Math 4330 Take-Home Exam 2

Monday, November 16 – Monday, November 23, 2015

Your work on this exam is to be done in accordance with the following:

1. You may use the handouts, your own class notes, but nothing else; e.g., no books (not even the ones you've been referred to) nor the internet.
2. You may not obtain aid nor discuss the exam with any other person.
3. If you have any questions, please send e-mail, call, or come by my office:
   Keith Dennis Malott 524 255-4027 **math4330@rkd.math.cornell.edu**
   Please do not ask the TA any questions about the exam as he has been instructed to refer all questions to me.
4. Please return the exam to me or to the receptionist in the Math Office (third floor, Malott) by 4:00 pm, Monday, November 23. You may instead submit your solutions as a pdf file by e-mail to the address given above (same deadline: 4:00 pm).

PLEASE WRITE YOUR ANSWERS VERY CAREFULLY, EXPLAINING EXACTLY WHAT YOU ARE DOING, AND SHOWING ALL OF THE COMPUTATIONS.

ALL PARTS OF ALL PROBLEMS REQUIRE PROOFS.

Problem 1. [5 points]
Let F be a field. For $a \in F$ let $E_a : F[x] \longrightarrow F$ be the "evaluation at a" map: $E_a(f) = f(a)$. Let $\mathcal{E} = \{ E_a \mid a \in F \} \subseteq (F[x])^*$. Prove that the set $\mathcal{E}$ is always linearly independent. [Note that $\mathcal{E}$ may be infinite. How do you prove an infinite set is linearly independent?] Conclude that $\dim_F (F[x])^*$ is at least $|F|$.

Problem 2. [20 points]
Let

$$0 \longrightarrow U \xrightarrow{S} V \xrightarrow{T} W \longrightarrow 0$$

be a short exact sequence of vector spaces over the field F. Show that

$$0 \longrightarrow W^* \xrightarrow{T^t} V^* \xrightarrow{S^t} U^* \longrightarrow 0$$

is a short exact sequence as well. Make no assumptions on the dimensions of the three vector spaces.

Problem 3. [25 points] Let F be a field and let F^F denote the set of all functions from F to F. Recall that this is a ring under the usual definition of addition and multiplication of functions (that is, add or multiply their values). As was shown earlier, there is a function $E : F[x] \longrightarrow F^F$ given by sending the formal polynomial in $F[x]$ to the function which is computed by using the given polynomial as the formula for computation. This function E preserves both addition and multiplication (it is what is called a *ring homomorphism*).

a. Further it was noted that this function is not always one-to-one. Prove that E is one-to-one if and only if F is an infinite field. In this case give a function in F^F which is not given by a polynomial.

b. Prove that E is onto if and only if F is a finite field. Show that the kernel of E (the polynomials that go to 0) is an ideal of $F[x]$. Give an explicit monic generator of this ideal.

c. **[Extra Credit]**
If F has q elements, show that the generator you found in the preceding part is equal to $x^q - x$.

Problem 4. [20 points]
Let F be an arbitrary field. Recall that $e_{i,j}$ means the matrix with 1 in the i-th row and j-th column.

a. Let $N = e_{2,1} + e_{3,2} + \cdots + e_{n,n-1} \in F^{n \times n}$. Determine the minimal polynomial of N.

b. Let $c \in F$. Let $S = cI + N \in F^{n \times n}$. Determine the minimal polynomial of S.

c. Let $M = N + e_{1,n} = e_{2,1} + e_{3,2} + \cdots + e_{n,n-1} + e_{1,n} \in F^{n \times n}$. Determine the minimal polynomial of M.

d. Let $c_1, \ldots, c_n \in F$ and let $D = \operatorname{diag}(c_1, \ldots, c_n) \in F^{n \times n}$ be the diagonal matrix with c_i in position (i, i). Determine the minimal polynomial of D.

e. Compute the minimal polynomial for the matrix

$$\begin{bmatrix} 0 & 0 & a \\ 1 & 0 & b \\ 0 & 1 & c \end{bmatrix}.$$

f. Let $A \in F^{n\times n}$ and let $B \in F^{m\times m}$. Assume f is the minimal polynomial of A and that g is the minimal polynomial of B. Let

$$D = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

be the block diagonal matrix in $F^{(n+m)\times(n+m)}$. Determine the minimal polynomial of D in terms of f and g.

Problem 5. [30 points]
Let V be a vector space over the field F. Let $T \in \mathrm{End}_F(V) = \mathrm{Hom}_F(V,V)$ be a linear transformation with minimal polynomial $\pi(x) = (x-c_1)(x-c_2)\cdots(x-c_k)$ where $c_1, c_2, \ldots, c_k \in F$ are distinct. Let $\pi_i(x) = \pi(x)/(x-c_i)$ and $p_i(x) = \pi_i(x)/\pi_i(c_i)$. Assume $k \geq 2$. [Note that $\pi(x)$ is the minimal polynomial of T means that $\pi(T) = 0$ but $\pi_j(T) \neq 0$ for all j; 0 means the linear transformation in $\mathrm{End}_F(V)$.]

