The Matrix of a Linear Transformation

We begin by discussing examples of linear transformations given by multiplication by matrices. Let F be an arbitrary field and let m, n, and p be positive integers. Let $B \in F^{m\times n}$ be a matrix and define the function $L_B : F^{n\times 1} \longrightarrow F^{m\times 1}$ by $L_B(C) = BC$. Then L_B ("left multiplication by B") is a linear transformation by the properties of matrix multiplication:
for any $C, C_1, C_2 \in F^{n\times 1}$

$$\begin{aligned} L_B(C_1 + C_2) &= B(C_1 + C_2) \\ &= BC_1 + BC_2 \\ &= L_B(C_1) + L_B(C_2) \end{aligned}$$

and for $\alpha \in F$

$$\begin{aligned} L_B(\alpha C) &= B(\alpha C) \\ &= \alpha(BC) \\ &= \alpha L_B(C) . \end{aligned}$$

Similarly, it is easy to check that if also $B_1, B_2 \in F^{m\times n}$ we have $L_{B_1+B_2} = L_{B_1} + L_{B_2}$ and $L_{\alpha B} = \alpha L_B$. It then follows that the map

$$\mathcal{L} : F^{m\times n} \longrightarrow \operatorname{Hom}_F(F^{n\times 1}, F^{m\times 1}) \tag{1}$$

given by $\mathcal{L}(B) = L_B$ is itself a linear transformation. In fact, this map is an isomorphism of vector spaces:

Note that if we compute $L_B(e_j)$ where e_j is the j-th element in the standard basis for $F^{n\times 1}$ we get the j-th column of B. Hence if $B \in \ker\mathcal{L}$, then L_B is the 0 transformation and the columns of B must all be 0. Thus B is the 0 matrix, $\ker\mathcal{L} = 0$, and so $\mathcal{L}$ is one-to-one.

Now $\dim_F F^{m\times n} = \dim_F \operatorname{Hom}_F(F^{n\times 1}, F^{m\times 1})$ as both are mn. Thus $\mathcal{L}$ being one-to-one is also onto and hence an isomorphism.

Next let $A \in F^{p\times m}$ and $L_A : F^{m\times 1} \longrightarrow F^{p\times 1}$. Then we may compose $L_A \circ L_B : F^{n\times 1} \longrightarrow F^{p\times 1}$. For $C \in F^{n\times 1}$ we have

$$\begin{aligned} (L_A \circ L_B)(C) &= L_A(L_B(C)) \\ &= A(BC) \\ &= (AB)C \\ &= L_{AB}(C) \end{aligned}$$

and hence

$$L_A \circ L_B = L_{AB} \tag{2}$$

We can rephrase this by saying that the following diagram commutes:

$$\begin{array}{ccc} F^{p\times m} \times F^{m\times n} & \xrightarrow{\mathcal{L}\times\mathcal{L}} & \mathrm{Hom}_F(F^{m\times 1}, F^{p\times 1}) \times \mathrm{Hom}_F(F^{n\times 1}, F^{m\times 1}) \\ \Big\downarrow \cdot & & \Big\downarrow \circ \\ F^{p\times n} & \xrightarrow{\qquad\mathcal{L}\qquad} & \mathrm{Hom}_F(F^{n\times 1}, F^{p\times 1}) \end{array}$$

where $(\mathcal{L}\times\mathcal{L})(A,B) = (\mathcal{L}(A), \mathcal{L}(B))$ and the vertical maps are multiplication and composition, respectively. Here, $\mathcal{L}$ is to be interpreted as meaning the appropriate map which depends on the size of the matrices.

Note that both horizontal maps are isomorphisms of vector spaces and both vertical map are bilinear functions: that is, they are functions of two variables which are linear transformations in each variable separately when the other variable is fixed. For example,

$$(A_1 + A_2)\cdot B \;=\; A_1B + A_2B\,.$$

We'll see such functions later when we look at determinants and tensor products.

Further note that if $n = m = p$, then this says that

$$\mathcal{L}:\; F^{n\times n} \longrightarrow \mathrm{Hom}_F(F^{n\times 1}, F^{n\times 1})$$

also satisfies $\mathcal{L}(AB) = \mathcal{L}(A)\circ\mathcal{L}(B)$, that is, $\mathcal{L}$ is a ring isomorphism since clearly $\mathcal{L}(I) = 1_{F^{n\times 1}}$ (the identity function on columns of length n).

Bases and the Matrix of a Linear Transformation

Let U and V be finite dimensional vector spaces over the same field F. Let $\mathcal{A} = \{u_1.\dots,u_n\}$ be an ordered basis for U and $\mathcal{B} = \{v_1,\dots,v_m\}$ an ordered basis for V. Let $T: U \longrightarrow V$ be a linear transformation. Then there exist unique scalars $a_{ij}\in F$ such that

$$T(u_j) \;=\; \sum_{i=1}^{m} a_{ij}v_i\,. \tag{3}$$

The *matrix of T with respect to the ordered bases $\mathcal{A}$ and $\mathcal{B}$* is the matrix

$$[T]_{\mathcal{A},\mathcal{B}} \;=\; \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \in F^{m\times n}\,. \tag{4}$$

Note that the columns of this matrix are just the coordinates of $T(u_j)$ with respect to $\mathcal{B}$, $\Big[T(u_j)\Big]_{\mathcal{B}}$, so we could have also described the matrix as

$$[T]_{\mathcal{A},\mathcal{B}} \;=\; \big[\,[T(u_1)]_{\mathcal{B}},\dots,[T(u_n)]_{\mathcal{B}}\,\big]\,. \tag{5}$$

Theorem 1. *Let U and V be finite dimensional vector spaces over the field F with ordered bases $\mathcal{A}$ and $\mathcal{B}$, respectively. Then the function*

$$[\quad]_{\mathcal{A},\mathcal{B}} : \mathrm{Hom}_F(U,V) \longrightarrow F^{m\times n}$$

is an isomorphism of vector spaces.

Proof. We give a few of the details.

Linearity: If $T : U \longrightarrow V$ and $\alpha \in F$, then in equation (3) for αT all entries get multiplied by α. Hence all entries in equation (4) get multiplied by α as well. That is, $[\alpha T]_{\mathcal{A},\mathcal{B}} = \alpha[T]_{\mathcal{A},\mathcal{B}}$. Similarly by explicitly writing $S(u_j)$ and adding the corresponding equation to that for T (equation (3)) gives $[S+T]_{\mathcal{A},\mathcal{B}} = [S]_{\mathcal{A},\mathcal{B}} + [T]_{\mathcal{A},\mathcal{B}}$.

Onto: First, given a matrix $M \in F^{m\times n}$ with entries m_{ij}, by the UMP for bases there exists a linear transformation $S : U \longrightarrow V$ given by $S(u_j) = \sum_{i=1}^{m} m_{ij} v_i$. Hence by equation (4) $[S]_{\mathcal{A},\mathcal{B}} = M$, that is, $[\quad]_{\mathcal{A},\mathcal{B}}$ is onto.

