Quotient Spaces

Let V be a vector space over the field F and let $W \subseteq V$ be a subspace. We now contruct a new vector space over F with a method analogous to that used for constructing the integers modulo n. We temporarily define a "congrunce relation" using the subspace W. We will say that two vectors $x, y \in V$ are *congruent modulo* W, and write $x \equiv y \bmod W$ when $x - y \in W$. This gives an equivalence relation on V:

r. $x \equiv x \bmod W$,

s. $x \equiv y \bmod W$ implies $y \equiv x \bmod W$,

t. $x \equiv y \bmod W$ and $y \equiv z \bmod W$ implies $x \equiv z \bmod W$.

The relation being reflexive is just $x - x \in W$ which holds since 0 is in a subspace. The relation is symmetric since $x - y \in W$ implies $y - x \in W$ since a subspace is closed under negation.
The transitivity of the relation holds as $x - y \in W$ and $y - z \in W$ implies that $x - z = (x - y) + (y - z) \in W$, since any subspace is closed under addition.

Next note that the equivalence class of $v \in V$ under this equivalence relation is just the set

$$\begin{aligned} \text{class}(v) &= \{ u \in V \mid u \equiv v \bmod W \} \\ &= \{ v + w \mid w \in W \} \\ &= v + W \end{aligned}$$

This follows as $u \equiv v \bmod W$ means $u - v \in W$, that is $u - v = w$ for some $w \in W$. Thus $u = v + w$. Conversely any such u is congruent to $v \bmod W$. We this define this set to be $v + W$ in analogy with our earlier notation for the sum of two subspaces. Such a subset is called a *coset* of W.

Recall that an equivalence relation on a set decomposes the set into a disjoint union of the equivalence classes. The quotient set is the set whose elements are these pieces. We now look at the set of these pieces:

We write V/W for the set of the equivalence classes of $V \bmod W$:

$$V/W = \{ v + W \mid v \in V \} .$$

We now make this set into a vector space over F by defining addition and scalar multiplication:

$$\begin{aligned} (v_1 + W) + (v_2 + W) &= (v_1 + v_2) + W \\ a(v + W) &= (av) + W \end{aligned}$$

We must first show that these definitions make sense, or as a mathematician says, "that they are well-defined" (see the remark near the end of the handout on equivalence relations). There is a problem – the v in the $v+W$ may be any element of the set $v+W$, that is any representative of the coset. That is, any element $v+w$ for any $w \in W$. Note that then $v+W = v'+W$ if and only if $v-v' \in W$ (if and only if $v \equiv v' \mod W$).

Hence

$$\begin{array}{rcccc} v_1 + W & = & v_1' + W & \iff & v_1 - v_1' \in W \\ v_2 + W & = & v_2' + W & \iff & v_2 - v_2' \in W \\ \text{and adding gives} & & & & \\ (v_1+v_2)+W & = & (v_1'+v_2')+W & \iff & (v_1+v_2)-(v_1'+v_2') \in W\ . \end{array}$$

A similar argument works for scalar multiplication:

$$\begin{array}{rcccc} v + W & = & v' + W & \iff & v - v' \in W \\ \text{and multiplying by } a \text{ gives} & & & & \\ (av)+W & = & (av')+W & \iff & av - av' \in W\ . \end{array}$$

It is now easy to check that V/W is a vector space over F using these definitions. For example, since $0 \in V$ is the zero in V, $0+W = W$ is the zero of V/W. It then follows that $-(v+W) = (-v)+W$, since it behaves the correct way, and by uniqueness must be the only element that does.

There are several axioms to check, but the philosophy is simple: The corresponding result holds for V/W by using the definition and putting a number of instances of "$+W$" on the axiom for V. For example

$$\begin{aligned} a\cdot(b\cdot(v+W)) &= a\cdot((bv)+W) \\ &= (a(bv))+W \\ &= ((ab)v)+W \\ &= (ab)\cdot(v+W) \end{aligned}$$

Example 1. Consider $\mathbb{R}^2$, the Euclidean plane, and W a one-dimensional subspace (geometrically a line passing through the origin). It is easy to check that for $v \in \mathbb{R}^2$, $v+W$ is the line parallel to W that passes through v. The quotient space $\mathbb{R}^2/W$ consists of the set of all lines in $\mathbb{R}^2$ which are parallel to W.

The same sort of thing happens in higher dimensions (e.g., for W a plane through the origin in $\mathbb{R}^3$ and $v \in \mathbb{R}^3$, $v+W$ is just the plane parallel to W which contains v). For that reason these equivalence classes are sometime called *affine subspaces.* The quotient space consists of the set of all such affine subspaces parallel to the given W.

We next look at the function

$$p:\ V \longrightarrow V/W$$

which sends each vector in V to the coset (equivalence class) in which it lies, $p(v) = v+W$. Note that

- p is onto
- p is a linear transformation
- $\ker p = W$.

