Kang-Li Cheng

ksc66

Math 4330 Homework Set 8

Due Monday, November 9, 2015

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NOTE: Late homework not accepted.

Read: "Bases and Coordinates", "The Matrix of a Linear Transformation" and "Dual Spaces".

Problems marked by $\boxed{\text{box}}$ or $\boxed{\star}$ are more challenging and may be turned in anytime during the semester. There will be several such problems assigned during the term. Please turn in *separately* from routine assignments – if incorrect or incomplete, they will be returned to you to complete correctly. Final deadline is Monday, Nov. 30, no exceptions.

Do the following problems from the handouts:

MatLinTrans 9

MatLinTrans 10
Note: The vector spaces in parts d, e, f are NOT assumed to be finite dimensional.

MatLinTrans 21

MatLinTrans 27

Ex07 1. This Counts as Two Problems
Let n be a positive integer.

a. Let $\mathrm{Tr} : F^{n\times n} \longrightarrow F$ denote the trace function on $n\times n$ matrices over the field F. Show $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ for any $A, B \in F^{n\times n}$. Show that Tr is never the zero linear functional.

b. Show that there do not exist $A, B \in F^{n\times n}$ for F the field of complex numbers, such that $AB - BA = I$. What happens for an arbitrary field? F^n

c. Let $T : V \longrightarrow V$ be a linear transformation where V is a finite dimensional vector space over a field F. Choose a basis $\mathcal{B}$ for V and define $\mathrm{Tr}(T) = \mathrm{Tr}([T]_{\mathcal{B},\mathcal{B}})$. Show that the definition does not depend on the choice of basis $\mathcal{B}$.

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d. Let $f : F^{n\times n} \longrightarrow F$ be a linear functional which satisfies $f(AB) = f(BA)$ for all $A, B \in F^{n\times n}$. Prove that $f = a \cdot \mathrm{Tr}$ for some scalar $a \in F$. Further show that if the characteristic of F is 0, then $f = \mathrm{Tr}$ precisely when $f(I) = n$. What happens if the characteristic is not 0? Can you find another way to decide if $f = \mathrm{Tr}$ by computing a single value?

e. Let $S = \mathrm{Span}_F(\{AB - BA \mid A, B \in F^{n\times n}\})$. Prove that $S = \ker \mathrm{Tr}$.
Hint: Compute the dimension of $\ker \mathrm{Tr}$, and find a basis for S using some well-known matrices.

f. Let $A \in \mathbb{R}^{n\times n}$. Show that $A = 0$ if and only if $A^t \cdot A = 0$.

Ex07 2. Let V be a finite-dimensional vector space over the field F and let W be a subspace of V. If f is a linear functional on W, prove that there is a linear functional h on V such that $h(w) = f(w)$ for all $w \in W$.

Ex07 3. Let n and m be positive integers.

a. Let F be a field. Let $A \in F^{m\times n}$ and let $B \in F^{n\times m}$. Then AB is an $m \times m$ matrix and BA is an $n \times n$ matrix. Is it always true that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$? (Note that there are two different trace functions used here.) If so, give a proof. If not, give a counterexample.

b. Replace the field by R a commutative ring with identity. Answer the same question.

c. Let R be a ring with identity, but do not assume it is commutative. Answer the same question.

$m\times n$ $\quad n\times m$

$n\times m \sim m\times n$

$\begin{bmatrix} a_{1,1} & \\ & a_{m,m} \end{bmatrix}$ $\quad \begin{bmatrix} a \end{bmatrix}$

$a_{1,1}$ $a_{1,2}$ $\quad \begin{bmatrix} b_1 & b_{23} \\ b_{2,1} & b_{22} \end{bmatrix}$

$a_{2,1}$ a_{22}

a_{31} a_{32}

$Tr: F^{n\times n} \to F$ Show $Tr(AB) = Tr(BA)\ \forall\ A, B \in F^{n\times n}$

a)   Tr is never zero linear functional

$Tr(AB)$

$= \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ji} b_{ji}$, since $(AB)_{ii} = \sum_{j=1}^{n} a_{ij} b_{ji}$

$Tr(BA) = \sum_{j=1}^{n} \sum_{i=1}^{n} b_{ji} a_{ij} = Tr(AB)$, this is just reindexing.

17   Trace is never zero functional since $Tr(E_{11}) = 1\ \forall n$.

show

b) $\nexists\ A, B \in F^{n\times n}$, $F = \mathbb{C}$ s.t. $AB - BA = I$

In the field $\mathbb{C}$, $Tr(AB) = Tr(BA)$ so there are no matrices s.t. $AB - BA$

In an arbitrary field, this may be true. Let $F = \mathbb{F}^2$. Then let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. $AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $BA = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, so $AB - BA = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = I \in \mathbb{F}^2$

since $-1 = 1$ in $\mathbb{F}^2$.

c) V is finite dimensional, ~~$\mathcal{B} = \mathcal{B}_n$~~ vsp over F. $T: V \to V$ is a linear transformation. Choose a basis $\mathcal{B}$ for V. define $Tr(T) = Tr([T]_{\mathcal{B},\mathcal{B}})$. Show this definition does not depend on choice of $\mathcal{B}$.

