Stat 310A/Math 230A Theory of Probability

Homework 7 Solutions

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Exercises on weak convergence of distributions

Exercise [3.2.8]

(a). It is easy to see that $Y_n \xrightarrow{\mathcal{D}} c$ if and only if $Y_n \xrightarrow{p} c$ (for example, this is a consequence of Theorem 3.2.10). Assuming the latter, we proceed to show that $X_n + Y_n \xrightarrow{\mathcal{D}} X_\infty + c$. To this end, fix continuity points $x < y < z$ of $F_{X_\infty}(\cdot)$ and let $\epsilon > 0$ be small enough for $x + \epsilon < y < z - \epsilon$. Then, the monotonicity of $\mathbf{P}$ yields that,

$$\mathbf{P}(X_n \le x) - \mathbf{P}(Y_n \ge c + \epsilon) \le \mathbf{P}(X_n + Y_n \le y + c) \le \mathbf{P}(X_n \le z) + \mathbf{P}(Y_n \le c - \epsilon)\,.$$

Since $X_n \xrightarrow{\mathcal{D}} X_\infty$, taking $n \to \infty$ gives

$$\mathbf{P}(X_\infty \le x) \le \liminf_{n\to\infty} \mathbf{P}(X_n + Y_n \le y + c) \le \limsup_{n\to\infty} \mathbf{P}(X_n + Y_n \le y + c) \le \mathbf{P}(X_\infty \le z)$$

Since continuity points of F_{X_∞} are a dense subset of $\mathbb{R}$, we can let $x \uparrow y$ and $z \downarrow y$, deducing out of the continuity of $F_{X_\infty}(\cdot)$ at y that $\mathbf{P}(X_n + Y_n \le y + c) \to \mathbf{P}(X_\infty \le y)$ when $n \to \infty$. This applies at every continuity point y of $F_{X_\infty}(\cdot)$, or equivalently, at every continuity point $y + c$ of $F_{X_\infty + c}(\cdot)$, as stated.

(b). Suppose now that $Y_n = Z_n - X_n \xrightarrow{\mathcal{D}} 0$. Then, by the preceding proof, $X_n \xrightarrow{\mathcal{D}} X$ implies that $Z_n = X_n + Y_n \xrightarrow{\mathcal{D}} X$. Further, since $Y_n \xrightarrow{\mathcal{D}} 0$, also $-Y_n \xrightarrow{\mathcal{D}} 0$, hence $Z_n \xrightarrow{\mathcal{D}} X$ implies that $X_n = Z_n - Y_n \xrightarrow{\mathcal{D}} X$, and we are done.

(c). With $X_n \xrightarrow{\mathcal{D}} X_\infty$ if follows from part (b) of the Portmanteau theorem that

$$\limsup_{n\to\infty} \mathbf{P}(|X_n| \ge M) \le \mathbf{P}(|X_\infty| \ge M) \downarrow 0$$

as $M \uparrow \infty$. That is, for any $\delta > 0$ there exists $M < \infty$ such that $\sup_{n\ge 1} \mathbf{P}(|X_n| \ge M) \le \delta$. Since Y_∞ is non-random, clearly $Y_\infty X_n \xrightarrow{\mathcal{D}} Y_\infty X_\infty$ and view of part (b), it suffices to prove that $V_n X_n \xrightarrow{\mathcal{D}} 0$ for $V_n = Y_n - Y_\infty$. Our assumption that $Y_n \xrightarrow{\mathcal{D}} Y_\infty$ implies that $V_n \xrightarrow{\mathcal{D}} 0$ and by Exercise 3.2.9, also $V_n \xrightarrow{p} 0$. Hence, for any $\varepsilon > 0$ and $n \ge 1$,

$$\mathbf{P}(|V_n X_n| \ge \varepsilon) = \mathbf{P}(|V_n X_n| \ge \varepsilon, |X_n| \ge M) + \mathbf{P}(|V_n X_n| \ge \varepsilon, |X_n| < M) \le \delta + \mathbf{P}(|V_n| \ge \varepsilon/M)\,.$$

Considering $n \to \infty$ followed by $\delta \downarrow 0$, we thus deduce that $\mathbf{P}(|V_n X_n| \ge \varepsilon) \to 0$ as $n \to \infty$, i.e. that $V_n X_n \xrightarrow{\mathcal{D}} 0$.

Exercise [3.2.9]

1. Fixing $\delta, \epsilon > 0$, Kolmogorov's maximal inequality implies that

$$\mathbf{P}\Big(\sup_{(1-\epsilon)b_m \le k \le (1+\epsilon)b_m} |S_k - S_{[(1-\epsilon)b_m]}| > \delta\sqrt{vb_m}\Big) \le 2\epsilon/\delta^2\,.$$

   Considering $Z_m = S_{N_m}/\sqrt{vb_m}$ and $Y_m = S_{b_m}/\sqrt{vb_m}$, with $N_m/b_m \xrightarrow{p} 1$, it thus follows that

$$\limsup_{m\to\infty} \mathbf{P}(|Z_m - Y_m| > \delta) \le 2\epsilon/\delta^2\,.$$

Since this holds for all $\epsilon > 0$ we have that $\mathbf{P}(|Z_m - Y_m| > \delta) \to 0$ for each $\delta > 0$, that is, $Z_m - Y_m \xrightarrow{p} 0$. In particular, $Z_m - Y_m \xrightarrow{\mathcal{D}} 0$. Since $Y_m \xrightarrow{\mathcal{D}} G$ by the CLT, the desired conclusion that $Z_m \xrightarrow{\mathcal{D}} G$ follows from Exercise 3.2.11.