The first follows from the definition of V/W: it is the set of all such $p(v) = v + W$. The second is equivalent to the definition of addition and scalar multiplication in V/W:

$$\begin{aligned} p(u+v) &= (u+v)+W \\ p(u)+p(v) &= (u+W)+(v+W) \end{aligned}$$

and

$$\begin{aligned} p(av) &= (av)+W \\ ap(v) &= a(v+W)\ . \end{aligned}$$

For the last we have $\ker p = \{ v \in V \mid p(v) = 0 \}$, but $p(v) = v + W = 0 + W$ just means that $v = v - 0 \in W$.

We finally consider the special role that this linear transformation $p: V \longrightarrow V/W$ plays for quotient spaces.

Theorem 2 (Universal Mapping Property for Quotient Spaces). *Let V and U be vector spaces over the field F and W a subspace of V. For every linear transformation $t: V \longrightarrow U$ which satisfies $W \subseteq \ker t$, there exists a unique linear transformation $T: V/W \longrightarrow U$ such that the following diagram commutes:*

$$\begin{tikzcd} V \arrow[r, "p"] \arrow[dr, "t"'] & V/W \arrow[d, dotted, "T"] \\ & U \end{tikzcd}$$

that is, $T \circ p = t$.

Remark 3. The term "commutes" will more generally mean that if one has a diagram with a number of objects (e.g., vector spaces, fields, or whatever) with a number of functions (arrows) between some of the objects, we say that the diagram *commutes* if for every pair of objects which can be connected by a path (all arrows pointing the same direction so that composition of the functions is possible) in more than one way, the compositions of the functions along the various possible paths must always be equal.

Hence an alternative way of stating the preceding theorem is

If the left triangle below commutes, then there exists a unique linear transformation $T:\ V/W \longrightarrow U$ making the right trangle commute.

$$\begin{tikzcd}
W \arrow[r] \arrow[dr, "0"'] & V \arrow[r, "p"] \arrow[d, "t"] & V/W \arrow[dl, dotted, "T"] \\
& U &
\end{tikzcd}$$

Proof. In outline, in almost all cases, proofs of universal mapping properties take the following form: first show uniqueness, next use the result of uniqueness (typically a formula) to show existence of the sought-after function, and finally verify that the function just constructed has all of the right properties.

Uniqueness: We show that there is only one linear transformation T that satisfies the equation $T \circ p = t$. If such a T exists, we have

$$T(v + W) = T(p(v)) = t(v) \tag{1}$$

by the definition of p and the definition of composition of functions. That is, the value of T is completely determined by the given function t.

Existence: The result of the previous part (equation (1)) is now used to define T. However, we must show that T is well-defined, since more than one vector can represent the coset $v + W$. If $v + W = v' + W$ then $v - v' \in W$, and we must show that $t(v)$ and $t(v')$ yield the same thing. But $v - v' \in W \subseteq \ker t$ so $t(v - v') = 0$ and as t is a linear transformation we do indeed have $t(v) = t(v')$, the value defined for T on the coset $v + W$.

Properties: First, T is a linear transformation:

$$\begin{aligned}
T((u + W) + (v + W)) &= T((u + v) + W) \\
&= t(u + v) \\
T(u + W) + T(v + W) &= t(u) + t(v)
\end{aligned}$$

and

$$\begin{aligned}
T(a(v + W)) &= T((av) + W) \\
&= t(av) \\
aT(v + W) &= at(v).
\end{aligned}$$

In each case the last two are equal since t is a linear transformation. Finally note that $T \circ p = t$ holds since the equation (1) used to define T was exactly that condition. $\square$

Remark 4. Upon comparing the Universal Mapping Property(UMP) for Quotient Sets to the UMP for Quotient Spaces, one sees that the condition 't is constant on the fibers of $p\colon V \to V/W$' has been replaced by '$W \subseteq \ker t$'. Because t is a linear transformation, verification of the condition for a single fiber suffices: t is constant on all fibers of p if and only if $W \subseteq \ker t = t^{-1}(0) =$ the fiber over 0. Verify and explain!

Remark 5. If we had already proven Proposition 1 (the Universal Mapping Property for Quotient Sets) of the handout on Equivalence Relations, we could simply quote that result, as well as Remark 4 to obtain the existence and uniqueness of the function T above. We would then only need to verify that T is a linear transformation.

For F a field and U, V vector spaces over F, $\mathrm{Hom}_F(U,V)$ denotes the set of all linear transformations $T: U \longrightarrow V$. This set is a vector space via the usual addition of linear transformations and multiplication by scalars:
For $T_1, T_2 : U \longrightarrow V$ linear transformations and $u \in U$, then $T_1 + T_2$ is defined by

$$(T_1 + T_2)(u) = T_1(u) + T_2(u) \ .$$

For $a \in F$, then aT is the linear transformation given by

$$(aT)(u) = aT(u) \ .$$

It is easy to check that these definitions make $\mathrm{Hom}_F(U,V)$ into a vector space over F.

