A Brief Introduction to Finite Element Methods for Fluid Flow Problems

James Percival

AMCG,
Department of Earth Science and Engineering
Imperial College London

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These slides

- Not an entire description of the finite element method
  - one hour lecture, not ten week course
- For mode details refer to the *Fluidity* manual.
- Slides available at jrper.github.io/2015/IntroToFEM.pdf
- Partial lecture notes are available at jrper.github.io/2015/FEMNotes.pdf

What We'll Cover

At the end of the hour we'll have discussed:

- Numerical discretizations in general.
- Overview of basic finite element method, applied to a simple ODE.
- Definitions of polynomial Lagrangian finite element basis functions, as used in the *Fluidity* solver
- Definitions of continuous and discontinuous finite element basis functions, as used in the *Fluidity* solver

Numerical Solutions to PDEs

Mathematical descriptions of physical processes are often as partial differential equations (PDEs). Variables are functions of time & space and satisfy relationships between their own partial derivatives.

Examples

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}, \qquad \text{(heat equation)}$$

$$\frac{\partial^2 a}{\partial t^2} = c^2 \frac{\partial^2 a}{\partial x^2} \qquad \text{(wave equation)}$$

$$\frac{\partial \tau}{\partial t} + u \frac{\partial \tau}{\partial x} = \kappa \frac{\partial^2 \tau}{\partial^2 x} \qquad \text{(advection-diffusion equation)}$$

Numerical Solutions to PDEs

Solving PDEs assumes knowledge of the variables at an infinite number of locations in space and time. However memory in computers (and humans) is finite. Equations must be discretized to create a smaller (and easier to solve) problem. Eg. to a matrix problem like

$$\begin{pmatrix} 3 & 2 & 0 \\ 4 & 4 & 1 \\ 0 & 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix},$$

or, in a more general form,

$$\mathrm{A}\boldsymbol{x} = \boldsymbol{b}.$$

Numerical Solutions to PDEs

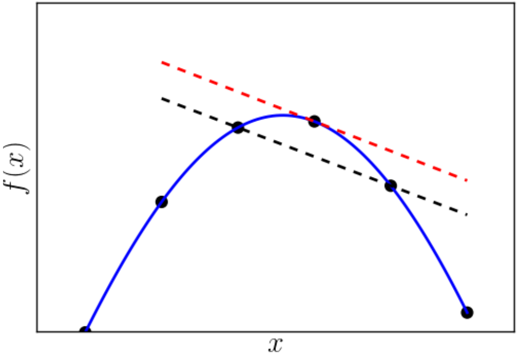
Some standard techniques for discretization are:

1. Finite difference methods.
2. Finite volume methods.
3. Spectral methods.
4. Finite element methods.

Fluidity (and Firedrake) implements the last of these.

Finite Difference Methods

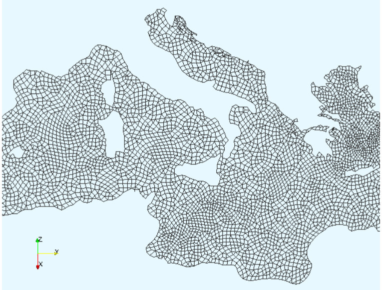
- Reduce problem domain to finite set of points.
- Replace exact derivatives with approximate difference equations.



$$\frac{df}{dx} \approx \frac{\Delta f}{\Delta x} = \frac{f(x_{i+1}) - f(x_{i-1})}{x_{i+1} - x_{i-1}}$$

Finite Volume Methods

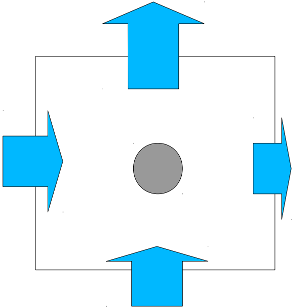
- Break problem domain into a finite set of sub-volumes.
- Volumes are typically simple geometric polyhedra (cuboids, wedges, tetrahedra)
- Forming a good mesh can be tricky.
- See meshing talk tomorrow by Dr. Alex Avdis for more on getting a good mesh.



Finite Volume Methods

- Solve for volume integral of quantities inside. Typically reduces problem into flux calculation across faces of the volume.
- If fluxes depend on derivatives then another method (e.g. finite differences) must be used to find them.

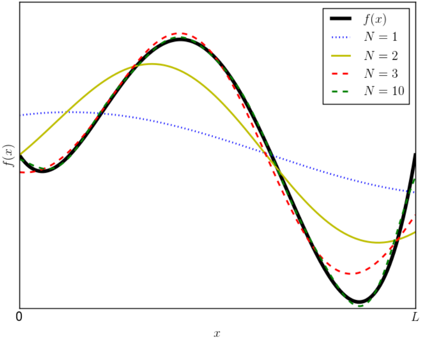
$$\frac{d}{dt}\int_{\Omega_i}\rho dV = \sum_{\text{faces}}\int_{\delta\Omega_i^{(j)}}\rho \boldsymbol{u}\cdot \boldsymbol{n} dS$$



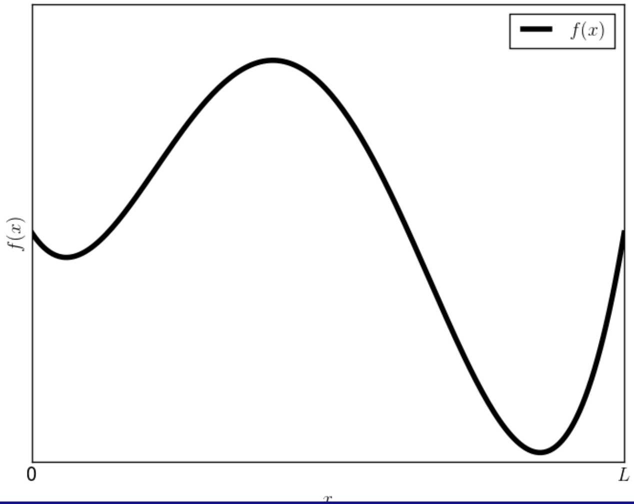
Spectral Methods

- Represent variables as (limit of) sum of orthogonal basis functions.
- Global basis functions vary over entire domain.
- Truncate infinite series and calculate behaviour of finite set of coefficients.

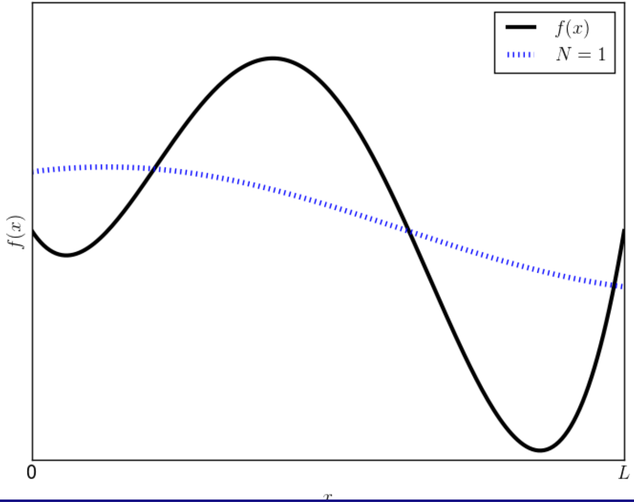
$$f\left(x\right) = \sum_{n=1}^{\infty} a_n \cos\left(n\frac{\pi x}{L}\right) + b_n \sin\left(n\frac{\pi x}{L}\right)$$
$$\approx \sum_{n=1}^{N} a_n \cos\left(n\frac{\pi x}{L}\right) + b_n \sin\left(n\frac{\pi x}{L}\right)$$



