System of linear equations MATH 4330 8/26/15

coefficients

F = field So F is a set with 2 functions (binary operations)

multiplication: $\cdot$ $F \times F \longrightarrow F$

$(a,b) \mapsto a \cdot b = ab$

addition: $+$ $F \times F \longrightarrow$

$(a,b) \longrightarrow a+b$

Satisfies

A1 associativity of addition $a+(b+c) = (a+b)+c$

A2 Commutativity of addition $a+b = b+a$

A3 0 exists $\exists\, 0 \in F$ $0+a = a+0$

A4 inverses exist $\forall a \in F$ $\exists -a \in F$ s.t $a+(-a) = 0$

M1 $a(bc) = (ab)c$

M2 $ab = ba$

M3 1 exists

M4 For $a \neq 0$ $a^{-1} \in F$ exists s.t. $a^{-1}a = 1$

Distributive Laws $a(b+c) = ab + ac$

$(a+b)c = \cancel{ab}\ ac + bc$

Non-Triviality $0 \neq 1$

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Fields

Remarks

1) $0 \neq 1$ $a \in F \rightarrow a = a \cdot 1 = a \cdot 0 = 0$

$0 = 1 \rightarrow F = \{0 = 1\}, |F| = 1$

2)

a) 0 is unique $0 = 0 + 0' = 0'$

b) 1 is unique $1 = 1 \cdot 1' = 1'$

3 a) additive inverses unique

$a + a' = 0 = a' + a$

$a + a'' = 0 = a'' + a$

$a'' + (a + a') = (a'' + a) + a'$

$a'' = a'$

multiplicative inverses unique

4) $a \in F$

then $a \cdot a = a \cdot a$

$0 = 0 + 0$

$0 \cdot a = (0 + a)a$

$= 0 \cdot a + a \cdot a$

$0 = 0 \cdot a$

5) $x, y \in F$ field

$xy = 0$

then $x = 0$ or $y = 0$

$x \neq 0$, $x^{-1} \in F$

$x^{-1} x y = x^{-1} 0$

$1 \cdot y = 0 \rightarrow y = 0$

8/31/15 Fields Cont'd

$F_p = \mathbb{Z}_p = \langle 0,1,2\dots,p-1\rangle = GF(p)$

compute mod p

p is prime

b) $F_2 \subseteq F_4 = \langle 0,1,\alpha,1+\alpha\rangle$

α

$0,1,\alpha$ all different

$= F_2[\alpha]$

$= [a+b\alpha \mid a,b\in F_2]$

+	0	1	α	$1+\alpha$
0	0	1	α	$1+\alpha$
1	1	0	$1+\alpha$	α
α	α	$1+\alpha$	0	1
$1+\alpha$	$1+\alpha$	α	1	0

in any row or column all 4 elements appear exactly once.

$\alpha^2 \neq 0,$

$\alpha^2 \neq 1$ $(\alpha+1)^2 = \alpha^2+2\alpha+1$

$= \alpha^2+1 = 1+1 = 0$

$\alpha^2 \neq \alpha$

$\alpha^2 = 1+\alpha \rightarrow$ F_4 is a field

characteristic of a ring

$1\in$

(Ring, add, mult,

omit mult comm

- $\neq 0$ if have mult. inverse
- $0 \neq 1$

eg. $\mathbb{Z}, \mathbb{Z}_n, \mathbb{R}^{2\times 2}, \mathbb{Z}^{3\times 3}$

$n\in\mathbb{Z},\ r\in R^{\text{ring}}$

$$n\cdot r = \begin{cases} \overbrace{r+r+\dots r}^{n} & \text{if } n>0 \\ 0 & \text{if } n=0 \\ -|n|\cdot r & \text{if } n<0 \end{cases}$$

$(1+1+\dots+1)\cdot r$

$h\cdot m\cdot r = (nm)(r)$

$h,m\in\mathbb{Z},\ r\in\mathbb{Z}$

$(n+m)r = nr+mr$

A ring R has positive characteristic

$1\in R$ if $\exists n\in\mathbb{Z},\ n>0$, with

$n\cdot 1 = 0,\ 1\in R.$

if no such $n>0$ exists, R has characteristic 0.