Remark 6. A universal mapping property such as the one just described always gives a one-to-one correspondence (bijection) between two collections of functions. In this case they are both sets of linear transformations (in fact, they are vector spaces over F):

$$\{\, t \in \mathrm{Hom}_F(V,U) \mid t(W) = 0 \,\} \longleftrightarrow \mathrm{Hom}_F(V/W,U)$$

where we write $\mathrm{Hom}_F(V,U)$ for the vector space of linear transformations from V to U.
The one-to-one arrow to the right is given by the theorem (existence and uniqueness). Further, it is onto, since given any $T \in \mathrm{Hom}_F(V/W,U)$ we can define the required t by $t = T \circ p$ (this gives the arrow pointing to the left). In fact, the given bijection is an isomorphism of vector spaces.

Exercises

All statements, answers to questions, etc. in the exercises require proofs (i.e., "give", "show", "verify", "compute", "what is", "describe", etc. require actual proofs, not just an answer). If an outline to solve a problem is given, follow it. Do not assume a vector space is finite-dimensional unless this is explicitly stated in the problem.

QuoSpace 1. Let V be a vector space over the field F. Let X be a non-empty subset of V with the property that the set

$$Y = \{\, x_1 - x_2 \mid x_i \in X \,\}$$

is closed under addition and scalar multiplication by elements of F. Show that this is equivalent to X being a coset of some subspace of V. What is Y in terms of this description? What is X?

QuoSpace 2. Let V be a vector space over the field F. Let v_1 and v_2 be two distinct elements of V. The *line through* v_1 *and* v_2 is the set $L \subseteq V$ given by

$$L = \{\, rv_1 + sv_2 \mid r, s \in F,\ r + s = 1 \,\} \ .$$

Let X be a non-empty subset of V which contains all lines through two distinct elements of X. Show that X is a coset of some subspace of V. Describe the subspace. Relate this exercise to the preceding exercise.
Cosets of a subspace are sometimes called *affine subspaces* of V in view of their geometric description. *Affine* indicates that geometrically the set is a translate of an actual subspace.

QuoSpace 3. Let V be a vector space over the field F. Let $\mathcal{A}(V)$ denote the set of all affine subspaces of V, that is, an element of $\mathcal{A}(V)$ is a coset $v + W$ of some subspace W of V. Note in particular that $V/W \subseteq \mathcal{A}(V)$ for every such subspace W. Prove the following statements:

a. If $\{\, A_i \mid i \in I \,\}$ is a collection of affine subspaces, then $A = \bigcap_{i\in I} A_i$ is either the empty set or an affine subspace of V.

b. If $A, B \in \mathcal{A}(V)$, then $A + B \in \mathcal{A}(V)$.

c. If $c \in F$ and $A \in \mathcal{A}(V)$, then $cA \in \mathcal{A}(V)$.

d. If $T : V \longrightarrow U$ is a linear transformation and $A \in \mathcal{A}(V)$, then $T(A) \in \mathcal{A}(U)$.

e. If $T : V \longrightarrow U$ is a linear transformation and $B \in \mathcal{A}(U)$, then either $T^{-1}(B) \in \mathcal{A}(V)$ or $T^{-1}(B)$ is empty.

QuoSpace 4. Let U, V be vector spaces over the field F. Let $x \in V$ be a fixed vector. The function $T_x : V \longrightarrow V$ given by $T_x(v) = x + v$, is called *translation by* x. Note that T_x is a linear transformation only when $x = 0$. For some $y \in U$ and linear transformation $T : V \longrightarrow U$, the function $f : V \longrightarrow U$ given by $f = T_y \circ T$ is called an *affine transformation*. The collection of all such is denoted by $\mathrm{Aff}_F(V, U)$. Prove that for $A \in \mathcal{A}(V)$ and $f \in \mathrm{Aff}_F(V, U)$, then $f(A) \in \mathcal{A}(U)$.

QuoSpace 5. Let V be a vector space over the field F. Give a careful (explicit) description of the following quotient spaces and an isomorphism with a more naturally described vector space.

a.

$$Q_1 = V/0$$

where 0 denotes the zero subspace of V.

b.

$$Q_2 = V/V .$$

QuoSpace 6. Let U, V, W be vector spaces over a field F with $W \subseteq V$ a subspace. Let $T : V \longrightarrow U$ be a linear transformation whose kernel contains W. Show that there is a well-defined linear transformation $S : V/W \longrightarrow U$ given by $S(v + W) = T(v)$. Note that $T = S \circ \pi$ where $\pi : V \longrightarrow V/W$ is the natural quotient map $\pi(v) = v + W$. Is S the only linear map satisfying this property?

QuoSpace 7. Let U and V be vector spaces over the field F. Let $T : V \longrightarrow U$ be a linear transformation with kernel Z. Show that the image of T is isomorphic to the quotient space V/Z. The isomorphism given is to be natural, that is, not depend on a choice of basis.