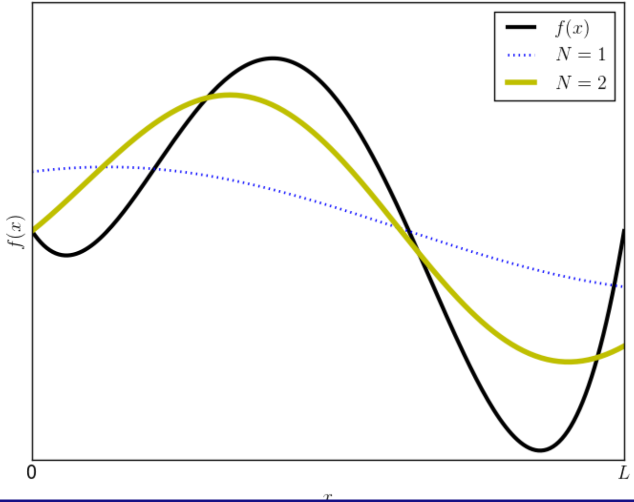
Spectral Methods



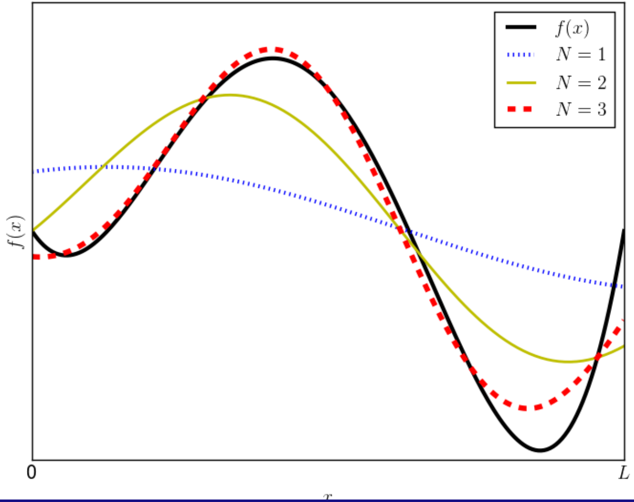
Spectral Methods



Spectral Methods



Spectral Methods



Finite Element Methods

- Combine parts of finite volume and spectral methods.
- Break domain into finite set of simple subvolumes (elements).
- Represent variables as a sum of simple basis functions.
- Basis functions non-zero only on local set of subvolumes.
- Discretize integral equation form of PDE.

Hybrid Methods

- Very common to combine multiple different approaches
  - Couple finite volume method (for global conservation) + finite difference method (for fluxes)
  - Couple finite volume method (for global conservation) + finite element method (for fluxes). "Control Volume" method
  - Fluidity: Finite elements (space) + finite differences (time)

Poisson Equation: Pressure in Navier-Stokes

Incompressible Navier-Stokes equations

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \nu \nabla^2 \boldsymbol{u}, \qquad \text{(momentum)}$$

$$\nabla \cdot \boldsymbol{u} = 0, \qquad \text{(continuity)}$$

Taking divergence

$$\underbrace{\frac{\partial}{\partial t}(\nabla \cdot \boldsymbol{u})}_{=0} + \nabla \cdot (\boldsymbol{u} \cdot \nabla \boldsymbol{u}) = -\nabla^2 p + \underbrace{\nu \nabla^2 \nabla \cdot \boldsymbol{u}}_{=0}.$$

I.e.

$$\nabla^2 p = -\nabla \cdot (\boldsymbol{u} \cdot \nabla \boldsymbol{u}).$$

Poisson Equation

General form of this equation

Poisson Equation

$$\nabla^2\psi + f(\boldsymbol{x}) = 0, \quad \forall \boldsymbol{x} \in \Omega$$

In 1D, setting Ω to the unit interval:

$$\frac{\partial^2\psi}{\partial x^2} + f(x) = 0 \quad \forall x \in (0,1). \tag{*}$$

This is the strong form of the Poisson equation.

Strong Form vs. Weak Form of an Equation

Strong form

Equation (*) true individually for each point in space,

$$\mathcal{L}_\psi(x) := \frac{d^2\psi}{dx^2} + f(x) = 0,$$

for all x in domain Ω.
Test a ψ for $\mathcal{L}_\psi$ by checking equation holds individually for each point in space.

Weak form

Integral equation holds for all choices of a 'test function', ϕ,

$$I_\psi(\phi) := \int_0^1 \phi\left(\frac{d^2\psi}{dx^2} + f\right)dx = 0,$$

where $\phi : \Omega \to \mathbb{R}$ is from a function space to be defined later.
Test possible ψ ('trial function') by checking integral equation holds for all test functions, ϕ.

Strong and Weak Forms

A solution to the strong form of the equations will be a solution to the weak form equations. A solution to the weak form of the equations may be a solution to the strong equations if it is smooth enough. The weak formulation extends the equations to allow non-smooth solutions which exist in a distributional sense.

Examples of common distributions

$$\delta\left(x\right), \quad \int_{-\infty}^{a} f\left(x\right)\delta\left(x\right) = \begin{cases} f\left(0\right), & a > 0, \\ 0, & a < 0. \end{cases} \qquad \text{(Dirac delta)}$$

$$H\left(x\right) := \int_{-\infty}^{x} \delta\left(s\right) ds = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases} \qquad \text{(Heaviside)}$$

Review of Vector Spaces

A set, $\mathcal{V}$, is a vector space if it has $+$ (addition) and $\cdot$(scalar multiplication) operators which satisfy the following eight rules:

$$\begin{aligned}
\boldsymbol{a}+(\boldsymbol{b}+\boldsymbol{c}) &= (\boldsymbol{a}+\boldsymbol{b})+\boldsymbol{c}, && (+\text{ associativity})\\
\boldsymbol{a}+\boldsymbol{b} &= \boldsymbol{b}+\boldsymbol{a}, && (\text{commutivity})\\
\text{there exists } \mathbf{0}\in\mathcal{V} \text{ s.t. } \boldsymbol{a}+\mathbf{0} &= \boldsymbol{a}, \forall \boldsymbol{a}\in\mathcal{V}, && (+\text{identity})\\
\forall \boldsymbol{a} \text{ there exists } -\boldsymbol{a}\in\mathcal{V} \text{ s.t. } \boldsymbol{a}+(-\boldsymbol{a}) &= \mathbf{0}, && (+\text{ inverse})\\
\alpha\cdot(\boldsymbol{a}+\boldsymbol{b}) &= \alpha\cdot\boldsymbol{a}+\alpha\cdot\boldsymbol{b}, && (\text{distributivity I})\\
(\alpha+\beta)\cdot\boldsymbol{a} &= \alpha\cdot\boldsymbol{a}+\beta\cdot\boldsymbol{a}, && (\text{distributivity II})\\
\alpha\cdot(\beta\cdot\boldsymbol{a}) &= (\alpha\times\beta)\cdot\boldsymbol{a}, && (\cdot\text{ associativity})\\
1\cdot\boldsymbol{a} &= \boldsymbol{a}. && (\cdot\text{ identity})
\end{aligned}$$

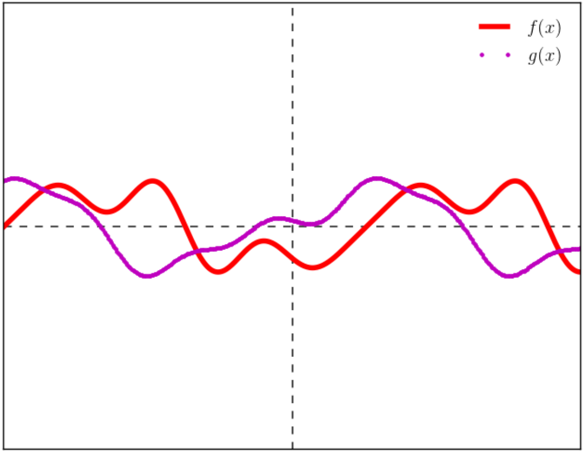
Review of Vector Spaces

Functions are a vector space under the following definitions of addition and scalar multiplication:

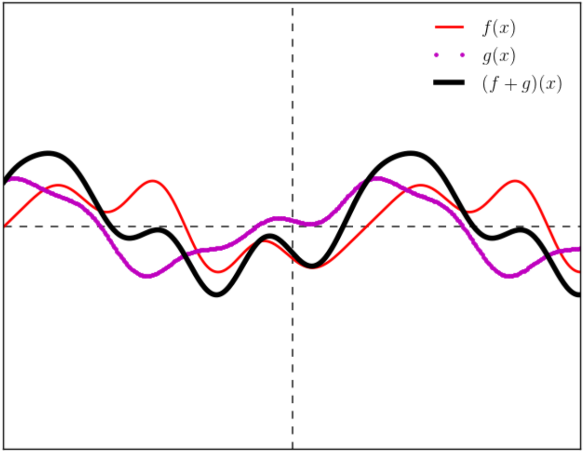
$$(f+g)(x) = f(x) + g(x),$$
$$(\alpha \cdot f)(x) = \alpha \times (f(x)).$$

I.e. functions are added/multiplied pointwise based on their result. Necessary axioms all follow from the normal rules of addition/multiplication and the zero function is just $f(x) = 0$.

