Math 4330 Homework Set 9

Due Monday, November 16, 2015

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NOTE: Late homework not accepted.

Read: "The Matrix of a Linear Transformation" "Dual Spaces" and the three handouts on Rings and Modules.

Problems marked by $\boxed{\text{box}}$ or $\boxed{\star}$ are more challenging and may be turned in anytime during the semester. There will be several such problems assigned during the term. Please turn in *separately* from routine assignments – if incorrect or incomplete, they will be returned to you to complete correctly. Final deadline is Monday, Nov. 30, no exceptions.

Do the following problems from the handouts:

DualSpace 7

DualSpace 8

DualSpace 16

DualSpace 25

DualSpace 33

DualSpace 34

DualSpace 36

Ex09 1. Let F be an arbitrary field.

a. Show that the intersection of an arbitrary number of ideals in $F[x]$ is an ideal in $F[x]$.

b. Let $f_1, \ldots, f_k \in F[x]$. The ideal generated by these is

$$(f_1, \ldots, f_k) = \{ g_1 f_1 + \cdots + g_k f_k \mid g_i \in F[x] \} ,$$

the set of all $F[x]$-linear combinations of $f_1, \ldots, f_k$. Show that this ideal is precisely the intersection of the ideals which contain all f_i, $1 \le i \le k$.

Ex09 2 (Exact Sequence of a Pair in a PID). Let R be a principal ideal domain (PID). Let $a, b \in R$, not both of which are 0. Define $f : R \times R \longrightarrow R$ by $f(s,t) = sa + tb$. Note that $R \times R$ is also a commutive ring with 1 when addition and multiplication are defined coordinate-wise:

(1) $(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$

(2) $(a_1, b_1) \cdot (a_2, b_2) = (a_1 a_2, b_1 b_2)$

Further note that $R \times R$ is an R-module with scalar multiplication defined by

(3) $r \cdot (a,b) = (ra, rb)$

a. Show that f satisfies

(i) $f(x+y) = f(x) + f(y)$ for all $x, y \in R \times R$.

(ii) $f(rx) = rf(x)$ for $r \in R$, $x \in R \times R$.

Hence f is an R-module homomorphism.

b. Show that $\operatorname{im} f \subseteq R$ is non-empty and is closed under addition and scalar multiplication; that is, $\operatorname{im} f$ is an R-submodule of R.

c. Compute $\operatorname{im} f$.

d. Show that $\ker f \subseteq R \times R$ is an R-submodule of $R \times R$.

e. Determine $\ker f$ explicitly: Show that there exists a function $g : R \longrightarrow R \times R$ of the form $g(r) = (r\alpha, r\beta)$ for some $\alpha, \beta \in R$ such that $\operatorname{im} g = \ker f$. Note that g satisfies the analogue of (i) and (ii) above (i.e., is an R-module homomorphism).

f. Show that there exists an exact sequence of R-modules

$$0 \longrightarrow X \xrightarrow{i} R \times R \xrightarrow{f} R \xrightarrow{p} Y \longrightarrow 0 .$$

What are X, i, Y, p?

g. Determine precisely all solutions (s,t), $s, t \in R$ of the equation $sa + tb = \gcd(a,b)$ where $\gcd(a,b)$ denotes the greatest common divisor of a and b.

Ex09 3. Let R be a PID and let $a, b \in R$ be two non-zero elements. Show that there exist elements $r, s, u, v \in R$ such that

a. $(a,b) = au + bv$,

b. $a = (a,b)r$, $b = (a,b)s$, $[a,b] = (a,b)rs$,

c. the matrices $A, B \in R^{2\times 2}$

$$A = \begin{bmatrix} u & v \\ -s & r \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 & -vs \\ 1 & ur \end{bmatrix}$$

are invertible and $\det A = \det B = 1$,

d. and further the following holds:

$$A \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} B = \begin{bmatrix} (a,b) & 0 \\ 0 & [a,b] \end{bmatrix}.$$