Stat 310B: Problem Set 2

Due in class on Tuesday, January 31

1. Let X_n be a martingale with $X_0 = 0$ and $E(X_n^2) < \infty$. Show that for any $\lambda > 0$,
$$P(\max_{1 \le m \le n} X_m \ge \lambda) \le \frac{E(X_n^2)}{E(X_n^2) + \lambda^2}.$$
Hint: Use the fact that $(X_n + c)^2$ is a submartingale and optimize over c.

2. Let S_n be a symmetric simple random walk starting at 0, and let $T = \inf\{n : S_n \notin (-a, a)\}$, where a is an integer. Find constants b and c so that $Y_n = S_n^4 - 6nS_n^2 + bn^2 + cn$ is a martingale, and use this to compute $E(T^2)$.

3. Let $S_n = \xi_1 + \cdots + \xi_n$ be a random walk with i.i.d. increments, with $S_0 = 0$. Suppose that $E(\xi_1) < 0$, $P(\xi_1 > 0) > 0$, and $E(e^{\theta \xi_1}) < \infty$ for all $\theta \in \mathbf{R}$. Show that there exists $\theta_0 > 0$ such that $E(e^{\theta_0 \xi_1}) = 1$, and that for any $a > 0$ and any such θ_0,
$$P(\sup_n S_n > a) \le e^{-\theta_0 a}.$$
(Hint: Use the exponential martingale briefly discussed in class.)

4. Let S_n be the total assets of an insurance company at the end of year n. In year n, premiums totaling $c > 0$ are received and claims ζ_n are paid where $\zeta_n \sim N(\mu, \sigma^2)$ and $\mu < c$. To be precise, if $\xi_n = c - \zeta_n$, then $S_n = S_{n-1} + \xi_n$. The company is ruined if its assets drop to 0 or less. Show that if $S_0 > 0$ is nonrandom, then
$$P(\text{ruin}) \le \exp(-2(c - \mu)S_0/\sigma^2).$$