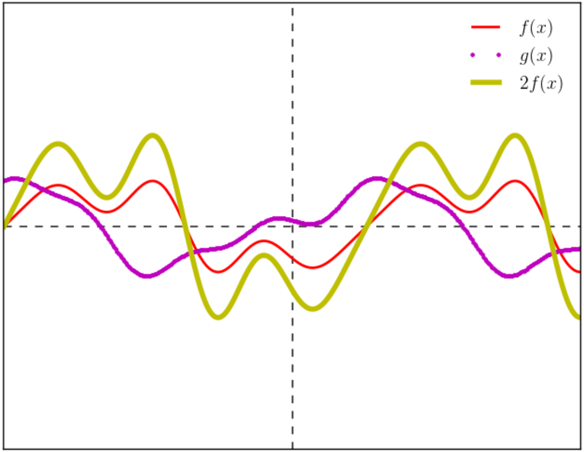
Review of Vector Spaces



Review of Vector Spaces



Review of Vector Spaces



Review of Vector Spaces

Within the space of functions there are many smaller subspaces. Eg:

Examples

Polynomials $f(x) = 1 + 3x + 4x^2 + 5x^3$

Functions on an interval $f(x) = \begin{cases} 0 & x < 0, \\ e^x & 0 \leq x \leq 1, \\ 0 & x > 1. \end{cases}$

Twice differentiable functions $f(x) = \begin{cases} x^2 + 2, & x < 0, \\ 2\left(e^x - x\right). & x \geq 0. \end{cases}$

Back to Finite Elements

Key idea of finite element method:

- Desire an exact solution ψ, from an infinite dimensional vector space, $\mathcal{V}$, which satisfies weak form equation, for test functions ϕ in $\mathcal{U}$.
- Find ψ^δ in approximate vector (sub)space $\mathcal{V}^\delta \subset \mathcal{V}$ with finite representation, which satisfies same weak form equation, for test functions ϕ^δ in approximate space $\mathcal{U}^\delta \subset \mathcal{U}$.
- Expectation that as $\delta \to 0$, $\mathcal{V}^\delta \to \mathcal{V}$ and $\psi^\delta \to \psi$.

Review of Section

- Strong form of PDEs prescribes behaviour pointwise
  - Finite difference methods work at a finite number of points
- Weak form of PDEs prescribes behaviour over intervals (areas, volumes etc.)
  - Finite element methods work over a finite number of intervals
- Functions live in vector spaces, which can be approximated.

Boundary Conditions for Weak Equations

Some possible forms of bc. for solution of Poisson equation to be well posed:

1. Dirichlet: $\psi(x) = a(x)$ for $x \in A \subset \delta\Omega$,
2. Neumann: $\frac{\partial \psi}{\partial x} = b(x)$ for $x \in B \subset \delta\Omega$.

In Galerkin FE formulation, Dirichlet boundary conditions typically require explicit modification to structure of the problem.
Neumann conditions are dealt with naturally as part of the formulation.
We'll use a Dirichlet boundary condition at $x = 0$, and a Neumann condition at $x = 1$ in our example.

Natural Boundary Conditions

Our weak form equation is

$$\int_0^1 \phi \left(\frac{\partial^2 \psi}{\partial x^2} + f \right) dx = 0,$$

Integrate by parts, putting boundary terms on RHS,

$$\int_0^1 \frac{\partial \phi}{\partial x} \frac{\partial \psi}{\partial x} \, dx - \int_0^1 \phi f \, dx = - \left[\phi \frac{\partial \psi}{\partial x} \right]_0^1 .$$

Chose ϕ to vanish on Dirichlet boundaries (and set $\psi = a(0)$)
Use our knowledge of $\frac{\partial \psi}{\partial x}$ on Neumann boundaries:

$$\sum_e \left[\int_{\Omega^{(e)}} \frac{\partial \phi}{\partial x} \frac{\partial \psi}{\partial x} \, dx \right] = \sum_e \left[\int_{\Omega^{(e)}} \phi f \, dx \right] - \phi(1)\, b(1) \qquad (\dagger)$$

Dirichlet Boundary Conditions

Dirichlet boundary conditions can be enforced through splitting solution into two parts

$$\psi = \psi_0 + \psi_d$$

where ψ_d is any (chosen) function satisfying Dirichlet bcs,

$$\psi_d(0) = a, \qquad \frac{\partial \psi_d}{\partial x}(1) = 0.$$

while the unknown ψ_0 satisfies a modified weak equation with bcs

$$\psi_0(0) = 0, \qquad \frac{\partial \psi_d}{\partial x}(1) = b$$

Boundary Conditions

Substitute back into weak form, putting ψ_0 on the left and known functions on the RHS.

$$\int_\Omega \frac{\partial \phi}{\partial x}\frac{\partial \psi_0}{\partial x}\,dx = \int_\Omega \phi f\,dx - \phi\left(1\right) b\left(1\right) - \int_\Omega \frac{\partial \phi}{\partial x}\frac{\partial \psi_d}{\partial x}\,dx \qquad (\dagger)$$

Note ψ_0 vanishes on Dirichlet boundaries, same as condition to ϕ. Important for existence.

Boundary Conditions: Strong vs. Weak

More generally, implementations of finite element boundary conditions come in two flavours, strong & weak.

Strong form bcs.

Information in boundary condition appears implicitly in weak form of PDE.
Solve by lifting method.

Weak form bcs.

Information in boundary condition appears explicitly in boundary integrals in weak form of PDE.
Solve by direct substitution.

Boundary Conditions: Strong vs. Weak

Which method to apply depends on both original PDE and weak form to be solved. Sometimes both are possible.
E.g., consider the advection equation,

$$\frac{\partial \tau}{\partial t} + \boldsymbol{a} \cdot \nabla \tau = 0,$$

for a tracer τ, given a known velocity field, $\boldsymbol{a}$ and Dirichlet boundary conditions at a inlet.
Note odd number of spatial derivatives, whereas even for Poisson equation.

Boundary Conditions: Strong vs. Weak

Strong form bcs.

Solve

$$\int_\Omega \phi \left(\frac{\partial \tau}{\partial t} + \boldsymbol{a} \cdot \nabla \tau \right) dx = 0,$$

Dirichlet bcs must be applied strongly.

Weak form bcs.

Integrate by parts,

$$\int_\Omega \phi \frac{\partial \tau}{\partial t} - \tau \nabla \cdot (\phi \boldsymbol{a})\, dx$$
$$= - \int_{\delta\Omega^+} \boldsymbol{a} \cdot \boldsymbol{n} \phi \tau_b \, dS,$$

where τ_b is the Dirichlet bc, applied weakly.

When using this sort of weak boundary condition, values may not be quite what you'd expect, however fluxes should be right.

Finite Element Basis Functions

Need discrete finite dimensional representation of problem to do numerical calculations on a computer. Set

$$\psi^{\delta}(x) = \sum_{i=1}^{N} \hat{\psi}_i N_i(x)$$

where $\hat{\psi}_i \in \mathbb{R}$ is a scalar parameter and $N_i : \Omega \to \mathbb{R}$ is a fixed shape function specifying spatial dependence. Can do the same for the space of test functions, ϕ.

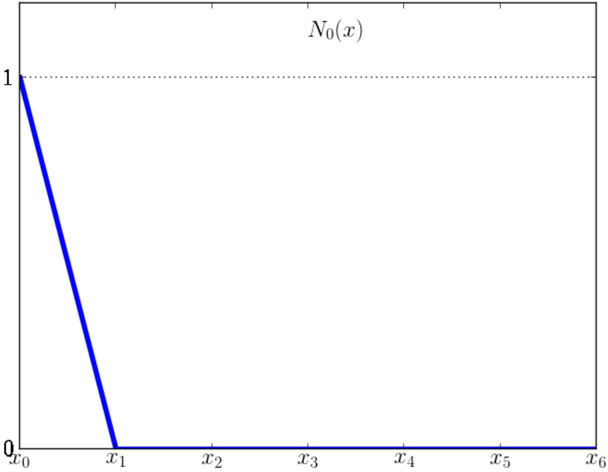
Finite Element Functions

For our 1D Poisson equation example we can choose to use the set of continuous, piecewise linear functions ('shape functions') on subdivisions of the unit interval.