a. Apply Lagrange Interpolation (i.e., the fact that $\{p_1, \ldots, p_k\}$ is the basis of $\mathcal{P}_k$ (all polynomials of degree less than k union $\{0\}$) which is dual to the set of evaluations $\{E_{c_1}, \ldots, E_{c_k}\} \subset \mathcal{P}_k{}^*$)
to write 1 as a linear combination of the p_i,
to write x as a linear combination of the p_i.

b. Let $T_i = p_i(T) \in \mathrm{End}_F(V)$. Prove that

$$I = T_1 + \cdots + T_k$$

holds in $\mathrm{End}_F(V)$. Here I denotes the identity linear transformation on V. Prove that

$$T_iT_j = 0$$

for $i \neq j$, and

$$T_i^2 = T_i$$

that is, T_i is idempotent.

c. In an analogous fashion write T as a sum of k linear transformations.

d. Prove that $\mathrm{im}\, T_i = \ker(T - c_iI)$.

e. Prove that part b. implies that

$$V = \mathrm{im}\, T_1 \oplus \cdots \oplus \mathrm{im}\, T_k .$$

f. Let V be finite dimensional. Let $d_i = \dim \operatorname{im} T_i$. Note that $d_i > 0$. Choose bases for $\operatorname{im} T_i$ and let $\mathcal{B}$ be the basis of V which is their union. Compute the matrix $[T]_{\mathcal{B}}$.

g. State the theorem that's has now been proved: "If $T \in \operatorname{End}_F(F)$ has minimal polynomial which has ———— and $\dim_F V$ ————, then there exists a basis for V for which the matrix of T is ————."

h. **[Extra Credit]**
Let $A, B \in F^{n\times n}$ each have minimal polynomial $\pi(x)$. Let $t = (d_1, \ldots, d_k)$ be the sequence of d_i as defined in the previous part for $T = L_A$. Let s be the corresponding sequence of integers for L_B. Prove that A and B are similar if and only if $t = s$.

i. **[Extra Credit]**
Let T be as in the second previous part. Prove that the characteristic polynomial of T is $(x - c_1)^{d_1} \cdots (x - c_k)^{d_k}$.

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5) $\sum c_i T_i$

$$D = \begin{pmatrix} \overbrace{c_1 \cdots c_1}^{d_1} & & & & \\ & \overbrace{c_2 \cdots c_2}^{d_2} & & & \\ & & \ddots & & \\ & & & \overbrace{c_k \cdots c_k}^{d_k} \end{pmatrix}$$

g. If $T \in \mathrm{End}_F(V)$ has minimal polynomial with simple roots and $\dim_F(V) < \infty$, $\exists$ basis for V which matrix of T is diagonalisable.

h). $P_1 A P_1^{-1} = D = P_2 B P_2^{-1}$

$P_2^{-1} P_1 A P^{-1} P_2^{-1} = B$ which can only be true if the sequence of integers d_i for D are the same.

i) Given $D = \begin{pmatrix} \overbrace{c_1 \cdots c_1}^{d_1} & & \\ & \overbrace{c_2 \cdots c_2}^{d_2} & \\ & & \overbrace{c_k \cdots c_k}^{d_k} \end{pmatrix}$

clearly char poly is standard formula $(x-c_1)^{d_1} \cdots (x-c_k)^{d_k}$

5. $T \in End_F(V) = Hom_F(V,V)$ is a linear transformation w/ minimal polynomial
$\pi(x) = (x-c_1)(x-c_2)\cdots(x-c_k)$, $c_1, \dots, c_k \in F$ are distinct.
Let $\pi_i(x) = \dfrac{\pi(x)}{x-c_i}$, $p_i(x) = \dfrac{\pi_i(x)}{\pi_i(c_i)}$, assume $k \geq 2$.

a) Apply Lagrange Interpolation to write 1 as linear combination of the p_i and x as a linear combination of the p_i.

$$\pi_1 = (x-c_2)\cdots(x-c_k)$$

$$\pi_2 = (x-c_1)(x-c_3)\cdots(x-c_k)$$

$$p_1(x) = \frac{(x-c_2)\cdots(x-c_k)}{(c_1-c_2)\cdots(c_1-c_k)}$$

So $p_1 + p_2 + \cdots p_k = \sum_{i=1}^{k} p_i = 1$

and $x = cp_1 + cp_2 + \cdots cp_k = \sum_{i=1}^{k} c_i p_i$

b) $I = (p_1 + \cdots + p_k)[T]$

$\sum p_i(T) = 1(T) = I$ $\square$

$i \neq j$, $T_i T_j = 0$ because $T_i = p_i(T)$ and $T_j = p_j(T)$, They are being divided by $\pi(x) = (x-c_1)\cdots(x-c_k)$ so $T_i \cdot T_j = 0$ if $i \neq j$.