One-to-one: If $T \in \ker[\quad]_{\mathcal{A},\mathcal{B}}$, then T is zero on each basis element of $\mathcal{A}$ and hence is the 0 linear transformation. □

Now take an arbitrary $u \in U$, write $u = \sum_{j=1}^{n} \alpha_j u_j$ and apply T:

$$\begin{aligned} T(u) &= \sum_{j=1}^{n} \alpha_j T(u_j) \\ &= \sum_{j=1}^{n} \alpha_j \sum_{i=1}^{m} a_{ij} v_i \\ &= \sum_{i=1}^{m} \Big(\sum_{j=1}^{n} a_{ij}\alpha_j\Big) v_i \end{aligned}$$

that is,

$$[T(u)]_{\mathcal{B}} = [T]_{\mathcal{A},\mathcal{B}}[u]_{\mathcal{A}} . \tag{6}$$

Note that this equation allows one to recover the definition of $[T]_{\mathcal{A},\mathcal{B}}$ if one has forgotten it: Applying (6) to a basis element u_j of $\mathcal{A}$ yields

$$\begin{aligned} \Big[T(u_j)\Big]_{\mathcal{B}} &= [T]_{\mathcal{A},\mathcal{B}}\Big[u_j\Big]_{\mathcal{A}} \\ &= [T]_{\mathcal{A},\mathcal{B}}\, e_j \end{aligned}$$

where e_j is the j-th element of the standard basis for $F^{n\times 1}$. Hence one obtains the j-th column of $[T]_{\mathcal{A},\mathcal{B}}$ as noted earlier.

We now give a description of this matrix as the solution of a universality question.

Theorem 2 (Universality of the Matrix of a Linear Transformation). *Let U and V be vector spaces over the field F with ordered bases $\mathcal{A}$ (of size n) and $\mathcal{B}$ (of size m), respectively. Then there exists a unique matrix $A \in F^{m\times n}$ such that the following diagram commutes:*

$$\begin{tikzcd} U \arrow[r, "T"] \arrow[d, "{[\ ]_{\mathcal{A}}}"] & V \arrow[d, "{[\ ]_{\mathcal{B}}}"] \\ F^{n\times 1} \arrow[r, "L_A"] & F^{m\times 1} \end{tikzcd}$$

that is, $[\ ]_{\mathcal{B}} \circ T = L_A \circ [\ ]_{\mathcal{A}}$ (i.e., $[T(u)]_{\mathcal{B}} = A[u]_{\mathcal{A}}$ for $u \in U$). Further, $A = [T]_{\mathcal{A},\mathcal{B}}$.

Proof. Note that by equation (6) $A = [T]_{\mathcal{A},\mathcal{B}}$ is one matrix which makes the diagram commute. But the remark after equation (6) asserts that the columns of this A are completely determined by the equation: that is, the matrix is unique. □

Proposition 3. *Let m and n be positive integers and F a field. Let the linear transformation*

$$[\ \ ]_{\mathcal{A},\mathcal{B}} : \operatorname{Hom}_F(F^{n\times 1}, F^{m\times 1}) \longrightarrow F^{m\times n}$$

be the matrix with respect to the standard bases $\mathcal{A}$ and $\mathcal{B}$. Then $[\,]_{\mathcal{A},\mathcal{B}}$ is the inverse of the isomorphism $\mathcal{L}$, that is

$$\mathcal{L} \circ [\ \ ]_{\mathcal{A},\mathcal{B}} = 1_{\operatorname{Hom}_F(F^{n\times 1}, F^{m\times 1})}$$

and

$$[\ \ ]_{\mathcal{A},\mathcal{B}} \circ \mathcal{L} = 1_{F^{m\times n}}$$

Change of Basis for Coordinates

We next determine precisely how things change when one chooses two different ordered bases for the same vector space. This is of interest for both theoretical and computational questions. Before we look at the matrix of a linear transformation, we first answer the question for coordinates with respect to a basis.

Theorem 4. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two ordered bases of size m for the vector space V over the field F. Then there exists a unique matrix $P = P(\mathcal{B}_1, \mathcal{B}_2) \in F^{m\times m}$ such that* *the following diagram commutes:*

$$\begin{tikzcd}
& F^{m\times 1} \arrow[dd, dotted, "L_P"] \\
V \arrow[ur, "{[\ \ ]_{\mathcal{B}_1}}"] \arrow[dr, "{[\ \ ]_{\mathcal{B}_2}}"'] & \\
& F^{m\times 1}
\end{tikzcd}$$

that is, $L_P \circ [\ \ ]_{\mathcal{B}_1} = [\ \ ]_{\mathcal{B}_2}$ *(i.e.,* $P\,[\,v\,]_{\mathcal{B}_1} = [\,v\,]_{\mathcal{B}_2}$ *for all* $v \in V$ *). Furthermore* P *is invertible.*

Proof. First note that there exists a unique linear transformation that makes the diagram commute and it is $[\ \ ]_{\mathcal{B}_2} \circ [\ \ ]_{\mathcal{B}_1}^{-1}$ (also note that these maps are both isomorphisms). Now any linear transformation from $F^{m\times 1}$ to itself is given by a unique L_P for some $P \in F^{m\times m}$ by the isomorphism of equation (1). Since $\mathcal{L}$ is an isomorphism of rings (see the last paragraph of the first section of these notes), L_P has an inverse if and only if P does. $\square$

Remark 5. 1. The argument used to determine the matrix of a linear transformation also works here to explicitly describe the columns of P: If $\mathcal{B}_1 = \{v_1, \ldots, v_m\}$ is the first basis, then

$$\begin{aligned} P\left[v_j\right]_{\mathcal{B}_1} &= \left[v_j\right]_{\mathcal{B}_2} \\ Pe_j &= \left[v_j\right]_{\mathcal{B}_2}. \end{aligned}$$

That is, the j-th column of P is just the column $\left[v_j\right]_{\mathcal{B}_2}$:

$$P = [\ [v_1]_{\mathcal{B}_2}, \ldots, [v_m]_{\mathcal{B}_2}\ ].$$

2. In many texts in Theorem 4 the matrix P is replaced by its inverse (and it's still called P). So the arrow for L_P points in the opposite direction; equivalently, in the equation form, P appears on the other side of the equation. That is, the matrix called the "change of basis matrix" elsewhere, may be the inverse of the one here. As usual, you need to be a bit cautious in comparing descriptions of the same material in different places, even though the exact same terminology is used.

We will now derive two conseqences of our discussion thus far. The following results can be thought of as studying what happens when
a) one fixes the bases of the vector spaces and composes linear transformations, or
b) one fixes the linear transformation and varies the bases of the domain and codomain.