QuoSpace 8. In this problem, you will show that the Universal Mapping Property characterizes the quotient space up to unique isomorphism. Let $W \subseteq V$ be vector spaces over a field F. Suppose we have a vector space Q and a linear transformation $\pi_Q : V \longrightarrow Q$ with $W \subseteq \ker(\pi_Q)$, and they have the property that for any linear transformation $T : V \longrightarrow U$ with $W \subseteq \ker(T)$ (where U is any vector space), there exists a unique linear transformation $T_Q : Q \longrightarrow U$ such that $T = T_Q \circ \pi_Q$. Prove that Q is isomorphic to the quotient space V/W.
(Hint: Use a Universal Mapping Property 4 times!)

QuoSpace 9. Let $W \subseteq V$ be vector spaces over a field F and $T : V \longrightarrow V$ be a linear transformation such that $T(W) \subseteq W$. Then T induces a linear transformation $\overline{T} : V/W \longrightarrow V/W$ given by $\overline{T}(v + W) = T(v) + W$.

a. Show $\overline{T}$ is a well-defined linear transformation on V/W. If V is finite-dimensional and T an isomorphism, prove that $\overline{T}$ is an isomorphism.

b. Is (a.) necessarily true if V is not assumed to be finite-dimensional? Prove or provide a counterexample.

QuoSpace 10. Let V be a vector space over F and $W \subseteq V$ a subspace. Let $p : V \longrightarrow V/W$ be the linear transformation given by $p(v) = v + W$. Let X be the set of all subspaces of V which contain W. Let Y be the set of all subspaces of V/W. Prove that p induces a one-to-one correspondence between these two sets as follows:

- $L \in X$ is mapped to $p(L) = \{ p(v) \mid v \in L \}$.
- $M \in Y$ is mapped to $p^{-1}(M) = \{ v \in V \mid p(v) \in M \}$.

That is, show that these two are inverse correspondences.

QuoSpace 11. Let U and V be vector spaces over the field F, and $W \subseteq V$ a subspace. No assumptions on dimensions. All isomorphisms given are to be natural.

a. Let $A = \{ T \in \mathrm{Hom}_F(V,U) \mid W \subseteq \ker(T) \}$. Show that A is a subspace of $\mathrm{Hom}_F(V,U)$. Prove $A \approx \mathrm{Hom}_F(V/W,U)$ and $\mathrm{Hom}_F(V,U)/A \approx \mathrm{Hom}_F(W,U)$ by constructing explicit isomorphisms.

b. Let $B = \{ T \in \mathrm{Hom}_F(U,V) \mid \mathrm{im}(T) \subseteq W \}$. Show that B is a subspace of $\mathrm{Hom}_F(U,V)$. Prove $B \approx \mathrm{Hom}_F(U,W)$ and $\mathrm{Hom}_F(U,V)/B \approx \mathrm{Hom}_F(U,V/W)$ by constructing explicit isomorphisms.

QuoSpace 12. Let V be vector spaces over a field F and let W be a subspace. By Exercise 10, we know that the subspaces of V/W are in one-to-one correspondence with the subspaces of V which contain W. Now suppose U is a subspace of V which contains W, so that U/W is a subspace of the vector space V/W. Give a description (a natural isomorphism) of the vector space $(V/W)/(U/W)$ in terms of yet another quotient.

QuoSpace 13. Let V be vector spaces over a field F and let W and U be subspaces. Prove that the quotient $U + W/W$ is naturally isomorphic to $U/(U \cap W)$.

QuoSpace 14. Let Z_1 and Z_2 be vector spaces over the field F. Assume there are subspaces $X_i \subseteq Y_i \subseteq Z_i$ for $i = 1, 2$. Define $Z = Z_1 \oplus Z_2$, $Y = Y_1 \oplus Y_2$ and $X = X_1 \oplus X_2$.

a. Show that there are natural inclusions (one-to-one linear transformations) $i : X \longrightarrow Y$ and $j : Y \longrightarrow Z$. Verify then that $k = j \circ i : X \longrightarrow Z$ is also a natural inclusion.

b. Using the Universal Mapping Property for Quotient Spaces, prove that there exists a natural linear transformation

$$q : Z/X \longrightarrow Z/Y .$$

(Hint: Start with a linear transformation $p : Z \longrightarrow ?$ which satisfies ...and conclude that there exists a unique linear transformation satisfying ....)

c. Compute $\ker q$.

d. Summarize the final result in the form of a natural short exact sequence.

History of the Notes

The Notes for the course *Math 4330, Honors Linear Algebra* at Cornell University have been developed over the last ten years or so mainly by the following (in chronological order):

Gerhard O. Michler

R. Keith Dennis

Martin Kassabov

W. Frank Moore

Yuri Berest.

Harrison Tsai also contributed a number of interesting exercises that appear at the ends of several sections of the notes.

Most sections have been revised so many times the original author may no longer recognize it. The intent is to provide a modern treatement of linear algebra using consistent terminology and notation. Some sections are written simply to provide a central source of information such as those on "Useful Definitions", "Subobjects", and "Universal Mapping Properties" rather than as a chapter as one might find in a traditional textbook. Additionally there are sections whose intent is to provide proofs of some results which are not given in the lectures, but rather provide them as part of a more thorough development of a tangential topic (e.g., Zorn's Lemma to develop cardinal numbers and the existence of bases and dimension in the general case).