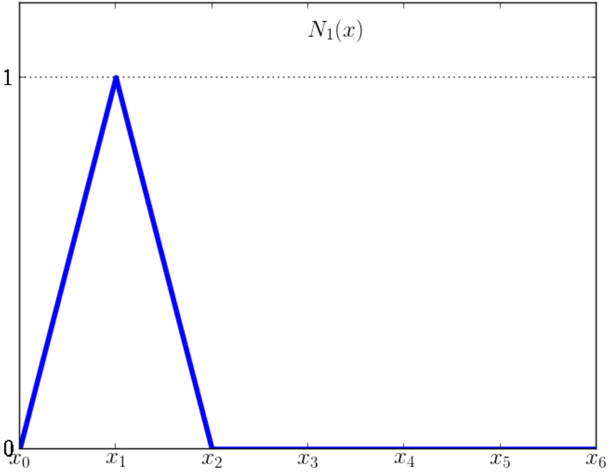
$$N_i = \begin{cases} 0, & x \leq x_{(i-1)}, \\ \frac{x - x_{(i-1)}}{x_i - x_{(i-1)}}, & x_{(i-1)} < x \leq x_i, \\ \frac{x_{(i+1)} - x}{x_{(i+1)} - x_i}, & x_i < x \leq x_{(i+1)}, \\ 0. & x > x_{(i+1)}. \end{cases}$$

Functions are equal to 1 at the set of points $[0, x_1, x_2, \ldots x_{n-1}, 1]$, sometimes called 'nodes' or 'degrees of freedom'. The subdivisions of Ω over which the N_is are smooth are often called elements.

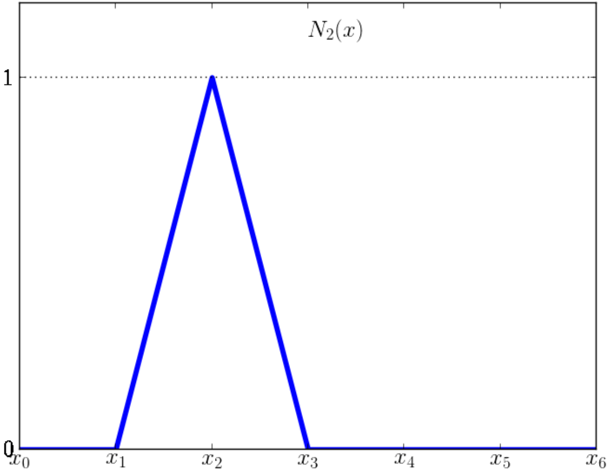
Finite Element Functions



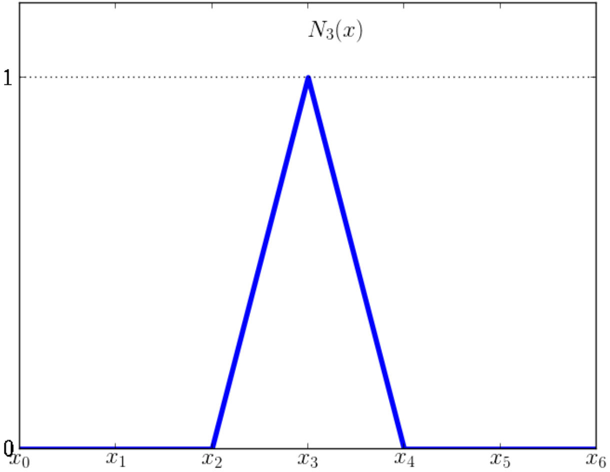
Finite Element Functions



Finite Element Functions



Finite Element Functions



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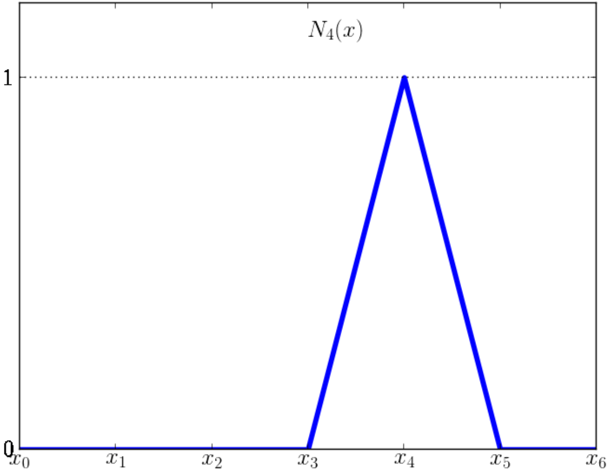

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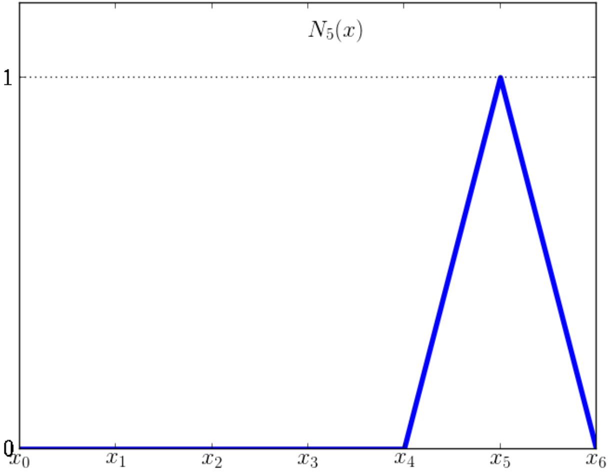
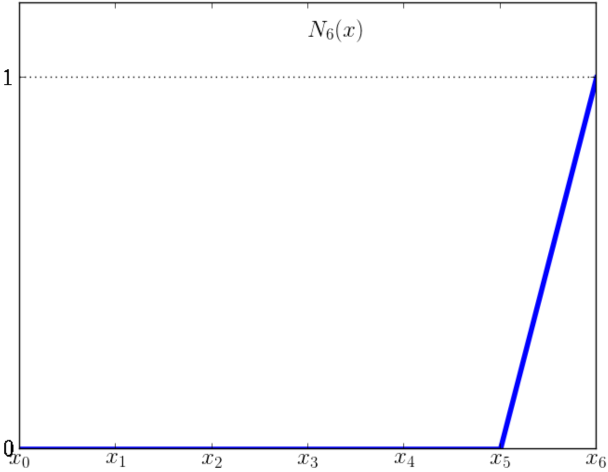


Finite Element Functions



Galerkin Approximation

To obtain the Galerkin approximation of the Poisson equation, we find the (unique) solution of the weak form equation when ψ and ϕ are both represented by our finite element expansions,

$$\psi_0^\delta\left(t,x\right)=\sum_{i=1}^{n}\hat{\psi}_i\left(t\right)N_i\left(x\right),\quad \phi^\delta\left(t,x\right)=\sum_{j=1}^{n}\hat{\phi}_j\left(t\right)N_j\left(x\right).$$

$$\psi_d=a\left(0\right)N_0$$

Note sums start from 0 due to bcs. The ψ^δ are trial functions. Function space they come from is the trial space. The $\phi^\delta$are test functions and live in the test space. Computation involves obtaining the finite number of $\hat{\psi}_i$. Can thus be solved on a computer.

Galerkin Approximation

Substituting the finite representations into (†) we get

$$\int \sum_{j=1}^{n} \hat{\phi}_j \frac{\partial N_j}{\partial x} \sum_{i=1}^{N} \hat{\psi}_i \frac{\partial N_i}{\partial x} \, dV + v_N^{\delta} b\left(x_N\right) = \int \sum_{j=1}^{n} \hat{\phi}_j^{\delta} N_j f \, dV,$$

$$\hat{\phi}_j \left(\underbrace{\left[\int \frac{\partial N_j}{\partial x} \frac{\partial N_i}{\partial x} dV \right]}_{\text{matrix}\, D_{ij}} \hat{\psi}_i - \int f N_j dV + \begin{cases} 0, & j = 1, \ldots n-1 \\ b\left(x_n\right), & j = n \end{cases} \right)$$

$$= 0$$

If bracket vanishes, solution applies for any $\hat{\phi}_j$, so we can drop them.