Smallest n with $n\cdot 1=0$, is called characteristic of R.

Char $= n$

AHI

char $\mathbb{R} = 0$

$F_2 = 2$

$F_3 = 3$

$F_p = p$

$\mathbb{Z}_n = n$

$\mathbb{Z} = 0$

$\mathbb{Q} = 0$

$\mathbb{R} = 0$

$\mathbb{C} = 0$

Examples of Fields

1. $\mathbb{Q}, \mathbb{R}, \mathbb{C}$

$\mathbb{Z}$ not a field

$\mathbb{R}^{2\times 2}$ Not

2. $\mathbb{Q}[i] = \{a+bi \mid a, b \in \mathbb{Q}\} \subseteq \mathbb{C}$

gaussian field

$z = a+bi \quad z \cdot \bar{z} = a^2+b^2 \geq 0 \quad = 0 \leftrightarrow a=0$ and $b=0$

if $z \neq 0 \leftrightarrow$ then $a \neq 0$ or $b \neq 0 \leftrightarrow a^2+b^2 > 0$

3. $1 < p \in \mathbb{Z}$ prime eg 5 $\sqrt{5} \in \mathbb{R}$ $\quad z^{-1} = \frac{\bar{z}}{a^2+b^2} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}$

$\mathbb{Q}[\sqrt{5}] = \{a + b\sqrt{5} \mid a, b \in \mathbb{Q}\} \subseteq \mathbb{R} \subseteq \mathbb{C}$

$(a+b\sqrt{5})(a-b\sqrt{5}) = a^2-5b^2 = 0 \Leftrightarrow a=0$ and $b=0$

need $\sqrt{5} \notin \mathbb{Q}$

if $a^2 = 5b^2$

$\left(\frac{a}{b}\right)^2 = 5$

4. Finite Fields

(a) $\mathbb{F}_2 = \{0,1\}$, $\mathbb{F}_3 = \{0,1,2\}$

+	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

$\mathbb{F}_5, \mathbb{F}_7, \mathbb{F}_p$

p = prime

b) $\mathbb{F}_2 \subseteq \mathbb{F}_4 = \{0, 1, \alpha, 1+\alpha\}$

+	0	1	α	$1+\alpha$
0	0	1	α	$1+\alpha$
1	1	0	$1+\alpha$	α
α	α	$1+\alpha$	0	1
$1+\alpha$				

$\alpha^2 \in \mathbb{F}_4$

$\alpha^2 = \alpha$

$\neq 0 \neq 1$

$\neq 1+\alpha$

9/2/15 Rings

$1 \in R$, R a ring

characteristic of R

smallest n, $n \in \mathbb{Z}$, $n > 0$

s.t. $\underbrace{1+1\ldots+1}_{n} = 0$ in R

if such n exists, else $char(R) = 0$

$1 \in \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{Q}[\sqrt{5}] \subseteq \mathbb{R}$ $\quad 0 \neq n$

$\subseteq \mathbb{Q}[i] \subseteq \mathbb{C}$ $\quad$ char $= 0$

2. $\mathbb{F}_2 = \mathbb{Z}_2 = \{0, 1\}$, $0 \neq 1$

$1+1 = 0$

$char\ \mathbb{F}_2 = 2$

3. $\mathbb{F}_2 \subseteq \mathbb{F}_4 = \{0, 1, \alpha, 1+\alpha\} = \{a + b\alpha \mid a, b \in \mathbb{F}_2\}$

$char\ \mathbb{F}_4 = 2$

4. $\mathbb{F}_3 = \{0, 1, 2\}$ $\quad 0 \neq 1$, $1+1 \neq 0$, $1+1+1 = 0$

$char\ \mathbb{F}_3 = 3$

5. $char\ \mathbb{F}_p = p$

6. $char\ \mathbb{Z}_n = n$

$n > 0$

Def: A ring $1 \in R$ is an integral domain

iff $\forall n, r, s \in R$, $r \cdot s = 0 \Leftrightarrow r = 0$ or $s = 0$

Lemma

If F is a field, then $char(F) = 0$, or $char(F) = p$

p is prime.

$r \neq 0$, $s \neq 0$ $\quad r \cdot s = 0$

Lemma: $1 \in R$ is a domain.

