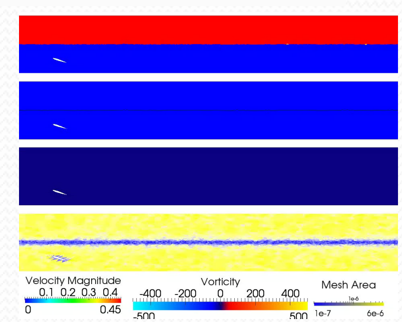
Fluidity – a finite element flow simulator

www.fluidity-project.org

- (CV)FEM based Navier-Stokes/Darcy flow solver framework
- Mesh adaptivity capability on unstructured meshes
- Parallelized
- Multimaterial & multiphase formulations (and both together)
- Used for CFD, GFD and oil reservoir simulations

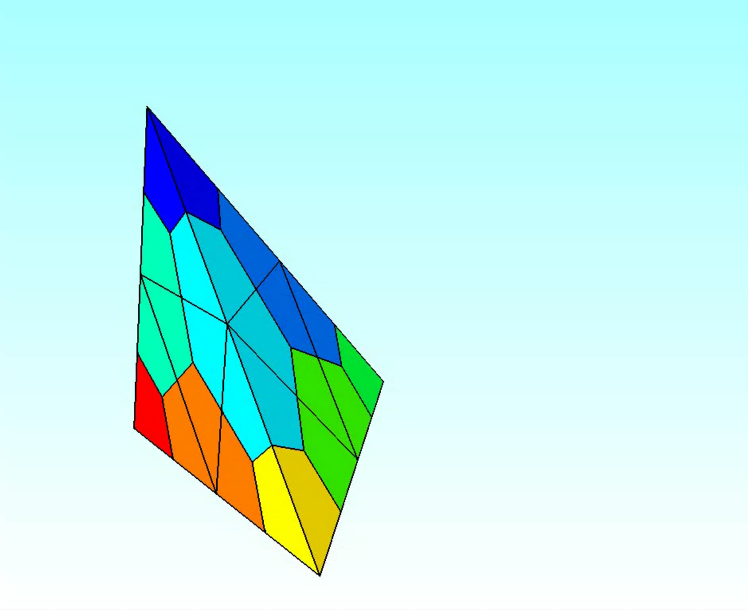


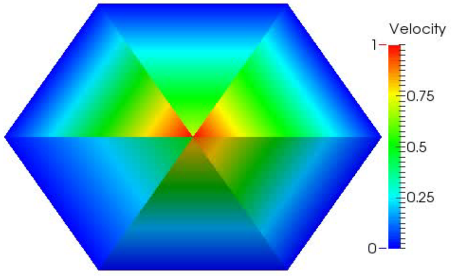




Fluidity – multiphase implementation

- Volume of fluid (VOF) approach for interface capturing.
- Compressible advection for two phase (n-phase) indicator functions
- Optimized for unstructured mesh modeling on simplices (triangles & tetrahedra.
- Control volume- finite element method
  - Control volumes [phase indicator functions/phase volume fraction]
  - Finite elements [pressure/velocity]
- Surface tension also implemented [Z. Xie. **R33.1**]
- Multiple Rheologies





Fluidity – multiphase implementation

Equation Set

momentum
$$\frac{\partial \rho \boldsymbol{u}}{\partial t} + \rho \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \rho \boldsymbol{g} + \boldsymbol{F}_{\text{visc}}$$

mass continuity
$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \boldsymbol{u} = 0$$

Indicator function
$$\rho = \rho(\boldsymbol{I}) \qquad \frac{\partial I_i}{\partial t} + \boldsymbol{u} \cdot \nabla I_i = 0$$

$$\boldsymbol{F}_{\text{visc}} = \boldsymbol{F}_{\text{visc}}(\boldsymbol{I})$$

Implying either:
$$\nabla \cdot \boldsymbol{u} = 0 \qquad p = \rho^2 \frac{\partial e}{\partial \rho}$$

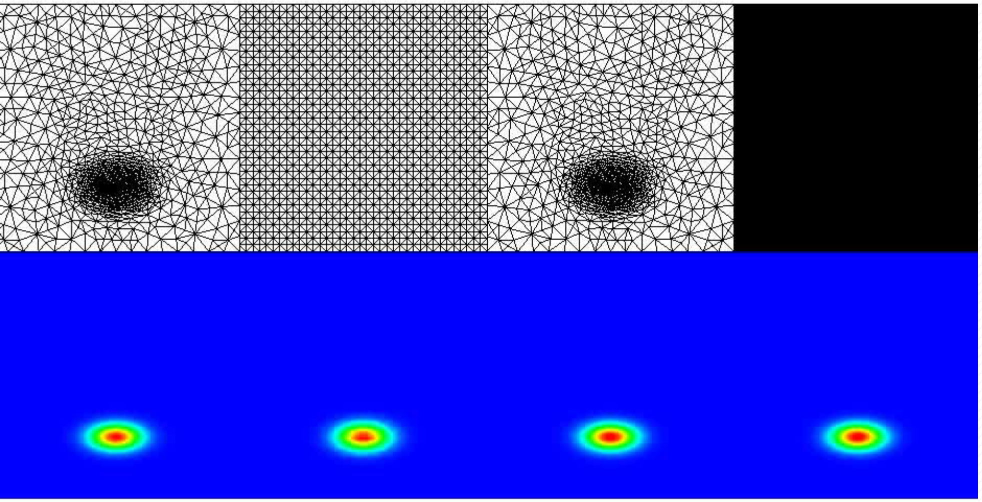
(incompressible) (compressible)

$$\mathcal{D}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2\delta_{ij}}{3}\nabla \cdot \boldsymbol{u}\right)$$

$$\boldsymbol{F}_{\text{visc}} = \begin{cases} \nabla \cdot \mu \mathcal{D} & \text{Newtonian} \\ \nabla \cdot \mu_{\text{eff}}(\mathcal{D})\mathcal{D} & \text{Shear Dependent} \\ \nabla \cdot \underline{\tau} \quad \underline{\tau} := \underline{\tau}(t, \boldsymbol{x}, \boldsymbol{u}) & \text{Viscoelastic} \end{cases}$$