Matrix Multiplication

We can now construct larger commutative diagrams by gluing together smaller ones, such as the ones we're already constructed. The situation is as follows: there are vector spaces U, V, W with finite ordered bases $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ of size n, m and p, respectively. Then one has a commutative diagram:

$$
\begin{tikzcd}
 & V \arrow[dr, "S"] \arrow[dd, "{[\ \ ]_{\mathcal{B}}}"] & \\
U \arrow[ur, "T"] \arrow[rr, "S \circ T"] \arrow[ddd, "{[\ \ ]_{\mathcal{A}}}"'] & & W \arrow[ddd, "{[\ \ ]_{\mathcal{C}}}"] \\
 & F^{m \times 1} \arrow[ddr, "L_{[S]_{\mathcal{B},\mathcal{C}}}"] & \\
 & & \\
F^{n \times 1} \arrow[uur, "L_{[T]_{\mathcal{A},\mathcal{B}}}"] \arrow[rr, "L_{[S \circ T]_{\mathcal{A},\mathcal{C}}}"'] & & F^{p \times 1}
\end{tikzcd}
$$

1. Theorem 2 gives a commuting left back square and a corresponding equation:

$$[\ \ ]_{\mathcal{B}} \circ T \; = \; L_{[T]_{\mathcal{A},\mathcal{B}}} \circ [\ \ ]_{\mathcal{A}}$$

2. Theorem 2 gives a commuting right back square and a corresponding equation:

$$[\ \ ]_{\mathcal{C}} \circ S \; = \; L_{[S]_{\mathcal{B},\mathcal{C}}} \circ [\ \ ]_{\mathcal{B}}$$

3. Theorem 2 gives a commuting front square and a corresponding equation:

$$[\ \ ]_{\mathcal{C}} \circ (S \circ T) \; = \; L_{[S \circ T]_{\mathcal{A},\mathcal{C}}} \circ [\ \ ]_{\mathcal{A}}$$

4. The top triangle commutes by definition.

5. Using the fact that all vertical maps in the diagram are isomorphisms, a simple computation (left to the reader) shows that the first 4 equations together imply that the bottom triangle commutes, that is

$$L_{[S \circ T]_{\mathcal{A},\mathcal{C}}} \; = \; L_{[S]_{\mathcal{B},\mathcal{C}}} \circ L_{[T]_{\mathcal{A},\mathcal{B}}} \, .$$

Recalling our initial discussion (applying the inverse of the isomorphism $\mathcal{L}$ to the last equation) yields

$$[S \circ T]_{\mathcal{A},\mathcal{C}} = [S]_{\mathcal{B},\mathcal{C}} \cdot [T]_{\mathcal{A},\mathcal{B}} . \tag{7}$$

This equation is the origin of matrix multiplication: that is, multiplication of matrices is **defined** to make this equation valid.

Proposition 6. *Under the same hypotheses the following diagram is commutative*

$$\begin{tikzcd}
\mathrm{Hom}_F(V,W) \times \mathrm{Hom}_F(U,V) \arrow[r, "\circ"] \arrow[d, "{[\ ]_{\mathcal{B},\mathcal{C}} \times [\ ]_{\mathcal{A},\mathcal{B}}}"] & \mathrm{Hom}_F(U,W) \arrow[d, "{[\ ]_{\mathcal{A},\mathcal{C}}}"] \\
F^{p\times m} \times F^{m\times n} \arrow[r, "\cdot"] & F^{p\times n}
\end{tikzcd}$$

where the top horizontal arrow denotes composition and the bottom horizonatal arrow denotes matrix multiplication. Further, the vertical arrows are isomorphisms of vector spaces and the horizontal arrows are bilinear functions.

Definition 7. Let U, V, W be vector spaces over the field F. A function

$$\beta : U \times V \longrightarrow W$$

is bilinear if for any $u_0 \in U$ and $v_0 \in V$ both of the restrictions $\beta(u_0, \ ) : V \longrightarrow W$ and $\beta(\ , v_0) : U \longrightarrow W$ are linear transformations.

Remark 8. Composition of linear transformations and multiplication of matrices are asserted to be examples of bilinear functions in the proposition. Other examples in linear algebra occur as inner (dot) products and tensor products. The latter will be constructed and discussed in the section "Tensor Products".

Proof. The fact that the diagram commutes is equivalent to the commutativity of the bottom rectangle in our large diagram (Equation 7). Matrix multiplication gives a bilinear function via the left and right distributive law for multiplication of matrices together with the fact that multiplication by a scalar commutes with matrix multiplication. A similar statement holds for composition of linear transformations. □

In the special case where $U = V$ and $n = m$, many times one calls a linear transformation in $\mathrm{Hom}_F(V,V)$ a *linear operator on* V or an *endomorphism of* V. We also write $\mathrm{End}_F(V)$ instead of $\mathrm{Hom}_F(V,V)$.

Theorem 9. *Let V be a vector space of dimension n over the field F with ordered basis $\mathcal{B}$. Then the function*

$$[\ \ ]_{\mathcal{B}} : \mathrm{End}_F(V) \longrightarrow F^{n\times n}$$

is an F-algebra isomorphism.

Proof. By Theorem 1 taking coordinates with respect to $\mathcal{B}$ is a vector space isomorphism. We have just shown that $[\ \ ]_{\mathcal{B}}$ respects multiplication as well. Further, $[1_V]_{\mathcal{B}} = I$ is easily checked. Finally, the ring multiplication and scalar multiplication are compatible follows by applying $[\ \ ]_{\mathcal{B}}$ to $\alpha(S \circ T) = (\alpha S) \circ T = S \circ (\alpha T)$ and using Equation 7 since $[\ \ ]_{\mathcal{B}}$ is a vector space homomorphism. □

Change of Basis for the Matrix of a Linear Transformation

Next we will determine precisely how the matrix of a linear transformation changes when we change the bases. Let U be a vector space over F of dimension n with ordered bases $\mathcal{A}_1$ and $\mathcal{A}_2$ having change of basis matrix $P = P(\mathcal{A}_1, \mathcal{A}_2)$. Let V be a vector space over F of dimension m with ordered bases $\mathcal{B}_1$ and $\mathcal{B}_2$ having change of basis matrix $Q = Q(\mathcal{B}_1, \mathcal{B}_2)$. Finally let $T \in \mathrm{Hom}_F(U, V)$. Then we obtain the following diagram:

$$
\begin{tikzcd}
U \arrow[rrr, "T"] \arrow[dr, "{[\ \ ]_{\mathcal{A}_1}}"] \arrow[dd, "{[\ \ ]_{\mathcal{A}_2}}"'] & & & V \arrow[dr, "{[\ \ ]_{\mathcal{B}_1}}"] \arrow[dd, "{[\ \ ]_{\mathcal{B}_2}}"'] & \\
& F^{n\times 1} \arrow[rrr, "L_{[T]_{\mathcal{A}_1,\mathcal{B}_1}}"] \arrow[dl, "L_P"] & & & F^{m\times 1} \arrow[dl, "L_Q"] \\
F^{n\times 1} \arrow[rrr, "L_{[T]_{\mathcal{A}_2,\mathcal{B}_2}}"'] & & & F^{m\times 1} &
\end{tikzcd}
$$

We now make an argument similar to the previous one:

1. The back square commutes (Theorem 2).

2. The front square commutes (Theorem 2).

3. The left end triangle commutes (Theorem 4).

4. The right end triangle commutes (Theorem 4).

5. Since all the maps in the two end triangles are isomorphisms, the 4 equations given by the preceding statements yield (please check!) that the bottom square commutes.