Suppose C is another basis for V. Then $\exists\ P$, an invertible matrix and

$[T]_{\mathcal{B},\mathcal{B}} = P[T]_{C,C}P^{-1}$. So $Tr([[T]_{\mathcal{B},\mathcal{B}}) = Tr(P[T]_{C,C}P^{-1}) = Tr(PP^{-1}[T]_{C,C})$

$= Tr([T]_{C,C})$ □

d) $f: F^{n\times n} \to F$ linear functional s.t. $f(AB) = f(BA)$ $\forall A, B \in F^{n\times n}$

Prove $f = a \cdot Tr$, $a \in F$

Let $\{AB - BA \mid A, B \in F^{n\times n}\} \subseteq F^{n\times n} = V$

and $\{M \in F^{n\times n}$ s.t. $Tr(M) = 0\} = W$ So $Tr(M)$ has codim of 1.

and $\dim\{f \in V^* \mid f(w) = 0\} = \operatorname{codim} W$. Since $Tr \in \{f \in V^* \mid f(w) = 0\}$,

we conclude $f = a \cdot Tr$. ✗

e) $S = \operatorname{Span}_F(\{AB - BA\} \mid A, B \in F^{n\times n})$. Prove $S = \ker Tr$

$0 \to \ker Tr \xrightarrow{i} F^{n\times n} \xrightarrow{Tr} F \to 0$ tells us $n^2 = \dim F^{n\times n} = \dim\ker Tr + \dim F$

$F = \dim\ker Tr + 1$, so $\dim\ker Tr = n^2 - 1$.

WTS $\dim S = n^2 - 1$. Let $A = E_{ij}$, $B = E_{jj}$, $i \neq j$. So $AB = E_{ij}$, $BA = 0$,

$AB - BA = AB$, $\Rightarrow E_{ij} \in S$. Suppose $C = E_{ij}$, $D = E_{ji}$. Clearly $E_{ii} - E_{jj}$

$= E_{ij} E_{ji} - E_{ji} E_{ij} \in S$. Note that $\dim(E_{ii} - E_{jj}) = n - 1$, so $\dim(S) =$

$n^2 - n + n - 1 = n^2 - 1$

This shows $S = \ker Tr$ □

f) Clearly $A = 0 \to A^t A = 0$. Suppose $A^t A = 0$. Then take the coordinate

(i,i) of $A^t A = 0$, which is $(a_{1i})^2 + (a_{2i})^2 + \dots + (a_{ni})^2 = 0$.

So $a_{ji} = 0$, $1 \leq j \leq n$. Therefore $A = 0$.

V finite dimensional vsp s.t. over F, $W \subseteq V$.

f is a linear functional on W, prove $\exists\, h$ on V s.t. $h(w) = f(w)$ $\forall w \in W$.

Define $\mathcal{W} = \{e_1, \dots, e_k\}$ as basis for $W \subseteq V$.

Then $f = \sum_{i=1}^{k} f(e_i) e_i^*$. We can extend $\mathcal{W}$ to a basis $\mathcal{B}$ for V by adding $\{e_{k+1}, \dots, e_n\}$ to $\mathcal{W}$. Define $h: V \to V$ as $h = \sum_{i=1}^{k} f(e_i) e_i^* + \sum_{j=k+1}^{k} e_j^*$. So $h(w) = \sum_{i=1}^{k} f(e_i) e_i^*(w) + \sum_{j=k+1}^{k} e_j^*(w)$

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$= \sum_{i=1}^{k} f(e_i) e_i^*(w)$, since $e_i^*(w) = 0$ when $k+1 \le i \le m$.

So $f(w) = h(w)$ $\forall w \in W$.

3) a) Let $A \in F^{m \times n}$, $B \in F^{n \times m}$. $Tr(AB) = \sum_{j=1}^{m} \sum_{i=1}^{n} a_{ji} b_{ij}$, $Tr(BA)$

$= \sum_{j=1}^{n} \sum_{i=1}^{m} b_{ji} a_{ij}$ but $\sum_{j=1}^{m} \sum_{i=1}^{n} a_{ji} b_{ij}$

$= \sum_{j=1}^{n} \sum_{i=1}^{m} b_{ji} a_{ij}$ by reindexing $\square$

) This holds if $F = R$,

R is commutative ring. Commutativity of multiplication is the key.

c) $Tr(AB) \neq Tr(BA)$ since we don't have commutativity.

why? example?

c) Cont'd WTS: if $\operatorname{im} T^n = \operatorname{im} T^{n+1}$ for some fixed $n > 0$, then it follows $\operatorname{im} T^{n+1} = \operatorname{im} T^{n+2}$ and $\operatorname{im} T^k = \operatorname{im} T^p$, $k \geq p$. We assume $\operatorname{im} T^n = \operatorname{im} T^{n+1}$ for some fixed $n > 0$. Using b) $\operatorname{im} T^{n+1} \supseteq \operatorname{im} T^{n+2}$. WTS: $\operatorname{im} T^{n+1} \subseteq \operatorname{im} T^{n+2}$. Since $\operatorname{im} T^{n+1} = \operatorname{im} T^n$, $\forall v \in V$, $\exists v' \in V$ s.t. $T^{n+1}(v) = T^{n+1}(v')$. So it follows that $\forall v \in V$, $\exists v' \in V$ s.t. $T^{n+1}(v) = T(T^{n+1}(v)) = T(T^{n+1}(v)) = T^{n+2}(v)$, so $\operatorname{im} T^{n+1} \subseteq \operatorname{im} T^{n+2}$, and $\operatorname{im} T^{n+1} = \operatorname{im} T^{n+2}$, $\operatorname{im} T^k = \operatorname{im} T^n$ $k \geq n$

9. $T: V \to V$ is a linear transformation. $B_n = \ker T^n$, $C_n = \operatorname{im} T^n$

a) $B_n \subseteq B_{n+1}$, $B = \bigcup B_i$ is subspace of V. ($C_1 \supseteq C_2 \supseteq \ldots C_n \supseteq \ldots C_{n+1}$ is stabilized and $\exists C_{k'} \ldots C = C_{k'}$ $1 \leq k' \leq \dim V$.)

