Ex 2 :

$$x(t) = e^{-at^2} \in L^1$$

$$t e^{-ct^2} \in L^1$$

$$X'(f) = \widehat{(-2j\pi t)x(t)}(f)$$

$$= \int_{\mathbb{R}} (-2j\pi t) e^{-at^2} e^{-2j\pi f t} dt$$

$$\frac{1}{a}\left(e^{-at^2}\right)' = -2t\, e^{-at^2}$$

$$= \int_{\mathbb{R}} \frac{j\pi}{a} \left(e^{-at^2}\right)' e^{-2j\pi f t} dt$$

$$= \frac{j\pi}{a} \left(\underbrace{\left[e^{-at^2} e^{-2j\pi f t} \right]_{-\infty}^{+\infty}}_{=0} - \underbrace{\int_{\mathbb{R}} e^{-at^2} (-2j\pi f) e^{-2j\pi f t} dt}_{} \right)$$

$$\boxed{X'(f) = -\frac{2\pi^2}{a} f \, X(f)}$$

$$\frac{X'(f)}{X(f)} = -\frac{2\pi^2}{a} f$$

$$\ln X(f) = -\frac{\pi^2}{a} f^2 + C$$

$$X(f) = K e^{-\frac{\pi^2}{a} f^2}$$

$$X(0) \begin{cases} = K \\ = \int_{\mathbb{R}} x(t)\, dt = \int_{\mathbb{R}} e^{-at^2} dt = \sqrt{\frac{\pi}{a}} \end{cases}$$

$$\boxed{X(f) = \sqrt{\frac{\pi}{a}} \, e^{-\frac{\pi^2}{a} f^2}}$$

Rq : $a = \pi$. $\quad x(t) = e^{-\pi t^2}$

$$X(f) = e^{-\pi f^2}$$

Ex 4 :

$$\boxed{\frac{d^2x(t)}{dt^2} - 4\pi^2 t^2 x(t) = 4\pi\, x(t)}$$

$x \in L^1$, $t x(t)$ et $t^2 x(t) \in L^1$, $x \in C^2$, $x', x'' \in L^1$

$$\widehat{x''}(f) + \widehat{-4\pi^2 t^2 x(t)}\,(f) = 4\pi\, \hat{x}(f)$$

$$\widehat{x''}(f) = (j2\pi f)^2\, \hat{x}(f) = -4\pi^2 f^2 \hat{x}(f)$$

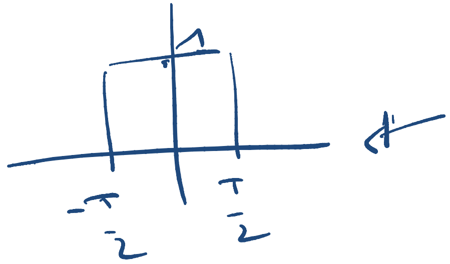
$$\widehat{-4\pi^2 t^2 x(t)}\,(f) = \widehat{(-2j\pi t)^2 x(t)}\,(f) = \hat{x}''(f)$$

$$\boxed{\hat{x}''(f) - 4\pi^2 f^2 \hat{x}(f) = 4\pi\, \hat{x}(f)}$$

Ex 6 :

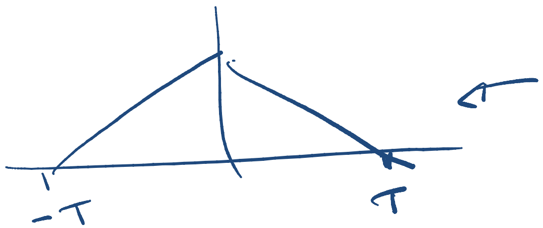
$$\int_{\mathbb{R}} \left(\frac{\sin t}{t}\right)^3 dt = \int_{\mathbb{R}} \underbrace{\left(\frac{\sin t}{t}\right)}_{x(t)} \underbrace{\left(\frac{\sin t}{t}\right)^2}_{y(t)} dt \underset{\text{Parseval-Plancherel}}{=} \int_{\mathbb{R}} \hat{x}(t)\,\hat{y}(t)\, dt$$