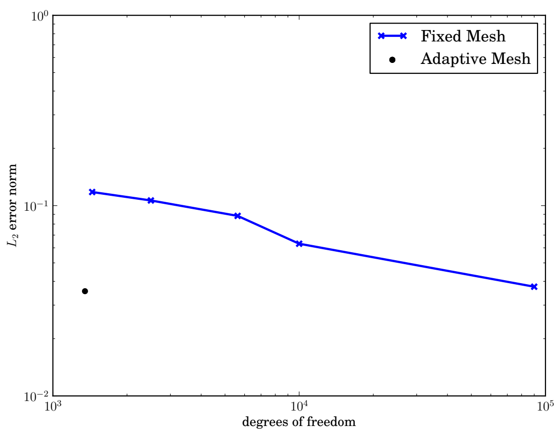
Mesh Adaptivity - examples

Initial adaptive mesh | Fixed mesh 1 | Adaptive mesh | Fixed mesh 2

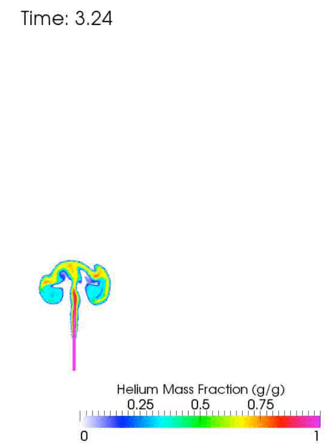


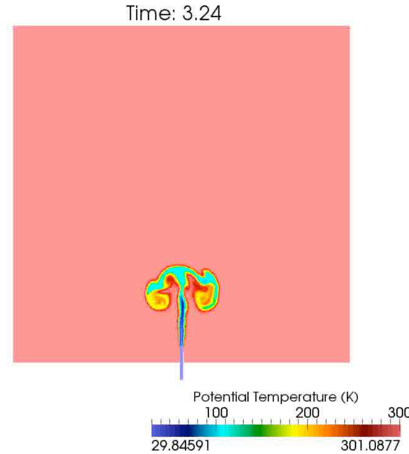
Analytic solution | Fixed mesh 1 | Adaptive mesh | Fixed mesh 2

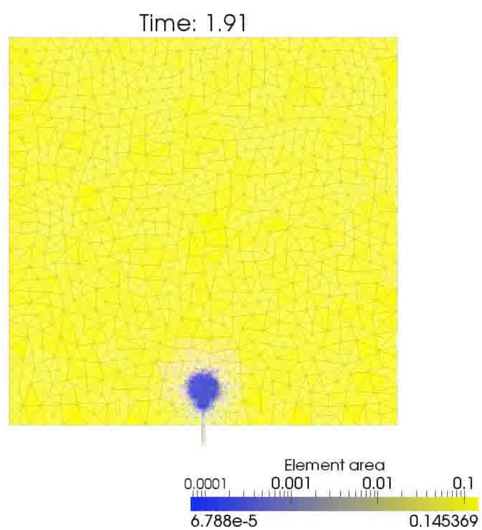
2 orders of magnitude smaller problem/half error

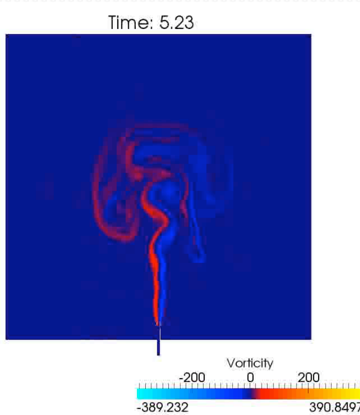


30x speed up









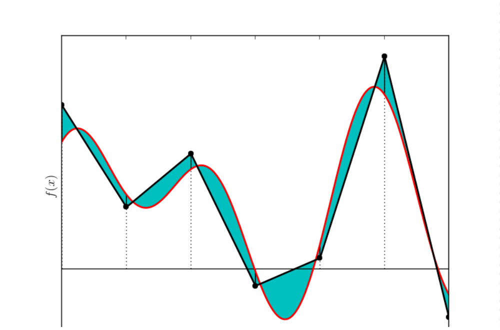
Mesh Adaptivity– Interpolation Error estimates

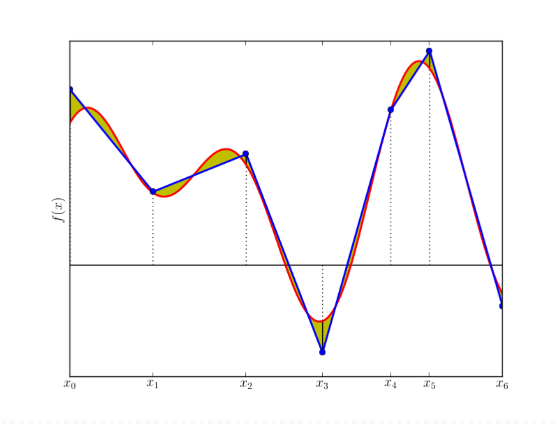
Motivation: Céa's Lemma

$$\text{error} := \left\| \psi^{\text{exact}} - \psi^{\delta} \right\| \leq C_1 \left\| \psi^{\text{exact}} - \psi^{\text{proj}} \right\|$$

$$\leq C_2 \sum_i h_i^2 \max_{x \in \Omega} \left| \frac{\partial^2 \psi}{\partial x^2} \right|$$

For sufficiently nice PDEs and linear (or better) elements





In higher dimensions, error estimate is a function of the Hessianmand the edge vectors.

$$\mathcal{H}(\psi) = \begin{pmatrix} \frac{\partial^2 \psi}{\partial x^2} & \frac{\partial^2 \psi}{\partial x \partial y} & \frac{\partial^2 \psi}{\partial x \partial z} \\ \frac{\partial^2 \psi}{\partial x \partial y} & \frac{\partial^2 \psi}{\partial y^2} & \frac{\partial^2 \psi}{\partial y \partial z} \\ \frac{\partial^2 \psi}{\partial x \partial z} & \frac{\partial^2 \psi}{\partial y \partial z} & \frac{\partial^2 \psi}{\partial z^2} \end{pmatrix}$$

$$\text{error} \sim \sum_k \boldsymbol{v}_k^T \mathcal{M} \boldsymbol{v}_k$$

$$\mathcal{M} = \mathcal{M}\left(\mathcal{H}(\psi)\right)$$

"Mesh metric"

Mesh Adaptivity – Interpolation Error Estimates

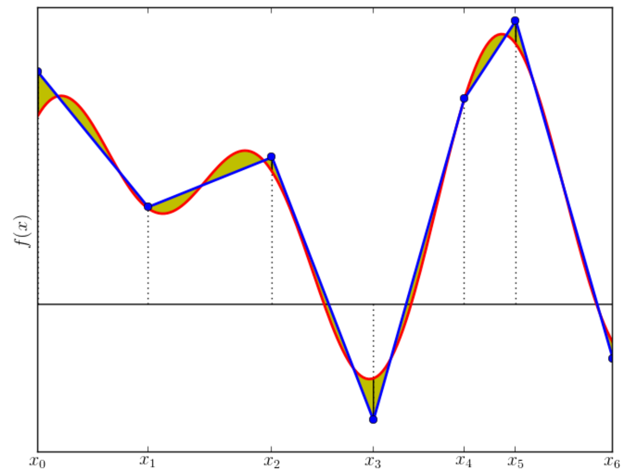
Problem: we don't know ψ^{exact}
Answer: Use old ψ^{δ} instead.