For $v \in \ker T^n$, note $T^{n+1}(v) = T(T^n(v)) = T(0) = 0 \to v \in \ker T^{n+1}$

So $B_n \subseteq B_{n+1}$. Suppose $v, w \in B$, then $v \in B_m$ and $w \in B_n$, m and n are arbitrary indices. Let $k = \max(m, n)$. Then $m, n \in B_k$. B_k is a subspace because

10 it is the kernel of a linear map and $v + w \in B_k \to v + w \in B$. So B is a subspace of V

b) Show $C_n \supseteq C_{n+1}$, $C = \bigcap C_i$ is a subspace of V

$\operatorname{im}(T) = \{T(v) \mid v \in V\}$. Let $v \in \operatorname{im}(T^{n+1})$. $T^{n+1}(v) = T(T^n(v)) = v$, $v \in T^n$.

So $\operatorname{im} T^n \supseteq T^{n+1}$. Since $\operatorname{im} T^n$ is a subspace of V and arbitrary intersection of subspaces is a subspace, $\bigcap C_i$ is a subspace of V.

c) Show $B = B_n$ for some n, if V is finite dimensional. Find a bound on n, depending on V and independent of T. Similarly, show $C = C_n$ for some n.

If $B_{i+1} = B_i$, then $B_n = B_i$ $\forall n > i$.

We know $B_{i+1} \supseteq B_i$. If $\ker T^{i+1} = \ker T^i \to B = \ker T$. $v \in \ker T^{i+k}$, then $T^{i+1}(T^{k-1}(v)) = 0$

$T^{k-1}(v) \in \ker T^{i+1} = \ker T^i$. $T^i(T^{k-1}(v)) = 0$ and $T^{i+k-1}(v) = 0$

$v \in \ker T^{i+k-1} \to v \in \ker(T^{i+k-2}) \to \ldots v \in \ker(T^i)$ $\quad \ker(T^{i+k}) = \ker(T^i)$ $\forall k > 1$

We repeat this process until $v \in \ker(T^i)$.

So the bound on n is at most $\dim(V)$. Cont'd above...

d) $V = B \oplus C$, V is finite dimensional (✱ using c))

1) First we show that $B \cap C = \{0\}$. $\exists n$ s.t. $B_n = B$ and m s.t. $C_m = C$. Let $N = \max(n,m)$, then $B_N = B$, $C_N = C$. Let $v \in B \cap C$ and $T^N(v) = 0$. Also $v = T^N(w)$, for some $w \in V$. So $T^{2N}(w) = 0$ $\rightarrow w \in B_{2N} = B = B_N \rightarrow T^N(w) = 0$. Conclude $v = 0$.

2) Now we show $B + C = V$. $B_N = \ker(T^N)$, $C_N = \operatorname{im}(T^N)$. $\dim B_N + \dim C_N = \dim V$ (using Nullity + Rank $= \dim V$)

$\dim B_n + \dim C_N = \dim V \rightarrow \dim B + \dim C = V$.

So showing 1) and 2) proves $V = B \oplus C$ when V is finite dimensional.

e) Not true if V is not finite dimensional

Let $V = F[x]$. Let $T: F[x] \rightarrow F[x]$ be defined by $T(x^i) = x^{i+1}$ $\forall i \geq 0$. So $\ker T^n = 0$ for $n \geq 1$, so $B = 0$ and $\bigcap_{n \in \mathbb{N}} \operatorname{im} T^n = 0$, since $\operatorname{im} T^n$ is the set of all polynomials of degree $\geq n$. Clearly $F[x] \neq 0 \oplus 0$ □

f) Show T maps B to B, C to C. If V is finite dimensional, T restricted to B is nilpotent. T restricted to C is an isomorphism.

$\forall v \in V$, $v \in B \leftrightarrow v \in \ker T^n$, $n > 1$. So $T(v) \in \ker T^{n-1}$, $T(v) \in B$.
We also see that $v \in C \leftrightarrow T^n(v') = v$, for sum $v' \in V$, $n > 1$. Therefore $T^{n+1}(v') = T(T^n(v'))$ $= T(v)$, so $T(v) \in C$. This shows T maps B to B, C to C.

When V is finite dim, $B = B_k$, $1 \leq k \leq \dim_F(V)$, result of c). So $B = \ker T^k$ and $T^k(b) = 0$ $\forall b \in B$. This shows T restricted to B is nilpotent. Similarly $C = C_n$ for $1 \leq n \leq \dim_F V$. $C = \operatorname{im} T^n$ and from c) $\operatorname{im} T^n = \operatorname{im} T^{n+1}$, and we conclude T restricted to C is an isomorphism.

g) Let $V = F^{n \times 1}$. $V = B \oplus C$ from d), since V is finite dimensional.
All matrices $A \in F^{n \times n}$ can be identified with $T: V \rightarrow V$, a linear transformation.
Let B_1 be a basis for B and C_1 be a basis for C. $\mathcal{B} = B_1 \cup C_1$

So $[T]_{\mathcal{B},\mathcal{B}} = \begin{bmatrix} [T]_{B_1,B_1} & 0 \\ 0 & [T]_{C_1,C_1} \end{bmatrix}$. Here $[T]_{B_1,B_1}$ is nilpotent, $[T]_{C_1,C_1}$ is invertible. Because the matrices are equivalent $\leftrightarrow$ they are the same linear transformation with respect to different bases, we are done □

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$S, T \in Hom_F(U,V)$, S and T are equivalent if $\exists$ invertible P, Q s.t. $S = PTQ$.

