Stat 310A/Math 230A Theory of Probability

Homework 6 Solutions

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Exercises on the strong law of large numbers and the central limit theorem

Exercise [2.3.13]

Clearly, $|X_n| = |X_{n-1}||U_n|$, resulting with

$$\log |X_n| = \sum_{k=1}^{n} \log |U_k| + \log |X_0| .$$

As $\mathbf{P}(|U_1| \le r) = r^2$ for $0 \le r \le 1$, it follows from Corollary 1.3.60 and integration by parts that $\mathbf{E} \log |U_1| = \int_0^1 2r \log r dr = -1/2$. Further, $\log |U_k|$ are i.i.d. so by the strong law of large numbers we have that $n^{-1} \log |X_n| \overset{a.s.}{\to} -1/2$.

Exercise [2.3.22]

Let $X_k \equiv Y_k/[k(\log k)^{1+\epsilon}]^{1/2}$. By Kronecker's Lemma it is suffcient to prove that the series $\sum_{k=1}^{\infty} X_k$ converges almost surely. For this we use Theorem 2.3.16 by noting that $\mathbb{E}(X_k) = 0$ and

$$\sum_{k=1}^{\infty} \mathrm{Var}(X_k) \le B \sum_{k=1}^{\infty} \frac{1}{k(\log k)^{1+\epsilon}} < \infty . \tag{1}$$

Exercise [2.3.24]

Let $Y_n = X_n I_{\{X_n<1\}}$ and $a_n = \mathbf{E}Y_n$, both in $[0, 1)$.

1. The assumed finiteness of the series whose terms are the sum of two non-negative quantities, implies that each of the corresponding series is finite. That is, both $\sum_n \mathbf{P}(X_n \ge 1) < \infty$ and $\sum_n a_n < \infty$. By the first Borel-Cantelli lemma, the finiteness of $\sum_n \mathbf{P}(X_n \ge 1)$ implies that $\mathbf{P}(X_n \ge 1 \text{ i.o. }) = 0$. Thus, to prove that $\sum_n X_n(\omega)$ converges w.p.1, it suffices to prove that the series $\sum_n Y_n(\omega)$ converges w.p.1. Since $Y_n \in [0, 1)$, it follows that $Y_n^2 \le Y_n$, hence $\mathrm{Var}(Y_n) \le \mathbf{E}Y_n^2 \le a_n$. Consequently, the finiteness of $\sum_n a_n$ implies that $\sum_n \mathrm{Var}(Y_n) < \infty$. Theorem 2.3.16 then results with the convergence w.p.1. of the random series $\sum_n (Y_n(\omega) - a_n)$. Since we know that the non-negative constant $\sum_n a_n$ is finite, this implies that the random series $\sum_n Y_n(\omega)$ also converges w.p.1.

2. We prove the converse by proving the contrapositive. If $\sum_n \mathbf{P}(X_n \ge 1)$ is infinite, then with $\{X_n\}$ independent, by the second Borel-Cantelli lemma we know that $\mathbf{P}(X_i \ge 1 \text{ i.o. }) = 1$, which implies that $\sum_n X_n(\omega)$ diverges w.p.1. Suppose next that $\sum_n a_n$ is infinite. Then, by the hint, $\prod_n (1 - a_n) = 0$, or equivalently, $e_k = \prod_{n=1}^{k} (1 - a_n) \downarrow 0$ as $k \to \infty$. Since Y_n are independent, we have that $e_k = \mathbf{E}Z_k$ for the non-negative random variable $Z_k = \prod_{n=1}^{k} (1 - Y_n) \le 1$. Further, $Z_k \downarrow Z_\infty = \prod_n (1 - Y_n) \ge 0$ for $k \to \infty$ and any $\omega \in \Omega$. By the bounded convergence theorem this implies that $e_k \to \mathbf{E}Z_\infty$. Consequently, $\mathbf{E}Z_\infty = 0$, hence also $Z_\infty = \prod_n (1 - Y_n) = 0$ w.p.1. Applying the hint in the converse direction we conclude that $\sum_n X_n \ge \sum_n Y_n = \infty$ w.p.1.

