Physics-Based Animation

Some Slides/Images adapted from Marschner and Shirley

Physics-Based Animation

Agenda

- Newton’s Laws of Motion
- The Mass-Spring System
- Implicit Integration via Optimization
- A Local-Global Solver for Fast-Mass Springs

Newton's Laws

1. Every object will remain at rest or in uniform motion in a straight line unless compelled to change its state by the action of an external force.

2. The force acting on an object is equal to the time rate-of-change of the momentum.

3. For every action there is an equal and opposite reaction.

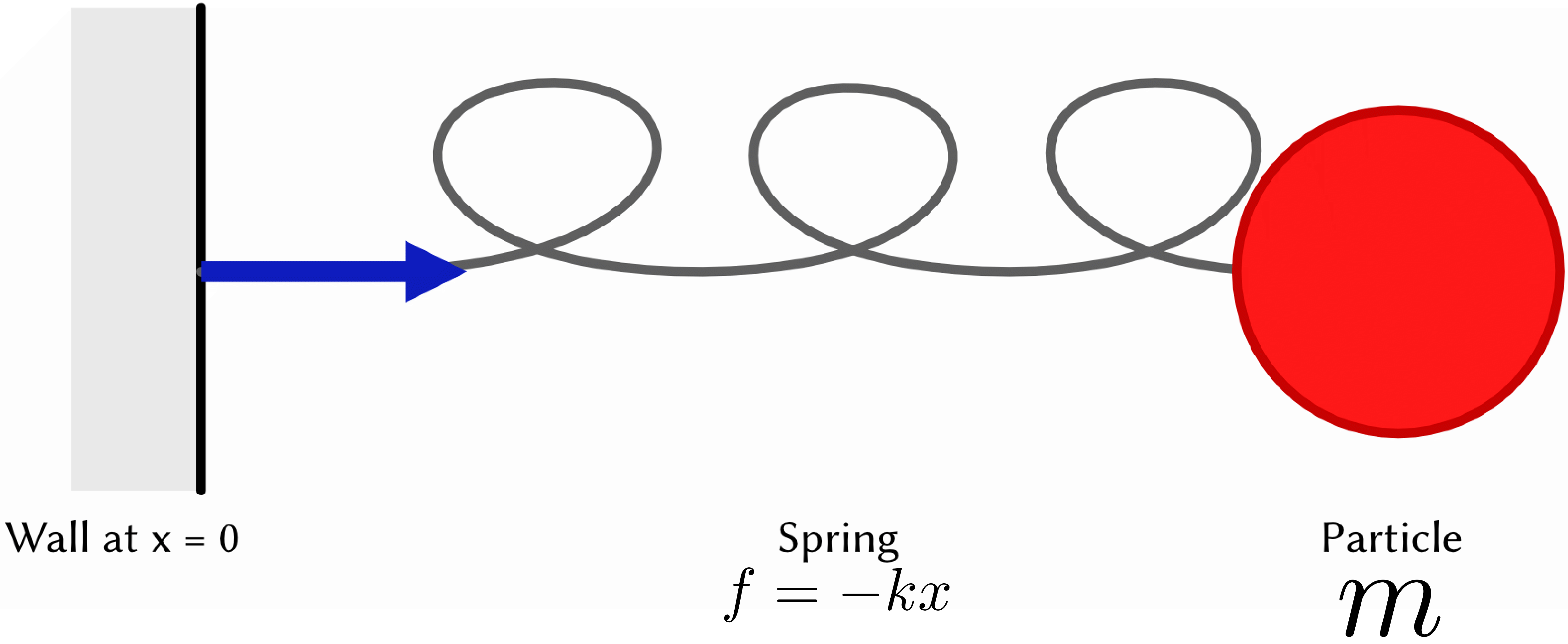
Newton's Second Law

Acceleration

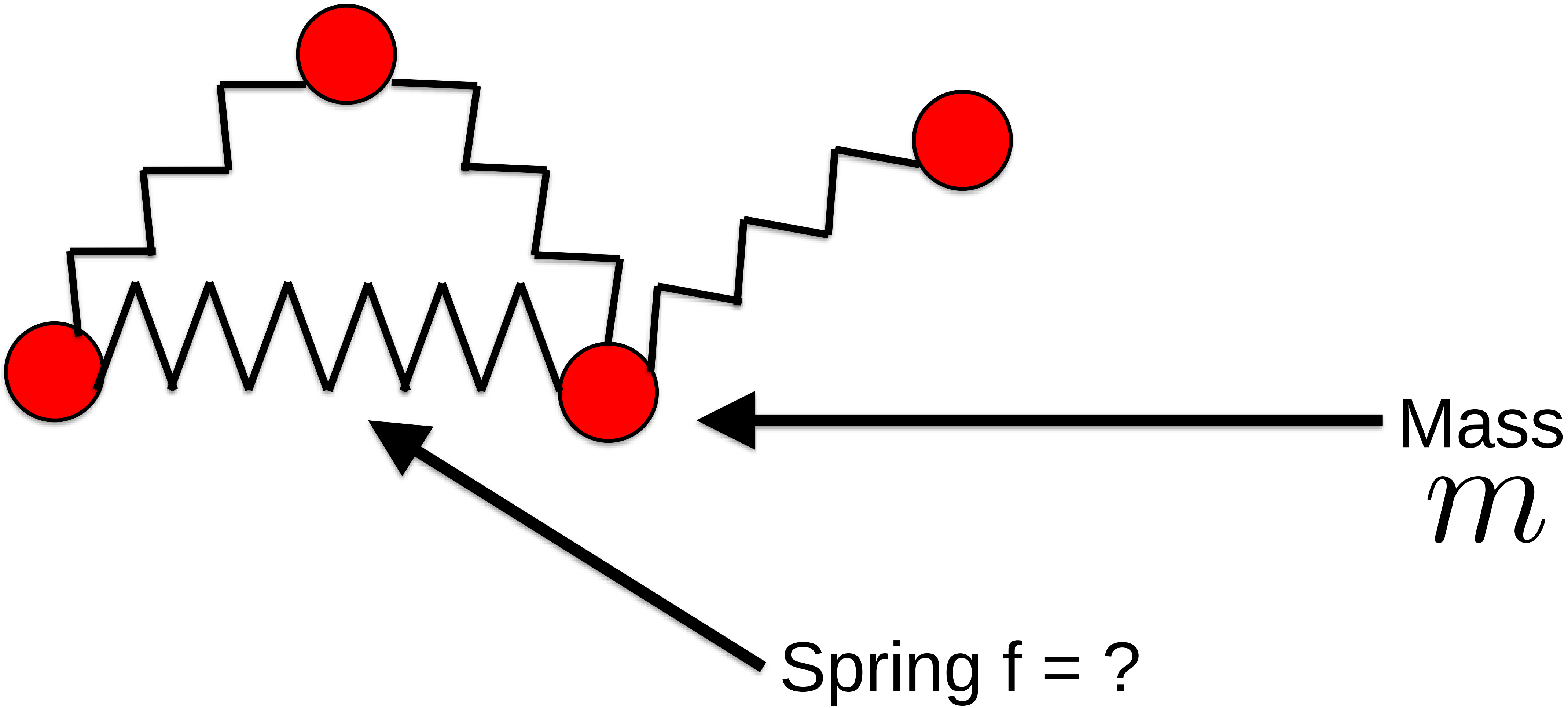
$$ma = f$$

Mass

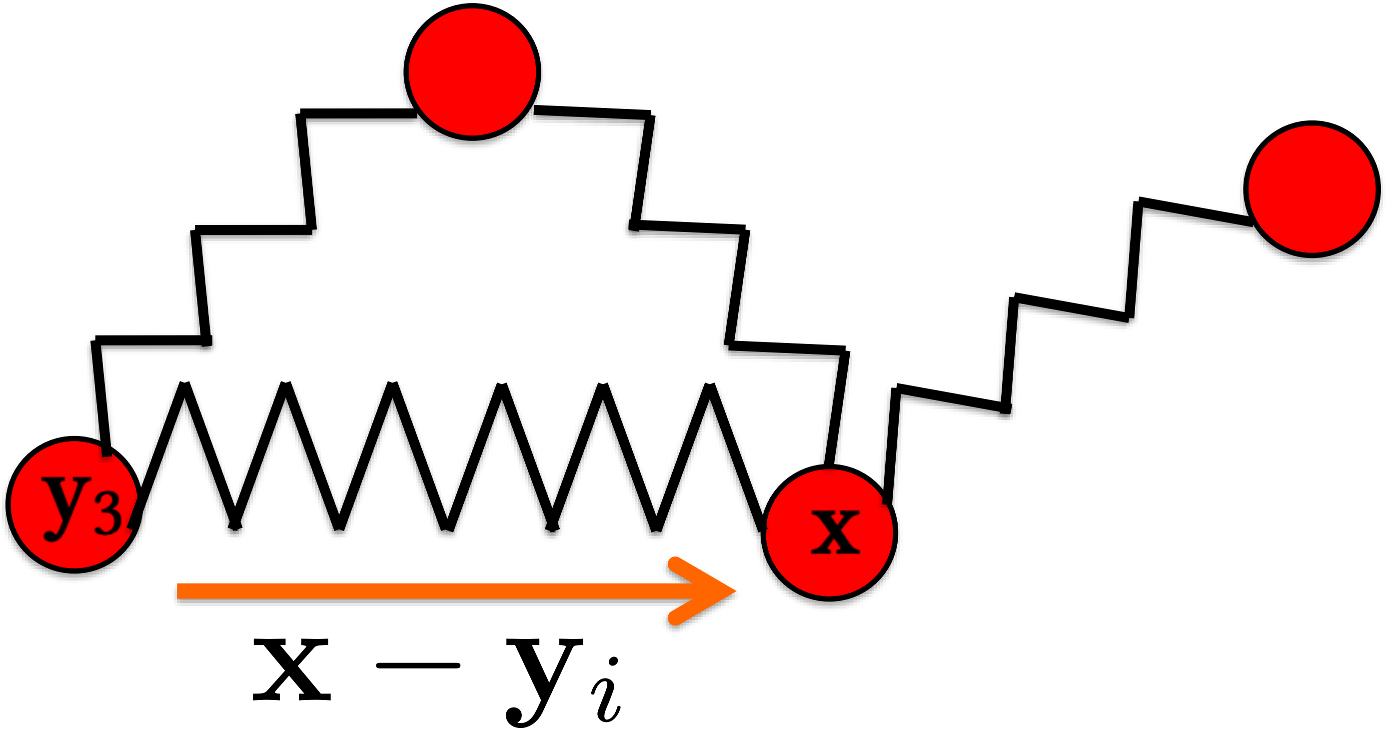
force


Wall at x = 0
Spring
$f = -kx$
Particle
m

The Mass-Spring System



The Mass-Spring System

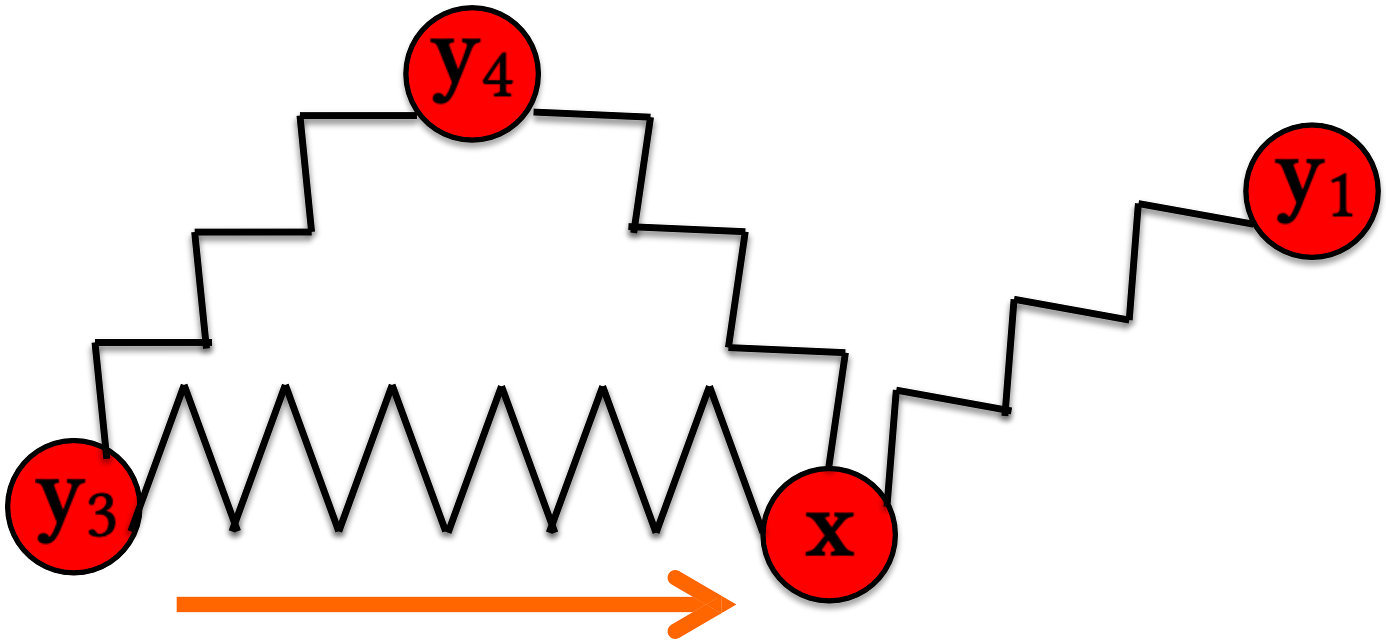


Direction of Spring

Rest Length

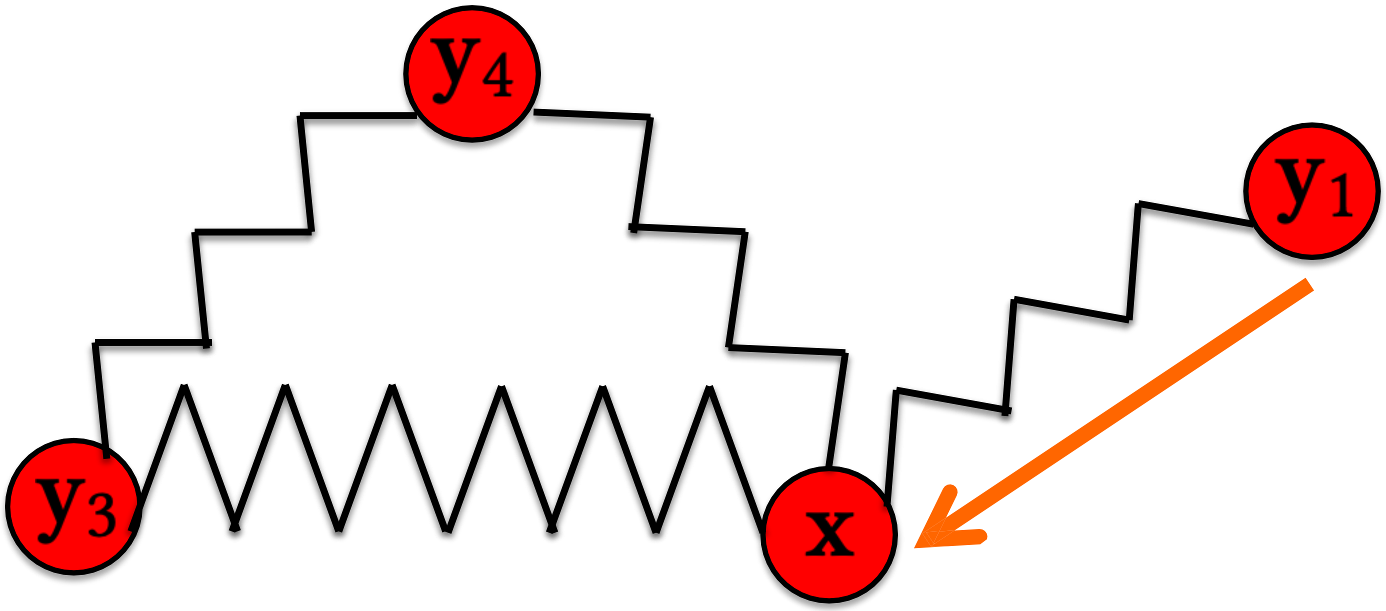
$$\mathbf{f}_x = -k\left(\underbrace{\|\mathbf{x}-\mathbf{y_i}\| - r_{ix}}_{\text{Change in Length}}\right)\frac{\mathbf{x}-\mathbf{y_i}}{\|\mathbf{x}-\mathbf{y_i}\|}$$