$char(R) = 0$, in $char(R) = p$, p a prime.

Pf: if $(1+1)(1+1+1) = 0$

R is a domain, then $char\ R = 2$ or 3

If $char\ R \neq 0$ and $char\ R$ is not prime

then $char\ R = n$, n is composite integer.

eg $n = r \cdot s$, $r, s \in \mathbb{Z}$

$+\ r > 1$, $s > 1$

if $n \cdot 1 = 0$

$(rs) \cdot 1 = 0$ $\quad$ R a domain

$(r \cdot 1)(s \cdot 1) = 0$ $\rightarrow r \cdot 1 = 0$ or $s \cdot 1 = 0$ $\quad$ but $1 < r < n$, $1 < s < n$

dist.

a contradiction.

$char(R) = n \rightarrow n \cdot r = 0 \quad \forall r \in R$

$n \cdot r = \underbrace{r + \ldots + r}_{n} = \underbrace{(1+1 \ldots)}_{n} r$

$= 0 \cdot r = 0$

Pf

$$\boxed{\begin{array}{l}\text{if } (1+1)(1+1+1) = 0 \\ R \text{ is a domain, then } char\ R = 2 \text{ or } char\ R = 3\end{array}}$$

$1 \in R$

$\langle 1 \rangle \overset{def}{=} \{n \cdot 1 \mid n \in \mathbb{Z}\} \subseteq R$

subring

$n \cdot 1 + m \cdot 1$

$(n+m) \cdot 1$

$n \cdot (m \cdot 1) = (nm) 1$

$1 \in F$

① $\mathrm{char}\, F = p$,

① $\dim F = p$

$\langle 1 \rangle = \{ n.1 \mid n \in \mathbb{Z} \} \subseteq F$

$= \{0, 1, 2, \ldots, p-1\} \subseteq F$

$\parallel$ F_p

$\mathrm{char}\, F = 0$, F

$\langle 1 \rangle \subseteq F$ $\mathbb{Z} \subseteq F$

$\mathbb{Z} \cong \{ n.1 \mid n \in \mathbb{Z} \}$

$n \to n.1$ $\quad n, m \in \mathbb{Z}, 0 < n < m$ $\quad n^{-1} = \frac{1}{n}$

$0 < n < m$

$n.1 \mid 1 + 1 \cdots + 1 = n.1$ $\quad \left(\{ n/m \mid n, m \in \mathbb{Z} \right)$

$m.1 \mid 1 + 1 \cdots 1 - 1 = m.1$ $\quad m \neq 0$

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Image and Kernel

Char 0, p, etc

$\mathbb{Z}$ "not modular"

$\mathbb{Q}(\sqrt{5})$ pf. requires unique factorization of integers.

homomorphism, $\langle d \rangle = I \subseteq \mathbb{Z}$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \| $

construction of finite fields $\{jd \mid j \in \mathbb{Z}\}$

$h: \mathbb{Z} \to R$

$1 \longmapsto 1$

h-ring homomorphism $\begin{cases} h(1) = 1 \\ h(x+y) = h(x) + h(y) \end{cases} \to h(0) = 0 \to h(-n) = -h(n)$

$h(n) = h(\overbrace{1+1\cdots+}^{n}) = h(1) + h(1) + \cdots + h(1) = n\, h(1) = n \cdot 1$

$\langle d \rangle = \ker h = \{z \in \mathbb{Z} \mid h(z) = 0\}$ ideal image $= \{h(x) \mid x \in \mathbb{Z}\}$

$= \{n \in \mathbb{Z} \mid n \cdot 1 = 0\} \subseteq \mathbb{Z}$ $= \{n1 \mid n \in \mathbb{Z}\}$

$= \langle 1 \rangle \leq R$

subring

In all cases $\mathrm{char}(R) = d$

$d \in \mathbb{Z}$, $d \geq 0$, $\langle d \rangle = \{jd \mid j \in \mathbb{Z}\} = \ker h$

F a field

V is a vector over F

if V is a set & then $\exists$ 2 functions (called binary operations)

addition $+: V \times V \to V$

$(u, v) \longmapsto u+v$

scalar mult $\cdot : F \times V \to V$

$(a, v) \mapsto a \cdot v = av$

which satisfy

Examples of Vector Spaces

9/9/15

F is a field

1. (a) F^n, $F^{n\times 1} = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \middle| \, a_i \in F \right\}$

b) $F^{1\times n} = \{ (a_1, \dots, a_n) \mid a_i \in F \}$

c) $F^{m\times n}$ $m, n \geq 1 \in \mathbb{Z}$

2. S a nonempty set

F^S = all functions from S to F.

