Math 547: Introduction to Mathematical Statistics 2
Midterm Exam

1. Let $X_1, \ldots, X_n$ be independent, identically distributed random variables, with the exponential density $f(x;\theta) = \theta e^{-\theta x}$, $x > 0$.

   Obtain the maximum likelihood estimator $\hat{\theta}$ of θ. What is the asymptotic distribution of $\sqrt{n}(\hat{\theta} - \theta)$?

   Show that $\hat{\theta}$ is biased as an estimator of θ.

   What is the minimum variance unbiased estimator of θ? Justify your answer carefully, stating clearly any results you use.

   What is the MLE of $e^{-\theta}$? What is the limiting distribution of the MLE of $e^{-\theta}$ (after standardizing)?

2. Let $X_1, \ldots, X_n$ be independent random variables with densities

$$f_{X_i}(x;\theta) = \begin{cases} e^{i\theta - x} & x \geq i\theta \\ 0 & x < i\theta. \end{cases} \tag{1}$$

   Prove that $T = \min_i(X_i/i)$ is a sufficient statistic for θ.

3. Let $X_1, \ldots, X_n$ be independent, identically distributed random variables, with the exponential density $f(x;\theta) = e^{-(\theta - x)}$, $x > 0$, $-\infty < \theta < \infty$.

   Let $X_{(1)} < \ldots < X_{(n)}$ be ordered sample, and define $Y_i = X_{(n)} - X_{(i)}$, $i = 1, \ldots, n-1$ and verify that the set $(Y_1, \ldots, Y_{n-1})$ is ancillary for θ.

   Find the best unbiased estimator of θ^r.

4. Let $X_1, \ldots, X_n$ be a random sample from a population with pmf

$$P_\theta(X = x) = \theta^x (1-\theta)^{1-x}, \quad x = 0 \text{ or } 1, \quad 0 \leq \theta \leq \frac{1}{2} \tag{2}$$

   Find the method of moments and the MLE of θ. Find the mean squared error for each of the estimators. Which is preferred? Justify your choice.

5. Let $X_1, \ldots, X_n$ be independent and identically distributed with pdf $f(x;\theta) = \theta e^{-\theta x}$ for $x > 0$ and zero otherwise. Find an unbiased estimator of $e^{-\theta} = P(X \geq 1)$. Find a sufficient statistic for θ. Find the minimum variance unbiased estimator of $e^{-\theta}$. (Hint: Use the Rao-Blackwell technique.)