1. (20 points)

$$\begin{aligned}
\pi_2(\theta \mid y_1, y_2) &= \frac{p(y_1, y_2 \mid \theta)\pi(\theta)}{\int_\Theta p(y_1, y_2 \mid \theta)\pi(\theta)d\theta} \\
&= \frac{p_1(y_1 \mid \theta)p_2(y_2 \mid \theta, y_1)\pi(\theta)}{\int_\Theta p_1(y_1 \mid \theta)p_2(y_2 \mid \theta, y_1)\pi(\theta)d\theta} \\
&= \frac{p_2(y_2 \mid \theta, y_1)\pi_1(\theta \mid y_1)m_1(y_1)}{\int_\Theta p_2(y_2 \mid \theta, y_1)\pi_1(\theta \mid y_1)m_1(y_1)d\theta} \\
&= \frac{p_2(y_2 \mid \theta, y_1)\pi_1(\theta \mid y_1)}{\int_\Theta p_2(y_2 \mid \theta, y_1)\pi_1(\theta \mid y_1)d\theta} \\
&= \frac{p_2(y_2 \mid \theta, y_1)\pi_1(\theta \mid y_1)}{m(y_1, y_2)}
\end{aligned}$$

where

$$m_1(y_1) = \int_\Theta p_1(y_1 \mid \theta)\pi(\theta)d\theta.$$

2. (30 points)

(a) (10 points) We have

$$\begin{aligned}
y_t \mid \theta &\sim Bin(n, \theta) \\
\theta &\sim Beta(x_t, m - x_t)
\end{aligned}$$

So,

$$\begin{aligned}
\theta \mid y_t &\sim Beta(x_t + y_t, m - x_t + n - y_t), \\
E(\theta \mid y_t) &= \frac{x_t + y_t}{m + n}, \\
Var(\theta \mid y_t) &= \frac{(x_t + y_t)(m + n - x_t - y_t)}{(m + n)^2(m + n + 1)}.
\end{aligned}$$

(b) (10 points) From part a), posterior mean of θ is $\tilde{\theta} = \frac{x_t + y_t}{m+n}$. Then,

$$\begin{aligned}
E(\tilde{\theta}) &= E\left(\frac{x_t + y_t}{m + n}\right) = \frac{m\theta + n\theta}{m + n} = \theta \Rightarrow bias = 0, \\
Var(\tilde{\theta}) &= Var\left(\frac{x_t + y_t}{m + n}\right) = \frac{m\theta(1 - \theta) + n\theta(1 - \theta)}{(m + n)^2} = \frac{\theta(1 - \theta)}{m + n}, \\
MSE(\tilde{\theta}) &= Var(\tilde{\theta}) + bias^2 = \frac{\theta(1 - \theta)}{m + n}.
\end{aligned}$$

(c) (10 points) MLE for θ is $\hat{\theta} = \frac{y_t}{n}$. Then,

$$\begin{aligned}
E(\hat{\theta}) &= \theta, \\
Var(\hat{\theta}) &= \frac{\theta(1 - \theta)}{n}, \\
MSE(\hat{\theta}) &= Var(\hat{\theta}) + bias^2 = \frac{\theta(1 - \theta)}{n} > MSE(\tilde{\theta}).
\end{aligned}$$

So $\tilde{\theta}$ is preferred.

2. (50 points)

(a) (7 points)
prior: $p_1(\theta) = e^{-\theta}$.
likelihood: $p(\boldsymbol{y}_{30} \mid \theta) = \prod_{i=1}^{30} \theta^2 y_i e^{-\theta y_i} = \prod_{i=1}^{30} y_i \theta^{60} e^{-45.6\theta}$
posterior:

$$\begin{aligned} p_1(\theta \mid \boldsymbol{y}_{30}) &\propto p(\boldsymbol{y}_{30} \mid \theta) p_1(\theta) \\ &\propto \theta^{60} e^{-46.6\theta} \end{aligned}$$

So, $\theta \mid \boldsymbol{y}_{30} \sim gamma(61, 46.6)$ and

$$\begin{aligned} E(\theta \mid \boldsymbol{y}_{30}) &= \frac{61}{46.6} = 1.309 \\ Var(\theta \mid \boldsymbol{y}_{30}) &= \frac{61}{46.6^2} = 0.028 \end{aligned}$$

(b) (7 points)

$$\begin{aligned} \log p(y \mid \theta) &= 2\log\theta + \log y - \theta y, \\ \frac{\partial}{\partial\theta} \log p(y \mid \theta) &= \frac{2}{\theta} - y, \\ \frac{\partial^2}{\partial\theta^2} \log p(y \mid \theta) &= -\frac{2}{\theta^2}, \\ I(\theta) &= -E\left(\frac{\partial^2}{\partial\theta^2} \log p(y \mid \theta)\right) = \frac{2}{\theta^2}. \end{aligned}$$

So the Jeffreys prior is

$$\begin{aligned} p_2(\theta) &\propto \sqrt{I(\theta)} \\ &\propto \frac{1}{\theta} \end{aligned}$$

(c) (7 points) posterior under $p_2(\theta)$:

$$\begin{aligned} p_2(\theta \mid \boldsymbol{y}_{30}) &\propto p(\boldsymbol{y}_{30} \mid \theta) p_2(\theta) \\ &\propto \theta^{59} e^{-45.6\theta} \end{aligned}$$

So, $p_2(\theta \mid \boldsymbol{y}_{30})$ is $gamma(60, 45.6)$.

