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ksc66

Math 4330 Homework Set 2

Due Friday, September 12, 2015

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Read: Handouts on "Some Useful Definitions", "Fields", "Examples of Vector Spaces", and "Subobjects".

Problems marked by $\boxed{\text{box}}$ or $\boxed{\star}$ are more challenging and may be turned in anytime during the semester. There will be several such problems assigned during the term. Please turn in *separately* from routine assignments – if incorrect or incomplete, they will be returned to you to complete correctly. Final deadline is Monday, Nov. 1, no exceptions.

Do the following problems from the handouts:

Fields 15

Fields 26

Fields 27

SubObj 3

SubObj 5

SubObj 6

SubObj 7 (compare to Fields 7)

due 9/11/15

Homework 2

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Fields 15

a. Let $Q[\sqrt[3]{2}]$ denote minimal subfield of C which contains $\sqrt[3]{2}$.

Explicit description as a set $Q[\sqrt[3]{2}] = \{ a + b\sqrt[3]{2} + c(\sqrt[3]{2})^2 \mid a, b, c \in Q\}$

1) $0 = 0 + 0\sqrt[3]{2} + 0(\sqrt[3]{2})^2 \to 0 \in Q[\sqrt[3]{2}]$

2) $1 = 1 + 0\sqrt[3]{2} + 0(\sqrt[3]{2})^2 \to 1 \in Q[\sqrt[3]{2}]$

3) Let $x = a + b\sqrt[3]{2} + c(\sqrt[3]{2})^2$ and $y = d + e\sqrt[3]{2} + f(\sqrt[3]{2})^2$

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$x + y = a + d + (b + e)\sqrt[3]{2} + c + f(\sqrt[3]{2})^2$, so closure under addition.

Closed under multiplication since $\{1, \sqrt[3]{2}, \sqrt[3]{2}^2\}$ is closed under multiplication

Let $f: \gamma \to \gamma \times \alpha$ be a linear map. Since F is a subobject of $\mathbb{R}$, it is a domain. So our map is injective and surjective, since F is finite vsp over Q. So $\exists\, \beta$ s.t. $\beta \times \alpha = 1$. $\beta = \alpha^{-1}$ □

b. $Q[\sqrt{2}, \sqrt{3}] = R = \{ a + b\sqrt{2} + c\sqrt{3} + d\sqrt{2}\sqrt{3} \mid a, b, c, d \in Q\}$

By computing sums and products our set R is a subring of C.

So we have identity, closure, associativity, commutativity and inverses for addition, identity for multiplication. Can easily check for associativity, commutativity, and identity for multiplication. α is injective and surjective because this is multiplication by some element a domain. Therefore $\exists\, \beta$ s.t. $\beta \times \alpha = 1$.

c. Assume σ is an automorphism. Then $\sigma(1) = 1$, $\sigma(n) = n$, $\sigma(n+1) = \sigma(n) + \sigma(1) = n+1$.

We have trivial automorphisms, and one negating $\sqrt{2}$ and another negating $\sqrt{3}$.

automorphism negating $\sqrt{2}$ is an automorphism.

$(a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6})(u + v\sqrt{2} + x\sqrt{3} + y\sqrt{6}) = (au + 2bv + 3cx + 6dy) + (av + bu + 3cy + 3dx)\sqrt{2} + (ax + 2by + cu + 2dv)\sqrt{3} + (ay + bx + cv + du)\sqrt{6}$

Similarly automorphism negating $\sqrt{3}$ and automorphism negating $\sqrt{6}$ are both automorphisms. So there are 4 total.

Fields 27. Let F be a field. Let R be the set of 2×2 matrices of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ for $a, b \in F$, usual matrix operations.

a) Show R is commutative ring with I and set of diagonal matrices are naturally isomorphic to F. Show.

1) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = 0 = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

By calculation multiplication is commutative and associative in R.

2) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} -a & b \\ -b & -a \end{bmatrix}$ and $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ Show.

3) $\forall\, r, s \in R$, $(-r)(s) = r(-s) = -(rs)$

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4) $xy = yx$

b) For some nonzero matrix $A \in R$, A has an inverse in $R \leftrightarrow \det A \neq 0$.

Let $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ $\det A = a^2 + b^2 \neq 0$

$\det A \neq 0$ in two $\mathbb{Q}, \mathbb{R}, F_7$ but in $\mathbb{C}$ $(i^2+1=0)$ and F_5 $1^2+2^2=0$.

c) All non zero matrices $A \in R$ have some $B \in R$ s.t. $AB = I$.

if $\det A \neq 0$, then $a^2+b^2 \neq 0$, so for fields where this holds.

d)

F is a field if $x^2+y^2=0$ has no solutions in F.

$3^2=9$, $4^2=16$ Cannot be squares of integers, which are 1 mod 4 or 0.

e)

Fermat's little thm says if $p \equiv 1 \bmod 4$, $p = a^2+b^2$.

So F is a field for $p \equiv 3 \pmod 4$.

2) $Span_F(Span_F(S)) = Span_F(S)$

This is true by same argument in 1). → Say $Span_F(S)$ is vector subspace and you are applying the previous part to this.

Not the same arguement as S is an arbitrary set.

3) $Span_F\left(\bigcup_{i\in I} S_i\right) = \sum_{i\in I} Span_F(S_i)$

Need not be a Vector space.

$\bigcup_{i\in I} Span_F(S_i) = Span_F(\bigcup_i S_i)$ ~~?~~. Not true

→ Vector space

Clearly $\bigcup_{i\in I} Span_F(S_i) \subseteq Span_F(\bigcup_i S_i)$ and $Span_F(\bigcup_i S_i) \subseteq \bigcup_{i\in I} Span_F(S_i)$.

Then the result follows from Subobject 5.

?

4) $Span_F\left(\bigcap_{i\in I} S_i\right) \subseteq \bigcap_{i\in I} Span_F(S_i)$

Let $v \in Span_F(\bigcap_{i\in I} S_i)$. Then $\exists\ x_1, x_2, \ldots, x_n \in Span_F(\bigcap_{i\in I} S_i)$ such that $v = a_1x_1 + a_2x_2 + \ldots + a_nx_n$, $a_1, \ldots, a_n \in F$. $x_1, x_2, \ldots, x_n \in S_i$, for $i = 1, \ldots, n$. So $v = a_1x_1 + a_2x_2 + \ldots + a_nx_n \in Span(S_i)$ for $i = 1, \ldots, n$. Since is in Span of each S_i, $v \in \bigcap_{i\in I} Span_F(S_i)$.

Therefore $Span_F(\bigcap_{i\in I} S_i) \subseteq \bigcap_{i\in I} Span_F(S_i)$.

Example of equality not holding.

Let $V = \mathbb{R}$, $S_1 = \{1\}$, $S_2 = \{2\}$. So $S_1 \cap S_2 = \emptyset$, $span_F(S_1 \cap S_2) = \{0\}$.

But $span(S_1) = span(S_2) = \mathbb{R} \rightarrow Span(S_1) \cap Span(S_2) = \mathbb{R}$.