A large number of challenging exercises from many different sources have been included. Although most should be readily solvable by students who have mastered the material, a few even more challenging ones still remain.

Much still remains to be done. Corrections and suggestions for additional exercises, topics and supplements are always welcome.

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Exact Sequences

We begin by recalling some definitions.

Definition 1. Let V and W be vector spaces over the field F. Let $f : V \longrightarrow W$ be a linear transformation. The *kernel* of f is the set of all vectors in V which are mapped to the zero vector in W, i.e.,

$$\ker f = \{ v \in V \mid f(v) = 0 \} .$$

It can easily be verified that this set is a subspace of V.

The *image* of f is the set of all vectors in W which have a preimage in V, i.e.,

$$\operatorname{im} f = \{ f(v) \mid v \in V \} .$$

It can easily be checked that this set is a subspace of W.

Definition 2. The quotient space $W/\operatorname{im} f$ is called the *cokernel* of the linear transformation f and is denoted by $\operatorname{coker} f$. The other quotient $V/\ker f$ is called the *coimage* of f and is denoted by $\operatorname{coim} f$.

Remark 3. Since $\ker f$ is a subspace of V, there is the inclusion $i : \ker f \longrightarrow V$. There is also the inclusion $j : \operatorname{im} f \longrightarrow W$. Since we defined $\operatorname{coker} f$ as the quotient space of V modulo W, there is a natural surjection $\pi : W \longrightarrow \operatorname{coker} f$. Analoguosly there is a surjection $\rho : V \longrightarrow \operatorname{coim} f$.

One of the first theorems about quotient spaces is the *First isomorphism theorem.* Using the language of images and coimages it can be stated as follows:

Theorem 4 (First Isomorphism Theorem). *For any linear transformation f there is a natural isomorphism from $\operatorname{coim} f$ to $\operatorname{im} f$.*

Remark 5. In most texts this is stated as $V/\ker f \approx \operatorname{im} f$.

Proof. Although the result should be clear, we will construct the desired isomorphism using the universal properties we have developed thus far.

The Universal Mapping Property for Quotient Spaces gives the existence of the linear transformation h and the following commutative diagram:

$$\begin{tikzcd}
\ker f \arrow[r, "i"] \arrow[dr, "0"'] & V \arrow[r, "p"] \arrow[d, "f"] & V/\ker f \arrow[dl, dotted, "h"] \\
 & \operatorname{im} f &
\end{tikzcd}$$

Thus $h \circ p = f$ and we claim that h is an isomorphism. Now h is onto by construction since $f : V \longrightarrow \operatorname{im} f$ is onto. Further, h is injective since its kernel is zero (the zero of $V/\ker f$ is $\ker f$). □

It is often useful to consider sequences of vector spaces connected by linear transformations with nice properties. The nicest such sequences are exact sequences, and will be our subject of study for the rest of this section.

Definition 6. Let V_i be vector spaces over the field F and let $f_i : V_i \longrightarrow V_{i+1}$ be linear transformations. Consider the sequence:

$$\cdots \longrightarrow V_{i-1} \xrightarrow{f_{i-1}} V_i \xrightarrow{f_i} V_{i+1} \longrightarrow \cdots$$

The sequence is called *exact at* V_i if $\operatorname{im}(f_{i-1}) = \ker(f_i)$. The sequence is called *exact* if it is exact at every V_i (for which the statment makes sense).

Remark 7. Note that the condition $\operatorname{im}(f_{i-1}) = \ker(f_i)$ implies that $f_i \circ f_{i-1} = 0$. However, this is *not* a sufficient condition for exactness (i.e., the two statements are not equivalent); see Exercise 4.

Example 8. The quintessential examples of exact sequences are the following:

1. The exact sequence of a quotient space for $W \subseteq V$ a subspace:

$$0 \longrightarrow W \xrightarrow{i} V \xrightarrow{p} V/W \longrightarrow 0 .$$

2. The exact sequence arising from a linear transformation $f : V \longrightarrow U$:

$$0 \longrightarrow \ker f \xrightarrow{i} V \xrightarrow{f} U \xrightarrow{\pi} \operatorname{coker} f \longrightarrow 0 .$$

Remark 9. Here the unlabelled arrows on the left and right ends are both 0 – there exists a unique linear transformation from 0 (the vector space with a single element) to any other vector space; similarly there is a unique linear transformation from any vector space to 0. In neither case does one normally label the arrows.

The language of exact sequences is very useful since many properties of linear transformations can be stated efficiently in terms of exact sequences. Exercise 1 provides a basic introduction to understanding this language. One can, for example, define $\ker f$ and $\operatorname{coker} f$ as the "unique" vector spaces (with the given maps) which make the sequence in second part of Example 8 above exact (see Exercise 10 and Exercise 11).