T_i, $T_i^2 = T_i$ because $p_i(T) = 1(T) = I$ and $I^2 = I$.

c) $I = T_1 + \cdots + T_k$

$T = c_1 T_1 + \cdots + c_k T_k$

$T_j \mid Im\, T_j = id$

$\rightarrow (T_j T_j = T_j)$

e) Already know $V = Im(I) = Im\, T_1 + \cdots Im\, T_k$ $(T_j T_i = 0)$

WTS $v_i \in Im\, T_i$, $\sum v_i = 0 \rightarrow v_i = 0 \ \forall i$ $T_j(\sum_i v_i) = 0 \rightarrow T_j v_j = 0$

$\rightarrow v_j = 0$

1. F is a field. For $a \in F$, let $E_a : F[x] \to F$ be "eval at a" map: $E_a(f) = f(a)$.
$\mathcal{E} = \{E_a \mid a \in F\} \subseteq (F[x])^*$. Prove $\mathcal{E}$ is always linearly independent

An infinite set is linearly independent $\leftrightarrow$ all of its finite subsets are linearly independent. This is clearly true for $\mathcal{E}$.

2. 1) $0 \to U \xrightarrow{S} V \xrightarrow{T} W \to 0$ short exact sequence of vector spaces over F

Show 2) $0 \to W^* \xrightarrow{T^t} V^* \xrightarrow{S^t} U^* \to 0$ is a short exact sequence.

Given that 1) is a short exact sequence, we know S is injective, T is surjective. Suppose $w, w' \in W^*$. Then let $w(T) = w'(T)$. Because T is surjective, $\exists v \in V$ s.t. $T(v) = x$, $x \in W$. Now $w(T(v)) = w'(T(v))$ and $w(x) = w'(x) \to w = w' \in W^*$.

This shows that T^t is injective.

To show that S^t is surjective, let $h \in U^*$. Want $f \in V^*$ s.t. $f(S) = h$. We know that f defined on the range of S can be extended to a linear functional on V from a previous homework problem. Let $f = h(S^{-1})$ on range of S.

Then $f(s) = h(S^{-1}(s)) = h$, $\to$ S^T is surjective $\square$

$$\begin{pmatrix} 0 & 0 & & \\ 1 & 0 & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ & & e_{n,n-1} & 0 \end{pmatrix} \in F^{n\times n}$$

$$\begin{vmatrix} -\lambda & 0 & a \\ 1 & -\lambda & b \\ 0 & 1 & c-\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -\lambda & b \\ 1 & c-\lambda \end{vmatrix}$$

$$-\lambda\left(\lambda^2 - c\lambda - b\right) - 0 + a \begin{vmatrix} 1 & -\lambda \\ 0 & 1 \end{vmatrix}$$

$$a$$

3 Let F be a field, F^F the set of all functions from F to F. $E: F[x] \to F^F$ is a function given by sending $f \in F[x]$ to the function computed using the given polynomial as formula for computation.

a) Prove E is one-to-one $\leftrightarrow$ F is an infinite field.

"$\rightarrow$" If E is one-to-one, then F is an infinite field. Suppose to the contrary that F is finite, and $|F| = n < \infty$. Then $(x^n - 1)x = 0$ $\forall x \in F_n$ but we know that the zero polynomial is an identical function but different from $(x^n - 1)x$. So E is not one-to-one, a contradiction $\square$

"$\leftarrow$" If F is an infinite field, then E is one-to-one.

Suppose E is not one-to-one. Then for $p(x)$ over F, $\exists$ $q(x)$ s.t. $p(x) = q(x)$ $\forall x$,

Then $(q-p)(x) = 0$ $\forall x \rightarrow q - p$ has all elements of F as root, which is impossible because degree of any polynomial is finite $\square$

b) E is onto $\leftrightarrow$ F is finite field. $\ker(E)$ is an ideal of $F[x]$. Suppose $x \in \ker(E)$ then $E(x) = 0$, $E(s) = 0$ $E(x \circ s) = E(x) \circ E(s) \circ E(s \circ x) = 0$ $\rightarrow \ker E$ is an ideal

"$\leftarrow$" If f is finite, then we can show $\forall f \in F^F$ are mapped to by $E: F[x] \to F^F$.

We use Lagrange Interpolation to define polynomials $p(x) = \dfrac{(x - a_1)(x - a_2)}{(a_n - a_1)(a_n - a_2)} z_n$

$+ \dfrac{(x - a_1)(x - a_n)}{(a_2 - a_1)(a_2 - a_n)} z_2 + \dfrac{(x - a_2)(x - a_3)}{(a_1 - a_2)(a_1 - a_n)} z_n$, where $\left.\begin{matrix} a_1 \mapsto z_1 \\ a_2 \mapsto z_2 \\ \vdots \\ a_n \mapsto z_n \end{matrix}\right\} \in F^F$

This $p(x)$ does the following, $a_1 \mapsto 0$, $a_2 \mapsto 0$, $a_n \mapsto z_n$ in the first term and so on for each z_i.

"$\leftarrow$" If E is onto, then F must be finite. Because $|F[x]| = |F|$ and $|F^F| \geq |2^F|$ and $|2^F| > |F| \rightarrow |F[x]| < |F^F|$. If F is infinite, clearly E cannot be onto.

c) If F has q elements, then $x^q - x$ is the generator of $\ker(E)$, an ideal.