Newton's Second Law for Each Particle



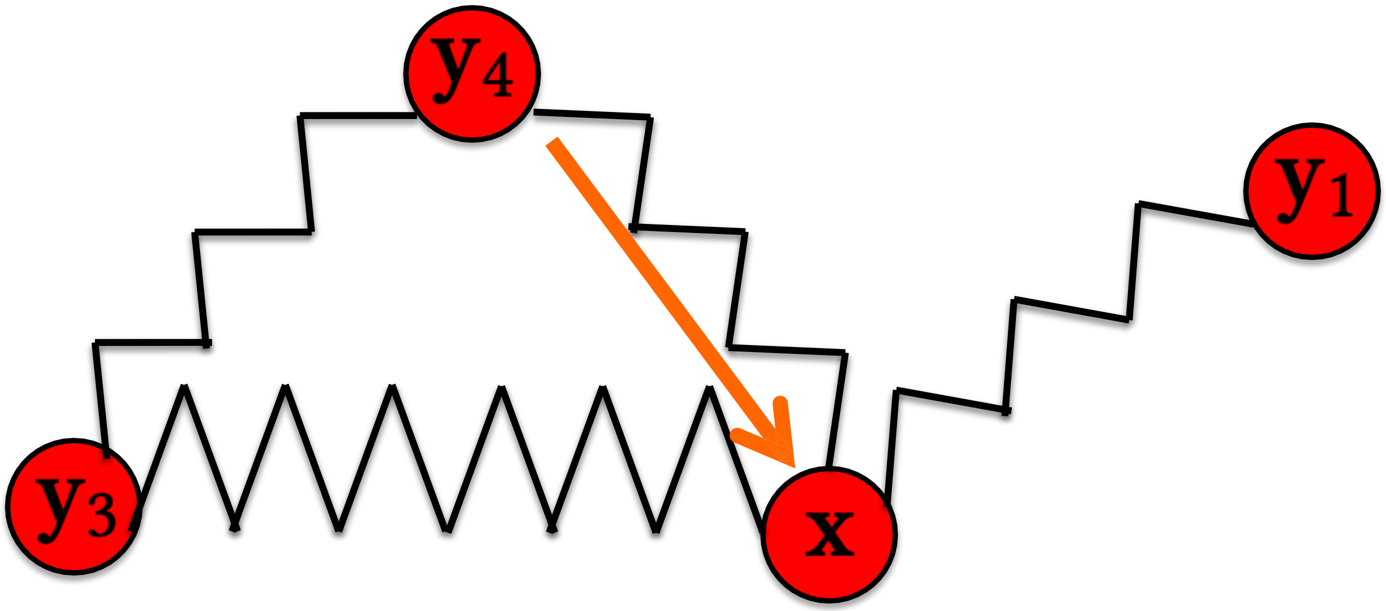
$$m_x \mathbf{a}_x = \sum_i \mathbf{f}_x(\mathbf{y}_i)$$

Newton's Second Law for Each Particle



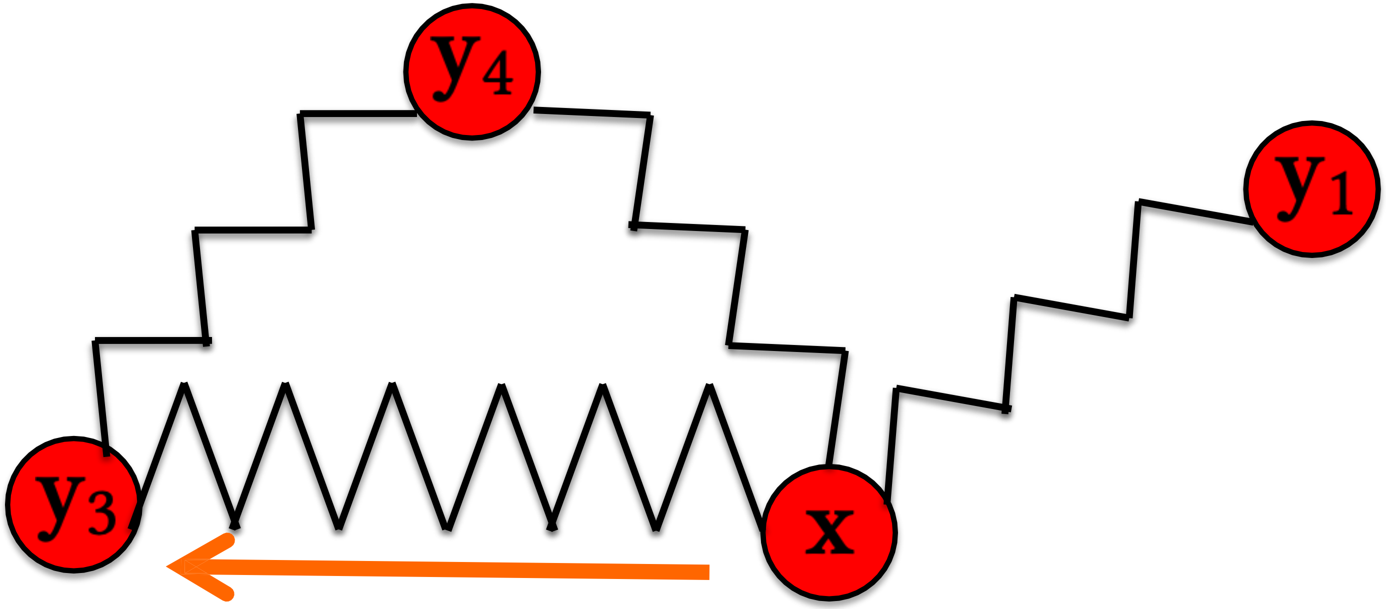
$$m_x \mathbf{a}_x = \sum_i \mathbf{f}_x(\mathbf{y}_i)$$

Newton's Second Law for Each Particle



$$m_x \mathbf{a}_x = \sum_i \mathbf{f}_x(\mathbf{y}_i)$$

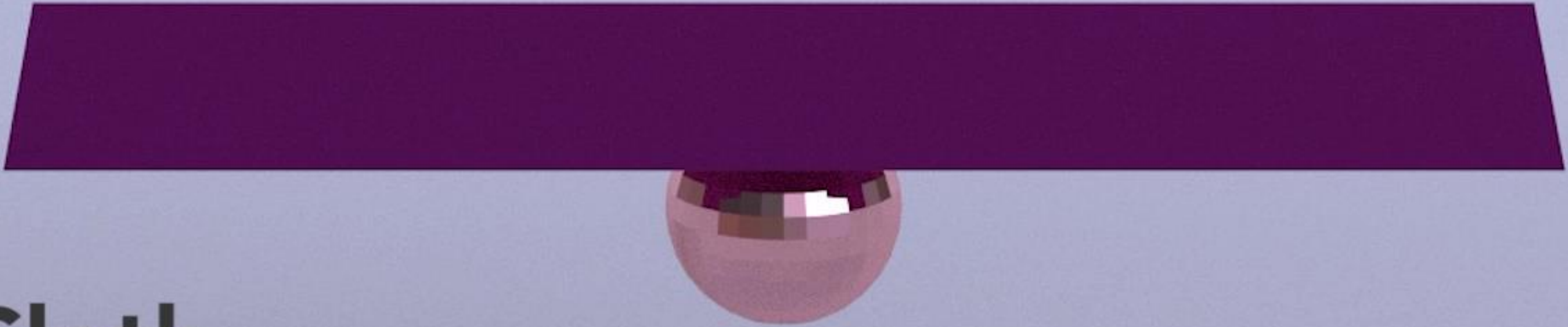
Newton's Second Law for Each Particle



$$m_x \mathbf{a}_x = \sum_i \mathbf{f}_x(\mathbf{y}_i)$$

One equation for each object/particle.

We will solve them all together.