Verify this gives an equivalence relation on $Hom_F(U,V)$.

a) 1) $S \sim S \to S = PSQ$

2) $S \sim T$ $S = PTQ$ and $T = Q^{-1} S P^{-1}$, since P and Q are invertible

3) $S \sim T$ and $T \sim R$, then $S \sim R$

$S \sim T \to S = PTQ$. $T \sim R = T = ARB$

$T = Q^{-1} S P^{-1} \to Q^{-1} S P^{-1} = ARB$

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$S = QARBP$, QA and BP are invertible since product of invertible matrices is invertible.

b) $S, T \in Hom_F(U,V)$ are equivalent $\leftrightarrow$ rank S = rank T

"$\to$" $S = PTQ$, $Im\, S = PT(Im\, Q) = PT(U) = P(Im\, T)$. $Im\, S \cong Im\, T$ since P is isomorphic.

"$\leftarrow$" If rank S = rank T, then show $S = PTQ$; P, Q invertible. Suppose S is full rank, rank $S = m$

Then $\exists P_1, Q_1$ invertible matrices such that $P_1 S Q_1$ = rref S = $P_2 T Q_2$. So $P_2^{-1} P_1 S \cdot Q_1 Q_2^{-1} = T$

So if rank S = rank T and they are full rank, we are done.

So if rank S = rank T, we can make them into full rank matrices.

Now WTS that if rank S = rank T, we can make them into full rank matrices.

$S: U \to V$, $T: U \to V$. Let $R \in Hom_F(V,V)$, now $RS: U \to V$ and $RS: U \to im\, T$.

$T: U \to im\, T$. $im\, RS = R\, Im\, S = Im\, T$. $Im\, S \cong Im\, T$ since if we can

$\cap \quad \cap$

$V \quad V$

choose a basis $s_1 \dots s_k$ and $t_1 \dots t_k$ for S and T respectively.

Since we are not given that V is finite dimensional we are implicitly invoking the axiom of choice.

c) U and V not finite dimensional, counterexamples to B.

$(a_1, a_2 \dots) \in \mathbb{R}^\infty$

Let S be a left shift map and T be identity map

So $S(a_1, a_2, \dots, a_k) = (a_2, a_3 \dots a_{k+1}) \in \mathbb{R}^\infty$ Then clearly the left shift is surjective but not injective and identity map is bijective □

d) 1) $\dim \ker S = \dim \ker T$
2) $\dim \operatorname{im} S = \dim \operatorname{im} T$
3) $\dim \operatorname{coker} S = \dim \operatorname{coker} T$

If $S, T \in \operatorname{Hom}_F(V)$ are equivalent, then $S = PTQ$. P and Q are isomorphic $\rightarrow$ there is an induced bijection between bases of $\ker S$ and $\ker T$, $\operatorname{im} S$ and $\operatorname{im} T$, and $\operatorname{coker} S$ and $\operatorname{coker} T$.

So we conclude $\dim \ker PTQ = \dim \ker T$, $\dim \operatorname{im} PTQ = \dim \operatorname{im} T$, and $\dim \operatorname{coker} PTQ = \dim \operatorname{coker} T$. Let S and T be s.t. $\dim \ker S = \dim \ker T$, $\dim \operatorname{im} S = \dim \operatorname{im} T$, $\dim \operatorname{coker} S = \dim \operatorname{coker} T$. Choose B_1 as a basis for $\ker S$, and let B_1' be a basis for $\ker T$. Then let $\mathcal{B} = B_1 \sqcup B_2$, $\mathcal{B}' = B_1' \sqcup B_2'$ both be basis of V. Now $|B_1| = B_2$. Let C_1 and C_1' be images of B_2 under S and T, respectively. Note $S: B_1 \to 0$ and $T: B_1' \to 0$. Recall $\dim \operatorname{im} S = \dim \operatorname{im} T$, so $|C_1| = |C_1'|$. $\mathcal{C} = C_1 \sqcup C_2$ and $\mathcal{C}' = C_1 \sqcup C_2'$ as defined be bases for V. Here $P: C_1 \to 0$ and $P: C_2 \to 0$. Now let D and D' be images of C_2 and C_2', respectively. So D and D' are bases for coker S, T respectively. So bijections between each pair of bases, $(B_1, B_2), (C_1, C_1'), (D, D')$ induce linear maps $P \in \operatorname{Hom}_F(V,V)$ and $Q \in \operatorname{Hom}_F(V,V)$ s.t. $S = PTQ$, $T = P^{-1}TQ$ □

e) The condition should be $\leftrightarrow$ $\dim \ker S = \dim \ker T$ and $\dim \operatorname{coker} S = \dim \operatorname{coker} T$ since we showed 1), 2), 3) to be TFAE.

f) Suppose $R, S, T, \in \operatorname{Hom}_F(V,V)$, R semi-equiv R since $R = I_V R I_V$

Suppose R semi equiv S, $R = P_1 S Q_1$, then $S = P_1' S Q_1'$ so S semi equiv R.

If R semi-equiv S, S semi-equiv T, $R = P_1 S Q_1$, $S = P_1' R Q_1'$, $S = P_1'' T Q_1''$ $T = P_1''' S Q_1'''$. So $R = (P_1' P_1'') T (Q_1'' Q_1)$ and $T = (P_1''' P_1' R Q_1' Q_1''')$ □

27. $T \in Hom_F(V, V)$ is idempotent linear transformation

a) Prove $V = im\, T \oplus ker\, T$

For $v \in V$, $v = Tv + (v - T(v)) \in im(T) + ker(T)$ since $T(v - Tv) = Tv - T^2 v = 0$

Since T is idempotent, $Tv \in ker(T) \cap im(T)$ and $T^2 v = Tv = 0 \to$ the sum is direct

b) $[T]_B = \begin{bmatrix} [T]_{B_1, B_1} & 0 \\ 0 & 0 \end{bmatrix}$, computed by definition.