The conclusion (after applying the inverse of the isomorphism $\mathcal{L}$ and multiplying on the right by P^{-1}) is that

$$[T]_{\mathcal{A}_2,\mathcal{B}_2} = Q\,[T]_{\mathcal{A}_1,\mathcal{B}_1}\,P^{-1} \tag{8}$$

This condition on a pair of matrices arises in a number of contexts and motivates the following definition.

Definition 10. Matrices $A, B \in F^{m\times n}$ are called *equivalent* if there exist invertible matrices $Q \in F^{m\times m}$ and $P \in F^{n\times n}$ such that $B = QAP$.

It is easy to check that equivalence of matrices gives an equivalence relation on matrices in $F^{m\times n}$. An equivalent definition would arise if there were an inverse on the P since an invertible matrix always has an inverse. Recall that a square matrix has and inverse if and only if it is a product of a finite number of row and column operations. This symmetry in the definition then makes the following clear: Two matrices $A, B \in F^{m\times n}$ are equivalent if and only if B can be obtained from A be a sequence of row and column operations. One can then easily show:

Theorem 11. *Two matrices A, $B \in F^{m\times n}$ are equivalent if and only if their row and column reduced echelon forms are equal.*

Proof. See Exercise 5. □

Note that by equation (8) two matrices arise as the matrix of the same linear transformation via different pairs of bases if they are equivalent. Conversely given $A, B \in F^{m\times n}$ which are equivalent, then one easily checks that the linear transformation $T = L_A$ has B as its matrix for appropriately chosen bases:

Theorem 12. *Two matrices A, $B \in F^{m\times n}$ are equivalent if and only if they arise as the matrix of the same linear transformation with respect to different pairs of bases.*

Proof. See Exercise 6 □

In the important special case of $m = n$, $\mathcal{A}_1 = \mathcal{B}_1$ and $\mathcal{A}_2 = \mathcal{B}_2$, the discussion above gives a more restrictive condition in (8) since in this case one has $P = Q$. We thus also make the following definition:

Definition 13. Matrices $A, B \in F^{n\times n}$ are called *similar* if there exists an invertible matrix $P \in F^{n\times n}$ such that $B = PAP^{-1}$.

In an analogous fashion one sees that if $U = V$ and we use the same basis at each end of T, two matrices are similar if and only if they are the matrix of the same linear transformation from U to itself with respect to different bases:

Theorem 14. *Two matrices A, $B \in F^{n\times n}$ are similar if and only if they arise as the matrix of the same linear transformation with respect to different bases.*

Definition 15. If $T : U \longrightarrow V$ is a linear transformation, then the *rank* of T, denoted $\operatorname{rank} T$, is the dimension of the vector space $\operatorname{im} T$.

Remark 16. Let $\mathcal{B}$ be a basis for V. Since $[\ \ ]_{\mathcal{B}}$ is an isomorphism, $\operatorname{im} T \approx \operatorname{im}([\ \ ]_{\mathcal{B}} \circ T)$ have the same dimension. If U is a finite dimensional vector space and $\mathcal{A} = \{u_1, \ldots, u_n\}$ is an ordered basis, we have the matrix of T described by equation (5) as

$$[T]_{\mathcal{A},\mathcal{B}} = [\,[T(u_1)]_{\mathcal{B}}, \ldots, [T(u_n)]_{\mathcal{B}}\,] .$$

If we write a vector $u \in U$ as a linear combination of elements of the ordered basis $\mathcal{A}$ and apply T, we see that $[T(u)]_{\mathcal{B}}$ is the corresponding linear combination of the columns of $[T]_{\mathcal{A},\mathcal{B}}$. That is, $\operatorname{Span}_F\left(\left\{ \left[T(u_j)\right]_{\mathcal{B}} \,\middle|\, 1 \le j \le n \right\}\right) = \operatorname{im}([\ \ ]_{\mathcal{B}} \circ T)$. Thus $\operatorname{rank} T$ is the dimension of the column space of $[T]_{\mathcal{A},\mathcal{B}}$. One can show (see for example the first chapter of Hoffman and Kunze) this is the same as the row rank of $[T]_{\mathcal{A},\mathcal{B}}$, that is, just the rank of the matrix $[T]_{\mathcal{A},\mathcal{B}}$. When we study dual spaces, we'll give a different proof that the row rank and column rank of a matrix are equal.

Remark 17. Let $T : U \longrightarrow V$ be a linear transformation with U and V finite dimensional. Choose a basis $\mathcal{A}_2$ for $\ker T$ and enlarge to a basis $\mathcal{A} = \mathcal{A}_1 \dot\cup \mathcal{A}_2$ for U. Then $\mathcal{B}_1 = T(\mathcal{A}_1)$ is a linearly independent subset of V and can be enlarged to a basis $\mathcal{B}$ for V. Let n_i be the size of $\mathcal{A}_i$ and m_i the size of $\mathcal{B}_i$. So $n = n_1 + n_2$, $m = m_1 + m_2$, with $n_1 = m_1$. One easily checks now that $[T]_{\mathcal{A},\mathcal{B}}$ is an $m \times n$ matrix with the upper left hand corner an $n_1 \times n_1$ identity matrix and all other entries 0. That is, the matrix is in both row reduced and column reduced echelon form. The number $n_1 = m_1$ is precisely the rank of the matrix. It follows that two $m \times n$ matrices are equivalent if and only if they have the same rank. See Exercise 7.

Theorem 18. *Two matrices A, $B \in F^{m \times n}$ have equal row and column reduced echelon forms if and only if* $\operatorname{rank} A = \operatorname{rank} B$.

Proof. See Exercise 7. □

Corollary 19. *Two matrices A, $B \in F^{m \times n}$ are equivalent if and only if* $\operatorname{rank} A = \operatorname{rank} B$.

Proof. Just combine Theorem 11 with Theorem 18. □

In summary then the three conditions on two matrices of the same size

- equivalence,
- same row and column reduced echelon form,
- equal ranks,

all mean the same thing.

Examples

In the case of a linear transformation $T: V \longrightarrow V$ where $\mathcal{B}$ is a basis for V, we abbreviate $[T]_{\mathcal{B},\mathcal{B}}$ by writing simply $[T]_{\mathcal{B}}$.

We now give a few examples.