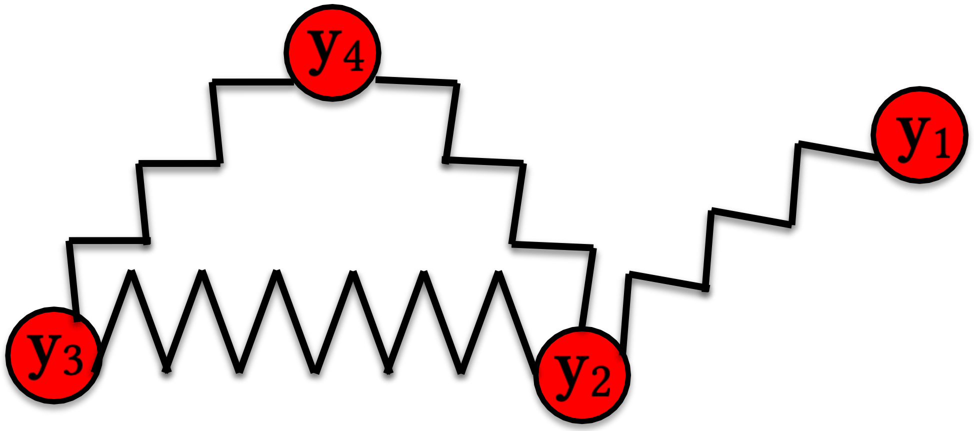


Cloth

SIMIT GPU

15,630 Triangles
7,988 Verts
14 FPS

Newton's Second Law: System of Equations



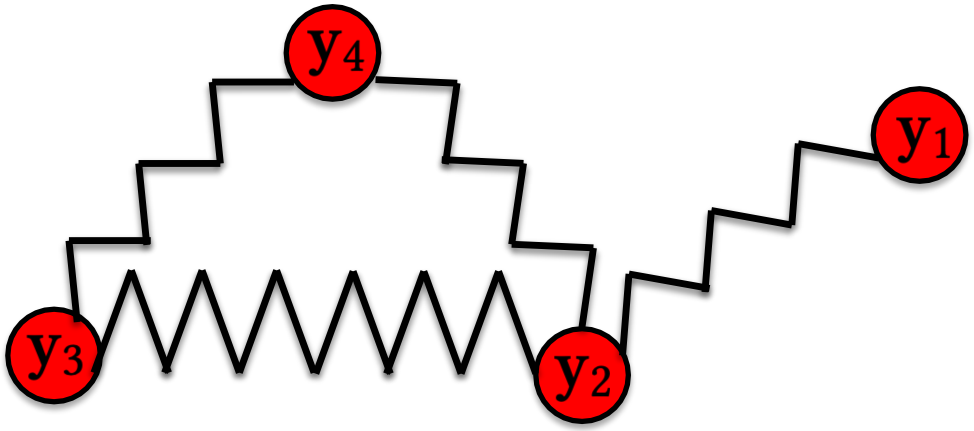
$$m_1\mathbf{a}_1 = \sum_i \mathbf{f}_1(\mathbf{y}_i)$$

$$m_2\mathbf{a}_2 = \sum_i \mathbf{f}_2(\mathbf{y}_i)$$

$$m_3\mathbf{a}_3 = \sum_i \mathbf{f}_3(\mathbf{y}_i)$$

$$m_4\mathbf{a}_4 = \sum_i \mathbf{f}_4(\mathbf{y}_i)$$

Newton's Second Law: System of Equations



$$\begin{pmatrix} m_1 \cdot I & 0 & 0 & 0 \\ 0 & m_2 \cdot I & 0 & 0 \\ 0 & 0 & m_3 \cdot I & 0 \\ 0 & 0 & 0 & m_4 \cdot I \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \\ \mathbf{a}_4 \end{pmatrix} = \begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \\ \mathbf{f}_4 \end{pmatrix}$$

Mass Matrix $\qquad \mathbf{a}(t) \qquad \mathbf{f}(t)$

Time Integration

$$M\mathbf{a}(t) = \mathbf{f}(\mathbf{y}(t))$$

$$M\frac{d^2\mathbf{y}(t)}{dt^2} = \mathbf{f}(\mathbf{y}(t))$$

Time Integration

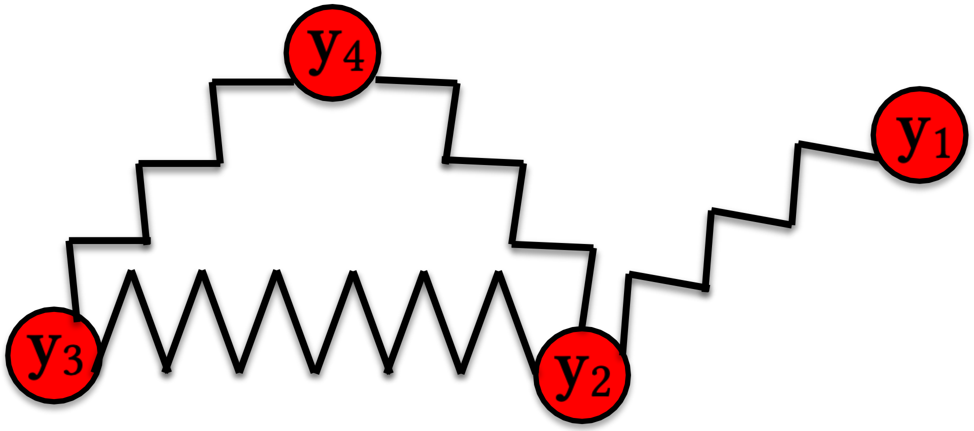
$$M \frac{d^2 \mathbf{y}(t)}{dt^2} = \mathbf{f}(\mathbf{y}(t))$$

Use Finite Differences!

$$\frac{d^2 \mathbf{y}(t)}{dt^2} \approx \frac{\mathbf{y}^{t+1} - 2\mathbf{y}^{t} + \mathbf{y}^{t-1}}{\Delta t^2}$$

https://en.wikipedia.org/wiki/Finite_difference#Higher-order_differences
https://en.wikipedia.org/wiki/Finite_difference_coefficient#Central_finite_difference

Time Integration



Need to Discretize!

$$M\frac{d^2\mathbf{y}(t)}{dt^2} = \boxed{\mathbf{f}(\mathbf{y}(t))}$$

Use Finite Differences!

$$\frac{d^2\mathbf{y}(t)}{dt^2} \approx \frac{\mathbf{y}^{t+1} - 2\mathbf{y}^{t} + \mathbf{y}^{t-1}}{\Delta t^2}$$

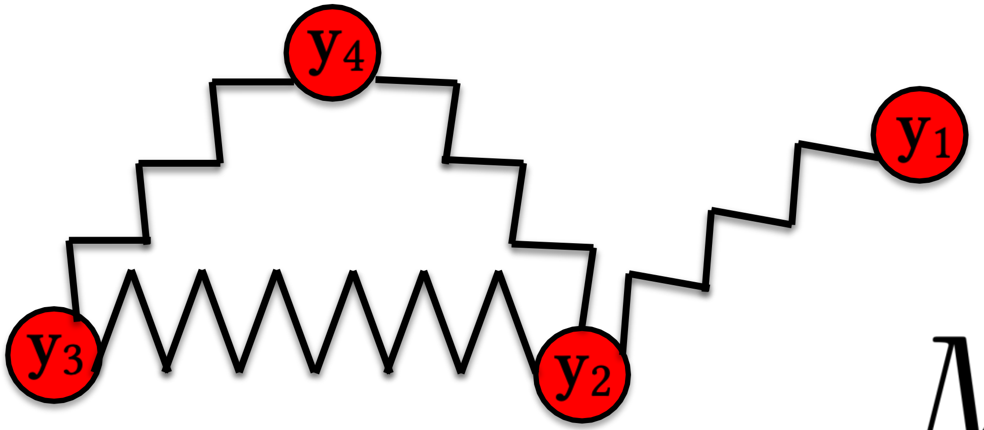
Time Integration: Explicit vs. Implicit

$$M \frac{d^2 \mathbf{y}(t)}{dt^2} = \mathbf{f}(\mathbf{y}(t))$$

Explicit: $y_{t+dt} = g(y_t)$. Future state y_{t+dt} is an explicit equation of current state y_t and dt.

Implicit: $h(y_t, y_{t+dt})=0$. Future state y_{t+dt} is an implicit equation.