Remark 10. Some exact sequences turn up very often and have a special name. An exact sequence of the form

$$0 \longrightarrow U \xrightarrow{f} V \xrightarrow{g} W \longrightarrow 0$$

is called a *short exact sequence.* There are shorter exact sequences, but this length is the first which is "interesting" (see Exercise 1).

We say that a short exact sequence *splits* if there exists a linear transformation $s : W \longrightarrow V$ such that $g \circ s = 1_W$ where 1_W is the identity map on W. Notice that the other composition $s \circ g$ is almost never the identity map on V.

Exercises

In the following exercises U, V, W, and V_i denote vector spaces over the same field F.

ExSeq 1. This exercise shows how one can extract (or include) information in an exact sequence . In particular we see that a short exact sequence is the first exact sequence that is long enough to contain any really interesting information.

a. Show that a linear transformation f is injective if and only if the sequence

$$0 \longrightarrow V \xrightarrow{f} W$$

is exact.

b. Show that a linear transformation f is surjective if and only if the sequence

$$V \xrightarrow{f} W \longrightarrow 0$$

is exact.

c. The vector space V is 0 if and only if the sequence

$$0 \longrightarrow V \longrightarrow 0$$

is exact.

d. From the previous parts it follows that f is an isomorphism if and only if the sequence

$$0 \longrightarrow V \xrightarrow{f} W \longrightarrow 0$$

is exact.

ExSeq 2. Verify that for $f : V \longrightarrow W$

$$0 \longrightarrow \ker f \xrightarrow{i} V \xrightarrow{\overline{\pi}} \operatorname{coim} f \longrightarrow 0$$

and

$$0 \longrightarrow \operatorname{im} f \xrightarrow{\overline{i}} W \xrightarrow{\pi} \operatorname{coker} f \longrightarrow 0$$

are short exact sequences.

ExSeq 3. Verify that

$$0 \longrightarrow V_1 \xrightarrow{\iota} V_1 \oplus V_2 \xrightarrow{\rho} V_2 \longrightarrow 0$$

is an exact sequence, where ι is inclusion into the first summand and ρ is projection onto the second.

ExSeq 4. Let $U \xrightarrow{f} V \xrightarrow{g} W$ be a sequence of linear transformations so that $gf = 0$ (such a sequence is called a *complex*). Construct a vector space H that is zero precisely when the sequence above is exact at V.

ExSeq 5. Show that giving an exact sequence

$$\cdots \longrightarrow V_{i-1} \xrightarrow{f_{i-1}} V_i \xrightarrow{f_i} V_{i+1} \longrightarrow \cdots$$

is the same as giving a collection of short exact sequences

$$0 \longrightarrow K_i \longrightarrow V_i \longrightarrow K_{i+1} \longrightarrow 0 ,$$

one for each i for which the sequence is exact at V_i. This fact is sometimes referred to by saying that the top exact sequence is constructed by *splicing* together the collection of short exact sequences. [First think of the case where the sequence is infinite in both directions. Then worry about what happens when it is shorter.]

ExSeq 6. Let

$$0 \longrightarrow U \xrightarrow{f} V \xrightarrow{g} W \longrightarrow 0$$

be a short exact sequence. Prove that the following are equivalent. Do not use the existence of bases, but only use the information given.

a. The sequence splits on the right, that is, there exists a linear transformation $s: W \longrightarrow V$ such that $g \circ s = 1_W$.

b. The sequence splits on the left, that is, there exists a linear transformation $t: V \longrightarrow U$ such that $t \circ f = 1_U$.

c. There exists an isomorphism $\gamma : V \longrightarrow U \oplus W$ which satisfies $\gamma \circ f = i_1$ and $p_2 \circ \gamma = g$ for i_1 and p_2 denoting inclusion into the first summand, and projection onto the second summand, respectively.

ExSeq 7. a. Show that any short exact sequence of vector spaces splits,

$$0 \longrightarrow U \xrightarrow{f} V \xrightarrow{g} W \longrightarrow 0$$

i.e., there exists a linear transformation $s: W \longrightarrow V$ such that $g \circ s = 1_W$.

b. Prove that for an exact sequence of the form

$$0 \longrightarrow U \xrightarrow{i} V \xrightarrow{\pi} W \longrightarrow 0$$

there exists an isomorphism between V and $U \oplus W$.

In categorical language this implies that *the category of vector spaces over a field is semisimple.*

Remark 11. The linear transformation s in part a. and the isomorphism in part b. in the exercise above are **NOT** canonical, i.e., in order to describe them one needs to make choices.

ExSeq 8. a. Let $0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$ be an exact sequence and assume the vector spaces have finite dimension. Show that $\dim V = \dim U + \dim W$.

b. Let

$$0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow \cdots \longrightarrow V_n \longrightarrow 0$$

be an exact sequence of vector spaces, where each V_i is finite-dimensional. Prove that

$$\sum_{i=1}^{n}(-1)^i \dim V_i = 0 \, .$$

ExSeq 9. Construct a short exact sequence of groups which does not split.