$\overset{?}{=}$

c) Suppose $A, B \in F^{n \times n}$ are idempotent. Then $\exists$ bases $B_1, B_2 \in F^{n \times n}$

s.t. $[L_A]_{B_1} = \begin{bmatrix} [L_A]_{B_1, B_1} & 0 \\ 0 & 0 \end{bmatrix}$, $[L_B]_{B_2} = \begin{bmatrix} [L_B]_{B_2, B_2} & 0 \\ 0 & 0 \end{bmatrix}$

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B_1 and B_2 are bases for $im\, L_A$ and $im\, L_B$ respectively.

So $[L_A]_{B_1}$ and $[L_B]_{B_2}$ are similar $\leftrightarrow$ rank $[L_A]_{B_1, B_1}$ = rank$[L_B]_{B_2, B_2}$).

However rank$([L_A]_{B_1 B_1})$ = rank $(im\, L_A)$ = rank(A) and also

rank $[L_B]_{B_2 B_2}$ = rank$(im\, L_B)$ = rank B. So A is similar to B

$\leftrightarrow$ rank(A) = rank(B) $\square$

21.)?. 0

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Math 4330 Homework Set 9

Due Monday, November 16, 2015

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NOTE: Late homework not accepted.

Read: "The Matrix of a Linear Transformation" "Dual Spaces" and the three handouts on Rings and Modules.

Problems marked by $\boxed{\text{box}}$ or $\boxed{\star}$ are more challenging and may be turned in anytime during the semester. There will be several such problems assigned during the term. Please turn in *separately* from routine assignments – if incorrect or incomplete, they will be returned to you to complete correctly. Final deadline is Monday, Nov. 30, no exceptions.

Do the following problems from the handouts:

DualSpace 7

DualSpace 8

DualSpace 16

DualSpace 25

DualSpace 33

DualSpace 34

DualSpace 36

Ex09 1. Let F be an arbitrary field.

a. Show that the intersection of an arbitrary number of ideals in $F[x]$ is an ideal in $F[x]$.

b. Let $f_1, \ldots, f_k \in F[x]$. The ideal generated by these is

$$(f_1, \ldots, f_k) = \{ g_1 f_1 + \cdots + g_k f_k \mid g_i \in F[x] \} ,$$

the set of all $F[x]$-linear combinations of $f_1, \ldots, f_k$. Show that this ideal is precisely the intersection of the ideals which contain all f_i, $1 \le i \le k$.

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Ex09 2 (Exact Sequence of a Pair in a PID). Let R be a principal ideal domain (PID). Let $a, b \in R$, not both of which are 0. Define $f : R \times R \longrightarrow R$ by $f(s,t) = sa + tb$. Note that $R \times R$ is also a commutive ring with 1 when addition and multiplication are defined coordinate-wise:

(1) $(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$

(2) $(a_1, b_1) \cdot (a_2, b_2) = (a_1 a_2, b_1 b_2)$

Further note that $R \times R$ is an R-module with scalar multiplication defined by

(3) $r \cdot (a, b) = (ra, rb)$

a. Show that f satisfies

(i) $f(x + y) = f(x) + f(y)$ for all $x, y \in R \times R$.

(ii) $f(rx) = rf(x)$ for $r \in R$, $x \in R \times R$.

Hence f is an R-module homomorphism.

b. Show that $\operatorname{im} f \subseteq R$ is non-empty and is closed under addition and scalar multiplication; that is, $\operatorname{im} f$ is an R-submodule of R.

c. Compute $\operatorname{im} f$.

d. Show that $\ker f \subseteq R \times R$ is an R-submodule of $R \times R$.

e. Determine $\ker f$ explicitly: Show that there exists a function $g : R \longrightarrow R \times R$ of the form $g(r) = (r\alpha, r\beta)$ for some $\alpha, \beta \in R$ such that $\operatorname{im} g = \ker f$. Note that g satisfies the analogue of (i) and (ii) above (i.e., is an R-module homomorphism).

f. Show that there exists an exact sequence of R-modules

$$0 \longrightarrow X \xrightarrow{i} R \times R \xrightarrow{f} R \xrightarrow{p} Y \longrightarrow 0\,.$$

What are X, i, Y, p?

g. Determine precisely all solutions (s,t), $s, t \in R$ of the equation $sa + tb = \gcd(a,b)$ where $\gcd(a,b)$ denotes the greatest common divisor of a and b.

Ex09 3. Let R be a PID and let a, $b \in R$ be two non-zero elements. Show that there exist elements $r, s, u, v \in R$ such that

a. $(a,b) = au + bv$,

b. $a = (a,b)r$, $b = (a,b)s$, $[a,b] = (a,b)rs$,

c. the matrices $A, B \in R^{2\times 2}$

$$A = \begin{bmatrix} u & v \\ -s & r \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 & -vs \\ 1 & ur \end{bmatrix}$$

are invertible and $\det A = \det B = 1$,

d. and further the following holds:

$$A \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} B = \begin{bmatrix} (a,b) & 0 \\ 0 & [a,b] \end{bmatrix}.$$

2a) Let $x, y \in R \times R$, $r \in R$. Let $x = (x_1, x_2)$, $y = (y_1, y_2)$.

Then $f(x+y) = f(x_1+y_1, x_2+y_2) = (x_1+y_1)a + (x_2+y_2)b$

$= x_1 a + y_1 a + x_2 b + y_2 b = (x_1 a + x_2 b) + (y_1 a + y_2 b) = f(x_1, x_2) + f(y_1, y_2)$

$= f(x) + f(y)$.