3. The series $S := \sum_n G_n^2$ of non-negative terms converges in $\overline{\mathbb{R}}$ so the question is merely when is $\mathbf{P}(S(\omega) < \infty) = 1$. Since $\mathbf{E}G_n^2 = \mu_n^2 + v_n$ for all n, we have that $e = \mathbf{E}S$. Consequently, if e is finite then $\mathbf{P}(S < \infty) = 1$. Conversely, assuming $\mathbf{P}(S < \infty) = 1$, upon applying part (b) for $X_n = G_n^2$ we find that $s := \sum_n \mathbf{E}[\min(G_n^2, 1)]$ must be finite. As $G_n \stackrel{\mathcal{D}}{=} \mu_n + \sqrt{v_n} Y$ for Y of a standard normal distribution, we deduce by linearity of the expectation that $s = \mathbf{E}f(Y)$. With $Y \stackrel{\mathcal{D}}{=} -Y$ we further find that $s = \frac{1}{2}\mathbf{E}[f(Y) + f(-Y)]$. Now, by the hint provided, $f(y) + f(-y) = \infty$ for all $y \neq 0$ in case $e = \infty$. In particular, if $e = \infty$ then also $s = \infty$, contradicting our assumption that $\mathbf{P}(S < \infty) = 1$ and thus proving our thesis.

Exercise [3.1.11]

1. We apply Lindeberg's CLT to the sum $\widehat{S}_n$ of the zero mean, mutually independent variables $X_{n,k} = v_n^{-1/2}(X_k - \mathbf{E}X_k)$. Since $\widehat{S}_n$ is then of unit variance, it suffices to check Lindeberg's condition

$$g_n(\varepsilon) = \sum_{k=1}^n \mathbf{E}[X_{n,k}^2; |X_{n,k}| \geq \varepsilon] = v_n^{-1} \sum_{k=1}^n \mathbf{E}[(X_k - \mathbf{E}X_k)^2; |X_k - \mathbf{E}X_k| \geq \varepsilon v_n^{1/2}]$$

$$\leq \varepsilon^{2-q} v_n^{-q/2} \sum_{k=1}^n \mathbf{E}[|X_k - \mathbf{E}X_k|^q] \to 0$$

in order to conclude with the stated CLT.

2. We have $v_n = n$ and for some $q > 2$,

$$v_n^{-q/2} \sum_{k=1}^n \mathbf{E}(|X_k - \mathbf{E}X_k|^q) \leq v_n^{-q/2} \sum_{k=1}^n C = Cn^{1-q/2} \to 0$$

as $n \to \infty$. The stated convergence in distribution thus follows from Lyapunov's theorem.

3. Here $\mathbf{E}X_k = 0$, $\mathrm{Var}(X_k) = 1/k$ and $\mathbf{E}(|X_k - \mathbf{E}X_k|^q) = \mathbf{E}(|X_k|^q) = 1/k$ for any $q > 0$. Therefore, $v_n = \mathrm{Var}(S_n) = \sum_{k=1}^n k^{-1}$ diverges and

$$v_n^{-q/2} \sum_{k=1}^n \mathbf{E}(|X_k - \mathbf{E}X_k|^q) = v_n^{1-q/2} \to 0$$

for any $q > 2$. So, with $\mathbf{E}S_n = 0$, by Lyapunov's theorem $v_n^{-1/2} S_n \xrightarrow{\mathcal{D}} G$. Further, $(\log n)^{-1} v_n \to 1$, and the normal distribution function is continuous, hence it follows that also $(\log n)^{-1/2} S_n \xrightarrow{\mathcal{D}} G$.

Exercise [3.1.10]

1. By independence,

$$b_n = \mathrm{Var}(R_n) = \sum_{k=1}^n \mathrm{Var}(B_k) = \sum_{k=1}^n k^{-1}(1 - k^{-1}) = \sum_{k=1}^n k^{-1} - \sum_{k=1}^n k^{-2}\,.$$

Further, since $\log n = \int_1^n x^{-1} dx$, it follows from the monotonicity of $x \mapsto x^{-1}$ that $\sum_{k=2}^n k^{-1} \leq \log n \leq \sum_{k=1}^n k^{-1}$. With $\sum_k k^{-2}$ finite and $\log n \to \infty$, we get that $b_n / \log n \to 1$ as claimed.

2. Since $|X_{n,k}| \leq (\log n)^{-1/2}$ for all n, k and ω, it follows that $g_n(\varepsilon)$ of (3.1.4) is zero as soon as $n > \exp(\varepsilon^{-2})$, so Lindeberg's condition is satisfied here. Further, by part (a) the zero-mean random variables $X_{n,k}$ are such that $v_n = \sum_{k=1}^n \mathbf{E}X_{n,k}^2 = b_n / \log n \to 1$ as $n \to \infty$.