ExSeq 10. The Universal Mapping Property of the kernel.
Let V and U be linear transformations over the field F and $T : V \longrightarrow U$ be a linear transformation.

a. Let $\ker T$ denote the kernel of T and let $\iota :\ \ker T \longrightarrow V$ be the usual inclusion. Show that the exact sequence

$$0 \longrightarrow \ker T \xrightarrow{\iota} V \xrightarrow{T} U$$

satisfies the following:
Given any linear transformation $j : L \longrightarrow V$ such that $T \circ j = 0$, then there exists a unique linear transformation $u : L \longrightarrow \ker T$ such that $j = \iota \circ u$.

b. Show that if

$$0 \longrightarrow K \xrightarrow{i} V \xrightarrow{T} U$$

is a short exact sequence, then it satisfies the Universal Mapping Property for the kernel of T.

c. Show that for any two short exact sequences

$$0 \longrightarrow K \xrightarrow{i} V \xrightarrow{T} U$$

and

$$0 \longrightarrow K' \xrightarrow{i'} V \xrightarrow{T} U$$

then there exists a unique isomorphism $u :\ K' \longrightarrow K$ satisfying $i' = i \circ u$.

ExSeq 11. The Universal Mapping Property of the cokernel.
State and prove an analogous ("dual") result for the cokernel of T. [Hint: Reverse all arrows.]

ExSeq 12. Consider the following commutative diagram of vector spaces with exact rows; i.e., the rows of the diagram are short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & V_1 \arrow[r, "j_1"] \arrow[d, "\varphi_1"'] & V_2 \arrow[r, "\pi_1"] \arrow[d, "\varphi_2"'] & V_3 \arrow[r] \arrow[d, "\varphi_3"'] & 0 \\
0 \arrow[r] & W_1 \arrow[r, "j_2"] & W_2 \arrow[r, "\pi_2"] & W_3 \arrow[r] & 0
\end{tikzcd}$$

a. Prove that there is an exact sequence

$$0 \longrightarrow \ker\varphi_1 \longrightarrow \ker\varphi_2 \longrightarrow \ker\varphi_3 \;.$$

where the maps in the exact sequence are induced by those in the diagram above.

b. Prove that there is an exact sequence

$$\operatorname{coker}\varphi_1 \longrightarrow \operatorname{coker}\varphi_2 \longrightarrow \operatorname{coker}\varphi_3 \longrightarrow 0 \;.$$

where the maps in the exact sequence are induced by those in the diagram above.

c. Consider the following construction concerning an element of $\ker\varphi_3$:

Step 1: Let $v_3 \in \ker\varphi_3$. Since π_1 is onto, there is some $v_2 \in V_2$ so that $\pi_1(v_2) = v_3$.

Step 2: Now $\pi_2(\varphi_2(v_2)) = \varphi_3(\pi_1(v_2)) = 0$ by commutativity of the right square and our choice of v_3.

Step 3: Therefore, $\varphi_2(v_2) \in \ker\pi_2 = \operatorname{im} j_2$, and so there is some element $w_1 \in W_1$ so that $j_1(w_1) = \varphi_2(v_2)$.

Show that this construction indeed defines a linear transformation (using the notation in the above construction):

$$\begin{aligned} \psi\colon \ker\varphi_3 &\longrightarrow \operatorname{coker}\varphi_1 \\ \psi(v_3) &= w_1 + \operatorname{im}\varphi_1 \in \operatorname{coker}\varphi_1 = W_1/\operatorname{im}\varphi_1 \end{aligned}$$

In other words, show that the choices made in the above construction only depend on the element chosen in the kernel of φ_3, and show that ψ is in fact a linear transformation.

d. Show that the sequence of maps from parts a. – c. fit into the following exact sequence:

$$0 \longrightarrow \ker\varphi_1 \longrightarrow \ker\varphi_2 \longrightarrow \ker\varphi_3 \xrightarrow{\psi} \operatorname{coker}\varphi_1 \longrightarrow \operatorname{coker}\varphi_2 \longrightarrow \operatorname{coker}\varphi_3 \longrightarrow 0 \;.$$

Note that given the first two parts, this amounts to showing only that the above sequence of maps is exact at $\ker\varphi_3$ and $\operatorname{coker}\varphi_1$.

ExSeq 13. a. Let V_i for $1 \le i \le k$ be k vector spaces over a field F. Let $W_i \subseteq V_i$ for $1 \le i \le k$ be subspaces of each. Let $V = V_1 \oplus V_2 \oplus \cdots \oplus V_k$ be the direct sum and $W = W_1 \oplus W_2 \oplus \cdots \oplus W_k$ be the direct sum of the subspaces. Give a natural inclusion $i : W \longrightarrow V$. Prove that there is a natural direct sum of k vector spaces Q_i such that there is a short exact sequence

$$0 \longrightarrow W \xrightarrow{i} V \xrightarrow{q} Q \longrightarrow 0 \;.$$

Explicitly give each of i, q, Q_i and Q and a proof that the sequence is exact.

b. State in words (with a minimum use of symbols) the assertion proved in the previous part.

c. Restate a third time as a formula of the form: "There is a natural isomorphism

$$\frac{V}{W} = ?$$

where the terms are written in such a way as to make the statement obvious, easy to remember, and easy to prove." Note that the symbolic statement just written is not part of the answer.