$f(rx) = f(r(x_1, x_2)) = f(rx_1, rx_2) = rx_1 a + rx_2 b = r(x_1 a + x_2 b)$

$= r f(x_1, x_2) = r \cdot f(x)$. So f is an R-module homomorphism.

Ex 09 1 a) Let F be an arbitrary field. Show intersection of ideals in $F[x]$ is an ideal in $F[x]$.

Let $\{I_\alpha\}_{\alpha \in A} \subseteq F[x]$ be a set of ideals. Let $p(x), q(x) \in \bigcap_\alpha I_\alpha$ and $c \in F$. $p(x)+q(x) \in I_\alpha$ and $c \cdot p(x) \in I_\alpha$ for each α, since I_α is an ideal $\forall \alpha$. Then $p(x)+q(x) \in \bigcap_\alpha I_\alpha$ and $c \cdot p(x) \in \bigcap_\alpha I_\alpha$. Then by Lemma 6 (Rings and Factorization), $\bigcap_\alpha I_\alpha$ is an ideal of $F[x]$ $\square$

b) Suppose $f_1, \dots, f_k \in F[x]$ and $(f_1, \dots, f_k) = \{g_1 f_1 + \dots + g_k f_k \mid g_i \in F[x]\}$ is the set of all polynomial linear combinations of $f_1, \dots, f_k$. This set is an ideal of $F[x]$. Suppose $I \subseteq F[x]$ is some ideal s.t. $f_1, \dots, f_k \in I$.

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So $g_1 f_1 + \dots + g_k f_k \in I$ $\forall \{g_1, \dots, g_k\} \in F$. Therefore $(f_1, \dots, f_k) \subseteq I$, and $(f_1, \dots, f_k) \in \bigcap I_\alpha$, the intersection of all ideals of $F[x]$ containing $f_1, \dots, f_k$. Now clearly $f_1, \dots, f_k = \bigcap I_\alpha$ since $f_1, \dots, f_k$ also contains $f_1, \dots, f_k$.

2. b) $f(1,0) = a$ and $f(0,1) = b$, at least one is non zero since $a, b \in R$ are not both 0. Let $x, y \in \operatorname{im} f$, $a \in F$. Then $\exists$ $p, q, s, t \in R$ s.t.

$\operatorname{im} f \neq \{0\}$.

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$f(p,q) = x$, $f(s,t) = y$. Using a), $f(p+s, q+t) = f(p,q) + f(s,t) = x+y$. $f(as, at) = af(s,t) = ax$, $\rightarrow$ $x+y, ax \in \operatorname{im} f$. Clearly $\operatorname{im} f$ has closure under addition and mult by scalars in F. So $\operatorname{im} f$ is an R-submodule of R $\square$

a)

c) Note that $\operatorname{im} f = \{sa + tb \mid s, t \in R\} \rightarrow \operatorname{im} f = (a, b)$, which is the ideal generated by $a, b \in R$.

d) Let $x, y \in \ker f$, $r \in R$.

$f(x+y) = f(x) + f(y) = 0 + 0 = 0$

$f(rx) = rf(x) = r0 = 0$. So $x+y, rx \in \ker f \to$ closure under addition and mult by scalars in R. So $\ker f$ is R-submodule of $R \times R$.

e) Let $(\alpha, \beta) \in \ker f$. Suppose $g : R \to R \times R$ is given by $g(r) = (r\alpha, r\beta)$. Then $\operatorname{im} g \subseteq \ker f$, since $\ker f$ is closed under scalar mult, and $(\alpha, \beta) \in \ker f$. Suppose $(c, d) \in \ker f$. Then $0 = f(c,d)$ $= ca + db$ and it follows that $\ker f \subseteq \operatorname{im} g$ □

f) From d) we have $0 \to X \to R \times R \xrightarrow{f} R \xrightarrow{p} \operatorname{coker} f \to 0$.

Here $\operatorname{coker} f = R/\operatorname{im} f$, $p : R \to R/\operatorname{im} f$ is a projection. Because (α, β) are nonzero. $g(r) = 0 \leftrightarrow r = 0 \to g$ is injective. So $\ker g = 0$, $\ker f = \operatorname{im} g$

$\ker p = \operatorname{Im} f$ and $\operatorname{Im} p = \operatorname{coker} f \to$ sequence is exact.

g) R is a PID and a, b not both 0, $\to \gcd(a,b)$ is unique up to units, and $\exists$ $p, q \in R$ such $\gcd(a,b) = pa + qb$. ✓ From Lemma 19.

2.

3. a) Again by Lemma 1a, $(a,b) = \gcd(a,b)$ exists and is unique up to units, $\exists\, u, v \in R$ s.t. $(a,b) = au + bv$.

b) By def of gcd, $(a,b) \mid a$, $\to \exists\, r \in R$, s.t. $a = (a,b)r$.
Also $(a,b) \mid b \to \exists\, s \in R$ s.t. $b = (a,b)s$.
$ab = (a,b)[a,b]$ by Lemma 1a, a and $b \neq 0$.
PIDs are commutative $\to$ $(a,b)[a,b] = ab = (a,b)r(a,b)s$
$= (a,b)(a,b)rs \to [a,b] = (a,b)rs$.

c) $A = \begin{pmatrix} u & v \\ -s & r \end{pmatrix}$, $B = \begin{pmatrix} 1 & -vs \\ 1 & ur \end{pmatrix}$

Using previous results a) and b), $\begin{pmatrix} u & v \\ -s & r \end{pmatrix}\begin{pmatrix} r & -v \\ s & u \end{pmatrix} = \begin{pmatrix} ur+vs & -uv+vu \\ -sr+rs & sv+ru \end{pmatrix}$

$= I \to A$ has an inverse. $|A| = 1$.

