Stat 310A/Math 230A Theory of Probability

Homework 8

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Solutions should be complete and concisely written. Please, use a separate sheet (or set of sheets) per each problem. Staple sheets referring to the same problem, and write your name on each sheet.

You are welcome to discuss problems with your colleagues, but should write and submit your own solution.

In some cases, multiple homework options will be proposed (and indicated as 'Option 1', 'Option 2', etc.). You are welcome to work on all the problems proposed (solutions will be posted), but should submit only those corresponding to one 'Option'.

Exercises on characteristic functions

Solve Exercises [3.3.10], [3.3.20], [3.3.21], in Amir Dembo's lecture notes.

An exercise on weak convergence of measures

Let $\Omega = \{0,1\}^{\mathbb{N}}$ be the set of (infinite) binary sequences $\omega = (\omega_1, \omega_2, \omega_3, \dots)$, and consider the topolgy generated by the following basis of neighborhoods of $\omega \in \Omega$:

$$N_\ell(\omega) = \left\{\xi \in \Omega\,:\, \xi_1^\ell = \omega_1^\ell\right\}, \tag{1}$$

with $\ell \in \mathbb{N}$ (here we use the notation $\omega_1^\ell = (\omega_1, \dots, \omega_\ell)$). Let $\mathcal{B}_\Omega$ be the Borel σ-algebra associated to this topology.

For n even, let A_n be the set of sequences defined as follows

$$A_n = \left\{\omega \in \Omega\,:\, \sum_{i=1}^{n} \omega_i = n/2,\ \omega_i = 0 \text{ for all } i > n\right\}. \tag{2}$$

Consider the sequence of probability measures $\{\nu_n\}_{n\in 2\mathbb{N}}$, with ν_n the uniform distribution over A_n, i.e.

$$\nu_n(\{\omega\}) = \begin{cases} \binom{n}{n/2}^{-1} & \text{if } \omega \in A_n, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

1. Show that, for each n, ν_n is indeed a measure over $\mathcal{B}_\Omega$.
2. What is the weak limit of ν_n as $n \to \infty$? Prove your answer.

In solving the last point you can assume the following

Fact 1. *Let $h : \Omega \to \mathbb{R}$ be a continuous function. Then h is uniformly continuous in the following sense. There exists a function $\delta : \mathbb{N} \to \mathbb{R}$, $\ell \mapsto \delta(\ell)$, with $\lim_{\ell\to\infty} \delta(\ell) = 0$, such that, for any $\omega \in \Omega$, $\omega' \in N_\ell(\omega)$, we have $|h(\omega') - h(\omega)| \le \delta(\ell)$.*