$f(x) = (x^q - x)g(x) + r(x)$, let $\deg r(x) < q$. If $f(a) = a$ $\forall a \in F$, $r(a) = 0$ $\forall a \in F$. Polynomial can only have finitely distinct roots

4. Let F be an arbitrary field

a) Let $N = e_{2,1} + e_{3,2} + \dots + e_{n,n-1} \in F^{n\times n}$. Determine minimal polynomial of N.

The minimal polynomial is x^n, since $N^{n-1} \neq 0$, but $N^n = 0$.

b) $c \in F$, $S = cI + N \in F^{n\times n}$, min. polynomial of S.

The minimal polynomial is $(x-c)^n$

$S^{n-1} \neq 0$, $S^n = 0$, $\rightarrow (x-c)^n$ must be the minimal polynomial.

c) $M = N + e_{1,n} = e_{2,1} + e_{3,2} + \dots + e_{n,n-1} + e_{1,n} \in F^{n\times n}$

min. polynomial of $M = x^n - 1$

The eigenvalues of M are $\pm 1, \pm i$ and have multiplicity of 1.

So $f \mid \pi_D$, $g \mid \pi_D$

and $p \mid \pi_D$

so $\pi_D = p = f$ or g □

d) $D = \mathrm{diag}(c_1, \dots, c_n)$, $c_1 \dots c_n \in F$

The minimal polynomial is $(x-c)$. Char poly is $(x-c)^n$ and

$(x-c)$ is the minimal divisor of $(x-c)^n$.

e) $\begin{bmatrix} 0 & 0 & a \\ 1 & 0 & b \\ 0 & 1 & c \end{bmatrix}$

$\det(A - \lambda I)$

$\begin{vmatrix} -\lambda & 0 & a \\ 1 & -\lambda & b \\ 0 & 1 & c-\lambda \end{vmatrix}$

$= -\lambda^3 + c\lambda^2 + b\lambda + a$

$\exists$, $D = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$

min poly $= \pi_D = \begin{pmatrix} p(A) & 0 \\ 0 & p(B) \end{pmatrix}$

p annihilates $D \leftrightarrow p$ annihilates A and B

$p = f \vee g$ annihilates D, so $\pi_D \mid p$. π_D annihilates $D \rightarrow \pi_D$ annihilates A and B.

Kang-Li Cheng

Math 4330 Homework Set ~~5~~ 6

Due Friday, October 9, 2015

Keith Dennis Malott 524 255-4027 **math4330@rkd.math.cornell.edu**

TA: Gautam Gopal Krishnan 120 Malott Hall gk379@cornell.edu

NOTE: Late homework not accepted.

Read: "Quotient Spaces", "Exact Sequences", "Bases and Coordinates" and "Universal Mapping Properties".

NOTE: Exam 1 will be Fri. Oct. 16 – Fri. Oct. 23

Problems marked by $\boxed{\text{box}}$ or $\boxed{\star}$ are more challenging and may be turned in anytime during the semester. There will be several such problems assigned during the term. Please turn in *separately* from routine assignments – if incorrect or incomplete, they will be returned to you to complete correctly. Final deadline is Monday, Nov. 30, no exceptions.

Do the following problems from the handouts:

ExSeq 1

ExSeq 3

ExSeq 5

ExSeq 6

ExSeq 14

33.5/50

due 10/14/15 Homework 5 Kang-Li Cheng

1. a) A linear transformation is injective $\leftrightarrow$ the sequence $0 \to V \xrightarrow{f} W$ is exact.

"$\leftarrow$" If the sequence is exact, then $\ker(f) = \mathrm{im}(0 \to V) = 0$, so f is injective.

"$\rightarrow$" $\mathrm{im}(0 \to V) = 0 \to \ker(f) = 0$ since f is injective. So the sequence is exact.

b) A linear transformation is surjective $\leftrightarrow$ $V \xrightarrow{f} W \to 0$ is exact.

"$\leftarrow$" If the sequence is exact, then $\ker(W \to 0) = \mathrm{im}(f)$. So f must be surjective.

"$\rightarrow$" $\ker(W \to 0) = W$ and if f is surjective, then $\mathrm{im}(f) = W$ so the sequence is exact.

c) The vector space V is 0 $\leftrightarrow$ $0 \to V \to 0$ is exact.

"$\rightarrow$" If V is 0, then $\ker(V \to 0) = 0$ and $\mathrm{im}(0 \to V)$ is 0, so the sequence is exact.

10 "$\leftarrow$" If the sequence is exact, then $\mathrm{im}(0 \to V) = \ker(V \to 0)$, so V must be 0.

d) f is an isomorphism $\leftrightarrow$ $0 \to V \xrightarrow{f} W \to 0$ is exact.

"$\rightarrow$" $\ker(W \to 0) = W$ and $\mathrm{im}(f) = W$ since f is surjective, f is also injective so $\mathrm{im}(0 \to V) = \ker(f) = 0$ and we conclude the sequence is exact.

"$\leftarrow$" The sequence is exact, so $\ker(W \to 0) = \mathrm{im}(f) \to f$ is surjective.
Also $\ker(f) = \mathrm{im}(0 \to V) \to f$ is injective.
So f is an isomorphism.

