Math 4330 Homework Set 7

Due Monday, November 2, 2015

Keith Dennis Malott 524 255-4027 **math4330@rkd.math.cornell.edu**

TA: Gautam Gopal Krishnan 120 Malott Hall gk379@cornell.edu

NOTE: Late homework not accepted.

Read: "Universal Mapping Properties", "Bases and Coordinates" and "The Matrix of a Linear Transformation".

Problems marked by $\boxed{\text{box}}$ or $\boxed{\star}$ are more challenging and may be turned in anytime during the semester. There will be several such problems assigned during the term. Please turn in *separately* from routine assignments – if incorrect or incomplete, they will be returned to you to complete correctly. Final deadline is Monday, Nov. 30, no exceptions.

Do the following problems from the handouts:

ExSeq 15

BaseCoord 8

BaseCoord 17

BaseCoord 21

BaseCoord 23a
Note: This is NOT a starred problem; only 23b is.

BaseCoord 25

BaseCoord 29

Ex07 1. Let V be a vector space of dimension n over the field F, and let $T : V \longrightarrow V$ be a linear transformation such that $T^n = 0$, so T is nilpotent. Assume also that $T^{n-1} \neq 0$. Suppose $v \in V$ is not in the kernel of T^{n-1}. Prove that $\mathcal{B} = \{v, T(v), \ldots, T^{n-1}(v)\}$ is an ordered basis for V. Compute the matrix of T with respect to the basis $\mathcal{B}$. Let $c \in F$ and define $S : V \longrightarrow V$ be given by $S(u) = cu + T(u)$. Compute the matrix of S with respect to $\mathcal{B}$. Compute the matrix of T^2 with respect to $\mathcal{B}$.

Ex07 2. Let V be finite dimensional over the field F. Let $S, T \in \mathrm{Hom}_F(V,V)$ be such that $ST = 1_V$. Show that there exists a polynomial $f \in F[x]$ such that $S = f(T)$.

Ex07 3. Let m, n, s be positive integers and F a field. For $A \in F^{m\times n}$ define $T\colon F^{n\times s} \longrightarrow F^{m\times s}$ by $T(M) = AM$ (matrix multiplication by A on the left).

a. If $\mathcal{A}$ is an ordered basis for $F^{n\times s}$ and $\mathcal{B}$ is an ordered basis for $F^{m\times s}$, give the dimensions (i.e., it is $k \times \ell$, for ?) of the matrix for $[T]_{\mathcal{A},\mathcal{B}}$.

b. Using the standard bases for $F^{n\times s}$ and $F^{m\times s}$, compute $[T]_{\mathcal{A},\mathcal{B}}$ in terms of A. Note that you MUST choose an ordering for the bases used; a nice choice will substantially simplify the problem.

c. Give, and prove, a formula for the rank and nullity of the matrix $[T]_{\mathcal{A},\mathcal{B}}$ in terms of A. Do this directly: do not use the theorem about the sum of the two.

d. Similarly $B \in F^{n\times m}$ define $S\colon F^{s\times n} \longrightarrow F^{s\times m}$ by $S(M) = MB$ (matrix multiplication by B on the right). Repeat all of the parts above for S, including finding a really nice matrix for S by ordering the standard bases carefully. Explain how you choose this ordering (which should be different from that chosen above for A), that is, what facts are you exploiting?