The Right Hand Side

Generally the right hand side of the equation is known explicitly as a function $f:\Omega\to\mathbb{R}$. Hence $\int_0^1 \phi^\delta f\,dx$ can be calculated exactly. In practice (especially for coupled problems,) it is usually represented in the approximate function space,

$$f^\delta(x) = \sum_{i=0}^{N} \hat{f}_i N_i(x),$$

where (for our choice of shape functions),

$$\hat{f}_i = f(x_i).$$

Finite Element Poisson Matrix Problem

Dirichlet condition: $\hat{\psi}_0 = a(0)$, turns up on right hand side:

$$
\begin{bmatrix}
-\frac{2}{h} & \frac{1}{h} & 0 & 0 & 0 & 0 \\
\frac{1}{h} & -\frac{2}{h} & \frac{1}{h} & 0 & 0 & 0 \\
0 & \frac{1}{h} & -\frac{2}{h} & \frac{1}{h} & 0 & 0 \\
0 & 0 & \frac{1}{h} & -\frac{2}{h} & \frac{1}{h} & 0 \\
0 & 0 & 0 & \frac{1}{h} & -\frac{2}{h} & \frac{1}{h} \\
0 & 0 & 0 & 0 & \frac{1}{h} & -\frac{1}{h}
\end{bmatrix}
\begin{bmatrix}
\psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6
\end{bmatrix}
=
\begin{bmatrix}
\frac{h}{6}\hat{f}_0 + \frac{2h}{3}\hat{f}_1 + \frac{h}{6}\hat{f}_2 - \frac{1}{h}a(0) \\
\frac{h}{6}\hat{f}_1 + \frac{2h}{3}\hat{f}_2 + \frac{h}{6}\hat{f}_3 \\
\frac{h}{6}\hat{f}_2 + \frac{2h}{3}\hat{f}_3 + \frac{h}{6}\hat{f}_4 \\
\frac{h}{6}\hat{f}_3 + \frac{2h}{3}\hat{f}_4 + \frac{h}{6}\hat{f}_5 \\
\frac{h}{6}\hat{f}_4 + \frac{2h}{3}\hat{f}_5 + \frac{h}{6}\hat{f}_6 \\
\frac{h}{6}\hat{f}_5 + \frac{h}{3}\hat{f}_6\text{-b(1)}
\end{bmatrix}
$$

Finite Difference Poisson Matrix Problem

Finite Difference discretization of same problem:

$$
\begin{bmatrix}
\frac{-2}{h^2} & \frac{1}{h^2} & 0 & 0 & 0 & 0 \\
\frac{1}{h^2} & \frac{-2}{h^2} & \frac{1}{h^2} & 0 & 0 & 0 \\
0 & \frac{1}{h^2} & \frac{-2}{h^2} & \frac{1}{h^2} & 0 & 0 \\
0 & 0 & \frac{1}{h^2} & \frac{-2}{h^2} & \frac{1}{h^2} & 0 \\
0 & 0 & 0 & \frac{1}{h^2} & \frac{-2}{h^2} & \frac{1}{h^2} \\
0 & 0 & 0 & 0 & \frac{1}{h^2} & \frac{-1}{h^2}
\end{bmatrix}
\begin{bmatrix}
\psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6
\end{bmatrix}
=
\begin{bmatrix}
\hat{f}_2 - \frac{a(0)}{h^2} \\ \hat{f}_3 \\ \hat{f}_4 \\ \hat{f}_5 \\ \hat{f}_6 \\ \frac{1}{2}\hat{f}_6\text{-b}(1)
\end{bmatrix}
$$

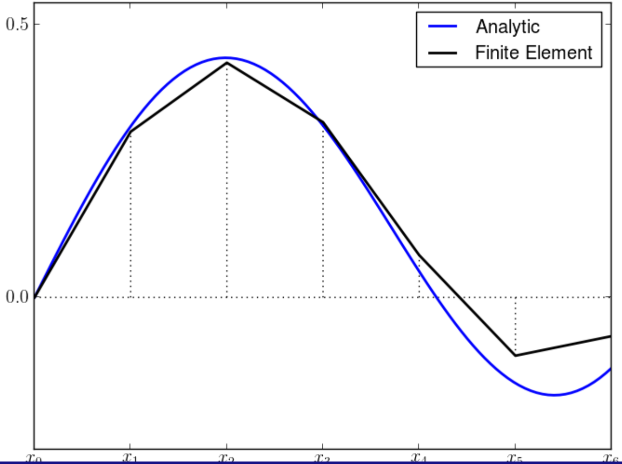
Finite Volume Poisson Matrix Problem

Finite volume discretization of same problem. 2 point finite difference approximation for flux terms:

$$
\begin{bmatrix}
\frac{-3}{h} & \frac{1}{h} & 0 & 0 & 0 & 0 \\
\frac{1}{h} & \frac{-2}{h} & \frac{1}{h} & 0 & 0 & 0 \\
0 & \frac{1}{h} & \frac{-2}{h} & \frac{1}{h} & 0 & 0 \\
0 & 0 & \frac{1}{h} & \frac{-2}{h} & \frac{1}{h} & 0 \\
0 & 0 & 0 & \frac{1}{h} & \frac{-2}{h} & \frac{1}{h} \\
0 & 0 & 0 & 0 & \frac{1}{h} & \frac{-1}{h}
\end{bmatrix}
\begin{bmatrix}
\psi_{\frac{1}{2}} \\ \psi_{\frac{3}{2}} \\ \psi_{\frac{5}{2}} \\ \psi_{\frac{7}{2}} \\ \psi_{\frac{9}{2}} \\ \psi_{\frac{11}{2}}
\end{bmatrix}
=
\begin{bmatrix}
h\hat{f}_{1/2} - 2\frac{a(0)}{h} \\ h\hat{f}_{3/2} \\ h\hat{f}_{5/2} \\ h\hat{f}_{7/2} \\ h\hat{f}_{9/2} \\ h\hat{f}_{11/2} - b(1)
\end{bmatrix}
$$

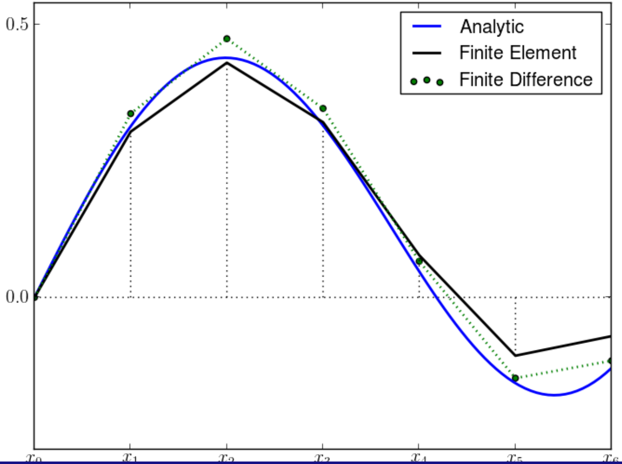
Solutions: Finite Element

$f = 10\sin(5x) + 1/2\cos(3(x+1/2))$, $\psi(0) = 0$, $\frac{d\psi}{dx} = 1$:



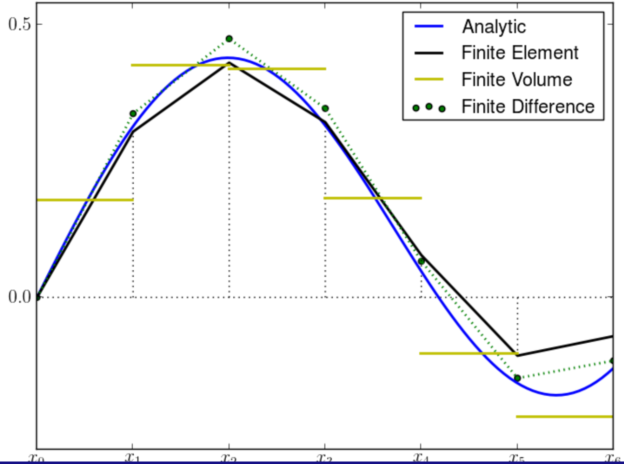
Solutions: Finite Difference

$f = 10\sin(5x) + {}^1\!/_2\cos(3(x + {}^1\!/_2))$, $\psi(0) = 0$, $\frac{d\psi}{dx} = 1$:



Solutions: Finite Volume

$f = 10\sin(5x) + 1/2\cos(3(x+1/2))$, $\psi(0) = 0$, $\frac{d\psi}{dx} = 1$:



Example II: Advection-Diffusion

Use same finite element framework for 1D advection diffusion equation:

$$\frac{\partial \tau}{\partial t} + \frac{\partial}{\partial x}(u\tau) = \frac{\partial}{\partial x}\left(\kappa \frac{\partial \tau}{\partial x}\right)$$

$$\int_0^1 \phi \frac{\partial \tau}{\partial t}\, dx = \int \frac{\partial \phi}{\partial x}\left(u\tau - \kappa \frac{\partial \tau}{\partial x}\right)\, dx$$