$f, g : S \to F$

$(f+g)(s) = f(s) + g(s)$

$a \in F$ $(af)s = af(s)$

Why? X, Y true finite sets

X^Y = all fns from Y to X

$X = \{1, 2, 3\}$ $Y = \{a, b, c, d\}$

$|X^Y| = 3^4 = |X|^Y$

$f: Y \to X$

S nonempty set

V vsp over F

V^S $f, g, : S \to V$

$(f+g)s = f(s) + g(s)$

$(af)(s) = af(s)$

2c $F_n^{(S)} \leq F^S$

all functions $f: S \to F$ which have finite support

$\mathrm{supp}(f) = \{ s \in S \mid f(s) \neq 0 \}$

2d $V^{(S)}$

3 (a) $F[x]$ = formal polynomial w/ coefficients in F.

$\{1, x, x^2, x^3\} = \{x^c \mid \in \mathbb{Z} \geq 0\}$

a basis for $F[x]$

$F[x] = \{ a_0 + a_1 x + \dots + a_n x^n \mid n \in \mathbb{Z}, n \geq 0, a_0, \dots, a_n \in F \}$

$a_0 + a_1 x + 0x^2$
$= a_0 + a_1 x$

$\left[\begin{array}{l} a_0 + a_1 x + \dots + a_n x^n \neq 0 \\ = b_0 + b_1 x + \quad + b_m x^m \end{array} \right.$

$n, m \in \mathbb{Z}, n, m \geq 0$

$\leftrightarrow a_0 = b_0$

$a_1 = b_1$

etc.

$ev_a : F[x] \xrightarrow{\text{formal}} F$

$\quad f$

$a \in F$ $ev_a(f) = a_0 + a_1 a + a_2 a^2 + \dots + a_n a^n$
$= f(a)$

$f = a_0 + a_1 x + \dots + a_n x^n$

$ev: \boxed{F[x] \to F^F}$

$f \to ev(f)$

$ev(f): \underset{a}{F} \to F$

$F = F_2 \quad F_2[x] = (0, 1, x, 1+x, x^2, 1+x+^2$

$|F_2^{F_2}| = |F_2|^{|F_2|} = 2^2 = 4$

b) formal power series

$F[[x]]$

$a_0 + a_1 x + a_2 x^2 + a_3 x^3$

4. $F \leq K$

fields

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$F \subseteq K$ K is a v sp over F

subfield, extension of F

Examples $\mathbb{F}_2 \subseteq \mathbb{F}_4$

$\mathbb{Q} \subseteq [\sqrt{5}] \subseteq \mathbb{R} \subseteq \mathbb{C}$

$\subseteq \mathbb{Q}[2]$ $\subseteq$

$M \in F^{m \times n}$

"solution space" $\{ C \in F^{n \times 1} \mid MC = 0 \}$ ($n \times 1$, $m \times n$, $n \times 1$, $m \times 1$)

$D \subseteq \mathbb{R}$ $D = (0,1)$

a) all functions D to $\mathbb{R}$ (ie $\mathbb{R}^D$)

b) all continuous fns D to $\mathbb{R}$

c) all differentiable fns

d) $\mathcal{C}^n(D), n \in \mathbb{Z}, n \geq 0$; $\mathcal{C}^\infty(D)$

Subspaces ("subobjects")

Def: V is a vector space over F

$W \subseteq V$ is a subspace of V if W is vector space

$w, w_2, u, v \in V$ $a, \in F$

$a \cdot w$

$w_1 + w_2$

$+$ and $\cdot$ for W given by restriction, induced

$+ : V \times V \to V$

$+ : W \times W \to W$

$F \times V \to V$

$F \times W \to W$

Lemma.