Estimate second derivatives from the finite element data

Try to extremize

$$I(\boldsymbol{v}) = \sum_k \frac{1}{\epsilon} \boldsymbol{v}_k^T \mathcal{M} \boldsymbol{v}_k - 1$$

Where the **v**s are the edges of the mesh



$$\mathcal{M} = \mathcal{M}\left(\mathcal{H}(\psi^{\delta})\right)$$

Additional constraints for

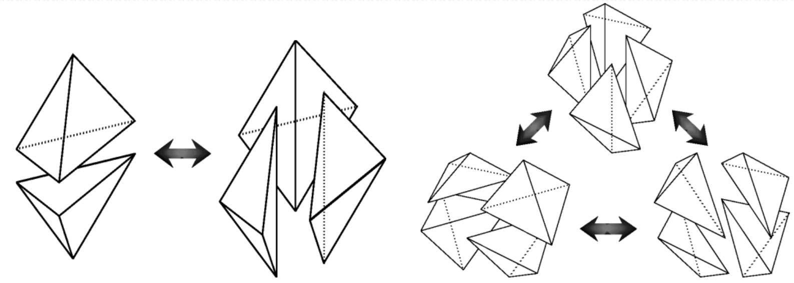
- Bounds on min/max edge lengths
- Gradation of the increase
- Aspect ratios
- Metric advection

Mesh Adaptvity: Error Estimate Optimization

Local remeshing algorithm. Preserve the old mesh in regions where it is adequate.

Apply *hr* adaptivity in the regions where fixing is necessary: Add, remove nodes, reconnect nodes and move nodes.

Algorithm works iteratively, attacking areas in which edge lengths are far from the ideal calculated from interpolation error estimate and freezing sufficently good elements.



(a) Face to edge (b) Edge swapping

Fig. 1. Digram showing: (a) edge to face and face to edge swapping; (b) edge to edge swapping with four elements.

Pain, Umpleby, de Oliveira & Goddard,(2001).
Tetrahedral mesh optimisation and adaptivity for steady-state and transient finite element calculations,
Computer Method in Applied Mechanics & Engineering

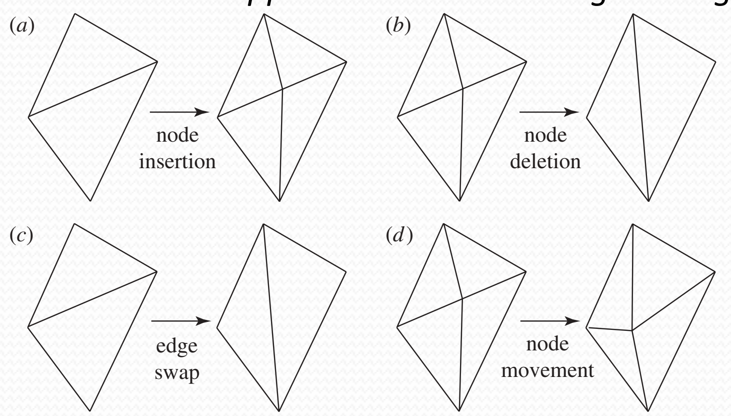
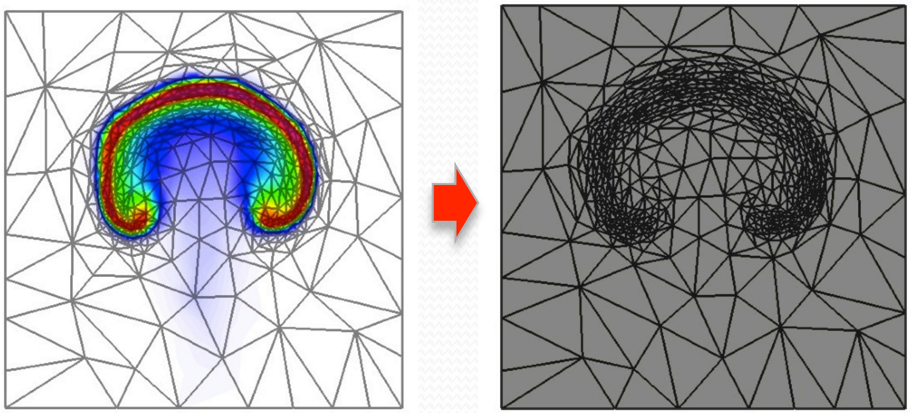
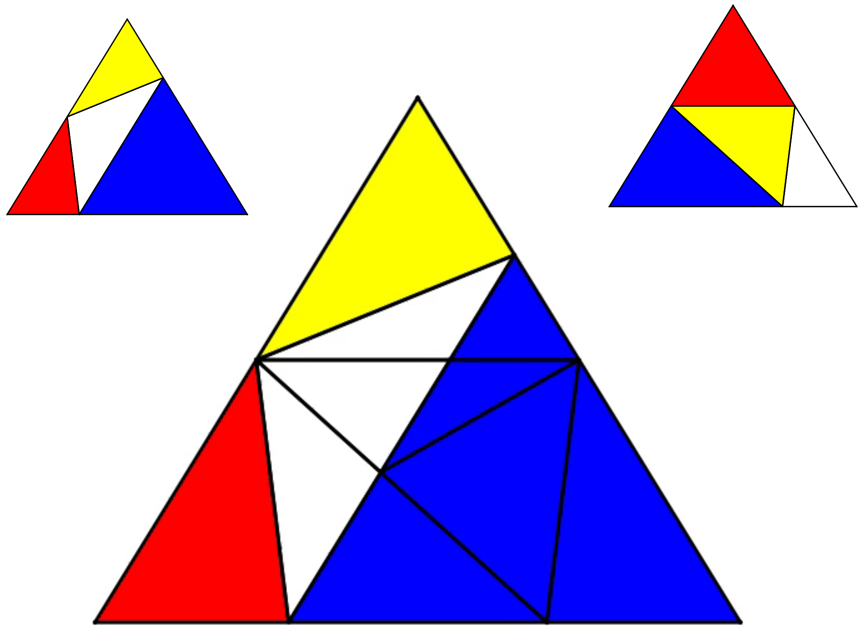


Figure 1. Local element operations used to optimize the mesh in two dimensions. (a) Node insertion or edge split. (b) Node deletion or edge collapse. (c) Edge swap. (d) Node movement.

Piggott, Farrell, Wilson, Gorman & Pain (2009)
Anisotropic mesh adaptivity for multi-scale ocean modelling
Philosophical Transactions of the Royal Society A

Mesh to mesh interpolation - supermeshing





FE solutions/test functions piecewise smooth over mesh elements

Elements of supermesh: old variables and new test fns both smooth.

No jumps.