Example II: Advection-Diffusion

Given FEM framework, crank the handle to reduce the problem;
Discretize through Finite Element Galerkin Method,

$$\tau^\delta = \sum_{i=0}^{n} \hat{\tau}_i N_i^\tau,$$

$$u^\delta = \sum_{i=0}^{n} \hat{u}_i N_i^u$$

$$\phi^\delta = \sum_{i=0}^{n} \hat{\phi}_i N_i^\tau,$$

Note that the method doesn't require $N_i^u = N_i^\tau$. Mixed formulations are possible

Example II: Advection-Diffusion

Following substitution, integrate by parts to obtain

$$\underbrace{\int_0^1 N_i N_j \, dx}_{\text{"Mass matrix"} M_{ij}} \quad \frac{\partial \hat{\tau}_j}{\partial t} - \int_0^1 \frac{\partial N_i}{\partial x} \left(N_j \sum_{k=0}^{n} u_k N_k^u - \kappa \frac{\partial N_j}{\partial x} \right) dx \hat{\tau}_j = 0,$$

or in matrix form,

$$\mathrm{M} \frac{\partial \boldsymbol{\tau}}{\partial t} + \mathrm{A}\left(\boldsymbol{u}\right)\boldsymbol{\tau} + \mathrm{D}\left(\kappa\right)\boldsymbol{\tau} = 0,$$

This is the FEM form of the tracer advection-diffusion equation. Further details will depend on the choice of shape functions and timestepping method.

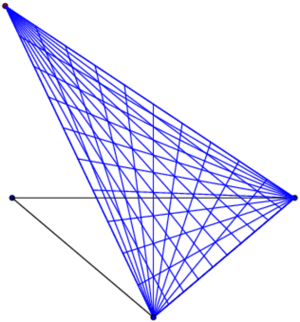
Review of Section

- Finite element methods solve weak (integral) equations
- Functions get approximated by finite dimensional sums of polynomial functions, non-zero over small regions of problem domain (elements)
- Linear PDE problem gives linear (matrix) problem for $\hat{\psi}_i$.

The efficient computational representation and solution of these sorts of problems will form the basis of other sessions.

Extensions: Higher dimensions

The linear finite element method extends naturally on simplicies [line elements, triangle elements, tetrahedrons]. Basis functions are set to 1 at one vertex and to zero on the others. E.g. for triangles:



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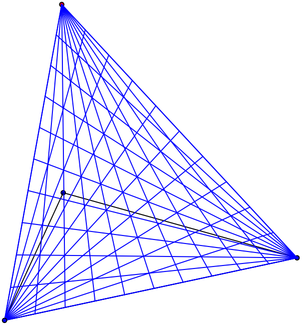

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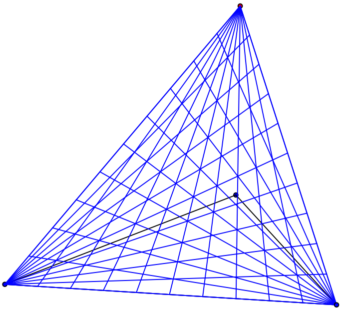

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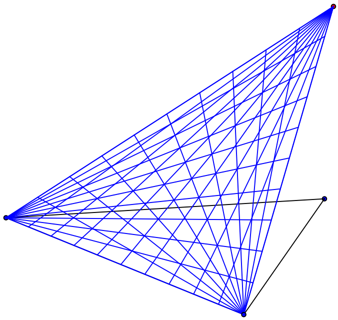


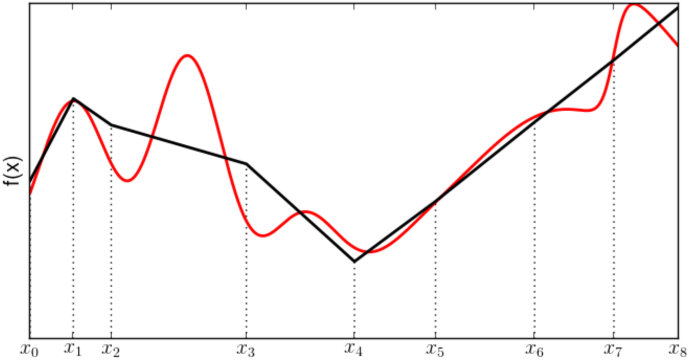
Increasing the degrees of freedom

To increase the number of free parameters in the approximate solution space (and thus attempt to get a more accurate solution) there are several options:

- More, smaller subdivisions [step size,h]
  - This is the system used in Fluidity's mesh adaptivity.
- Use higher order polynomials, e.g. quadratic functions [polynomial order, p]
- Use discontinuous functions [Discontinuous Galerkin formulation]

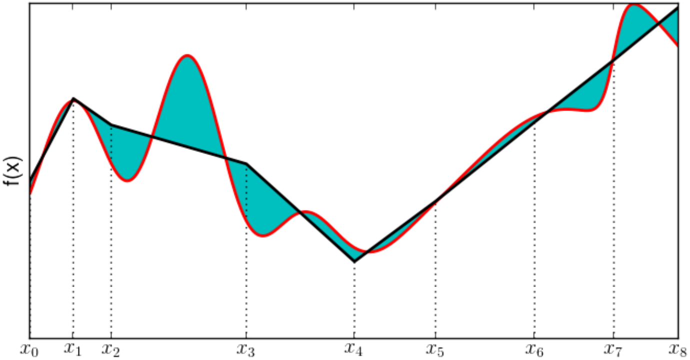
Increasing the degrees of freedom

- Projection (in black) of smooth function (red).
- Linear, continuous basis, Galerkin method (P1 CG).
- In 1d degrees of freedom $\approx$ no. of elements



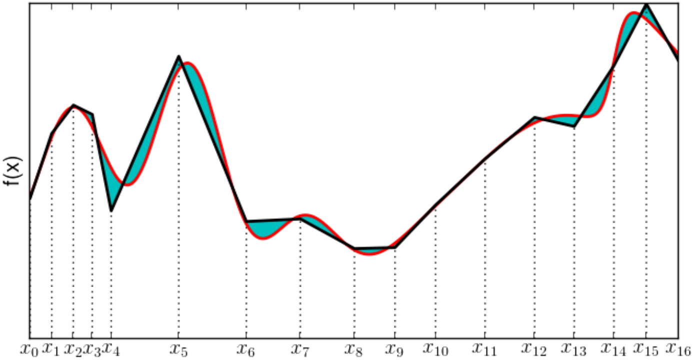
Increasing the degrees of freedom

- Projection (in black) of smooth function (red), & error (blue).
- Linear, continuous basis, Galerkin method (P1 CG).
- In 1d degrees of freedom $\approx$ no. of elements



Increasing the degrees of freedom

- May increase the number of elements
- More elements mean more degrees of freedom
- One of the methods used in Fluidity adaptivity routines