Implicit Time Integration

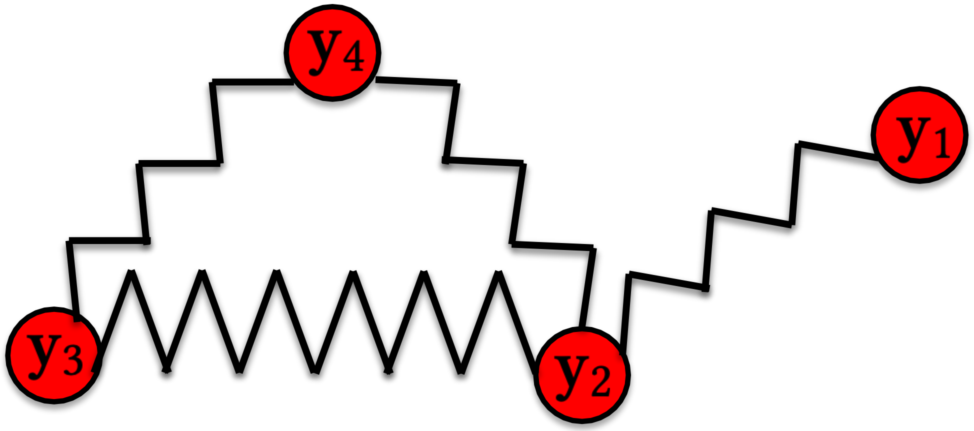


$$M\frac{d^2\mathbf{y}}{dt^2}(t) = \mathbf{f}\left(\mathbf{y}^{t+1}\right)$$

Use Finite Differences!

$$\frac{d^2\mathbf{y}(t)}{dt^2} \approx \frac{\mathbf{y}^{t+1} - 2\mathbf{y}^{t} + \mathbf{y}^{t-1}}{\Delta t^2}$$

Implicit Time Integration



$$M\left(\frac{\mathbf{y}^{t+1}-2\mathbf{y}^{t}+\mathbf{y}^{t-1}}{\Delta t^2}\right)=\mathbf{f}(\mathbf{y}^{t+1})$$

$$M\mathbf{y}^{t+1}=M\left(2\mathbf{y}^{t}-\mathbf{y}^{t-1}\right)+\Delta t^2\mathbf{f}\left(\mathbf{y}^{t+1}\right)$$

Goal: Solve for $\mathbf{y}^{t+1}$

Implicit Integration as Optimization

Rather than directly solve:

$$M\mathbf{y}^{t+1} - M(2\mathbf{y}^{t} - \mathbf{y}^{t-1}) - \Delta t^2 \mathbf{f}(\mathbf{y}^{t+1}) = 0$$

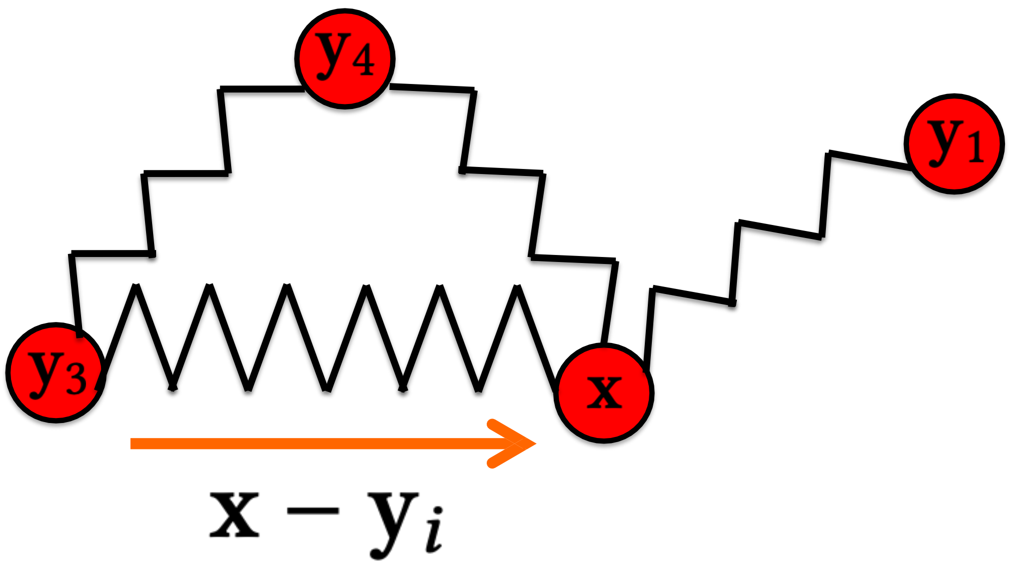
We can view the mass-force equtions as an energy function **E(q)** whose gradient vanishes as above:

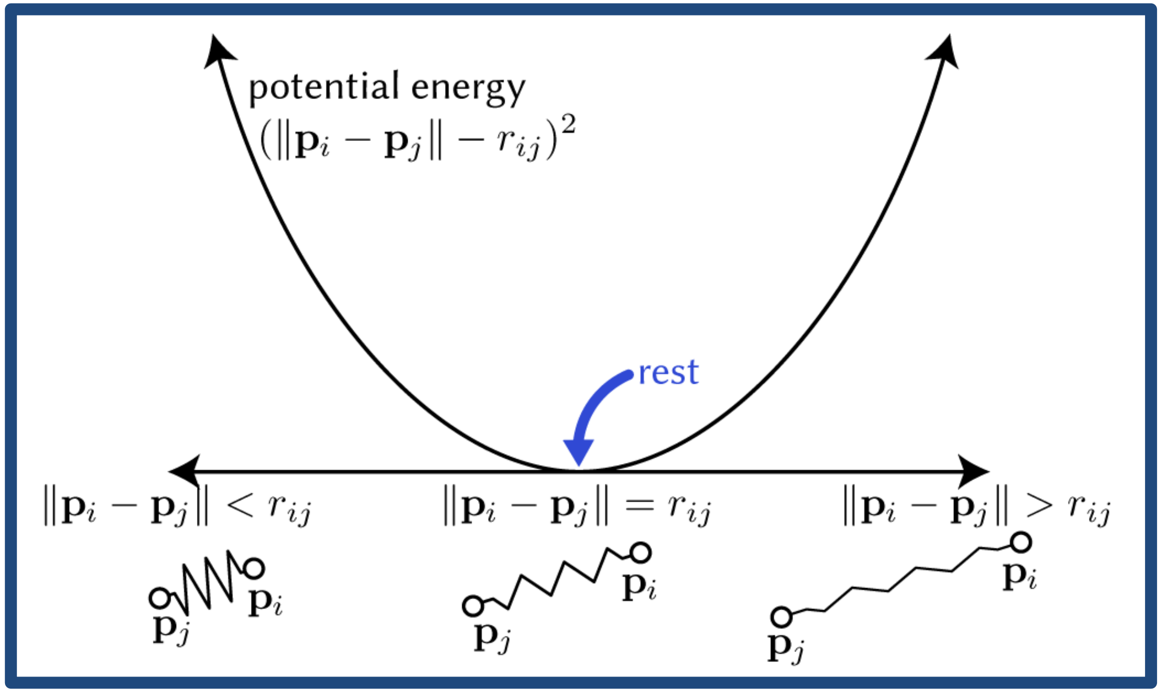
$$\nabla_{\mathbf{q}} E\left(\mathbf{y}^{t+1}\right) = 0$$

Turning integration into an optimization problem:

$$\mathbf{y}^{t+1} = \arg\min_{\mathbf{q}} E\left(\mathbf{q}\right)$$

Mass-Spring Potential Energy





$$\boxed{V(\mathbf{y}_i, \mathbf{x})} = \frac{1}{2}k\big(\underbrace{\|\mathbf{y}_i - \mathbf{x}\| - \overset{\text{Rest Length}}{r_{ix}}}_{\text{Change in Length}}\big)^2$$

$\boxed{-\nabla V}$

$$\mathbf{f}_x(\mathbf{y}_i) = -k(\|\mathbf{x} - \mathbf{y}_i\| - r_{ix})\frac{\mathbf{x} - \mathbf{y}_i}{\|\mathbf{x} - \mathbf{y}_i\|}$$

Energy

$$\mathbf{y}^{t+1} = \operatorname*{argmin}_{\mathbf{y}} \underbrace{\boxed{\left(\sum_{ij} \frac{1}{2} k(\|\mathbf{y}_i - \mathbf{y}_j\| - r_{ij})^2\right)} - \frac{\Delta t^2}{2}\left(\sum_i m_i \left(\frac{\mathbf{y}_i - 2\mathbf{y}_i^t + \mathbf{y}_i^{t-1}}{\Delta t^2}\right)^2\right) - \left(\sum_i \mathbf{y}_i^\top \mathbf{f}_i^{\text{ext}}\right)}_{E(\mathbf{y})}$$