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$\begin{pmatrix} 1 & -vs \\ 1 & ur \end{pmatrix}\begin{pmatrix} ur & vs \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} ur+vs & -urvs+vsur \\ ur-ur & vs+ur \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \to B$ has inverse,

$|B| = 1$.

d). $A\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}B = \begin{pmatrix} u & v \\ -s & r \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} 1 & -vs \\ 1 & ur \end{pmatrix} = \begin{pmatrix} au & bv \\ -as & br \end{pmatrix}\begin{pmatrix} 1 & -vs \\ 1 & ur \end{pmatrix}$

$= \begin{pmatrix} au+bv & -auvs+bvur \\ -as+br & ars^2+bur^2 \end{pmatrix} = \begin{pmatrix} (a,b) & 0 \\ 0 & (a,b)rsvs + (a,b)rsur \end{pmatrix}$

$= \begin{pmatrix} (a,b) & 0 \\ 0 & [a,b](vs+ur) \end{pmatrix} = \begin{pmatrix} (a,b) & 0 \\ 0 & [a,b] \end{pmatrix}$.

17. V vsp over field F, finite dimension $n>0$.

Let $B=\{f_1,\dots,f_n\} \subset V^*$. Assume $\exists\, v\in V,\ v\neq\vec{0}$ s.t. $f_i(v)=0\ \forall\ 1\leq i\leq n$.

Prove B is linearly independent. If $\exists\, v\in V$ s.t. $f_i(v)=0\ \forall i,\ v\neq 0$, then

$\{f_1,\dots,f_n\} \subseteq (\mathrm{Span}_F(v))^0$, and $\dim(\mathrm{Span}_F(v))=1$ since $v\neq 0$. We use

Thm 16 $\dim_F W+\dim_F W^0=\dim_F V$ where $W\subseteq V$. So $\dim(\mathrm{Span}_F(v))^0=\dim V-\dim(\mathrm{Span}_F(v))=n-1<n$.

So we conclude $\{f_1,\dots,f_n\}$ is in $n-1$ dimensional subspace of V^*.

V^* has dimension n, so there is some linear dependence in $\{f_1,\dots,f_n\}$. $\square$

Let V be finite-dim vsp over field F and let $T:V\to V$ be a linear transformation.

$c\in F$, suppose $\exists\, v\in V$ s.t. $T(v)=cv,\ v\neq\vec{0}$. Prove $\exists$ non-zero linear functional f on

s.t. $T^t(f)=cf$.

f For every $f\in V^*$, $(T^t f-cf)(v)=f(Tv-cv)=0$. Since $v\neq\vec{0}$, we know

$\exists$ some $g\in V^*$ s.t. $g(v)\neq 0$. So $\mathrm{im}(T^t-cI)\neq$ all of V^*. Then because V^* is a finite

mensional space, $\ker(T^t-cI)\neq\{0\}$, $\rightarrow \exists\, f\in V^*$ s.t. $T^t f-cf=0$ $\square$

. a) $(V/W)^* \longrightarrow W$ Show there exists a natural isomorphism.

3.

33. Let $W_1, W_2 \subseteq V$ be subspaces of V over F.

1) If $W_1 \neq W_2$, show $\exists\, f \in V^*$ s.t. $f = 0$ on either W_1 or W_2, but not both.

Let $W_1, W_2 \subseteq V$ be subspaces of V. Suppose $\mathcal{B} = \{ v_\alpha : \alpha \in A \}$ is a basis for V and $W_1 \neq W_2$. Then there $\exists$ $v_\alpha \in B$ s.t. v_α is in W_1 or W_2 but not in both. Not true. WLOG, we say $v_\alpha \in W_1$ and $v_\alpha \notin W_2$. Let $f \in V^*$ be defined as $f(v_\beta) = \begin{cases} 1 & \text{if } \beta = \alpha \\ 0 & \text{if } \beta \neq \alpha \end{cases}$ Then since $v_\alpha \notin W_2$, $f(w) = 0$ $\forall w \in W_2$, so f is zero on W_2. Clearly $f \neq 0$ on W_1 because $v_\alpha \in W_1$, $f(v_\alpha) = 1$.

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2) $W_1 = W_2 \Leftrightarrow W_1^0 = W_2^0$

$S^0 = \{ f \in V^* \mid f(s) = 0 \ \forall s \in S \}$

"$\rightarrow$" $W_1 = W_2 \rightarrow W_1^0 = W_2^0$ by definition.

"$\leftarrow$" Let $B = \{v_1, \ldots, v_n\}$ be a basis for V and $B^* = \{\phi_1, \ldots, \phi_n\}$ a dual basis.

$\phi_i(v_j) = \delta_{ij}$ (Kronecker Delta)

X

Let $W_1 = \mathrm{span}(v_k, \ldots v_e) = W_2$

$W_1^0 = \mathrm{span}(\phi_1, \ldots, \phi_{k-1}, \phi_{e+1} \ldots \phi_n) = W_2^0$

If $W_1^0 = W_2^0$, they are spanned by some elements of B^*. $W_1^0 = \mathrm{span}(\phi_1, \ldots, \phi_{k-1}, \phi_{e+1}, \ldots, \phi_n) = W_2^0$

$W_1 = \mathrm{span}(v_k \ldots v_e) = W_2$ □

34. V and W vsp over F, $T: V \to W$ linear transformation

a) $\ker T^t = (\operatorname{im} T)^0$

Note that $\ker T^t$ is $\{f \in W^* \mid T^t(f)(v) = f(T(v)) = 0\}$ $\forall v \in V$.