2. Fixing $x \in \mathbb{R}$, let $\ell(t) = \lfloor t + x\sqrt{vt} \rfloor + 1$, noting that $(N_t - t)/\sqrt{vt} \leq x$ if and only if $S_{\ell(t)} > t$. Further, $\ell(t) \to \infty$ as $t \to \infty$ and it is not hard to verify that $(t - \ell(t))/\sqrt{v\ell(t)} \to x$. By the CLT we know that $(S_n - n)/\sqrt{vn} \xrightarrow{\mathcal{D}} G$, hence for any $\epsilon > 0$,

$$\mathbf{P}(G \leq x - \epsilon) \leq \liminf_{t \to \infty} \mathbf{P}(S_{\ell(t)} > t) \leq \limsup_{t \to \infty} \mathbf{P}(S_{\ell(t)} > t) \leq \mathbf{P}(G \leq x + \epsilon)\,.$$

Considering $\epsilon \to 0$ we deduce that $\mathbf{P}(N_t - t \leq x\sqrt{vt}) = \mathbf{P}(S_{\ell(t)} > t) \to F_G(x)$, for $t \to \infty$ and any fixed $x \in \mathbb{R}$, as stated.

Exercise [3.2.13]

1. $F_T(\log n + y)^n = (1 - e^{-y}n^{-1})^n \to \exp(-e^{-y})$ as $n \to \infty$, for any $y \in \mathbb{R}$.
2. $F_T(yn^{1/\alpha})^n = (1 - y^{-\alpha}n^{-1})_+^n \to \exp(-y^{-\alpha})$ as $n \to \infty$, for any $y > 0$.
3. $F_T(yn^{-1/\alpha})^n = (1 - |y|^\alpha n^{-1})_+^n \to \exp(-|y|^\alpha)$ as $n \to \infty$, for any $y \leq 0$.

Exercise [3.2.26]

We check that the total variation distance is a metric. Recall that by Proposition 3.2.21, $d(\mu, \nu) = \sum_{x \in \mathbf{Z}} |\mu(x) - \nu(x)|$ for any two probability measure μ, ν on $\mathbf{Z}$. Hence, clearly $d(\mu, \nu) = d(\nu, \mu)$ and $d(\mu, \nu) = 0$ if and only if $\mu(x) = \nu(x)$ for any $x \in \mathbf{Z}$, i.e. $\mu = \nu$. To check the triangle inequality we note that the triangle inequality for real numbers implies

$$|\mu(x) - \nu(x)| + |\nu(x) - \pi(x)| \geq |\mu(x) - \pi(x)|$$

then sum over $x \in \mathbf{Z}$. In this space, the convergence in total variation is merely the convergence of $\sum_x |\mu_n(x) - \mu_\infty(x)|$ to zero. As explained in Example 3.2.23, this is equivalent to the pointwise convergence of $\mu_n(x)$ to $\mu_\infty(x)$, which in turn is equivalent to the weak convergence of μ_n to μ_∞.

Exercise [3.2.14]

1. From Exercise 2.2.23, it follows that

$$\lim_{t \to \infty} \frac{\mathbf{P}(G > t + y/t)}{\mathbf{P}(G > t)} = \lim_{t \to \infty} \frac{t}{t + y/t} e^{-y - y^2/(2t^2)} = e^{-y}.$$

2. Setting $b_n \uparrow \infty$ such that $1 - F_G(b_n) = n^{-1}$, let $p_n = \mathbf{P}(G > b_n + y/b_n)$ noting that $np_n \to e^{-y}$ when $n \to \infty$ (by part (a)). Since G_i are i.i.d. this implies that for $n \to \infty$,

$$\mathbf{P}(b_n(M_n - b_n) \leq y) = \prod_{i=1}^{n} \mathbf{P}(G_i \leq b_n + y/b_n) = (1 - p_n)^n \to \exp(-e^{-y})\,.$$

3. From the upper bound of part (a) of Exercise 2.2.23 we have that for some $C < \infty$ and all n,

$$\mathbf{P}(G > \sqrt{2\log n}) \leq \frac{C}{n\sqrt{\log n}}\,,$$

whereas from the corresponding lower bound we have that for some $c > 0$ and all n large enough,

$$\mathbf{P}(G > \sqrt{2\log n - 2\log\log n}) \geq \frac{c\log n}{n\sqrt{\log n}}\,.$$

Since $\mathbf{P}(G > b_n) = 1/n$, it follows that for all n large enough

$$\sqrt{2\log n} \geq b_n \geq \sqrt{2\log n - 2\log\log n}\,.$$

In particular, $b_n/\sqrt{2\log n} \to 1$ as $n \to \infty$. Further, from part (b) we easily deduce that for any $y_n \to \infty$,

$$\lim_{n\to\infty} \mathbf{P}(|M_n - b_n| > y_n/b_n) = 0\,.$$

Thus, considering $y_n \to \infty$ such that $y_n/b_n \to 0$, we see that $M_n/b_n \xrightarrow{p} 1$, and consequently also $M_n/\sqrt{2\log n} \xrightarrow{p} 1$.