- $\frac{1}{T}\,\widehat{\Pi_T}(f) = \operatorname{sinc}(\pi T f)$, $T = \frac{1}{\pi}$

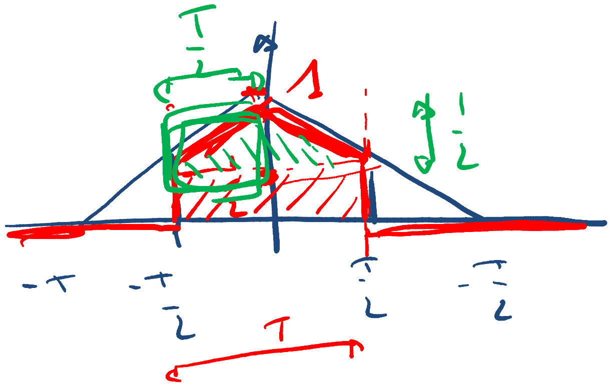


$$\pi\,\widehat{\Pi_{\frac{1}{\pi}}}(f) = \operatorname{sinc}(f) = \frac{\sin f}{f}$$

- $\frac{1}{T}\,\widehat{\wedge_T}(f) = \operatorname{sinc}^2(\pi T f)$



$T = \frac{1}{\pi}$ $\qquad \pi\,\widehat{\wedge_{\frac{1}{\pi}}}(f) = \operatorname{sinc}^2(f)$



$$\int \left(\frac{\sin t}{t}\right)^3 dt = \int x(t)\, y(t)\, dt$$

$$= \pi^2 \underbrace{\int \Pi_T(t)\, \Lambda_T(t)\, dt}_{= \frac{T}{2} + \frac{T}{4}} \quad , \quad T = \frac{1}{\pi}$$

$$= \frac{\pi^2}{\pi} \left(\frac{1}{2} + \frac{1}{4} \right)$$

$$= \frac{3\pi}{4}$$

$$\Pi_T \xrightarrow{\text{TF}} T \operatorname{sinc}(\pi T f)$$

$$\xleftarrow{\text{TF}^{-1}} \quad (\text{ds } L^2)$$

$$\int x(t)\, e^{-2j\pi f t}\, dt$$

$$= \int x(t) \left(\cos 2\pi f t + j \sin 2\pi f t \right) dt$$

$$\int \left|\frac{\sin t}{t}\right|^3 dt = \int \underbrace{\left(\frac{\sin t}{t}\right)^{3/2}}_{x(t)} \underbrace{\left(\frac{\sin t}{t}\right)^{3/2}}_{x(t)} dt \qquad \in L^2$$

Ex 8:

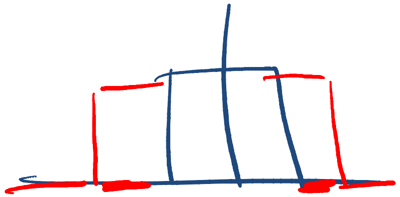
$$x * y(0) = \int_{\mathbb{R}} \frac{\sin \pi a u}{\pi u} \, \frac{\sin \pi b(0-u)}{\pi(0-u)} \, du$$

1) $$x(t) = \frac{\sin \pi a t}{\pi t} = a \operatorname{sin_c}(\pi a t) = \widehat{\Pi_a}(t)$$

$$y(t) = \frac{\sin \pi b t}{\pi t} = b \operatorname{sin_c}(\pi b t) = \widehat{\Pi_b}(t)$$

$$x * y(t) = \widehat{\Pi_a} * \widehat{\Pi_b}(t) = \widehat{\Pi_a \Pi_b}(t) = \widehat{\Pi_c}(t)$$

$c = \min(a, b)$, $\Pi_a \Pi_b = \Pi_c$



Donc : $x * y(t) = c \operatorname{sinc}(\pi c t) = \dfrac{\sin(\pi c t)}{\pi t}$

2) $\displaystyle\int_{\mathbb{R}} \frac{(\sin \pi a t)(\sin \pi b t)}{\pi^2 t^2}\, dt = x * y(0) = c = \min(a,b)$

$\displaystyle\int_{\mathbb{R}} \frac{\pi^2 \sin^2 t}{\pi^2 t^2}\, dt = \pi^2 c \quad , \quad c = a = b = \frac{1}{\pi}$