3. Applying Lindeberg's CLT we have that $(R_n - \mathbf{E}R_n)/\sqrt{\log n} \xrightarrow{\mathcal{D}} G$. It is easy to check that such convergence in distribution remains in effect even after adding the non-random $(\mathbf{E}R_n - \log n)/\sqrt{\log n} \to 0$.

Exercise [2.3.14]

1. By induction, $\log W_n = \sum_{i=1}^n X_i$ for the i.i.d. random variables $X_i = \log(qr + (1-q)V_i)$. As $\{X_i\}$ are bounded below by $\log(qr) > -\infty$, it follows that $\mathbf{E}[(X_1)_-]$ is finite, so the strong law of large numbers implies that $n^{-1}\log W_n \stackrel{a.s.}{\to} w(q)$, as stated.

2. Since $q \mapsto (qr + (1-q)V_1(\omega))$ is linear and $\log x$ is concave, it follows that $q \mapsto \log(qr + (1-q)V_1)$ is concave on $(0,1]$, per $\omega \in \Omega$. The expectation preserves the concavity, hence $q \mapsto w(q)$ is concave on $(0,1]$.

3. By Jensen's inequality for the concave function $g(x) = \log x$, $x > 0$, we have that

$$w(q) = \mathbf{E}\log(qr + (1-q)V_1) \le \log(qr + (1-q)\mathbf{E}V_1)\,.$$

   Hence, if $\mathbf{E}V_1 \le r$ then $w(q) \le \log(qr + (1-q)r) = \log r = w(1)$.
   Recall that $(\log x)_- \le 1/(ex)$ for all $x \ge 0$. Hence, if $\mathbf{E}V_1^{-1}$ is finite, then so is $\mathbf{E}[(\log V_1)_-]$. Consequently, the strong law of large numbers of part (a) also applies for $n^{-1}\log W_n$ in case $q = 0$ (i.e., for $X_i = \log V_i$). Further, when $\mathbf{E}[(\log V_1)_-]$ is finite, $w(q) = w(0) + \mathbf{E}\log(qrV_1^{-1} + 1 - q)$ and by Jensen's inequality

$$\mathbf{E}\log(qrV_1^{-1} + 1 - q) \le \log(qr\mathbf{E}V_1^{-1} + 1 - q) \le 0$$

   if $\mathbf{E}V_1^{-1} \le r^{-1}$, implying that then $w(q) \le w(0)$.

4. Our assumption that $\mathbf{E}V_1^2 < \infty$ and $\mathbf{E}V_1^{-2} < \infty$ implies that $\mathbf{E}V_1 < \infty$ and $\mathbf{E}V_1^{-1} < \infty$. Further, $w(0) = \mathbf{E}\log V_1 \le \mathbf{E}V_1$ is then also finite. We have shown in part (c) that $w(q) \le w(1) = \log r$ in case $\mathbf{E}V_1 \le r$ and that $w(q) \le w(0)$ in case $\mathbf{E}V_1^{-1} \le r^{-1}$. Consequently, if suffices to show that if $\mathbf{E}V_1 > r > 1/\mathbf{E}V_1^{-1}$, then there exists $q^\star \in (0,1)$ where $w(\cdot)$ reaches its supremum (which is hence finite). The former condition is equivalent to $\mathbf{E}Y > 0$ and $\mathbf{E}Z > 0$ for $Y = rV_1^{-1} - 1 \ge -1$ and $Z = r^{-1}V_1 - 1 \ge -1$, both of which are in L^2. Further, since $q \mapsto w(q) : [0,1] \to \mathbb{R}$ is a concave function, the existence of such $q^\star \in (0,1)$ follows as soon as we check that $w(\epsilon) - w(0) = \mathbf{E}\log(1+\epsilon Y) > 0$ and $w(1-\epsilon) - w(1) = \mathbf{E}\log(1+\epsilon Z) > 0$ when $\epsilon > 0$ is small enough. To this end, note that $\log(1+x) \ge x - x^2$ for all $x \ge -1/2$. Hence, $\mathbf{E}\log(1+\epsilon Y) \ge \epsilon\mathbf{E}Y - \epsilon^2\mathbf{E}Y^2 > 0$ for $\epsilon \in (0,1/2)$ small enough. As the same applies for $\mathbf{E}\log(1+\epsilon Z)$, we are done.
   We see that one should invest only in risky assets whose expected annual growth factor $\mathbf{E}V_1$ exceeds that of the risk-less asset, and that if in addition $\mathbf{E}V_1^{-1} > r^{-1}$, then a unique optimal fraction $q^\star \in (0,1)$ should be re-invested each year in the risky asset.