Example 20. Let F be a field and n a non-negative integer. We let $\mathcal{P}_n \subseteq F[x]$ be the subspace of all polynomials of degree less than n together with 0. So $\dim \mathcal{P}_n = n$. Now $\mathcal{B} = \{1, x, x^2, \ldots, x^{n-1}\}$ is an ordered basis for $\mathcal{P}_n$. If we consider $D: \mathcal{P}_n \longrightarrow \mathcal{P}_n$ to be the restriction to $\mathcal{P}_n$ of the usual derivative, it is given on this basis by $D(x^j) = jx^{j-1}$. Using bases $\mathcal{B}$ and $\mathcal{B}$ we easily obtain the matrix of D as having entry i in position $(i, i+1)$ and 0 elsewhere:

$$[D]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & n-2 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & n-1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Example 21. We continue with the same notation as in the previous example. Define the linear transformation $\Delta: \mathcal{P}_n \longrightarrow \mathcal{P}_n$ by $\Delta(f) = f(x+1) - f(x)$. If we compute on elements of the basis $\mathcal{B}$ we obtain

$$\begin{aligned} \Delta(x^j) &= (x+1)^j - x^j \\ &= \sum_{i=0}^{j-1} \binom{j}{i} x^i \end{aligned}$$

Hence the (i,j) entry of $[\Delta]_{\mathcal{B}}$ is the binomial coefficient $\binom{j-1}{i-1}$ for $i < j$ and 0 otherwise. For example if $n = 6$ we obtain

$$[\Delta]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & 3 & 6 & 10 \\ 0 & 0 & 0 & 0 & 4 & 10 \\ 0 & 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We will now give another version of this example, but with respect to a different basis. First a simple lemma.

Lemma 22. *Let $S = \{ f_i \mid i \in I \}$ be a subset of $F[x]$. Assume*

1. *$f_i \neq 0$ for $i \in I$,*

2. $\deg f_i \neq \deg f_j$ *for* $i \neq j$.

Then S *is linearly independent over* F .

This Lemma will in particular apply to collections of polynomials of the form $x^{(i)}$ given below. Let $i \geq 0$ be an integer. Define

$$x^{(0)} = 1$$

and for $i > 0$

$$x^{(i)} = x(x-1)\cdots(x-i+1)$$

(i factors) since $\deg x^{(i)} = i$.

Example 23. Take $\mathcal{A} = \{ x^{(0)}, x^{(1)}, x^{(2)}, \ldots, x^{(n-1)} \}$ which is a basis for $\mathcal{P}_n$ by the Lemma. We compute

$$\begin{aligned} \Delta(x^{(j)}) &= (x+1)^{(j)} - x^{(j)} \\ &= (x+1)x(x-1)\cdots(x+1-j+1) - x(x-1)\cdots(x-j+1) \\ &= x^{(j-1)}[(x+1)-(x-j+1)] \\ &= jx^{(j-1)} . \end{aligned}$$

Hence we obtain

$$[\Delta]_{\mathcal{A}} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & n-2 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & n-1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} .$$

Note that this is exactly the same matrix which was obtained for D using the standard basis $\mathcal{B}$:

$$[D]_{\mathcal{B}} = [\Delta]_{\mathcal{A}} .$$

Remark 24. The relationship between the matrices for D and Δ obtained in the examples should not be too surprising once one recalls the definition of the derivative in calculus – take the limit of $\frac{f(x+t)-f(x)}{t}$ as t goes to 0.

The operator Δ defined above can be defined on any vector space of functions by the same formula: $\Delta(f) = f(x+1) - f(x)$ (some use $f(x) - f(x-1)$ instead). It is usually referred to as a finite difference operator. Is is studied for continuous functions, differentiable functions, etc. in what is called the "calculus of finite differences" or the study of "difference equations". Results of this area of study are useful in both pure and applied mathematics.

Remark 25. One of the major problems studied and some of the most useful results in linear algebra involve finding a basis (or bases) so that the matrix of a given linear transformation takes a particularly nice form. Earlier you've seen the row reduced echelon form as well as the row and column reduced echelon form, both of which can be described this way. There are other such nice forms for matrices which arise: the Jordan canonical form, the rational canonical form, and the Smith normal form for example. We'll study some of these later in the course.

Exercises

MatLinTrans 1. Prove Lemma 22.

MatLinTrans 2. Compute the change of basis matrix $P = P(\mathcal{A}, \mathcal{B})$ and its inverse for the two bases $\mathcal{A}$ and $\mathcal{B}$ given for $\mathcal{P}_n$ in the Examples 20 and 23 above.

MatLinTrans 3. a) Let $T : \mathcal{P}_4 \longrightarrow \mathcal{P}_4$ be the linear transformation given by $T(f) = f'' + f' - 2f$. Using the computation in Example 20, compute the matrix associated to T with respect to the basis $\{1, x, x^2, x^3\}$.

b) Let $T : \mathcal{P}_5 \longrightarrow \mathcal{P}_5$ be the linear transformation given by $T(f) = f^{\langle 3\rangle} + 3f^{\langle 2\rangle} - f^{\langle 1\rangle} - 2f$, where $f^{\langle i\rangle} = \Delta f^{\langle i-1\rangle}$, and $f^{\langle 0\rangle} = f$. Using the computation in Example 23, compute the matrix associated to T with respect to the basis $\{1, x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)}\}$.

MatLinTrans 4. a) Show that equivalence and similarity of matrices are equivalence relations.

b) Show by example that equivalent matrices need not be similar.

MatLinTrans 5. Prove Theorem 11.

MatLinTrans 6. Complete the proof of Theorem 12

MatLinTrans 7. Using elementary operations prove Theorem 18:
Two matrices A, $B \in F^{m\times n}$ have the same row and column reduced echelon form if and only if they have the same rank.

MatLinTrans 8. This is a more detailed version of Exercise 2 above arising from the examples 20 and 23 given earlier. Define the entries in the change of basis matrices for the two different bases for $\mathcal{P}_n$ by the formulas:

$$x^{(m)} = \sum_{k=0}^{m} s(m,k)x^k$$

and

$$x^m = \sum_{k=0}^{m} S(m,k)x^{(k)} .$$

a) Verify that the $S(m,k)$ satisfy the recurrence relation

$$S(m,k) = kS(m-1,k) + S(m-1,k-1) .$$

b) Find a recurrence relation for the $s(m,k)$, and verify that it is correct.

c) Show that $S(m,k)$ equals the number of partitions of a set of size m into k non-empty pieces.

d) For each m and n compute the sum

$$\sum_{k=m}^{n} S(n,k)s(k,m) .$$

[Hint: What is this number?]

MatLinTrans 9. Let $T: V \longrightarrow V$ be a linear transformation. Let $B_n = \ker T^n$, and let $C_n = \operatorname{im} T^n$.

a) Show that $B_n \subseteq B_{n+1}$, and that $B = \bigcup B_i$ is a subspace of V.

b) Show that $C_n \supseteq C_{n+1}$, and that $C = \bigcap C_i$ is a subspace of V.

c) Show that if V is finite dimensional, $B = B_n$ for some n, and find a bound on n (depending on V) that is independent of T. Similarly, show that $C = C_n$ for some (possibly different) n.

d) Show that if V is finite dimensional, $V = B \oplus C$.

e) Is part (d) true if V is not finite dimensional?

f) Show that T maps B to B and C to C. Furthermore, show that if V is finite dimensional, T restricted to B is nilpotent (that is, $T^k = 0$ for some k) and T restricted to C is an isomorphism.

g) Use part (f) to show that any matrix in $F^{n\times n}$ is similar to a matrix of the form

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$$

where X is nilpotent (that is, $X^k = 0$ for some k), and Y is invertible.