3. Verify that $0 \to V_1 \xrightarrow{\iota} V_1 \oplus V_2 \xrightarrow{\rho} V_2 \to 0$ is an exact sequence.

ι is an inclusion and ρ is a projection.

Clearly $\ell : V_1 \to V_1 \oplus V_2$ is injective and $\rho : V_1 \oplus V_2 \to V_2$ is surjective.

WTS: $\ker(p : V_1 \oplus V_2 \to V_2) = \mathrm{im}(\ell : V_1 \to V_1 \oplus V_2)$

9 Suppose $(v_1, v_2) \in \ker(p)$ so $p(\ell(v_1)) = v_2 = 0$. Then $(v_1, v_2) = (v_1, 0) = \ell(v_1)$ $\in \mathrm{im}(\ell)$.

If $\ell(v_1) \in \mathrm{im}(\ell)$, then $p(\ell(v_1)) = p(v_1, 0) = 0 \to \ell(v_1) \in \ker(p)$ $\square$

5. Show that giving an exact sequence $\cdots V_{i-1} \xrightarrow{f_{i-1}} V_i \xrightarrow{f_i} V_{i+1} \rightarrow \cdots$ is the same as giving a collection of short exact sequences $0 \rightarrow K_i \rightarrow V_i \rightarrow K_{i+1} \rightarrow 0$

$\ker f_i = \operatorname{Im} f_{i-1} \ \forall_i \iff V_{i-1} \rightarrow V_i \rightarrow V_{i+1} \rightarrow$

$\Updownarrow$

$$0 \rightarrow \operatorname{Im} f_{i-1} \xrightarrow{\text{inclusion}} V \xrightarrow{f_i} \operatorname{Im} f_i \rightarrow 0 \quad \underset{\sim}{\forall i}$$

$K_i = \operatorname{Im} f_{i-1}$ $\qquad$ $\operatorname{Im} f_i = K_{i+1}$

3

Proof ?

6. Let $0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$ be a short exact sequence.

Prove a, b, c are equivalent

a) The sequence splits on the right, $\exists\ s: W \to V$ s.t. $g \circ s = 1_W$.

b) The sequence splits on the left, $\exists\ t: V \to U$ s.t $t \circ f = 1_U$.

c) $\exists$ isomorphism $\gamma: V \to U \oplus W$ s.t. $\gamma \circ f = i_1$ and $p_2 \circ \gamma = g$ for i_1 and p_2 denoting inclusion into the first summand and projection onto the second summand, respectively.

a) $\to$ c) Let $x \in V$, then $g(x - sg(x)) = g(x) - g(x) = 0$. This implies $\exists\ y \in U$ s.t. $x - sg(x) = f(y)$. So $V = f(U) + s(W)$. Now WTS $V = f(U) \oplus s(W)$. Let $k \in f(U) \cap s(W)$, then $\exists\ u \in U$ and $w \in W$ s.t. $k = f(u) = s(w)$ and $g(k) = g(f(u)) = gs(w) = w \to w = 0$ and $k = s(w) = 0$. $\square$

4

b) $\to$ c) ?

14. Diagram of vector spaces and linear transformations

a)

$$\begin{tikzcd}
A \arrow[rr, "t"] \arrow[dr, "i"] \arrow[ddr, bend right, "\ell"'] & & B \arrow[ddl, bend left, "r"] \\
& D \arrow[ur, "o"] \arrow[d, "v"] & \\
& C &
\end{tikzcd}$$

7.5

Assume $t = o \circ i$

$v = r \circ o$

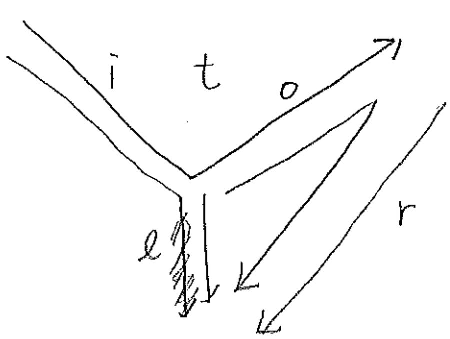
$\ell = v \circ i$

Show these three imply $\ell = r \circ t$

$\ell = r \circ \underbrace{(o \circ i)}_{= t} = v \circ i$ which is true.

So $\ell = r \circ t$ follows $\square$

b)



Case 1

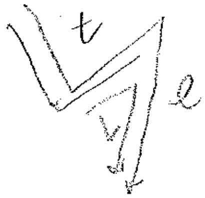
Assume

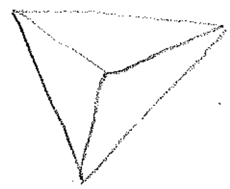
$\boxed{\begin{array}{l} t = o \circ i \\ \ell = v \circ i \\ \ell = r \circ t \end{array}}$

$v \circ i = r \circ o \circ i \longrightarrow v = r \circ o$?

Implication does not hold, consider i is the zero map, then nothing can be said about what v, r, or o are.