1) V is a vector space over F, $W \subseteq V$ is a subset

Then W is a subspace of $V \iff$ 1, $W \neq \emptyset$

2) if $w_1, w_2 \in W$, then $w_1 + w_2 \in W$

3) If $a \in F, w \in W$, then $aw \in W$

$\to$ ✓

$\leftarrow$ given 1, 2, 3

$W_0 \in W$ by 1)

$0 \cdot W_0 = 0 \in W$ by 3)

$w \in W$: $-1 \cdot w = -w$

Examples of subspaces

$0 = \{0\} \subseteq V$ trivial

$V \subseteq V$

1. $\{(a, b, 0, 0) \mid a \in F\}$ $F^{1 \times n}$

2. $\left\{ \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ a_n & 0 & & 0 \end{pmatrix} \middle| \, a_1, \ldots, a_n \in F \right\} \subseteq F^{m \times n}$

$\mathcal{A} \subseteq F^{m \times n}$

$\mathcal{A}$ = all symmetric matrices

$\begin{pmatrix} \ddots & & \\ & \ddots & \\ & & \ddots \end{pmatrix}$ $\begin{pmatrix} a & c \\ c & b \end{pmatrix}$

$M = (a_{ij})_{ij}$

$M^t = (a_{ji})_{ij}$

$M^T = M$

$(M_1 + M_2)^T = M_1^t + M_2^t$

3. T, S nonempty subset

$T \subseteq S$

$W \subseteq F^S$

$F^T \to F^S$

$f \to \hat{f}$

$\hat{f}(s) = \begin{cases} f(s) & s \in T \\ 0 & s \notin T \end{cases}$

$\{g \in F^S \mid g(s) = 0,\ s \notin T\}$

$F^T \to F^S$

Lemma F $V_i \subseteq V$ vsp subspace, $i \in I$.

Then $\bigcap_{i \in I} V_i$ is a subspace of V.

4. $F[x]$

$n \geq 0$

$\mathcal{P}_n = \{$ all polynomials in $F[x]$ deg $< n$

$a_n \neq 0$

$h = a_0 + a_1 x + \cdots + a_n x^n \neq 0$

$a \in F$ deg $h = n$

$h \neq 0$

$W\{0\}$

5. $F_2 \subseteq F_4$ If vector space over

$Q \subseteq R \subseteq C$ then ...

etc

Vector Space, Span

9/14/15

Lemma

V vsp ~~is a~~ over a field F

$V_i \subseteq V$, $i \in I$, subspace.

Then $W = \bigcap_{i\in I} V_i \subseteq V$ is a subspace

i) $0 \in V_i, \forall i \in I \to 0 \in W$

ii) $x, y \in W \to x, y \in V_i, \forall i \to x+y \in V_i, \forall i$
$\to x+y \in W$

iii) $a \in F, x \in W \to x \;\; x \in V_i$
$\to ax \in V_i \to ax \in W$

Ex. 1. $I = \emptyset \to W = V$ by det.

2. collection of all subspaces of V
$W = \{0\} = 0$

V a vsp. over field F, $S \subseteq V$ (arbitrary subset)

$\langle S \rangle$ def. subspace spanned by S.

$= \bigcap_{\substack{W \text{ subspace of } V \\ S \subseteq W \subseteq V}} W$

$S \subseteq \langle S \rangle$ ← This is a subspace of V

So 2) $\langle S \rangle$ is the unique smallest subspace of ~~V~~ which contains S.

Construction

3) a) $S = \emptyset$
$\langle S \rangle = \{0\}$

$\langle S \rangle$ = all finite linear combinations of finite subsets of S

that is $= \left\{ a_1 s_1 + a_2 s_2 + \dots + a_n s_n \;\middle|\; \begin{array}{l} n \geq 1 \\ s_1, \dots, s_n \in S \\ a_1, \dots, a_n \in F \end{array} \right.$

$\shortparallel$ RHS

note: $S \subseteq \langle S \rangle$ = RHS

RHS $\subseteq S$

RHS is a subspace of V

A. V vector space over F

$W_1, W_2 \subseteq V$ subspaces

def $W_1 + W_2 = \{w_1 + w_2 \mid w_i \in W_i\} \subseteq V$
$= \mathrm{Span}_F \{W_1 \cup W_2\}$