MatLinTrans 10. Let U, V be vector spaces over the field F. Let $\operatorname{Hom}_F(U,V)$ be the set of all linear transformations $T: U \longrightarrow V$. Recall that $\operatorname{coker} T = V/\operatorname{im} T$ and T determines an exact sequence:

$$0 \longrightarrow \ker T \xrightarrow{i} U \xrightarrow{T} V \xrightarrow{p} \operatorname{coker} T \longrightarrow 0 .$$

i denotes the inclusion map and $p: V \longrightarrow V/\operatorname{im} T$ the natural surjective linear transformation. The rank of T is the dimension of $\operatorname{im} T$, that is, the cardinality of any basis for $\operatorname{im} T$.

a) For $S, T \in \operatorname{Hom}_F(U,V)$ we will say that S and T are *equivalent* if there exist invertible $P \in \operatorname{Hom}_F(V,V)$ and $Q \in \operatorname{Hom}_F(U,U)$ so that $S = PTQ$. Verify that this gives an equivalence relation on $\operatorname{Hom}_F(U,V)$.

b) Assume U is finite dimensional. Show that $S, T \in \operatorname{Hom}_F(U,V)$ are equivalent if and only if $\operatorname{rank} S = \operatorname{rank} T$.

c) If U and V are not finite dimensional, show that there are counter-examples to the previous statement.

d) For arbitrary vector spaces U, V over F, show that two linear transformations $S, T \in \mathrm{Hom}_F(U, V)$ are equivalent, if and only if all three of the following hold:

(1) $\dim \ker S = \dim \ker T$,

(2) $\dim \operatorname{im} S = \dim \operatorname{im} T$,

(3) $\dim \operatorname{coker} S = \dim \operatorname{coker} T$.

[Hint: Construct a basis for $\ker S$ and enlarge to a basis for U carefully. Similarly for T.]

e) In view of the previous part, characterize precisely when equivalence is determined by the rank.

f) In view of the previous parts, one is forced to find a characterization of rank different from that of part (a) for vector spaces of arbitary dimension. Let $S, T \in \mathrm{Hom}_F(U, V)$. We will say that S and T are *semi-equivalent*, if there exist $P_1, P_2 \in \mathrm{Hom}_F(V, V)$ and $Q_1, Q_2 \in \mathrm{Hom}_F(U, U)$ such that both of the following hold:

(1) $S = P_1 T Q_1$,

(2) $T = P_2 S Q_2$.

Show that semi-equivalence defines an equivalence relation. Further show that S and T are semi-equivalent if and only if $\operatorname{rank} S = \operatorname{rank} T$. [Note that two equations were necessary here as we do not assume that P_i or Q_i are invertible.]

MatLinTrans 11. a) Let R be a commutative ring. Matrices $A, B \in R^{m\times n}$ are said to be *equivalent* if there exist invertible matrices $Q \in R^{m\times m}$ and $P \in R^{n\times n}$ so that $B = QAP$. $C, D \in R^{n\times n}$ are *similar* if there exists $P \in R^{n\times n}$ such that $D = PCP^{-1}$. Show that these yield equivalence relations.

b) Let F be a field and $F \subseteq F[X]$. Show that if A, $B \in F^{n\times n}$ are similar, then $xI - A$, $xI - B \in F[x]^{n\times n}$ are equivalent.

Remark: It is true, but much more difficult to prove, that the converse is true as well.

MatLinTrans 12. Let V be a vector space of dimension n over the field F, and let $T : V \longrightarrow V$ be a linear transformation such that $T^n = 0$, so T is nilpotent. Assume also that $T^{n-1} \neq 0$. Suppose $v \in V$ is not in the kernel of T^{n-1}. Prove that $\mathcal{B} = \{ v, T(v), \ldots, T^{n-1}(v) \}$ is a basis for V. Compute the matrix of T with respect to the basis $\mathcal{B}$. Let $c \in F$ and define $S : V \longrightarrow V$ be given by $S(u) = cu + T(u)$. Compute the matrix of S with respect to $\mathcal{B}$. Compute the matrix of T^2 with respect to $\mathcal{B}$.

MatLinTrans 13. Let V be finite dimensional over the field F. Let $S, T \in \mathrm{Hom}_F(V,V)$ be such that $ST = 1_V$. Show that there exists a polynomial $f \in F[x]$ such that $S = f(T)$.

MatLinTrans 14. Let $V = U = \mathbb{R}^{3\times 2}$. Let $T : V \longrightarrow U$ be the linear transformation given by the formula $T(B) = AB$ where

$$A = \begin{bmatrix} 2 & 5 & 0 \\ 0 & 1 & -1 \\ 2 & 1 & 4 \end{bmatrix}.$$

Choose ordered bases for V, U and compute the matrix of T with respect to these bases. Find the rank of T and find a basis for the kernel of T.

MatLinTrans 15. Let F be a field and let V be the vector space consisting of 0 together with all polynomials of degree n or less. Define the function α on elements of V by

$$\alpha(p(x)) = \frac{d}{dx}(x^n \cdot p(\frac{1}{x}))$$

for $p(x) \in V$. Show the following:

a. $\alpha(p(x)) \in V$ for $p(x) \in V$.

b. α is a linear transformation. Compute the matrix $[\alpha]_{\mathcal{B},\mathcal{B}}$ with respect to the standard basis $\mathcal{B} = \{1, x, \dots, x^n\}$.

MatLinTrans 16. Let V be the vector space over the complex numbers $\mathbb{C}$ of all functions from $\mathbb{R}$ into $\mathbb{C}$, i.e., the space of all complex-valued functions on the real line. Let $f_1(x) = 1$, $f_2(x) = e^{ix}$, $f_3(x) = e^{-ix}$.

a. Prove that f_1, f_2, and f_3 are linearly independent over $\mathbb{R}$.

b. Let $g_1(x) = 1$, $g_2(x) = \cos x$, $g_3(x) = \sin x$. Show that g_1, g_2, and g_3 are linearly independent over $\mathbb{R}$. Further find an invertible 3×3 matrix P satisfying

$$g_j = \sum_{i=1}^{3} P_{ij} f_i$$

MatLinTrans 17. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation which rotates a vector counterclockwise by $\pi/2$ around the axis given by the vector $(1,1,1)^t$. Write the matrix $[T]$ of T with respect to the standard basis.

MatLinTrans 18. Let T denote the linear transformation from $F^{2\times 2}$ to $F^{2\times 2}$ defined by $T(X) = AX - XA$, where $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Find the matrix of T with respect to the standard basis of $F^{2\times 2}$. What are $\dim \ker T$ and $\dim \operatorname{im} T$?

MatLinTrans 19. Let θ be a real number. Prove that the following two matrices are similar over the field of complex numbers:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}.$$

Hint: Let $T : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ be the linear transformation which is represented by the first matrix with respect to the standard ordered basis. Find vectors v_1 and v_2 such that $T(v_1) = e^{i\theta}v_1$, $T(v_2) = e^{-i\theta}v_2$ and $\{v_1, v_2\}$ is a basis for $\mathbb{C}^2$.