Case 2)





The case where we assume

$$\left.\begin{aligned} t &= o \circ i \\ v &= r \circ o \\ \ell &= r \circ t \end{aligned}\right\} \rightarrow \ell = v \circ i$$

$$\ell = r \circ (o \circ i)$$

$$\ell = v \circ i$$

implication holds

Case 3

Assume

$$\left.\begin{aligned} v &= r \circ o \\ \ell &= v \circ i \\ \ell &= r \circ t \end{aligned}\right\} \rightarrow t = o \circ i$$

$$\ell = r \circ (o \circ i)$$

$$r \circ t = r \circ (o \circ i)$$

implication holds

We also showed in a) that commutativity of the 3 small triangles implies commutativity of the large triangle, so 3 out of 4 cases hold.

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ksc66

Math 4330 Homework Set 3

Due Monday, September 21, 2015

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Read: Notes on "Fields", "Some Useful Definitions", "Subobjects", "Direct Sums and Products", and "Equivalence Relations".

Problems marked by $\boxed{\text{box}}$ or $\boxed{\star}$ are more challenging and may be turned in anytime during the semester. There will be several such problems assigned during the term. Please turn in *separately* from routine assignments – if incorrect or incomplete, they will be returned to you to complete correctly. Final deadline is Monday, Nov. 30, no exceptions.

Do the following problems from the handouts:

SumProd 3

SumProd 4 ✓

SumProd 5 ✓

SumProd 6

SumProd 16

and also

Hw03 1. a. Let W_1 and W_2 be subspaces of a vector space V such that their set-theoretic union is also a subspace. Prove that one of the spaces W_i is contained in the other.

$\boxed{\text{b.}}$ Prove this generalization of the first part. Assume that F is an infinite field. Let $V_1, \ldots, V_n$ be subspaces of a vector space V over F. Prove that $V_1 \cup \cdots \cup V_n$ is a subspace if and only if some V_i contains all the others. What happens if F is a finite field? [Give an exact statement relating n and the size of F in case that is necessary.]

43/70

Hw03 2. Let A be an abelian group (that is, addition $+$ is defined and it satisfies exactly the same 4 properties that are satisfied by the addition of fields or vector spaces). For $a \in A$ and $n > 0$ an integer, we define $n \cdot a = a + \cdots + a$ (n terms in sum) exactly as we did for fields. The *exponent* of A is the smallest positive integer n, if it exists, such that $n \cdot a = 0$ for all $a \in A$. Otherwise we say the exponent of A is infinite. We write $\exp(A) = n$ or $\exp(A) = \infty$. Note the similarity to (and difference from!) the definition of characteristic for a field. This exercise determines which abelian groups can be made into a vector space over the prime fields $\mathbb{F}_p$ or $\mathbb{Q}$.

a. Let F be a field and let V be a non-zero vector space over F. Are $\exp(V)$ and char(F) related? If so, how? Prove your statement.

b. Let $\mathbb{F}_p$ be the finite field of integers modulo p for p a prime. Let A be an abelian group. Determine precisely when A can be made into a vector space over $\mathbb{F}_p$, define the scalar multiplication in that case, and verify that indeed A is a vector space over $\mathbb{F}_p$ with the definition you have given.

$\boxed{\text{c.}}$ Let A be an abelian group. Give necessary and sufficient conditions on A such that it can be made into a vector space over $\mathbb{Q}$. [Hint: For n a non-zero integer and $a \in A$, what can one say about $\frac{1}{n} \cdot a$?]

Homework 3

Kang-Li Cheng

! ~~SumProd 3~~ HW3 1(a)

Given $W_1 \cup W_2$ is a subspace of V

Proof: Suppose $W_1 \not\subset W_2$ and $W_2 \not\subset W_1$. Then $\exists$ $\vec{w_1} \in W_1$, $\vec{w_2} \in W_2$ but $\vec{w_1} \notin W_2$ and $\vec{w_2} \notin W_1$. So $\vec{w_1} + \vec{w_2} \notin W_1$ and $\vec{w_1} + \vec{w_2} \notin W_2$, so $W_1 \cup W_2$ is not closed under addition and $W_1 \cup W_2$ is not a subspace.

10 Want to prove $\vec{w_1} + \vec{w_2} \notin W_1$. Suppose the contrary, then $-x \in W_1$ and $(x+y)+(-x) \in W_1$ $\rightarrow$ $y \in W_1$, a contradiction. By the same argument, $\vec{w_1} + \vec{w_2} \notin W_2$. □

SumProd 4

Given $T: V \to V$ is a linear transformation from V to itself, $T^2 = T$.

Proof: Show $V = im(T) \oplus ker(T)$

First we show that $V = im(T) + ker(T)$. Suppose $v \in V$, v is an arbitrary element. Since T is a linear transformation and $T^2 = I$, $v - T(v) = T(v) - T(T(v)) = T(v) - T(v) = 0$.

9 So $v - T(v) \in ker(T)$. Now let $u = (v - T(v)) + T(v)$, u is the sum of an element in $ker(T)$ and another element in $im(T)$.

Now we want to show $ker(T) \cap im(T) = \{0\}$.