B. V vector space over F

$V_i \subseteq V, i \in I$

$\sum_{i\in I} V_i = \mathrm{Span}_F \left\{ \bigcup_{i\in I} V_i \right\}$

V_1, V_2 vsp over F

$T: V_1 \to V_2$ def: $T(x+y) = T(x) + T(y)$ $\forall x, y \in V_1$

$T(ax) = aT(x)$ $\forall a \in F$

Linear transformation

image of $T = \operatorname{im} T$
$= \{T(x) \mid x \in V_1\} \subseteq V_2$ subspace

kernel of $T = \ker T = \{x \in V_1 \mid T(x) = 0\} \subseteq V_1$ subspace

Lemma: $T: V_1 \to V_2$ linear transformation

T is one to one $\leftrightarrow \ker T = \{0\}$

T is onto $\leftrightarrow \operatorname{im} T = V_2$

Direct Sum

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$\subseteq$ ( ) direct sum

V vsp over F

$W_1, W_2 \subseteq V$ subspaces s.t.

1) $W_1 + W_2 = V$

2) $W_1 \cap W_2 = 0$

note $v \in V$, then $\exists$ $w_1 \in W_1$, $w_2 \in W_2$ s.t. $v = w_1 + w_2$

existence $v \in V$

by #1 $\exists w_1 \in W_1$, $\exists w_2 \in W_2$

$w_1 + w_2 = v$

uniqueness $v = w_1 + w_2$ $\quad w_1, w_1' \in W_1$

$w_2, w_2' \in W_2$

$W_1 \ni w_1 - w_1' = w_2' - w_2 \in W_2$

so $= \langle 0 \rangle$

$0 = w_1 - w_1'$

$0 = w_2 - w_2'$

$w_1 = w_1'$

$w_2 = w_2'$

Given an internal direct sum:

$W_1, W_2 \subseteq V$ satisfying 1) $W_1 + W_2 = V$

2) $W_1 \cap W_2 = 0$

$T: W_1 \oplus W_2 \to V$

$(w_1, w_2) \mapsto w_1 + w_2$

3) 1to1 by #2

2) onto by #1

T is a linear transformations

$T(a(w_1, w_2)) = T((aw_1, aw_2))$

$a \in F$

$= aw_1 + aw_2 \in V$

$= a(w_1 + w_2) = aT(w_1, w_2)$

$T((v_1, v_2) + (w_1, w_2))$

$= T(v_1 + w_1, v_2 + w_2)$

$= (v_1 + w_1 + v_2 + w_2)$

$= (v_1 + v_2) + (w_1 + w_2)$

$= T(v_1, v_2) + T(w_1, w_2)$

$\ker T - \{(w_1, w_2) \mid \underset{=0}{T(w_1, w_2)}\} =$

$\downarrow$

$\{(0,0)\}$

Thm - The external direct sum and the internal direct sum are (canonically) naturally isomorphic vector spaces

F is a field

1) $V_1, V_2 \ldots V_k$ vsp over F

$V_1 \oplus V_2 \oplus \ldots \oplus V_k = \{(v_1, v_2, \ldots, v_k) \mid v_i \in V_i\}$

2) $W_i \leq V$ vsp over F

$1 \leq i \leq k$

1) $W_1 + W_2 + \ldots + W_k = V$

for each i, $1 \leq i \leq k$

$W_i \cap \left(\sum_{j \neq i} W_j\right) = 0$

1) $W_1 + W_2 + W_3 = k^2$

2) $W_1 \cap (W_2 + W_3) = W_1$

$w_i = -w_1 - w_i - \cdots - w_{i-1} - w_{i+1} \cdots - w_r$

$W_i \cap \left(\sum_{j \neq i} W_j \right) = 0$

$V_1 \oplus V_2 \oplus \cdots \oplus V_k$

$v = (v_1, v_2, \ldots, v_k)$

$f : \{1, 2, \ldots, k\} \to \bigcup_{i=1}^{k} V_i$

Condition $f(i) \in V_i$

$f \leftrightarrow (f(1), f(2) \ldots f(k))$

$f : I \to \bigcup_{i \in I} V_i \quad f(i) \in V_i$

$supp(f) = \{ i \in I \mid f(i) \neq 0 \}$

↑ finite set