MatLinTrans 20. Let V be a finite-dimensional vector space over the field F and let S and T be linear transformations from V to V. When do there exist ordered bases $\mathcal{B}$ and $\mathcal{B}'$ for V such that

$$[S]_{\mathcal{B},\mathcal{B}} = [T]_{\mathcal{B}',\mathcal{B}'}\ ? \qquad (\star)$$

Prove that such bases exist if and only if there is an invertible linear transformation $U : V \longrightarrow V$ such that $T = USU^{-1}$.

Outline of proof: If the equation $(\star)$ holds, let U be the linear transformation which sends $\mathcal{B}$ to $\mathcal{B}'$, and show that $S = UTU^{-1}$. Conversely, if $T = USU^{-1}$ for some invertible U, let $\mathcal{B}$ be any ordered basis for V and let $\mathcal{B}'$ be its image under U. Then show that the required equation $(\star)$ holds.

MatLinTrans 21. a. Show that the linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $T(x_1, x_2) = (x_1, 0)$ is represented with respect to the standard ordered basis of $\mathbb{R}^2$ by the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

and satisfies $T^2 = T$.

b. Prove that if $S : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a linear transformation which satisfies $S^2 = S$, then $S = 0$, or $S = I$, or there is an ordered basis $\mathcal{B}$ for $\mathbb{R}^2$ such that $[S]_{\mathcal{B},\mathcal{B}} = A$.

MatLinTrans 22. Let V be an n-dimensional vector space over the field F, and let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be an ordered basis for F.

a. By the Universal Mapping Property of bases there exists a unique linear transformation $T : V \longrightarrow V$ such that $T(v_j) = v_{j+1}$ for $1 \le j \le n-1$, and $T(v_n) = 0$. Give the matrix of T with respect to the ordered basis $\mathcal{B}$ (used as the basis at each end of T).

b. Prove that $T^n = 0$ but $T^{n-1} \ne 0$.

c. Let $S : V \longrightarrow V$ be any linear transformation such that $S^n = 0$ but $S^{n-1} \ne 0$. Prove that there exists an ordered basis $\mathcal{B}'$ so the the matrix of S with respect to this basis is the same matrix as found in part a.

d. Prove that if M and N in $F^{n\times n}$ are such that $M^n = N^n = 0$, but $M^{n-1} \neq 0$ and $N^{n-1} \neq 0$, then M and N are similar matrices.

MatLinTrans 23. Let $T : V \longrightarrow V$ be a linear transformation on a finite dimensional vector space V of dimension n. For $i \geq 0$, let $W_i = \ker(T^i)$ and $k_i = \dim(W_i)$, where $T^0 = I$. In this problem, you will investigate possibilities for the sequence $(k_0, k_1, k_2, \ldots)$. In particular, you will show that successive differences cannot increase. In other words, if the dimension of the kernel increases by some amount m at a particular step, then at each future step, it cannot increase by more than m.

a. A Simple Example: Assume T is nilpotent with $T^{n-1} \neq 0$. Compute the sequence k_i for T. (The extra assumption on T holds ONLY in this part.)

b. Prove that $k_{i+1} \geq k_i$ for $i \geq 0$.

c. Prove that $k_2 - k_1 \leq k_1 - k_0$.

d. Prove that $k_{i+2} - k_{i+1} \leq k_{i+1} - k_i$ in general. (Hint: Induction is not necessary. Consider induced maps on appropriate quotient spaces such as W_{i+1}/W_i or V/W_i.)

e. Let $T_i : V_i \longrightarrow V_i$ be linear transformations on the finite-dimensional vector spaces V_i, for $i = 1, 2$. Determine the sequence for $T_1 \oplus T_2 : V_1 \oplus V_2 \longrightarrow V_1 \oplus V_2$ in terms of the sequences for T_i. [Here $(T_1 \oplus T_2)(v_1, v_2) = (T_1(v_1), T_2(v_2))$.]

f. There is a sort of converse which states that if $(k_0, k_1, k_2, \ldots)$ is a sequence of non-negative integers with $k_{i+1} \geq k_i$, $k_{i+2} - k_{i+1} \leq k_{i+1} - k_i$, and $k_i \leq n$ for $i \geq 0$, and also $k_0 = 0$, then there exists a linear transformation $T : F^n \longrightarrow F^n$ with $\dim(\ker(T^i)) = k_i$ for $i \geq 0$. Find a 6×6 matrix in row reduced echelon form which gives the sequence $(0, 3, 5, 5, 5, \ldots)$?

$\boxed{\text{g.}}$ Carefully state and prove the converse.

MatLinTrans 24. A linear transformation $T : V \longrightarrow V$ is called *nilpotent* if $T^i = 0$ for some $i > 0$. If $\dim(V) = n < \infty$ and T is nilpotent, prove that $T^n = 0$.

MatLinTrans 25. Let $E = \mathrm{Hom}_F(V, V)$ where V is a vector space of dimension n over the field F. An element $\alpha \in E$ is called *nilpotent* if $\alpha^k = 0$ for some integer $k \geq 1$. The smallest integer k such that $\alpha^k = 0$ but $\alpha^{k-1} \neq 0$ is called the *nilpotency index* of α. Show that:

a. The nilpotency index of a nilpotent $\alpha \in E$ is at most n.

b. If $\alpha, \beta \in E$ are both nilpotent and $\alpha\beta = \beta\alpha$, show that $\alpha + \beta$ and $\alpha\beta$ are both nilpotent.

c. Give an example where $\alpha, \beta \in E$ are nilpotent, but do not commute and $\alpha + \beta$ is not nilpotent.

d. If $\alpha \in E$ is nilpotent with nilpotency index k, I is the identity element of E and $f_i \in F$, show that

$$\gamma = f_0 I + f_1 \alpha + f_2 \alpha^2 + \cdots + f_{k-a} \alpha^{k-1}$$

is invertible if and only if $f_0 \neq 0$.

e. If $f_0 \neq 0$, compute γ^{-1}.

MatLinTrans 26. Let m, n, s be positive integers and F a field. For $A \in F^{m\times n}$ define $T : F^{n\times s} \longrightarrow F^{m\times s}$ by $T(M) = AM$ (matrix multiplication by A on the left).

a. If $\mathcal{A}$ is an ordered basis for $F^{n\times s}$ and $\mathcal{B}$ is an ordered basis for $F^{m\times s}$, give the dimensions of the matrix for $[T]_{\mathcal{A},\mathcal{B}}$.

b. Using the standard bases for $F^{n\times s}$ and $F^{m\times s}$, compute $[T]_{\mathcal{A},\mathcal{B}}$ in terms of A. Note that you MUST choose an ordering for the bases used; a nice choice will substantially simplify the problem.

c. Give, and prove, a formula for the rank and nullity of the matrix $[T]_{\mathcal{A},\mathcal{B}}$ in terms of A.

d. Similarly $B \in F^{n\times m}$ define $S : F^{s\times n} \longrightarrow F^{s\times m}$ by $S(M) = MB$ (matrix multiplication by B on the right). Repeat all of the parts above for S, including finding a really nice matrix for S by ordering the standard bases carefully. Explain how you choose this ordering (which should be different from that chosen above for A), that is, what facts are you exploiting?