Allows efficient conservative mesh to mesh interpolation via projection methods

P. E. Farrell & J. R. Maddison (2011) Computer Methods in Applied Mechanics and Engineering

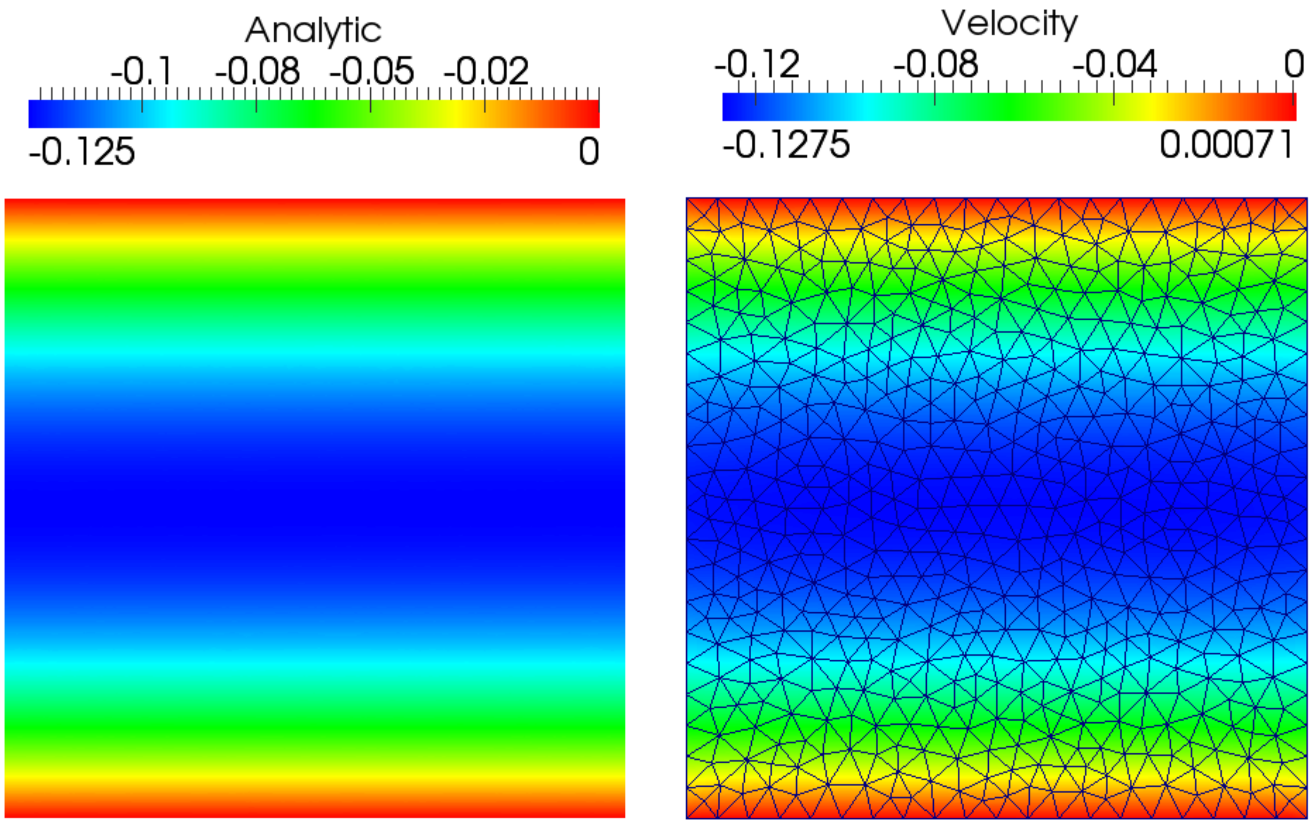
$$\sum_j \int_\Omega N_i^{\rm new} N_j^{\rm new} \psi_j^{\rm new} dV = \sum_k \int_\Omega N_i^{\rm new} N_k^{\rm old} \psi_k^{\rm old} dV$$

Single Phase Newtonian flow

Single phase laminar channel flow

Spin up from
Pressure differential

$$u(y) = -\frac{1}{2\mu}\frac{\partial p}{\partial x}(h^2 - y^2)$$



Two phase Newtonian flow

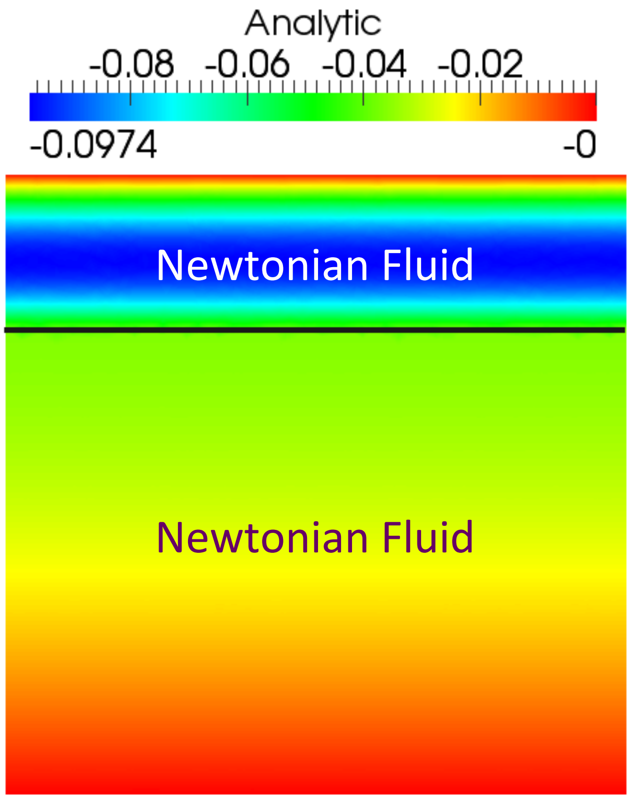
multiphase laminar channel flow

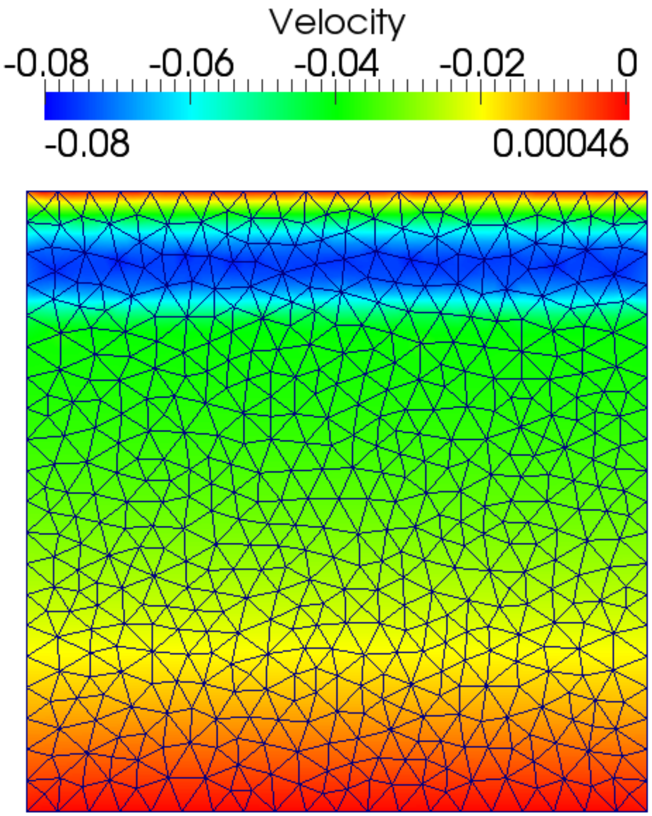
Spin up from
Pressure differential

$$c_0 = \frac{1}{2}\left[\frac{(\mu_1 - \mu_0)\, h_c^2 + \mu_0 h^2}{(\mu_1 - \mu_0)\, h_c + \mu_0 h}\right]$$

$$u(y) = -\frac{1}{2\mu_0}\frac{\partial p}{\partial x}\left[2c_0 y - y^2\right]$$

$$u(y) = -\frac{1}{2\mu_1}\frac{\partial p}{\partial x}\left[2c_0(y-h) + h^2 - y^2\right]$$