Potential energy

$$V(\mathbf{y}_i, \mathbf{x}) = \frac{1}{2} k(\|\mathbf{y}_i - \mathbf{x}\| - r_{ix})^2$$

Energy

$$\mathbf{y}^{t+1} = \underset{\mathbf{y}}{\operatorname{argmin}} \underbrace{\left(\sum_{ij} \frac{1}{2} k(\|\mathbf{y}_i - \mathbf{y}_j\| - r_{ij})^2\right) - \frac{\Delta t^2}{2} \boxed{\left(\sum_i m_i \left(\frac{\mathbf{y}_i - 2\mathbf{y}_i^t + \mathbf{y}_i^{t-1}}{\Delta t^2}\right)^2\right)} - \left(\sum_i \mathbf{y}_i^\top \mathbf{f}_i^{\text{ext}}\right)}_{E(\mathbf{y})}$$

Potential energy

$$V(\mathbf{y}_i, \mathbf{x}) = \frac{1}{2} k(\|\mathbf{y}_i - \mathbf{x}\| - r_{ix})^2$$

0.5*ma^2
Kinetic energy-like

$$\mathbf{a}_i^t = \ddot{\mathbf{y}}_i^t = \frac{d^2 \mathbf{y}_i(t)}{dt^2} \approx \frac{\mathbf{y}_i^{t+1} - 2\mathbf{y}_i^t + \mathbf{y}_i^{t-1}}{\Delta t^2}$$

Energy

$$\mathbf{y}^{t+1} = \underset{\mathbf{y}}{\operatorname{argmin}} \underbrace{\left(\sum_{ij} \frac{1}{2} k(\|\mathbf{y}_i - \mathbf{y}_j\| - r_{ij})^2\right) - \frac{\Delta t^2}{2} \left(\sum_i m_i \left(\frac{\mathbf{y}_i - 2\mathbf{y}_i^t + \mathbf{y}_i^{t-1}}{\Delta t^2}\right)^2\right) - \boxed{\left(\sum_i \mathbf{y}_i^\top \mathbf{f}_i^{\text{ext}}\right)}}_{E(\mathbf{y})}$$

External forces

Potential energy force

$$V(\mathbf{y}_i, \mathbf{x}) = \frac{1}{2} k(\|\mathbf{y}_i - \mathbf{x}\| - r_{ix})^2$$

0.5*ma^2

Kinetic energy-like

$$\mathbf{a}_i^t = \ddot{\mathbf{y}}_i^t = \frac{d^2 \mathbf{y}_i(t)}{dt^2} \approx \frac{\mathbf{y}_i^{t+1} - 2\mathbf{y}_i^t + \mathbf{y}_i^{t-1}}{\Delta t^2}$$

Energy

Construct a function ***E***, such that its minimizer is a simulation solution ***∇E = f-ma***

$$\mathbf{y}^{t+1} = \operatorname*{argmin}_{\mathbf{y}} \underbrace{\left(\sum_{ij} \frac{1}{2} k(\|\mathbf{y}_i - \mathbf{y}_j\| - r_{ij})^2\right) - \frac{\Delta t^2}{2} \left(\sum_i m_i \left(\frac{\mathbf{y}_i - 2\mathbf{y}_i^t + \mathbf{y}_i^{t-1}}{\Delta t^2}\right)^2\right) - \left(\sum_i \mathbf{y}_i^\top \mathbf{f}_i^{\text{ext}}\right)}_{E(\mathbf{y})}$$

…verify that ***∇E = 0*** is indeed the force equation below.

$$M\left(\frac{\mathbf{y}^{t+1} - 2\mathbf{y}^t + \mathbf{y}^{t-1}}{\Delta t^2}\right) = \mathbf{f}(\mathbf{y}^{t+1})$$

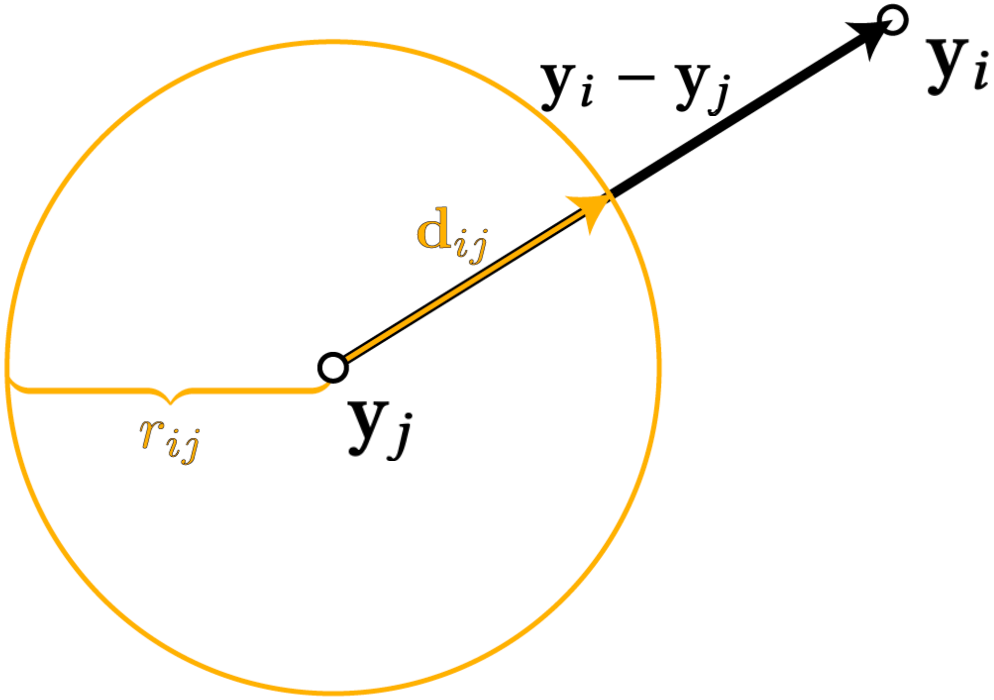
Energy

$$\mathbf{y}^{t+1} = \underset{\mathbf{y}}{\mathrm{argmin}} \underbrace{\left(\sum_{ij} \frac{1}{2} k \boxed{(\|\mathbf{y}_i - \mathbf{y}_j\| - r_{ij})}^2 \right) - \frac{\Delta t^2}{2} \left(\sum_i m_i \left(\frac{\mathbf{y}_i - 2\mathbf{y}_i^t + \mathbf{y}_i^{t-1}}{\Delta t^2} \right)^2 \right) - \left(\sum_i \mathbf{y}_i^\top \mathbf{f}_i^{\mathrm{ext}} \right)}_{E(\mathbf{y})}$$

Non linear :(

Observation!

$$(\|\mathbf{y}_i - \mathbf{y}_j\| - r_{ij})^2 = \min_{\mathbf{d}_{ij} \in \mathbb{R}^3, \|\mathbf{d}\| = r_{ij}} \|(\mathbf{y}_i - \mathbf{y}_j) - \mathbf{d}_{ij}\|^2$$



Observation!

$$(\|\mathbf{y}_i - \mathbf{y}_j\| - r_{ij})^2 = \min_{\mathbf{d}_{ij} \in \mathbb{R}^3,\, \|\mathbf{d}\| = r_{ij}} \|(\mathbf{y}_i - \mathbf{y}_j) - \mathbf{d}_{ij}\|^2$$

$$\mathbf{y}^{t+1} = \operatorname*{argmin}_{\mathbf{y}} \underbrace{\underbrace{\left(\sum_{ij} \frac{1}{2} k \|(\mathbf{y}_i - \mathbf{y}_j) - \mathbf{d}_{ij}\|^2\right)}_{E_2(\mathbf{y})} - \underbrace{\frac{\Delta t^2}{2} \left(\sum_i m_i \left(\frac{\mathbf{y}_i - 2\mathbf{y}_i^t + \mathbf{y}_i^{t-1}}{\Delta t^2}\right)^2\right)}_{E_1(\mathbf{y})} - \left(\sum_i \mathbf{y}_i^\top \mathbf{f}_i^{\text{ext}}\right)}_{\tilde{E}(\mathbf{y})}$$

Quadratic!