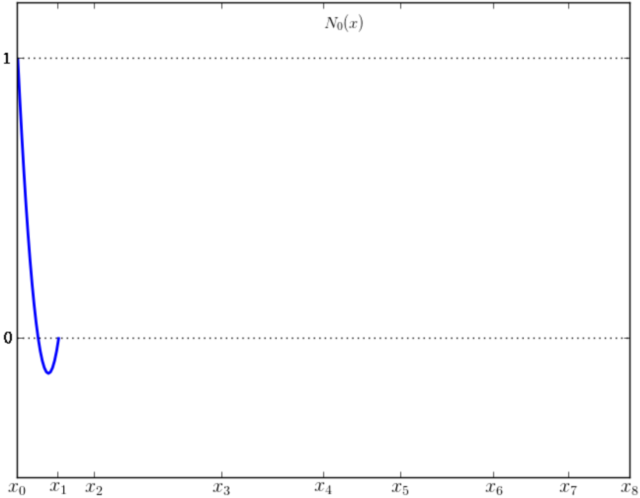
Increasing the degrees of freedom

Quadratic shape functions

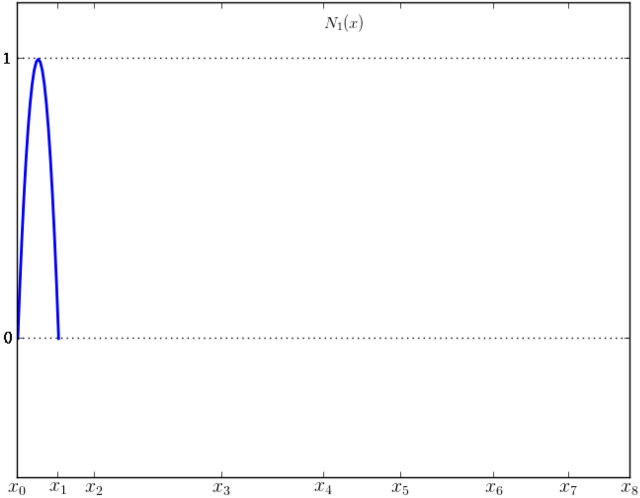
$$N_{2i} = \begin{cases} 0, & x \leq x_{i-1}, \\ \frac{2x^2-(3x_{i-1}+x_i)x+(x_i+x_{i-1})x_{i-1}}{(x_i-x_{i-1})^2}, & x_{i-1} < x \leq x_i, \\ \frac{2x^2-(3x_{i+1}+x_i)x+(x_i+x_{i+1})x_{i+1}}{(x_{i+1}-x_i)^2}, & x_i < x \leq x_{i+1}, \\ 0. & x > x_{i+1}. \end{cases}$$

$$N_{2i+1} = \begin{cases} 0, & x < x_i, \\ -\frac{x^2-(x_{i+1}+x_i)x+x_ix_{i+1}}{(x_{i+1}-x_i)^2}, & x_i < x \leq x_{i+1}, \\ 0. & x > x_{i+1}. \end{cases}$$

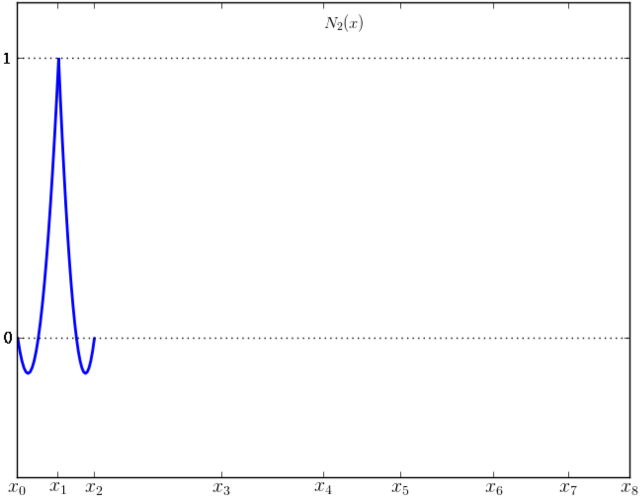
Finite Element Functions



Finite Element Functions



Finite Element Functions



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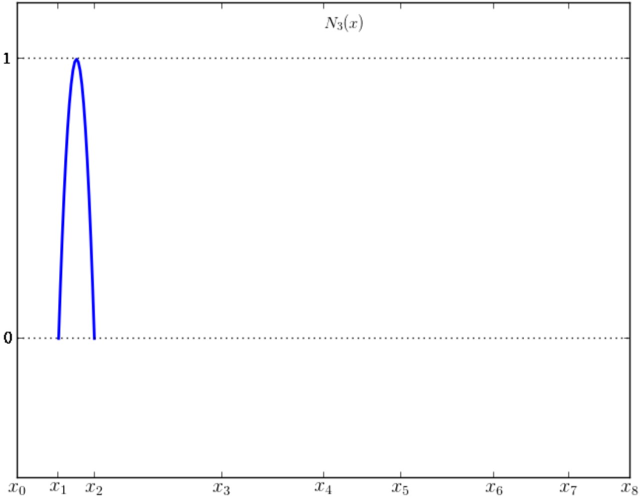
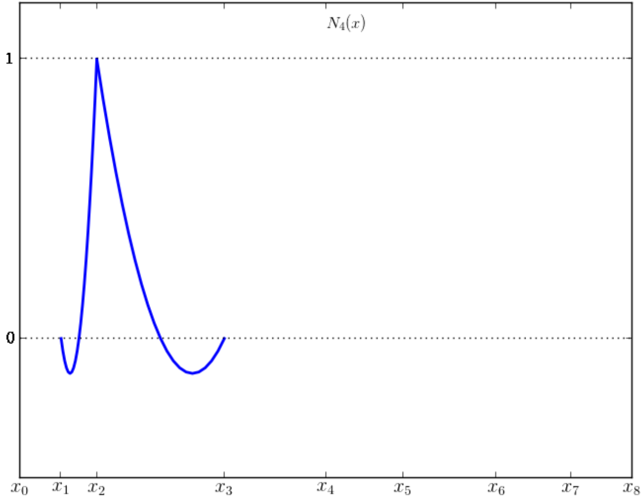
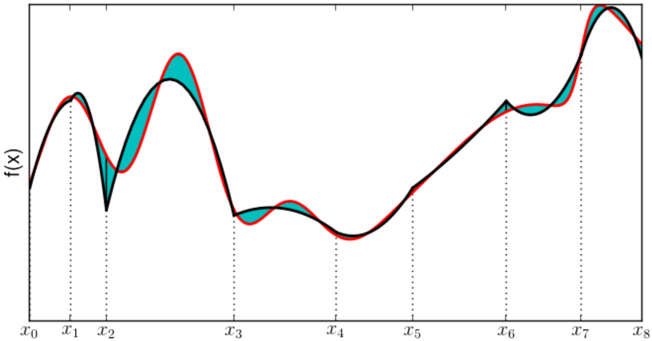


Finite Element Functions



Increasing the degrees of freedom

- Projection of a smooth function.
- Quadratic, continuous basis, Galerkin method (P2 CG).
- In 1d degrees of freedom $\approx 2 \times$ no. of elements.
- Good representation of slowly varying functions



Increasing the degrees of freedom

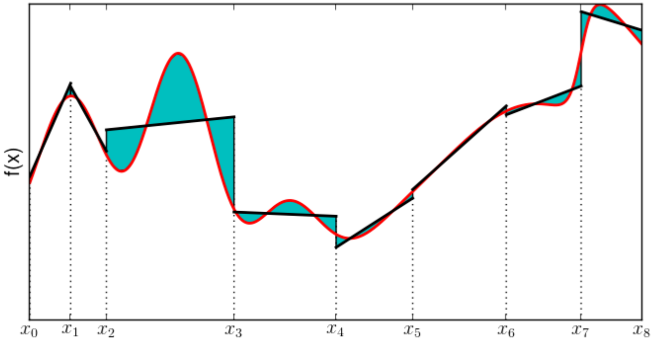
Discontinuous linear shape functions

$$N_{2i} = \begin{cases} 0, & x \leq x_i, \\ \frac{(x_{i+1}-x)}{(x_{i+1}-x_i)}, & x_i < x \leq x_{i+1}, \\ 0. & x > x_{i+1}. \end{cases}$$

$$N_{2i+1} = \begin{cases} 0, & x < x_i, \\ -\frac{(x-x_i)}{(x_{i+1}-x_i)}, & x_i < x \leq x_{i+1}, \\ 0. & x > x_{i+1}. \end{cases}$$

Increasing the degrees of freedom

- Projection of a smooth function.
- Linear, discontinuous basis, Galerkin method (P1 DG).
- In 1d degrees of freedom $\approx 2 \times$ no. of elements.
- Good representation of discontinuities/fronts/large gradients.



Review of Section

- The degrees of freedom (and thus size) of a problem can be increased by:
  - increasing the number of element subdivisions (making step size h smaller)
  - increasing the order of the shape functions applied on elements (increasing polynomial degree, p)
  - relaxing continuity constraints at the interface between elements (discontinuous Galerkin method, nonconforming elements)

Reassuring Mathematics

This section summarises some useful results from mathematical analysis for finite element problems. In particular, we note that results exists to show that, under certain provisos finite element solutions to a given problem

- exist
- are unique
- converge
- converge to the right answer.

Choice of Vector Spaces

Going back to the weak form for the original infinite dimensional problem,

$$\int_\Omega \nabla\phi\cdot\nabla\psi\,dV = \int_\Omega \phi f\,dV + \int_{\delta\Omega^N}\phi b\,dV,$$

it is obvious that ψ and $\nabla\psi$ must be well behaved enough for these integrals to exist.