D'où :
$$\int_{\mathbb{R}} \frac{\sin^2 u}{u^2}\, du = \pi$$

$a = b = \frac{1}{\pi}$ $\quad \displaystyle\frac{\pi^2}{\pi} \int \frac{\sin^2 t}{\pi^2 t^2} = c\pi^2$

Ex 9:

$$RCv'(t) + v(t) = x(t)$$

$$x \in L^1, \quad v \in L^1, \quad v \in C^1, \quad v' \in L^1$$

$$\hat{v}(f) = \dots$$

1) $z(t) = e^{-\lambda t} \mathbb{1}_{\mathbb{R}^+}(t), \quad \lambda > 0$

$$\hat{z}(f) = \int_{\mathbb{R}^+} e^{-(\lambda + j2\pi f)t} \, dt = \frac{1}{\lambda + j2\pi f}$$

2) $RC \, \widehat{v'}(f) + \hat{v}(f) = \hat{x}(f)$

$$RC(j2\pi f)\hat{v}(f) + \hat{v}(f) = \hat{x}(f)$$

$$\hat{v}(f)(1 + j2\pi RCf) = \hat{x}(f)$$

$$\Rightarrow \quad \hat{v}(f) = \frac{1}{1 + j2\pi RCf} \hat{x}(f)$$

$$\boxed{\hat{v}(f) = \frac{1}{RC} \frac{1}{\left(\frac{1}{RC} + j2\pi f\right)} \hat{x}(f)}$$

$$h(t) = \frac{1}{RC} e^{-\frac{t}{RC}} \mathbb{1}_{\mathbb{R}^+}(t) \qquad \text{causale}$$

$$\hat{h}(f) = \frac{1}{RC\left(\frac{1}{RC} + 2j\pi f\right)}$$

$$\hat{v}(f) = \hat{h}(f)\,\hat{x}(f) = \widehat{h*x}(f)$$

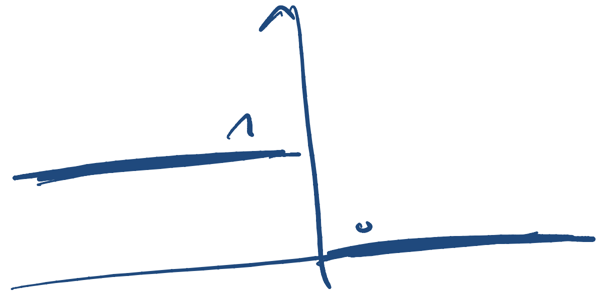
Par injectivité de la T.F. : $\boxed{v(t) = h*x(t) \quad \text{p.p.}}$

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow v(t) \qquad\qquad \underbrace{\in L^\infty \qquad \in L^1}_{\text{Continue}}$$

$$\left.\begin{array}{l} v \text{ continue} \\ h*x \text{ continue} \end{array}\right\} \Longrightarrow v(t) = h*x(t) \quad \forall t \in \mathbb{R}$$

Ex 2 :

1) · $h' = \delta$ $\quad (T_h' = \delta)$



· $h_\sigma' =$

$$\boxed{T_f' = T_{f'} + \sum \sigma_n \delta_{a_n}}$$

$$T_{h_\sigma}' = \underline{T_{h_\sigma'}} + \sigma_0 \delta_0 = 0 + (-1)\,\delta_0 = -\delta$$

ou $\langle h_\sigma', \varphi \rangle = -\langle h_\sigma, \varphi' \rangle$

$$= -\int_{\mathbb{R}} h_\sigma(x)\,\varphi'(x)\,dx$$

$$= -\int_{-\infty}^{0} \varphi'(x)\,dx = -\left[\varphi(x)\right]_{-\infty}^{0} = -\varphi(0) = \langle -\delta, \varphi \rangle$$

$$f(x) = |x| \longrightarrow f'(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases} = h(x) - h_\sigma(x)$$

$$\langle |x|', \varphi \rangle = -\langle |x|, \varphi' \rangle$$

$$= -\int_{\mathbb{R}} |x| \, \varphi'(x) \, dx$$

$$= -\int_{-\infty}^{0} -x \, \varphi'(x) \, dx - \int_{0}^{+\infty} x \, \varphi'(x) \, dx$$