??

5. Let V be vectorspace of all functions from $\mathbb{R}$ to $\mathbb{R}$. V_e is subset of even functions, V_o is subset of odd functions.

a) V_e is a subspace of V

not empty, contains zero function.

Suppose $f \in V_e$, $g \in V_e$, $c \in \mathbb{R}$. Then $(f+cg)(-x) = f(-x) + c \cdot g(-x)$

8

Similarly V_o is a subspace of V. $= f(x) + c \cdot g(x) = (f+cg)(x)$

b) Prove $V_e + V_o = V$

Suppose $f \in V$, f is an arbitrary function. Then let $f_e(x) = \frac{1}{2}(f(x) + f(-x))$ and $f_o(x) = \frac{1}{2}(f(x) - f(-x))$. Now f_e is an even function, f_o is an odd function, so $f = f_e + f_o$. Verify $f_e(x)$ is indeed even. $f_e(-x) = \frac{1}{2}(f(-x) + f(-(-x)))$ $= \frac{1}{2}(f(x) + f(-x))$.

Similarly $f_o(x)$ must be odd.

c) Suppose $f \in V_e \cap V_o$ and $x \in \mathbb{R}$. WTS f is zero function.

$f \in V_e \cap V_o \rightarrow f \in V_e$ so $f(x) = f(-x)$. But f is also in V_o, so $f(x) = -f(-x)$. Now $f(x) = -f(x)$, so $2f(x) = 0$. f must be zero function.

d) Conclusion: $V = V_e \oplus V_o$ by definition. We have shown $V_e \cap V_o = 0$ function and $V_e + V_o = V$

e) The functions from $F \rightarrow F$ must be well defined for the

x

conditions $\underbrace{V_e + V_o = V \text{ and } V_e \cap V_o = \{0\}}_{\text{What happens in char 2?}}$ to hold.

6. F is a field, $\text{char}(F) \neq 2$. V is vector space over F. T is a linear $V \to V$ transformation s.t. $T^2 = I$. $V^+ = \{v \in V \mid T(v) = +v\}$, $V^- = \{v \in V \mid T(v) = -v\}$
Show $V = V^+ \oplus V^-$

Suppos $v = \frac{T(v)+v}{2} + \frac{v-(T(v))}{2}$. Then $T\left(\frac{v+(Tv)}{2}\right) = \frac{v+(T(v))}{2}$

and $T\left(\frac{v-(T(v))}{2}\right) = \left(\frac{T(v)-T(T(v))}{2}\right) = \frac{T(v)-v}{2} = -\frac{v-T(v)}{2}$

10 If $v \in V^+ \cap V^-$, then $T(v) = v = -v \to 2v = 0$, which contradicts the condition that $\text{char}(F) \neq 2$. So $V^- \cap V^+ = \{0\}$.
Therefore $V = V^+ \oplus V^-$.

Scm Prod 16

F^s ?

0

due 11/30/15 Math 4330 boxed * problems

Kang-Li Cheng

Hw 03 1 $\boxed{b}$ Let F be an infinite field. Let $V_1, \dots, V_n$ be subspaces of a vector space V over F. Prove $V_1 \cup \dots \cup V_n$ is a subspace $\Longleftrightarrow$ some V_i contains all the others. Consider if F is a finite field.

Suppose $V = V_1 \cup \dots \cup V_n$, where n is the smallest possible. Let $x \in V_1$, x not in any other V_j, $j > 1$. Find $y \notin V_1$. We now construct a set $A = \{y + kx : k \in F\}$. Note $A \cap V_j$ is at most one point $\forall j > 1$, otherwise we will have $k \neq k'$ and $(k-k')x \in V_j$, a contradiction to $x \notin V_j$. So A intersects $\bigcup_{j=1}^{n} V_j$ in at most $n-1$ points. This is another contradiction, since A is infinite. So if no V_j contains all other V_i, $i \neq j$, V is not a subspace.

8

$|F| < n^2$

If F is a finite field, the above proposition is true if $|F| \geq n$.

Clearly if $\exists V_j$ s.t. $V_1, \dots, V_n$ contained in V_j, then clearly $V_j = V_1 \cup \dots \cup V_n$ is a subspace.

resp.

Hw 03 2 $\boxed{c}$ Let A be an abelian group w/ addition. When can A be made a vector space over $\mathbb{Q}$? Consider $n \in \mathbb{Z}$, $n \neq 0$, $a \in A$, what is $\frac{1}{n} \cdot a$?

8 For A to be a vector space, 1) need $\forall a \in A$, $n \in \mathbb{N}$, $\exists b \in A$ s.t. $nb = a$.

2) And if $\exists n \in \mathbb{N}$, $a \in A$ s.t. $na = 0$, then $a = 0$.

"$\rightarrow$" If A is a vector space, $n \cdot b = 0 \rightarrow \frac{1}{n}(n \cdot b) = 0 \rightarrow b = 0$

Let $a = \frac{1}{n}(b)$, $\rightarrow b = n \cdot (\frac{1}{n} \cdot b)$

"$\leftarrow$" For any $\frac{p}{q} \in \mathbb{Q}$, $b \in A$, $\frac{p}{q} b = a$ as $q \cdot a = p \cdot b$.

if $q a = pb$, $qa' = pb \rightarrow q(a_1 - a_2) = 0$, $q \neq 0 \rightarrow a = a'$ and therefore a is unique.