Choice of Vector Spaces

We require the function is square integrable,

$$\|\psi\|^2 := \int_\Omega \psi^2 \, dV < \infty. \tag{1}$$

(The space of function which satisfy this is normally called $\mathcal{L}_2(\Omega)$) and also that

$$\|\nabla\psi\|^2 := \int_\Omega \nabla\psi \cdot \nabla\psi \, dV < \infty, \tag{2}$$

Functions which satisfy both (1) & (2) are in the space of square integrable functions with square integrable derivatives, denoted $\mathcal{H}^1(\Omega)$. This is a Sobolev space.

Weak equations and Bilinear forms

The volume integral in (*) defines a symmetric bilinear form, $a : \mathcal{H}^1(\Omega) \times \mathcal{H}^1(\Omega) \to \mathbb{R}$,

$$a(\phi, \psi) := \int_\Omega \nabla\psi \cdot \nabla\phi \, d^n x.$$

where

$$a(\phi, \psi) = a(\psi, \phi),$$
$$a(c_1\phi + c_2\xi, \psi) = c_1 a(\phi, \psi) + c_2 a(\xi, \psi).$$

Lax-Milgram Theorem

Two properties of the bilinear form, a, are used to show well-posedness:

$$a(\phi,\psi) \leq C\,\|\psi\|\,\|\phi\| \text{ for some C} > 0, \qquad \text{(continuity)}$$

$$a(\psi,\psi) \geq c\,\|\psi\|^2 \text{ for some c>0.} \qquad \text{(coercive/elliptic)}$$

Well posedness - existence

Existence follows from application of Riesz representation theorem to the Hilbert space problem

$$a\left(u,v\right)=f\left(v\right),$$

Precise details lie outside of the scope of this lecture, but effectively guarantees an "inverse" to the map

$$\phi_v\left(u\right)=a\left(u,v\right),$$

so that for sufficiently smooth data we can always get a solution.

Well posedness - uniqueness:

Suppose there are two different solutions, ψ_1 and ψ_2, i.e.

$$a\left(\phi, \psi_1\right) = a\left(\phi, \psi_2\right) = \int_\Omega \phi f \, d^n x, \text{ for all } \phi \in \mathcal{H}^1\left(\Omega\right).$$

Then

$$a\left(\phi, \psi_1 - \psi_2\right) = 0$$

but $\psi_1 - \psi_2 \in \mathcal{H}^1\left(\Omega\right)$, so can choose to test $\phi = \psi_1 - \psi_2$ Then

$$a\left(\psi_1 - \psi_2, \psi_1 - \psi_2\right) = 0 \geq c \left\|\psi_1 - \psi_2\right\|^2,$$

So $\psi_1 = \psi_2$, hence solution is unique.

Well posedness - convergence

Let $\psi \in \mathcal{V}$ be exact solution, $\psi^\delta \in \mathcal{V}^\delta \subset \mathcal{V}$ be the finite element solution $\xi \in \mathcal{V}^\delta$ be an arbitrary function. Then $\psi^\delta - \xi \in \mathcal{V}$ and $\psi^\delta - \xi \in \mathcal{V}^\delta$ and

$$a(\underbrace{\psi^\delta - \xi}_{\in\mathcal{V}}, \psi) = \int_\Omega \left(\psi^\delta - \xi\right) f d^n x \qquad \text{(From PDE)}$$

$$a(\underbrace{\psi^\delta - \xi}_{\in\mathcal{V}^\circ}, \psi^\delta) = \int_\Omega \left(\psi^\delta - \xi\right) f d^n x \qquad \text{(FEM)}$$

Well posedness - convergence

$$
\begin{aligned}
c\left\|\psi-\psi^{\delta}\right\|^{2} \leq & a\left(\psi-\psi^{\delta}, \psi-\psi^{\delta}\right), \\
& = a\left(\psi-\psi^{\delta}, \psi-\psi^{\delta}\right)+a\left(\psi-\psi^{\delta}, \psi^{\delta}-\xi\right) \\
& \qquad + a\left(\psi^{\delta}-\xi, \psi^{\delta}\right)-a\left(\psi^{\delta}-\xi, \psi\right), \\
& = a\left(\psi-\psi^{\delta}, \psi-\psi^{\delta}+\psi^{\delta}-\xi\right) \\
& \qquad -\int_{\Omega}\left(\psi^{\delta}-\xi\right) f d^{n} x+\int_{\Omega}\left(\psi^{\delta}-\xi\right) f d^{n} x, \\
& = a\left(\psi-\psi^{\delta}, \psi-\xi\right) \leq C\left\|\psi-\psi^{\delta}\right\|\|\psi-\xi\| .
\end{aligned}
$$

Well posedness - convergence

Hence it is guaranteed (Cea's lemma)

$$\left\|\psi - \psi^{\delta}\right\| \leq \frac{C}{c} \inf_{\xi \in \mathcal{V}^{\delta}} \|\psi - \xi\| .$$

Choose ξ to be linear projection of ψ, i.e $\xi\left(x_i\right) = \psi\left(x_i\right)$, $\frac{\partial^2 \xi}{\partial x^2} = 0$, then

$$\|\psi - P\psi\| \leq \alpha \sup_{\Omega_i} h_i \sup_{x \in \Omega} \left|\frac{\partial^2 \psi}{\partial x^2}\right|$$

Hence

$$\left\|\psi - \psi^{\delta}\right\| \leq \frac{\alpha C}{c} \sup_{\Omega_i} h_i \sup_{x \in \Omega} \left|\frac{\partial^2 \psi}{\partial x^2}\right|$$

$$\left\|\psi - \psi^{\delta}\right\| \to 0 \text{ as } \sup_{\Omega_i} h_i \to 0.$$

Review of Section

We have shown that finite element approximations to the solutions to PDEs

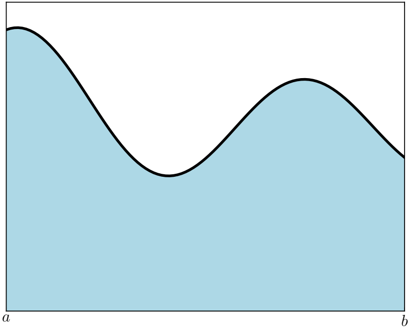
- are unique,
- converge,
- and converge to the right answer.

We have also given a hint that they exist. We have also shown that by using knowledge about the form of the solution we can choose elements to minimize the estimated error for a given number of degrees of freedom.

Quadrature

Numerical method to calculate/approximate integrals:

$$\int_a^b f(\boldsymbol{x})\, d^n x \approx \sum_{x=1}^{N} w_i f(\boldsymbol{p}_i)$$



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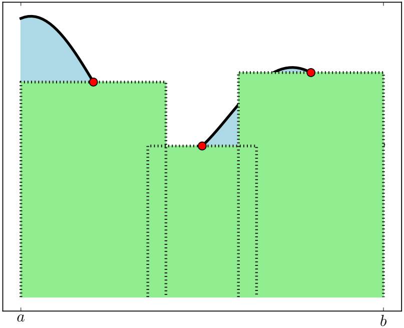

Quadrature

Some famous quadratures:

1. Midpoint rule [one point method]

$$w = a - b, \quad p = \frac{a+b}{2},$$

2. Simpson's rule [3 point method]

$$w_1 = \frac{a-b}{6}, \quad w_2 = \frac{4\,(a-b)}{6}, \quad w_3 = \frac{a-b}{6},$$

$$p_1 = a, \quad p_2 = \frac{a+b}{2}, \quad p_3 = b,$$

Quadrature

A quadrature is called exact for functions where

$$\int_a^b f(\boldsymbol{x})\, d^n x = \sum_{x=1}^{N} w_i f(\boldsymbol{p}_i)\,.$$

The degree (or order) of a quadrature rule over an interval is the largest integer, n such that method is exact for all

$$p_n = a_0 x^n + a_1 x^{n-1} + \ldots a_n \quad a_i \in \mathbb{R}.$$

For a FE method, want a quadrature which captures the “worst” order of the terms in the integral, eg

$$\int_{x_j}^{x_{j+1}} \nabla N_i^{(\tau)} N_j^{(u)} N_j^{(\tau)} dx.$$

Final Summary

- Finite element methods solve a weak form of the exact equations in an approximate solution space.
- The approximate solution is defined (almost) everywhere.
- Basis functions have order and continuity criteria.
- Neuman conditions dealt with implicitly inside formulation
- Dirichlet conditions appear in right hand side (as in finite difference methods).

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