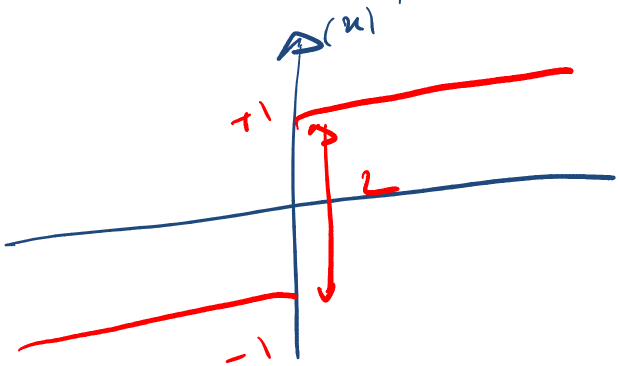
$$= \underbrace{\Big[x\varphi(x)\Big]_{-\infty}^{0}}_{=0} - \underbrace{\int_{-\infty}^{0} \varphi(x) \, dx}_{\int_{\mathbb{R}} h_\sigma(x)\varphi(x)\,dx} - \underbrace{\Big[x\varphi(x)\Big]_{0}^{+\infty}}_{=0} + \underbrace{\int_{0}^{+\infty} \varphi(x) \, dx}_{=\int_{\mathbb{R}} h(x)\varphi(x)\,dx}$$

$$= \int_{\mathbb{R}} \underbrace{\big(h(x) - h_\sigma(x)\big)} \varphi(x) \, dx = \langle h - h_\sigma, \varphi \rangle$$

$$|x|' = h - h_\sigma \qquad \left(T'_{|x|} = T_{h - h_\sigma} \right)$$

$$|x|' = h - h_\sigma$$

$$\Rightarrow |x|'' = (|x|')' = h' - h_\sigma' = \delta - (-\delta) = 2\delta$$



2) $(fg)' = f'g + fg' \Rightarrow (f\varphi)' = f'\varphi + f\varphi'$

$\Rightarrow f\varphi' = (f\varphi)' - f'\varphi$

$f \in C^\infty, \quad T \in \mathcal{D}'$

$$\langle (fT)', \varphi \rangle = -\langle fT, \varphi' \rangle$$

$$= -\langle T, f\varphi' \rangle$$

$\langle T', \varphi \rangle = -\langle T, \varphi' \rangle$

$$= -\langle T, (f\varphi)' - f'\varphi \rangle$$

$$= -\langle T, (f\varphi)' \rangle + \langle T, f'\varphi \rangle$$

$$= \langle T', f\varphi \rangle + \langle f'T, \varphi \rangle$$

$$= \langle fT', \varphi \rangle + \langle f'T, \varphi \rangle = \langle fT' + f'T, \varphi \rangle$$

$$\boxed{(fT)' = fT' + f'T}$$

3) $\left(\underbrace{|x|}_{T}\,\underbrace{\sin(x)}_{f}\right)' = (\sin x)'\,|x| + (\sin x)\,|x|'$