Two phase Newtonian flow

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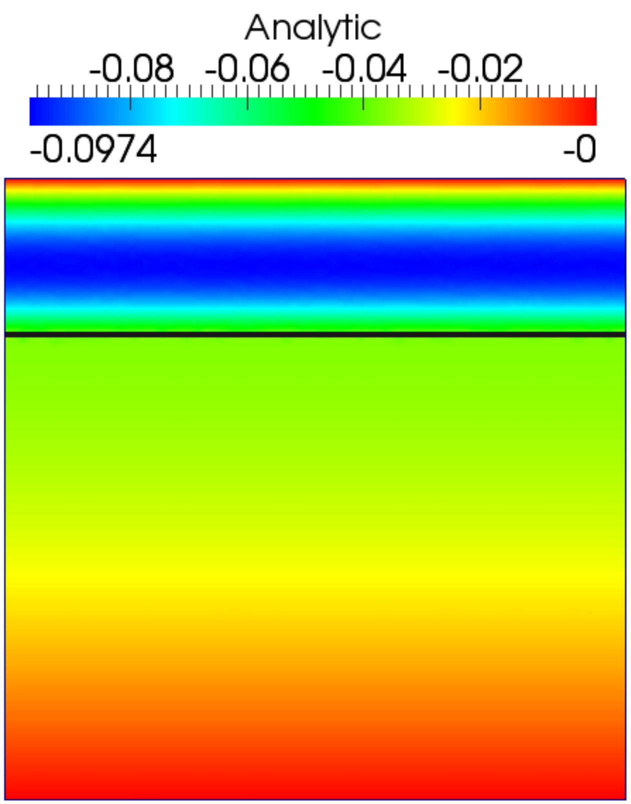
