Stat 310A/Math 230A Theory of Probability

Homework 3 Solutions

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Exercises on convergence theorems

Exercise [1.3.7]

If $h = I_B$ is an indicator function, this follows from the definition. Linearity of the integration extends the result to simple functions, and then the monotone convergence theorem gives the result for nonnegative functions. Finally, by taking positive and negative parts we get the result for all integrable functions.

Exercise [1.3.21]

(a). Cauchy-Schwarz implies

$$(\mathbf{E}YI_{(Y>a)})^2 \le \mathbf{E}Y^2\mathbf{P}(Y>a)$$

For $\mathbf{E}Y > a \ge 0$ the left hand side is larger than $(\mathbf{E}Y - a)^2$ so rearranging gives the desired result.
(b). This follows again from Cauchy-Schwarz noting that $\{\mathbb{E}|Y^2 - v|\}^2 \le \mathbb{E}\{(Y-\sqrt{v})^2\}\mathbb{E}\{(Y+\sqrt{v})^2\}$.
(c). Let $B_1 = \emptyset$ and $B_i = \bigcup_{j=1}^{i-1} A_j$ for $i = 2, \dots, n$. Noting that $C_i = A_i \cap B_i^c$ are disjoint sets, such that $\bigcup_{i=1}^n A_i = \bigcup_{i=1}^n C_i$, we have by the additivity of probability measures that

$$\mathbf{P}(\bigcup_{i=1}^n A_i) = \mathbf{P}(\bigcup_{i=1}^n C_i) = \sum_{i=1}^n \mathbf{P}(C_i) = \sum_{i=1}^n \mathbf{P}(A_i) - \sum_{i=2}^n \mathbf{P}(A_i \cap B_i)$$

(omitting the zero probability of $A_1 \cap B_1 = \emptyset$ on the right side). Since $A_i \cap B_i \subseteq \bigcup_{j<i} A_i \cap A_j$, we get by sub-additivity of probability measures that for $i = 2, \dots, n$,

$$\mathbf{P}(A_i \cap B_i) \le \sum_{j=1}^{i-1} \mathbf{P}(A_i \cap A_j)\,,$$

which by the preceeding identity results with the second Bonferroni inequality.

For $Y = \sum_{i=1}^n I_{A_i}$ we have that $\{Y > 0\} = \bigcup_{i=1}^n A_i$ whereas

$$m = \mathbf{E}Y = \sum_{i=1}^n \mathbf{P}(A_i)\;, \qquad v = \mathbf{E}Y^2 = \sum_{i,j=1}^n \mathbf{P}(A_i \cap A_j)\;.$$

Excluding the trivial case where $\mathbf{P}(A_i) = 0$ for all i, we have from part (a) that $\mathbf{P}(Y > 0) \ge m^2/v \ge 2m - v$. That is,

$$\mathbf{P}(\bigcup_{i=1}^n A_i) \ge \sum_{i=1}^n \mathbf{P}(A_i) - 2\sum_{1\le j<i\le n} \mathbf{P}(A_i \cap A_j)\,.$$

Part (b) improves on the latter bound by removing the factor 2 in the right-most correlation term. However, this improvement is somewhat irrelevant if one uses such bounds to approximate $\mathbf{P}(Y > 0)$ by $\mathbf{E}Y = \sum_{i=1}^n \mathbf{P}(A_i)$ upon showing that the correlation term is *much smaller* than $\mathbf{E}Y$.

Exercise [1.3.36]

Let $Z_n(\omega) := \sup_{m\geq n} |X_m(\omega)|$ noting that $Z_n \downarrow Z$ for every ω, with $Z \geq 0$. Since $X_n(\omega) \to 0$ if and only if $Z(\omega) = 0$, the a.s. convergence of X_n to 0 is equivalent to $\mathbf{P}(Z > \epsilon) = 0$ for each $\epsilon > 0$. Note that $\{Z_n > \epsilon\} \downarrow \{Z > \epsilon\}$, hence $X_n \overset{a.s.}{\to} 0$ if and only if for each $\epsilon > 0$ there is n such that $\mathbf{P}(Z_n > \epsilon) < \epsilon$. To complete the proof observe next that $\{|X_M| > \epsilon\} \subseteq \{Z_n > \epsilon\}$ for any random integer $M(\omega) \geq n$, with set equality for

$$M(\omega) = \inf\{m \geq n \colon |X_m(\omega)| > \epsilon\}$$

in case $Z_n(\omega) > \epsilon$ and $M(\omega) = n$ otherwise.

Exercise [1.3.37]

1. Fix a Borel set $B \subseteq \mathcal{B}$. Note that

   $$\{\omega : Y_N(\omega) \in B\} = \bigcup_{n=1}^{\infty} (Y_n^{-1}(B) \cap N^{-1}(\{n\}))\ .$$

   Since Y_n and N are random variables, $Y_n^{-1}(B) \in \mathcal{F}$ and $N^{-1}(\{n\}) \in \mathcal{F}$, implying that $\{\omega : Y_N(\omega) \in B\} \in \mathcal{F}$. With B arbitrary Borel set, we see that Y_N is a random variable.

2. Note that $|Y_{N_k}(\omega) - Y_\infty(\omega)| \to 0$ for any ω such that $N_k(\omega) \to \infty$ and $|Y_n(\omega) - Y_\infty(\omega)| \to 0$. The result follows by the definition of a.s. convergence.

3. Take as Y_n the random variable X_n of Example **??**. Then, as shown there $Y_n \overset{p}{\to} 0$ while for each $\omega \in (0, 1]$ we have that $Y_n(\omega) = 1$ for infinitely many values of n. The latter property implies that $N_k = \min\{\ell \geq k : Y_\ell = 1\}$ are finite. Further, $N_k(\omega) \to \infty$ (since $N_k \geq k$) and by definition $Y_{N_k} = 1$ for all k and ω, as needed.

4. Let $Z_r = \max_{n>r} |Y_n - Y_\infty|$. For any $\epsilon > 0$ and positive integers k, r,

   $$\mathbf{P}(|Y_{N_k} - Y_\infty| > \epsilon) \leq \mathbf{P}(Z_r > \epsilon) + \mathbf{P}(N_k \leq r)\ .$$

   Fix $\epsilon > 0$ and $r < \infty$. Considering $k \to \infty$ you have

   $$\limsup_{k\to\infty} \mathbf{P}(|Y_{N_k} - Y_\infty| > \epsilon) \leq \mathbf{P}(Z_r > \epsilon)\ .$$

   Since $Y_n \overset{a.s.}{\to} Y_\infty$ also $Z_r \overset{a.s.}{\to} 0$, implying that $\mathbf{P}(|Y_{N_k} - Y_\infty| > \epsilon) \to 0$ as needed.