MatLinTrans 27. Let R be a ring. An element $e \in R$ is called *idempotent* if $e^2 = e$. Let V be a vector space over the field F and let $T \in \operatorname{Hom}_F(V,V)$ be an idempotent linear transformation.

a. Prove that $V = \operatorname{im} T \oplus \ker T$.

b. If V is finite dimensional, let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ be a basis for V where $\mathcal{B}_1$ is a basis for $\operatorname{im} T$ and $\mathcal{B}_2$ is a basis for $\ker T$. Compute the matrix $[T]_{\mathcal{B}}$.

c. Let $A, B \in F^{n\times n}$ be two idempotent matrices. Prove that A and B are similar if and only if they have the same rank by considering the linear transformations L_A and L_B and applying the previous parts.

MatLinTrans 28. Let V be a finite dimensional vector space over the field F. Let $T : V \longrightarrow V$ be a linear transformation and let W be a T-invariant subspace of V (i.e., $T(w) \in W$ for all $w \in W$). Let $T_0 : W \longrightarrow W$ be the restriction of T to W. Let $\mathcal{A} = \{ w_1, \ldots, w_m \}$ be a basis for W and enlarge this to a basis $\mathcal{B} = \{ w_1, \ldots, w_m, v_1, \ldots, v_n \}$ for V. Then $\mathcal{C} = \{ v_1 + W, \ldots, v_n + W \}$ is a basis for V/W. Let $T_1 : V/W \longrightarrow V/W$ be the linear transformation given by $T_1(v + W) = T(v) + W$.

a. Show that

$$[T]_{\mathcal{B},\mathcal{B}} = \begin{bmatrix} [T_0]_{\mathcal{A},\mathcal{A}} & J \\ 0 & [T_1]_{\mathcal{C},\mathcal{C}} \end{bmatrix}$$

where 0 denotes the $n \times m$ matrix of zeros and J is some $m \times n$ matrix.

b. Now assume that the field F has the following special property:
For every linear transformation $T : V \longrightarrow V$ *on a finite dimensional vector space, there exists a scalar* $c \in F$ *and some non-zero vector* $v \in V$ *so that* $T(v) = cv$ *.*

For example, the complex numbers have this property as we will see later. In this situation, prove (by giving an induction argument and quotient spaces) that for any linear transformation there exists a basis so that the matrix of the linear transformation with respect to this basis is upper triangular. (A matrix is *upper triangular* if every entry below the main diagonal is 0.)

MatLinTrans 29. Let U, V, U_1, U_2, V_1, V_2 be arbitrary vector spaces over the field F (Note that their dimensions need not be finite).

a. Show that there is an isomorphism

$$\mathrm{Hom}_F(V, U_1 \oplus U_2) \longrightarrow \mathrm{Hom}_F(V, U_1) \oplus \mathrm{Hom}_F(V, U_2) \quad .$$

b. Show that there is an isomorphism

$$\mathrm{Hom}_F(V_1 \oplus V_2, U) \longrightarrow \mathrm{Hom}_F(V_1, U) \oplus \mathrm{Hom}_F(V_2, U) \quad .$$

c. Put parts (a) and (b) together and give a description of $\mathrm{Hom}_F(V_1 \oplus V_2, U_1 \oplus U_2)$ as a direct sum of 4 vector spaces.

d. Let $f_{ij} : V_j \longrightarrow U_i$ be a linear transformation. Then by part (c) we can use these four linear transformations to describe a linear transformation from $V_1 \oplus V_2$ to $U_1 \oplus U_2$. Write these in the form of a matrix:

$$\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \quad .$$

We will call this matrix M the *decomposition matrix* of the linear transformation. Note that it depends on the particular way in which we write the vector spaces as direct sums. If $T : V_1 \oplus V_2 \longrightarrow U_1 \oplus U_2$ is a linear transformation, give explicit formulas for the f_{ij} in terms of the canonical injections and surjections (i.e., $p_1 : V_1 \oplus V_2 \longrightarrow V_1$ defined by $p_1(v_1, v_2) = v_1$, etc.). We will denote this matrix by $[T]$ (it will be too cumbersome to include its dependency on the direct sums, but keep in mind that it does). Now if we write elements of the direct sums in columns we can apply these matrices of linear transformations as follows:

$$M \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} f_{11}(v_1) + f_{12}(v_2) \\ f_{21}(v_1) + f_{22}(v_2) \end{pmatrix} \quad .$$

e. If W_1, W_2 are also vector spaces over F, consider

$$\mathrm{Hom}_F(V_1 \oplus V_2, U_1 \oplus U_2) \times \mathrm{Hom}_F(W_1 \oplus W_2, V_1 \oplus V_2) \longrightarrow \mathrm{Hom}_F(W_1 \oplus W_2, U_1 \oplus U_2)$$

given by composition, that is, (T, S) goes to TS. Show that in terms of our matrix notation above

$$[TS] = [T][S]$$

where the operation on the right-hand side is given by (you guessed it) matrix multiplication.

f. Assume these vector spaces have finite dimension over F. Choosing bases $\mathcal{A}_j$ for V_j and $\mathcal{B}_i$ for U_i construct bases $\mathcal{A}$ for $V_1 \oplus V_2$ and $\mathcal{B}$ for $U_1 \oplus U_2$ (as was done in class to prove the formula for the dimension of the direct sum). Let $T : V_1 \oplus V_2 \longrightarrow U_1 \oplus U_2$ be a linear transformation. Give a formula relating the matrix $[T]_{\mathcal{A},\mathcal{B}}$ to the matrices of the f_{ij} (see part (d) above) given in terms of $\mathcal{A}_j, \mathcal{B}_i$. What can you deduce applying this to the formula in part (e)?

g. State (but do not prove) the generalization of (a)-(f) to direct sums with a larger finite number of summands.

History of the Notes

The Notes for the course *Math 4330, Honors Linear Algebra* at Cornell University have been developed over the last ten years or so mainly by the following (in chronological order):

Gerhard O. Michler

R. Keith Dennis

Martin Kassabov

W. Frank Moore

Yuri Berest.

Harrison Tsai also contributed a number of interesting exercises that appear at the ends of several sections of the notes.

Most sections have been revised so many times the original author may no longer recognize it. The intent is to provide a modern treatement of linear algebra using consistent terminology and notation. Some sections are written simply to provide a central source of information such as those on "Useful Definitions", "Subobjects", and "Universal Mapping Properties" rather than as a chapter as one might find in a traditional textbook. Additionally there are sections whose intent is to provide proofs of some results which are not given in the lectures, but rather provide them as part of a more thorough development of a tangential topic (e.g., Zorn's Lemma to develop cardinal numbers and the existence of bases and dimension in the general case).

A large number of challenging exercises from many different sources have been included. Although most should be readily solvable by students who have mastered the material, a few even more challenging ones still remain.

Much still remains to be done. Corrections and suggestions for additional exercises, topics and supplements are always welcome.

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