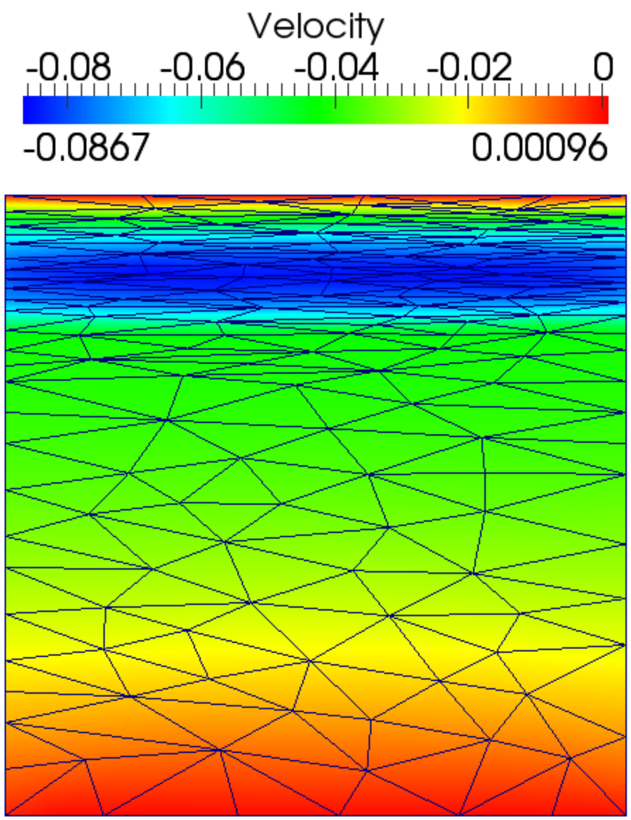
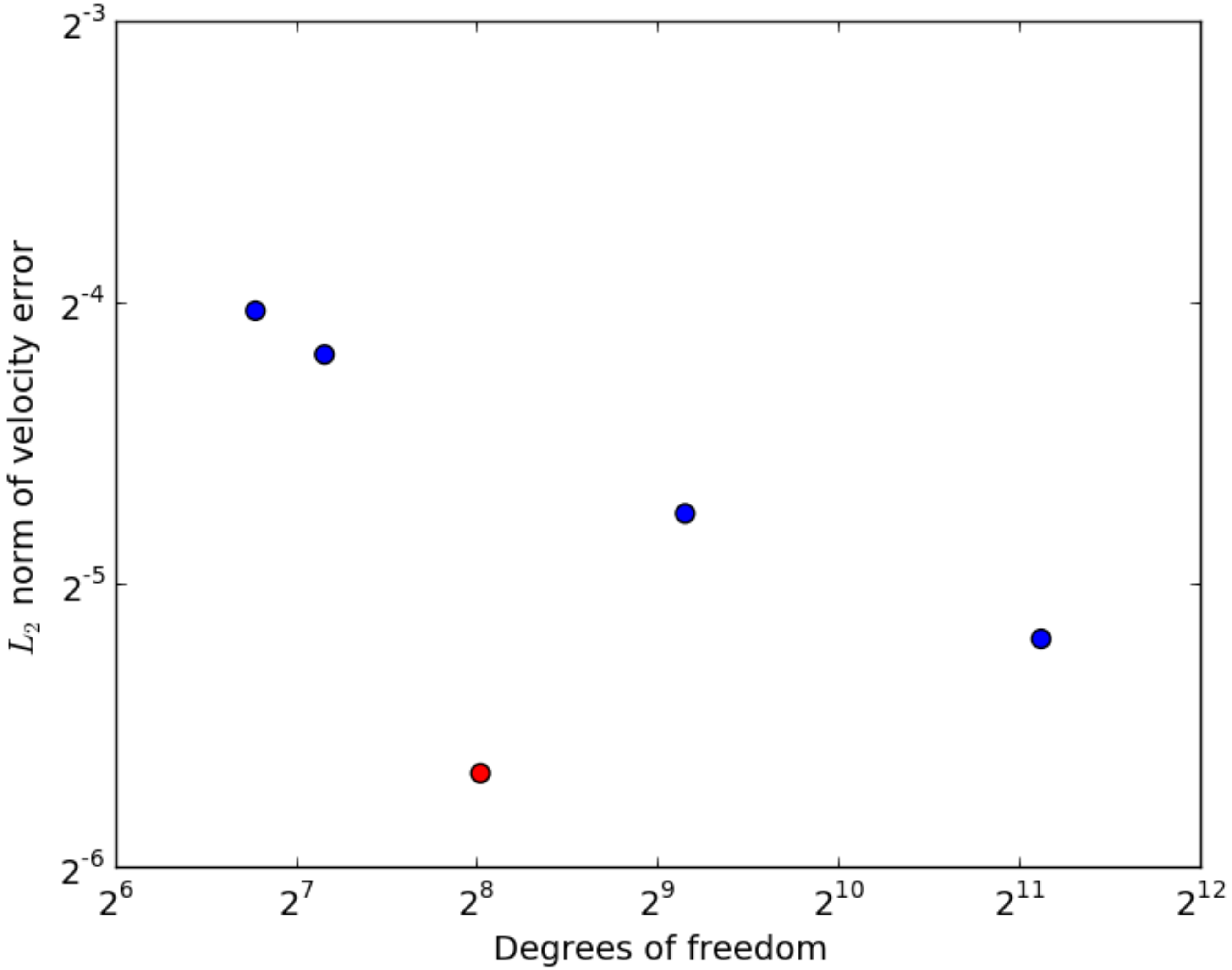
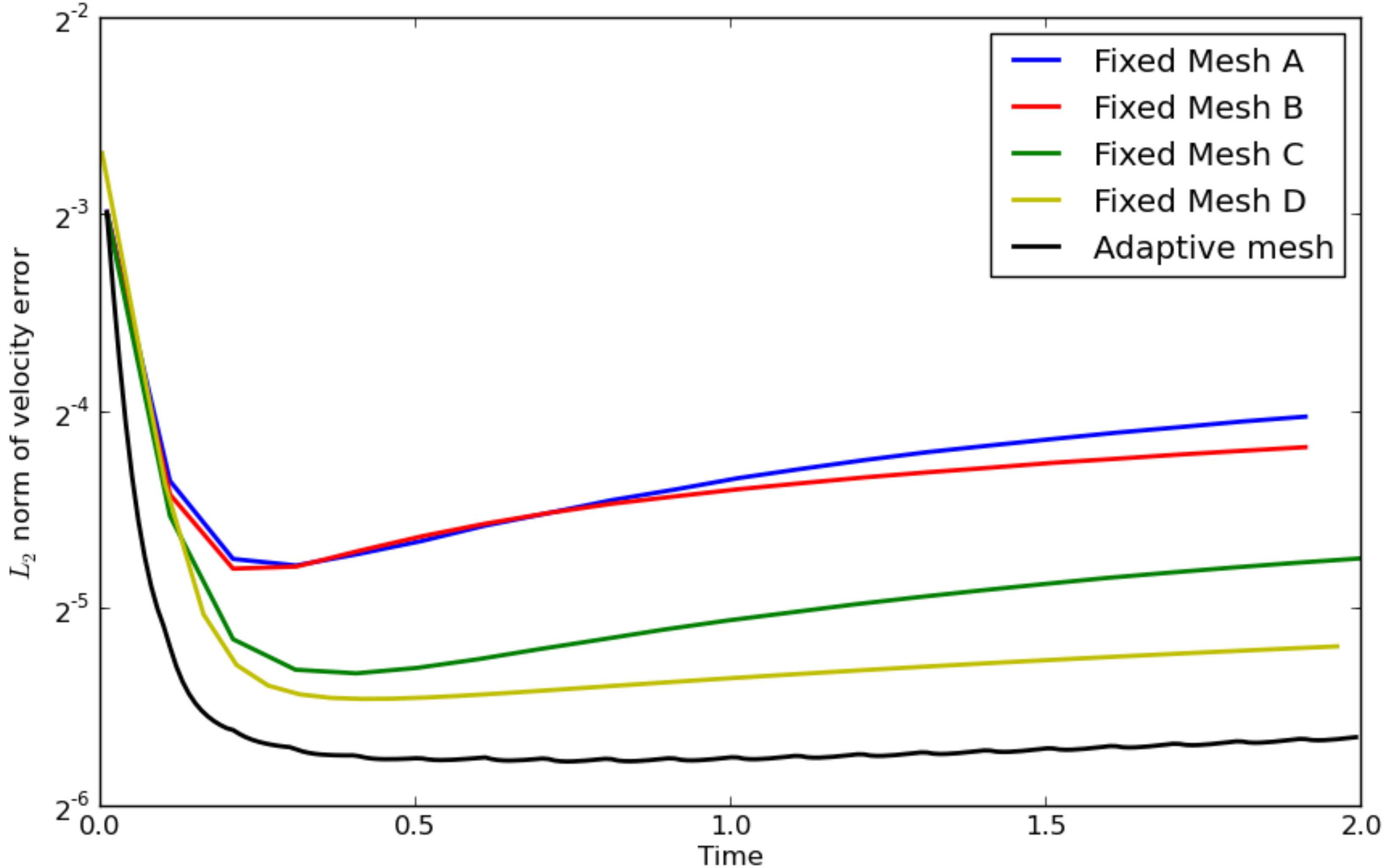