So $\ker T^t \subseteq (\operatorname{im} T)^0$. We can show the reverse inclusion. Let $f \in (\operatorname{im} T)^0$.

Then $f(w) = 0$ $\forall$ $w \in \operatorname{im}(T) \to f(T(v)) = 0$ $\forall v \in V$. So $(\operatorname{im}(T))^0 \subseteq \ker T^t$ $\square$

b) V and W have finite dimension.

Show i) $\operatorname{rank} T^t = \operatorname{rank} T$

ii) $\operatorname{im} T^t = (\ker T)^0$

For i) express T as $L_M: F^{n \times 1} \to F^{m \times 1}$ defined by $L_M(C) = MC$, where C is a column vector in $F^{n \times 1}$ and $M \in F^{m \times n}$. We use the result from pg. 13 "Dual Spaces" to see that $\operatorname{im} L_m \subseteq F^{m \times 1}$ is $\operatorname{Colspace}(M)$, and $\operatorname{rank} L_m = \dim(\operatorname{colspace}(M))$.

So apply standard bases and $[L]_{A,B} = M$, $[L^t]_{B,A} = M^t$

From Thm 24, $M \cong M^t$, $\to$ $\operatorname{rank} M = \operatorname{rank} M^t = \operatorname{rank} T = \operatorname{rank} T^t$.

ii) Suppose $f \in (\ker T)^0$. Then $f(v) = 0$ $\forall$ $v \in \ker T$. Since $T^t(f)(v) = f(T(v)) = f(0) = 0$

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$\forall$ $v \in \ker T$, $(\ker T)^0 \subseteq \operatorname{im} T^t$. Rank-nullity says $\dim(\operatorname{im} T) + \dim(\ker T) = n$.

Apply Thm 16 and clearly $\dim(\operatorname{im} T) + \dim(\ker T) = \dim(\ker T) + \dim((\ker T)^0)$

$\to \dim((\ker T)^0) = \dim(\operatorname{im} T) = \dim(\operatorname{im} T^t)$. Finally, because $(\ker T)^0 \subseteq \operatorname{im} T^t$, $(\ker T)^0 = \operatorname{im} T^t$ $\square$

c) If V, W are infinite dim, $(\ker T)^0 \subseteq \operatorname{im} T^t$ but equality is not necessarily true.

X

36. V is finite dim vsp over F.

$W_1, W_2 \subseteq V$ subspaces.

a) Show $(W_1 + W_2)^0 = W_1^0 \cap W_2^0$

Let $f \in (W_1 + W_2)^0$, then $f(w_1 + w_2) = 0$, $w_1 \in W_1$, $w_2 \in W_2$.
So $f(w_1) = 0$ and $f(w_2) = 0$ $\forall w_1 \in W_1, w_2 \in W_2 \rightarrow f \in W_1^0 \cap W_2^0$.
Conclude $(W_1 + W_2)^0 \subseteq W_1^0 \cap W_2^0$ □
Now suppose $f \in W_1^0 \cap W_2^0$. So $f(w_1) = 0$, $f(w_2) = 0$, $w_1 \in W_1$, $w_2 \in W_2$.
Then $f(w_1 + w_2) = 0$ and $f \in (W_1 + W_2)^0$. Therefore $W_1^0 \cap W_2^0 \subseteq (W_1 + W_2)^0$
and $W_1^0 \cap W_2^0 = (W_1 + W_2)^0$

b) Let $f \in (W_1 \cap W_2)^0$. So $f(w) = 0$ $\forall$ $w \in W_1 \cap W_2$.
Let $\mathcal{B}_1 = \{v_a, \ldots, v_h\}$ be a basis for W_1, $\mathcal{B}_2 = \{v_g, \ldots, v_m\}$ be a basis for W_2. Now $\mathcal{B}_1 \cap \mathcal{B}_2 = \{v_g, \ldots, v_h\}$ is a basis for $W_1 \cap W_2$. We can extend this to $\mathcal{B} = \{v_1, \ldots, v_h\}$ a basis of V, so $W_1 \cap \mathcal{B} = \mathcal{B}_1$ and $W_2 \cap \mathcal{B} = \mathcal{B}_2$. Define $g, h \in V^*$ by $g(v_i) = \begin{cases} 0 & \text{if } 1 \le i \le h \\ f(v_i) & \text{if } h+1 \le i \le n \end{cases}$

$h(v_i) = \begin{cases} 0 & \text{if } g \le i \le n \\ f(v_i) & \text{if } 1 \le i \le g-1 \end{cases}$

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Now $g \in W_1^0$, $h \in W_2^0$ and $f(v_i) = g(v_i) + h(v_i)$ for $1 \le i \le n$
$\rightarrow f = g + h$. Therefore $(W_1 \cap W_2)^0 \subseteq W_1^0 + W_2^0$.
Conversely, let $f \in W_1^0 + W_2^0$. Then $\exists$ $g, h \in W_1^0$ and W_2^0 resp. s.t. $f = g + h$. So $f(w) = g(w) + h(w) = 0 + 0 = 0$, $\forall w \in W_1 \cap W_2$.
Then $W_1^0 + W_2^0 \subseteq (W_1 \cap W_2)^0 \rightarrow W_1^0 + W_2^0 = (W_1 \cap W_2)^0$

c) If V is inf. dim. part a) still holds but b needs V to be finite dim. We can only say $W_1^0 + W_2^0 \subseteq (W_1 \cap W_2)^0$ but not conversely.

Why? Example?