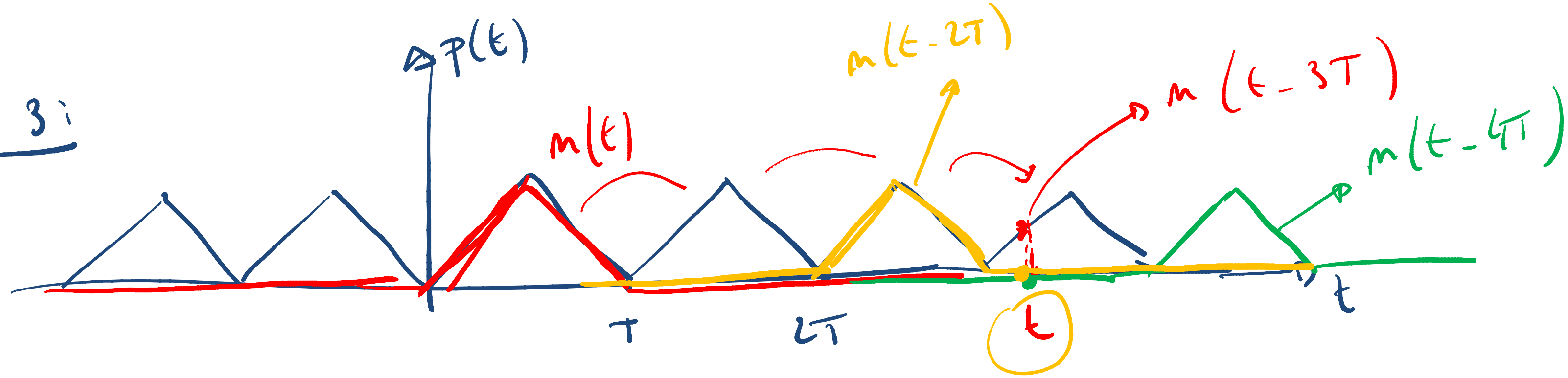
$= \cos x \cdot |x| + (\sin x)\left(h - h_\sigma\right)$

$\left(|x|\sin(x)\right)'' = (\cos x)'\,|x| + \cos x\,|x|' + (\sin x)'(h - h_\sigma) + (\sin x)(h - h_\sigma)'$

$= -\sin x\,|x| + \cos x\left(h - h_\sigma\right) + \cos x\left(h - h_\sigma\right) + \underbrace{\sin x \cdot 2\delta}_{= 2\sin 0\,\delta \,=\, 0}$

$= -\sin x\,|x| + 2\cos(x)\left(h - h_\sigma\right)$

Pb 3:



$$p(t) = \sum_{k=-\infty}^{+\infty} m(t-kT)$$

$$m(t-kT) = (m * \delta_{kT})(t)$$

$$p(t) = \sum_k m * \delta_{kT}(t) = \left(m * \sum_k \delta_{kT}\right)(t)$$

$$= (m * \Delta_T)(t)$$

Ш_T

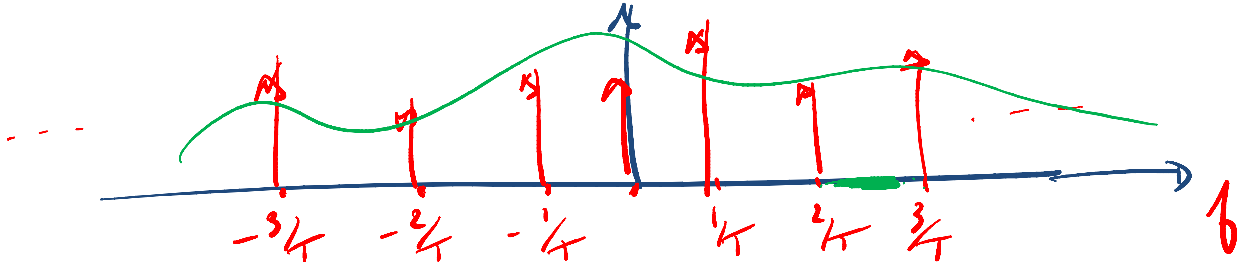
$$p = m \cdot \Delta_T$$

$$\hat{p} = \widehat{m \cdot \Delta_T} = \hat{m} * \hat{\Delta}_T = \frac{1}{T}\, \hat{m} * \Delta_{1/T}$$

$$\hat{\Delta}_T = \frac{1}{T}\Delta_{1/T} \qquad = \frac{1}{T}\, \hat{m} * \sum_k \delta_{\frac{k}{T}}$$

$$\hat{p} = \frac{1}{T} \sum_k \underbrace{\hat{m} * \delta_{k/T}}_{= \hat{m}\left(\frac{k}{T}\right)\delta_{k/T}}$$

$$\boxed{\hat{p} = \frac{1}{T} \sum_k \hat{m}\left(\frac{k}{T}\right) \delta_{k/T}}$$



1) $R_{xy}(\tau) = \frac{1}{T}\int_0^T x(t)\, y^*(t-\tau)\, dt = \frac{1}{T}\, \tilde{x} * \tilde{y}_\sigma$ $\longrightarrow$ défini partout borné