Convergence rates and error reduction



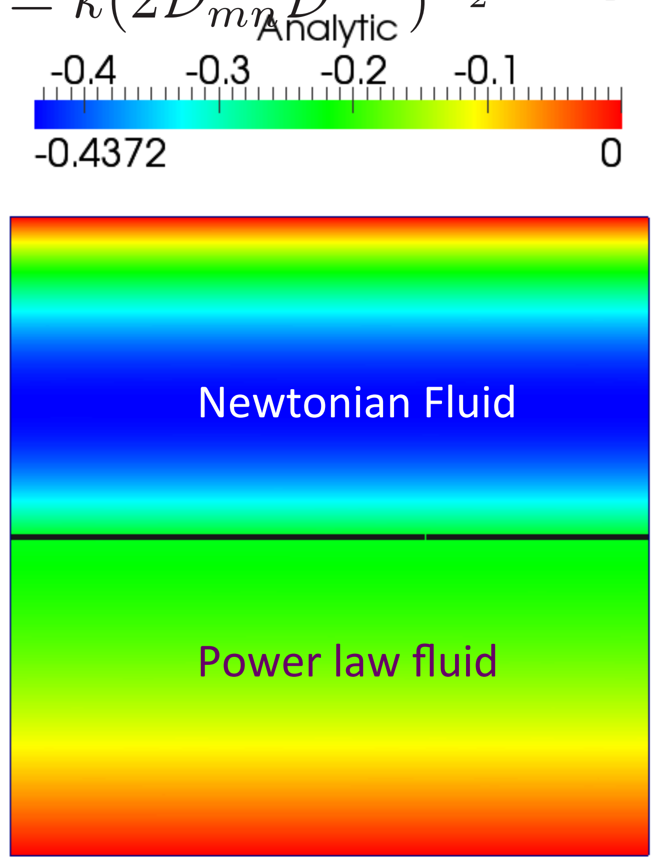
Temporal convergence to steady state

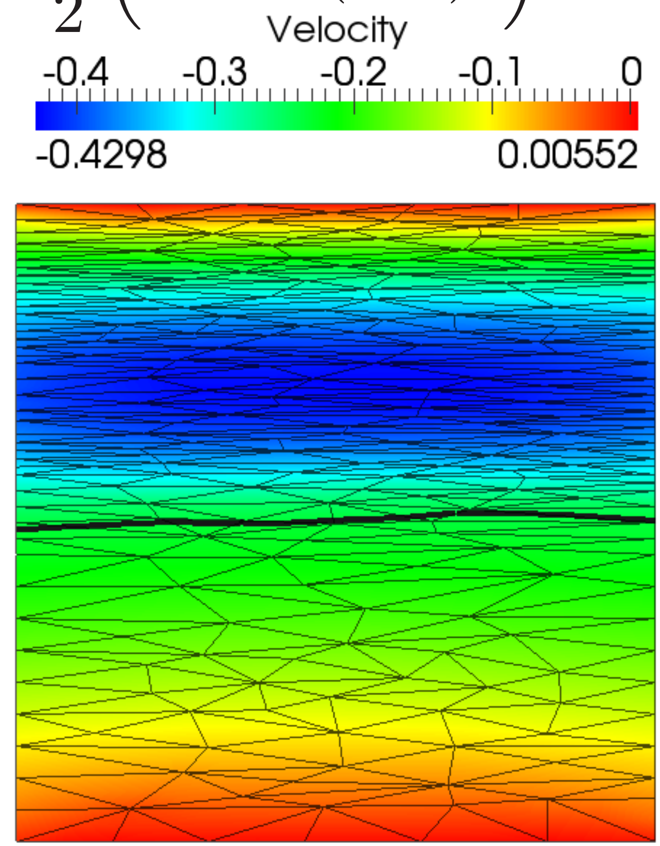


Non-Newtonian Rheologies

multiphase laminar channel flow : power law

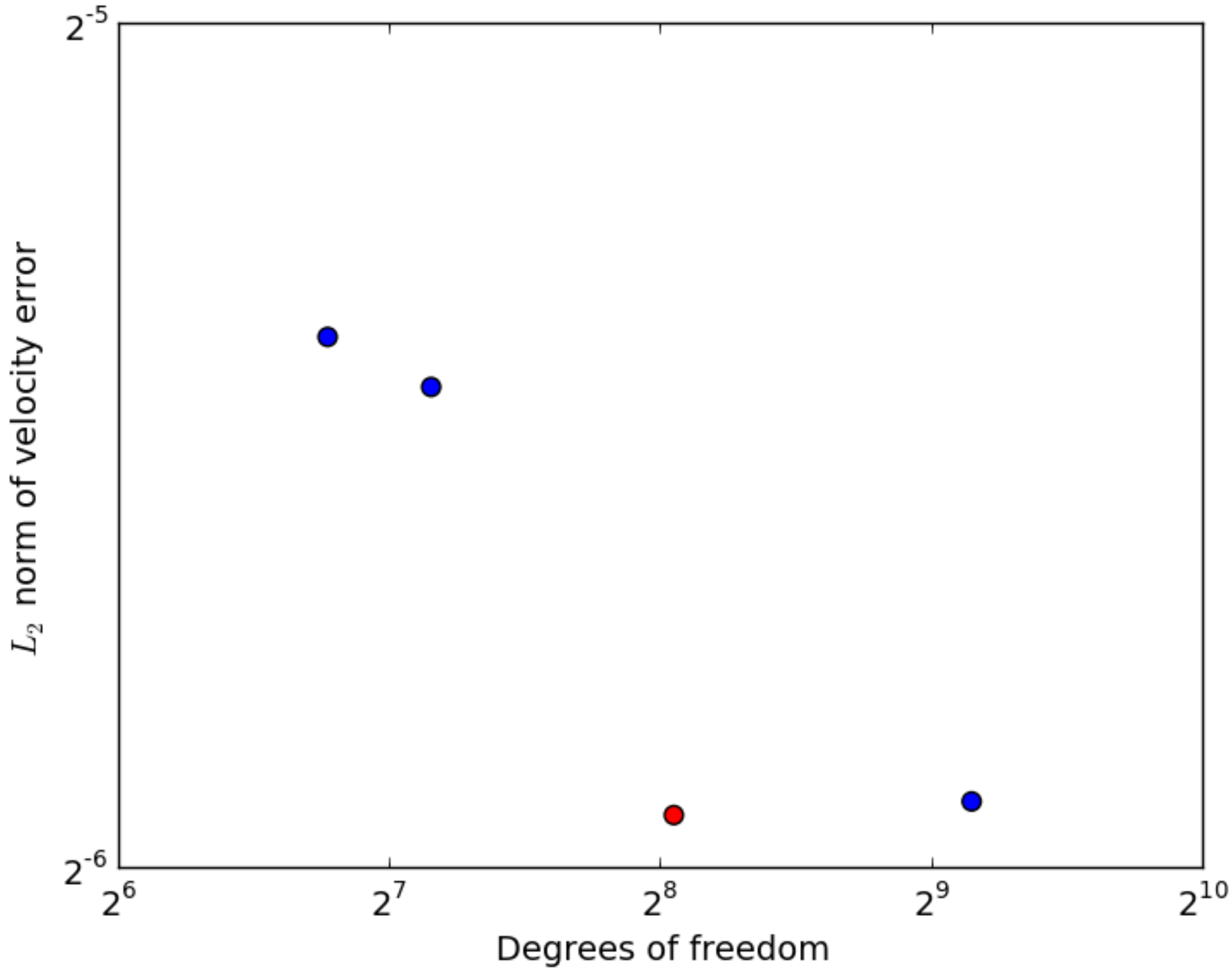
$$\mu_{\text{eff}} = k(2\mathcal{D}_{mn}\mathcal{D}^{mn})^{\frac{n-1}{2}} \qquad \mathcal{D} = \frac{1}{2}\left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^2\right)$$





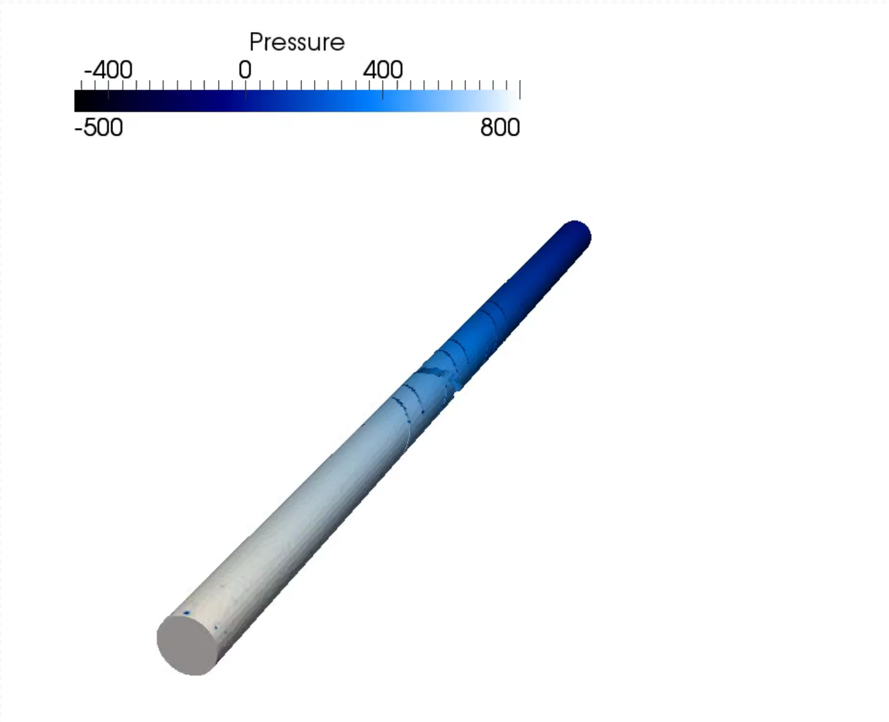
$$u(y) = \begin{cases} \frac{n}{n+1}\left[\frac{1}{k}\frac{dp}{dx}\right]^{\frac{1}{n}}\left(|y-c|^{\frac{n+1}{n}} - |c|^{\frac{n+1}{n}}\right) \\ \quad y \geq h_c \\ \frac{1}{\mu_b}\frac{dp}{dx}\left(\frac{(y^2-h^2)}{2} - c\,(y-h)\right) \\ \quad y < h_c \end{cases}$$