Local-Global Solvers for Mass-Spring Systems

$$\mathbf{E}_1\left(\mathbf{y}^{t+1}\right) = \frac{1}{2}\left(\mathbf{y}^{t+1}\right)^T M\mathbf{y}^{t+1} - \left(\mathbf{y}^{t+1}\right)^T M\mathbf{b}$$

where $\mathbf{b} = 2\mathbf{y}^t - \mathbf{y}^{t-1}$

$$E_2 = \sum_{ij} \frac{k}{2}\left(\|\mathbf{y}_i - \mathbf{y_j}\|^2 - 2(\mathbf{y}_i - \mathbf{y}_j)^T\mathbf{d}_{ij} + \mathbf{d}_{ij}^T\mathbf{d}_{ij}\right)$$

Both energies are quadratic now. This will let us build a fast algorithm

We will do this using block coordinate descent. First optimize over one set of variables (the d’s) then the second set (the y’s) …. Rinse and repeat!

Local-Global Solvers for Mass-Spring Systems

WHILE Not done

 For Each Spring

  Local Optimization

 Global Optimization

END

Now we can start defining these steps for mass-springs

The Local Step

Hold **y** constant and optimize each spring vector **d**

$$\arg\min_{\mathbf{d}_{ij}, |\mathbf{d}_{ij}|=r_{ij}} \sum_{ij} \frac{k}{2}\|\mathbf{y}_i - \mathbf{y_j}\|^2 - 2(\mathbf{y}_i - \mathbf{y}_j)^T \mathbf{d}_{ij} + \mathbf{d}_{ij}^T \mathbf{d}_{ij}$$

Rotate **d**’s to align with current **y**’s.

$$E_{ij} = \arg\min_{\mathbf{d}_{ij}, |\mathbf{d}_{ij}|=r_{ij}} \underbrace{\frac{k}{2}\|\mathbf{y}_i - \mathbf{y_j} - \mathbf{d}_{ij}\|^2}_{\text{No sum anymore!}}$$

The Global Step

Minimizing wrt to **y** requires us to find

$$\mathbf{y}^{t+1} \text{ s.t. } \nabla_{\mathbf{y}}(E_1(\mathbf{y}) + \Delta t^2 E_2(\mathbf{y}, \mathbf{d}_{ij})) = \mathbf{0}$$

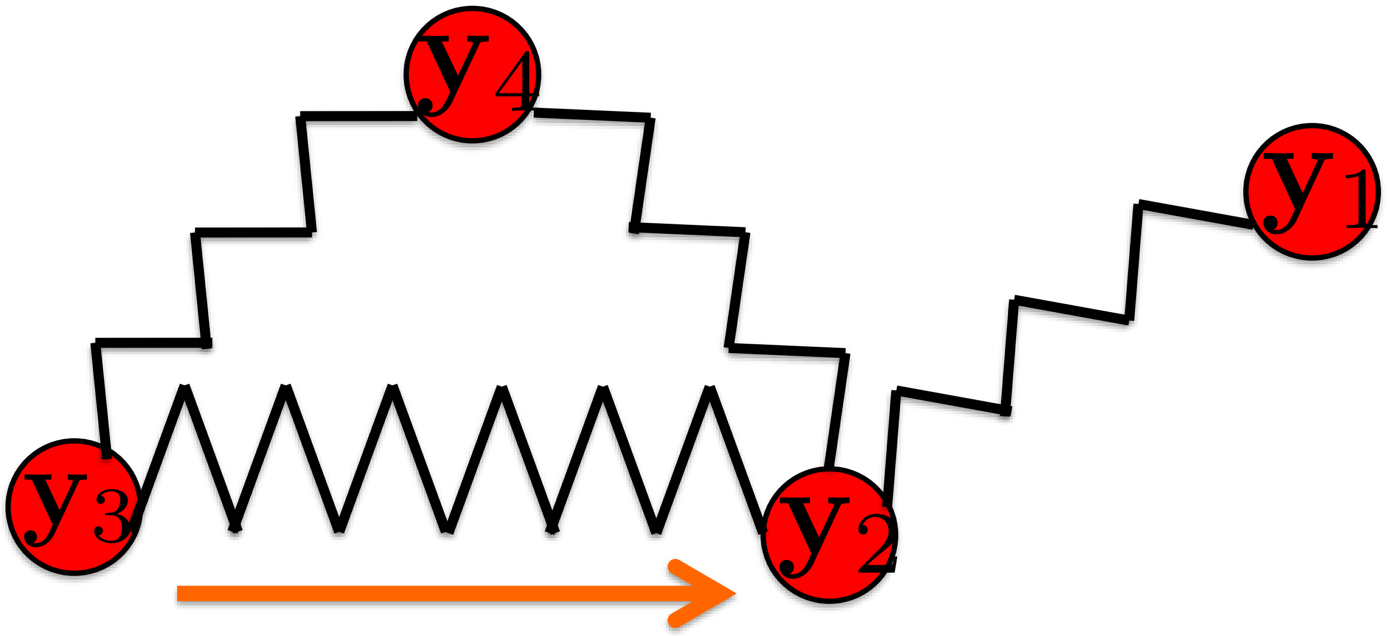
Recall $$\mathbf{E}_1(\mathbf{y}) = \frac{1}{2}\mathbf{y}^T M\mathbf{y} - \mathbf{y}^T M\mathbf{b}$$

$$\nabla \mathbf{E}_1 = M\mathbf{y} - M\mathbf{b}$$

$$\mathbf{b} = 2\mathbf{y}^t - \mathbf{y}^{t-1}$$

$$E_2 = \sum_{ij} \frac{k}{2} \|\mathbf{y}_i - \mathbf{y_j}\|^2 - 2(\mathbf{y}_i - \mathbf{y}_j)^T \mathbf{d}_{ij} + \mathbf{d}_{ij}^T \mathbf{d}_{ij}$$

The Global Step



$$\mathbf{y} = \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \mathbf{y}_3 \\ \mathbf{y}_4 \end{pmatrix}$$

$$\Delta \mathbf{y} = \underbrace{\begin{pmatrix} I & -I & 0 & 0 \\ 0 & I & -I & 0 \\ 0 & I & 0 & -I \\ 0 & -I & I & -I \end{pmatrix}}_{G} \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \mathbf{y}_3 \\ \mathbf{y}_4 \end{pmatrix}$$

Each row is a spring

The Global Step

Using this we can rewrite the second energy as

$$E_2 = \frac{k}{2}\left(\mathbf{y}G^T G\mathbf{y} - 2\mathbf{y}^T G^T \mathbf{d} + \mathbf{d}^T \mathbf{d}\right)$$

$$\mathbf{d} = \begin{pmatrix} \mathbf{d}_{12} \\ \mathbf{d}_{23} \\ \mathbf{d}_{24} \\ \mathbf{d}_{34} \end{pmatrix}$$

So the gradient becomes

$$\nabla E_2 = kG^T G\mathbf{y} - k\mathbf{G}^T \mathbf{d}$$

And the total global step finds y so that

$$\nabla(E_1 + \Delta t^2 E_2) = (M + \Delta t^2 k G^T G)\mathbf{y} - (M\mathbf{b} - \Delta t^2 k\mathbf{G}^T \mathbf{d}) = 0$$

The Global Step

And the total global step finds y so that

$$\nabla(E_1 + \Delta t^2 E_2) = (M + \Delta t^2 k G^T G)\mathbf{y} - (M\mathbf{b} - \Delta t^2 k \mathbf{G}^T \mathbf{d}) = 0$$

or

$$(M + \Delta t^2 k G^T G)\mathbf{y} = (M\mathbf{b} - \Delta t^2 k \mathbf{G}^T \mathbf{d})$$

You can solve this linear system using the Cholesky Solver in Eigen

Local-Global Solvers for Mass-Spring Systems

WHILE Not done

//Local Steps

For Each Spring

$$E_{ij} = \arg\min_{\mathbf{d}_{ij}, |\mathbf{d}_{ij}|=r_{ij}} \frac{k}{2}\|\mathbf{y}_i - \mathbf{y_j} - \mathbf{d}_{ij}\|^2$$

