**Stat 310A/Math 230A Theory of Probability**

# Homework 4

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Solutions should be complete and concisely written. Please, use a separate sheet (or set of sheets) per each problem. Staple sheets, and write your name on each sheet.

You are welcome to discuss problems with your colleagues, but should write and submit your own solution. In some cases, multiple homework options will be proposed (and indicated as 'Option 1, 'Option 2, etc.).

You are welcome to work on all the problems proposed (solutions will be posted), but should submit only those corresponding to one 'Option.

Exercises' numbers refer to the version of the notes posted on the class webpage.

## Exercises on independent random variables and product measures

Solve Exercises [1.3.65], [1.4.15], [1.4.18] in Amir Dembo's lecture notes.

## Exercises on $L_p$ spaces

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Throughout this exercise, we will say that two random variables $X, Y$ are equivalent if $\mathbb{P}(\{\omega\ :\ X(\omega) = Y(\omega)\}) = 1$.

1. Show that the above indeed defines an equivalence relation.

Let, for $p > 0$, $L_p(\Omega, \mathcal{F}, \mathbb{P})$ denote the space of (equivalence classes of) random variables $X$ such that $\mathbb{E}\{|X|^p\} < \infty$.

2. Show that, for $p \geq 1$, $\|X\|_p \equiv \mathbb{E}\{|X|^p\}^{1/p}$ is a norm on this space.

For $p = \infty$, $L_\infty(\Omega, \mathcal{F}, \mathbb{P})$ is the space of equivalence classes of random variables $X$ such that there exists $M < \infty$ such that $\mathbb{P}(\{\omega\ :\ |X(\omega)| \leq M\}) = 1$.

3. Show that $\|X\|_\infty \equiv \inf\{M\ :\ \mathbb{P}(\{\omega\ :\ |X(\omega)| \leq M\}) = 1$ is a norm on $L_\infty(\Omega, \mathcal{F}, \mathbb{P})$.
4. For $X \in L_\infty(\Omega, \mathcal{F}, \mathbb{P})$, show that $X \in L_p(\Omega, \mathcal{F}, \mathbb{P})$ for any $p > 0$, and that $\|X\|_p \to \|X\|_\infty$ as $p \to \infty$.
5. For $X$ a random variable, let $S(X) \equiv \mathbb{P}(\{\omega\ :\ X(\omega) \neq 0\})$. Show that, if $X \in L_q(\Omega, \mathcal{F}, \mathbb{P})$ for some $q > 0$, then $\lim_{p\to 0} \|X\|_p^p = S(X)$.
6. Show that the space of simple functions SF is dense in $L_p(\Omega, \mathcal{F}, \mathbb{P})$ for any $0 < p \leq \infty$.

   [Note that for $0 < p < 1$ the space $L_p(\Omega, \mathcal{F}, \mathbb{P})$ is not a normed space. The statement has to be interpreted in the following sense. For any random variable $X$ with $\mathbb{E}\{|X|^p\} < \infty$, there exist a sequence of simple functions $X_n$ such that $\mathbb{E}\{|X_n - X|^q\} \to 0$.]

## Optional: For the enthusiasts

This will not be graded, but is an interesting and well known fact: $L_p(\Omega, \mathcal{F}, \mathbb{P})$ is a Banach space for all $1 \leq p \leq \infty$. This means (beyond the fact of being a normed space) the following. If $\{X_n\}$ is a Cauchy sequence of (equivalence classes of) random variables in $L_p(\Omega, \mathcal{F}, \mathbb{P})$, then $\{X_n\}$ converges to a limit $X_\infty \in L_p(\Omega, \mathcal{F}, \mathbb{P})$. *Cauchy* means that, for any $\epsilon > 0$ there exists $N(\epsilon) < \infty$ such that, if $m, n \geq N(\epsilon)$, then $\|X_m X_n\|_p \leq \epsilon$.

At this point of the course, you know all that is needed to prove this (or to read a proof ;-).

[Hint: Every Cauchy sequence converges if the following happens: for any sequence $\{Z_n\}$ such that $\sum_{k=1}^{\infty} \|Zn\|_p < 1$ the sums $W_n = \sum_{k=1}^{n} Z_k$ converge.]