ExSeq 14. Consider the following diagram of vector spaces and linear transformations:

$$\begin{tikzcd}
A \arrow[rr, "t"] \arrow[dr, "i"] \arrow[ddr, "l"'] & & B \arrow[ddl, "r"] \\
& D \arrow[ur, "o"'] \arrow[d, "v"] & \\
& C &
\end{tikzcd}$$

a. Show that if the three small (inner) triangles commute, then the large outside triangle commutes. That is, conclude that $l = r \circ t$ follows from the three other equations given by the hypothesis.

b. There are 4 triangles in this diagram: three smaller inner triangles and the large outer triangle. Determine for which subsets of 3 triangles, their commutativity implies the commutativity of the fourth.

ExSeq 15. Consider the following diagram of vector spaces and linear transformations:

$$\begin{tikzcd}
& V \arrow[dr, "s"] \arrow[dd, "b"] & \\
U \arrow[ur, "t"] \arrow[rr, "u"] \arrow[dd, "a"'] & & W \arrow[dd, "c"] \\
& Y \arrow[dr, "\alpha"] & \\
X \arrow[ur, "\beta"] \arrow[rr, "\gamma"'] & & Z
\end{tikzcd}$$

[VISO] The three vertical arrows are isomorphisms.

a. Assuming the statement [VISO], show that if the three (left back, right back, front) rectangles and top triangle commute, then the bottom triangle commutes. That is, conclude that $\gamma = \alpha \circ \beta$ follows from the four other equations given by the hypothesis.

b. Is the result still true if the statement [VISO] does not hold?

ExSeq 16. Consider the same diagram that appears in the previous problem:

$$\begin{tikzcd}
 & V \arrow[dr, "s"] \arrow[dd, "b"] & \\
U \arrow[ur, "t"] \arrow[rr, "u"] \arrow[dd, "a"'] & & W \arrow[dd, "c"] \\
 & Y \arrow[dr, "\alpha"] & \\
X \arrow[ur, "\beta"] \arrow[rr, "\gamma"'] & & Z
\end{tikzcd}$$

[TISO] All arrows in the two triangles are isomorphisms.

a. Assuming the statement [TISO], show that if the two (left back and front) rectangles and the two (top and bottom) triangles commute, then the right back rectangle commutes. That is, conclude that $c \circ s = \alpha \circ b$ follows from the four other equations given by the hypothesis.

b. Is the result still true if the statement [TISO] does not hold?

History of the Notes

The Notes for the course *Math 4330, Honors Linear Algebra* at Cornell University have been developed over the last ten years or so mainly by the following (in chronological order):

Gerhard O. Michler

R. Keith Dennis

Martin Kassabov

W. Frank Moore

Yuri Berest.

Harrison Tsai also contributed a number of interesting exercises that appear at the ends of several sections of the notes.

Most sections have been revised so many times the original author may no longer recognize it. The intent is to provide a modern treatement of linear algebra using consistent terminology and notation. Some sections are written simply to provide a central source of information such as those on "Useful Definitions", "Subobjects", and "Universal Mapping Properties" rather than as a chapter as one might find in a traditional textbook. Additionally there are sections whose intent is to provide proofs of some results which are not given in the lectures, but rather provide them as part of a more thorough development of a tangential topic (e.g., Zorn's Lemma to develop cardinal numbers and the existence of bases and dimension in the general case).

A large number of challenging exercises from many different sources have been included. Although most should be readily solvable by students who have mastered the material, a few even more challenging ones still remain.

Much still remains to be done. Corrections and suggestions for additional exercises, topics and supplements are always welcome.

Keith Dennis

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Math 4330 Homework Set 5

Due Monday, October 5, 2014

Keith Dennis Malott 524 255-4027 **math4330@rkd.math.cornell.edu**

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NOTE: Late homework not accepted.

Read: "Equivalence Relations", "Quotient Spaces", "Exact Sequences", and "Bases and Coordinates".

NOTE: Exam 1 will be Fri. Oct. 16 – Fri. Oct. 23

Problems marked by $\boxed{\text{box}}$ or $\boxed{\star}$ are more challenging and may be turned in anytime during the semester. There will be several such problems assigned during the term. Please turn in *separately* from routine assignments – if incorrect or incomplete, they will be returned to you to complete correctly. Final deadline is Monday, Nov. 30, no exceptions.

Do the following problems from the handouts:

QuoSpace 1

QuoSpace 2

QuoSpace 8

QuoSpace 10

QuoSpace 11

QuoSpace 12

QuoSpace 13