Well-definedness?

A is closed under addition since it is a group. $\forall \frac{p}{q}, \frac{a}{b} \in \mathbb{Q}$, $c, d \in A$

$\frac{p}{q}(c+d) = \frac{p}{q} c + \frac{p}{q} d$. $\left(\frac{p}{q} + \frac{a}{b}\right) c = \frac{p}{q} c + \frac{a}{b} c$

$\frac{p}{q}\left(\frac{a}{b} \cdot c\right) = \frac{pa}{qb} c = \left(\frac{p}{q} \cdot \frac{a}{b}\right) c$. Finally $\frac{a}{a} \cdot c = 1 \cdot c = c$.

↑ assoc.

So A is a vector space.

4. $\boxed{c.}$ $F^{m\times m}$ for F a field, $m>1$

$U(R)$ for $R=F^{m\times m}$ is the set of invertible $m\times m$ matrices.

8 left associate $[A]=\{$ matrices found by applying elementary row operations on $A\}$

row equivalent class$(R)=\{E_1\dots E_nR : n\geq 1\}$

Suppose $B=UA$, U is invertible

$E_1E_2\dots E_nU=I$

$U=E_n^{-1}\dots E_2^{-1}E_1^{-1}$

$\rightarrow B=E_n^{-1}\dots E_2^{-1}E_1^{-1}A$

$\rightarrow B=A$ w/ series of row operations

zero matrix is an equivalence class.

System of unique representatives

$\{0\}\cup$ matrices in $F^{m\times m}$ in rref.

"changing rows by cols"

right associate $[A]=\{AE_n\dots E_2E_1 \mid E_i$ are elementary$\}$

System of representatives, column reduced ef $\in F^{m\times m}$.

Associate $[A]=\{E_1E_2\dots E_sAE_{s+1}E_{s+2}\dots E_{s+n} \mid E_i$ are elementary$\}$

Representatives $\{0\}\cup\{$ matrices in cref and rref$\}$

Conjugate $[A]=\{E_1E_2\dots E_nAE_n^{-1}E_{n-1}^{-1}\dots E_1^{-1} \mid E_i$ are elementary$\}$

Square matrix $A\sim$ its Jordan normal form, $\times$

$\{0\}\cup\{$Jordan normal form in $F^{m\times m}\}$ $J_i=\begin{bmatrix}\lambda_i & & 0\\ & \ddots & \\ 0 & & \lambda_i\end{bmatrix}$

$J=\begin{bmatrix}J_1 & & \\ & \ddots & \\ & & J_m\end{bmatrix}$

nilp variety
dual
polis algebra
Kittens
House of Leaves
Borges
Turnstile
[illegible]

~~HW03~~ 1a.

Sumprod 3

Let V_1, V_2 be vsp over $\mathbb{F}$, $V_1 \oplus V_2$. $i_j: V_j \to V_1 \oplus V_2$

$p_j: V_1 \oplus V_2 \to V_j$.

$(i_1 \circ p_1)(v_1, v_2) + (i_2 \circ p_2)(v_1, v_2) = i_1(v_1) + i_2(v_2) = (v_1, 0) + (0, v_2) = (v_1, v_2)$.

So $i_1 \circ p_1 + i_2 \circ p_2$ is id map on $V_1 \oplus V_2$. $i_1 \circ p_1 + \cdots + i_k \circ p_k$ for finite $k \in \mathbb{Z}_+$.

Since only finitely many $v_j \in \bigoplus_{j=1}^{\infty} V_j \neq 0$, the sum has finitely many nonzero

10 summands, so well defined. So $\sum_{j=1}^{\infty} i_j \circ p_j$ is id map on infinite direct sum.

No analogy for infinite direct product. May have infinitely many nonzero summands in.

Hw03 2b) Let F_p be a finite field of integers mod p, where p is a prime.

Let A be an abelian group.

So $\forall a \in A$, $pa = 0$ must be true, as shown in 1a). So $\forall a \in A$ is

in a subgroup of $A \cong \mathbb{Z}_p$. A can be a vector space $\Leftrightarrow$ exp $A = \bigoplus_{i \in I} \mathbb{Z}_p$.

$x \cdot \left(\sum_{i \in I} a_i \right) = \sum_{i \in I} x a_i$, This is well defined since all but a finite number of a_i's are nonzero.

$0 \in A$ since A is a group. $x \cdot \left(\sum_{i \in I} y a_i \right) = x \left(y \sum_{i \in I} a_i \right)$

10 $= (xy) \cdot \left(\sum_{i \in I} a_i \right)$, $x, y \in F$, $\sum_{i \in I} a_i \in A$. Distribution follows since

F_p is a field, A is a group. Clearly $1 \in F_p$. So $A = \bigoplus_{i \in I} \mathbb{Z}_p$ is vsp over F_p.