$y(u-T) = y(u)$

- $R_{xy}(T+\tau) = \frac{1}{T}\int_0^T x(t)\, \underbrace{y^*(t-(\tau+T))}_{y^*(t-\tau-T) = y^*(t-\tau)}\, dt = R_{xy}(\tau)$

$\Rightarrow R_{xy}$ périodique de période T

- $x \in L^2([0,T])$ et y $\Rightarrow$ $x * y$ continue et borné

$$x * y(\tau) = \int_{\mathbb{R}} x(t)\, y(\tau - t)\, dt$$

Soit $y_\sigma(t) = y(-t)$

$$\tilde{x}(t) = \begin{cases} x(t) & \text{sur } [0,T] \\ 0 & \text{sinon} \end{cases}, \quad \tilde{y}_\sigma(t) = \begin{cases} y_\sigma(t) & \text{sur } [0,T] \\ 0 & \text{sinon} \end{cases}$$

$\tilde{x}, \tilde{y}_\sigma \in L^2(\mathbb{R})$

2)

$$R_{xy}(\tau) = \frac{1}{T}\int_0^T x(t)\, y^*(t-\tau)\, dt$$

$$= \frac{1}{T}\int_0^{T} \sum_{k=-\infty} c_k^x e^{2j\pi\frac{k t}{T}} \sum_{\ell=-\infty} (c_\ell^y)^* e^{-2j\pi\frac{\ell}{T}(t-\tau)}\, dt$$

$$= \frac{1}{T}\int_0^T \sum_k \sum_\ell \underbrace{c_k^x (c_\ell^y)^* e^{2j\frac{\pi}{T}(kt - \ell(t-\tau))}}_{f(t,k,\ell)}\, dt$$

$$\frac{1}{T}\int_0^T \sum_k \sum_\ell |f(t,k,\ell)|\, dt$$

$$= \frac{1}{T}\int_0^T \sum_k \sum_\ell |c_k^x| |c_\ell^y| \cdot 1 \, dt$$

$$= \frac{1}{T}\sum_k \sum_\ell |c_k^x| |c_\ell^y| \underbrace{\int_0^T dt}_{T} = \underbrace{\left(\sum_k |c_k^x|\right)}_{<\infty} \underbrace{\left(\sum_\ell |c_\ell^y|\right)}_{<\infty}$$

D'après Fubini :

$$R_{xy}(\tau) = \frac{1}{T} \sum_k \sum_l c_k^x (c_l^y)^* e^{2j\pi \frac{l\tau}{T}} \underbrace{\int_0^T e^{2j\pi \frac{(k-l)t}{T}} dt}$$

$$\int_0^T e^{2j\pi \frac{(k-l)t}{T}} dt = \begin{cases} \int_0^T 1 \, dt = T & \text{si } k = l \\ \left[\dfrac{e^{2j\pi \frac{(k-l)t}{T}}}{2j\pi \frac{k-l}{T}} \right]_0^T = \dfrac{1}{2j\pi \frac{k-l}{T}} \left(e^{2j\pi(k-l)} - 1 \right) = 0 & k \neq l \end{cases}$$

$$\Rightarrow \boxed{R_{xy}(\tau) = \sum_k c_k^x (c_k^y)^* e^{2j\pi \frac{k\tau}{T}}}$$

$$R_x(\tau) = \sum_k |c_k^x|^2 e^{2j\pi \frac{k\tau}{T}}$$

DSP de x

$$S_{xy} = TF(R_{xy})$$

$$= TF\left(\sum_k c_k^x (c_k^y)^* e^{2j\pi\frac{k\tau}{T}}\right)$$

$$= \sum_k c_k^x (c_k^y)^* \underbrace{TF\left(e^{2j\pi\frac{k\tau}{T}}\right)}_{\delta_{k/T}}$$

$$S_{xy} = \sum_k c_k^x (c_k^y)^* \delta_{k/T}$$

$x=y$: $$S_x = \sum_k |c_k^x|^2 \delta_{k/T}$$ $\Rightarrow$ Spectre de raies

$$P_x = \sum |c_k^x|^2$$