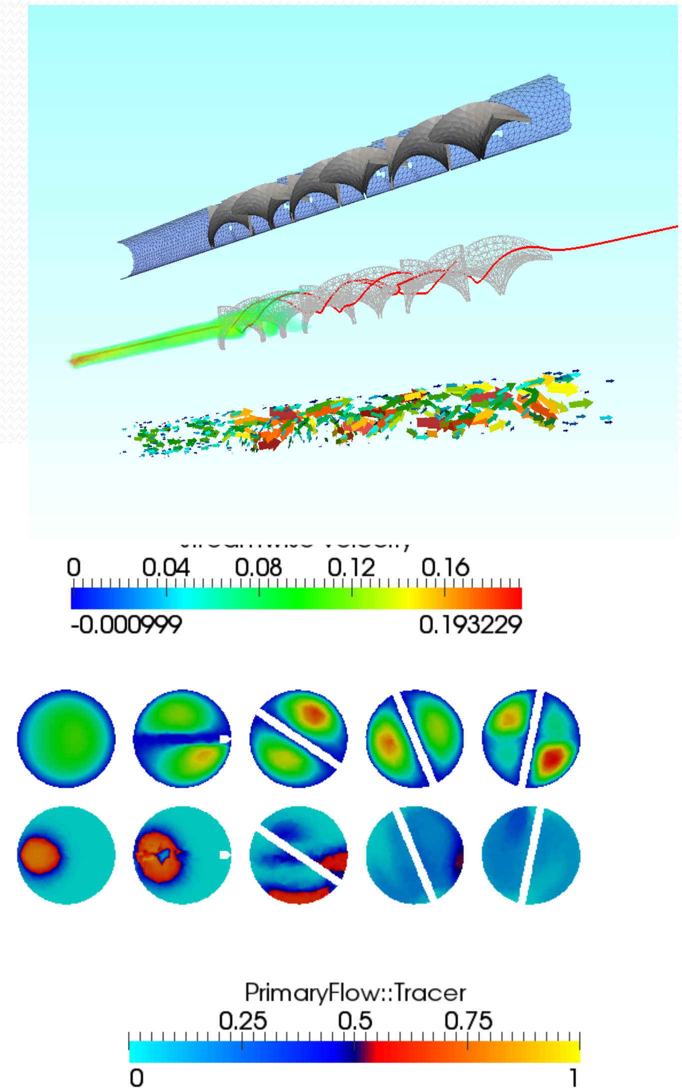
Convergence rates and error reduction



Carreau fluids

3D problems in non-idealized geometries





Other Rheologies

- viscoelastic stress model
  - Oldroyd B
    - Adds fluid memory term
    - Includes rotational terms for convection of local coordinate
  - Polymers & Boger fluids

$$\frac{\partial}{\partial t}(\rho \boldsymbol{u}) + \nabla \cdot \rho \boldsymbol{u}\boldsymbol{u} = -\nabla p - \rho \boldsymbol{g} + \nabla \cdot \left(\mu_s \left[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T - \frac{2}{3} \underline{\boldsymbol{I}} \nabla \cdot \boldsymbol{u} \right] + \underline{\boldsymbol{\tau}_p} \right)$$

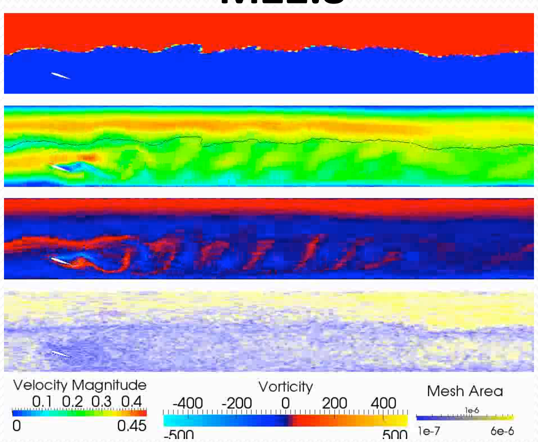
$$\frac{\partial \underline{\boldsymbol{\tau}_p}}{\partial t} + \boldsymbol{u} \cdot \nabla \underline{\boldsymbol{\tau}_p} - \left[(\nabla \boldsymbol{u})^T \cdot \underline{\boldsymbol{\tau}_p} + \underline{\boldsymbol{\tau}_p} \cdot \nabla \boldsymbol{u} \right] = \frac{1}{\lambda_1} \left[\mu_p \left[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T - \frac{2}{3} \underline{\boldsymbol{I}} \nabla \cdot \boldsymbol{u} \right] - \underline{\boldsymbol{\tau}_{\boldsymbol{p}}} \right]$$

Upcoming MEMPHIS talks @ APS-DFD

Studies of Interfacial Perturbations in Two Phase Oil-Water Pipe Flows Induced by a Transverse Cylinder

Dr. Maxime Chinaud

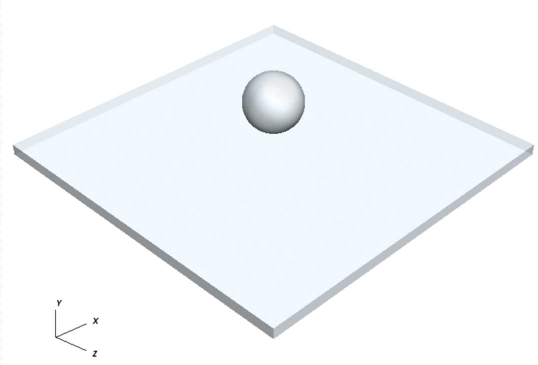
M22.8



Numerical study of Taylor bubbles with adaptive unstructured meshes

Dr. Zhihua Xie

R33.1



Optimisation of sensor locations for falling film problems based on importance maps

Dr. Zhizhao Che

R35.9

Slides available from http://jrper.github.io/talks/APS-DFD2014