(d) (7 points)
From part a), we have $p_1(\theta \mid \boldsymbol{y}_{30}) \propto \theta^{60} e^{-46.6\theta}$. Then,

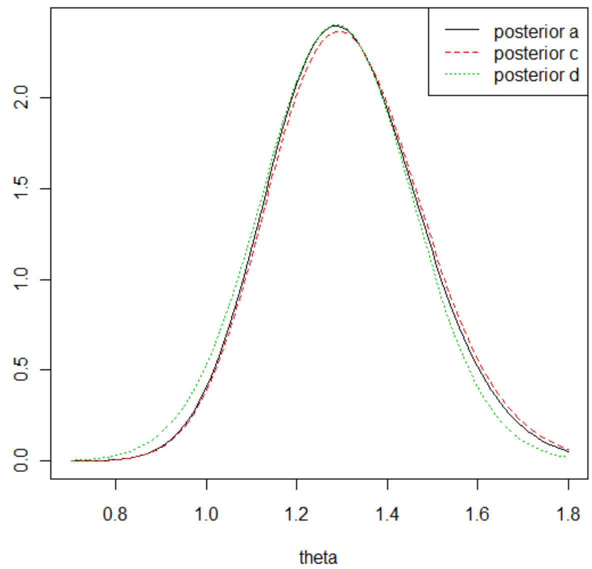
$$\begin{aligned} \log p_1(\theta \mid \boldsymbol{y}_{30}) &= constant + 60\log\theta - 46.6\theta, \\ \frac{\partial}{\partial\theta} \log p_1(\theta \mid \boldsymbol{y}_{30}) &= \frac{60}{\theta} - 46.6 = 0 \Rightarrow \hat{\theta} = \frac{60}{46.6} = 1.288, \\ \frac{\partial^2}{\partial\theta^2} \log p_1(\theta \mid \boldsymbol{y}_{30}) &= -\frac{60}{\theta^2}, \\ I(\hat{\theta}) = -\frac{\partial^2}{\partial\theta^2} \log p_1(\theta \mid \boldsymbol{y}_{30})|_{\theta=\hat{\theta}} &= \frac{60}{1.288^2} = 36.168. \end{aligned}$$

So, under p_1, $\theta \mid \boldsymbol{y}_{30} \approx N(\hat{\theta}, I(\hat{\theta})^{-1}) = N(1.288, 0.0276)$.

(e) (7 points)

```
theta <- seq(0.7,1.8,0.01)
post_a = dgamma(theta,61,46.6)
post_c = dgamma(theta,60,45.6)
post_d = dnorm(theta,1.288,sqrt(0.0276))

plot(theta,post_a,type='l',ylab='',ylim=c(0,14))
lines(theta,post_c,lty=2,col=2)
lines(theta,post_d,lty=3,col=3)
legend('topright',legend=c('posterior a','posterior c','posterior d'),
       lty=c(1,2,3),col=c(1,2,3))
```



(f) (7 points)

```
qgamma(c(0.025,0.975),61,46.6)
qgamma(c(0.025,0.975),60,45.6)
```

Under p_1: $[1, 1.657]$
Under p_2: $[1, 1.669]$

(g) (8 points) Under p_1 and p_2, the posteriors are $gamma(\alpha, \beta)$. Then,

$$
\begin{aligned}
p(y_{31} \mid \boldsymbol{y}_{30}) &= \int_0^\infty p(y_{31} \mid \theta) p(\theta \mid \boldsymbol{y}_{30}) d\theta \\
&= \int_0^\infty \theta^2 y_{31} e^{-\theta y_{31}} \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta} d\theta \\
&= y_{31} \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty \theta^{\alpha+2-1} e^{-(\beta+y_{31})\theta} d\theta \\
&= y_{31} \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+2)}{(\beta+y_{31})^{(\alpha+2)}} \\
&= \alpha(\alpha+1)\beta^\alpha \frac{y_{31}}{(\beta+y_{31})^{(\alpha+2)}}
\end{aligned}
$$

Under p_1 with $\alpha = 61, \beta = 46.6$:

$$
p_1(y_{31} \mid \boldsymbol{y}_{30}) = (61)(62)46.6^{61} \frac{y_{31}}{(46.6 + y_{31})^{63}}
$$

Under p_1 with $\alpha = 60, \beta = 45.6$:

$$
p_1(y_{31} \mid \boldsymbol{y}_{30}) = (60)(61)45.6^{60} \frac{y_{31}}{(45.6 + y_{31})^{62}}
$$

```
y <- seq(0,8,0.01)
plot(y,61*62*46.6^61*y/(y+46.6)^63,type='l',ylab='',main='posterior predictive density')
lines(y,60*61*45.6^60*y/(y+45.6)^62,lty=2)
legend('topright',legend=c('under p_1','under p_2'))
```