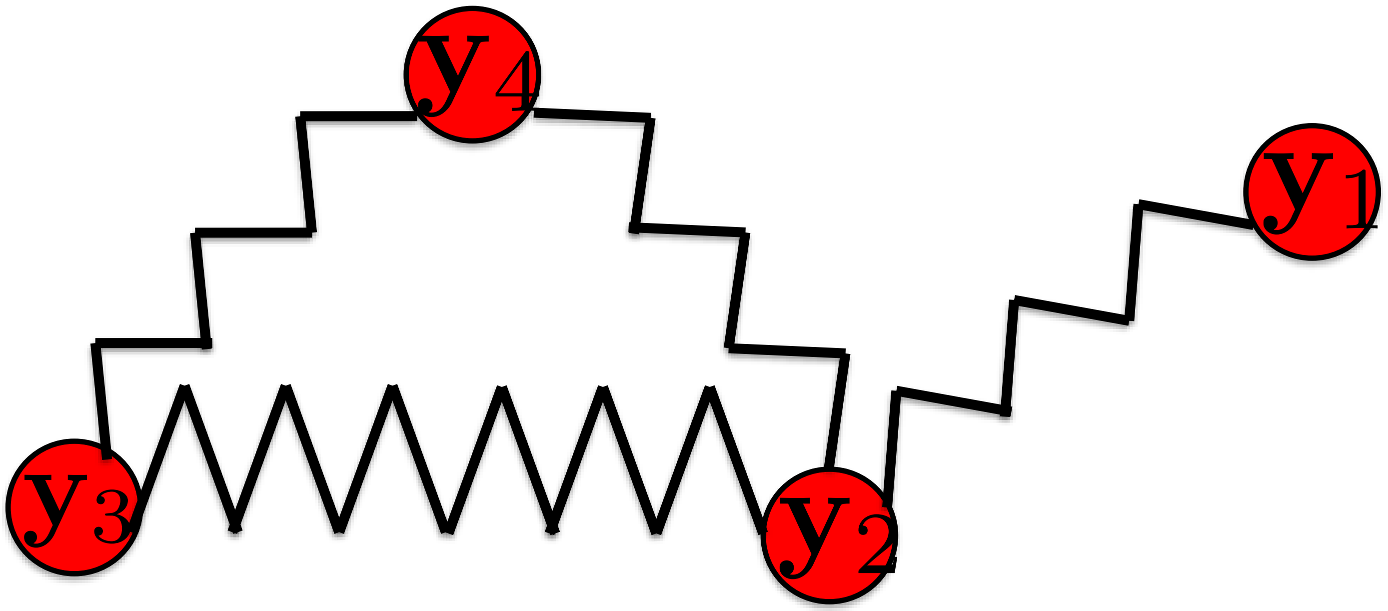
//Global Step

Solve $(M + \Delta t^2 k G^T G)\mathbf{y} = (M\mathbf{b} + \Delta t^2 k \mathbf{G}^T \mathbf{d})$

END


libigl viewer

Fixed Points

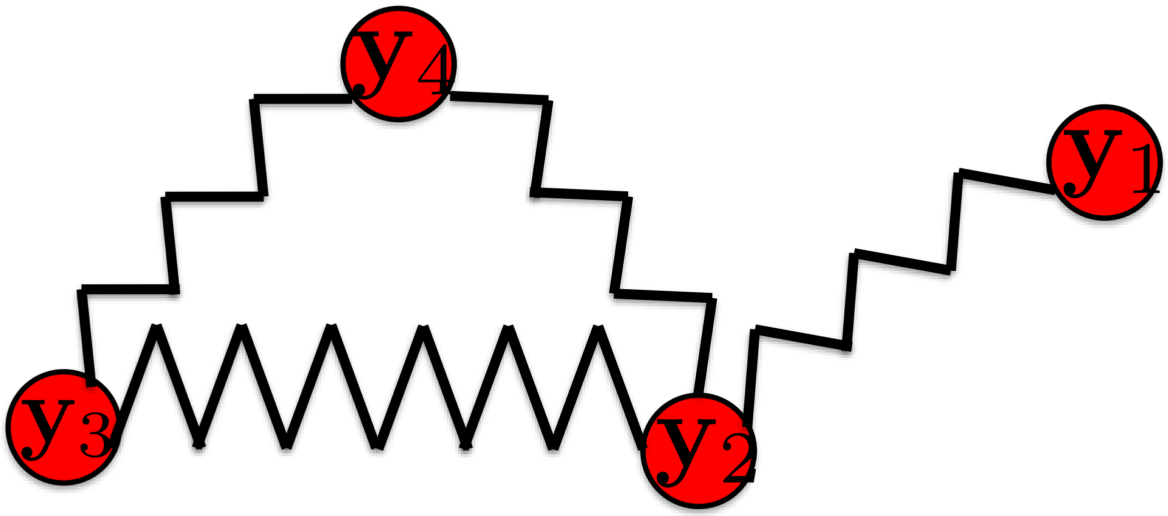


$$\mathbf{y} = \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \mathbf{y}_3 \\ \mathbf{y}_4 \end{pmatrix}$$

Let's say we never want $\mathbf{y}_3$ to move:

i.e $\mathbf{y}_3 = \mathbf{c}$ forever and always

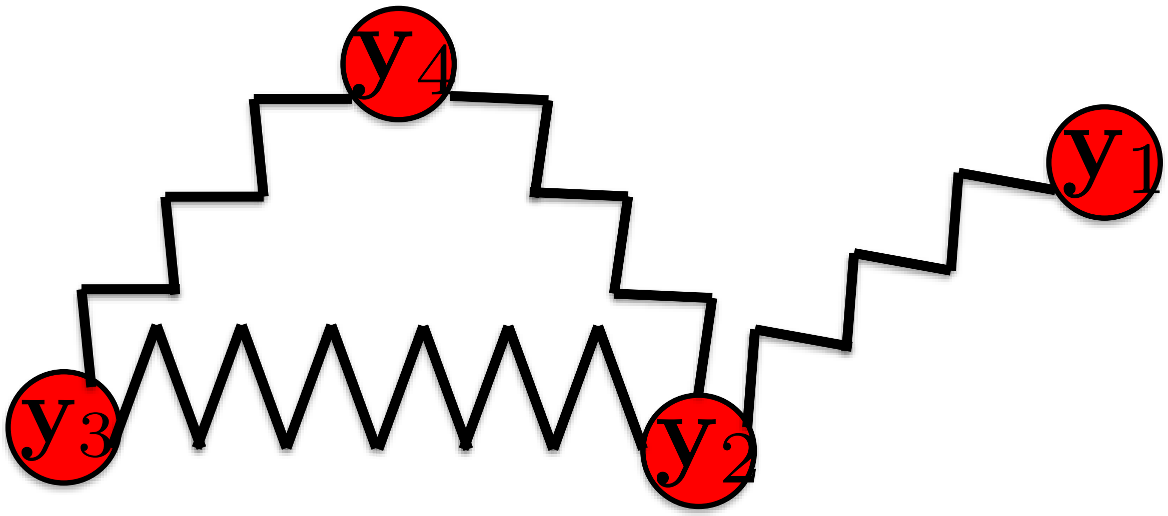
Fixed Points via Projection



$$\mathbf{y} = \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \mathbf{y}_3 \\ \mathbf{y}_4 \end{pmatrix}$$

$$\mathbf{y} = \underbrace{\begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{pmatrix}}_{\mathrm{P}} \underbrace{\begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \mathbf{y}_4 \end{pmatrix}}_{\tilde{\mathbf{y}}} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ \mathbf{c} \\ 0 \end{pmatrix}}_{\mathrm{c}}$$

Fixed Points via Projection



$$(M + \Delta t^2 k G^T G)\mathbf{y} = (M\mathbf{b} - \Delta t^2 k \mathbf{G}^T \mathbf{d})$$

Substituting $\mathbf{y} = P\tilde{\mathbf{y}} + \mathbf{c}$ in $A\mathbf{y} = \mathbf{f}$

Too many rows now … $AP\tilde{\mathbf{y}} = \mathbf{f} - A\mathbf{c}$

$$P^T A P\tilde{\mathbf{y}} = P^T(\mathbf{f} - A\mathbf{c})$$

…Rebuild $\mathbf{y}$

Next: AR | VR